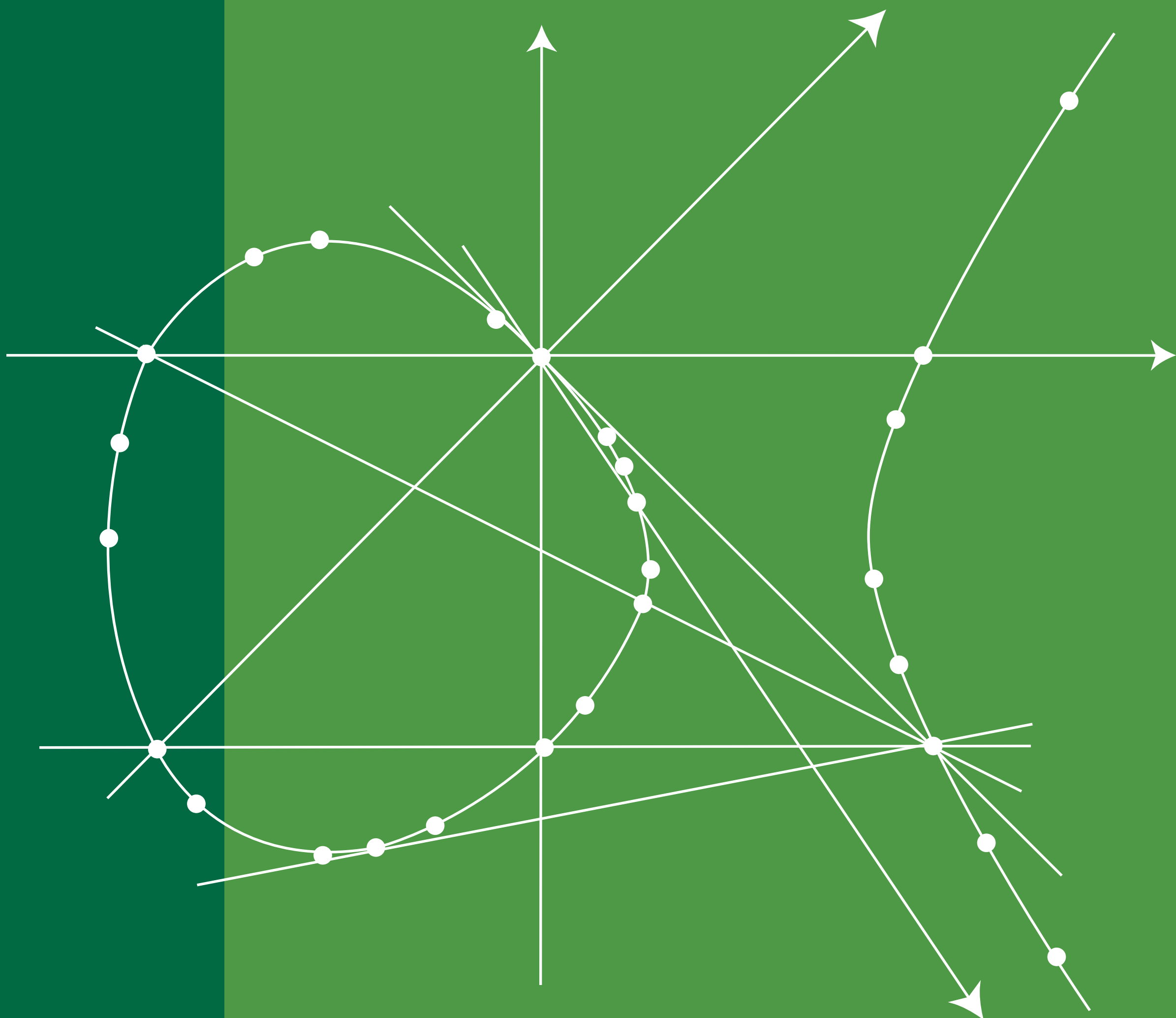
European Women in Mathematics

Proceedings of the eighth general meeting

International Centre for Theoretical Physics
Trieste, Italy, 12–17 December 1997



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PREFACE

During the meeting of European Women in Mathematics in Madrid 1995 it was decided to organise the next general meeting of EWM in 1997, continuing the bi-annual rhythm, but without knowing where the meeting actually would take place. We are extremely grateful to Professor Narasimhan, director of the Mathematics Group at the International Centre of Theoretical Physics in Trieste, for accepting to house the meeting and to have it organised in collaboration with ICTP.

The Centre has a long and fruitful tradition for encouraging people—from developing countries in particular—to take up or continue a research career in mathematics or physics. The Centre provides wonderful working conditions and organises international meetings.

With the help of ICTP the information about the EWM meeting was widely distributed and we reached out to many more women mathematicians around the world than ever before. Besides the usual channels for information—the regional coordinators, the EWM web-page, the EWM Newsletter and the EWM e-mail network—a poster was produced at ICTP and the announcement was mailed to about 1500 addresses around the world and also put on the ICTP web-page.

The interest in the meeting was overwhelming. More than 150 mathematicians applied to participate. The meeting was attended by about 100 participants from about 30 countries, most European countries were represented as well as Chile, Egypt, India, Iran, Kyrgyzstan, Nepal, Tunisia, Uzbekistan and West Bank. The total list of mathematical fields represented was also quite impressive. The research topics and fields of interests of the participants covered mathematics in the broadest sense, from pure mathematics to all kinds of applied mathematics, history of mathematics and didactic methods.

Over several years we have experimented with improvements in ways of communicating mathematics to each other at EWM meetings. As a result we have sessions with different topics, starting with an introductory talk. It is a big challenge to organise a session and to be a speaker addressing such a broad audience. At the Trieste meeting one of the sessions was on *p-adic Numbers*, organised by Catherine Goldstein, another one on *Representation of Groups*, organised by Michele Vergne, and the third one an interdisciplinary session on *Symmetries*, organised by Ina Kersten and Sylvie Paycha. While the speakers of the two first sessions had all been invited ahead of time, only two speakers were invited for the last, the extra activities that were decided during the meeting involved inputs from participants and thereby broadened the interdisciplinary aspect.

For the first time a *Poster Session* was included in an EWM meeting. It was organised by Capi Corrales and Laura Fainsilber who encouraged the participants to make non traditional posters, also including some personal information. Indeed, the poster session became a very colourful and important part of the conference. Besides the mathematical activities an EWM meeting usually contains a topic for general discussion. At this meeting the topic was *Women and Mathematics: East-West-North-South*, organised by Marjatta Näätänen and Marie Demlova. As an introduction the video *Women and Mathematics across Cultures* was shown. This video includes four interviews that were filmed during the previous EWM meeting in Madrid.

It is always difficult to obtain sufficient funding. We did not obtain enough to support all of those who were dependent on financial support from outside and who could not participate without. We are grateful for the support we received from the European Union, the ICTP, UNESCO, University of Trieste, University of Gothenburg and Chalmers Technical University, the European Mathematical Society, and a private donation of Else Hoeyrup. ICTP offered to fully support two women mathematicians (later on this became three) coming from developing countries to take part in the EWM meeting and furthermore to stay as visitors at the Centre for a period of two months. We received more than 50 applications for these special stipends. It was extremely difficult to choose among the many well qualified women. Besides the three chosen who spent two months at the Centre, three other participants benefitted from joint agreements between their university and ICTP. Of great value was of course the general support of ICTP by letting us use their facilities: conference and meeting rooms, library, computers, photocopying equipment, and guest houses. Moreover, we had the precious help of two of its staff members; Sharon Laurenti collaborated with us until October when Livia Zetto took over, she became responsible for most of the local organisation.

In our experience it is a different task to organise an EWM meeting than to organise any other mathematical conference. From the many letters and applications we received and from our personal experience we know that EWM can make a difference. On top of the mathematical concerns and practical matters it also becomes a more personal project to make the meeting a success. We shall never forget how Livia Zetto participated whole-hearted in our goals of reaching out and making the meeting a success. The intensive collaboration between the three of us over the last couple of months before the meeting took place worked out very well. Only when it was over we had time to reflect on how special this collaboration had been to all of us.

In our opinion, the meeting was a success, both from a mathematical and a non-mathematical point of view. We wish to thank all the organisers, the speakers, the participants and the ICTP staff for letting this happen.

Bodil Branner and Emilia Mezzetti.

The complete EWM organising committee of the Trieste meeting consisted of Christine Bessenrodt (Germany), Bodil Branner (Denmark), Marie Demlova (Czech Republic), Emilia Mezetti (Italy), Rosa-Maria Miro Roig (Spain), Marjatta Näätänen (Finland), Sylvie Paycha (France), Ragni Piene (Norway), Caroline Series (United Kingdom), Inna Yemelyanova (Russia).

These proceedings contain reports on all the mathematical talks that were held at the meeting, as well as articles on the other events: the poster session, the EWM video, the discussion on women in mathematics, and information about the life and structure of the association. They are available both on paper and in electronic form, via the web page of EWM:

http://www.math.helsinki.fi/EWM/

or of the Electronic Publishing House

http://math.hindawi.com/ewm-97

We wish to thank the organisers and all those who gave talks in Trieste and wrote articles for these proceedings, as well as those who worked on the Proceedings of the 1995 EWM meeting in Madrid, namely Bodil Branner, Núria Fagella, and Christian Mannes, for providing us with an inspiring precedent and with a TeX style.

The editors, Laura Fainsilber and Catherine Hobbs

EWM and the 8th General Meeting

EUROPEAN WOMEN IN MATHEMATICS

EWM is an affiliation of women bound by a common interest in the position of women in mathematics. Our purposes are:

- To encourage women to take up and continue their studies in mathematics.
- To support women with or desiring careers in research in mathematics or mathematics related fields.
- To provide a meeting place for these women.
- To foster international scientific communication among women and men in the mathematical community.
- To cooperate with groups and organizations, in Europe and elsewhere, with similar goals.

Our organization was conceived at the International Congress of Mathematicians in Berkeley, August 1986, as a result of a panel discussion organized by the Association for Women in Mathematics, in which several European women mathematicians took part. There have since been seven European meetings: in Paris (1986), in Copenhagen (1987), in Warwick (England) (1988), in Lisbon (1990), in Marseilles (1991), in Warsaw (1993), in Madrid (1995) and in Trieste (1997). The next meeting will be in 1999 in Hannover.

At the time of writing, there are participating members in the following countries: Belgium, Bulgaria, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Italy, Latvia, Lithuania, Malta, the Netherlands, Norway, Poland, Portugal, Romania, Russia, Spain, Sweden, Switzerland, Turkey, Ukraine, and the United Kingdom; contacts in Albania, Brazil, Chile, Egypt, India, Iran, Khirghistan, Nepal, Tunisia, Uzbekistan, the West Bank. Activities and publicity within each country are organized by regional co-ordinators. Each country or region is free to form its own regional or national organization, taking whatever organizational or legal form is appropriate to the local circumstances. Such an organization, Femmes et Mathematiques, already exists in France. Other members are encouraged to consider the possibility of forming such local, regional or national groups themselves.

There is also an e-mail network and a web page:

http://www.math.helsinki.fi/EWM,

where you will find this report as well as the proceedings of the previous general meeting in Madrid in 95, the yearly Newsletters, access to a bibliography on women mathematicians, and more. To subscribe to the *ewm-all* e-mail network send the following command (typing your own personal names instead of firstname(s) and lastname): *join ewm-all firstname(s) lastname* as the only text in the body of a message addressed to:

mailbase@mailbase.ac.uk.

You will then receive confirmation of your subscription.

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May, 1998.

OPENING WORDS OF PROFESSOR NARASIMHAN

Director of ICTP

On behalf of the International Centre for Theoretical Physics, I would like to welcome all the participants in the 8th General Meeting of European Women in Mathematics. The ICTP is particularly happy to host this activity and be associated with it.

Since for many of you this is the first visit to ICTP, I would like to give some information about the activities in Mathematics at the ICTP. As you may be aware, the main function of the ICTP is to foster the development of science in Third World countries and to help the scientists in these countries who work under various handicaps, like isolation.

The Mathematics Section of the ICTP organises 3 to 4 conferences/schools each year in fields of Mathematics cultivated in the developing countries. We have 300 to 400 participants in these conferences/schools.

The other major activity of the Mathematics Section is the visiting programme for research. Each year a field of Mathematics is chosen as a theme for emphasis, and around half the number of visitors would be in the field of emphasis. At the same time we invite 3 to 4 established mathematicians in this field to visit the Centre during the year, so that the younger mathematicians staying here can interact with them and profit by it. We have approximately 100 visitors each year under this programme.

Each year, a generous contribution from the Commission of the European Communities—within the framework of the “Training and Mobility of Researchers: Euroconferences” programme—makes it possible to provide financial support for young mathematicians from European countries to participate in the conferences in Mathematics organized by ICTP. This has been mutually beneficial for mathematicians from these countries and those from the developing countries. In addition to visitors from developing countries, we have also a number of mathematicians coming to the Centre from Europe. This year the Mathematics Section had 151 visitors from Europe.

I would particularly like to thank the staff of the Mathematics Section for all the committed help they have given in organizing the Meeting.

I wish you a pleasant stay in Trieste and a fruitful meeting.

MEDIA COVERAGE OF THE EWM TRIESTE MEETING

Transcripts of press cuttings about the Meeting

The 8th General Meeting of EWM in Trieste was mentioned in the Italian press and in the ICTP's own news service. The following articles are transcripts of some of the articles which appeared.

"We Are Women, We Can Count," from ICTP Monthly update of activities and events, December 1997

The 8th General Meeting of European Women in Mathematics will be held at the Centre between 12–16 December. More than 150 mathematicians are expected to attend, including three men. The meeting, which takes place every other year, is being co-sponsored by ICTP, European Mathematical Society, UNESCO Venice Office for Science and Technology in Europe, Universita di Trieste and Ent regionale per il diretto all studio universitario. Previous meetings of European Women in Mathematics have been held in Madrid, Warsaw, Copenhagen and Paris.

"Centre Hosts Women Mathematicians," from ICTP Monthly update of activities and events, January 1998

Last month, more than 100 women gathered at the ICTP for the 8th General Meeting of European Women in Mathematics (EWM). Among the technical subjects discussed were symmetries, group theories and p-adic numbers. In addition, general and group sessions—as well as a video-focussed on the plight of women in this largely male-dominated discipline. While women mathematicians continue to confront problems of isolation and serious roadblocks to promotion, not all the news is bad. Although the percentage of women mathematicians teaching in many northern European universities—including those in Austria and Germany—remains below 10 percent, the percentage of women mathematicians in many southern European universities—including those in Italy, Portugal and Spain—now ranges between 40 and 50 percent. The next meeting of the EWM will be held in 1999 in Germany.

"Questa è matematica per l'altra metà del cielo," from *Il Piccolo*, Thursday 11th December

Donne e matematica a Trieste. Bel trinomio, è proprio il caso di dirlo. Da domani fino al 16 dicembre, il Centro internazionale di fisica teorica di Miramare ospiterà l'ottavo meeting di European Women in Mathematics.

La scelta della città per l'edizione italiana di questo appuntamento biennale non è casuale. La coordinatrice italiana dell'associazione è, infatti, Emilia Mezzetti, una triestina doc. Nata a Trieste, studi classici al Liceo Dante, si laurea in matematica all'Università della stessa città nel 1973. È ora docente di Geometria presso il Dipartimento di scienze matematiche del medesimo ateneo. L'abbiamo incontrata per farci spiegare cos'è Ewm. "È un'associazione di donne interessate alla situazione femminile nella comunità matematica. Nel 1986, a Berkeley, al Congresso internazionale dei matematici, un'assemblea dell'Association for Women in Mathematics, attiva in america da oltre vent'anni, valutava la presenza delle donne matematiche nel mondo. Lì, alcune europee si confrontarono sulla situazione nel vecchio continente. Fu, in pratica, la nascita di Ewm, sancita ufficialmente ad Helsinki nel 1993."

Perché una società matematica di sole donne? La ricerca di qualità, in ogni disciplina, non prescinde forse dal sesso? "A parte il fatto che gli uomini possono farne parte come 'supporting members,' il problema è-dice-l'enorme divario delle percentuali di donne attive nelle varie comunità matematiche europee. Per esempio, malgrado il successo dei movimenti femministi, pochissime donne nord-europee iniziano una carriera matematica (circa 4 per cento contro il 31–40 per cento in Italia).

"Situazione di quasi parità, invece nei paesi slavi. Di fatto, però, le giovani matematiche nord-europee vedono pochi modelli da imitare e non hanno spesso forza e determinazione sufficienti per continuare la carriera. Non a caso il meeting di Trieste si concluderà con un dibattito sul tema: 'Donne e matematica: Est-Ovest-Nord-Sud.' "

Quante donne parteciperanno? "Più di 150. Circa una trentina italiane. Alcune, giovanissime, ancora indecise se intraprendere la carriere scientifica. Qualche curiosità: une donna arriverà dal Nepal, una dal Kirghizistan e una dall'Uzbekistan. Rimarreno a Trieste per circa due mesi."

WOMEN AND MATHEMATICS: EAST-WEST-NORTH-SOUTH

MARJATTA NÄÄTÄNEN AND MARIE DEMLOVÁ

University of Helsinki, Finland and Czech Technical University, Czech Republic

Introduction. Before the discussion the video “Women and Mathematics across Cultures, EWM—European Women in Mathematics” was shown (a description of which is given in the next article). After the video the participants were divided into seven groups:

- France, Italy, Portugal, Spain and Turkey;
- Russia, Ukraine, Georgia, Romania and Bulgaria;
- Germany and Switzerland;
- Scandinavia, Estonia, Czech Republic, Poland and the Netherlands;
- Great Britain, Malta and Greece;
- non-European countries.

In general discussion suggestions, ideas and experiences were collected. The general discussion was led by Rosa Maria Miro Roig and Marjatta Näätänen.

Outline of the Report.

1. General situation.
2. Suggestions.
3. Situation in non-European countries.
4. Russia.
5. Two short reports (Great Britain, Malta, Greece and Germany, Switzerland).

1. General situation. The problem of women mathematicians turned out to be very similar regardless of culture. The general opinion was that there is no equality between women and men. Women are struggling to meet several fulltime commitments and a lot of expectations from the society.

For women, a friendly supportive atmosphere is important. Women mathematicians are often isolated, hence a supportive network of women is of great importance.

Many countries, for example those of the third world, share lack of resources, problems with communication (e-mail not available or filtered, lack of travelling possibilities and low salaries of scientists). In those countries women mathematicians are even more isolated and family commitments heavier than in western European countries. Another matter of great common concern was the diminishing number of mathematics students—mathematics is loosing its attraction even if its importance in most fields is growing.

As the situation stands outstanding women do manage to arrive at high positions but they have to be of a much higher calibre that their male counterparts.

Because of—rather that in spite of—the fact that we have gone a long way to improve the conditions of female mathematicians we realise that there is still a long way to go.

2. Suggestions. General suggestions that arose form the general discussion in Trieste.

- Women scientists should be visible.
  - Women should be scientifically visible, for example they should be invited to give talks. Several successful attempts were carried out, e.g., in Norway.
  - Women should be visible in the media. In some countries women seem to have a negative image in the media. One problem with the media is that they do not always respect the private life of the interviewed persons.
- Access to scientific information should be guaranteed. This will help women mathematicians to feel less isolated, especially women from the countries with lack of resources. Women should get a fair share of funding.
- Increase the number of women mathematicians in higher positions.
  - Flexible paths for career development are recommended; also the possibility of early permanent jobs seems to support women.
  - Recommended quota to ensure that a fair percentage of qualified women rise at each stage to the next (for example: are nominated professors).
  - Special positions for women; this requires positive atmosphere in public opinion and not too tight a job market.
  - Special lecture series and awards for women.
  - Making it easier for women to obtain part-time employment, even as scientists on jobs with large responsibilities.
  - Pregnancy should be treated on the same level as military service i.e. time spent on maternity leave should not be counted against a woman when it comes to comparing the number of papers she has published with a person who has not had time away from their studies.
  - Financial help for graduate students with children.
  - Try to change attitudes of employees to choose math graduates, try to make mathematics more attractive to students.
- Increase of the number of female students of mathematics.
  - Attitudes of teachers at primary and secondary schools should be changed into encouraging girls to study mathematics and not discouraging them.
  - Organize special meetings, programmes, summer schools for girls.
  - Women scientists could meet with female students to encourage and inspire them. Several successful and unsuccessful attempts were carried out, e.g., in Norway.
  - The mathematical level of primary school teachers is very important.

Attempts that have been tried with success/without success in some countries.

- Special positions for women. It was successfully done in Sweden but requires special conditions: e.g., a good atmosphere in the society, not too tight a job market. The success was sometimes limited by the lack of female candidates.
- Women's counsellor at universities to guarantee that women are treated fairly. This has been introduced in Germany; at some universities with success, in others without success.

- Scholarships for women trying to get back into scientific life, e.g., for women there should be no age limit. In France several women mathematicians have restarted research. In Germany women prefer not to apply for such grants, there is a general fear that having such a grant in a CV makes a bad impression.
- Change of curricula at all levels.
  - University: combining mathematics with other fields like computer science, management, information technology, economics, natural sciences, biology and languages was tried with success in Poland.
  - High school: different streams (for example math and technical science, natural science and biology, economics, culture and languages) all with at least some mathematics, is being tested in the Netherlands and has been successful in Estonia and Poland.
- Early mathematics education with good teachers and good books is advisable. In Ukraina they start with children of age 4.
- Campaigns for girls to choose mathematics and exact sciences. Open hours for girls to visit technical universities. This was tried in several countries with bigger or smaller success.
- Single sex schools seem to encourage girls to better achievements in mathematics, this is an experience obtained in UK.
- Girls prefer to compete either in groups or for example by e-mail (then they are not stressed by time-limits). This is a widely obtained experience.
- Trying to improve the conditions of women via political parties was tried in Greece with some success.

Suggestions for the work of EWM arising from the general discussion.

- EWM should be more active in spreading out information about positions that women can apply for in different countries.
- For the third world countries EWM could give moral support, spread information, for example the Newsletter, and to give contacts.
- EWM should try to create scholarships for women in isolated places.
- EWM can serve to create supportive networks of women.
- EWM should ensure a good proportion of female speakers at conferences when possible.
- EWM could use its influence to promote exchange visits between foreign universities especially for researchers that find they cannot share their interests with their colleagues in their departments.

More information about the situation of women in mathematics in different countries together with statistical data can be found in "Round Table D: Women and Mathematics" edited by Kari Hag, contained in the Proceedings of the European Mathematical Congress, Budapest 1996.

3. The non-European countries: Chile/Brazil, India, Iran, Kyrgystan and Uzbekistan, Nepal, Palestine, Tunisia. The common problems (social, family, etc.) for women with a career all over the world are shared by all. In these discussions additional problems were brought to light that are particular to each country. These problems are not exclusively women's problems, but are rather problems that stand in the way of the development of mathematics in general in these countries. We can sum up the problems as arising from the limited availability of resources, the difficulty in communication and travel and the limited provision made by the institutions and/or goverments for the advancement of education.

The problem arising from the availability of very few resources poses a greater difficulty to women. Few resources can be partially compensated for by better communication, which implies the presence of good means for it and often the need for travel. For those women who have families, we know how difficult travel can be. Men are at an advantage here and hence can develop better in their fields. Moreover in some countries travel is more difficult because of the expense and the difficulties in getting visas.

The limited means of communication makes an additional obstacle in some countries. Also, the use of e-mail can be limited for political reasons. The filtering of incoming e-mail often results in never receiving the messages. In Nepal e-mail services are provided by private companies and its use can be quite costly on the individual. Institutions do not provide e-mail service. These limitations in communication pose an additional difficulty of acquiring information about activities in mathematics around the world.

In what follows is a presentation of the different situations in each country and the conditions of mathematical activity and of women in mathematics there.

Chile/Brazil. The conditions in Chile are rather poor. The situation of Brazil is better but still not a good one. Many women in Brazil tend to study mathematics up to the masters degree and do not continue. Often they end up being teachers in schools. Getting a position at a university with a Ph.D. is not difficult for women but attaining higher positions is.

India. The situation of women in mathematics in India is worsened by the atmosphere in the Indian society. It is considered fair and necessary for a wife to follow her husband and leave her research career. There are examples where a wife was not allowed to defend her Ph.D. thesis because her husband did not approve of it.

Iran. The introduction of a Ph.D. programme in mathematics about ten years ago (after the Iran/Iraq war) has been a very important factor in the development of mathematics in Iran. Women are encouraged to seek a programme of study in mathematics and do not feel discriminated against. Their attaining of higher positions is solely hindered by their commitments to their family. The resources in mathematics are adequate. The presence of a resource center for scientific research in Tehran helps a lot in providing and finding material upon the request of the individual researcher.

Kyrgystan and Uzbekistan. As in other countries of the former Soviet Union the situation is Kyrgystan and Uzbekistan is very difficult for there is a lack of all sorts of resources. Salaries of all scientists are very low, especially of women. There is no possibility of getting literature, no journals are ordered by libraries. The situation of women scientists is even worse for usually women have the duty to make their families survive.

Nepal. Up to 50 years ago there was no provision made by the goverment of Nepal for the education of girls. The rate of illiteracy in Nepal stands at 60% now. These factors have had adverse effect on the development of women in many fields, also in mathematics. In the past 50 years improvements were introduced and many programmes of scholarships for girls have been developed. The Ph.D. programme in Nepal is 20 years old. At the present there are only 4000 women in Nepal holding graduate degrees. Of these 30 are in mathematics. Of these 30 women 50% are working in schools and 12 of them are working at the university. Only 2 of the 12 hold Ph.D. degree. There is no woman in full professor postion. Resources are inadequate and for any advancement in mathematical research outside contacts have to be sought.

Palestine. The situation in Palestine is quite bleak. It is worth noting that the universities in Palestine offer undergraduate programmes in almost all fields. Although there have been some attempts, only very few fields have started graduate programmes. Only one graduate programme in mathematics leading to a masters degree is offered and it is a joint programme with a university in Britain. In universities a lot of emphasis is placed on teaching (an average teaching load is 12 hours per week), although research is expected. Research on teaching and community development is preferred. This places an extra burden on those who want to do research in the pure fields. Also the fields of interest of mathematicians are very diverse, hence everyone feels isolated. The only way to develop is to seek outside contacts.

Tunisia. There is mathematical activity in Tunis. The mathematics department of the university in Tunis has more than 100 faculty members of which 20% are women. It is not a problem for women to choose an education in mathematics nor is it difficult to acquire a position at the university. The problem is in their ability to get into higher positions. For 20 years only one woman was able to become a full professor and now there are only 4 in that position. The difficulty lies in the many commitments a woman has in her family and it is not a problem of discrimination. The Tunisian goverment strongly encourages education and makes special provisions for women in education. School and university education in Tunisia is free.

4. Russia. The situation of women in mathematics in general does not differ much from the situation in other countries. One of the main problems is the lack of resources. The latest research shows that in general there is no equality between women and men in mathematics in Russia. Women suffer more from the lack of scientific information and the lack of adequate equipment. Most women think that they have smaller chance of publication. Another problem, typical of Russia, arises from the fact that Russia is a very large country and to be in contact is rather difficult.

Research was done by Vitalina Koval (1989) to map the situation of women in science. There are some facts appearing in her report: The percentage of women in science and scientific service has increased from 42% in 1940 to 53% nowadays. At the same time the salaries in science have decreased: In 1940 the salary of a scientist was 142% of the average salary level, in 1997 it was only 75% of the average salary level. There are only 34.4% women among the scientists with Ph.D., and there are only 14.9% women among the scientists who achieved the highest scientific degree in Russia—Doctor of Sciences. Only 7.7% of full professors in mathematics are women, but almost half (41.7%) of women among scientists without Ph.D degrees are women.

Last year an International conference of women–mathematicians was held on the Black Sea near the city of Novorossiysk. A questionnaire was spread among the participants and the following facts have come up from it (more than 50 participants responded). The respondents had rather high qualification, 76% of them had a scientific degree; more than a half of them had at least ten publications during the last 5 years. Since the economic situation of women mathematicians is very difficult 64% of women have an extra job. In spite of this 56% of them are satisfied with their work, practically all (96%) want to keep their work. Due to bad conditions 43% would be ready to work abroad. Almost all respondents stressed that in the whole there was no equality of men and women in mathematics in Russia.

5. Two short reports characterizing the situation in Europe. Two short reports were chosen to show the typical situation in Europe; one describes UK together with Malta and Greece and the second one deals with the situation in German speaking countries—Germany and Switzerland.

5.1. Report of the group Great Britain, Malta and Greece.

Irene Sciriha, Malta

Our group consisted of four members from British universities, three from Malta and on from Greece. We started by comparing the percentages of female Ph.D. graduates, of those qualifying in math and of those reaching the grade of full professors in the various countries. The fraction in Britain is very low with 17% female Ph.D.s, 7% of the Ph.D.s in math are female and only 3 out of 267 full professors are female. In Malta the fractions are close to zero in each category. In Greece the number of women compares well with that of men and the problem lies elsewhere since mathematicians find it hard to find employment because there are too many qualified mathematicians. So many have to do unrelated jobs like driving taxis and working in restaurants.

This data led us to search the reasons causing these differences. Why is mathematics so popular in Greece but the number of undergraduates is dwindling in many universities of the UK and in Malta? One possible reason is the importance given to the subject at secondary level in Greece. The proliferation of new degree courses in nearly related subjects like Computer Science, Information Technology, Business Studies and Accountancy in Malta may have contributed to the reduction in the number of undergraduates opting to study mathematics. However even in Greece, the number of women qualified in Mathematics that actually make it to the higher grades is low. Besides, men seem to find it much easier to obtain financial aid than women do.

To encourage more people to take up Maths in Malta and the UK, fun problem sessions for pre-university students should be organised. Besides efforts should be made to dispel the fear often expressed for the subject by most pre-university students.

In Malta, a problem facing Math graduates is that the type of occupation offered to them is mainly teaching. Whereas in the UK there are many math graduates working as statisticians, in the Accountancy, Financial, Insurance and Nuclear Fuel fields, these occupations seem to be filled with graduates in Engineering and Economics in Malta. It was suggested that besides a PR exercise to alter the attitude of employers towards the Math graduates to obtain more attractive job openings presently taken up by graduates in other fields, a restructuring of the degrees directed towards numerate Maths, say, could help to offer more options.

The ethics by which employers should be guided was next discussed. It was generally agreed that a woman faces a number of drawbacks mainly because of the many full-time commitments she is expected to see to. Whereas asking a female candidate whether she is married and has children is considered unethical, some members felt that asking how long the candidate intended to stay is acceptable. Besides, pregnancy should be treated on the same level as a service to the country like 'military service.'

In the UK there was no discrimination positive or otherwise with female undergraduates. The negative bias starts when looking for an occupation or for a post-doc when male counterparts seem to be favoured. Whereas a woman having children should be given all the legal support to obtain optimum conditions, care should be taken to prevent a social problem: that of parents working long hours and neglecting their children. We should expect as a fundamental right legislation enabling the time available to work to be co-ordinated with the time that parents need to be at home with their children. It is important that females are not penalised because of unfair expectations by society. Positive discrimination in favour of women is not very flattering and the ones who benefit from it may tend to lose as regards the prestige they enjoy. The opinion that the best person for a particular job should be chosen was expressed, however the action adopted in Sweden to appoint a female professor in every department to promote the idea of a role model was considered as a positive step that should be copied.

A problem encountered in Malta is due to the small size of its population and the isolation of the island that inhibits cross fertilisation of ideas. There is only one University and so there are restrictions that deter women from proceeding with their studies. Among these are the lack of diversity of branches of specialisation, the attitude of predominantly male selection boards that give male candidates bonus points (perhaps not openly) and the lack of financial support.

5.2. Minutes of the session on Germany and Switzerland.

Anke Wich, Germany

1. Prologue. There were participants form Germany and Switzerland and the problems and attempts to overcome them seemed very similar in both countries.

In order to understand the below discussion one had better know what stages there are in a "typical" German curriculum vitae. We usually attend school for 13 years, starting at the age of six, till we get our *Abitur*, which enables us to go to university. What may happen there till one finally reaches the tenured position of a professor is explained in the following table. In particular note that the *Habilitation* is a precondition of reaching professorship. Periods are to be understood as counted in years.

period	certificate/title aspired to	typical position	period of payment
5–6	Diplom	student	–
2–5	Doktor	assistant (BAT2)	5
3–7	Habilitation	assistant (C1, C2)	7
		Professor	permanent

Moreover, it might be of interest that Germany is a federal republic and cultural and educational decisions are usually taken on a county *(Land)* level. Consequently situations and policies might vary largely from county to county. The counties mentioned here are those represented by the participants.

2. Situation-statistics. In the latest EWM statistics on women engaged in mathematics Switzerland occupies the very last place, Germany the second last.

The WWW-site of the DMV (German Mathematical Society) recently published a survey (http://www-dmv.math.tu-berlin.de/archiv/memoranda/statistikMW.html) yielding numbers of women involved in mathematics at German universities. This survey basically was an initiative of C. Bessenrodt as the EWM regional coordinator in Germany and as the representative of the EMS Committee on Women and Mathematics and was only rudimentarily supported by the DMV.

By an unofficial list of female habilitations since 1919, the first woman ever to receive her habilitation in Germany was Emmy Noether (Göttingen) in 1919, and there have been 90 since. This is to be compared to an average number of 40 men per annum in recent years. The percentage of women to receive their habilitation has not significantly increased within the last decade.

3. Attempted Remedies. A brief outline of some of the initiatives taken by German universities and counties to increase the percentage of their female professors.

Funding-Habilitationsstipendium and Wiedereinstiegsförderung. These are grants given to scientists holding a doctoral degree and striving for a habilitation.

The *Habilitiationsstipendium* (grant for a habilitation) is given to applicants of both sexes for a period of three years, and it amounts to DM 3000.- per month. Yet for instance in the county of Sachsen-Anhalt the aim is to have a percentage of 65% of women among the scientists receiving it, and since at the moment the committee giving the grants consists of equally many women and men they manage to realize that aim.

The *Wiedereinstiegsförderung* is meant for women who due to family reasons had to interrupt their careers and now wish to (re-)start their habilitation. It is paid for a period of two years and amounts to DM 2000.- per month.

Yet women prefer not to apply for these grants, in particular the *Wiedereinstiegsförderung*, as they are problematic, seen from various angles: they are not generally respected; there is a general fear that having them in your CV makes a bad impression. The problem is that they are just temporary fundings; once terminated they leave you to a still uncertain future, just some two or three years older than before. Moreover they require the habilitation to take place within the period the grant is being paid and thus are only appropriate during the final period of a habilitation process. We are still lacking appropriate funding for the beginning post doc period.

