Research Article

Bounded Subsets of Smirnov and Privalov Classes on the Upper Half Plane

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Some characterizations of boundedness in $N^*(D)$ and $N^p(D)$ ($1 < p < \infty$) will be described, where $N^*(D)$ denote the Smirnov class and $N^p(D)$ the Privalov class on the upper half plane $D = \{z \in \mathbb{C} | \text{Im} z > 0\}$, respectively.

1. Introduction

Let $U$ and $T$ denote the unit disk and the unit circle in $\mathbb{C}$, respectively. The Privalov class $N^p(U), 1 < p < \infty$, is defined as the set of all holomorphic functions $f$ on $U$, satisfying

$$\sup_{0<\rho<1} \int_T (\log(1 + |f(\rho \zeta)|))^p \, d\sigma(\zeta) < +\infty,$$

where $d\sigma$ denotes normalized Lebesgue measure on $T$. The notion of $N^p(U)$ was introduced by Privalov [1] and has been explored by several authors (see [2–4]). Letting $p = 1$, we have the Nevanlinna class $N(U)$. It is well-known that each function $f$ in $N(U)$ has the nontangential limit $f^*(\zeta) = \lim_{\rho \to 1-} f(\rho \zeta)$ (a.e. $\zeta \in T$) and that $\log(1 + |f|)$ (and, hence, $(\log(1 + |f|))^p$ for $p > 1$) is subharmonic if $f$ is holomorphic. Define a metric

$$d_{N(U)}(f, g) = \int_T (\log(1 + |f^*(\zeta) - g^*(\zeta)|))^p \, d\sigma(\zeta),$$

for $f, g \in N^p(U)$. With the metric $d_{N(U)}(\cdot, \cdot)$ $N^p(U)$ becomes an $F$-algebra [2]. Recall that an $F$-algebra is a topological algebra in which the topology arises from a complete metric.

We denote the Smirnov class by $N_s(U)$, which consists of all holomorphic functions $f$ on $U$ such that $\log(1 + |f(z)|) \leq Q(\phi)(z)$ ($z \in U$) for some $\phi \in L^1(T), \phi \geq 0$, where the right side denotes the Poisson integral of $\phi$ on $U$. It is known that if $f \in N(U), \ f$ belongs to $N_s(U)$ if and only if

$$\lim_{\rho \to 1-} \int_T \log(1 + |f(\rho \zeta)|) \, d\sigma(\zeta) = \int_T \log(1 + |f^*(\zeta)|) \, d\sigma(\zeta).$$

Under the metric $d_{N_s(U)}(f, g) = \int_T (\log(1 + |f^*(\zeta) - g^*(\zeta)|))^p \, d\sigma(\zeta)$ for $f, g \in N_s(U)$, the class is also an $F$-algebra (see [5]).

For $0 < p < \infty$, the class $M^p(U)$ is defined as the set of all holomorphic functions $f$ on $U$ such that

$$\int_T (\log(1 + Mf(\zeta)))^p \, d\sigma(\zeta) < +\infty,$$

where $Mf(\zeta) = \sup_{0<\rho<1} |f(\rho \zeta)|$ is the maximal function. The class $M^1(U)$ was introduced by Kim in [6]. As for $p > 0$, the class was considered in [7, 8]. For $f, g \in M^p(U)$, define a metric

$$d_{M^p(U)}(f, g) = \left\{ \int_T (\log(1 + M(f - g)(\zeta)))^p \, d\sigma(\zeta) \right\}^{\alpha_p},$$

where $\alpha_p = \min(1, p)$. With this metric $M^p(U)$ is also an $F$-algebra (see [9]).

Hindawi
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It is well-known that $H^p(U) \subset N^p(U) \subset M^1(U) \subset N_s(U) \subset N(U)$ ($0 < q \leq \infty$, $p > 1$), where $H^p(U)$ denotes the Hardy space on $U$. Moreover, it is known that $N(U) \subset M^p(U)$ ($0 < p < 1$) [6].

Mochizuki [10] introduced the Nevanlinna class $N_s(D)$ and the Smirnov class $N_s(D)$ on the upper half plane $D = \{z \in \mathbb{C} | \text{Im} z > 0\}$; the class $N_s(D)$ is the set of all holomorphic functions $f$ on $D$ satisfying

$$\sup_{y \to 0} \int_{\mathbb{R}} (\log (1 + |f(x + iy)|)) \, dx < +\infty$$

and $N_s(D)$ the set of all holomorphic functions $f$ on $D$ satisfying log$(1 + |f(z)|) \leq P(\phi)(z)$ ($z \in D$) for some $\phi \in L^1(\mathbb{R})$, $\phi \geq 0$, where the right side denotes the Poisson integral of $\phi$ on $D$. It is well-known that each function $f$ in $N_s(D)$ has the nontangential limit $f^*(x) = \lim_{y \to 0^+} f(x+iy)$ (a.e. $x \in \mathbb{R}$). Let $f \in N_s(D)$. Then $f \in N_s(D)$ if and only if

$$\lim_{y \to 0^+} \int_{\mathbb{R}} \log (1 + |f(x + iy)|) \, dx$$

(see [10]). Moreover, under the metric

$$d_{N_s(D)} (f, g) = \int_{\mathbb{R}} (\log (1 + |f^*(x) - g^*(x)|)) \, dx,$$

the class $N_s(D)$ becomes an $F$-algebra [10].

