Research Article

The Configuration Space of $n$-Tuples of Equiangular Unit Vectors for $n = 3$, 4, and 5

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Abstract

Let $M_n(\theta)$ be the configuration space of $n$-tuples of unit vectors in $\mathbb{R}^3$ such that all interior angles are $\theta$. The space $M_n(\theta)$ is an $(n-3)$-dimensional space. This paper determines the topological type of $M_n(\theta)$ for $n = 3, 4, \text{and} 5$. A definition is given as follows. For $n$ considered by many authors. In the equilateral case, the space of spatial polygons of arbitrary edge lengths has been recently starting in [1], the topology of the configuration space of a polygon; the group $SO(3)$ acts diagonally on $(a_1, \ldots, a_n)$. Many topological properties of $P_n(\ell)$ are already known: First, it is clear that there is a homeomorphism

$$P_n(\ell) \cong P_n(1) \quad \forall \ell.$$

Second, it is proved in [2] that $P_3(1)$ is homeomorphic to the Pezzo surface of degree 5.

Third, when $n$ is odd, the integral cohomology ring $H^*(P_n(1); \mathbb{Z})$ was determined in [3]. We refer to [4] for other properties of $P_n(1)$, which is an excellent survey of linkages.

In another direction, we consider the space of $n$-tuples of equiangular unit vectors in $\mathbb{R}^3$. More precisely, we define the following: We fix $\theta \in [0, \pi]$ and set

$$A_n(\theta) = \{(a_1, \ldots, a_n) \in (S^2)^n \mid \langle a_i, a_{i+1} \rangle = \cos \theta \quad \text{for} \quad 1 \leq i \leq n-1, \quad \langle a_n, a_1 \rangle = \cos \theta\},$$

where $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^3$. Using (3), we define

$$M_n(\theta) = \frac{A_n(\theta)}{SO(3)}.$$  (4)

It is expected that the space $M_n(\theta)$ is much more difficult than $P_n(\ell)$. For example, the following trivial observation shows that $M_n(\theta)$ does not admit a similar property to (2): when $n$ is odd, we have $M_n(0) = \{ \text{one point} \}$ and $M_n(\pi) = \emptyset$. We claim that $M_5(\theta)$ is a hypersurface of the torus $T^{n-2}$. In fact, if we forget the condition $\langle a_n, a_1 \rangle = \cos \theta$ in (3), the space corresponding to (4) is $T^{n-2}$ as observed in [5, 6]. Hence the claim follows.

We recall previous results on $M_n(\theta)$. First, [7] considered the case for $\theta = \pi/2$. The main result is that, realizing $M_n(\pi/2)$ as a homotopy colimit of a diagram involving $M_{n-2}(\pi/2)$ and $M_{n-1}(\pi/2)$, we inductively computed $\chi(M_n(\pi/2))$. In particular, we obtained a homeomorphism $M_3(\pi/2) \cong \Sigma_3$, where $\Sigma_3$ denotes a connected closed orientable surface of genus 5.

Second, we set

$$X_n(\theta) = P_n(1) \cap M_n(\theta) .$$  (5)

Note that $X_n(\theta)$ is the configuration space of equilateral and equiangular $n$-gons. Crippen [8] studied the topological type of $X_n(\theta)$ for $n = 3, 4, \text{and} 5$. The result is that $X_3(\theta)$ is either $\emptyset$, one point, or two points depending on $\theta$. Later, O’Hara [9] studied the topological type of $X_5(\theta)$. The result is that $X_5(\theta)$ is disjoint union of a certain number of $S^1$'s and points.
The purpose of this paper is to determine the topological type of $M_n(\theta)$ for $n = 3, 4,$ and $5$. In contrast to the fact that at most one-dimensional spaces appear in the results of $[8, 9]$, surfaces appear in our results.

This paper is organized as follows. In Section 2, we state our main results and in Section 3 we prove them.

2. Main Results

Theorem A. The topological type of $M_3(\theta)$ is given in Table 1.

Table 1: The topological type of $M_3(\theta)$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Topological type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2\pi/3 &lt; \theta \leq \pi$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>one point</td>
</tr>
<tr>
<td>$0 &lt; \theta &lt; 2\pi/3$</td>
<td>two points</td>
</tr>
<tr>
<td>$0$</td>
<td>one point</td>
</tr>
</tbody>
</table>

Theorem B. (i) The topological type of $M_4(\theta)$ is given in Table 2.

(ii) As $\theta$ approaches $\pi/2$, point A in Figure 1(a) approaches point B.