Thus all in all a permanent solution would be highly preferable.

Women's Counsellor—Frauenbeauftragte. In the meantime most German and Swiss universities have *Frauenbeauftragte*, at least one, in most cases one per faculty plus an overall one for the whole university. They act as a counsellor and active support to women who come up with particular problems and generally watch out that women are not discriminated against. There are faculties, for instance at the TU Darmstadt, where the *Frauenbeauftragte* has a right to take the part of a counsellor in every employment procedure, be it for assistant jobs or professorships (cf. *Frauenförderplan*).

Results seem to vary widely. For women at those universities it is certainly a great relief just to know there is someone whom they can contact in cases of emergency. And there have been many cases where the *Frauenbeauftragte* was able to help where there would have been no solution without her (e.g., discrimination against female students during oral examinations). For the *Frauenbeauftragten* themselves life sometimes is not so pleasant since (male) reactions to their existence—as holders of that very job, not personally of course—are still ambiguous. Yet acceptance seems to have been improving.

There also have been complaints about rather, to say the least, inefficient *Frauenbeauftragte*, who then, by generalisation, might endanger the whole concept. On the other hand a devoted *Frauenbeauftragte* will invest a lot of time she might have turned into academic qualification otherwise.

Quota—The Hessian Frauenförderplan. In the county of Hessen there is a quota on the employment of women in public institutions. At universities the aim is to have the same percentage of female assistants at a faculty as of women receiving their diplomas from that faculty. Moreover the percentage of female professors at a faculty should equal the percentage of women holding a habilitation all over Germany.

Up to now it does work out very well for the assistant level, but not at all for the professoral level. In Darmstadt for example, there is not a single female professor. Rumour has it that the committees who have to work out the ranking of applicants for a professorship never place a woman among the top three of them—unless they want to have her in the first place—for fear she might be appointed by the ministry (who takes the final decision) for political reasons (Being pro-women is politically correct and might make a nice feature for the next election …).

4. East-West. The one thing we could definitely say here is that the system in the former GDR much more encouraged and enabled women to pursue both, family life and their careers. It was considered normal that a woman had a full time job as well and, at the same time, that state facilities should take care of her children. Hence day nurseries and schools were provided, offering three meals, educational and entertainment programs in extremely small groups (sometimes 5 children per tutor). Nowadays, if you want to have your children to be taken care of during th whole day, you have to find a private nurse and pay an enormous amount of money. Moreover, the former GDR provided permanent post doc jobs, and less mobility was required from women who wanted to pursue their academic careers.

5. Immediate actions to be taken.

- Convince the DMV to order a survey on what has become of those women who have benefitted from the *Wiedereinstiegs-* or *Habilitationsstipendien*: Were they able to still pursue their mathematical careers?
- Encourage a doctoral thesis (statistics/history) on the development of women in Swiss and German mathematics.

THE EWM VIDEO: WOMEN AND MATHEMATICS ACROSS CULTURES

MARJATTA NÄÄTÄNEN

University of Helsinki, Finland

1. How it came about. The video "Women and mathematics across cultures" came out in 1996. The EWM video was made by the initiative of Marjatta Näätänen (Finland), in collaboration with many people, especially Bodil Branner (Denmark), Kari Hag (Norway) and Caroline Series (UK). The filming was done mainly in Madrid with the indispensable help of Capi Corrales (Spain). The project started in 1995 and ended up with a 25-minute video.

Why a video?

The push to make the video came from my 3-fold frustration

- the experience from a predominantly male group making a video on a Finnish mathematician and ending up with Finland looking like a country with hardly any women
- experiences with people in womens' studies coming to us with a long set of questions expecting us to provide them with their research material, questions we often do not even find to be relevant. The last "straw" was when they refused, not wanting to "risk their careers," to work on the concrete case of extremely bad and unfair publicity a female candidate for a Rectorship got in Finland
- like all of us, I am extremely busy, and do not have time to go around talking about our situation. I wanted to get our own voice heard without "interpreters."

I thought that if we made a video, then despite our busy-ness, using modern technology we could just send the video to convey the message for those who want to hear.

How was it made?

There was no money, I had very little experience, but I succeeded in getting the small but very important initial support—a sympathetic and influential person from the predominantly male video group mentioned above helped to get the small initial funding, and Bodil Branner and Kari Hag agreed to join the project. Ilona Ikonen, a video student, came to the Madrid EWM meeting to do the interviews. We had problems all the time, the rented equipment did not work, too few people to do the job etc. Fortunately Capi Corrales was good in creative problem solving. The women who kindly agreed to tell their story all did it very well and many people helped in many ways.

The preliminary editing was done in Copenhagen by Bodil Branner, Kari Hag and myself, again with minimal cost at Bodil's home. Kari had undertaken the organizing of the round table in Budapest EMS-meeting and she got financing from her university to complete part of the video which could be used at the roundtable. The biggest single task was the coloured map of Europe, to get the statistics and the colors I wanted. Later, I got money from the Finnish Cultural Foundation to make the part introducing EWM and to complete the project. Caroline Series joined our group via e-mail to write good texts in proper English.

Since we had problems with the sound, finally I had to add a full written text for the interviews so that people can easily understand what is being said. This I made by listening several times to the video, almost learning it by heart.

How has it been used?

The video has been mailed to all EWM coordinators and contact people in about 30 countries. It has been shown in different countries in mathematical meetings, congresses, some schools, teachers' meetings, science meetings, in series of mathematical videos, to some journalists, diplomats, politicians. The response has been quite positive. The best comment was when a woman came to me and said that earlier she did not understand what women mathematicians were talking about, why should they have special problems? After watching the video she felt moved and changed her opinion, seeing how hard these women are trying to be able to pursue their work in mathematics, the subject they love.

2. The full text of the video. The idea of EWM—European Women in Mathematics—began in 1986 at the International Congress of Mathematicians in Berkeley, California, where several women mathematicians from Europe were taking part in a panel discussion organised by AWM, the Association for Women in Mathematics.

EWM started to grow as a network and over the next years organised meetings in Paris, Copenhagen, Warwick, Lisbon, Marseilles, Warsaw and Madrid. The meetings involved mathematical talks and general discussions and the network provided a meeting place for women mathematicians right across Europe, including the east.

We started to collect statistics and found surprising facts about the uneven spread of women mathematicians in different countries. In 1993, EWM was legally established with its base and main office in Helsinki.

There are, in 1996, over 200 members and 23 countries are represented. Each country or region is free to form its own organisation appropriate to local circumstances where activities are organised by a regional coordinator. EWM acts as a coordinating umbrella. Secretarial work is handled from Helsinki, mainly by e-mail.

EWM also has an e-mail network which enables its members to keep in touch, a newsletter and a homepage on the internet.

http://www.math.helsinki.fi/EWM

The purposes of EWM are to encourage women to take up careers in research in mathematics, to foster international scientific communication among such women, and to promote equal opportunity and equal treatment of women in the mathematical community.

It is an organisation for women, but men are welcome as supporting members. EWM organises meetings every other year. Meetings involve mathematical talks and discussions on topics, for example "creativity" and "family versus career," of general interest to women mathematicians. It is an organisation for women, but men are welcome as supporting members. We have been experimenting with new, non traditional and more "user friendly" methods of giving talks and learning mathematics. At the Madrid meeting in 1995, women who have studied and worked as mathematicians in different cultures described some of their personal experiences. It is dangerous to make generalisations, but there are interesting differences and indications as to why for example Latin countries have more women mathematicians than Northern European countries and why an organization like EWM is needed even nowadays in societies like ours.

The coloured map of Europe illustrates that the Latin countries have many more women mathematicians than the Scandinavian ones.

The map would change if the percentage of full professors in mathematics were used as criterion. Countries like France, Georgia, Italy and Poland would stand out with percentages ranging from 8 to 16.

The Madrid interviews.

Laura Fainsilber, France

Algebra and Number Theory

Born 1965, University studies USA and France: MIT, UC Berkeley, Paris 6, Ph.D. 1994 Besancon.

(end of written text)

"So I have travelled a lot, I have been in different departments with very different atmospheres and at first I did not feel that being a woman in mathematics was at all an issue and I did not think that I should be singled out or anything. I knew I was in a minority but I did not want to be treated separately or anything like that.

And then I started thinking it was a problem in Berkeley when I saw that most of my friends who were women were flunking the exams and were dropping out of the graduate programs.

Then when I came back to France I felt the atmosphere was very different because the way people interact was very different and my situation was also different.

When I came back I was amongst students who knew less than I did and I was being paid more attention and that made a difference I think in the way I felt about doing math and I felt much more confident because I was getting a lot more contact with professors than I had otherwise and so I decided I would rather stay in France and not go back to the States."

(Written) *Post-doc: Switzerland 1994–95, Besançon (France)1995–96 Single, no children.*

"I think the difference I felt between Besançon and Geneva is similar to the difference between Southern countries like Spain and Italy where there are a lot of women; they do not feel isolated but there are other problems that come up and women in Northern countries where they are very few and their problems are not the same but the isolation is tremendous and because there are these two extremes it makes it difficult but interesting for a society like EWM to bring these people together. The people from the North need to see groups of women who work together, it is impressive.

I do not know, one thing I was thinking of was the structure of academic careers. I think in countries that give permanent positions rather early women have a better chance than in countries where you go from postdoc to postdoc and have to wait a long time while you are having children and while raising them at a time that is really crucial for women—it is a difficult time for men too.

I found this year just after the Ph.D. very difficult partially because I was isolated in Geneva, partially finding my bearings and my motivation and getting to start to work on my own after the Ph.D., and a lot of women do get lost at that stage—in addition to the ones that got lost before—so I think that areas where they already have a permanent position at that stage, even if they fumble around for a few years it does not matter as much because they have a permanent position. It is clear in France of all the women I know that some do excellent work and are quite impressive but maybe a lot of them do have a few years where things do not always fall together."

How would you compare working in mathematics and in other fields?

"In math the type of work we do is different from other fields in that it is very individual. It is not ill defined but there isn't a lab and an experiment that is going on, the teaching is a continuous thing and that is well defined and it's clearly visible from the outside. The mathematics depends so much on the concentration we can give to it and the sort of quiet atmosphere and the self-confidence of the individual that it makes us probably more sensitive to anything that goes wrong.

There is clearly, at least in France there is clearly the fact that it is essentially a masculine field so you can position yourself in different ways—go into it because it is a masculine field and you want to, or you can try to ignore it. But it is clear that when you meet people they say: "You are a mathematician?! You don't look like a mathematician" you: "Really?" They are very surprised and I do not think they have the same reaction towards male mathematicians. You go around in the departments except in Besançon and a few other places you see people around and you go to conferences there are one or two women in the room, that makes the atmosphere different from other fields.

(written) *Mara D. Neusel, Germany*
Invariant Theory
Born 1964, University studies Germany: Doktor 1992 Göttingen (geometric topology) mid 1992–1995 unemployed; stipends and/or visiting positions at Pedagogicheskii Universitet Yaroslavl (Russia), University of Kassel (Germany), University of Zürich (Switzerland), University of Minnesota Minneapolis (USA), Institut für experimentelle Mathematik Essen (Germany), Yale University (USA), MIT (USA) since mid 1995 assistant at University of Magdeburg (Germany). Married, no children.

"What I want to discuss is I think the most powerful weapon against women in math in Germany:

In Germany women have been allowed to study at the university for about 90–100 years, so this means they have the right to enter the university to attend the lectures and so on. This does not mean that the professors are aware of them, it is quite the contrary, they ignore them. For example, I spent after my Ph.D. a little bit of time in German-speaking Switzerland and it is the same there, women are just ignored, and I spent a little bit of time in Russia and in the United States and there it is quite different. At the departments I visited there were at least some women and the atmosphere was not so chilly, so hostile.

And after studies?

"To come back to this young female Ph.D., suppose against all probability she is stubborn enough to try to find her way into the mathematical community, to stick it out, then she has to face the problem that usually vacant jobs are not announced publicly in Germany. The area is so narrow and the deadline is two weeks after the announcement appears, so that it is obvious that the decision is already made. Without definite details, I want to remark that at the university where I studied, Göttingen, only two women achieved their Habilitation Thesis, this was Emmy Noether and Helene Braun and the last one 53 years ago.

I think that is quite impressive.

At the stage for applying for professorships in addition to all committees being dominated by men, the committees who choose referees whose opinions are solicited for ranking the candidates are also dominated by men so it is little wonder that so few women appear at the top of such lists, in Germany!

It seems to me that the status of position correlates inversely to the number of women who hold this position, so status times number of women is constant. And Germans believe that the worth of position is founded on its technological achievements, scientific advances, so of course Germany is incapable of allowing women to hold such important positions as full professorships at university.

I think this is quite an interesting question for a social historian.

When I think about that old boys' club in Germany I think we do need strong young womens' club to combat it."

Why did you choose mathematics?

"I have a very simple answer to this question: I just love it."

(written) *Marjorie Batchelor, USA, lives in UK,*
Coalgebras and Supergeometry

"You have invited me to come and say a little bit of my experiences of being a woman mathematician in America and in England, particularly comparing how women are received in those two countries."

(written) *Born 1952, University studies USA and England: AB Smith College, Warwick University, Ph.D. 1978 MIT (USA)*

Research Fellow, New Hall, Cambridge, 1979–1982. SERC Advanced Fellowship 1985–1993. Visiting Professor, Tufts University, 1989–1990. SERC Research Assistant, 1993–1994. Retired March 1994. Began informal apprenticeship as violin maker. Began part-time work for local music shop March 1996. Continues to work on the research as an amateur. Married, 3 sons (born 1980, 1982, 1985).

"I think it is always dangerous to attempt to make comparisons on how women are treated in country X and country Y on basis of personal experience.

For example I was always treated with great courtesy both as a student and as a visitor at MIT and certainly I was treated with great courtesy at Tufts and I think it is very dangerous to compare that particularly since at Cambridge I was hunting for jobs and I never actually looked for jobs or applied for jobs in the States, and I have no doubt whatsoever that my reception would have been different in the States if I had been hunting for jobs—but I think there are some differences, some real differences.

I am particularly aware in Britain of the absence of concern about actually making sure that equal opportunities and affirmative action are in fact implemented.

In the States it has been a law for some time and it is a law which seems to bite, not in that it is universally respected but at least those people who are hiring are frightened of the law, frightened enough to make it affect their judgement, moreover and more important than that, frightened of being taken to task for not observing the law.

They would very much like to make a good impression, it is politically correct and they would like to be seen to be favouring women or at least giving them a fair chance whenever possible which has a very positive affect on hiring I think, they listen to you, quite keen to see if there is any chance that you would be a candidate for them.

Now that does not mean that there aren't any number of examples in the States where women have been very badly done by, just as there are many examples I am sure of men who have been very badly done by as well, but there is the consideration at least that they are concerned that they should look good.

For example when MIT lost Michele Vergne not only the math department but the entire university was in great pains to at least be seen to be doing something."

Did you have the same experience in England?

"Now that has not been my experience in Britain, and I would say that in other respects I found reactions at least in England to be substantially behind reactions that I have found in the States.

For example when it became clear that my own future was in danger I decided to go round and ask my senior colleagues for help. Had they any ideas what I might apply for or how I might find further work, and I was dismayed that half of them should look at me with some surprise and say: "Well, does it matter? Your childern are not in private education, do you need the money? Your husband has a job."

Now these people were not unkind and I counted them as my friends and I still do, it just did not occur to them that that would not be the done thing to say.

The other statement that I got back from them was: "Well, gee, I'd love to help you but I can't lie for you, your CV just does not compare with that of the 30-year-old competitors for this job (30 year-old men, implied).

Now that I found also something that I think would not have been said, it might have been thought but it would not have been openly said in the States, given that I have been working halftime for most of the last 10 years and it is not entirely reasonable to compare performance in 10 years working halftime with performance of 10 years working fulltime.

It was not so much that it was said but that the problem of comparing CV's for people who have worked halftime just had not occurred to the people who judged me.

I am unhappy just saying negative things I've found and I'd like to take the opportunity to say also some of the positive things.

There is a very great deal that people can do in a small way to make things much easier for women who are trying to survive in math, and that is what I call the transitivity of respect:

If mathematician A respects mathematician B and mathematician B respects mathematician C then mathematician A will also respect mathematician C and if C is a young woman trying to make a name and B happens to be one of the big names in the field then you'll find that most of the other mathematicians will indeed respect the woman in question.

That is of enormous help and I have been on the receiving end of that sort of help a number of times.

(written) *Isabel Salgado Labouriau, Brazil, lives in Portugal.*

Singularity theory, Dynamical systems, applications to Biology

Born 1954, grew up in Brazil. University studies: Brazil and England, Universidade de Brasilia, Warwick. Ph.D. 1982 Warwick

Universidade do Porto (Portugal) since 1982, 1 year visits to Warwick 1989–90, Sao Carlos (Brasil) 1995–96. Married, no children.

"I studied in Brazil, started studying math in Brazil and then later on went to do graduate work in England. And that was the first time someone told me that there was any problem with being a woman and being a mathematician, until that moment it was a profession a woman could choose or not.

Back in Portugal it's very similar to Brazil, you have lots of women doing math. So it didn't strike me as very important except everything that is international related to your career, then you notice that being a woman is an issue.

What is your impression of the situation in England?

"I have this feeling that it works on the guilty feelings of these women for instance in England that are trying to have a mathematical career and it makes it very hard for them, much harder than it was for me.

You have more women doing math in all the latin countries. I have no idea why that is, it is an observation of fact, but I have no explanation for that.

You certainly have a different way of organizings the society. Maybe you have noticeably, comparing for example with England, more women who have a career, so naturally you have more that do math, but I have no idea why is that so.

I mean one would expect the opposite and then you discover, when I went to England, you see it was very funny, I was going to this 1st world country coming from the 3rd world, everything was going to be much better and then I realized that it was not, it was the opposite, in social terms it was much worse in England than it was in Brazil.

In Brazil if you want to work you work and it is natural.

It is difficult to find out, I think that would be interesting for a sociologist to study but I am not a sociologist.

After England, Portugal was like going home in many respects, and one of them was working in this department where you have more women than men which was a sort of feeling of coming back to normal.”

(End of video)

The video “Women and Mathematics across Cultures, EWM—European Women in Mathematics,” now equipped with subtitles, is available from the EWM office in Helsinki. The cassettes are in VHS. The cost varies due to the system (PAL, SECAM, NTSC) and the mailing cost.

For details, please contact:
Riitta Ulmanen,
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REPORT ON THE POSTER SESSION

LAURA FAINSILBER

1. The plan. As an introduction, here is a part of the message that was sent to participants before the meeting:

"We hope that we can turn the Poster Session into something more inspiring and creative than is usually seen. We are depending on all of you to make this a success.

We encourage everyone of you to make a poster. Really everyone: no matter if you are a senior researcher having obtained a lot of results already or if you are a beginner still in the process of picking a topic to work on. We believe that this is the best way in which such a large and broad group of mathematicians can communicate their mathematical interests to each other and quickly make contacts.

Please keep in mind that the aim of the posters, beyond the presentation of results, is to introduce ourselves, explain the type of problems we are working on, and to give a context in which others can ask questions. We would also like the posters to contain a photo of you, and some information about yourself.

Please, make your poster up to 60 cm high and 70 cm wide (at most six A4 pages). You can either prepare your poster in advance or you can produce your poster after arrival. But be aware that there is not much time for preparation. We will provide paper, pens and glue; please bring the photo and ideas.

On Thursday evening, there will be a poster preparation workshop. We plan to form small groups according to fields of interest. Each group will make a large poster to serve as a general map of the mathematical area, in which the particular interests of the individual members of the group will appear in more detail. The idea is both to learn about each other's mathematics and to understand relationships between specialities.

On Friday evening, we will formally open the poster session and it will remain open for the rest of the meeting.

Title and abstract. Many of you have already sent a title and a short abstract of a poster. If you have not already done so we ask you to give us this information as soon as possible. Please sent the information to Livia Zetto (zetto@ictp.trieste.it)*, the secretary at ICTP who has been associated with our meeting. This will help us make a pre-plan for how to order the posters and you will make our planning easier by providing us with this information. But we will not be excluding anybody from contributing to the meeting by a poster in the very last minute".*

2. Goals and results. The poster session had several functions, for each participant and for the group as a whole.

One was to reflect on making good communicative posters. We often have to present our work in this form at conferences, we practice giving talks, but have little training in turning our work into an informative 60 cm $\times$ 70 cm board. The exercise here was particularly difficult since the posters were meant for non-specialists. At the same time, there was less pressure than at a specialised meeting, so we had a good opportunity to practice and experiment.

It was important for each participant, not only the speakers, to have an opportunity to expose her work. One effect was to spark contact between participants with close research interests who may not otherwise know of each others work since the talks are meant for and attended by a general audience. Another was for the readers to come in contact with fields far from their own. This consolidates the interdisciplinary aspect of EWM meetings, and enables us to get to know each other mathematically as well as socially.

For the group as a whole, the richness and diversity of all the posters gave a both global and detailed picture of who constitutes EWM. It was quite impressive for each person, whether or not she had contributed a poster, to realize how much we represent.

3. What actually happened. The evening before the start of the conference, we held a poster preparation meeting. We first made a list of topics on a blackboard, and tried to group participants according to specialities. The idea was to make group posters that would represent rather wide areas. In fact, the specialities were quite spread out and this was difficult to realize. Many had work to do on their personal posters and found an opportunity to improve the graphics with some of the material available. Others discussed their field with their "mathematical neighbors" but the evening was barely enough to share our understanding of the objects we deal with, let alone put it on paper. One team did manage to produce a collective poster.

After the first day of the conference, and a little extra time to finish up our posters, the exhibition was assembled in the main hall. We tried to arrange the posters according to fields. The poster session was officially opened after dinner and continued until the end of the week. Since our lecture room opened into the main hall, there was ample opportunity to see the posters (though maybe not enough time to read them). All in all, 60 posters were presented (for 98 participants).

Most posters gave a combination of personal and mathematical information. There were a few posters on women in mathematics and on educational themes, as well as four panels on mathematicians from an Italian exhibition "Women scientists of the Occident. Two centuries of History."

4. What makes a good poster? The first ingredient of course is the mathematics. Then there are many things we can do to help get it across and to introduce ourselves.

Many of the posters were both informative and pleasant to read. No single recipe came out of the session, not even agreement on what may have been "the best" posters. What did come out, is that it is difficult and takes care and time to express one's work in very few very simple words and images, and that there are several different ways to do it very well. Different approaches work well for different fields of course, depending on the availability of pictures and on the accessibility of the material.

Most posters were based on a paper, or on a shortened and simplified version of the written exposition of a result. Of course, this is the format in which we are used to presenting mathematics, and it is probably the most informative for those who are already interested in a subject and have time to read. But some other ingredients can also attract a wider audience and make it more accessible.

The clarity of the poster was important: that it be written big enough (even by hand) and that the general layout be easy to see from a distance made the posters more accessible.

Some of the successful recipes included graphics, for instance pictures of the author, drawings, computer-generated illustrations of mathematical objects, a diagram of the mathematical field, with connections to other fields and problems.

A mixture of personal and professional information made them attractive and facilitated contact with the author. One participant told the story, with photographs, of how she had convinced her family to let her come.

Strategies also depended on the level of advancement of the authors. Some posters, especially by Ph.D. students, concentrated on stating a problem, by defining the objects and stating some important properties. One experienced mathematician mentioned her most important result, gave a list of good reference texts in her field, and indicated how to contact her, in a very simple and effective poster.

5. Next time. Here are a few suggestions to improve on the poster session for the next meeting.

We start with a practical issue, very simple but not so easy to achieve in practice: to have good lighting of the posters. Since the hall at ICTP was not meant as a place to read, the lighting was not sufficient to take a good look at the posters in the evening.

It is good that they be in a place where one naturally spends time between lectures, and where one has room to walk around and chat. It would be better to have more of an opening ceremony, when everyone is there and takes a first look at the posters, as well as an occasion later on to meet with the authors, once participants have had time to read some posters and get interested (since the authors and the participants are the same people, this could involve several shifts where some of the authors stand nearby their poster).

Deciding how to arrange the posters, and in particular trying to gather posters that were “mathematically close”, turned out to be a challenge, overcome only with the help of guessing and random decisions. A possible activity for the first evening, as a way for the participants to introduce themselves, would be to try to draw collectively a “map” of the specialities represented at the meeting. It would be interesting both to discover one’s neighbours and to understand links between fields that are further away from one’s own. This map could then be posted, along with a geographical map showing where participants come from, and also help to lay out the posters.

Given the success of the poster session in Trieste, the effort we had put into our posters, and the richness represented by the whole, the question came up of how we could give the posters a more permanent form and preserve them. One suggestion was to form a booklet with reduced versions of the posters, a sort of “poster session proceedings,” another was to put them on the web. For this time, the poster session remained an ephemeral event, some authors took their posters home, the others were carried to Helsinki, and we publish here abstracts of the ones that had to do with

p-adic methods, in relation with the session on the p-adics. Maybe a poster session is essentially an ephemeral form of math presentation, as some of its interest comes from the combination of all the independent contributions. Still, it would be worthwhile to think about ways to make it durable by reflecting it in another medium.

6. List of posters.

- *Polina Agranovich*, *Ukraine*, Polynomial representation of subharmonic functions in the half-plane with masses on a finite system of rays.
- *Saloua Aouadi Mani*, *Tunisia*, DKT finite element approximation of geometrically exact shell models.
- *Irina Astashova*, *Russia*, On asymptotic properties of one-dimensional Schrödinger equations.
- *Shanti Bajracharya*, *Nepal*, Analytical and group theoretic study of special functions.
- *Karin Baur*, *Switzerland*, Construction of a covering in complex projective space.
- *Eva Bergkvist*, *Sweden*, Polynomial convexity.
- *Eva Bergqvist and Catarina Rudälv*, *Sweden*, The University of Umeaa (without polar bears and reindeer).
- *Eva Bergqvist and Catarina Rudälv*, *Sweden*, Women in mathematics in Sweden.
- *Audrienne Bezzina*, *Malta*, Women in mathematics.
- *Andrea Blunck*, *Germany*, The projective line over a ring.
- *Larissa Bourlakova*, *Russia*, The first integrals and the Lyapunov functions.
- *Rachel Camina*, *UK*, p-adics and pro-p-groups.
- *Constanta-Dana Constantinescu*, *Rumania*, Applications of p-adic numbers in the theory of dynamical systems.
- *Claudia-Paula Curt*, *Rumania*, Starlike and convex mappings of order p defined on the unit ball of $\mathbb{C}^n$.
- *Susanna De Maron*, *Italy*, Great mathematicians, from Hypatia to Florence Nightingale—Italian women mathematicians. (Part of an exhibition on women scientists of the Occident).
- *Marie Demlova Czech Republic* Research activities—Teaching—References.
- *Laura Fainsilber*, *Sweden*, Quadratic forms over the p-adics.
- *Lisbeth Fajstrup*, *Denmark*, Geometrical methods in computer science.
- *Barbara Fantechi*, *Italy*, Deformation theory; Gromov–Witten invariants.
- *Cettina Gauci*, *Malta*, Chaos theory.
- *Fateme-Helen Ghane*, *Iran*, Degree one maps of the circle with non-trivial and non-persistent rotation sets.
- *Danielle Gondard–Cozette*, *France*, Fields of interest—Methods of proofs—Sample of results—Papers.
- *Sandra Hayes*, *Germany*, The real dynamics of Bieberbach's example.
- *Helle Hein*, *Estonia*, Optimisation of geometrically nonlinear plastic shallow shells.
- *Shirin Hejazian*, *Iran*, Derivations of JB^*-algebras.
- *Catherine Hobbs*, *UK*, Applications of singularity theory to oscillating integrals.
- *Tatiana Ivanova*, *Russia*, The infinitesimal symmetries of the self-dual Yang-Mills

equations.

- *Magdalena Jaroszewska*, *Poland*, Quality assessment in higher education.
- *Sudesh Kaur Khanduja*, *India*, Generalised Schönemann-Eisenstein irreducibility criterion.
- *Lyudmila Kirichenko*, *Ukraine*, The sufficient conditions of transition to the beforehand given limit distribution.
- *Bettina Kuerner*, *Germany*, Construction of a bilinear form f for a linear group G such that $G \subseteq O^*(V, f)$.
- *Olga Kuznetsova*, *Ukraine*, Strong summability. Sidon type inequalities. Integrability of multiple trigonometric series.
- *Nadia Larsen*, *Denmark*, Faithful representations of crossed products by actions of $\mathbb{N}^k$.
- *Maria Leftaki*, *Greece*, Some periodic and symmetric orbits of a charged particle moving in the field of two revolving parallel magnetic dipoles.
- *Gulbadan Matieva*, *Kyrgystan*, Teaching experience—Research—Publications. "On the geometry of partial mappings of Euclidean space."
- *Emilia Mezzetti*, *Italy*, On threefolds which are covered by a family of lines of dimension two.
- *Clementina Mladenova*, *Bulgaria*, Formulation of multibody system dynamics on a Lie group.
- *Sanghamitra Mohanty*, *India*, Fractal geometry of circular mappings.
- *Marjatta Näätänen*, *Finland*, Examples of symmetric Fuchsian groups.
- *Constanta Olteanu*, *Sweden*, Particular case of the movement of electroconductor viscous fluids around the plane board with incidence.
- *Luisa Paoluzzi*, *Italy*, Determining 3-orbifolds and singular sets via Heegaand diagrams.
- *Sylvie Paycha*, *France*, Some mathematics around path integrals.
- *Emilia Viorica Petrisor*, *Rumania*, Symmetric periodic orbits in the dynamics of reversible diffeomorphisms.
- *Dorina Raducanu*, *Rumania*, On some classes of holomorphic functions of $\mathcal{C}^n$ into the complex plane.
- *Mukhaya Rasulova*, *Uzbekistan*, The solution of the Poisson-Boltzmann equation for self-consistent potential of infinite, random, nonlinear and non-uniform systems.
- *Helen Robinson*, *UK*, Division algebras and fibrations of spheres by great spheres.
- *Jacqueline Rojas Arancibia*, *Chile*, From conical sextuplets to canonical curves in $\mathbb{P}^3$.
- *Olga Rozanova*, *Russia*, Energy estimations and blow–up of solutions in a system of atmosphere dynamics.
- *Catarina Rudälv*, *Sweden*, Rational approximation in $\mathbb{C}^n$, $n \geq 2$.
- *Irene Sciriha*, *Malta*, Nut graphs-maximally extending cores.
- *Silke Slembek*, *Germany*, Constructivist tendencies in algebra—The algorithm of Grete Henry-Hermann.
- *Tamara Stryzhak*, *Ukraine*.
- *Sorayya Talebi*, *Iran*, Derivations of reversible JC-algebras.

- *Betul Tanbay*, *Turkey*, M-ideals and the Schur property.
- *Ufuk Taneri*, *Turkey*, Exploitation of the symbolic computation in the evaluation of the group theoretical characteristics.
- *Rodica Tomescu*, *Rumania*, A limit theorem for the specialty sequence.
- *Lyudmila Turowska*, *Ukraine*, Representations of twisted generalised Weyl constructions.
- *Tatiana Vasilieva*, *Russia*, The use of regularisation methods for an aerodynamics inverse problem.
- *Hilda Irene van der Veen*, *the Netherlands*, Soil plasticity and Eigenproblems.
- *Anke Wich*, *Germany*, Sketched geometries.
- *Corinna Wiedhorn*, *Germany*, Groups, parabolic systems and flag–transitive geometries.
- *Inna Yemelyanova*, *Russia*, Symmetries and differential equations.

OTHER ACTIVITIES OF EWM SINCE THE MADRID MEETING

BODIL BRANNER, LAURA FAINSILBER AND SYLVIE PAYCHA

Here is a list of activities that EWM has been involved with in the last few years. We refer to specific reports for more details on each of them. Some of the information is available on the web site.

1. Meetings and events

1.1. Interdisciplinary workshops.

• *Renormalization*: Paris, June 14 and 15, 1996. Organized with *femmes et mathematiques* by Sylvie Paycha. See the 4th EWM Newsletter.

• *Moduli spaces in mathematics and physics*: Oxford, July 2 and 3, 1998. Organized by Frances Kirwan, Sylvie Paycha, Tsou Sheung Tsun. See the report in the following pages.

1.2. International events.

• Round table at the 2nd *European Congress of Mathematicians*, Budapest, July 1996. Organized by Kari Hag. See the proceedings of the congress.

• Round table at the *International Congress of Mathematicians*, Berlin, August 1998.

Organized and moderated by Srinivasan, Bhama, University of Illinois at Chicago, U.S.; Christine Bessenrodt, University of Madgeburg, Germany; Bettye Anne Case, Florida State University, U.S. In collaboration with AWM (Association for Women in Mathematics) and the committee on women in mathematics of the European Mathematical Society.

Events and policies: Effects on women in mathematics.

The panellists are women in mathematics from several different countries. Each will discuss impacts she has noted on the work and lives of women in the mathematical sciences which may result from various national policies, practices, and events. Do some of these events cause more women or fewer to participate in mathematics?

1.3. Regional associations and meetings.

• In France, *femmes et mathematiques* runs one or two-day meetings every few months, with mathematical talks for a general mathematical audience, and lectures and debates on themes related to women in mathematics, often in dialogue with women outside mathematics, for example other educators, sociologists, psychoanalysts, musicians. Most of the meetings take place in Paris; once a year a major meeting is organized in another city (Bordeaux in November 1998, with mathematics and musical composition.)

Every January since 1996, *femmes et mathematiques* gathers women graduate students and recent Ph.D.'s for the *forum des jeunes mathematiciennes* where they give short talks on their research. This is an opportunity for many young women who are not usually in contact with the association to present their work, gain experience and visibility, meet other women mathematicians, and become aware of issues, in particular about recruitment and careers of women mathematicians in France. It takes place right before the start of the 'recruitment season' and has been very successful. Short write-ups of the talks are published in the journal of *femmes et mathematiques*.

• *RAWM*: The Russian Association for Women in Mathematics has organized severalinternational conferences in the last few years. See the EWM newsletter number 5 and the web site of *Women in Science and Education*,

http://mars.biophys.msu.ru/awse.

• The *Franco-Russian Meeting* was organized jointly by *femmes et mathematiques* and *RAWM* in Luminy, France, Dec 2–6 1996. It gathered 11 Russian and 20 French women mathematicians around mathematical talks, introductory and specialized, and discussions. See the report in Newsletter number 4.

• *BWM*: British Women in Mathematics have organised one-day workshops roughly every 15 months, featuring women mathematicians as speakers. The next BWM day is planned for September 16th 1999 in Edinburgh. See the last 3 EWM Newsletters for brief reports on the BWM days which have been held so far.