The class $M^p(D)$ ($0 < p < \infty$) is defined as the set of all holomorphic functions $f$ on $D$ such that

$$\int_{\mathbb{R}} (\log (1 + Mf(x)))^p \, dx < +\infty,$$

where $Mf(x) = \sup_{y \to 0} |f(x + iy)|$. The class $M^p(\mathbb{X})$ with $p = 1$ was introduced by Ganzhula in [11]. As for $p > 1$, Efimov and Subbotin investigated this class [12]. For $f, g \in M^p(D)$, define a metric

$$d_{M^p(D)} (f, g) = \left\{ \int_{\mathbb{R}} (\log (1 + M(f - g)(x)))^p \, dx \right\}^{1/p},$$

(11)

where $\alpha_p = \min(1, p)$. With this metric $M^p(D)$ is also an $F$-algebra (see [11, 12]).

In [13], the class $N^p(D)$ was introduced, analogous to $N^p(\mathbb{X})$; that is, we denote by $N^p(D)$ ($p > 1$) the set of all holomorphic functions $f$ on $D$ such that

$$\sup_{y \to 0} \int_{\mathbb{R}} (\log (1 + |f(x + iy)|))^p \, dx < +\infty.$$

(12)

Each $f \in N^p(D)$ has the nontangential limit $f^*(x)$ for a.e. $x \in \mathbb{R}$, and under the metric,

$$d_{N^p(D)} (f, g) = \left\{ \int_{\mathbb{R}} (\log (1 + |f^*(x) - g^*(x)|))^p \, dx \right\}^{1/p},$$

(13)

the class $N^p(D)$ becomes an $F$-algebra [13].

A subset $L$ of a linear topological space $A$ is said to be bounded if for any neighborhood $U$ of zero in $A$ there exists a real number $\alpha$, $0 < \alpha < 1$, such that $\alpha L = \{\alpha f; f \in L\} \subset U$. Yanagihara characterized bounded subsets of $N_s(U)$ [14]. As for $M^p(D)$ with $p = 1$, Kim described some characterizations of boundedness (see [6]). For $p > 1$, these characterizations were considered by Meštrović [15]. As for $M^p(D)$ with $p = 1$, Ganzhula investigated the properties of boundedness [11] and Efimov characterized bounded subsets of $M^p(D)$ in the case $0 < p < \infty$ [16]. In recent paper [17], the author described bounded subsets of $M^p(U)$ in the case $0 < p < 1$.

In this paper, we consider some characterizations of boundedness in $N_s(D)$ and $N^p(D)$ ($p > 1$).

2. The Results

**Theorem 1.** Let $p > 1$. $L \subset N^p(D)$ is bounded if and only if

(i) there exists a $K < \infty$ such that

$$\int_{\mathbb{R}} (\log (1 + |f^*(x)|))^p \, dx < K$$

(14)

for all $f \in L$;

(ii) for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{E} (\log (1 + |f^*(x)|))^p \, dx < \epsilon, \quad \forall f \in L,$$

(15)

for any measurable set $E \subset \mathbb{R}$ with the Lebesgue measure $|E| < \delta$.

**Proof.** We follow [16, Theorem 1].

**Necessity.** Let $L$ be a bounded subset of $N^p(D)$.

(i) For any number $\eta > 0$ there exists $\alpha > 0$, $0 < \alpha < 1$, such that

$$\int_{\mathbb{R}} (\log (1 + |f(x)|))^p \, dx < \eta^p$$

(16)

for all $f \in L$. Utilizing the inequality $(1 + x)\alpha \leq 1 + \alpha x$ ($0 < \alpha < 1, \ x \geq 0$), it follows that, from (16),

$$\int_{\mathbb{R}} (\log (1 + |f^*(x)|))^p \, dx$$

$$\leq \int_{\mathbb{R}} \left( \log (1 + \alpha |f^*(x)|)^{1/\alpha} \right)^p \, dx$$

(17)

$$= \frac{1}{\alpha^p} \int_{\mathbb{R}} (\log (1 + \alpha |f^*(x)|))^p \, dx < \left( \frac{n}{\alpha} \right)^p = K$$

$$= \text{constant}$$

for all $f \in L$. Therefore, condition (i) holds.

(ii) For any number $\epsilon > 0$, we take $\eta = \epsilon^{1/p}/2$. Choose a number $\alpha = \alpha(\epsilon)$, $0 < \alpha < 1$, such that equality (16) holds for all $f \in L$. Then for any measurable set $E \subset \mathbb{R}$, using Minkowski's inequality, we have the estimate

$$\int_{E} (\log (1 + |f^*(x)|))^p \, dx$$

$$< \int_{E} \left( \log \left( \frac{1}{\alpha} + |f^*(x)| \right) \right)^p \, dx$$

$$< \int_{E} \left( \log \left( \frac{1}{\alpha} + |f^*(x)| \right) \right)^p \, dx$$
If we take \( \delta > 0 \) as \( \delta < \varepsilon / (2^p(\log(1/\alpha))^p) \), then
\[
\int_E (\log(1 + |f^*(x)|))^p \, dx < \left( \frac{\varepsilon^{1/p} - \varepsilon/2}{2} \right)^p = \varepsilon
\]
for all \( f \in L \) and any measurable set \( E \subset \mathbb{R} \), \( |E| < \delta \). Thus condition (ii) holds.