Table 2: The topological type of $M_4(\theta)$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Topological type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/2 &lt; \theta &lt; \pi$</td>
<td>Figure 1(a)</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>Figure 1(b)</td>
</tr>
<tr>
<td>$0 &lt; \theta &lt; \pi/2$</td>
<td>Figure 1(a)</td>
</tr>
<tr>
<td>$0$</td>
<td>one point</td>
</tr>
</tbody>
</table>

Theorem C. (i) The topological type of $M_5(\theta)$ is given in Table 3. Let $\Sigma_g$ be a connected closed orientable surface of genus $g$.

(ii) (a) Let $\theta$ satisfy that $2\pi/5 < \theta < 2\pi/3$. We study the situation where $\theta$ approaches $2\pi/3$. We identify the torus $\Sigma_1$ with the Dupin cyclide, which we denote by $D$. (See Figure 2.) Using this, we identify $\Sigma_5$ with $\#_3 D$, where the connected sum is formed by cutting a small circular hole away from the narrow part of $D$. As $\theta$ approaches $2\pi/3$, the center of each narrow part pinches to a point. Thus the five singular points appear.

(b) We consider the situation where $\theta$ increases from $2\pi/3$. Then each pinched point of $M_5(2\pi/3)$ separates. Thus we obtain $S^2$.

(c) Let $\theta$ satisfy that $2\pi/5 < \theta < 2\pi/3$. We consider the situation where $\theta$ approaches $2\pi/5$. In contrast to (a), the center of exactly one narrow part pinches to a point. Thus one singular point appears.

Corollary D. As a subspace of $(S^2)^5 \times [0, \pi)/SO(3)$, we define the space

$$Y = \bigcup_{\theta \in [0, \pi]} M_5(\theta).$$

Then $M_5(0)$ is a singular point of $Y$ and has a neighborhood $C \Sigma_4$, where $C$ denotes the cone.

Remark 1. Cone-type singularities appear in Theorems B and C and Corollary D. We note that singularities of configuration spaces of mechanical linkages have been studied extensively by Blanc and Shvalb [10].

3. Proofs of the Main Results

We fix $\theta \in [0, \pi]$ and set

$$e_1 = (1, 0, 0),$$

$$p = (\cos \theta, \sin \theta, 0).$$

Normalizing $a_1$ and $a_2$ to be $e_1$ and $p$, respectively, we have the following description:

$$M_n(\theta) = \{(a_1, \ldots, a_n) \in (S^2)^n \mid a_i = e_1, a_2 = p, \langle a_i, a_{i+1} \rangle = \cos \theta \text{ for } 2 \leq i \leq n-1, \langle a_n, a_1 \rangle = \cos \theta \}.\tag{8}$$

Hereafter we use (8).

In order to prove our main results, we use the following fact, whose proof is left to the reader.

Fact 2. Let $(\alpha, \beta, \gamma) \in (S^2)^3$ satisfy that

$$\langle \alpha, \beta \rangle = 0,$$

$$\langle \alpha, \gamma \rangle = \cos \theta.$$ (9)

Then, there exists $\phi \in \mathbb{R}$ such that

$$y = (\cos \theta) \alpha + (\sin \theta \cos \phi) \beta + (\sin \theta \sin \phi) (\alpha \times \beta).\tag{10}$$

Now we first consider the case $n = 5$. Consider Fact 2 for $\alpha = p$, $\beta = (-\sin \theta, \cos \theta, 0)$, and $\gamma = a_5$. Then there exists $x \in \mathbb{R}$ such that

$$a_5 = (\cos \theta) p + (\sin \theta \cos x) (-\sin \theta, \cos \theta, 0) + (\sin \theta \sin x) (0, 0, 1).\tag{11}$$

Next, we consider Fact 2 for $\alpha = e_1$, $\beta = (0, 1, 0)$, and $\gamma = a_5$. Then there exists $z \in \mathbb{R}$ such that

$$a_5 = (\cos \theta, \sin \theta \cos z, \sin \theta \sin z).\tag{12}$$

Finally, we consider Fact 2 for $\alpha = a_4$ in (12),

$$\beta = (-\sin \theta, \cos \theta \cos z, \cos \theta \sin z),\tag{13}$$

and $\gamma = a_5$. Then there exists $y \in \mathbb{R}$ such that

$$a_4 = (\cos \theta) \alpha + (\sin \theta \cos y) \beta + (\sin \theta \sin y) (\alpha \times \beta).\tag{14}$$

Now we define the function $f : (\mathbb{R}/2\pi \mathbb{Z})^3 \times [0, \pi] \to \mathbb{R}$ by

$$f(x, y, z, \theta) = (\langle 11, 14 \rangle - \cos \theta).\tag{15}$$
We can understand \( M_5(\theta) \) as a level set. More precisely, we define the function
\[
h : f^{-1}(0) \to \mathbb{R}
\]
by \( h(x, y, z, \theta) = \theta \). Then we have
\[
M_5(\theta) = h^{-1}(\theta)
\]
if \( 0 < \theta \leq \pi \).