• In Germany, members of EWM have been working with the womens' delegate (*frauenbeauftragten*) at several universities (see Newsletter 4). A meeting of German women mathematicians: *Tagung deutscher Matematekerinnen*, will take place October 16th and 17th in Darmstadt, with talks in pure and applied mathematics, and a discussion on the theme of the role model function of women in mathematics and the natural sciences. It is organized by Christine Bessenrodt, Andrea Blunck, Roxana Brechner, Eva Hermann, and Bettina Kuerner, with both public and industry funding.

Bettina Kuerner is also building a web site for German Women in Mathematics at

http://www.mathematik.tu-darmstadt.de/ewm/

• A *Nordic Summer school* for female Ph.D. students, was organized by Gerd Brandell at the mathematics department of Luleaa university in Sweden, June 15–20, 1996. The Program included three minicourses of 6 hours each, on *Holomorphic Dynamics in the Complex Plane* by Bodil Branner, *Enumerative Algebraic Geometry* by Ragni Piene, and *On the Foundations of Mathematics, from Set Theory to Constructivism* by Jan Smith, as well as lectures and seminar talks given by the participants.

2. Publications and videos

• *Women and mathematics across cultures*. This video film, produced in 1996, briefly introduces EWM, provides some statistics, and allows four women mathematicians to share their personal experiences about the impact of cultural differences on the status of women in the profession. Its making was motivated by questions such as: Why are there many more women mathematicians in Italy and Brazil than in Norway, England or Germany? What is it about Latin culture that encourages women mathematicians, while that of Northern Europe mitigates against them?

The film was directed by Marjatta Näätänen in collaboration with Bodil Branner

(Technical University of Denmark), Kari Hag (UNIT-NTH, Trondheim, Norway), and Caroline Series (University of Warwick, UK).

See the article and the full text of the video in these proceedings.

• *Video on the Franco-Russian conference* in Luminy: produced with femmes et mathematiques by Sylvie Paycha and Christine Charreton.

• *Proceedings of the Madrid meeting*. Editors: Bodil Branner and Nuria Fagella. Available from the regional coordinators or on the web site of EWM.

• *Proceedings of the workshop on renormalization*, edited by Sylvie Paycha, with *femmes et mathematiques*.

• *Newsletter* number 4 was edited by Cathy Hobbs and came out in January 1997, number 5 was edited by Ewa Bergqvist, Nadia Larsen, Catarina Rudälv, Ufuk Taneri, and Anke Wich and came out in April 1998. They were distributed via e-mail and are available from the web page. They both contain information on the life of the association, reports on general and regional meetings of EWM, on events and policies in various countries regarding women in mathematics.

• *femmes et mathematiques* now publishes a journal, since 1996. It contains news of the association, mathematical survey articles (written versions of the talks given at meetings), and articles on education and women.

• *RAWM* has published articles, abstracts and proceedings of the conferences of Women Mathematicians that took place in Voronezh in 1995 and in Nizhny Novgorod in 1996.

• A *bibliography* of math books written or edited by women, gathered by Mara Neusel and containing about 1200 references, is accessible from the web page. Raphaële Supper has gathered a bibliography of ressources on women in mathematics, which appeared as supplement to number 1 of the journal of *femmes et mathematiques*.

3. Communication and information.

• Olga Caprotti has developped a *web site*

http://www.math.helsinki.fi/EWM/

which makes a lot of information about EWM easily available, both general information for people outside the association, and detailed reports such as these proceedings. See the article by Olga in Newsletter 5.

• The *e-mail network* reaches 245 people, some members, some not. It is used to circulate EWM information, such as the newsletters, job and conference announcements, and sometimes for questions and discussion.

4. Projects for the future.

• The *9th General meeting* is being planned by Christine Bessenrodt, Polina Agranovich, Ina Kersten, Olga Kounakovskaya, Irene Pieper-Seier, Ufuk Taneri and Tsou Sheung Tsun. It will take place at Kloster Loccum near Hannover, Germany August 30-Sept. 5, 1999.

• Developing communication with the regional coordinators. One way to do that is to start a news bulletin: every 3 months the international coordinators could write to all the regional coordinators and ask for brief news about recent local events. The idea is to hear about activities but also what new policies or events may affect women in mathematics in particular regions or universities.

• *femmes et mathematiques* has two projects related to World Mathematical Year 2000: a book entitled “Regards de mathematiciennes” and a video entitled “Mathematiciennes aux quatre coins du monde.”

• We are open to suggestions for *WMY2000*. Contact Kari Hag.

• We would like to develop more “networking” structure: links and exchange programs between women mathematicians, between Europeans and non-European, mentoring for younger women.

• We would like to find funding sources for the association, in addition to looking for funding for each separate event.

EWM WORKSHOP ON MODULI SPACES IN MATHEMATICS AND PHYSICS

Oxford, 2 and 3 July 1998

Supported by the London Mathematical Society and 'Algebraic Geometry in Europe.'

Organising committee: Frances Kirwan (Oxford), Sylvie Paycha (Clermont-Ferrand), Tsou Sheung Tsun (Oxford).

This interdisciplinary workshop was organized around 7 talks giving different points of view on the notion and use of moduli spaces. Various areas of mathematics and mathematical physics were represented: algebraic geometry, quantum field and gauge theory, and dynamical systems. The different perspectives presented here contributed to the richness of the meeting which was attended by about 20 mathematicians, including a good number of graduate students, from various countries in Europe and another 20 from Oxford and other universities in Britain. Special efforts were made by the speakers to present their topic in a form accessible to non-specialists.

Many participants expressed their wish for other such topical small scale meetings to take place in the future and some concrete proposals were made during the meeting.

The small scale of this meeting made possible many informal discussions among participants between the talks. At the end of the meeting, one of them nearly forgot her train, so engrossed was she with the discussions!

The contents of the talks will appear in proceedings which we hope will be readable by non-specialists who wish to have an idea what moduli spaces are.

The following are the abstracts of the talks.

1. Frances Kirwan (Oxford): Introduction to moduli spaces. Classification problems in algebraic geometry (and other parts of geometry) often break down into two steps. The first step is to find as many discrete invariants as possible (for example, if we want to classify compact Riemann surfaces then the obvious discrete invariant is the genus). The second step is to fix values of the discrete invariants and to try to construct a moduli space; that is, an algebraic variety (or other appropriate space in other parts of geometry) whose points correspond to the equivalence classes of the objects to be classified, in some natural way. This talk will attempt to explain how this idea can be made more precise, and to describe some ways to construct moduli spaces.

2. Claire Voisin (Paris): Hodge theory and deformations of complex structure. This talk will introduce to the theory of variations of Hodge structure, that is the way the Hodge decomposition of a projective or Kähler compact variety varies with the complex structure, and its applications: in one direction, the theory of periods helps understanding properties of the moduli space (Torelli type theorems, obstructions, curvature properties flatness ...). In the other direction, deforming the variety allows to establish strong Hodge theoretic statements for the generic fiber (Noether-Lefschetz type theorems, (non)-triviality of the Abel-Jacobi map, Nori's connectivity theorem).

3. Rosa-Maria Miro-Roig (Barcelona): Moduli spaces of vector bundles on algebraic varieties. Moduli spaces are one of the fundamental constructions of Algebraic Geometry and they arise in connection with classification problems. In my talk, I will restrict my attention to moduli spaces of stable vector bundles on smooth algebraic projective varieties. Roughly speaking a moduli space of stable vector bundles on an algebraic projective variety X is a scheme whose points are in "natural bijection" to isomorphic classes of stable vector bundles on X.

Once the existence of the moduli space is established, the question arises as what can be said about its local and global structure. More precisely, what does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry? Until now, there is no a general answer to these questions.

The goal of my talk is to review some of the known results about moduli spaces of H-stable vector bundles on a smooth, irreducible, projective, algebraic variety (X, H). In particular, the properties which nicely reflect the general philosophy that moduli spaces inherit a lot of geometrical properties of the underlying variety.

4. Tsou Sheung Tsun (Oxford): Some uses of moduli spaces in particle and field theory. In this talk I shall try to give an elementary introduction to certain areas of mathematical physics where the idea of moduli space is used to help solve problems or to further our understanding. In the wide area of gauge theory, I shall mention instantons, monopoles and duality. Then, under the general heading of string theory, I shall indicate briefly the use of moduli space in conformal field theory and M-theory.

5. Ragni Piene (Oslo): On the use of moduli spaces in curve counting. In enumerative algebraic geometry one works with various kinds of parameter spaces—Chow varieties, Hilbert schemes, moduli spaces of maps. We shall discuss these spaces and how they can be used to attack curve counting problems—in particular the problem of counting curves on a surface. This classical problem turns out to be of interest to theoretical physicists. Their interest has triggered quite a lot of work on the problem, in the context of both algebraic and symplectic geometry, but even more, their point of view has provided the mathematicians with new insight and new methods.

6. Mary Rees (Liverpool): Teichmüller distance and meromorphic 1-forms. This work arose out of a need to analyse a function of the form $d(x, \tau(x))$. (It uses some quite ancient theory, which was nontheless new to me.) Here, d is the Teichmüller distance function on a Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ of a surface S, and $\tau : \mathcal{T} \to \mathcal{T}$ is a function. An example is given by $\tau(x) = x \cdot g$, where g is an element of the modular group of S, which acts on $\mathcal{T}$. (I was actually motivated by a different, less classical example.) After introducing Teichmüller space (mostly for marked spheres), and

Teichmüller distance, I shall show a connection with holomorphic and meromorphic 1-forms (on a different surface S': a hyperelliptic curve in the case when S is a marked sphere). I shall look at bases of the first cohomology of a surface S' in terms of homolomorphic and meromorphic 1-forms. I shall use this to show how to find the second derivative of the Teichmüller distance function on $\mathcal{T}(S)$.

7. Tatiana Ivanova (Dubna): Moduli space of self-dual Gauge fields, holomorphic bundles and cohomology sets. The solution space of the self-dual Yang-Mills equations in Euclidean four-dimensional space R^4 is considered. We discuss the Penrose-Ward correspondence between complex vector bundles over R^4 with self-dual connections and holomorphic bundles over the twistor space of R^4. The moduli space of self-dual Yang-Mills fields is described in terms of Cech and Dolbeault cohomology sets.

THE MATHEMATICAL PART OF EWM MEETINGS

CAPI CORRALES AND LAURA TEDESCHINI LALLI

"We reproduce here an article originally written for the proceedings of the Madrid meeting in 1995, in the hope that it will continue to inspire those preparing presentations for EWM meetings and other occasions."

The organization of the scientific part of an EWM meeting is quite different from that of most mathematical meetings. Starting at the EWM meeting in Luminy in 1991 we decided to experiment with the format trying to reach the following main goals: to learn mathematics which is new to us; to learn how to transmit mathematics; to learn how to discuss mathematics with other mathematicians not necessarily specialists in the same field as we are; and finally to be able to establish scientific links which women, isolated for a number of reasons, can refer to at any stage in their professional career. We have been using the following structure as a model.

1. Before the meeting

Step 1. A scientific committee, chosen by the standing committee of EWM, selects three topics in mathematics. Several considerations are taken into account when choosing the topics:

- the topics should be in the avantgarde of current research;
- the topics should involve beautiful mathematics;
- the topics should try to include also branches of mathematics where, historically, for whatever reasons, the presence of women seems more difficult to detect.

Step 2. Once the topics are chosen, the scientific committee chooses a coordinator for each topic. Several considerations are taken into account when choosing the coordinators:

- their knowledge of the field;
- their commitment to the project of making the transmission of mathematics a main goal of their work;
- their ability and will to work in team with others.

Step 3. The coordinator selects the speakers for her topic. Several considerations are taken into account when choosing the speakers:

- their knowledge of the field;
- their ability, or their will to improve their ability, to transmit knowledge.

Step 4. Coordinators and speaker work together as a team in preparing the talks. The different talks form a whole, and the level of difficulty should be progressive. Once a speaker has been assigned a talk she is invited to give a written draft of her lecture to the coordinator. To ensure crossfield dissemination, and, above all, understandability, the coordinator then distributes these drafts among a few women mathematicians NOT specialists in the topic, who will read them and point out passages where assumptions are taken for granted, or needing an example, or otherwise remaining obscure, etc. We call the crucial function of these professionals "stupid readers," or "naive readers." The coordinator sends the comments of the non-specialists back to the speakers. The speakers make the appropriate corrections and changes and return the text to the coordinators, who send them again to the readers for a final check.

2. The lectures. Many are the questions that frame our work within the mathematical talks. Here are a few of them:

- how do we create an atmosphere in which the audience feels free to ask questions?
- how do we balance the inevitably different levels of knowledge about the topic in a general mathematical audience?
- how do we balance the flow of questions with the flow of the speaker?
- how do we manage to be understood by non-specialists without decaffeinating our expositions?
- mathematics is difficult; how can we make something clear and at the same time keep its richness, depth and not hide its difficulties?

Common sense is a main tool we count on, but we know it is not sufficient. Common sense, patience, and, as scientists, the will and inclination to experiment, try and find by searching. Several strategies have been tested, and as our experience develops, so does the number of strategies that we see work adequately towards answering the above questions. Here are a few:

- one or two women volunteer to concentrate to their fullest ability in the talk and ask questions when they do not follow the speaker, or think this is the case for many in the audience. We label this other crucial function "planted idiots." We think it works best if the planted idiot is actually naive in the field. Other questions are welcome as always;
- the speaker knows ahead of time that when a question is posed by someone in the audience, if someone else knows a more clear or direct way of answering it, this person will speak up. In this way the flow and rhythm of the talk is easier to mantain; and, since the speaker knows this might happen, she does not feel intruded or judged when it does;
- if interdisciplinary connections or other interesting discussions start taking place along the course of a talk, the coordinator of that topic should channel it into organizing a side discussion later, making sure there is a time and a space allowed for it and announced.

3. Writing and publishing the lectures. It is our experience that this is the step where we should be more cautious, since mathematicians have the habit of writing only for specialists. Hence, a process analogue to that of step 4 (before the meeting) is followed: each speaker sends a draft of the text to the coordinator. The coordinator should distribute this draft again among "naive readers," who will make sure the text is faithful to the version and comments agreed on before the actual talk. The coordinator receives the comments of the non-specialists and sends them back to the speaker. The speaker makes the appropriate corrections and returns the new corrected text to the coordinator, who, in turn, sends them again to the readers for a final check.

4. Conclusion. As one can deduce from the above summary, the mathematical part of an EWM meeting is conceived as a learning experience for ALL THE PERSONS taking part in it. Ideally:

- everyone will learn new mathematics, even the specialists. The advantage of speaking clearly to an interdisciplinary audience of mathematicians is that such situation rarely fails to give as fruit the bringing out of connections or points of view thus far unknown to us;
- the speakers will improve their ability as lecturers and mathematical writers;
- everyone will improve her ability to speak about what she works on.

Unfortunately, it is still the case in many European universities that women are singularities within the mathematical departments. Frequently this has a well known inhibiting effect on us, resulting in lack of self-consciousness or defensiveness, both particularly negative when we start our professional path. And if we are inhibited, we do not speak about mathematics, and if we do not speak about mathematics we do not learn how to speak about mathematics, and the loop traps us. The vicious circle of communication, well-known to many, creates a steady isolation which becomes sterile and depressing, as opposed to the temporary isolation which is necessary to all creative work. In fact, we think many problems arise for women in mathematical research from the different types of isolation (communication, life passages …) adding to the second, necessary one, and making it seem unbearable.

5. Other forms: The interdisciplinary workshops. As we went on planning this EWM meeting we came across words which seem to have different meaning in different branches of mathematics. But often the use of the same words in mathematics points to a common root, a core idea. We think (!) it is one of our original contributions to organize workshops around a word, or an idea, to re-walk paths and rediscover, if not build, common ground on both language and conceptual basis. The first such encounter took place in Madrid, on “Moduli spaces,” with speakers from algebraic geometry, number theory, hyperbolic geometry and quantum field theory. In Madrid the next interdisciplinary workshop was put forward, on the words “Renormalization Group.” It will hopefully take place in June, 1996 in Paris, with contributions from statistical physics, quantum field theory, markov processes, holomorphic dynamics and real dynamical systems. These workshops are kept more informal, with several persons responsible for illustrating what they deem necessary to the core idea, or the strength of the results that follow in their field. Everybody else is welcome to “pitch in” in workshop style.

“The *Renormalization* workshop did indeed take place, as did a workshop on moduli spaces in mathematics and physics in Oxford in July 1998. See the report in these proceedings.”

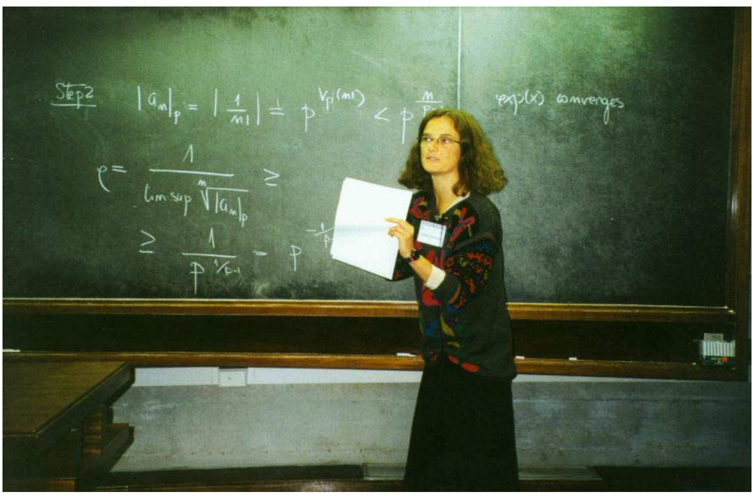
6. Poster sessions. Up to now, we have only once experienced a poster session. We think it is quite a challenge to our creativity to rethink poster sessions in a way that makes them a good communication tool. We are working on it.

"Since this article was written, we have had another poster session, in Trieste. See the report in these Proceedings."

PHOTO GALLERY

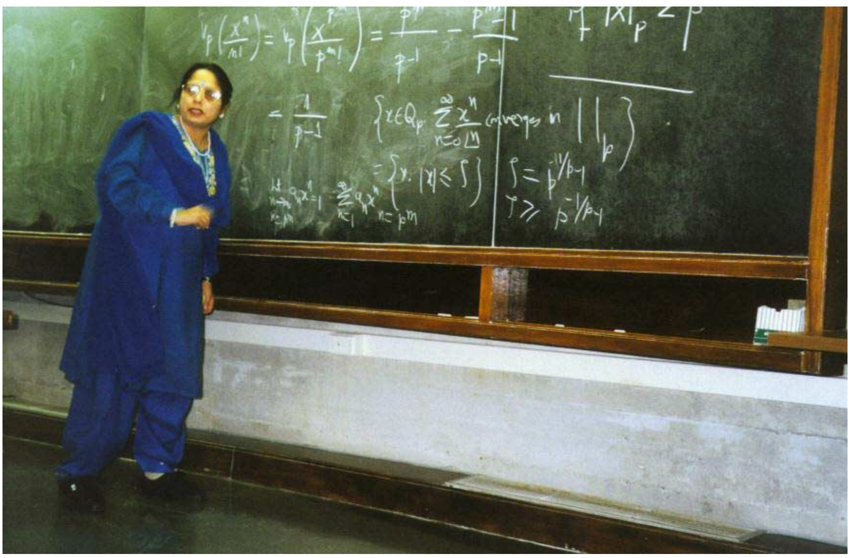
THE P-ADIC NUMBERS SESSION:

Catherine Goldstein, Capi Corrales and Francoise Delon



Capi Corrales

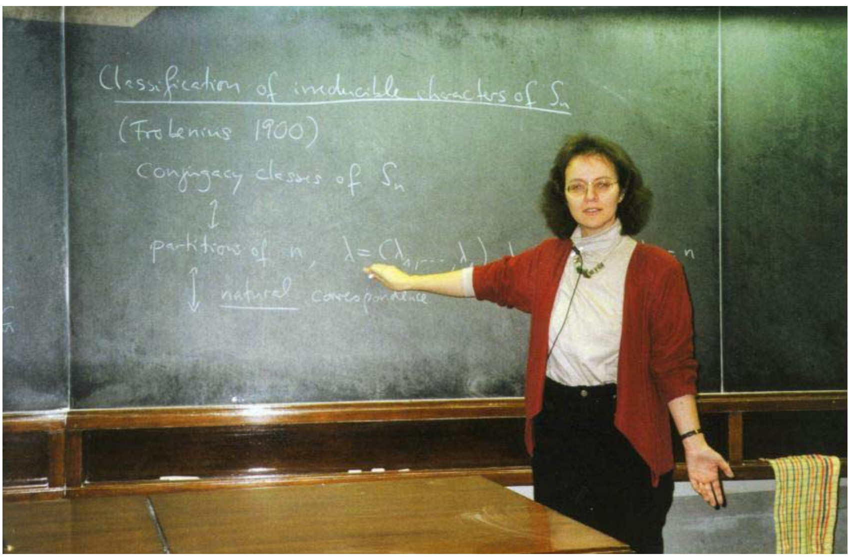
Catherine Goldstein



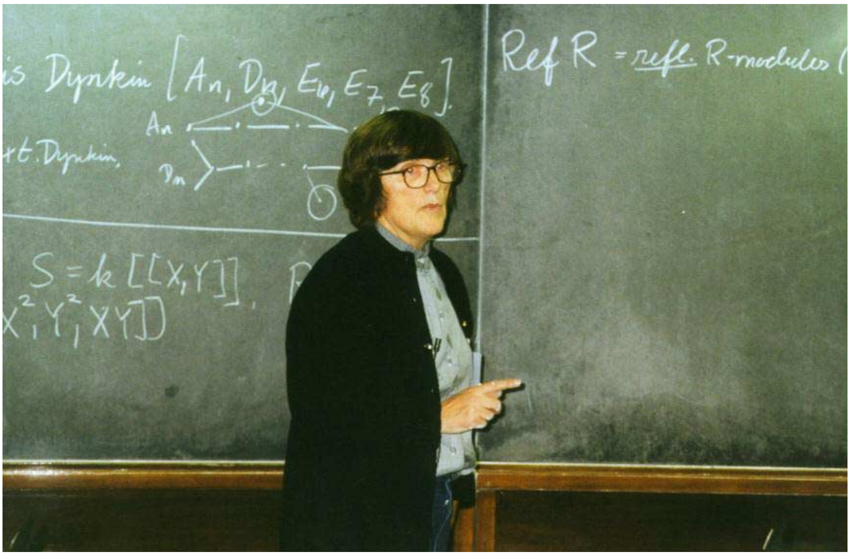
Sudesh Kaur Khanduja

THE REPRESENTATIONS SESSION:

Michele Vergne

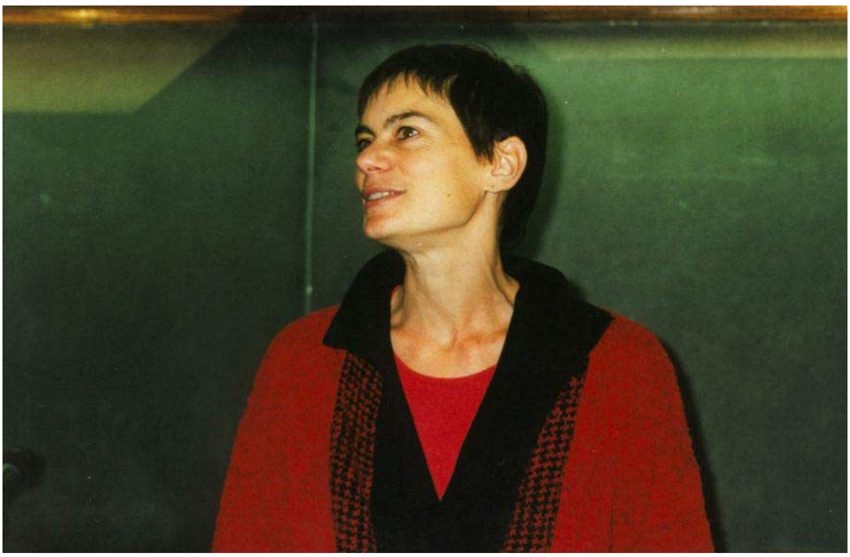


Christine Bessenrodt



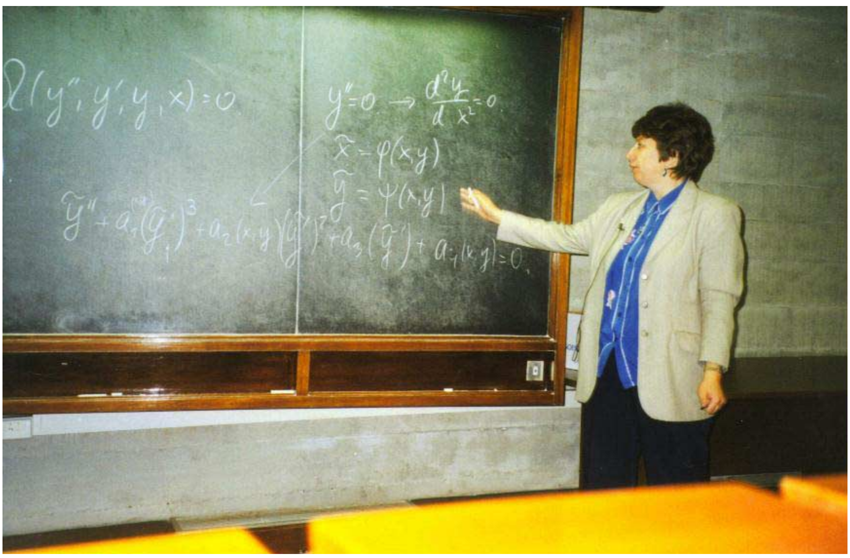
Idun Reiten

THE SYMMETRIES SESSION:



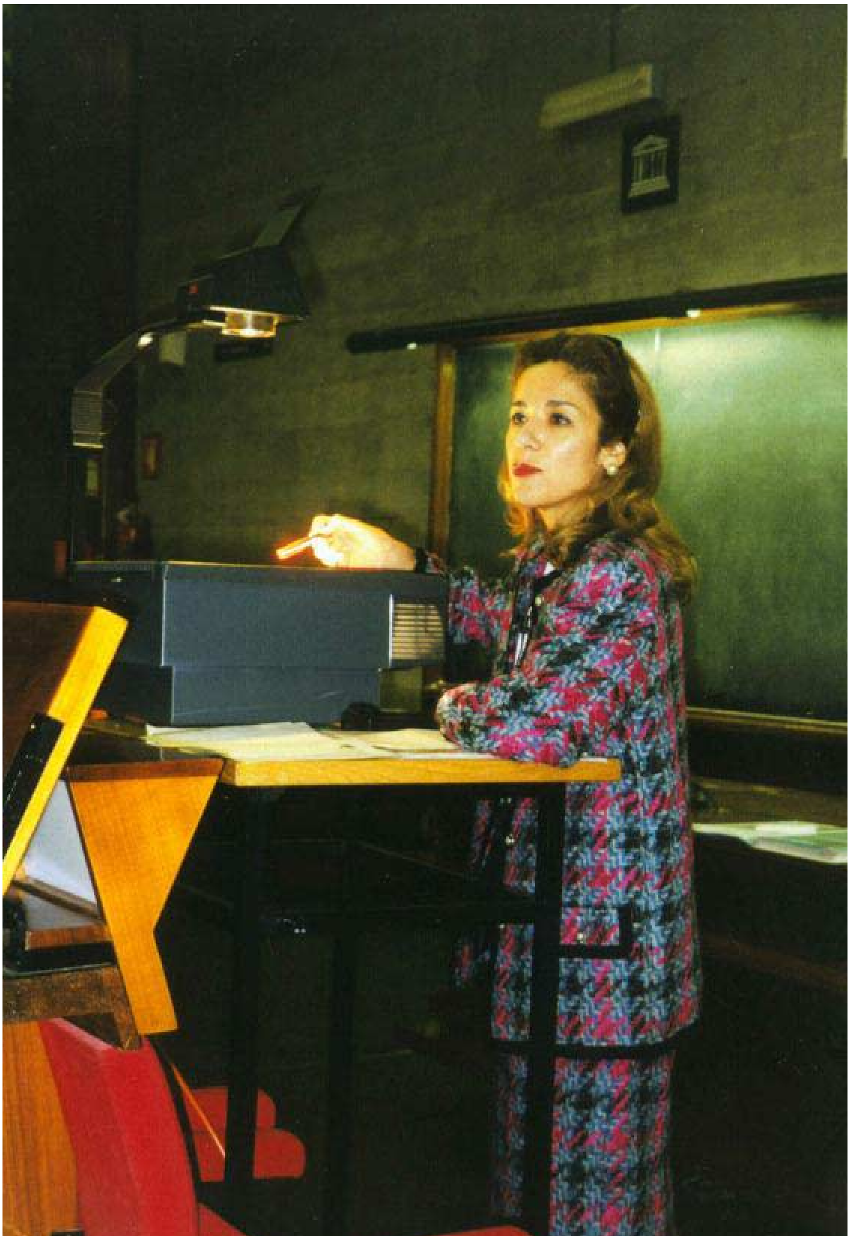
Sylvie Paycha

Tsou Sheung Tsun



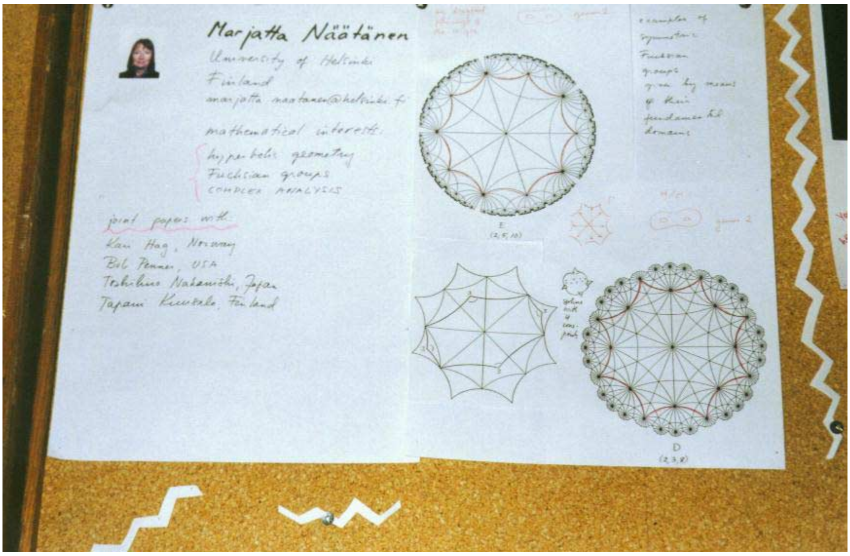
Lyudmilla Bordag

Valentina Golubeva

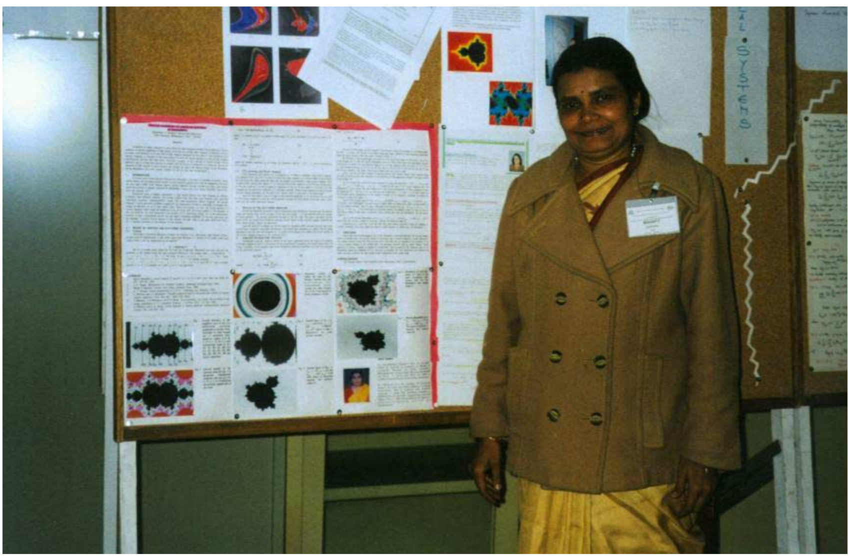


Ufuk Taneri

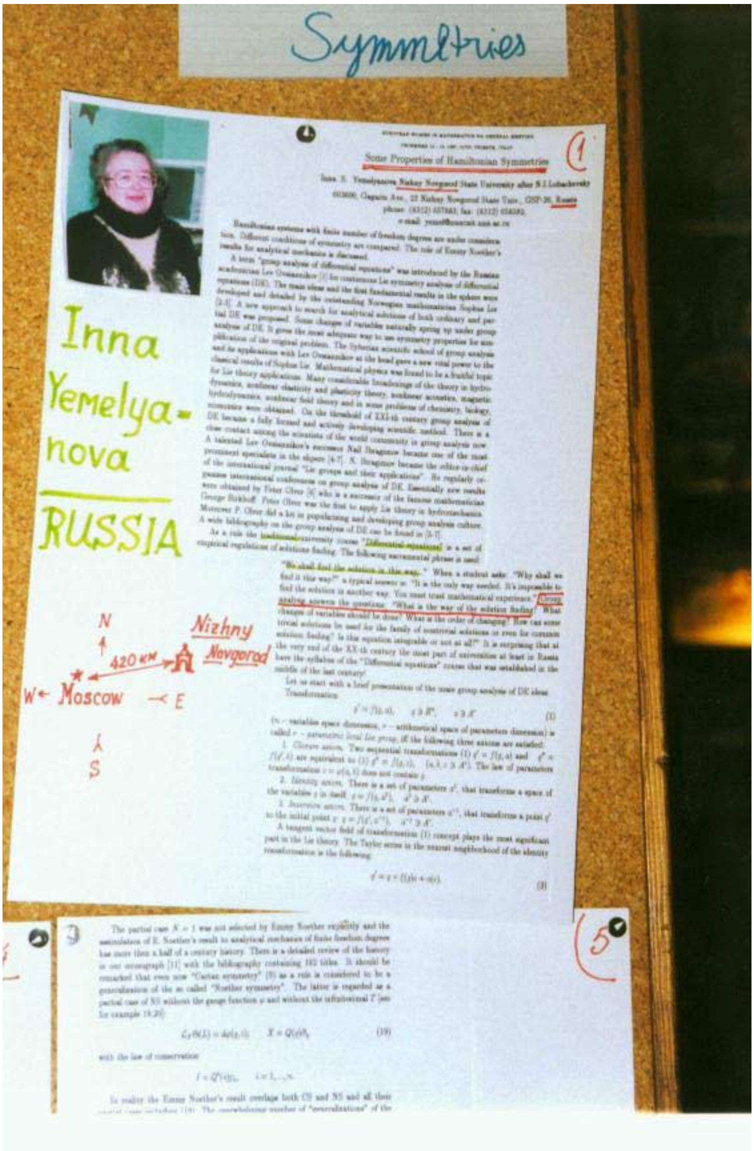
POSTERS



Poster by Marjatta Naatanen



Sanghamitra Mohanty with her poster



Poster by Inna Yemelyanova

DURING THE MEETING

Bodil Branner and Professor Narasimhan, of ICTP.