**Sufficiency.** Let conditions (i) and (ii) hold for a subset \( L \) of \( N_\mathbb{R}^P(D) \), \( p > 1 \). Consider a neighborhood
\[
V = \{ g \in N_\mathbb{R}^P (D) : d_{N(D)}(g, 0) < \eta \}.
\]
Take \( \varepsilon > 0 \) as \( \varepsilon < \eta^p/3 \). According to (ii), there exists a number \( \delta > 0 \) such that
\[
\int_E (\log(1 + |f^*(x)|))^p \, dx < \varepsilon < \frac{\eta^p}{3}
\]
for all \( f \in L \) and any measurable set \( E \subset \mathbb{R} \), \( |E| < \delta \). Next there exists a finite constant \( K > 0 \) such that condition (i) holds for all \( f \in L \). Applying Chebyshev’s inequality to the Lebesgue measure of the set \( \mathcal{E}_f = \{ x \in \mathbb{R} \mid (\log(1 + |f^*(x)|))^p > K/\delta \} \) for \( f \in L \), the following estimate is valid:
\[
|\mathcal{E}_f| \leq \frac{\delta}{K} \int_\mathbb{R} (\log(1 + |f^*(x)|))^p \, dx < \delta.
\]
Then we may assume \( E = \mathcal{E}_f \) and \( f^*(x) > \exp(K/\delta)^{1/p} - 1 = C \) in inequality (24); that is, \( |f^*(x)|/C < 1 \) for all \( x \in \mathbb{R} \setminus \mathcal{E}_f \). Therefore, for any number \( \alpha \) (0 < \( \alpha < 1 \)) and all \( f \in L \), we have the following:
\[
\int_\mathbb{R} (\log(1 + \alpha |f^*(x)|))^p \, dx
\]
\[
= \int_{\mathcal{E}_f} (\log(1 + \alpha |f^*(x)|))^p \, dx
\]
\[
+ \int_{\mathbb{R} \setminus \mathcal{E}_f} (\log(1 + \alpha |f^*(x)|))^p \, dx
\]
\[
< \int_{\mathcal{E}_f} (\log(1 + |f^*(x)|))^p \, dx
\]
\[
+ \int_{\mathcal{E}_f} (\log(1 + \alpha |f^*(x)|))^p \, dx
\]
\[
+ \int_{\mathbb{R} \setminus \mathcal{E}_f} (\log(1 + \alpha |f^*(x)|))^p \, dx,
\]
where \( \mathbb{R} |E| = E_1 \cup E_2, E_1 = \{ x \in \mathbb{R} \mid |f^*(x)| < 1 \}, \) and \( E_2 = \{ x \in \mathbb{R} \mid 1 \leq |f^*(x)| < C \} \). By using the elementary inequality \( 1 + \beta t \leq (1 + t)^{2\beta} \) (0 \( \leq t < 1 \), 0 \( < \beta < 1/2 \)) to the second integral in (23), using (21) and taking
\[
\alpha = \min \left( \frac{1}{2}, \frac{1}{2}, \frac{\eta^p}{3K}, \frac{1}{C} \left( \frac{2^{p/2}(1 + \alpha)}{3K} - 1 \right) \right),
\]
we have the following estimate
\[
\int_\mathbb{R} (\log(1 + \alpha |f^*(x)|))^p \, dx
\]
\[
< \frac{\eta^p}{3} + (2\alpha)^p K + \frac{\eta^p}{3K} \int_{E_1} (\log(1 + 1))^p \, dx
\]
\[
\leq \frac{\eta^p}{3} + \frac{\eta^p}{3}
\]
\[
\int_\mathbb{R} (\log(1 + |f^*(x)|))^p \, dx < \eta^p.
\]
Therefore, \( \alpha L \subset V \) and the set \( L \) is bounded in \( N_\mathbb{R}^P (D) \) by definition.

The proof of the theorem is complete. \( \square \)

Next we consider some characterizations of boundedness in \( N_\mathbb{R}^P (D) \). Proof of the following theorem can be obtained by taking \( p = 1 \) in the whole proof of Theorem 1; therefore, this proof may be omitted.

**Theorem 2.** \( L \subset N_\mathbb{R}^P (D) \) is bounded if and only if
(i) there exists \( K < \infty \) such that
\[
\int_\mathbb{R} (\log(1 + f^*(x))) \, dx < K
\]
for all \( f \in L \);
(ii) for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
\int_E (\log(1 + f^*(x))) \, dx < \varepsilon, \quad \forall f \in L,
\]
for any measurable set \( E \subset \mathbb{R} \) with the Lebesgue measure \( |E| < \delta \).

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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