**Remark 3.** Since \( f(x, y, z, 0) = 0 \) for all \( x, y, \) and \( z \), we have \( h^{-1}(0) = (\mathbb{R}/2\pi\mathbb{Z})^3 \). On the other hand, it is clear that \( M_5(0) = \{ \text{one point} \} \). Hence (17) does not hold for \( \theta = 0 \). Apart from this point, there is an identification
\[
Y \setminus M_5(0) = f^{-1}(0) \setminus h^{-1}(0),
\]
where \( Y \) is defined in (6).

**Lemma 4.** We set
\[
S := \left\{ (x, y, z, \theta) \in \left( \mathbb{R}/2\pi\mathbb{Z} \right)^3 \times (0, \pi] \mid f(x, y, z, \theta) = 0, \left( \frac{\partial f}{\partial x}(x, y, z, \theta), \frac{\partial f}{\partial y}(x, y, z, \theta), \frac{\partial f}{\partial z}(x, y, z, \theta) \right) = (0, 0, 0) \right\}.
\]
Then \( S \) is given in Table 4.

**Proof.** The lemma is proved by direct computations. \( \square \)

**Proof of Theorem C.** We consider \( h \) in (16) as a Morse function on \( f^{-1}(0) \). First, Table 4 and (17) show that \( M_5(4\pi/5) = \{ \text{one point} \} \).
Second, direct computation shows that
\[
\frac{\partial f}{\partial \theta}(0, 0, \pi, \frac{4\pi}{5}) = \frac{5}{2} \sqrt{\frac{5 - \sqrt{5}}{2}}.
\]
Since this is nonzero, the space \( f^{-1}(0) \) is smooth at \( (0, 0, \pi, 4\pi/5) \). Actually, we can prove that the point is a nondegenerate critical point of the function \( h \). Hence Morse lemma shows that there is a homeomorphism \( M_5(\theta) \cong S^2 \) for \( 2\pi/3 < \theta < 4\pi/5 \). But if we use [11, Corollary B], we need not check that \( h \) is nondegenerate at \( (0, 0, \pi, 4\pi/5) \). For our reference, we draw the figure of \( M_5(4\pi/5 - 0.1) \) in Figure 3.

Third, the other parts of Table 3 follow from Table 4. This completes the proof of Theorem C. \( \square \)

**Proof of Corollary D.** The corollary is an immediate consequence of Theorem C. \( \square \)
Figure 3: $M_5(4\pi/5 - 0.1)$.

Table 3: The topological type of $M_5(\theta)$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Topological type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4\pi/5 &lt; \theta \leq \pi$</td>
<td>$\varnothing$</td>
</tr>
<tr>
<td>$4\pi/5$</td>
<td>{one point}</td>
</tr>
<tr>
<td>$2\pi/3 &lt; \theta &lt; 4\pi/5$</td>
<td>$S^2$</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>Contains five singular points</td>
</tr>
<tr>
<td>$2\pi/5 &lt; \theta &lt; 2\pi/3$</td>
<td>$\Sigma_5$</td>
</tr>
<tr>
<td>$2\pi/5$</td>
<td>Contains one singular point</td>
</tr>
<tr>
<td>$0 &lt; \theta &lt; 2\pi/5$</td>
<td>$\Sigma_4$</td>
</tr>
<tr>
<td>$0$</td>
<td>{one point}</td>
</tr>
</tbody>
</table>

Table 4: The set $S$.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$(x, y, z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4\pi/5$</td>
<td>$(0, 0, \pi)$</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>$(\pi, 0, 0), (0, \pi, 0), (0, 0, \pi), (\pi, 0, \pi), (0, \pi, \pi)$</td>
</tr>
<tr>
<td>$2\pi/5$</td>
<td>$(0, 0, \pi)$</td>
</tr>
</tbody>
</table>

Proof of Theorem B. We define $a_3$ as in (11). We also define $a_4$ to be the right-hand side of (12). We define the function $f : (\mathbb{R}/2\pi \mathbb{Z})^2 \times [0, \pi] \to \mathbb{R}$ by

$$f(x, z, \theta) = \langle a_3, a_4 \rangle - \cos \theta. \quad (21)$$

Similarly to (17), we have $M_4(\theta) = h^{-1}(\theta)$. Since $h^{-1}(\theta)$ is one-dimensional, it is easy to draw its figure. Thus Theorem B follows.

Proof of Theorem A. We define the function $f : (\mathbb{R}/2\pi \mathbb{Z}) \times [0, \pi] \to \mathbb{R}$ by $f(x, \theta) = \langle a_3, e_1 \rangle - \cos \theta$. Since $M_3(\theta) = h^{-1}(\theta)$, Theorem A follows.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References