Listening to a lecture

Riitta Ulmanen and Laura Fainsilber

Participants from fSU

Bodil Branner, Mukhaya Rasulova and Gulbadan Matieva

Catherine Goldstein and Hilda Van der Veen

THE MATHEMATICAL PART

The mathematical programme constituted the main part of the EWM meeting and the mathematical papers form the main part of these Proceedings.

Included are edited versions of seven lectures given on the two chosen topics: Representations and p-Adic Numbers, six contributions which formed the basis of an interdisciplinary discussion on Symmetries and the abstracts of four of the posters which were on display at the meeting.

The talks were meant for a wide audience of mathematicians from all fields. The intent was both to introduce classical tools and fundamental questions in the fields, and to explain current research topics to non-specialists.

p-ADIC NUMBERS

A short course organized by CATHERINE GOLDSTEIN

The p-adic numbers were introduced at the turn of the century by the mathematician Kurt Hensel. From the beginning, they were seen both as useful tools (for instance in proofs) and as the missing pieces in order to get a more organized and coherent picture of the (mathematical) world. Hensel constructed them partly to provide arithmetics with as powerful a technique as power series expansion in function theory, but the mere possibility of this transfer was suggested by a broader program, for the promotion of which Leopold Kronecker, the advisor of Hensel, among others, was particularly important: to include in the same framework the theories of algebraic numbers, that is numbers z which are solutions of polynomial equations with rational coefficients $P(z) = 0$, and of algebraic functions, that is functions f which are solutions of polynomial equations $P(z, f(z)) = 0$. The first offered links to arithmetics, the second to complex analysis and Riemann surfaces.

Both the use of p-adic numbers for techniques and their role in an unified perspective are still operative today. The field of rational numbers can be completed either in the well-known manner to give rise to the real numbers or, for each prime p, to the various fields of p-adic numbers. An interesting suggestion is then to use simultaneously all the completions in order to enrich our understanding of the rational. Precise versions of this principle, some of its failure and adaptations will be discussed in the following papers. It suggests in any case to develop for each p-adic field methods, objects, tools which are used in the real field: one can hope for p-adic geometry, for p-adic differential equations, for p-adic physics. All these projects have received some attention from mathematicians, although with very different levels of depth and success.

In what follows, we won't be able to do justice to all these works, but we hope to give some incentive to explore them more closely. The first paper provides the basic definitions and intuitions which can help us understand what can be kept from our real habits, and what should be revised, sometimes drastically—making us in turn reconsider with a new curiosity the real situation. The second paper takes the point of view of model theory to give a more precise meaning to the idea that global situations (typically on the field of rational numbers) are more complicated than the local ones (typically on the field of real numbers or of p-adic numbers). The third paper discusses some applications of p-adic ideas in a geometrical environment, with, as a goal, an accessible introduction to some features of the recent proof of Fermat last theorem. In all three papers, examples will also be given to show how p-adic techniques are used in a variety of problems, classical or recent, p-adic or not. The missing pieces here are the more analytic aspects and the recent connections with physics. For some insights on these aspects, see G. Christol, p-adic Numbers and Ultrametricity, In *From*

Number Theory to Physics, 440–475, Springer, 1992.

We were very pleased to discover during the poster session several devoted to some aspects of p-adic numbers. They witness other aspects of the work on p-adics and, in particular, of on-going research activities linked to them in various areas, here: group theory, quadratic forms, valuation theory, dynamical systems. The corresponding abstracts will be found at the end of this chapter.

CATHERINE GOLDSTEIN

VIII^th EWM Meeting Proceedings
Trieste, Italy, 1997, 49–62

p-ADIC NUMBERS AND NON-ARCHIMEDEAN VALUATIONS

CAPI CORRALES RODRIGÁÑEZ

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Contents

1. Brief historical introduction. Over the last century, p-adic numbers and p-adic analysis have come to play a central role in number theory. This is due mainly to two reasons: they provide a very adequate language to describe congruences among integers, and they allow us to use the tools of analysis and calculus in the study of rational numbers, without having to choose between $\mathbb{R}$ and $\mathbb{C}$ as frameworks.

Out of the many ways of looking at p-adic numbers, perhaps the most direct one is to consider them as analogous to the real numbers. To do this, all we need is to reconsider our idea of what an absolute value is. From an algebraic point of view, there is no reason, for example, to consider the usual absolute value on $\mathbb{Q}$ as a given, that is, as the only one. Any function which assigns to each pair of rational numbers a third one and satisfies the same basic properties as the regular absolute value, should be just as good. If we start with the usual absolute value on $\mathbb{Q}$ and complete $\mathbb{Q}$ as a metric space by adding the limits of all Cauchy sequences, we obtain the field of real numbers $\mathbb{R}$; starting with a different absolute value, we get something else. What this something else is will be the subject of this talk.

This idea of first considering new ways to measure the "distance" between two rational numbers, and then constructing the corresponding completions, did not arise from a theoretical desire to generalize, but from several concrete situations in algebra and number theory. It turns out that each of the new metrics we will be able to construct on $\mathbb{Q}$ will be connected to a specific rational prime p, and will codify a great deal of arithmetical information related to that prime p. The first mathematician to introduce the p-adic numbers was Hensel in the 1920's, although E. Kummer used already the p-adic methods from 1894 on, as André Weil beautifully explains in his introduction to the complete works of E. Kummer (see [4] in the bibliography).

Kummer kept all through his life an epistolar relation with his student Kronecker, who wrote a thesis under his direction. Kronecker, in turn, was the teacher of Hensel. Hensel not only studied with Kronecker and Kummer, but he was also a student of Weierstrass and he knew well Cantor's definition of the real numbers (which was not the case for many of the mathematicians of his time), and Weber and Dedekind's ideas on the analogies between number fields and fields of functions. I explain all this because you will soon see it makes a lot of sense that it would be precisely a mathematician knowing all these tools who would introduce the notions and notations of p-adic numbers and p-adic methods.

Let us start with the simplest analogy, the one which probably motivated Hensel: the analogy between the ring of integers $\mathbb{Z}$ with its field of fractions $\mathbb{Q}$, and the ring of polynomials with complex coefficients $\mathbb{C}[X]$ with its field of fractions $\mathbb{C}(X)$.

An element $f(X)$ of $\mathbb{C}[X]$ is a rational function, that is, a quotient of two polynomials, $f(X) = P(X)/Q(X)$, with $P(X)$, $Q(X) \in \mathbb{C}[X]$ and $Q(X) \neq 0$. Similarly, a rational number $r \in \mathbb{Q}$ is a quotient of two integers, $r = a/b$, with a, $b \in \mathbb{Z}$, $b \neq 0$. The properties of both rings are very similar in terms of *factorization*: any polynomial can be expressed uniquely as $P(X) = \alpha(X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_n)$, with α , $\alpha_1,\ldots,\alpha_n \in \mathbb{C}$, while any integer can be expressed uniquely as ± 1 times a product of primes. This is precisely the analogy Hensel investigated: that between the primes $p \in \mathbb{Z}$ and the polynomials $(X-\alpha) \in \mathbb{C}[X]$.

Suppose we are given a specific polynomial $P(X)$ and an element $\alpha \in \mathbb{C}$. We can use a Taylor expansion to write the polynomial in the form

$$P(X) = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + \cdots + a_n(X-\alpha)^n \quad \text{with } a_i \in \mathbb{C};$$

the same strategy works for integers—w.l.o.g., we consider only positive integers—given a positive integer m and a prime p, we can always write m in base p, say,

$$m = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n, \quad \text{with } a_i \in \mathbb{Z}, 0 \leq a_i \leq p-1.$$

Both expansions give us "local" information. The expansion of $P(X)$ in powers of $(X-\alpha)$ tells us if $P(X)$ vanishes at α, and to what order. The expansion of m in base p shows if m is divisible by p and to what order.

Now, for polynomials, this can all be pushed to quotients. Taking $f(X) = P(X)/Q(X)$ in $\mathbb{C}(X)$ and $\alpha \in \mathbb{C}$, there is always a Laurent expansion—which can be obtained by division of the expansions of $P(X)$ and $Q(X)$—of the type

$$f(X) = a_N(x-\alpha)^N + a_{N+1}(x-\alpha)^{N+1} + \cdots, \quad \text{with } a_i \in \mathbb{C}.$$

This last object is more complicated than the previous Taylor expansion, since

(1) we can have $N < 0$, which would indicate that the multiplicity of α as a root of $Q(X)$ is larger than its multiplicity as a root of $P(X)$, or, in the language of analysis, that $f(X)$ has a pole at α of order $-N$; and
(2) the expansion will usually not be finite. Nevertheless it can be shown that the series $f(x)$ converges when x is close enough to, but different from, α.

Let us see an example. We consider the rational function $f(X) = X/(X-1)$, and we consider its expansions for different values of α. For $\alpha = 0$, we get

$$\frac{X}{(X-1)} = -X - X^2 - X^3 - X^4 - \cdots$$

which means that $f(0) = 0$ with multiplicity one. For $\alpha = 1$,

$$\frac{X}{(X-1)} = \frac{(1+X-1)}{(X-1)} = (X-1)^{-1} + 1,$$

so $f(X)$ has a pole of order one at $\alpha = 1$. Finally, for $\alpha = 2$ we have neither a pole nor a zero, since

$$\frac{X}{(X-1)} = 2 - (X-2) + (X-2)^2 + \cdots$$

The core of the matter is that every rational function can be expanded into a series of this type in terms of each of the "primes" $(X - \alpha)$. On the other hand, not every series comes from the expansion of a rational function, as we learned in calculus. As examples we have the expansions of $\sin x, \cos x$ or $\exp(x)$.

From an algebraic point of view, we read the situation in the following way: we have two fields, namely $\mathbb{C}(X)$ and the field which consists of all Laurent series in $(X - \alpha)$ (it turns out that the set of all Laurent series in $(X - \alpha)$ is also a field), which we call $\mathbb{C}((X-\alpha))$, and an *inclusion* or injective application $\mathbb{C}(X) \hookrightarrow \mathbb{C}((X-\alpha))$ between these two fields given by: $f(x) \mapsto$(expansion of $f(X)$ in powers of $(X - \alpha)$). There are infinitely many of these inclusions, one for each α , and each of them gives us information about the behaviour of $f(X)$ near α. Hensel's idea was to extend the analogy between $\mathbb{Z}$ and $\mathbb{C}[X]$ in such a way that it would include the construction of expansions for the rational numbers analogous to the ones we have constructed for the rational functions.

As was seen before, we already know the expansion for a positive integer m: it is just the base p representation of p,

$$m = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n, \quad \text{with } a_i \in \mathbb{Z},\ 0 \le a_i \le p-1,$$

which, as in the case of polynomials, is a finite expression.

To pass from positive integers to positive rationals, we just do as in the case of polynomials: we find the expansions for numerator and denominator and we divide formally. Let us see an example: we take $p = 3$, and consider the rational number 24/17. Since

$$\begin{aligned} 24 &= 0 + 2.3 + 2.3^2 \\ &= 2p + 2p^2 \end{aligned}$$

and

$$\begin{aligned} 17 &= 2 + 2.3 + 1.3^2 \\ &= 2 + 2p + p^2, \end{aligned}$$

we get

$$\begin{aligned}24/17 &= (2p+2p^2)/(2+2p+p^2)\\ &= p+p^3+2p^5+p^7+p^8+2p^9+\cdots.\end{aligned}$$

This process works for any positive rational number a/b, and the resulting series $a/b = \Sigma_{i\geq N} a_i p^i$, with $a_i \in \mathbb{Z}, 0 \leq a_i \leq p-1$, reflects the properties of a/b with respect to the prime p, in the sense that if a/b is in its lowest terms, $N < 0 \Rightarrow p \mid b$ and $p \nmid a$; $N = 0 \Rightarrow p \nmid ab$; $N > 0 \Rightarrow p \mid a$ and $p \nmid b$. This is called the *local behaviour* of a/b at p or near p.

Now, since our formal expressions can be multiplied, all we are missing in order to get the expansion of any rational number is the expansion of -1. We find that for any p,

$$-1 = (p-1) + (p-1)p + (p-1)p^2 + \cdots.$$

In this way we have obtained, in a formal way—i.e., we have no idea of whether these things converge in any form, or even if they make sense—, that every rational number x can be written as a Laurent series in powers of p truncated on the left,

$$a/b = a_N p^N + a_{N+1} p^{N+1} + \cdots$$

which we call the p-adic expansion of x (which, if x is an integer, is just its expansion in base p). It turns out that the set of all Laurent series in powers of p truncated on the left, with the operations of sum and multiplication, is a field larger than $\mathbb{Q}$, just as $\mathbb{C}((X-\alpha))$ was a field. We call it the field $\mathbb{Q}_p$ of p-adic numbers, and now we are ready to complete the analogy we had before, since what we have done is just to define an inclusion of fields $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ by means of the application $x \mapsto$ (p-adic expansion of x).

Before going on, let us see in a chart a summary of the analogy studied by Hensel:

Summary of the analogy studied by Hensel.

$\mathbb{Z} \subset \mathbb{Q} \hookrightarrow \mathbb{Q}_p$	$\mathbb{C}[X] \subset \mathbb{C}(X) \hookrightarrow \mathbb{C}((X-\alpha))$
$r \in \mathbb{Q}$ is $r = a/b$, with $a, b \in \mathbb{Z}, b \neq 0$.	$f(X) \in \mathbb{C}[X]$ is $f(X) = P(X)/Q(X)$, $P(X), Q(X) \in \mathbb{C}[X], \quad Q(X) \neq 0$

Factorization

$a = \pm p_1 \cdot p_2 \cdots p_n, p_i$ primes in $\mathbb{Z}$	$P(X) = \alpha(X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_n)$ with $\alpha, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$

Analogy studied by Hensel

primes $p \in \mathbb{Z}$	polynomials $(X-\alpha) \in \mathbb{C}[X]$.
Given a prime $p \in \mathbb{Z}$ $m = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n$	Given $\alpha \in \mathbb{C}$, $P(X) = a_0 + a_1(x-\alpha) + a_2(x-\alpha)^2 + \cdots + a_n(X-\alpha)^n$
with $a_i \in \mathbb{Z}$, $0 \le a_i \le p-1$ (w.l.o.g., $m > 0$)	with $a_i \in \mathbb{C}$.
Taking $a/b > 0$ in $\mathbb{Q}$	Taking $f(X) = P(X)/Q(X) \in \mathbb{C}(X)$ and $\alpha \in \mathbb{C}$,
$a/b = \sum_{i \ge N} a_i p^i$,	$f(X) = a_N(x-\alpha)^N + a_{N+1}(x-\alpha)^{N+1} + \ldots,$
with $a_i \in \mathbb{Z}, 0 \le a_i \le p-1$	with $a_i \in \mathbb{C} \ldots$

The above analogy has led us to a definition of a p-adic number (an element of the field $\mathbb{Q}_p$) as a formal object, something which is not very satisfactory. We will put remedy to this by showing how to construct $\mathbb{Q}_p$ as an analogue to the field of real numbers.

2. p-adic distances and valuations. We just mentioned that the definition of a p-adic number as a formal object is not very satisfactory. Let us explain with an example why. Formally, the number $\zeta = 4 + 5.7 + 4.7^2 + 0.7^3 + \cdots$, would give us, in some way, a solution to the equation $x^2 - 2 = 0$ in the sense that if we multiply the series by itself (as if it were an absolutely convergent series), we obtain

$$\begin{aligned}
\zeta^2 &= 16 + 40.7 + 57.7^2 + 40.7^3 + \cdots \\
\zeta^2 - 2 &= 14 + 40.7 + 57.7^2 + 40.7^3 + \cdots \\
&= (2+40)7 + 57.7^2 + 40.7^3 + \cdots \\
&= 0 + 42.7 + 57.7^2 + 40.7^3 + \cdots \\
&= 0 + 0.7 + (6+57)7^2 + 40.7^3 + \cdots \\
&= 0 + 0.7 + 0.7^2 + (9+40)7^3 + \cdots \\
&= \cdots \\
&= 0
\end{aligned}$$

But we cannot avoid asking the natural question: does it make sense to say that "were we to continue this process indefinitely, the number $\zeta = 4 + 5.7 + 4.7^2 + 0.7^3 + \cdots$, would give us a solution to the equation $x^2 - 2 = 0$?" To give this sentence some meaning, we need a new notion of convergence, according to which the sequence $x_0 = 4$, $x_1 = 4 + 5.7$, $x_2 = 4 + 5.7 + 4.7^2$, $x_3 = 4 + 5.7 + 4.7^2 + 0.7^3$, etc., will converge to ζ. What we have in our example is a sequence of integers $x_0, x_1, \ldots, x_n, \ldots$ in which, for each fixed n, $x_n^2 - 2$ is divisible by 7^{n+1}. The analogy with real numbers lead us to say that *two integers are 7-adically "close"* if their difference is divisible by a large exponent of 7. With this notion of "being close," we can now say that the squares of the integers in the sequence $x_0, x_1, \ldots, x_n, \ldots$ obtained above converge 7-adically to 2 when n grows.

Let us replace 7 by an arbitrary prime p, and then analyze for a minute what we just did. We have introduced, for each fixed prime p, what seems to be a new notion of distance between rational numbers: two rational numbers are considered "p-close" if their difference is divisible by a large power of p. How can we know whether this makes sense? Intuitively we see that if these new notions were in one way or another measures of distances between integers, associated to each of them we would also have a new "type" of absolute value, given by how far an integer is from 0 under each of these new "distances." This open a doorway for us, indicating the way to proceed.

Let us start with the definition of an absolute value on a field k, and then explore the possibilities, from our point of view, of such a definition in the concrete case of $\mathbb{Q}$.

Definition. An absolute value on a field k is a function $|\cdot| : k \to \mathbb{R}^+$ that satisfies the following conditions,

(i) $|x| = 0$ if and only if $x = 0$;
(ii) $|xy| = |x| \cdot |y|$ for all x, y in k;
(iii) $|x + y| \leq |x| + |y|$, for all x, y in k.

We say that an absolute value is *non-archimedean* if it satisfies the additional condition:

(iv) $|x + y| \leq \max\{|x|, |y|\}$ for all x, y in k;
otherwise we say it is *archimedean*.

Example 1. We take the field $\mathbb{Q}$ of rational numbers with the ordinary absolute value defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

This value is archimedean, since for $x = y = 1$, for example, condition (iv) is not satisfied.

Example 2. Let us try to construct an absolute value on $\mathbb{Q}$ associated with the notion of p-closeness defined before. From the defnition we gave of p-closeness, we deduce that a rational number x in smallest terms will be p-small (or p-close to zero) if its numerator is divided by a large power of p, and it will be p-large if its denominator is divided by a large power of p. Hence, all we need to focus on is the power of p "dividing" x (numerator or denominator). We do it the following way.

• **Step 1**: each rational number x can be written as $x = p^r(a/b)$, with $a, b, r \in \mathbb{Z}, a, b$ relatively prime. Since the integer r is determined by p and x, it makes sense to denote it by $v_p(x)$. Hence, we write $x = p^{v_p(x)}(a/b)$, with $(p, ab) = 1$, and we set $v_p(0) = \infty$.

• **Step 2**: we study the basic properties of the function $v_p : \mathbb{Q} \to \mathbb{R}$ we just defined. They are two: for all rational x, y we have

(a) $v_p(xy) = v_p(x) + v_p(y)$, and
(b) $v_p(x + y) \geq \min\{v_p(x), v_p(y)\}$.

Functions from a field into de real numbers satisfying these two properties are called *valuations* on the field. The valuation v_p is called the p-adic valuation on $\mathbb{Q}$.

• **Step 3**: this is the really astute step. If we compare the two properties a) and b) of the valuation with conditions ii) and iv) in the definition of absolute value, we see that they are very similar, except that in one the product has been turned into a sum (as when taking a logarithm), and in the other the inequality appears reversed. We can "unreverse" the inequality by changing the sign of v_p, and then change the sum into a product by putting it into a exponent. This suggests the following definitions, the crucial ones:

• For any rational number x we define the p-adic absolute value by

$$|x|_p := p^{-v_p(x)}, \quad \text{if } x \neq 0, \text{ and we set } |0|_p = 0.$$

• For any two rational numbers x, y we define their p-adic distance by

$$|x-y|_p := p^{-v_p(x-y)}, \qquad |0|_p = 0.$$

It is not difficult to check that $|x-y|_p$ is indeed a non-archimedean absolute value on $\mathbb{Q}$, that it satisfies conditions (i)–(iv) in the definition. Hence, we have constructed infinitely many new notions of absolute value for the field $\mathbb{Q}$, one for each choice of prime p. Around 1920, Ostrowski and Artin proved that the only absolute values (up to equivalence, where two absolute values are equivalent if they give way to the same metric on the field) one can define on the field $\mathbb{Q}$ are precisely the ordinary absolute value (which we will denote by $|\cdot|_\infty$) and the p-adic absolute values $|\cdot|_p$.

Example 3. This third example will serve both to show the generality of the theory we are developing, and to confirm Hensel's intuition on the similarity between $\mathbb{Q}$ and the field of rational functions.

• **Step 1**: First, for each polynomial $P(X) \in \mathbb{C}[X]$ (or $k[X], k$ any field) we define the valuation $v_\infty(P) = -\deg(P)$, and we extend this definition to rational functions by $v_\infty(f(X)) = v_\infty(P(X)/Q(X)) = v_\infty(P) - v_\infty(Q) = \deg(Q) - \deg(P)$. As in the p-adic case, using this valuation we can construct a non-archimedean absolute value on the field $\mathbb{C}(X)$ defined by $|f|_\infty = e^{-v_\infty(f)}$ for each $f(X) \in \mathbb{C}(X)$.

• **Step 2**: Now we can get other non-archimedean absolute values on $k(X)$ by imitating the definition of the p-adic absolute values, since $k[X]$ is a unique factorization domain. Just choose any irreducible polynomial $P(X)$ in $k[X]$ and proceed as before, defining first a valuation on $k(X)$ by counting the multiplicity of $P(X)$ as a factor of the different polynomials in $k[X]$, and then constructing from it the corresponding absolute value on $k(X)$.

Example 4. Let $\mathbb{Q}(i)$ be the field obtained by adjoining $i = \sqrt{-1}$ to the rational numbers, so $\mathbb{Q}(i) = \{(a+bi)/(c+di) \mid a,b,c,d \in \mathbb{Z}\} = \{a+bi \mid a,b \in \mathbb{Q}\}$. The "integers" in this field are the elements of $\mathbb{Z}[i] = \{a+bi \mid a,b \in Z\}$. It is not too difficult to check that this ring is a unique factorization domain with three types of primes:

(i) $1+i$ is prime (since $1+i = i(1-i)$ and i is a unit in $\mathbb{Z}[i], 1+i$ and $1-i$ give rise to the same prime in the ring $\mathbb{Z}[i]$);

(ii) if $p \in \mathbb{Z}$ is a prime number and $p \equiv 3 \pmod 4$, then p is a prime in $\mathbb{Z}[i]$;

(iii) if $p \in \mathbb{Z}$ is a prime number and $p \equiv 1 \pmod 4$, then there are two primes $x+iy$ and $x-iy$ in $\mathbb{Z}[i]$ satisfying $(x+iy)(x-iy) = p$.

In each of the three cases, we can use a prime $\pi \in \mathbb{Z}[i]$ to construct a π-adic valuation v_π and from it a π-adic absolute value $|\cdot|_\pi$ on $\mathbb{Q}(i)$ as before: $|\alpha|_\pi = c^{-v_\pi(\alpha)}$ for some fixed constant $c > 1$ and each $\alpha \in \mathbb{Q}(i)$.

Keeping in mind these four examples, an in particular the second one, the p-adic absolute values and distances, we return to our task of viewing the p-adic fields $\mathbb{Q}_p$ as analogues of the real numbers.

3. The p-adic numbers. When we complete $\mathbb{Q}$ with respect to the ordinary absolute value—by adding all the limits of Cauchy sequences—we know we obtain the field of real numbers $\mathbb{R}$. When we complete $\mathbb{Q}$ with respect to a p-adic distance—by adding also the limits of all Cauchy sequence with distances taken p-adically—we obtain precisely the p-adic field $\mathbb{Q}_p$ defined in section 1, namely

$$\mathbb{Q}_p = \left\{a_N p^N + a_{N+1} p^{N+1} + \cdots \mid a_i \in \mathbb{Z},\ 0 \le a_i \le p-1,\ N \in \mathbb{Z}\right\}$$

4. An application: the study of diophantine equations. A simple but quite spectacular example of how powerful the introduction of p-adic tools can be, is found in the study of diophantine equations. We all know that finding the integer solutions to diophantine equations—equations given by polynomials with integer coefficients—is a problem central in number theory and that it can be very difficult—think, for example, of the amount of time and work it has taken to solve Fermat's equation. Since the time of Kummer and Hensel, we know that when looking for the integer solutions of a polynomial equation, it can be very useful to search first its possible solutions mod m for different integer values of m, a problem that, for each value of m consists of checking finitely many possibilities. What do we mean by "working modulo m"?

Using the notation introduced by Gauss, we will say that two integers a and b *are congruent modulo* a third positive integer m, and we write $a \equiv b \pmod{m}$, if a and b produce the same remainder when divided by m. Another way of saying it is that m divides $a - b$. For example, we write $31 \equiv 3 \pmod{4}$—"31 is congruent to 3 modulo 4"—and we read "31 and 3 produce the same remainder when divided by 4," or "4 divides $31 - 3$."

Example. Let us look for the integer solutions to the equation $x^2 + y^2 + z^2 = N$, where N is an integer of the form $8k + 7$. In fact, let us see that this equation has no solutions in integers.

Any hypothetical solution (a, b, c) would produce a solution mod m for each m, since if we were to have the numerical equality $a^2 + b^2 + c^2 = N$, then, dividing by m and keeping the remainder in both sides of the equation would give us

$$a^2 + b^2 + c^2 \equiv N \pmod{m},$$

that is, (a, b, c) would also be a solution mod m for each integer choice of m. But, as it happens, modulo 8 our equation has no solutions, said differently,

$$x^2 + y^2 + z^2 \equiv N \pmod{8}$$

is not solvable, since if we plug into our equation the 8 possible values for x, y and z, we see that none of the resulting possibilities leads to a solution (the only squares mod 8 are 0, 1, 4, and so the possibilities we obtain combining them are $0, 1, 2, 3, 4, 5, , 6 \pmod 8$, none of them possibly congruent modulo 8 to 7, and thus to $N = 8k + 7$).

Hence, the hypothetical solution (a, b, c) cannot exist, and $x^2 + y^2 + z^2 = N$ is not solvable (which tells us that no integer of the from $8k + 7$ can be expressed as a sum of three squares).

In this way, we have replaced the infinite problem of finding the integer solutions to a diophantine equation by the problem—finite for each fixed choice of m—of finding its solutions modulo m. Also, thanks to the Chinese remainder theorem, we know that finding the solutions mod m to a diophantine equation can in turn be replaced by the easier one of finding the solutions to the equation modulo the different prime powers dividing m. "What an economy!," you may say. We have replaced one single equation by infinitely many equations! But, we answer, this new equations are all solvable. To solve the first single equation one has to be very, very clever. To solve these new ones, one only has to be very, very patient!

Now, Hensel's methods go further: he chooses not to be either stupid and patient nor quick and clever (and probably unsuccesful) but clever and lazy, so … he keeps thinking. In the above examples of polynomials we have used congruences only to find negative answers, that is, to show when a diophantine equation will not have solutions. The p-adic methods allow us to find solutions to a given equation modulo different primes and prime powers in a coherent way, and then use them to construct solutions in integers. This is known as a local-global method.

Clearly, since $\mathbb{Q} \subset \mathbb{Q}_p$ for all p, if a diophantine equation with coefficients in $\mathbb{Q}$ has no solutions in some of the p-adic fields $\mathbb{Q}_p$, it can't have them either in $\mathbb{Q}$. We now ask the reciprocal question: given a diophantine equation, or a system of diphantine equations with coefficients in $\mathbb{Q}$, can we get any information about its solutions in $\mathbb{Q}$ if we know its solutions in the different $\mathbb{Q}_p$ and $\mathbb{R}$?

The answer to this question relies largely on one of the most important algebraic properties of the p-adic numbers (and of other fields that, like $\mathbb{Q}_p$, are complete with respect to a non-archimedean valuation). It basically says that in many circumstances one can decide quite easily whether a polynomial has roots in $\mathbb{Z}_p$. The test involves first the construction of an "aproximate" root to the polynomial, and then verifying a condition on its derivative, and is called *Hensel's lemma.*

Hensel's lemma. *Let $P(X)$ be a polynomial with coefficients in $\mathbb{Z}_p$. Suppose that there exists a p-adic integer $a \in \mathbb{Z}_p$ such that $P(a) \equiv 0 (\bmod p\mathbb{Z}_p)$, and $P'(a) \pm 0 (\bmod p\mathbb{Z}_p)$. Then there exists a p-adic integer $\alpha \in \mathbb{Z}_p$ such that $a \equiv \alpha (\bmod p\mathbb{Z}_p)$ and $P(\alpha) = 0$.*

Example. Let us see whether 6 does have a square root in $\mathbb{Q}_5$, that is solve the equation $x^2 - 6 = 0$ in $\mathbb{Q}_5$. In this case, $P(X) = x^2 - 6, P'(X) = 2x$, and $a = 1$. We want to find integers $a_0, a_1, a_2, \ldots, 0 \le a_i \le 4$ such that

$$(a_0 + a_1 x 5 + a_2 x 5^2 + \cdots)^2 = 1 + 1x 5.$$

Comparing coefficients of $1 = 5^0$ on both sides gives $a_0^2 \equiv 1 \pmod 5$, and hence $a_0 = 1$ or 4. Let us take $a_0 = 1 = a$. Then, comparing coefficients of 5 on both sides gives $2a_1 x 5 \equiv 1x 5 \pmod{5^2}$, and so $2a_1 \equiv 1 \pmod 5$, and $a_1 = 3$. Proceeding this way we get a series

$$\alpha = 1 + 3.5 + 0.5^2 + 4.5^3 + \cdots,$$

where, after the choice of a_0, each a_i is uniquely determined. Had we chosen $a_0 = 4$, we would have gotten

$$-\alpha = 4 + 2.5 + 0.5^2 + 1.5^3 + \cdots$$

which reflects the fact that an element has exactly two square roots in $\mathbb{Q}_5$ if it has any.

This method of solving the equation $x^2 - 6 = 0$ in $\mathbb{Q}_5$ by first solving the congruence $a_0^2 - 6 \equiv 0 \pmod 5$, and then finding the remaining in a step-by-step process is so general that is precisely what forms the proof of Hensel's lemma.

such an approximation technique is essentially the same as Newton's method for finding a real root of a polynomial equation with real coefficients. That is why Hensel's lemma is often called the p-adic Newton's lemma.

In one respect, the p-adic Newton's method (Hensel's lemma) is better than Newton's method in the real case. The p-adic methos is guaranteed to converge, while Newton's real method often converges, but not always. For example if we take $P(X) = x^3 - x$, and make the choice $a = 1/\sqrt{5}$, then the situation we get is given by the figure, a situation which is impossible in $\mathbb{Q}_p$.

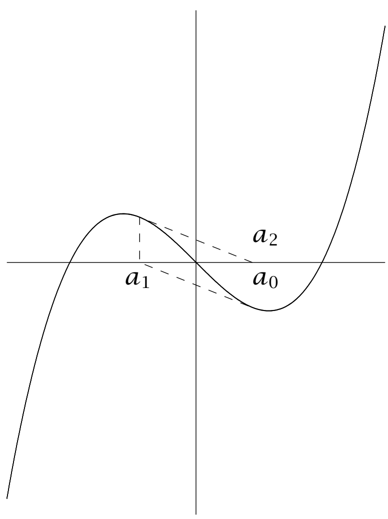

FIGURE 1. Failure of Newton's method in the real case (after N. Koblitz [3]).

Hensel's lemma introduces us to the theory of polynomials on the p-adic fields. As a consequence of this lemma, given a polynomial with integer coefficients one can decide easily whether it has roots in $\mathbb{Z}_p$, since it is enough to find roots mod p, a finite verification. The "same" is true for $\mathbb{R}$, where we can usually decide whether there are roots by sign considerations. Suppose, however, that we want to look for roots in $\mathbb{Q}$.

If we agree to call $\mathbb{R} = \mathbb{Q}_\infty$, we have already noticed that there are no rational roots if there is some $p \leq \infty$ for which there are no p-adic roots (∞-adic means real here). One way of reading this situation is following Hensel's original analogy: the p-adic fields (including $\mathbb{R}$) are analogous to fields of Laurent expansions, and correspond to "local" information about the prime "p." The fact that roots in $\mathbb{Q}$ automatically mean roots in $\mathbb{Q}_p$ for all p, means that a "global" root is also a "local" root at each p, that is, "everywhere."

Now, the converse question is the interesting one: could "local" roots be kind of "patched together" to give a "global" root?

That is, could the existence of solutions in $\mathbb{Q}_p$ for all primes p guarantee the existence of a solution on $\mathbb{Q}$? This question suggests what is known as the *Local-Global principle or Hasse-Minkowski principle.*

Local-Global principle. The existence or non-existence of solutions in $\mathbb{Q}$ (global solutions) of a diophantine equation can be detected by studying, for each prime $p \leq \infty$, the solutions of the equation in $\mathbb{Q}_p$ (local solutions).

The simplest example of diophantine equations in which this principle hold is that of quadratic forms over $\mathbb{Q}$. (We recall here that at the Warwick meeting of EWM in 1988, Eva Bayer spoke to us on the Local-Global principle for different fields.)

Hasse-Minkowski theorem. *Let F be a quadratic form in n variables (that is, a homogeneous polynomial of degree 2 in n variables). The equation $F = 0$ has non-trivial solutions in $\mathbb{Q}$ if and only if it has non-trivial solutions in $\mathbb{Q}_p$ for each $p \leq \infty$.*

Example 1. Let a, b and c be rational numbers, square-free and pairwise relatively prime. Then the equation

$$ax^2 + by^2 + cx^2 = 0$$

has non-trivial solutions in $\mathbb{Q}$ if and only if the following conditions are satisfied:

(i) a, b and c are not all positive or all negative;

(ii) for each odd prime p dividing a (or b, or c) there exists an integer r such that $b + r^2c \equiv 0 \pmod{p}$;

(iii) if a, b and c are all odd, then there are two of them whose sum is divisible by 4;

(iv) if a is even (or b, or c) then either $b + c$ or $a + b + c$ is divisible by 8.

Example 2. The Hasse-Minkowski principle does not hold for cubic equations.

Selmer gave the example

$$3x^3 + 4y^3 + 5x^3 = 0.$$

He showed that this equation has no integer solutions other than $(0, 0, 0)$. However, it can be checked that for every integer m, the congruence

$$3x^3 + 4y^3 + 5x^3 \equiv 0 \pmod{m}$$

has a solution in integers with no common factors.

5. The peculiarities of the p-adics. Let us now point out some of the more characteristic (and counter-intuitive!) properties of the p-adic valuations.

Algebraic peculiarities.

(1) Every p-adic number $a \in \mathbb{Q}_p$ has a unique expansion of the type

$$a = p^m(a_0 + a_1 p + \cdots + a_n p^n + \cdots),$$

with $m = v_p(a)$ and $1 \le a_0 \le p-1$, $0 \le a_n \le p-1$, $n = 1, 2, 3, \ldots$.

(2) We call $\mathbb{Z}_p$ the set of all $a \in \mathbb{Q}_p$ such that $|a|_p \le 1$. This is the set of all elements in $\mathbb{Q}_p$ whose p-adic expansion has no negative powers, and it is called the ring of p-adic integers or the ring associated to the p-adic valuation. Its subset $p\mathbb{Z}_p$ consisting of those elements in $\mathbb{Z}_p$ of the form $a = a_1 p + \cdots + a_n p^n + \cdots$ is an ideal, in fact its only maximal ideal. In general, given a valuation on a field k, the set of all elements in k with valuation ≥ 0 forms a ring, called the ring of the valuation; this ring has a unique maximal ideal consisting of the elements with valuation > 0. Reciprocally, an arbitrary ring is called a valuation ring if it coïncides with the ring of some valuation of its field of fractions. In our case, the set of integers $\mathbb{Z}$ form a dense subset of the ring $\mathbb{Z}_p$.

(3) In the case of the ordinary absolute value, once we obtain $\mathbb{R}$, we return to the original question: the resolution of diophantine equations. At this point it seems a good idea to add to $\mathbb{R}$ "numbers" that would provide solutions to equations of the type of $x^2 + 1 = 0$. And then something wonderful happens: once we have introduced $i = (-1)^{1/2}$ and we have defined the field of complex numbers $\mathbb{C}$, the following properties are verified:

(i) every polynomial equation with coefficients in $\mathbb{C}$ (so in particular in $\mathbb{Z}$) has all of its solutions in $\mathbb{C}$ (we say that $\mathbb{C}$ is an algebraically closed field), and

(ii) there is only one way of extending the ordinary absolute value on $\mathbb{R}$ to $\mathbb{C}$; with respect to this absolute value $\mathbb{C}$ is complete, i.e., every complex Cauchy sequence has a limit in $\mathbb{C}$.

Hence, the algebraic process ends in $\mathbb{C}$, algebraic extension of $\mathbb{R}$ of degree 2 (meaning that it is obtained by adding to R the solutions to a polynomial equation of degree 2, $x^2 + 1 = 0$). $\mathbb{C}$ is a field which is algebraically closed and complete with respect to the ordinary distance (Archimedean). Unfortunately, the case of the p-adic absolute values is much more complicated. When we add to $\mathbb{Q}$ all the limits to Cauchy sequences with respect to each $|\cdot|_p$, we obtain the field $\mathbb{Q}_p$ of the p-adic numbers, which is not algebraically closed (as R was not) (see, for example, [3] in the bibliography). Next, starting with each $\mathbb{Q}_p$ in order to obtain an extension which is algebraically closed we need to add an infinity of fields obtained from solutions to polynomial equations of degree larger and larger. And not only this. Once we obtain an algebraically closed extension of $\mathbb{Q}_p$, which we denote by $\mathbb{Q}_p^{al}$, it turns out that such an extension is not complete with respect to $|\cdot|_p$. We must add once more the limits of all Cauchy sequences (this time with their elements in $\mathbb{Q}_p^{al}$), in order to obtain a huge field that will be, finally, both algebraically closed and complete, and which is denoted by Ωp.

Geometric peculiarities.

(1) *Every triangle is isosceles with respect to any p-adic* valuation.

(2) We define

$$S_r(a) = \{x \in \mathbb{Q}_p; |x-a| = r\};$$
$$D_r(a) = \{x \in \mathbb{Q}_p; |x-a| \leq r\};$$
$$D_r^-(a) = \{x \in \mathbb{Q}_p; |x-a| < r\}.$$

Its turns out that every point of $D_r(a)$ functions as center, that is, $D_r(a) = D_r(b)$, for all $b \in D_r(a)$. Said differently with respect to the p-adic distance, *every point of a closed disc can be chosen as center*. This is not the only peculiar behaviour of p-adic discs. It also happens that given two different discs, they are either disjoint, or one is striclty contained in the other.

Analytic/topological peculiarities.

(1) In general, if we take a point x in a field K, we define the connected component of x in K to be the union of all connected sets that contain x. It can be described as the largest connected set containing x. For example, if $K = \mathbb{R}$, then the connected component of any point x in $\mathbb{R}$ is all of $\mathbb{R}$, simply because $\mathbb{R}$ is connected. Things are very different in $\mathbb{Q}_p$: the connected component of any point x in $\mathbb{Q}_p$ is the set $\{x\}$ consisting of only that point. What this says is that there are really no interesting connected sets in $\mathbb{Q}_p$: only the sets with one single element are connected. In fact, we have even more peculiar behaviours in $\mathbb{Q}_p$. For example, every open ball is the disjoint union of open balls. So that open balls are disconnected in a rather dramatic way in $\mathbb{Q}_p$!

(2) The set $S_r(a)$ is open in a topological sense, because every point x in it has a disc about it, for example $D_r^-(x)$ contained in $S_r(a)$. But then, any union of S's is open. Both $D_r(a)$ and $D_r^-(a)$, as well as their complements, are such unions: for example,

$$D_r^-(a) = \cup_{c<r} S_a(c),$$
$$D_r(a) = S_r(a) \cup D_r^-(a).$$

Hence both $D_r(a)$ and $D_r^-(a)$ are simultaneously open and closed.

(3) A set X is called compact if any collection of open sets which covers x has a finite subcollection which also covers X. We know that this rather un-intuitive definition is very important in classical analysis. In $\mathbb{R}$, for example, compact sets are precisely the closed and bounded sets. Another important property in classical analysis is the local compactness of $\mathbb{R}$: a set X is locally compact if every point has a neighbourhood which is compact. In the p-adic situation we have that $\mathbb{Z}_p$ is compact and is $\mathbb{Q}_p$ locally compact. Thus $\mathbb{Q}_p$ is a locally compact field (so we can have analysis in it) but at the same time totally discrete (so it will be a different type of analysis).

6. Generalizations to arbitrary fields, rings and groups. All that has been done can be generalized to arbitrary fields, rings and groups.

6.1. Extensions of valuations and valuation rings to larger fields. Some types of rings have essentially the same properties as the rings $\mathbb{Z}_p$ and we are going to call them Henselian rings, because Hensel's techniques can be carried on to this type of rings. These properties are two, and so we say that a ring R is *Henselian* if it satisfies:

(i) It has a unique maximal ideal $\mathcal{M}$ (or equivalently, the set of all non-invertible elements in R form an ideal different from R), and

(ii) Hensel's lemma: Let $P(X)$ be a monic polynomial of degree n with coefficients in R, and $P(X)$ the polynomial we obtain when we consider the coefficients of P in $R/\mathcal{M}$. If there exist coprime monic polynomials $g(X)$, $h(X)$ with coefficients in $R/\mathcal{M}$ with degrees r, $n-r$ such that $P(X) = g(X)h(X)$, then we can "lift" $g(X)$ and $h(X)$ back to polynomials $G(X)$, $H(X)$ with coefficients in R such that $P(X) = G(X)H(X)$.

6.2. General definition of a valuation. The general valuations on fields generalize the simple facts that we have observed for the field of rational functions and the field of rational numbers. A one-valued function v on a field k upon a simply ordered group G is called a *valuation* if

(i) $v(\alpha\beta) = v(\alpha) + v(\beta)$, and

(ii) $v(\alpha+\beta) \geq \min\{v(\alpha), v(\beta)\}$.

We make the additional convention that $v(0) = \infty$. Associated to the valuation v and the field k we have the following objects:

(a) Valuation group: $\mathrm{Im}(v)$, a subgroup of G.

(b) Valuation ring $\mathcal{O}(v)$: set of elements x in K with $v(x) \geq 0$ (example: $\mathbb{Z}_p$).

(c) Valuation ideal $\mathcal{P}(v)$: set of elements x in K with $v(x) > 0$, unique maximal ideal in the valuation ring (example: $p\mathbb{Z}_p$).

(d) Residue class field: $\mathcal{O}(v)/\mathcal{P}(v)$ (example: $\mathbb{Z}_p/p\mathbb{Z}_p$, isomorphic to $\mathbb{F}_p$, the finite field with p elements)

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SOME p-ADIC MODEL THEORY

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Introduction. One could think that $\mathbb{Z}$ or $\mathbb{Q}$ are "simple" objects, and $\mathbb{R}$ or the $\mathbb{Q}_p$'s complicated, as they are constructed starting from $\mathbb{Q}$. But if we think algebraically, the situation is reversed, $\mathbb{Q}$ is complicated while $\mathbb{R}$ and the $\mathbb{Q}_p$'s are simple. For instance we are able to solve polynomial equations in $\mathbb{R}$ or $\mathbb{Q}_p$, while we are unable to decide whether such equations have an integral solution. This last point is the content of the negative answer given by Matiyasevich to Hilbert's tenth problem; in other words the Diophantine problem over $\mathbb{Z}$ is undecidable ([29], see also [28]) and it is conjectured that it is impossible over $\mathbb{Q}$ as well. On the other hand, $\mathbb{R}$ and the $\mathbb{Q}_p$'s are "decidable:" there is an algorithm recognizing, after finitely many steps, not only whether a system of polynomial equations has a solution, but much more generally whether certain kinds of assertions, the "first-order sentences" which we are going to define below, are true or not (see [30]). The first-order logic is the mathematical theory taking the first-order sentences as its basic objects. In this framework (in which we will work from now on), the $\mathbb{Q}_p$'s are seen as examples of Henselian valued fields with residue field $\mathbb{F}_p$ and value group "almost" $\mathbb{Z}$, (again we are going to define this "almost"). Other properties of $\mathbb{Q}_p$, like local compactness or the fact that the value group is exactly $\mathbb{Z}$, will not be taken into account in this context.

1. True sentences in $\mathbb{Q}_p$. Typical fundamental objects will be expressions, which will be called formulas, of the form

$$\forall x_1 \exists x_2 \exists x_3 \left[3x_3x_4^2 + x_2x_1 + x_2^3 - x_4 = 0 \wedge x_1x_2x_3 \neq 0\right].$$

How are they done? We see that there are variables (x_1, x_2, x_3, x_4), which will here represent elements of $\mathbb{Q}_p$ (not subsets, nor functions on $\mathbb{Q}_p$) and there are quantifiers ($\forall$ or $\exists$) which are applied to some of the variables. Since we are considering $\mathbb{Q}_p$ as a ring, these variables can be added or multiplied; also there is a "1," hence an "n" for each integer n. This explains for example the coefficient "3" in the monomial $3x_3x_4^2$. Last we have the symbol $\wedge$ (which should be understood as a conjunction), we will also authorize $\vee$ (which represents a disjunction), $\neg$ (negation), $\bigwedge$ (finite conjunction) and $\bigvee$ (finite disjunction) (one can think of the relation between $\wedge$ and $\bigwedge$, or between $\vee$ and $\bigvee$, as similar to the relation between $+$ and Σ, or between $\cdot$ and Π).

Let us sum up this construction.

DEFINITIONS (comments included!).

(1) A (first-order) *formula in the ring language* has the following form (up to logical equivalence)

$$Q_1x_1Q_2x_2\cdots Q_nx_n \bigvee_{i=1}^{s}\left[\bigwedge_{i=1}^{r} P_{ij}(x_1,\dots,x_n)=0 \wedge \bigwedge_{i=1}^{r} R_{ij}(x_1,\dots,x_n)\neq 0\right],$$

where $Q_1, Q_2, \dots, Q_n$ are existential or universal quantifiers and the P_{ij}'s and the R_{ij}'s are polynomials with abstract integer coefficients.

The coefficients are abstract integers in the sense that, in a field of positive characteristic, it may happen that $n = 0$ for a non zero integer n. "Up to logical equivalence" means that we have made use of some logical properties in order to get this simple form of formulas: distributivity of $\wedge$ on $\vee$ in order to put first all disjunctions, then all conjunctions, commutativity and associativity of $\wedge$ in order to regroup first all equations then all inequations, and some extra rules in order to put all quantifiers first. We also made use of properties of $+$ and $\cdot$ in rings. Indeed, the non quantified part of the formula consists of polynomial equations because (first) equality is an allowed symbol, (secondly) the ring language is $\{0, 1, +, -, \cdot\}$, and we know that addition and multiplication, starting with variables or constants, with the usual rules of associativity, commutativity and distributivity, give rise to polynomials. Following commun usage in maths, we study

- groups with the language $\{e, \cdot, ^{-1}\}$,
- Abelian groups, with the language $\{0, +, -\}$,
- ordered Abelian groups with the language $\{0, +, -, \leq\}$.

(2) A general language has the following form

$$\mathcal{L} = \{a_1, a_2, \dots, f_1, f_2, \dots, R_1, R_2, \dots\}$$

where

- each a_i is a symbol for a constant,
- each f_i a symbol for a function, with a given number n_i of variables,
- each R_i a symbol for a relation, with a given number m_i of variables.

A formula of $\mathcal{L}$ has the following form[1]

$$Q_1x_1Q_2x_2\cdots Q_nx_n \bigvee_{i}\bigwedge_{j} P_{ij},$$

where P_{ij} = (possibly $\neg$) Q_{ij} and each Q_{ij} has one of the following forms (these are examples!)

$$x_{i_1} = x_{i_2}$$
$$f_1(x_{i_1}, f_2(a_1, a_5), \dots, x_{n_1}) = x_{i_2} \quad \text{(here } n_2 = 2 \text{ and } n_1 \geq 3)$$
$$R_1(x_{i_1}, x_{i_2}, a_1) \quad \text{(here } m_1 = 3)$$
$$R_2(f_4(x_{i_n}, x_1), a_3) \quad \text{(here } m_2 = n_4 = 2)$$
$$\dots$$

[1] see the exact definition in Appendix A.

(Note the special status of the equality =, whose use is always allowed in formulas.)

So a general language consists of a certain number of symbols, for constants, functions or relations. Choosing in which language to consider a structure is an important operation, which determines what substructures or homomorphisms are. For example, depending whether we consider $\mathbb{R}$ as a field, as an additive ordered group or as an ordered set, the automorphisms are the identity, positive homotheties or increasing bijections. Another example: In the ring language, a substructure is a subring, in the field language, a subfield. For all this see [26].

Valued fields are usually studied in the language $\{0, 1, +, -, \cdot, A\}$ where A is a unary symbol of relation interpreted as $A(x)$ iff $v(x) \geq 0$ (see Appendix B). In general the valuation gives additional information over the field, but in the case of $\mathbb{Q}_p$, an interesting fact occurs.

Fact. *The structures of field and valued field of $\mathbb{Q}_p$ are equivalent.*

Proof. If $p \neq 2$, then $v(x) \geq 0$ iff $\exists y\, 1 + px^2 = y^2$. If $p = 2$, then $v(x) \geq 0$ iff $\exists y\, 1 + px^3 = y^3$. Indeed, if $v(x) \geq 0$ then $1 + px^2 \equiv 1 \mod p$, which is a square in $\mathbb{F}_p$, hence lifts to a square in $\mathbb{Q}_p$ if $p \neq 2$. If $v(x) < 0$ then $v(1+px^2) = v(px^2) = 1 + 2v(x)$ not even, and $1 + px^2$ is definitely not a square. Same argument for $p = 2$. This shows that a formula in the language of valued field can be translated in a formula only in the language of ring. □

(3) A *sentence* or *axiom* (in a certain language) is a formula without free (i.e. non quantified) variable.

(4) A property is *axiomatizable* (in a certain language) if it is equivalent to the satisfaction of a family of axioms. To *axiomatize* a structure $\mathbb{M}$ is to find an *axiom system* for $\mathbb{M}$, i.e. some list of sentences

- which are true in $\mathbb{M}$,
- which we are able to enumerate.
- Finally we want this system to be *complete*, which means that any other structure satisfying it will satisfy exactly the same sentences as $\mathbb{M}$ does.

The delicate point is the second one: What does an infinite list mean? How can we describe it, without using " ... "? If we are looking for an axiom system for $\mathbb{M}$, why not take the set of all true sentences of $\mathbb{M}$? Precisely because this is a pile: it is not an acceptable list for which we can explicitely write the n-th term. I am not going to define a "right" list (this is the first task of the theory of recursivity, see for example [27]) but will only give examples:

"Being a field" can be said with an axiom in the ring language (each usual rule defining a field is expressed by an axiom).

"Being a field of characteristic zero" is equivalent to the conjunction of the previous axioms plus an axiom scheme, whose n-th term is $n \neq 0$, for each positive integer n. You should be aware that the "n" in the sentence above contains hidden " ... ": $n = 1 + \cdots + 1, n$ times. An exact definition must proceed inductively, defining the $(n+1)$-th axiom from the n-th one.

(5) Two structures in the same language are *elementarily equivalent* if they satisfy the same sentences. Notation: $M \equiv N$.

(6) If M is a structure, $a \in M$ and ϕ is a formula, the notation "$M \vDash \phi(a)$" means that, in M, $\phi(a)$ holds.

Theorem (Ax-Kochen-Eršov, [1, 2, 5]). (1) *A valued field (K, v) is elementarily equivalent to $(\mathbb{Q}_p, v_p)$ iff*

- *K has characteristic zero*
- *$K/v \simeq \mathbb{F}_p$*
- *v is Henselian*
- *$(vK, 0, v(p), +, -, \leq) \equiv (\mathbb{Z}, 0, 1, +, -, \leq)$.*

We have previously seen how it is possible to express that the characteristic is zero by means of infinitely many first-order sentences. The fact $K/v \simeq \mathbb{F}_p$ is axiomatized by the following axioms: $v(p) > 0$ and

$$\forall x \{v(x) \geq 0 \Rightarrow [v(x) > 0 \vee v(x-1) > 0 \vee \cdots \vee v(x-(p-1)) > 0]\}$$

and Hensel's Lemma by a scheme of axioms, one for each degree of the polynomial.

An equivalent formulation of the result is that the list above axiomatises $\mathbb{Q}_p$.

(2) *A valued field (K, v) with residual characteristic zero is axiomatized by*

- *the axioms expressing Hensel's Lemma,*
- *the axioms satisfied by the residue field*
- *the axioms satisfied by the valuation group (as an ordered group).*

Note that axioms over K/v or vK can easily be translated into axioms over (K, v).

2. An application: the asymptotic solution of Artin's conjecture

Definition. Let i and d be integers. A field K is called $C_i(d)$ if every homogeneous polynomial of degree d in at least $d^i + 1$ variables has a non trivial zero. A field K is called C_i if it is $C_i(d)$ for all integers d.

Examples.

(1) By definition C_0 means algebraically closed.

(2) An orderable field cannot be $C_2(d)$ for any d: consider the form $X_1^2 + X_2^2 + \cdots + X_{d^2+1}^2$.

(3) Every finite field is C_1 (see [33]).

(4) If K is C_i, then $K(X)$ and the formal power series field $K((X))$ are C_{i+1} (see [33]).

By (2) and (3), $\mathbb{F}_p((X))$ is C_2. Now $\mathbb{F}_p((X))$ and $\mathbb{Q}_p$ are very similar: they are both complete valued fields, valued in $\mathbb{Z}$ and with residue field $\mathbb{F}_p$. But the first one is of characteristic p and the second one characteristic 0. Artin conjectured that $\mathbb{Q}_p$ is C_2 too. Ax and Kochen, using the theorem stated in the first section, gave an asymptotic positive answer.

Theorem. *Let us fix d. Then all but finitely many $\mathbb{Q}_p$'s are $C_2(d)$.*

The proof uses the important tool of *ultraproducts* (see [26]).

Let $(\mathbb{M}_i)_{i \in I}$ be a family of structures in a same language $\mathcal{L}$, for examples all rings, or all ordered groups, and U an ultrafilter over the index family I. We consider

- the Cartesian product $P := \Pi_{i \in I} M_i$

- the equivalence relation on P

$$(x_i) \sim (y_i) \quad \text{iff } \{i \in I; x_i = y_i\} \in U.$$

The ultraproduct of the $(\mathbb{M}_i)$'s relative to U is defined to be the structure $\mathbb{M}$ of $\mathscr{L}$

- with underlying set $M := P/\sim$
- with functions of $\mathscr{L}$ defined as follows

$$f_{\mathbb{M}}((x_i)_{i\in I}) \text{ is the class of } \left(f_{\mathbb{M}_i}(x_i)\right)_{i\in I} \text{ modulo } U$$

- and relations

$$R_{\mathbb{M}}((x_i)_{i\in I}) \quad \text{iff } \{i \in I; \mathbb{M}_i \models R(x_i)\} \in U.$$

We then have

Łoš's Theorem. *Let ϕ be a formula in $\mathscr{L}$ and $x \in M$, then*

$$M \models \phi(x) \quad \textit{iff } \{i; M_i \models \phi(x_i)\} \in U.$$

Examples of ultraproducts. (1) Let U be a non trivial ultrafilter over $I = P :=$ {prime numbers}. Then the ultraproduct F of the $\mathbb{F}_p$'s relative to U has characteristic zero.

Proof. Let us fix an arbitrary q. The axiom $q = 0$ is satisfied only in $\mathbb{F}_q$, i.e.

$$\{p \in P; \mathbb{F}_p \models q \neq 0\} = P \setminus \{q\},$$

a set which belongs to U since this ultrafilter is non trivial. Hence $F \models q \neq 0$ for each prime q. □

(2) If U is the principal ultrafilter generated by i_0, then the ultraproduct of the M_i's relative to U is isomorphic to M_{i_0}.

Proof of the theorem. Let us consider a non trivial ultrafilter U over the set of prime numbers and the ultraproducts $\mathbb{F}_p((X))^U$ and $\mathbb{Q}_p^U$. They are both Henselian valued fields, with value group $\mathbb{Z}^U$ and residue field $\mathbb{F}_p^U$. The second one has characteristic zero, as does the first one if U is non trivial. Therefore they are elementarily equivalent for any such U, which implies that the property C_2 can be transfered from $\mathbb{F}_p((X))^U$ to $\mathbb{Q}_p^U$. We also know that

$$\mathbb{Q}_p^U \models C_2(d) \quad \text{iff } \{p; \mathbb{Q}_p \models C_2(d)\} \in U.$$

Consequently

$$\{p; \mathbb{Q}_p \models C_2(d)\} \in U$$

for any non trivial U, which means that the complement of this subset in I is finite. □

Artin's Conjecture turned out to be false when Terjanian (see Appendix C) and other people proved: no $\mathbb{Q}_p$ is C_2. This shows the interest of the asymptotic positive answer by Ax and Kochen. Further their proof gives a precise and exact content to the intuition Artin had: Certainly $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ are very similar, but they have different characteristic; this problem disappears if we compare the classes of all $\mathbb{Q}_p$'s and all $\mathbb{F}_p((t))$'s, for all prime p's.

3. Formulas with free variables or definable subsets of $\mathbb{Q}_p^m$. Let us consider the correspondence which, to a formula, associates the subset of points satisfying it, in the same way as in geometry we associate to an equation the geometrical locus of its solutions. Let us call such a subset "definable." The set of all definable subsets is clearly a Boolean algebra, since

- the intersection of two such subsets is defined by the conjunction of the formulas defining each of them
- the union of subsets corresponds to the disjunction of formulas,
- the complementation to the negation.

Further, if we let dimensions vary,

- the class of all definable subsets is closed under projection, which corresponds to existential quantification.
- It contains hyperplanes which are parallel to coordinate hyperplanes, defined by formulas $X_i = a$,
- and hyperplanes parallel to diagonal hyperplanes, defined by formulas $X_i = X_j$.

As we see on the example of the hyperplanes of equation $X_i = a$, we allow parameters from the structure.

The other definable subsets depend on the structure we are considering. For example, the graph of any function of the language is definable, as is the set of elements satisfying any relation of the language. We have to add some other subsets and then close this family under Boolean combination and projection. Is there any possibilty of describing this class in a simple way? Algebraically closed fields supply a nice example.

Example. Let K be an algebraically closed field. Call a subset of some K^n *Zariski closed* if it is the set of solution of some equational system, and *constructible* if it is a Boolean combination of Zariski closed subsets. Now a theorem of Chevalley asserts that the projection of any constructible set is constructible. So definable and constructible subsets are the same, and we do not need projection in order to generate the class of definable subsets: The model theoretical formulation of this is that algebraically closed fields "eliminate quantifiers" in the language of rings.

In $\mathbb{Q}_p$ we also have a very nice characterization of definable subsets.

Theorem. *In the valued field $\mathbb{Q}_p$ the definable subsets are Boolean combinations of Zariski closed subsets and subsets defined by formulas $P_n(f(x))$, where n is an integer, f a polynomial with integral coefficients and P_n is defined by the equivalence $P_n(y) \leftrightarrow \exists z\ z^n = y$.*

In other words we are allowed to consider only existential quantification applied to very simple formulas. But it is also to be noted that the language allowing quantifier elimination is now $\{0, 1, +, -, \cdot, (P_n)_{n\in\mathbb{N}^*}\}$, which is infinite, while the ring language is finite.

We are now going to give two very different applications of this result, one is a local application used to solve a conjecture in number theory, the other one makes a repetitive use of the result in order to develop a wide area of research.

3.1. The solution by Denef of a conjecture of Serre and Oesterlé. Let $f(X_1, \dots, X_m) \in \mathbb{Z}_p[X_1, \dots, X_m]$. For each $n \in \mathbb{N}$, we define two natural integers

$$M_n := \#\Big\{(a_1, \dots, a_m) \in \mathbb{Z}_p^m / (p^n \cdot \mathbb{Z}_p)^m ; f(a_1, \dots, a_m) = 0 \text{ as an element of } \mathbb{Z}_p/(p^n \cdot \mathbb{Z}_p)\Big\}$$

(M_n is the number of approximations modulo p^n of zeros of f)

$$N_n := \#\Big(\big\{(a_1, \dots, a_m) \in \mathbb{Z}_p^m ; f(a_1, \dots, a_m) = 0\big\} / (p^n \cdot \mathbb{Z}_p)^m\Big)$$

(N_n is the number of (exact) zeros of f which are distinct modulo p^n). As an example, if f has a single variable, M_n and N_n are eventually constant, becoming both equal to the number of zeros of f. We define the Poincaré power series

$$\mathcal{M}(T) = \Sigma_{n\in\mathbb{N}} M_n T^n,$$

and

$$\mathcal{N}(T) = \Sigma_{n\in\mathbb{N}} N_n T^n.$$

Theorem (Igusa, conjectured by Borewicz and Šafarevič). *$\mathcal{M}$ is a rational function.*

Theorem (Denef, conjectured by Serre and Oesterlé). *$\mathcal{N}$ is a rational function.*

We sketch here Denef's proof [4].

(1) Using some integral representation, $\mathcal{N}$ is a rational iff $I(S) := \int_D |w|^s |dx||dw|$ is, where $D = \{(x, w) \in \mathbb{Z}_p^m \times \mathbb{Z}_p ; \exists y \in \mathbb{Z}_p^m \ [\ x \equiv y \mod w \wedge f(y) = 0\ ]\}$.
(2) By quantifier elimination, D can be rewritten as a Boolean combination of expressions of the type $g(x) = 0$, or $P_n(h(x))$, for $n \in \mathbb{N}$ and polynomials g and $h \in \mathbb{Z}_p[X_1, \dots, X_m]$.
(3) It remains then to prove the rationality of the integral above for these particular domains, which means much further (but possible) work ....

3.2. p-adic semi-algebraic geometry and spectrum. In the thirties Tarski proved the decidability of $\mathbb{R}$ as a field. During the following two decades, the field of real numbers was systematically studied by Abraham Robinson. He was able to abstract, from this case, some general notions which are now fundamental in model theory. He also got new insights on classical results on $\mathbb{R}$ such as the Artin-Lang theorem and Hilbert's 17th problem; these lead him to astonishing simplifications in the proofs, as well as to qualitative improvements of the results themselves. He was the precursor of a simultaneous, algebraic and model theoretic, treatment of real algebra and geometry, which is one of the pillars of modern real semi-algebraic geometry (these historical developments are described and analysed in [31]) Now, the analogy between $\mathbb{R}$ and $\mathbb{Q}_p$ was first noticed by Kochen: He gave it as a motivation for studying Hilbert's 17th problem over $\mathbb{Q}_p$. This analogy has been then systematically developped (see [7]). In particular there is now a p-adic semi-algebraic geometry and almost all classical real results (see [16]) find their analogue in this context. We quote below the most important facts

order	p-adic valuation, i.e. satisfying $v(p) = 1$ (first positive element of valuation group) and $K/v \simeq \mathbb{F}_p$
formally real (:= orderable) field	formally p-adic (:= which can be equipped with a p-adic valuation) field
real closed field := real field without non trivial real algebraic extension; K real closed $\Leftrightarrow [K^a : K] = 2$, where K^a is the algebraic closure of $K \Leftrightarrow K \equiv \mathbb{R}$	p-adically closed field := p-adic field without non trivial algebraic p-adic extension; K p-adically closed $\Leftrightarrow$ K carries a Henselian p-adic valuation v satisfying $vK \equiv \mathbb{Z} \Leftrightarrow K \equiv \mathbb{Q}_p$
Hilbert's 17th problem: over $\mathbb{R}$ every positive definite rational function is a sum of squares of rational functions.	Description of rational function with range in $\mathbb{Z}_p$, i.e. "integral definite," by means of "Kochen's operator" $\gamma(t) = \frac{1}{2p}((t^p - t + 1)^{-1} + (t^p - t - 1)^{-1})$. Define $R := \mathbb{Z}_p[\gamma(K[X])]$, for $X = (X_1, \ldots, X_m)$, and $T := 1 + p.R$. Then $f \in \mathbb{Q}_p(X)$ is integral definite iff it belongs to the quotient ring $T^{-1}R$. ([13] strengthened by [8])

Quantifier Elimination

$\{0, 1, +, -, \cdot, \leq\}$ ("Tarski-Seidenberg")	$\{0, 1, +, -, \cdot, (P_n)_{n \in \mathbb{N}^*}\}$ where $P_n(x) \leftrightarrow [\exists y \; y^n = x]$ ([6])

Reformulation

(in both cases, semi-algebraic = definable)

Any semi-algebraic subset of $\mathbb{R}^m$ is a Boolean combination of sets of the form $\{x; f(x) > 0\}$ for $f \in \mathbb{R}[X]$, where $X = (X_1, \ldots, X_m)$.	Any semi-algebraic set is a Boolean combination of sets of the form $\{x; f(x) = 0\}$ or $\{x; P_n(f(x))\}$.
Let K be a real closed extension of $\mathbb{R}$ and $A \subseteq K^m$ definable with parameters from K. Then $A \cap \mathbb{R}^m$ is semi-algebraic in $\mathbb{R}$.	The same with p-adically closed instead of real closed and $\mathbb{Q}_p$ instead of $\mathbb{R}$ ([19])

Open Quantifier Elimination

Any open definable subset of $\mathbb{R}^m$ is of the form $\cap\cup\{x; f_{ij}(x) > 0\}$ (note that the negation is no longer allowed as the complement of an open set is in general not open).	Any open definable subset of $\mathbb{Q}_p^m$ is of the form $\cap\cup\{x; \dot{P}_{n_{ij}}(f_{ij}(x))\}$ where $\dot{P}_n(x) \leftrightarrow [P_n(x) \wedge x \neq 0]$.([23])
$\|x\|$	Over $\mathbb{Q}_p$ we have the norm $\| \ \|_p$ (see Appendix B). But there is no equivalent over p-adically closed field K such that $vK \neq \mathbb{Z}$. Further, even over $\mathbb{Q}_p$, $\| \ \|_p$ is NOT a definable function.

The adequate reformulation is as follows:

For any definable closed subset F there is a definable continuous fonction with F as zero-set.	Same as on the left [15]
(Łojasiewicz inequality) A semi-alg. closed, bounded set $\subseteq \mathbb{R}^m$, $f, g : A \to \mathbb{R}$ semi-alg., continuous $Z(f) \subseteq Z(g) \Rightarrow \exists N \in \mathbb{N}, c \in \mathbb{R}, \|g\|^N \leq c \cdot \|f\|$ over A (a function is semi-algebraic $:\Leftrightarrow$ its graph is, and $Z(f) := \{x \in \mathbb{R}^m; f(x) = 0\}$)	Same as on the left with $\mathbb{Q}_p$ instead of $\mathbb{R}$ [17]
semi-algebraic connected components (a semi-algebraic set has finitely many connected components, each of them semi-algebraic)	NO EQUIVALENT (as far as I know …)

Classical spectrum of a ring = Space of prime ideals

Real spectrum = Space {prime ideal + an ordering over the fraction field of the quotient ring }. In particular, for a field, Real Spectrum = Space of all orders	p-adic Spectrum = Space {prime ideal + over the fraction field of the quotient ring, an additional structure given by some choice for the P_n's }. In particular, for a field K, p-adic Spectrum = Space of all embeddings in a p-adically closed field [22]
If V is a real variety and $\mathscr{C}(V)$:= {continuous definable real functions on V}, then the classical spectrum of $\mathscr{C}(V)$ is homeomorphic to the real spectrum of $K[V]$. [18]	Same as on the left with “p-adic” instead of “real” [15]

There are also for the p-adic case partial analogues to the "continuous solution to Hilbert's 17th problem" ([10, 11, 12, 9]). More generally, the references quoted above provide a considerable amount of results.

4. Appendix A: Inductive definition of formulas.

- The set of terms of a language $\mathscr{L}$ is the smallest set containing the variables and constants of $\mathscr{L}$, and closed under functions from $\mathscr{L}$.
- Atomic formulas have the following form

$$t_1 = t_2$$
$$R(t_1, \dots, t_n)$$

  for $t_1, \dots, t_n$ terms and R a relation from $\mathscr{L}$.
- The set of formulas is the smallest set containing atomic formulas and closed under $\vee, \wedge, \neg$ and quantification over some of the variables.

By applying this inductive process, we get formulas which do not have the nice form given in the first section. In order to rewrite a formula, with first all quantifiers, then all disjunctions and then all conjunctions, you need a notion of logical equivalence between formulas, which is also inductively defined. This is then a theorem that any formula has a "prenex disjunctive normal form," i.e. is logically equivalent to a formula of the form given in definition 1 of Section 1. See [27].

5. Appendix B: Valued Fields. For all this, see also the contribution of Capi Corrales in this volume.

A *valued field* (K, v) is a field K equipped with a map $v : K^* \to vK \cup \{\infty\}$ where vK is an ordered abelian group, written additively, and ∞ an extra element satisfying $\forall x \in vK, x + \infty = \infty + \infty = \infty > x$. Further v is required to satisfy

- $v(x) = \infty$ iff $x = 0$,
- $v(x + y) \geq \min\{v(x), v(y)\}$, inequality known as "ultrametric inequality,"
- $v(x \cdot y) = v(x) + v(y)$.

  The *valuation group* is vK.

  The *valuation ring* is

$$A_v := \{x \in K; v(x) \geq 0\},$$

  the *valuation ideal*

$$M_v := \{x \in K; v(x) > 0\}.$$

A_v is a local ring, with unique maximal ideal M_v. The *residue field* is

$$K/v := A_v/M_v.$$

Note that the characteristic of the residue field of a valued field of characteristic zero is either zero (example: $\mathbb{C}((t))$ with v the valuation relative to t) or any p (example: $\mathbb{Q}_p$). The residual characteristic of a valued field of characteristic p is p.

Another (equivalent) presentation is the following. We give ourselves in K a "big" subring A, big in the sense that $\forall x \in K$, either $x \in A$ or $x \neq 0$ and $x^{-1} \in A$.

We can then define over K a valuation for which A is the valuation ring. Let us first define

$$A^* := \{\text{units of } A\} = \{x \in A; \exists y \in A, xy = 1\}$$
$$G := K^*/A^* \text{all considered as multiplicative groups}$$

for $x, y \in K^*$, in G we set $\overline{x} \leq \overline{y}$ iff $x^{-1}y \in A$. (This is clearly an order, which is total by the condition on A.)

We now define $v : K \to G \cup \{\infty\}$ by setting

$$v(x) = x \mod G \text{ for } x \in K^*$$
$$v(0) = \infty.$$

In other words, in order to define a valuation over a field, it is enough to say when the valuation of an element is positive.

In this context of general valuation, we have the same formal definition of *Henselian field.* (K, v) is Henselian if it satisfies Hensel Lemma, that is: for any monic polynomial $f \in A_v[X]$, if f has a simple residual root $\alpha \in K/v$ then α lifts up in K to a root of f. (Which means: if for an $a \in A_v, v(f(a)) > 0 = v(f'(a))$ then there is a $b \in K$ satisfying $f(b) = 0$ and $v(b - a) > 0$).

A valuation determines a distance with range in $vK \cup \{\infty\}$, we measure the distance between two points x and y by $v(x - y)$. Note that $v(x - y) = \infty$ iff $x = y$, which means that we should have to do something like a "negative exponentiation," as we do in p-adic numbers

$$|x|_p := \left(-\frac{1}{p}\right)^{v_p(x)}.$$

But, as in general vK is arbitrary, we don't have an exponential function and we work with v, but we often have to reverse unequalities in our head. This metric space has the following feature: any triangle is isoceles, the two equal side being the big ones (indeed, it follows from the ultrametric unequality that, if $v(x - y)$ and $v(y - z)$ are distinct, then $v(x - z) = \min\{v(x - y), v(y - z)\}$). As a consequence, each point in an open or closed ball $\{x; v(x - a) \geq \text{ or } > \rho\}$ is a center!

Since we have a distance, we have a topology, therefore the Implicit Function Theorem makes sense. Now, Hensel Lemma is a strong form of Implicit Function Theorem for polynomials.

6. Appendix C: The counter-example of Terjanian. We will construct in $\mathbb{Q}_2$ a homogeneous form of degree 4, with 18 $(> 4^2 = 16)$ variables and without non trivial zeroes.

(1) It is enough to construct over $\mathbb{Z}$ a homogeneous f of degree 4, with 9 variables and such that

$$\text{for all } \overline{x} \in \mathbb{Z},\ f(\overline{x}) \equiv 0 \mod 4 \ \Rightarrow\ 2 \text{ divides } \overline{x}.$$

Then $h(\overline{x},\overline{y}) := f(\overline{x}) + 4f(\overline{y})$ will be a solution.

Proof. If h has a non trivial zero over $\mathbb{Q}_2$, it has one $(\overline{a}',\overline{b}') \in \mathbb{Z}_2$ having at least one coordinate of v_2-valuation 0. Since $\mathbb{Z}_2$ is the completion of $\mathbb{Z}$ for v_2, there is $(\overline{a},\overline{b}) \in \mathbb{Z}$ arbitrarily close to $(\overline{a}',\overline{b}')$, i.e. for arbitrarily big $N \in \mathbb{N}$,

$$v_2(\overline{a}-\overline{a}'), v_2\left(\overline{b}-\overline{b}'\right) > N,$$

hence, by Taylor's formula,

$$v_2\left(h\left(\overline{a},\overline{b}\right) - h\left(\overline{a}',\overline{b}'\right)\right) > N,$$

hence, as $h(\overline{a}',\overline{b}') = 0$,

$$h\left(\overline{a},\overline{b}\right) \equiv 0 \mod 16.$$

Now

$$\begin{aligned} h\left(\overline{a},\overline{b}\right) \equiv 0 \mod 16 &\Rightarrow f(\overline{a}) \equiv 0 \mod 4 \Rightarrow 2 \mid \overline{a} \text{ by the hypothesis on } f \\ &\Rightarrow 16 \mid f(\overline{a}) \text{ as } f \text{ is homogeneous of degree 4} \\ &\Rightarrow 16 \mid 4f\left(\overline{b}\right) \Rightarrow 4 \mid f\left(\overline{b}\right) \Rightarrow 2 \mid \overline{b}, \text{ by the hypothesis on } f, \end{aligned}$$

which contradicts the choice of $(\overline{a},\overline{b})$ having a coordinate of v_2-valuation 0. □

(2) Let us define

$$n(X,Y,Z) := X^2YZ + XY^2Z + XYZ^2 + X^2Y^2 + X^2Z^2 + Y^2Z^2 - X^4 - Y^4 - Z^4.$$

It is easy to verify that, for $x, y, z \in \mathbb{Z}$,

- if 2 divides x, y, z then $16 | n(x, y, z)$, and
- if 2 divides exactly two elements among x, y, z, or one, or none of them, then $n(x, y, z) \equiv 3 \mod 4$.

(3) Now let $f := n(X,Y,Z) + n(U,V,W) + n(A,B,C)$. By exhausting all possible residues modulo 4 we see that, for $X,Y,Z,U,V,W,A,B,C \in \mathbb{Z}$

$$\begin{aligned} &\text{the condition } f(X,Y,Z,U,V,W,A,B,C) \equiv 0 \mod 4 \\ &\Rightarrow n(X,Y,Z), n(U,V,W), n(A,B,C) \equiv 0 \mod 4 \\ &(\text{as } 3+3+3,\ 3+3+0 \text{ and } 3+0+0 \text{ are all } \not\equiv 0 \mod 4) \\ &\Rightarrow 2 \text{ divides } X,Y,Z,U,V,W,A,B,C \text{ (by the 2. above)}, \end{aligned}$$

hence f has the property of 1. above, which finishes the proof.

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SOME APPLICATIONS OF p-ADIC POINTS OF VIEW TO ELLIPTIC CURVES

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A crucial incentive for the development of p-adic techniques is provided by the so-called *local-global principle*: in order to study the rational solutions of an arithmetical problem—a global problem, thus in general difficult—, one might begin with a study of the solutions in the various p-adic completions, including the real or the complex ones, and then try to recover information on the initial problem from the data obtained in studying the second problems. As Capi Corrales explained in her article, this principle works well for instance for quadratic forms (Hasse-Minkowski theorem) and in other situations (see [1]), but already fails for cubic forms. Nevertheless, it is possible to adapt these ideas in some cases where the local-global principle strictly does not apply and this paper will deal with some of these adaptations in the case of cubic equations associated with elliptic curves. These curves have been studied since the nineteenth century at least—and particular cases much before—, and they have appeared on the front page of mathematical newspapers recently in connection with the proof of Fermat's last theorem, stating that for $n > 2$, $a^n + b^n = c^n$ has no rational solution with $abc \neq 0$. p-adic techniques play an important role in the actual proof and the present article could also been considered as an elementary invitation to this topic. I will first explain briefly the setting of elliptic curves, then show how various p-adic approaches can be used to grasp the rational solutions, even if the local-global principle does not apply directly any more. As is most usual in contemporary arithmetics, the end of this article will mainly deal with conjectures, the partial proof of one of them (the Shimura-Taniyama-Weil conjecture) being fundamental in the completion of the proof of Fermat's theorem. Some excursions are proposed to related topics or generalizations or complements: they can of course be left aside.

1. A pragmatic briefing on elliptic curves

1.1. Definition. In concrete terms, an elliptic curve E (defined over $\mathbf{Q}$) is defined by an equation in the projective plane (the numbering of the coefficients, although a bit disconcerting at first sight, is traditional)

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

where the a_i are rational numbers (in fact, they can be chosen to be integers), such that the curve E is smooth, that is: the tangent is well defined in every point. This condition can be expressed by saying that a certain polynomial in the coefficients a_i, the discriminant Δ, is not zero. For instance, if $a_1 = a_2 = a_3 = 0$, $\Delta(a_i) = -16(27a_6^2 + 4a_4^3)$, and the smoothness of the curve is connected to the fact that the third-degree polynomial on the right side of the equation of the curve has no multiple root.

One can point out that there is a single point at infinity, that is for which $Z = O$, and that it has rational coordinates. In what follows, I will often forget it and refer to the curve by its affine equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. In fact, as soon as a plane smooth curve has a cubic equation and at least one point with rational coordinates, a rational transformation of coordinates can be found such that the equation is of the above type (the given point with rational coordinates is taken as the point at infinity).

Here are a few classical examples of elliptic curves.

$$y^2 = x^3 - x \qquad \Delta = 2^6$$

$$y^2 + y = x^3 - x^2 \qquad \Delta = 11$$

$$y^2 + y = x^3 - x \qquad \Delta = 37$$

$$y^2 = x(x - a^r)(x + b^r) \qquad \Delta = (abc)^{2r}/256$$

with r prime and $a, b \in \mathbf{Z}$ such that there exists an integer c with $a^r + b^r = c^r$.

This last example will be used in the proof of Fermat's theorem. The equation, indeed, defines an elliptic curve as soon as it is smooth, and one can check easily that this amounts to saying that $abc \neq 0$, in other words that there is a counterexample to Fermat's theorem.

The points with real coordinates of the third curve are drawn in Figure 1, as an example. The points denoted by **1**, **2**, etc. have rational coordinates.

Excursion to rational points on curves. One might wonder about the points with rational coordinates on general curves defined over $\mathbf{Q}$. We will follow the usual terminolgy and call these points *rational points*. As suggested by David Hilbert and Adolf Hurwitz in 1890, then again by Henri Poincaré in 1901, the problem can be tackled through a classification of curves up to birational transformations, that is, transformations defined, everywhere except maybe in a finite numbers of points, by rational functions on the curve and such that the same is true of their inverse. (Up to a finite number of points), these transformations evidently do not alter the rationality properties of the points.

One can notice that the degree of a defining equation of the curve is not an invariant for this kind of transformation: the curve $y^2 = x^3$ for instance is birationally equivalent to a line, through the transformation $x = t^2$, $y = t^3$ and its inverse $t = y/x$. An important invariant is the genus—if you are used to dealing with algebraic complex curves (otherwise called Riemann surfaces), the genus is for instance the number of "holes." The genus takes into account not only the degree, but also the singular points of the curve. The genus of a smooth curve defined by an equation of degree n is $n(n-1)/2$ and the genus decreases if there are singular points: for instance, smooth cubic curves have genus 1, but the cubic curve just mentionned is of genus 0, because of the cusp at the origin.

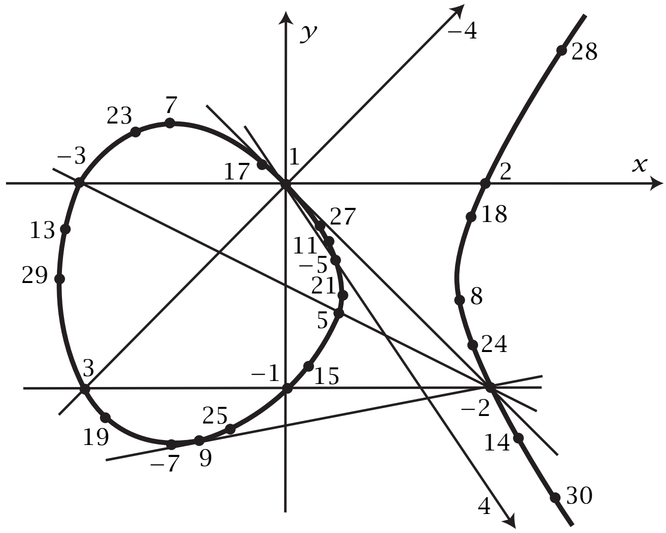

FIGURE 1. Rational points on the curve $y^2 + y = x^3 - x$. After R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

Curves of genus zero are birationally equivalent to the line or the conics; in particular, if they have any rational point, they have infinitely many of them. Curves of genus one may have no rational points, as the Selmer cubic example given in Capi Corrales' talk shows; if they have at least one rational point, they are elliptic curves: we will discuss in detail their rational points a bit later. Curves of genus greater than one have only a finite number of rational points: this very difficult result was conjectured in the twenties by Louis Mordell and proved in 1982 by Gert Faltings.

We will need more arithmetical information on the elliptic curves before turning to the p-adic aspects, but let me give already a glimpse into this direction, by letting p-adics appear through their first-order approximation, so to speak, that is through reductions modulo p.

1.2. Reduction modulo p. Let p be a prime number, or equivalently as seen in previous talks, let v be the associated discrete valuation. One can choose the equation of the curve E in such a way that it is minimal with respect to v, that is such that the coefficients a_i are of positive valuation and that $v(\Delta)$ is minimal among the possible $v(\Delta)$. When the curve is defined over $\mathbf{Q}$, one can in fact choose such a minimal equation globally: with the same integers a_i, the equation is minimal for every v (this would not be true over a number field for instance).

One can then reduce the coefficients of the equation of E modulo p, which gives the equation of a curve $\tilde{E}$ on the finite field $\mathbf{F}_p$. Three cases are possible:

- The reduced curve $\tilde{E}$ on $\mathbf{F}_p$ is smooth (this occurs when $v(\Delta) = 0$). One says that the curve has good reduction.
- The reduced curve $\tilde{E}$ has a node (e.g., the equation reduces to $y^2 = x^3 - x^2$). One says that there is semi-stable (bad) reduction.

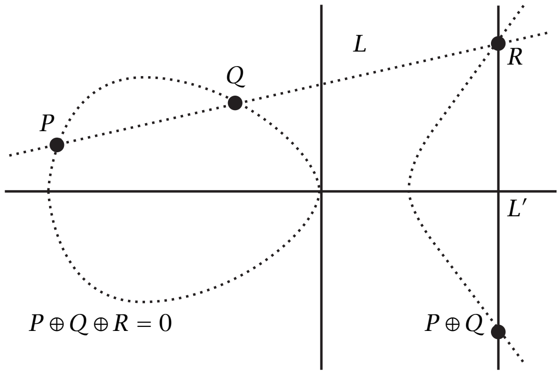

FIGURE 2. After J. Silverman, *The Arithmetic of Elliptic curves*, Springer, 1986.

- The reduced curve $\tilde{E}$ has a cusp (e.g., the equation reduces to $y^2 = x^3$). One says that the reduction is bad and unstable.

In the last two cases, $v(\Delta) > 0$; one can distinguish between them by looking at the valuation of a certain polynomial combination c_4 of the coefficients a_i.

This information is encapsulated in an integer, called the conductor N of the curve. It is divisible only by the primes of bad reduction (in particular, it has the same prime divisors as the disciriminant Δ). Moreover, the primes of semistable bad reduction (resp. unstable bad reduction) appear in N with the power 1 (resp. strictly greater than 1). For example, in the elliptic curves associated with Fermat's theorem seen above, the conductor is equal to $\prod_{p|abc} p$.

1.3. The group law on an elliptic curve. The main feature concerning the points of an elliptic curve is that they form an abelian group. The group law is constructed in the following way: If two distinct points P and Q are given on the curve, the secant through P and Q cuts again the curve in a third point R (well-defined, because the curve is defined by a cubic equation). One defines the addition on the curve by the property that $P \oplus Q \oplus R = O$. One can check that it amounts to saying that the sum of P and Q is the point $P \oplus Q$ obtained as the third point of intersection of the secant going through R and the point at infinity. If the two points P and Q are the same, the secant becomes the tangent at the curve and the same construction applies, *mutatis mutandis*. The group structure can be proved for instance on the cartesian coordinates, by computing the equations of the tangents or of the secants. To fix the ideas, for instance, the double of a point $P = (x, y, 1)$ on the curve $y^2 = x^3 + 1$ is given by

$$\begin{aligned} 2(x,y,1) &= (x,y,1) \oplus (x,y,1) \\ &= \left(\frac{x^4 - 8x}{4x^3 + 4}, -\left(\frac{3x^2}{2y}\right)\left(\frac{x^4 - 8x}{4x^3 + 4}\right) + \frac{x^3 - 2}{2y}, 1 \right). \end{aligned}$$

The neutral element of the group is the point at infinity, we will note it from now on O_E. One can check on the figure above that the points **2**, **3**, etc. are obtained by iteration for this law from the point **1**—in particular, for example, $-$**3**, **2** and **1** are on a line.

The 2-torsion points (the points such that $2P = O_E$) are the points with "vertical" tangents (that is tangents going through the point at infinity). There are four of them (including O_E) and they form a subgroup of the type $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$. The 3-torsion points are the inflexion points, there are nine of them and they form a subgroup of the type $\mathbf{Z}/3\mathbf{Z}\times\mathbf{Z}/3\mathbf{Z}$. The same is true in general for the n-torsion points: they will play an important role in the construction of representations associated to the elliptic curve, as we will see later.

The rational points obviously form a subgroup, because all the constructions given above preserve the rationality: for instance the secant through two rational points has an equation with rational coefficients and the third point of intersection with the elliptic curve is still rational. This group is called the Mordell-Weil group of the curve E and its structure is described in the following crucial theorem.

Theorem 1 (Mordell-Weil)**.** *The abelian group $E(\mathbf{Q})$ of the rational points on an elliptic curve E defined over $\mathbf{Q}$ is of finite type.*

In other words, $E(\mathbf{Q}) \simeq$ finite group$\times\mathbf{Z}\times\cdots\times\mathbf{Z}$. The number r of copies of $\mathbf{Z}$, that is the number of independant generators of infinite order, is called the rank of the elliptic curve.

This theorem was proved by Mordell in 1922 and generalized, in André Weil's 1928 thesis, to number fields and cases associated with curves of higher genus. As far as the torsion part is concerned, only a finite number of possibilities may occur, as proved by Barry Mazur in 1974: it can be a cyclic group $\mathbf{Z}/n\mathbf{Z}$, with $1 \le n \le 10$ or $n = 12$ or it can be a group of the type $\mathbf{Z}/2\mathbf{Z}\times\mathbf{Z}/2n\mathbf{Z}$ with $n = 1,2,3,4$. For each possibility, one knows families of examples. On the other hand, we still do not know how high the rank can be or if there exists elliptic curves of arbitrarily high rank, and the determination of actual generators of infinite order, even for not too big ranks, is far from being easy.

Let me give some examples

- The curve $y^2 = x^3 - x$ has rank 0 and its Mordell-Weil group has four elements, all of order 2.

$$E(\mathbf{Q}) = \{(0,0),(\pm1,0),O_E\} \simeq \mathbf{Z}/\mathbf{2Z}\times\mathbf{Z}/\mathbf{2Z}.$$

- The curve $y^2 = x^3 - 43x + 166$ has also rank 0 and its Mordell-Weil group is cyclic of order 7;

$$E(\mathbf{Q}) = \{(3,\pm8),(-5,\pm16),(11,\pm32),O_E\} \simeq \mathbf{Z}/\mathbf{7Z}.$$

- The curve $y^2 = x^3 + 877x$ has been studied by Cassels and Bremmer. Its rank is one and the Mordell-Weil group is

$$E(\mathbf{Q}) \simeq \mathbf{Z}\times\mathbf{Z}/\mathbf{2Z},$$

  that is, every rational point P can be written either as mP_0 or as $mP_0 \oplus (0,0)$, where P_0 is a generator of the infinite cyclic part, for instance,

$$P_0 = \left(\frac{375494528127162193105504069942092792346201}{6315987776871054254632207806972380441 00},y_0\right).$$

This example shows that the determination of the generators is not quite trivial!

- The curve $y^2 = x^3 - 226x$ has rank 3.

$$E(\mathbf{Q}) \simeq \mathbf{Z}^3 \times \mathbf{Z}/2\mathbf{Z},$$

where three independent generators of the infinite part are

$$\begin{aligned} P_1 &= (-1, 13), \\ P_2 &= (5121/4, 1155/8); \\ P_3 &= (-8, 36). \end{aligned}$$

- The curve $y^2 = x^3 + 16D$, with $D = -408368221541174183$ (studied by J. Quer) has rank 12.

2. A first use of the p-adics: the computation of the torsion. Mazur's proof concerning the torsion is quite difficult. But it is often possible to detect impossibilities through a p-adic analysis, exactly as it is often possible to prove some impossibilities for integral solutions of an equation by arguments modulo p, or more generally p-adic ones. Thus, let us denote by p a prime number and let us suppose that the elliptic curve E has good reduction modulo p. Let n be an nonzero integer, prime to p. The fundamental result for our purpose is then

Theorem 2. *The group of p-adic points on E of n-torsion (denoted by $E(\mathbf{Q}_p)[n]$) can be injected into the group of $\mathbf{F}_p$-rational points of the reduced curve $\tilde{E}$ modulo p.*

What does this statement mean and why is it useful? The minimal equation of E has integral coefficients, and one can look for solutions with values in $\mathbf{Q}_p$, or, to say it briefly, for p-adic points on E. Through the same secant-tangent procedure described earlier, these points form a group (which contains the Mordell-Weil group of the rational points) and one can investigate its elements of n-torsion. On the other hand, by reduction modulo p of the coefficients, one obtains as explained above a smooth cubic curve $\tilde{E}$ defined on $\mathbf{F}_p$—do not forget that one has assumed that E has good reduction modulo p. It is then legitimate to consider its solutions in $\mathbf{F}_p$, which, with the same procedure, form an abelian group. The theorem states than there is an injection between these groups. The proof comes from the following exact sequence

$$0 \to E_1(\mathbf{Q}_p) \to E(\mathbf{Q}_p) \to \tilde{E}(\mathbf{F}_p) \to 0,$$

where the third arrow represents the reduction modulo p and $E_1(\mathbf{Q}_p)$ its kernel. The key is that $E_1(\mathbf{Q}_p)$ has at most p-torsion; the n-torsion, for n prime to p, of the second group injects then into the third. The necessary description of $E_1(\mathbf{Q}_p)$ and of its torsion is by no means trivial, see for instance [20].

The theorem can be used to prove results on the p-adic, thus on the global torsion. Here are two examples taken from [20].

Example 1.

$$y^2 + y = x^3 - x + 1, \qquad \Delta = -611 = -13.47.$$

It is easy to check that the reduced curve modulo 2 has no solution modulo 2—except of course the solution at infinity: namely, the left hand side of the equation is always equal to 0 modulo 2, and the right hand side always equal to 1. Thus, for every odd n, $E(\mathbf{Q}_p)[n]$ injects into $\tilde{E}(\mathbf{F}_2) = 0_{\tilde{E}}$ and the Mordell-Weil group of the curve E itself has no odd torsion.

Example 2.

$$y^2 = x^3 + 3, \qquad \Delta = -3^5 2^4.$$

One checks here easily that the cardinality of $\tilde{E}(\mathbf{F}_5)$ is 6 and the cardinality of $\tilde{E}(\mathbf{F}_7)$ is 13. As the n-torsion of the Mordell-Weil group for $n \neq 5$ or 7 should inject in both these groups, it is trivial. On the other hand, the cardinality of the 5-torsion (respectively of the 7-torsion) divides 25 (resp. 49) as mentionned earlier and explained below, thus there is no non trivial torsion. Moreover, there is an evident rational solution (1,2) on the curve. This point is thus of infinite order.

3. The l-adic representations associated with an elliptic curve. We will see now how the infinite part of the Mordell-Weil group can also be studied more closely with p-adic ideas. This includes the Selmer and Tate-Safarevic groups attached to an elliptic curve on one hand, and, on the other, L-functions and representations. The first topic will be briefly explained at the end of this paper. I will mainly explain here the construction and use of representations associated to an elliptic curves. Each of them is indexed by a prime number, traditionnaly denoted by l and not by p (we will have occasion to understand why). Thus let us fix such a prime l greater than or equal to 3 and such that E has good reduction at l. We will need to study the points on E of l^r-torsion for various r. The first results being valid for $n-$torsion points, for any integer $n \geq 2$, I will give them in this situation. Thus, let as always $E[n] = \{P \mid \overbrace{P + \cdots + P}^{n\text{times}} = 0\}$ be the set of the n-torsion points of the curve E (including, of course, the "zero" element, that is the point at infinity). These $E[n]$ are obviously $\mathbf{Z}/n\mathbf{Z}$-modules. From the expression of the coordinates of the sum of several points, it is not difficult to see that the x-coordinates of the points in $E[n]$ satisfy an equation of degree $(n^2-1)/2$ with rational coefficients; the y-coordinates are then given by the equation for E (thus there are in general two values of y for each x). Thus, as already mentionned earlier, there are n^2 points in $E[n]$ (including the point 0_E); $E[n]$ is in fact a free $\mathbf{Z}/n\mathbf{Z}$-module of rank 2. Moreover, the Galois group of $\mathbf{Q}$ acts on it. The following excursion is intended to provide the necessary background and can be skipped by people already familiar with Galois theory.

Excursion to Galois groups and Galois representations. The Galois group of a polynomial equation with rational coefficients is simply the group of permutations of the roots of this equation which take into account all the rational relations between these roots, that is the relations expressible by means of polynomials with rational coefficients. In other words, if (some of) the roots $x_1, \ldots, x_n$ of an equation satisfy $Q(x_i) = 0$, with Q a polynomial with rational coefficients, the images $s(x_i)$ for a permutation s belonging to the Galois group of the equation also satisfy

$Q(s(x_i)) = 0$. In particular, the Galois group fixes all the rational roots (if $x_1 = a$ with a rational, one also has $s(x_1) = a$).

For example, let us consider the equation $x^4 - x^3 - 2x^2 + 3x - 1 = 0$. It has four roots, $x_1 = 1$, $x_2 = -1$, $x_3 = (-1+\sqrt{5})/2$, $x_4 = (-1-\sqrt{5})/2$. There are a priori $4! = 24$ permutations between these four roots, but the only ones to be taken into account should preserve for instance rational relations such as $x_1 - 1 = 0$, $x_2 + 1 = 0$, $x_3 + x_4 + 1 = 0$, $x_3 x_4 - 1 = 0$. Thus they fix x_1 and x_2, and either also fix the two other roots or exchange them. The Galois group has two elements: the identity, and the permutation which exchanges x_3 and x_4, corresponding to the transformation $\sqrt{5} \to -\sqrt{5}$.

Consider now the equation $x^4 + x^3 + x^2 + x + 1 = 0$, whose roots are the four non-trivial 5-th roots of unity. Each root is a power of one of them, for example of x_1, and this (rational) relation should be kept by the authorized permutations. Such a permutation is thus determined as soon as one knows the image de x_1: there are four possibilities for it, x_1, x_2, x_3, x_4, and the Galois group has four elements, it is in fact the cyclic group of order 4.

Consider finally the equation $3x^4 - 6x^2 + x - 1 = 0$, no polynomial relation exists between the roots, except of course the equation itself, the Galois group of the equation is here the group S_4 of all the permutations between the four roots.

The absolute Galois group, $G_{\mathbf{Q}}$, gathers all these pieces of information for all the polynomial equations with rational coefficients: an element of the absolute Galois group determines on each equation a particular permutation of the roots (which of course could be simply the identity). A good and simple introduction to these issues is for instance [22]. This Galois group is very important for arithmeticians, because it controls for instance how to go from the algebraic closure of $\mathbf{Q}$—which contains the roots of all the polynomial equations with rational coefficients and on which phenomena are often simpler or at least can be dealt with through a lot of tools, for instance algebraic geometry or complex analysis—to $\mathbf{Q}$ itself. The absolute Galois group is infinite. For recent work in order to understand better the structure and the properties of the absolute Galois group, see [8].

The absolute Galois group fixes in particular the equation determining the x-coordinates of the points in $E[n]$ and the equations giving the y-coordinates of the points with a given x-coordinate. Thus, it permutes the points of $E[n]$ among themselves. If one fixes a basis (P_1, P_2) of the $\mathbf{Z}/n\mathbf{Z}$-module $E[n]$, for each element σ of the Galois group, $\sigma(P_1)$ and $\sigma(P_2)$ can be expressed in this basis, as a linear combination with coefficients in $\mathbf{Z}/n\mathbf{Z}$: $\sigma(P_i) = r_i P_1 + s_i P_2$. Thus one obtains a representation—it is a continuous homomorphism between topological groups, but I won't describe here the relevant topologies

$$\rho_n : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z}),$$

where to each element σ of $G_{\mathbf{Q}}$ is associated the matrix $\left(\begin{smallmatrix} r_1 & r_2 \\ s_1 & s_2 \end{smallmatrix}\right)$.

Example. For the curve $y^2 - y = x^3 - x$ and $n = 3$, the equation giving the x-coordinates of the points in $E[3]$ is $3x^4 - 6x^2 + x - 1 = 0$ and the Galois group of the equation is here the group S_4. The action of the absolute Galois group is determined by this action on the x- and y-coordinates of the points. One could describe explicitely the representation ρ_3 above through the choice of a basis of $E[3]$ on $\mathbf{Z}/3\mathbf{Z}$.

There exists a natural map from $E[n]$ to $E[m]$, as soon as m divides n (if $P \in E[n]$, then $(n/m)P \in E[m]$) and the associated representations are compatible. In particular, it is possible to define a unique Galois representation gathering the Galois representations associated to $E[l]$, $E[l^2]$, $E[l^3]$, etc. This representation is the l-adic representation associated to E, denoted from now on by ρ_{l^∞},

$$\rho_{l^\infty} : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{Z}_l).$$

Theorem 3. *Each l-adic representation as above determines the curve E, up to isogeny.*

This means that if two curves have a common l-adic representation, each is the quotient of the other by a finite group. In fact, the l-adic representation allows us to recover a fundamental object associated with the elliptic curve, its L-function.

4. The L-function of an elliptic curve. These functions are analogous to the celebrated Riemann Zeta-function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_{p\,\text{prime}} (1 - 1/p^s)^{-1}.$$

They encapsulate the "modulo p" pieces of information for the curve. More precisely, we have seen at the beginning that the reduction $\tilde{E}$ of the elliptic curve modulo a prime p can be still a smooth curve (if p does not divide the discriminant Δ or equivalently the conductor N) or have one singularity, either a cusp or a node (if p is a divisor of N or Δ). Let us define a_p such that the number of points on the reduced curve $\tilde{E}$ modulo p is $p+1-a_p$ (if, instead of E, one looks at the projective line, one would have $p+1$ points, thus the term a_p is a kind of correcting term). It was proved by Helmut Hasse that $\mid a_p \mid < 2\sqrt{p}$. Now, let us define

$$L_p(s) = (1 - a_p p^{-s} + p^{1-2s})^{-1} \quad \text{for } p \text{ not dividing } N$$
$$L_p(s) = 1 \quad \text{if } \tilde{E} \text{ has a cusp}$$
$$L_p(s) = (1 \pm p^{-s})^{-1} \quad \text{if } \tilde{E} \text{ has a node}$$

(I won't discuss here the determination of the sign in the last case) and, for $\mathrm{Re}(s) > 3/2$,

$$L(E/\mathbf{Q}, s) = \prod_p L_p(s).$$

The product defining the L-function does not converge in general, but one has the following conjecture

Conjecture 1. *The L-function has an analytic continuation to the whole complex plane. Moreover, it satisfies a functional equation. Let*

$$\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(E/\mathbf{Q}, s),$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^s dt/t$ *is the usual* Γ*-function, then one has*

$$\Lambda(s) = \pm\Lambda(2-s).$$

The definition and the properties (conjectural or not) of the L-function of an elliptic curve are coherent with those of the zeta-functions attached to curves or varieties over a finite field on one hand and, on the other, with those of the $L-$functions associated with Galois representations (for a leisurely introduction to these topics, see [6]). In particular,

$$L(E/\mathbf{Q}, s) = \prod_p \det\left(1 - p^{-s}\rho_{l^\infty}(\mathrm{Frob}_p) \mid V_l^{I_p}\right)^{-1},$$

where, for $l \neq p$ (thus the choice of the notation!), ρ_{l^∞} is the l-adic representation defined earlier, V_l is a $\mathbf{Q}_l$-vector space called the Tate space, constructed from the points of l, l^2, etc. -torsion on the curve and on which the absolute Galois group acts via the l-adic representation, I_p is a subgroup of the Galois group called the inertia group at p and Frob_p a specific (conjugacy class of) elements, the Frobenius at p, in the Galois group. It has to be noticed that these last objects have nothing to do with the elliptic curve, they are defined in the framework of algebraic number theory (see for instance [21]) and they will be used exactly in the same way for another Galois representation. The curve appears here only through the l-adic representation (and the Tate space): remarkably enough, one l (of good reduction) is sufficient to recover most of the picture.

Unfortunately, the above conjecture is not known, except if some cases where an extra structure is provided on the curve: in the complex multiplication case and the modular case. In the first case, one can define on the curve not only the multiplication of a point by a usual integer (through the group structure), but also the multiplication by integers in a quadratic field. The simplest example is the one of the curve $y^2 = x^3 - x$ where one can define the multiplication by $i = \sqrt{-1}$ (and then by every integer in $\mathbf{Q}[i]$), as $i \cdot (x, y) = (ix, -y)$. In the complex multiplication case, the L-function of the curve is related to L-functions associated with characters of the quadratic field, for which analytic continuation has been proved. The second case corresponds to the possibility of parametrizing the points on the curve through the so-called *modular functions*. We will devote the next section to this issue, in particular because (conjecturally, as we will see), this case should be in fact general: every elliptic curve defined over $\mathbf{Q}$ should admit such a parametrization. The partial proof of this conjecture constitutes the main result of Wiles's work of 1994–1995 and implies in particular Fermat's theorem.

Before turning to this topic, let me state another conjecture which shows how this L-function, and thus the l-adic representation, give access to the infinite part of the Mordell-Weil group. The starting point, due to Birch and Swinnerton-Dyer in the sixties, was the following idea: if the curve has strictly positive rank, that is a rational point which is not a torsion point, then this will always provide a non-trivial contribution when one considers the reduction of the curve modulo a prime number. Thus the product of the $(1+p-a_p)/p$ will diverge and the L-function will have a zero at $s=1$. More precisely:

CONJECTURE 2 (Birch and Swinnerton-Dyer). *The order of vanishing of the L-function at $s=1$ is exactly the rank r of the curve.*

The conjecture also gives an expression for the leading coefficient, in terms of various objects associated to the elliptic curve. For the sake of space, I will omit this part.

5. The Shimura-Taniyama-Weil conjecture. As explained above, this conjecture, if proved, would allow us to know the analytic continuation and the functional equation for the L-function of any elliptic curve defined over $\mathbf{Q}$. It is also connected to the recent proof of Fermat's theorem. Although its expression is not p-adic per se, its (partial) proof by Wiles relies heavily on p-adic and l-adic techniques. I won't be able here to give its due to the beautiful, but technical work involved and will only try to convey an idea of the conjecture itself and at least of the range of methods used in Wiles' proof.

The Shimura-Taniyama-Weil conjecture (STW conjecture in what follows), in its crudest form, states that every elliptic curve defined over $\mathbf{Q}$ admits a parametrization by modular functions, in some respects a result analogous to the parametrization of the circle by circular functions. I will first define these functions, then give several, more precise, forms of the conjecture, and finally give some hints about Wiles' proof and the link with Fermat's theorem.

5.1. Modular functions and forms. The definition of the modular functions (and forms) depends on two integers N (the level) and k (the weight; we will here mainly refer to $k=0$ or 2). The impression of carelessness which my notations might give, where two quantities are denoted by N (the conductor of an elliptic curve and the level of a modular function), will disappear as soon as the conjecture will be stated properly.

DEFINITION. A modular function h of level N and of weight k is a meromorphic function defined on the complex half-plane $H=\{z\in\mathbf{C}\mid \mathrm{Im}(z)>0\}$, such that

$$h\left(\frac{az+b}{cz+d}\right)=(cz+d)^k h(z),$$

for every matrix $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ belonging to the group

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z})\quad \text{with } c\equiv 0 \bmod N\right\}.$$

As usual, $SL_2(\mathbf{Z})$ is the group of matrices with integral coefficients and determinant 1. The definition can be extended to transformations belonging to an arbitrary congruence group Γ, that is a subgroup of finite index in $SL_2(\mathbf{Z})$ and containing the group

$$\Gamma(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbf{Z}),\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod N\right\}.$$

The group $\Gamma_0(N)$ is only an important particular example of a congruence group and we will speak only of this one here; Wiles uses several others in his work.

Let me remark that the matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_0(N)$ and hence that $h(z+1) = h(z)$: the property of the definition generalizes the notion of a period. Being periodical with period 1, the modular functions have a Fourier development

$$h(z) = \sum_{-\infty}^{\infty} c_n \exp(2\pi i n z),$$

and we also require that this development have at most finitely many nonzero coefficients c_n for $n < 0$. This condition is in fact a condition of meromorphy at infinity. Other analogous conditions of regularity for other points depending on N are also necessary, see [10] or [19].

Example. The formal series obtained by developping the produit

$$q \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{11n})^2,$$

with $q = \exp(2\pi i z)$, is a modular function of weight 2 and of level 11. All coefficients for $n < 0$ and $n = 0$ are zero.

5.2. A formulation of the STW conjecture. We can now give a first formulation of the STW conjecture.

Conjecture 3. *Let E be an elliptic curve defined over $\mathbf{Q}$, with conductor N, there exists a parametrization by modular functions h et g of weight 0 and of level N,*

$$x = g(z), \qquad y = h(z),$$

for every point $P = (x, y)$ of E (up to a finite number).

Remarks.

- As promised, the two N should coincide. The problems related to the conductor (or to the level of the associated functions) are technically very delicate and a great part of the work consists of dealing adequately with these levels.
- Every curve defined by a cubic equation, even with complex coefficients, admits a parametrization by periodic functions, the Weierstrass functions; this is a classical result well-known since the nineteenth century. But that modular functions for a congruence subgroup provide a parametrization is narrowly linked to the fact that the coefficients of the cubic equation are *rational*: the STW conjecture is an arithmetical conjecture, cf. [11].

Example. A modular parametrization (by functions of weight 0 and of level N=37) for the curve $y^2 + y = x^3 - x$ is given by

$$x(z) = q^{-2} + 2q^{-1} + 5 + 9q + 18q^2 + 29q^3 + \cdots$$
$$y(z) = q^{-3} + 3q^{-2} + 9q^{-1} + 21 + 46q + 92q^2 + \cdots,$$

with $q = \exp(2\pi i z)$. The detailed computations can be found in [12] or [26].

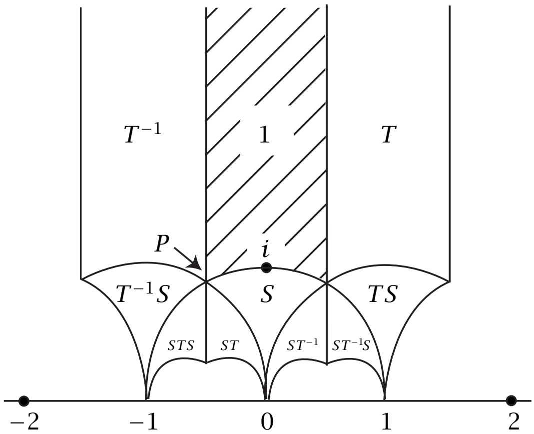

FIGURE 3. Fundamental domain for modular functions of level 1

This relatively simple formulation is neither the only one, nor the most widely used. The conjecture provides in fact a dictionary between "elliptic" objects (defined from the curve E) and "modular" objects (defined from the congruence groups and the modular functions). I will explain here some of these correspondances: that the STW conjecture can be proved arbitrarily from each of them is not at all trivial and relies on numerous previous works.

5.3. A geometrical interpretation. For a fixed integer N, the group $\Gamma_0(N)$ defines a tiling of the upper half-plane: one can namely cut this plane in infinitely many domains, which are deducible from each other by a homographic transformation, of the type $z \to (az+b)/(cz+d)$, with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Since a modular function of weight 0 is invariant by such a transformation, it is defined everywhere as soon as one knows it on one of these domains. The boundaries of a domain (up to a finite number of point) can also be glued together by identifying those which differ only by a transformation of $\Gamma_0(N)$. We draw underneath the case of $N = 1$. The transformation S is $z \to -1/z$, the transformation T is the translation by 1. One has noted on each domain which transformation would derive it from the domain **1**.

With these identifications, one obtains a real surface (which can be compactified), called $X_0(N)$. This is a Riemann surface and the modular functions of weight 0 are simply the functions on this surface. One can prove that the curve $X_0(N)$ has an equation defined on $\mathbf{Q}$; $X_0(37)$, for example, is, as a surface, a double tire, and as a curve, given by the equation $V^2 = 37 - U^6 - 9U^4 - 11U^2$, in the plane with coordinates U and V.

The existence of a modular parametrization for an elliptic curve E defined on $\mathbf{Q}$ is then equivalent to the existence of a non-constant, holomorphic mapping between Riemann surfaces (or of a rational map between algebraic curves).

$$X_0(N) \to E.$$

5.4. Modular functions. An other important form of the conjecture is that it is possible to associate to E a modular function f with weight 2 and level N with the following properties:

(i) $c_1 = 1$, $c_n = 0$ for $n \leq 0$ (such a function is said to be a normalized parabolic modular form).

(ii) For every prime l not dividing N,

$$c_l = 1 + l - \operatorname{card}(\tilde{E} \bmod l).$$

In other words, the l-th Fourier coefficient of the modular form is equal to the term a_l which appeared above in the definition of the L-function of the elliptic curve. The formula (ii) is crucial and will reappear in what follows.

Remarks. Modular forms of weight 2 give rise to differential forms on the surface $X_0(N)$; the one associated to E is the inverse image by the mapping $X_0(N) \to E$ of the differential form $dx/(2y + a_1)$ on E. The space of parabolic modular forms of weight 2 and level N is a finite-dimensional vector space. Its dimension is 0 for $N = 2, 1$ for $N = 11$, 2 for $N = 37$, for example. More generally, as could be expected, this dimension is the genus of Riemann surface $X_0(N)$.

The form f associated to E also satisfies other crucial properties: it is an eigenvector for the action of specific operators, and among them, of the so-called Hecke operators, T_n ($0 \neq n \in \mathbf{N}$). Taking them into account is in fact necessary to make the definitions and the proofs precise, see [10], but I won't do it here. The coefficients c_n are the corresponding eigenvalues for each T_n. Simple recurrence formulas allow us to determine every c_n as soon as the various c_l, l prime, are known.

Example. We have already exhibited a parametrization for the curve $y^2 - y = x^3 - x$. One deduces from it a parabolic normalized modular form

$$f(z)\,dz = \frac{dx(z)}{2y(z) - 1},$$

with development

$$f(z) = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 + \cdots,$$

where $q = \exp(2\pi i z)$ as always. One can check property (ii) on the first coefficients. For example, the solutions modulo 3 of the equation of the curve are: (0,0), (0,1), (0,2), (1, 0), (1,1), (1, 2), that is 7 solutions with the one at infinity, thus one should have $c_3 = 3 + 1 - 7 = -3$ (it works!).

To a modular form f, parabolic with weight 2 and level N as above, one can associate its Mellin tranform

$$\int_0^\infty f(it)t^s \frac{dt}{t},$$

which is also equal to

$$(2\pi)^{-s}\Gamma(s)L(f,s),$$

with $\Gamma(s) = \int_0^\infty e^{-t} t^s dt/t$ as earlier and $L(f,s) = \sum c_n n^{-s}$. This series L converges for $\mathrm{Re}(s) > 3/2$ and admits an analytic continuation to the whole complex plane; moreover it satisfies a functional equation relating its value at s to its value at $2-s$.

The STW conjecture also promised that this L-function coincides with the L-function of the elliptic curve defined earlier.

Remark. It seems that the mathematician Tajeki Taniyama was the first to suggest, during a problem session at the end of a conference in Tokyo in 1955, a weak version of STW: according to the english version (not published, but largely diffused), Taniyama asked if it was possible to find an "automorphic form" (a priori something a bit more general than modular functions) whose Mellin transform would give the L-function of the elliptic curve. In a series of papers, Goro Shimura constructed in particular for each modular form of weight 2 (of level N, parabolic, etc. as above) whose development in Fourier series has rational coefficients, an associated elliptic curve—if the coefficients belong to an extension of finite degree, one obtains varieties of higher dimension—and showed that these curves were modular in the sense of the geometrical interpretation given above. This work establishes half of the dictionary "elliptic" and "modular" dictionary. André Weil showed that it is sufficient to prove that the L-function of the curve (and a family of other analogous functions deduced from it) admits an analytic continuation and satisfies a functional equation of the required form to obtain the STW conjecture (see [24], tome 3, p. 165 for an exact statement). This statement has much contributed to make the STW conjecture convincing, because one expected such continuations and functional equations for all the L-functions associated to curves or algebraic varieties.

The interesting feature of this dictionary is that the modular functions are more concrete and accessible to computations than geometrical objects: for example, one can find rather easily estimates on the c_n which enable us to prove the analytic continuation of the modular L-function. Also, because the dimension of the space of the modular forms of weight 2 and of level N is finite, the STW conjecture provides directly the information that there are only finitely many elliptic curves with a given conductor.

5.5. Galois representations. The last form of the conjecture I want to explain is in fact the crucial one in Wiles' proof: It is expressible in terms of Galois representations. I do not want to explain how to construct such representations associated to modular forms, because it is quite technical (see [18] or [8]), but I will at least describe a characteristic property. As one can guess, the coefficients c_p play an important role here. For every prime p, we mentionned earlier a particular (class of) element(s) of the absolute Galois group, denoted by Frob_p. To each modular form f as above, one associates a representation

$$\rho_{l^\infty} : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{Z}_l),$$

such that the trace of the matrix $\rho_{l^\infty}(\mathrm{Frob}_p)$ is exactly the coefficient c_p (for p not dividing lN). One could also define analogously the representations $\rho_l, \rho_{l^2}, \rho_{l^3}$, etc., the corresponding traces being respectively c_p modulo l, c_p modulo l^2, c_p modulo l^3, etc.

Another form of the STW conjecture is that the l-adic representation ρ_{l^∞} associated to E is isomorphic to a representation arising from a modular form. This formulation is in particular compatible with formula (ii) seen above. Let us remark that while a single l is sufficient, the entire ρ_{l^∞} should be modular, the modularity of one of the representations ρ_l, ρ_{l^2}, etc. would not be sufficient.

In what follows, I will denote by $\rho_{l,E}$, $\rho_{l^\infty,E}$, etc., the representations constructed from an elliptic curve, and $\rho_{l,f}$, $\rho_{l^\infty,f}$, etc. those constructed from the coefficients of a modular form of weight 2, parabolic, etc.

Excursion to Langlands program. One can in fact associate an L-function to every adequate Galois representation. The Langlands program predicts that these L-functions coincide with the L-functions of automorphic forms (generalizing the modular forms).

Excursion to the notion of Galois representation. There are two slightly different vantage points on these representations. Either the emphasis is put on the Galois group itself, the various target spaces are as many ways of obtaining pieces of information about it; or the emphasis is put on the target spaces or modules, and the representation can be thought of as a extra structure on them ("action of the Galois group"). In this last perspective, which is ours here, the tendency is to get rid of the notation

$$\rho_{l,E}:\ G_{\mathbf{Q}} \longrightarrow \mathrm{GL}\,(E[l]) = \mathrm{GL}_2(\mathbf{Z}/l\mathbf{Z}),$$

to speak only of $E[l]$ as $G_{\mathbf{Q}}$-module of rank 2 (meaning: $E[l]$ with the extra structure given by the representation ρ_l of $G_{\mathbf{Q}}$). The difference is only a way of speaking, but it has a pedagogical advantage (at least at the level of this article): more complicated rings than $\mathbf{Z}/l\mathbf{Z}$ or $\mathbf{Z}_l$ appear in fact as ground rings for the $G_{\mathbf{Q}}$-modules used by Wiles and others, and I won't have to make them precise.

It is time to close these preliminary remarks and to turn to proofs.

5.6. The link with Fermat's theorem. Let me begin by a caution about the notations. There are a lot of primes entering these questions: the exponent in Fermat's equation, the index of the representations, the index of the Galois elements or the coefficients of the modular forms, etc. Generally, I have reserved r for Fermat's equation, l for the representation and p elsewhere (different choices are made in the relevant litterature). The problem is that in the proofs, some special identifications are made, for instance, one will choose sometimes $l = r$. Some caution is useful to avoid misunderstanding what is going on.

The elliptic curves to be considered are the ones defined by an equation

$$y^2 = x(x - a^r)(x - b^r),$$

where a, b, c are not all trivial and satisfy Fermat's theorem $a^r + b^r = c^r$, for a certain prime $r > 2$. These curves have been introduced and studied by Yves Hellegouarch in the sixties, but came back into focus in the mid-eighties through a suggestion of Gerhard Frey. The link with Fermat's theorem was in the end proved by Ken Ribet.

If these curves were modular, as promised by the STW conjecture, the representations $\rho_{l,E}$ would all be modular in the sense above, in particular this would be true of $\rho_{r,E}$ (r being the prime entering the equation of the elliptic curve). That is, for every prime p not dividing rN,

$$\operatorname{Tr}\rho_{r,E}(\mathrm{Frob}_p) \equiv c_p \,\mathrm{modulo}\, r,$$

where the c_p are the Fourier coefficients of a modular form of level N. Ken Ribet has shown that for this representation, it would be possible to decrease the level—a very hard game consisting in eliminating one by one every odd factor in the conductor of the curve, see [14] and [16, 15]. But there doesn't exist any non trivial parabolic modular form of weight 2 and of level 2, hence a contradiction. This proves that if an Hellegouarch curve did exist (that is if Fermat's theorem were false for an r), it could not be modular: the STW conjecture thus implies Fermat's theorem.

The next step then is the proof of the STW conjecture, at least for Hellegouarch curves. In fact, Andrew Wiles succeeded in proving it for all semi-stable elliptic curves. Recall that this means that the curves never reduce to a curve with a cusp, or equivalently, that no square divides their conductor. Here is a brief sketch of the main steps of the proof.

5.7. A proof of the STW conjecture for semi-stable ellitpic curves. The first step consists of proving that for many cases, $\rho_{3,E}$ is modular, that is there exists a modular form f of level N and of weight 2, $\sum c_n q^n$, such that

$$\operatorname{Tr}\left(\rho_{3,E}(\mathrm{Frob}_p)\right) \equiv c_p \,\mathrm{modulo}\, 3,$$

for all p not dividing $3N$.

The existence of such a form comes from the fact that, when the Galois group of the x-coordinate equation for $E[3]$ is the whole group S_4 of permutations of the four roots (as in the case of the curve $y^3 + y = x^3 - x$ seen above), the representation $\rho_{3,E}$ is a representation for which one can prove part of Langlands program (works by Langlands and Tunnell, see [17]); results by Deligne and Serre allow us then to produce the modular form f of weight 2. Notice that the Langlands program is known in very few cases: the fact that one can choose $l = 3$ (for which the equation for the x-coordinates is only of degree 4) is crucial in order that the proof can begin.

We now have at our disposal two representations ρ_{3^∞}, one associated to E, one to the form f obtained at the previous step (they only coincide a priori modulo 3). The whole question is then: can we garantee that the one associated to E, which is modular at the first order, so to speak, is modular as a whole?

In order to prove this point, Wiles uses a theory (deformation theory) which describes the possible liftings of representations modulo 3 into 3-adic representations (or, in other words, of $\mathbf{Z}/3\mathbf{Z}$-Galois modules of rank 2 to $\mathbf{Z}_3$-Galois modules of rank 2). One can think of it as a (very!) sophisticated version of Hensel's lemma. This theory has been elaborated during the last decade by Mazur, Hida, Tilouine, etc. Wiles constructed a *universal* modular Galois representation, such that all the modular representations lifting $\rho_{3,f}$ can be naturally deduced from this one (or, equivalently, a Galois-module $\mathbf{T}$ on an adequate ring, which is "modular-universal," in the sense that all the Galois modules on $\mathbf{Z}_3$ associated with modular forms congruent to f modulo 3 are images of $\mathbf{T}$ by adequate maps). He also constructed a *geometric universal* representation, $\mathbf{U}$, which plays the same role for the representations associated with curves. This very vague presentation masks important work necessary in order to control precisely for instance what happens for the divisors of $3N$.

The last step consists of a comparison between $\mathbf{T}$ and $\mathbf{U}$, which relies on an algebraic examination of the structure of these objects. Here was the flaw of Wiles' first proof, which was soon corrected in colloboration with Richard Taylor (see[25] and [23]). One can then conclude that $\rho_{3^\infty,E}$ also is modular.

If the representation $\rho_{3,E}$ is not adequate to catch immediately a modular form at the first step, Wiles uses the representation $\rho_{3,E'}$ for another auxiliary curve E' and compares then $\rho_{5,E}$ et $\rho_{5,E}$ in order to conclude.

6. Selmer groups and related topics. I will conclude this very schematic introduction with some hints about another important tool, Selmer groups. They occur in fact also in Wiles' proof, but I will describe here only the simplest cases.

A convenient way to introduce them is to go back to the Mordell-Weil theorem. Recall that it states that the group of rational points on the elliptic curve E is of finite type. Let me give you an idea of a possible proof of this theorem.

One fixes an integer $n \geq 2$ (the original choice was $n = 2$, but it works just as well for any number). Our aim is to express any rational point as a linear combination with integral coefficients of finitely many of them. Thus, let us consider one $P \in E(\mathbf{Q})$. One will write the group law as the usual addition on numbers and try to apply a kind of Euclidean algorithm. More precisely, one can prove that it is possible to write

$$P = nP' \oplus P_1,$$

where P' and P_1 belong to $E(\mathbf{Q})$, P' is smaller than P and P_1 belongs to a finite set of remainders modulo n.

What do I mean by smaller? One can give a naive definition of the size of a rational point: $h(P) = 0$ if P is the point at infinity, $h(P) = \log \max\{$numerator of x_P, denominator of $x_P\}$ if $P = (x_P, y_P, 1)$. This idea can be refined in order to obtain a quadratic function on points (the height) such that such that (as in the naive definition), given a constant A, only finitely many points are of height less than A.

The proof then clearly proceeds to its completion

$$\begin{aligned} P &= nP' \oplus P_1 \\ &= n^2P'' \oplus nP_2 \oplus P_1 \\ &= n^3P''' \oplus n^2P_3 \oplus nP_2 \oplus P_1 \\ &= \cdots, \end{aligned}$$

where the height of the rational points P, P', P'', etc. decreases. Then, after a finite number of steps, this height has become smaller than a constant, say A, fixed in advance, and one knows that there are a finite number of such small points. In particular, the point P has been expressed as a combination of a finite set of rational points, the "small" points just defined and the P_i (the remainders, so to speak, in the process of this Euclidean division). And the Mordell-Weil theorem is proved.

The main issue is of course to prove that the Euclidean algorithm really applies, that is that there is a finite set of remainders P_i. In other words, one wants to describe the group $E(\mathbf{Q})/nE(\mathbf{Q})$ and to prove that it is finite. The key idea is to interpret it as a subgroup of a cohomolgy group. One has the exact sequence

$$0 \to E(\bar{\mathbf{Q}})_n \to E(\bar{\mathbf{Q}}) \to E(\bar{\mathbf{Q}}) \to 0,$$

where the the third arrow is multiplication by n on the points of E with algebraic coordinates. By standard procedures, one can deduce from it a cohomology sequence

$$0 \to E(\mathbf{Q})_n \to E(\mathbf{Q}) \to E(\mathbf{Q}) \to H^1(G_{\mathbf{Q}}, E_n) \to H^1\Big(G_{\mathbf{Q}}, E(\bar{\mathbf{Q}})\Big) \to \cdots,$$

and from it the short exact sequence

$$0 \to E(\mathbf{Q})/nE(\mathbf{Q}) \to H^1(G_{\mathbf{Q}}, E_n) \to H^1\Big(G_{\mathbf{Q}}, E(\bar{\mathbf{Q}})\Big)_n \to 0.$$

These constructions are classical. The elements of the first cohomology groups H^1 are crossed homomorphisms, that is homomorphisms which take into account the action of the Galois group on the image; for instance, if the action of the Galois group on the n-torsion points is trivial (we have seen examples for $n = 2$, where the four 2-torsion points are rational, thus fixed by the Galois group), the group $H^1(G_{\mathbf{Q}}, E_n)$ is simply the group of homomorphisms from the Galois group to the n-torsion points. This group is still too big to be used adequately, but one can repeat the same constructions and obtain analogous exact sequences, not only on $\mathbf{Q}$, but also on every $\mathbf{Q}_p$. One obtains then the cohomology groups $H^1(G_{\mathbf{Q}_p}, E_n)$, which contain the global one $H^1(G_{\mathbf{Q}}, E_n)$. Now, it is possible to consider the kernel of the mapping

$$H^1(G_{\mathbf{Q}}, E_n) \to \prod_p H^1(G_{\mathbf{Q}_p}, E_n),$$

it is called the Selmer group of the elliptic curve E relative to the integer n and denoted by $S(E/\mathbf{Q}, n)$. From the various sequences explained above, one deduces that

$$0 \to E(\mathbf{Q})/nE(\mathbf{Q}) \to S(E/\mathbf{Q}, n).$$

Now, algebraic number theory can be used to prove that the Selmer group is finite: the idea is that homomorphisms on the Galois group can be expressed in terms of algebraic extensions and the fact that the Selmer group is the kernel of a kind of "local-global" situation is expressed by strong local conditions on these extensions, so strong that only finitely many of them can exist. This proves the finiteness of the "remainder" group and concludes the proof of the Mordell-Weil theorem.

The Selmer group is in fact easy to compute, which makes us hope that it can be used to provide an effective set of remainders. Unfortunately, the cokernel in the sequence

$$0 \to E(\mathbf{Q})/nE(\mathbf{Q}) \to S(E/\mathbf{Q}, n)$$

is not well-known. It is the set of n-torsion points of the so-called Tate-Safarevic group. This group measures, so to speak, the defect in the local-global principle, but one does not even know in general that it is finite. If it were the case, one would be able to obtain an effective version of the Mordell-Weil theorem, which is still not achieved.

However, a lot of work has been done in this direction in the last two decades. The Tate-Safarevic group enters also in the precise determination of the leading coefficient of the L-function of the elliptic curve at $s = 1$, according to the Birch and Swinnerton-Dyer conjecture described above. Compatible sequences of Selmer groups for $n = p$, p^2, etc. also give p-adic Selmer groups which can be related, at least conjecturally, to p-adic L-functions. Selmer groups play a decisive role in the few proofs available related to the Birch and Swinnerton-Dyer conjecture. More information on these issues can be found for instance in [3].

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Poster Abstract
p-ADICS AND PRO-p GROUPS

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A pro-p group can be defined in two ways
(i) as a special type of topological group
(ii) via a constructive definition which strings together finite p-groups.

Definition 1. A pro-p group is a compact Hausdorff topological group whose open subgroups form a base for the neighbourhoods of the identity and in which every open normal subgroup has index equal to some power of p.

Definition 2. A pro-p group is an inverse limit of finite p-groups.
See [2] for details.

We can view the p-adic integers as a pro-p group via their construction as an inverse limit of finite cyclic groups: $\mathbf{Z}_p = \lim_{\infty \leftarrow n} \mathbf{Z}/p^n\mathbf{Z}$. The p-adic integers play a rôle in the theory of pro-p groups, similar to that played by the cyclic groups in abstract group theory—in fact the p-adic integers form a pro-cyclic group. Over the last decade interest in pro-p groups has grown. Number theorists have shown a continued interest in pro-p groups due to their natural appearance as Galois groups of infinite field extensions. Group theorists have shown a growing awareness of their uses and there now exist many results about abstract groups proved with the help of pro-p groups. Thus pro-p groups have begun to generate interest in their own right. One of the most interesting pro-p groups is the Nottingham group which may be described as the group of normalised automorphisms of the ring $\mathbf{F}_p[[t]]$, namely those automorphisms acting trivially on $t\mathbf{F}_p[[t]]/t^2\mathbf{F}_p[[t]]$. Using work of Witt dating from the 1930s [3, 4], A. Weiss and C. Leedham-Green proved the following result about the Nottingham group.

Theorem 1 [1]**.** *The Nottingham group contains every finite p-group as a subgroup.*

After a careful analysis of Witt's methods it became clear that these finite p-groups could be linked together to prove the following surprising result.

Theorem 2 [1]**.** *The Nottingham group contains every finitely generated pro-p group as a closed subgroup.*

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Poster Abstract
SCHÖNEMANN-EISENSTEIN IRREDUCIBILITY CRITERION

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Area of interest: Valuations and their applications.

The earliest recorded irreducibility criterion is the one proved by Schönemann in 1846.

Schönemann Criterion. Suppose that p is prime and that the polynomial $F(x) \in \mathbb{Z}[x]$ has the form $F(x) = [f(x)]^s + pM(x)$ where $f(x)$ is irreducible modulo p, $M(x)$ is relatively prime to $f(x)$ modulo p and the degree of $M(x)$ is less than that of $F(x)$. Then $F(x)$ is irreducible in $\mathbb{Q}[x]$.

The Eisenstein criterion is a special case of the above criterion with $f(x) = x$.

We have given an irreducibility criterion for polynomials with coefficients in a valued field (K, v) where v is a valuation of any rank, which generalizes Schönemann Criterion. In particular when v is a valuation of any rank of a field K with value group G and

$$f(x) = x^m + a_1 x^{m-1} + \cdots + a_m$$

is a polynomial over K, using prolongations of v to a simple transcendental extension of K, it has been shown that if

$$(v(a_i)/i) \geq (v(a_m)/m) \quad \text{for } 1 \leq i \leq m,$$

and there does not exist any integer $r > 1$ dividing m such that $(v(a_m)/r)$ is in G, then $f(x)$ is irreducible over K.

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Poster Abstract
QUADRATIC FORMS OVER THE p-ADICS

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We fix a ring R and consider the question of classifiying all quadratic forms, or rather all symmetric bilinear forms over R up to isometries.

In the *classical case*, where R is a field of characteristic different from 2, a form is given by a diagonal matrix. For $R = \mathbb{C}$, the forms are classified by their rank, for $R = \mathbb{R}$, by the rank and signature. for finite fields by the rank and determinant of the corresponding matrix. For the p-adic fields $\mathbb{Q}_p$, we reduce modulo p and use Hensel's lemma to write every form as a direct sum $q = q_0 \perp p q_1$ where the determinants of q_0 and q_1 are units in $\mathbb{Z}_p$, and to classify forms by the rank and determinant of q_0 and q_1. For the field $R = \mathbb{Q}$, the situation is much more complicated since there are infinitely many square classes; the Hasse Minkowski theorem enables us to get information over $\mathbb{Q}$ using local information over all the $\mathbb{Q}_p$: forms can be classified by rank, determinant, and all the local Hasse-Witt invariants, at all primes.

In the *integral case*, R can be for instance the ring of integers of a number field. For $R = \mathbb{Z}$ forms are not diagonalizable in general and the situation is very complicated. We can get local information, since the classification over the local rings $\mathbb{Z}_p$ is easy (it looks like that over $\mathbb{Q}_p$), but the local-global principle does not hold.

I am interested in showing that for some rings, field isometries induce ring isometries. For example if $R = \mathbb{Z}_p[G]$ and $K = \mathbb{Q}_p[G]$ are group rings for a finite group G, and the canonical involution induced by $g \mapsto g^{-1}$, we showed.

Theorem 1 (see [1]). *If G is of odd order or if its p-Sylow subgroup is normal and h and h' are two hermitian forms over R that are K-isometric, then they are R-isometric.*

This has applications to the study of G-isometries of G-equivariant quadratic forms over rings of integers, and of self-dual normal integral bases.

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Poster Abstract

p-ADIC NUMBERS IN DYNAMICAL SYSTEMS

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Some applications of algebraic and topological properties of the set of p-adic numbers in the theory of dynamical systems are presented. These applications concern the chaotic behaviour of an Iterated Function System on its attractor and some topological properties of the open basin attractor of a dynamical system. We propose and we construct effectively an invariant measure on an I.F.S.

The general way to use some properties of p-adic numbers in the study of a discrete dynamical system S is to observe a topological conjugation of S with another discrete dynamical system S' whose phase space is the set of p-adic integers.

In this case some properties of S' are transferred to S.

$$\begin{array}{c} p\text{-adic integers} \\ \downarrow \\ \text{symbolic dynamics} \\ \downarrow \\ \text{dynamical systems} \end{array}$$

Contents of the poster:
Definitions and basic results;
Shadowing properties;
Stability;
Iterated function systems;
Measures on fractals.

REPRESENTATIONS

A short course organized by MICHÈLE VERGNE

The subject of group representations arose from two sources: the theory of finite group representations, intimately related to combinatorics, and the theory of infinite dimensional representations of Lie groups in Hilbert spaces, arising in the context of quantum mechanics. Nowadays, group representations are in connection with many different branches of Mathematics.

Talks in this conference gave an idea of the interrelation of group representations with various other mathematical topics. Quantum groups are related to the Yang-Baxter equation by the R-matrix, as described by Welleda Baldoni-Silva in her talk (reproduced only in brief form here). Infinite dimensional representations of real semi-simple Lie groups are connected with symplectic geometry via the orbit method, as shown in the talk of Pascale Harinck. Representations of graphs are related with algebras of invariants and singularity theory as shown in the talk of Idun Reiten. Modular representations of the symmetric group are related to combinatorics as shown in the talk of Christine Bessenrodt.

We also learned from these talks that, if there are very beautiful results obtained, many problems remain open, and many new theories emerge and need to be developed.

MICHÈLE VERGNE

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QUANTIZATION: MOTIVATIONS, CONSTRUCTIONS AND EXAMPLES

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These notes are just a short summary of a lecture given at the ICTP in Trieste on December 12, 1997, at the meeting of EWM, focussing only on some of the basic ideas that are involved in the actual construction of a deformation.

A Hopf algebra $A = A(\mu,\eta,\Delta,\epsilon,S)$ (over a ring K with unit) is an algebra with multiplication $\mu : A\otimes A \longrightarrow A$ (and unit ϵ), a coalgebra with comultiplication $\Delta : A \longrightarrow A\otimes A$ (and counit η) and an antipode map $S : A \longrightarrow A$. All the maps are assumed to be K-linear and one imposes obvious compatibility conditions between the operations.

Suppose that G is a finite group, then $C[G]$, the group algebra, is a cocommutative Hopf algebra and $F(G) = \{f : G \to \mathbb{C}\}$, the function algebra, is a commutative Hopf algebra, moreover $F(G) \simeq C[G]^*$.

We can easily generalize the previous examples by considering $U(\mathfrak{g})$, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, and $F(G)$ the ring of regular function of an algebraic group G.

$U(\mathfrak{g})$ is a cocommutative Hopf algebra and $F(G)$ is a commutative Hopf algebra. Since we are dealing with infinite dimensional Hopf algebras we don't have a duality result like in the finite group case. To restore the duality we are led to the notion of restricted dual. To illustrate the main ideas of the constructions, from now on, we will restrict our attention to the group $G = SL_2(\mathbb{C})$ and to the corresponding Lie algebra $\mathfrak{g} = sl_2(\mathbb{C})$. We refer the reader to the references for the precise statements concerning the results in a more general setting. We also refer to [1] for an extensive list of references in the literature.

$F(G)$ can be easily described as the ring of polynomials in a, b, c, d with complex coefficients modulo the two sided ideal I generated by $\det T - 1$. We think of a, b, c, d, as functions of the matrix entries and we write $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$F(G)$ is the restricted dual of $U(\mathfrak{g})$. The idea is indeed that $F(G)$ is generated by the matrix coefficients of the natural representation of G on $\mathbb{C}^2$ and $U^0(\mathfrak{g})$ (the restricted dual) by those of the corresponding Lie algebra representation.

The term *quantum group* is used in different contexts, but it certainly includes *deformations* of the classical objects associated to an algebraic group and introduced in the previous examples. Starting from the notion of topological Hopf algebra one arrives to the concept of deformation as follows: A topological Hopf algebra $A_h(\mu_h,\eta_h,\Delta_h,\epsilon_h,S_h)$ is a *deformation* of the Hopf algebra $A(\mu,\eta,\Delta,\epsilon,S)$ if the following holds: $A_h \simeq A[[h]]$ as K-module and $\mu_h = \mu(\operatorname{mod} h)$, $\Delta_h = \Delta(\operatorname{mod} h)$.

In particular a *quantum universal enveloping algebra*, denoted by QUE, is a deformation of $U(\mathfrak{g})$ and a *quantum function algebra*, denoted by QF, is a deformation of $F(G)$.

Because of cohomological obstructions it is not difficult to show by direct computations that a QUE is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra and a QF is isomorphic to $F(G)[[h]]$ as a coalgebra. Because of the above in the explicit construction of a QUE or a QF for our examples we deform only one type of structure.

Let U_h be a QUE for $\mathfrak{g} = sl(2,\mathbb{C})$. One starts by observing that a QUE induces a Lie bialgebra structure on $\mathfrak{g}$, defined by

$$\delta(x) = \frac{\Delta_h(a) - \Delta_h^{\mathrm{opp}}(a)}{h} \bmod h,$$

where $a = x \quad \bmod h$ and Δ_h^{opp} is the opposite comultiplication. Thus, in reverse, it can be shown how to construct a QUE, U_h by using the standard Lie bialgebra structure. The constructive idea is that the Lie bialgebra structure on $\mathfrak{g}$ gives informations on the first degree order component of the coalgebra structure that has to be defined. Further, the QUE so constructed is a quasi triangular Hopf algebra, and thus in particular there exists an element $R_h \in U_h \otimes U_h$, called universal R-matrix, satisfying the QYBE (quantum Yang Baxter equation):

$$(R_h)_{12}(R_h)_{13}(R_h)_{23} = (R_h)_{23}(R_h)_{13}(R_h)_{12}.$$

The previous equation at the limit, that is $\bmod h^2$, gives a solution r of the CYBE (classical Yang Baxter equation), where r determines the Lie bialgebra structure that we have deformed. Explicitely if we denote by H, X, Y the standard generator of $\mathfrak{g} = sl(2,\mathbb{C})$, then we arrive at the following.

Definition-Proposition. Let $K = \mathbb{C}[[h]]$ and let $\mathbb{P} = \mathbb{C}\{H, X, Y\}$ be the free algebra of noncommutative polynomials in three generators H, X, Y. We denote by I the h-adic closure in $\mathbb{P}[[h]]$ of the two sided ideal generated by

$$[X,X] - 2X, \qquad [H,Y] + 2Y, \qquad [X,Y] - \frac{e^{hH} - e^{-hH}}{e^h - e^{-h}}.$$

Then $U_h(sl_2(\mathbb{C})) = \mathbb{P}[[h]]/I$ is a QUE of $U(sl_2(\mathbb{C}))$, via the following structures maps

$$\Delta_h(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta_h(X) = X \otimes e^{hH} + 1 \otimes X, \qquad \Delta_h(Y) = Y \otimes 1 + e^{-hH} \otimes Y,$$
$$\epsilon_h(X) = \epsilon_h(Y) = \epsilon_h(H) = 0,$$
$$S_h(X) = -Xe^{-hH}, \qquad S_h(Y) = -e^{-hH}Y, \qquad S_h(H) = -H.$$

As we observed before, the QUE constructed has an extra structure that makes it a *quasitriangular* Hopf algebra, thus in particular there exists a universal R-matrix satisfying the QYBE.

Explicitely in our example we have

$$R_h = \sum_{n \geq 0} R_n(h) e^{h(H \otimes H)} X^n \otimes Y^n,$$

where

$$R_n(h) = \frac{q^{(1/2)n(n+1)}(1-q^2)^n}{[n]_q!}, \quad q = e^h.$$

Let $V = \{v_0, v_1\}$ be the two dimensional $U_h(sl_2(\mathbb{C}))$ module defined by $X \cdot v_0 = 0$, $X \cdot v_1 = v_0$, $Y \cdot v_0 = v_1$, $Y \cdot v_1 = 0$, $H \cdot v_0 = v_0$, $H \cdot v_1 = -v_1$. Let R be R_h composed with the action on $V \otimes V$. Then

$$R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

with respect to the base $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ and

$$\hat{R} = P \cdot R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

with respect to the same base, (P is the permutation operator).

$\hat{R}$ is an intertwining operator from $V \otimes V \to V \otimes V$ and satysfies the QYBE

$$(\hat{R})_{12}(\hat{R})_{23}(\hat{R})_{12} = (\hat{R})_{23}(\hat{R})_{12}(\hat{R})_{23}.$$

The fact that $\hat{R}$ is an intertwining operator is the key point to describe a deformation $F_h(G)$ of $F(G)$ and the above construction can be easily generalized to the setting of a quasi triangular Hopf algebra. The construction of QF is done in two steps, one first deforms $F(M_2)$ (M_2 is the algebra of 2×2 matrices) and then arrives to a deformation of $F(Sl_2(\mathbb{C}))$, by deforming the determinant condition. The first step can be done in a completely general way, that is one can always construct an Hopf algebra (it is in fact *cobraided*) starting from a matrix solution of the QYBE. This is the content of the Faddaev-Reshetikin-Takhtajan construction.

More precisely

$$F_h(M_2) = \mathbb{C}\{a,b,c,d\}[[h]]/I_h,$$

where $\mathbb{C}\{a,b,c,d\}$ is the free algebra generated by the noncommutative polynomials in a,b,c,d and I_h is the closure in the h-adic topology of the two sided ideal generated by

$$\hat{R}(T \otimes T) = (T \otimes T)\hat{R},$$

i.e., I_h is the ideal generated by the relations

$$qab = ba, \quad qac = ca, \quad bc = cb, \quad qbd = db, \quad qcd = dc, \quad ad - da = (q-q^{-1})bc.$$

$F_h(M_2)$ is a bialgebra via pointwise multiplication and

$$\Delta_h(T) = T \otimes T, \qquad \epsilon_h(T) = 1, \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is easy to see that $F(M_2)$ can be described as $F(M_2) = \mathbb{C}\{a,b,c,d\}/I_0$, ($I_0$ the ideal I_h for $h = 0$), thus at least formally justifying the use of I_h for the deformation. In effect the relations described by the ideal are exactly the ones to be imposed on matrix coefficients if we want $F_h(G)$ to be the restricted dual of U_h, as in the classical case; and these conditions are easily obtained by using the intertwining properties of $\hat{R}$.

Finally, $F_h(Sl_2) = F_h(M_2)/J_h$ where J_h is the h-adic closure of the two sided ideal generated by $\det_q T - 1$, where $\det_q T := ad - q^{-1}bc$. With this definition of determinant the structure maps pass to the quotient.

Define

$$S_h : F_h(Sl_2) \to F_h(Sl_2), \qquad S_h(T) = T^{-1}$$

then $F_h(Sl_2)$ is a QF for $F(Sl_2)$.

To conclude it would be interesting to show how the QYBE both in the constant matrix form or with spectral parameters, comes out naturally from the study of integrable lattices models.

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ORBIT METHOD FOR $Sl(2,\mathbb{R})$

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1. Introduction. An important problem in harmonic analysis on Lie groups or symmetric spaces is to describe the Plancherel formula. It is a generalization of the classical Plancherel theorem on $\mathbb{R}$ which says that the Fourier transform extends to an isometry of $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$. For f a function of class C^∞ with compact support, we define its Fourier transform $\hat{f}$ by $\hat{f}(x) = 1/(\sqrt{2\pi})\int_{\mathbb{R}} f(y)e^{ixy}dy$ and we have the inversion formula $f(x) = 1/(\sqrt{2\pi})\int_{\mathbb{R}} e^{-ixy}\hat{f}(y)dy$. The functions $x \to e^{ixy}$ are exactly the irreducible unitary representations of $\mathbb{R}$.

A beautiful method to obtain the Plancherel formula for groups is the orbit method which consist of relating irreducible unitary representations of the group with orbits of the coadjoint representation of the group. This method was first developped by A.A. Kirillov for nilpotent Lie groups ([11]).

I want to explain this on the example $Sl(2,\mathbb{R})$.

2. Representations of $Sl(2,\mathbb{R})$

2.1. Preliminaries on Lie groups. We say that a group G is a Lie group if it is an analytic manifold such that the group operations are analytic. Let e be the identity element of G. Let γg be the tangent space to G at e.

The group G acts on itself by inner automorphism $\varphi_x(g) = xgx^{-1}$. The differential $\mathrm{Ad}(x)$ of φ_x at e is called the adjoint action of G on γg.

The differential $\mathrm{ad}(X)$ of Ad at e is a map from γg to $\mathrm{End}(\gamma g)$ called the adjoint action of γg. We put $\mathrm{ad}(X)(Y) = [X,Y]$. This bracket gives γg a Lie algebra structure, which means that we have the two following properties:

(i) The bracket is antisymmetric: $[X,Y] = -[Y,X]$

(ii) The bracket satisfies the Jacobi relation: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[Y,X]] = 0$.

The group G acts on the dual γg^* of γg by the coadjoint action given by $(g\cdot\lambda)(X) = \lambda(\mathrm{Ad}(g^{-1})X)$.

We define on γg the G-invariant bilinear form κ by $\kappa(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$. It is called the Killing form of γg. We say that G is semisimple if κ is non degenerate.

Example. The group $Sl(n,\mathbb{R})$ is a semisimple Lie group. We have $\kappa(X,Y) = 2n\,\mathrm{tr}(XY)$. The adjoint action is given by $\mathrm{Ad}(g)X = gXg^{-1}$ and the bracket is $[X,Y] = XY - YX$.

2.2. Generalities on representations. Let G be a Lie group.

A *representation* π of G in a Banach space V is a group homomorphism π from G to $\mathrm{End}(V)$ such that the map $(g,v) \to \pi(g)v$ is continuous from $G\times V$ to V.

We say that (π, V) is *unitary* if V is a Hilbert space and the operators $\pi(g)$ are unitary.

Two unitary representations (π, V) and (π', V') are called unitarily equivalent if there exist an unitary operator $L : V \to V'$ such that, for all $g \in G$, we have $L \circ \pi(g) = \pi'(g) \circ L$. Such operators are called intertwining operators. We denote by $L_G(V, V')$ the set of intertwining operators between π and π'.

We say that (π, V) is *irreducible* if V admits no non-trivial closed subspace, stable by the action of all $\pi(g)$.

Lemma 1 (Schur's Lemma). *Let (π, V) be a unitary representation of G. Then (π, V) is irreducible if and only if $L_G(V, V) = \mathbb{C}\,\mathrm{Id}_V$.*

2.3. Finite representations of $Sl(2, \mathbb{R})$. Let $G = Sl(2, \mathbb{R}) = \{g \in M(2, \mathbb{R}); \det(g) = 1\}$ and $\gamma g = sl(2, \mathbb{R}) = \{X \in M(2, \mathbb{R}); \mathrm{tr}(X) = 0\}$. A natural basis of γg is given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and we have $[h, e] = 2e$; $[h, f] = -2f$; $[e, f] = h$.

We begin with an example.

Fix a positive integer n. Let V_n be the complex vector space of homogeneous polynomials of degree n in two variables z_1 and z_2. We have $\dim V_n = n + 1$.

We consider the representation Φ_n given by

$$\Phi_n(g) P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(g^{-1} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right).$$

Proposition 2. *The representation Φ_n is irreducible. If (π, V) is an irreducible representation of G with $\dim V = n + 1$ then (π, V) is equivalent to Φ_n.*

Proposition 3. (1) *Every finite dimensional representation of $Sl(2, \mathbb{R})$ is a direct sum of irreducible representations.*

(2) *Every finite dimensional unitary representation of $Sl(2, \mathbb{R})$ is trivial.*

Proof of Proposition 2. Let(π, V) be an irreducible representation of G with $\dim V = n + 1$. In such a case, we can define an irreducible representation $d\pi$ of the Lie algebra γg by $d\pi(X) = \frac{d}{dt}[\pi(\exp tX)v]_{/t=0}$ where $\exp X = \sum_{n \geq 0} \frac{X^n}{n!}$ is the usual exponential map. The map $d\pi$ satisfies: $d\pi([X, Y]) = d\pi(X) d\pi(Y) - d\pi(Y) d\pi(X)$. Using the bracket relations on γg, we can prove that there exist a basis $(v_0, \ldots, v_n)$ of V such that we have

$$\begin{aligned} &d\pi(h) v_j = (n - 2j) v_j \quad \text{for all } j \\ &d\pi(e) v_0 = 0, \qquad d\pi(e) v_j = j(n - j + 1) v_{j-1} \quad \text{for } j > 0 \\ &d\pi(f) v_n = 0, \qquad d\pi(f) v_j = v_{j+1} \quad \text{for } j < n. \end{aligned}$$

The representation Φ_n is a realization of such a representation (π, V). □

2.4. Irreducible unitary representations of $Sl(2,\mathbb{R})$. We have three series of unitary irreducible representations of $Sl(2,\mathbb{R})$.

The discrete series. Consider the upper-half plane $P^+ = \{x+iy; x, y \in \mathbb{R}$ and $y > 0\}$.

Let n be a strictly positive integer and consider

$$H_n = \left\{\text{holomorphic functions } \phi \text{ on } P^+ \text{ such that } \int_{P^+} |\, \phi\, |^2\, y^{n-1} dx\, dy < \infty\right\}.$$

It is a non empty Hilbert space.

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, we define $(\mathcal{D}_n(g^{-1})\phi)(z) = (cz+d)^{-(n+1)}\phi((az+b)/(cz+d))$.

The map $g \to \mathcal{D}_n(g)$ defines a unitary irreducible representation of G which is called the holomorphic discrete series.

For n a negative integer, we consider the space

$$\tilde{H}_n = \left\{\text{antiholomorphic functions } \phi \text{ on } P^+ \text{ such that } \int_{P^+} |\, \phi\, |^2\, y^{|n|-1} dx\, dy < \infty\right\}.$$

It is a non empty Hilbert space.

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, we define $(\mathcal{D}_n(g^{-1})\phi)(z) = (\overline{cz+d})^{-(1-n)}\phi((az+b)/(cz+d))$.

The map $g \to \mathcal{D}_n(g)$ defines a unitary irreducible representation of G which is called the antiholomorphic discrete series.

The principal series. Let $H = L^2(\mathbb{R})$ and let s be a real number. We define the representation $\mathcal{P}_s^\pm$ of G in $L^2(\mathbb{R})$ as follow: if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$\left(\mathcal{P}_s^+(g^{-1})f\right)(x) = |\, cx+d\, |^{-1+is} f\left(\frac{ax+b}{cx+d}\right)$$

and

$$\left(\mathcal{P}_s^-(g^{-1})f\right)(x) = \operatorname{sign}(cx+d)\ |\, cx+d\, |^{-1+is} f\left(\frac{ax+b}{cx+d}\right).$$

For $s \neq 0$, the representations $\mathcal{P}_s^+$ and $\mathcal{P}_s^-$ are unitary and irreducible. The representation $\mathcal{P}_s^\pm$ is equivalent to $\mathcal{P}_{-s}^\pm$. We call it the principal series of G.

The representation $\mathcal{P}_0^+$ is irreducible but $\mathcal{P}_0^-$ is the sum of two irreducible unitary representations $\mathcal{D}_0^+$ and $\mathcal{D}_0^-$.

The representation $\mathcal{D}_0^+$ acts on the set

$$\left\{\text{holomorphic functions } \phi \text{ on } P^+ \text{ s. t. } \sup_{y>0} \int_{\mathbb{R}} |\, \phi\, |^2\, dx < \infty\right\}$$

by $(\mathcal{D}_0^+(g^{-1})\phi)(z) = (cz+d)^{-1}\phi((az+b)/(cz+d))$ and the representation $\mathcal{D}_0^-$ acts on the set

$$\left\{\text{antiholomorphic functions } \phi \text{ on } P^+ \text{ s. t. } \sup_{y>0} \int_{\mathbb{R}} |\, \phi\, |^2\, dx < \infty\right\}$$

by $(\mathcal{D}_0^-(g^{-1})\phi)(z) = (\overline{cz+d})^{-1}\phi((az+b)/(cz+d))$.

The complementary series $\mathscr{C}^u$. We take $0 < u < 1$ and we define the Hilbert space

$$H_u = \left\{ f : \mathbb{R} \to \mathbb{C}; \| f \|^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(x)\overline{f}(y)}{|x-y|^{1-u}} dx\, dy < \infty \right\}.$$

The group G acts on H_u as follows: for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, we have

$$\mathscr{C}^u(g^{-1}) f(x) = | cx + d |^{-1-u} f\left(\frac{ax+b}{cx+d}\right).$$

These representations are unitary and irreducible and the series $\mathscr{C}^u$ is called the complementary series.

Theorem 4. *Each irreducible unitary representation of* $Sl(2,\mathbb{R})$ *is equivalent to one of the following type:*

(1) *The trivial representation,*
(2) $\mathscr{D}_n$ *for a non-zero integer* n,
(3) $\mathscr{D}_0^+$ *or* $\mathscr{D}_0^-$,
(4) $\mathscr{P}_s^{\pm}$ *for a non-zero real number* s,
(5) $\mathscr{P}_0^+$,
(6) $\mathscr{C}^u$ *for* $0 < u < 1$.

Proof. The idea of the proof is the following: Fix $(\pi, \mathscr{H})$ a unitary irreducible representation of G. Let $K = SO(2) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. We consider the space $\mathscr{H}^K$ of vectors v such that the space generated by the $\pi(k)v$ for $k \in K$, is a finite dimensional vector space. In this case, we can define a representation $d\pi$ of γg in $\mathscr{H}^K$ by differentiation. The study of this representation gives the theorem. □

3. Fourier transforms of coadjoint orbits

3.1. The coadjoint orbits of $sl(2,\mathbb{R})$. Let $G = SL(2,\mathbb{R})$ and let $\gamma g = sl(2,\mathbb{R})$ be the Lie algebra of G. We have

$$\gamma g = \left\{ X = \begin{pmatrix} x_1 & x_2 + x_3 \\ x_2 - x_3 & -x_1 \end{pmatrix}; x_j \in \mathbb{R} \right\}.$$

We want to study the coadjoint orbits of G on γg^*.

We identify γg and γg^* via the G-invariant form $(X, Y) \to \frac{1}{2}\operatorname{tr}(XY)$.

The function $\det X = x_3^2 - (x_1^2 + x_2^2)$ is invariant by the action of G on γg, and so we can describe the orbits of the coadjoint action as folllow:

(1) The orbit $\mathbb{O}_\lambda^d$ of an element $f_\lambda = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ with $\lambda \neq 0$: When $\lambda > 0$, we obtain the upper sheet $x_3 \geq 0$ of the two-sheeted hyperboloid

$$x_3^2 - \left(x_1^2 + x_2^2\right) = \lambda^2 \quad (\lambda \neq 0).$$

When $\lambda < 0$, we obtain the lower sheet $x_3 \leq 0$ of the two-sheeted hyperboloid

$$x_3^2 - \left(x_1^2 + x_2^2\right) = \lambda^2 \quad (\lambda \neq 0).$$

(2) The orbit $\mathbb{O}_s^p$ of an element $g_s = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$ with $s \neq 0$: It is the one-sheeted hyperboloid $x_3^2 - (x_1^2 + x_2^2) = -s^2$ $(s \neq 0)$.

(3) The point $\{0\}$ and the two connected components of the light cone $x_3^2 - (x_1^2 + x_2^2) = 0$ which correspond to the orbits of the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

We say that an element f is regular if $\det(f) \neq 0$ and nilpotent if $\det(f) = 0$. Let γg_{reg} denote the set of regular element of γg.

Put $\gamma t = \{f = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}; \theta \in \mathbb{R}\}$ and $\gamma a = \{f = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}; t \in \mathbb{R}\}$. These two algebras are commutative and consist of semisimple elements. They are maximal for these two properties. We call them Cartan subalgebras.

Every semisimple element of γg is G-conjugate to an element of γt or γa and we have the following decomposition

$$\gamma g_{\mathrm{reg}} = \left(\cup_{\lambda \in \mathbb{R}-\{0\}} \mathbb{O}_\lambda^d\right) \cup \left(\cup_{s>0} \mathbb{O}_s^p\right).$$

3.2. The Liouville measure on an orbit. We fix a regular element $f \in \gamma g$.

We consider the orbit $\Omega = G \cdot f = G/G(f)$ where $G(f) = \{g \in G; g \cdot f = f\}$ is the stabilizer of f in G. The tangent space of Ω is then isomorphic to the space $\gamma g/\gamma g(f)$ where $\gamma g(f) = \{X \in \gamma g; f([X,Y]) = 0 \text{ for all } Y \in \gamma g\}$.

On this space, we consider the form σ_f given by $\sigma_f(X \cdot f, Y \cdot f) = f([X,Y])$. It is an alternate non-degenerate closed 2-form on the tangent space of Ω and so it gives Ω the structure of a symplectic manifold.

We define the Liouville measure β_Ω on Ω by

$$\beta_\Omega = \frac{\sigma_f}{2\pi}.$$

With our choice of coordinate, for $f = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ we have $\beta_\Omega = (dx_1\, dx_2)/(\mid x_3 \mid)$, and for $f = \begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix}$, we have $\beta_\Omega = (dx_2 dx_3)/(\mid x_1 \mid)$.

We can choose canonically a G-invariant measure $d\dot{g}$ on $G/G(f)$ such that, for all function $\varphi \in \mathfrak{D}(\gamma g)$ (which means that φ is of class C^∞ with compact support) and for all $X \in \gamma g(f)_{\mathrm{reg}}$, we have

$$\beta_\Omega(\varphi) = \frac{1}{2\pi} \mid \det(\mathrm{ad}\, X)_{/\gamma g/\gamma g(f)} \mid^{1/2} \int_{G/G(f)} \varphi(g \cdot X) d\dot{g}.$$

The map $\mathcal{M}(\varphi)$ defined by $\mathcal{M}(\varphi)(X) = |\det(\mathrm{ad}\, X)_{/\gamma g/\gamma g(X)}|^{1/2} \int_{G/G(X)} \varphi(g \cdot X) d\dot{g}$ on γg_{reg} is called the orbital integral of φ.

We put $\gamma b = \gamma g(f)$. Let $S(\gamma b_{\mathbb{C}})$ be the symmetric algebra of $\gamma b_{\mathbb{C}}$. Each element $u \in S(\gamma b_{\mathbb{C}})$ gives rise to a differential operator $\partial(u)$ on γb. (For $X \in \gamma g$, we have $\partial(X) \cdot \varphi(Y) = \frac{d}{dt} \varphi(Y + tX)_{t=0}$).

For $\varphi \in \mathfrak{D}(\gamma g)$, the orbital integral $\mathcal{M}(\varphi)$ satisfies the following properties: ([7, 8])

(1) For $\gamma b = \gamma a$ or γt, there exists a compact set U in γb such that for all $u \in S(\gamma b_{\mathbb{C}})$, the map $\partial(u)\mathcal{M}(\varphi)$ is zero on $(\gamma b - U)_{\mathrm{reg}}$,

(2) The map $X \to \mathcal{M}(\varphi)(X)$ extends to a C^∞ function on γa,

(3) for all strictly positive integers n, we have

$$\lim_{\theta \to 0^+} \left(\frac{id}{d\theta}\right)^n (\mathcal{M}(\varphi))(f_\theta) + \lim_{\theta \to 0^-} \left(\frac{id}{d\theta}\right)^n (\mathcal{M}(\varphi))(f_\theta) = \lim_{t \to 0} \left(\frac{d}{dt}\right)^n (\mathcal{M}(\varphi))(f_t).$$

This relation is called the “jump relation.”

(4) $\lim_{\theta\to 0} \dfrac{d}{d\theta}(\operatorname{sign}(\theta)\mathcal{M}(\varphi))(f_\theta) = -2\varphi(0)$.

This relation is called the limit formula of Harish-Chandra.

Let $I(\gamma g)$ the set of G-invariant function on γg_{reg} satisfying this three conditions. We consider the topology defined by the seminorms $p_{\gamma b,u,U}(F) = \sup_{x\in U_{\mathrm{reg}}} |\partial(u)F(x)|$, where γb is a Cartan subalgebra of γg, U is a compact set in γb and $u \in S(\gamma b_{\mathbb{C}})$. This gives $I(\gamma g)$ a structure of inductive limit of Frechet space.

Theorem 5 [1]. *The map $\mathcal{M}$ is surjective from $\mathcal{D}(\gamma g)$ onto $I(\gamma g)$ and its transposed map is a bijection between the dual $I(\gamma g)'$ of $I(\gamma g)$ and the space of G-invariant distribution on γg.*

3.3. The Fourier transforms of orbits. The first important result due to Harish-Chandra is the following:

Theorem 6 [7, 8]. *The Liouville measure is tempered. (This means that there exist $r > 0$ such that $\int_\Omega (1+ \| \xi \|^2)^{-r} d\beta_\Omega(\xi) < \infty$.)*

So we can consider its Fourier transform, defined by $\hat{\beta}_\Omega(\varphi) = \beta_\Omega(\hat{\varphi})$ where $\varphi \in \mathcal{D}(\gamma g)$ and $\hat{\varphi}$ is its usual Fourier transform on the vector space γg. We obtain a G-invariant tempered distribution on γg.

Let $S(\gamma g_{\mathbb{C}})$ be the symmetric algebra of $\gamma g_{\mathbb{C}}$. We can see this algebra as the algebra of polynomials functions on γg^*. The algebra $S(\gamma g_{\mathbb{C}})^G$ of G-invariant polynomials on γg^* is isomorphic to the algebra of G-invariant differential operators with constant coefficients on γg by the map $X \to \partial(X)$ defined by $\partial(X)\varphi(Y) = \frac{d}{dt}(\varphi(X+tY))_{t=0}$. So for $p \in S(\gamma g_{\mathbb{C}})^G$, we have $\partial(p)\hat{\beta}_\Omega = p(if)\hat{\beta}_\Omega$.

Theorem 7 [9]. *The distribution $\hat{\beta}_\Omega$ is a locally integrable function on γg whose restriction to the set of regular elements is analytic.*

A general formula due to Rossman and results of Harish-Chandra enables us to calculate the Fourier transform of orbits on γg_{reg} ([14] and [9]). In our example, a simple calculation gives the following result

$$\hat{\beta}_{\mathbb{O}^d_\lambda}\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \frac{e^{-i\lambda\theta}}{2i\theta}\operatorname{sign}(\lambda)$$
$$\hat{\beta}_{\mathbb{O}^d_\lambda}\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \frac{e^{-|t\lambda|}}{|2t|}\operatorname{sign}(\lambda)$$

and

$$\hat{\beta}_{\mathbb{O}^p_s}\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = 0$$
$$\hat{\beta}_{\mathbb{O}^p_s}\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \frac{e^{\mathrm{ist}} + e^{-\mathrm{ist}}}{|2t|}.$$

4. Representations and Fourier transforms of orbits. The orbit method consists of relating the orbits of the coadjoint representation of G and the irreducible unitary representations of G.

We now introduce the notion of character of a representation.

Let (π,V) be a finite dimensionnal representation of G. The character of π is the map $\chi_\pi(g) = \operatorname{Tr}\pi(g)$ where Tr denote the trace.

If (π,V) is an infinite dimensionnal reprsentation, we can define, for $\varphi \in \mathfrak{D}(G)$, the operator $\pi(\varphi)$ by $\pi(\varphi) = \int_G \varphi(g)\pi(g)dg$ where dg is a Haar measure on G.

When (π,V) is irreducible and unitary and $\varphi \in \mathfrak{D}(G)$, we can define $\operatorname{Tr}\pi(\varphi)$ as follows: Let $(e_i)_{i\in I}$ be an orthonormal basis of V. Then $\operatorname{Tr}(\pi(\varphi)) = \sum_{i\in I}\langle \pi(\varphi)e_i, e_i\rangle$. The map $\varphi \to \operatorname{Tr}(\pi(f))$ defines a distribution on G which is invariant under inner automorphisms.

The distribution $\operatorname{Tr}(\pi(\varphi))$ is called the character of π. Such distributions were studied by Harish-Chandra who obtained the following main results.

Theorem 8 [9]. *Let $(\pi,\mathcal{H})$ be a unitary irreducible representation of G. Then there exists a locally integrable function Θ_π on G such that for all $\varphi \in \mathfrak{D}(G)$, we have*

$$\operatorname{Tr}(\pi(\varphi)) = \int_G \Theta_\pi(g)\varphi(g)\,dg.$$

Let G_{reg} be the set of regular element of G ($x \in G_{\mathrm{reg}}$ means that its stabilizer in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$). The restriction of Θ_π to G_{reg} is analytic and determines Θ_π.

We say that $(\pi,\mathcal{H})$ is in the discrete series if its character is given by a square integrable function.

Let

$$T = SO(2) = \left\{ y_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}; \theta \in \mathbb{R} \right\}$$

and

$$A = \left\{ \varepsilon x_t = \begin{pmatrix} \varepsilon e^t & 0 \\ 0 & \varepsilon e^{-t} \end{pmatrix}; t \in \mathbb{R} \text{ and } \varepsilon = \pm 1 \right\}.$$

Each regular element of G is G-conjugate to a regular element in T or in A.

We will now give the character of the discrete and principal series in terms of Fourier transforms of orbits.

Let $j(X)$ be the jacobian of the exponential map. We have $j\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right)^{1/2} = \sinh t/t$ and $j\left(\begin{smallmatrix} 0 & \theta \\ -\theta & 0 \end{smallmatrix}\right)^{1/2} = \sin\theta/\theta$. We obtain for $X \in \mathfrak{g}_{\mathrm{reg}}$ and $\varepsilon = \pm 1$

$$\operatorname{Tr}\mathfrak{D}_n(\varepsilon \exp X) = \varepsilon^{1+n}\hat{\beta}_{\mathbb{O}_n^d}(X)j(X)^{1/2}$$

and

$$\operatorname{Tr}\mathcal{P}_s^+(\varepsilon \exp X) = \hat{\beta}_{\mathbb{O}_s^p}(X)j(X)^{1/2},$$
$$\operatorname{Tr}\mathcal{P}_s^-(\varepsilon \exp X) = \varepsilon\hat{\beta}_{\mathbb{O}_s^p}(X)j(X)^{1/2}.$$

4.1. Inversion of orbital integrals and the Plancherel formula. For $\varphi \in \mathcal{D}(G)$, we can define the orbital integral $\mathcal{M}(\varphi)$ on G_{reg} as follow

$$\mathcal{M}(\varphi)(x) = \left| \det(1 - \mathrm{Ad}(x))_{/\gamma g/\gamma g(x)} \right|^{1/2} \int_{G/G(x)} \varphi(gxg^{-1})\, d\dot{g}$$

As on the Lie algebra, thus function satisfies jump relations on T and C^∞ continuation on A. We also have the limit formula of Harish-Chandra

$$\varphi(e) = -\frac{d}{d\theta}(\mathrm{sign}(\theta)\mathcal{M}(\varphi)(y_\theta))_{\theta=0}.$$

For $\varphi \in \mathcal{D}(G)$, we want to describe $\mathcal{M}(\varphi)$ and $\varphi(e)$ in terms of the distributions $\mathrm{Tr}(\pi(\varphi))$ for $\pi \in \hat{G}$. On $Sl(2,\mathbb{R})$ only the discrete and the principal series contribute to the Plancherel formula.

We introduce new functions. Let $X \in \gamma g_{\text{reg}}$: For a non zero integer n, we set

$$F_n(\varepsilon \exp X) = \varepsilon \hat{\beta}_{G\cdot X}\begin{pmatrix} 0 & n \\ -n & 0 \end{pmatrix} \mid 2n \mid$$

and $F_0^\pm = \lim_{n\to 0^\pm} F_n$.

For $s \in \mathbb{R}$ and $\varepsilon = \pm 1$, we define

$$F_{\varepsilon,s}(\varepsilon \exp X) = \sum_{Y \in \gamma g(X); \exp Y = 1} \hat{\beta}_{H\cdot Y}\begin{pmatrix} s & 0 \\ 0 & -s \end{pmatrix} |2s|.$$

We set $F_{+,s} = -(F_{1,s} + F_{-1,s})$ and $F_{-,s} = F_{1,s} - F_{-1,s}$.

Theorem 9 (Inversion formula for orbital integrals)**.** [2] *Let $I(G) = \mathcal{M}(\mathcal{D}(G))$.*

(1) *the functions F_n, $F_O^\pm$ and $F_{\pm,s}$ are in $I(G)$ and they are eigenfunctions under the action of left and right G-invariant differential operators on G,*

(2) *for all $\varphi \in \mathcal{D}(G)$ and $x \in G_{\text{reg}}$, we have*

$$2\pi\mathcal{M}(\varphi)(x) = \sum_{n\in\mathbb{Z}; n\neq 0} F_n(x)\,\mathrm{Tr}\mathcal{D}_n(\varphi) - i\big(\mathrm{Tr}\mathcal{D}_O^+(\varphi) - \mathrm{Tr}\mathcal{D}_O^-(\varphi)\big) + \frac{1}{2}\int_{s>0} \big(F_{+,s}(x)\,\mathrm{Tr}\mathcal{P}_s^+(\varphi) + F_{-,s}(x)\,\mathrm{Tr}\mathcal{P}_s^-(\varphi)\big)\, ds$$

Corollary 10 (Plancherel formula)**.** *For all $\varphi \in \mathcal{D}(G)$, we have*

$$2\pi\varphi(e) = \sum_{n\in\mathbb{Z}; n\neq 0} \mid n \mid \mathrm{Tr}\mathcal{D}_n(\varphi) + \frac{1}{2}\int_{s>0} s\tanh\left(\frac{\pi s}{2}\right)\mathrm{Tr}\mathcal{P}_s^+(\varphi)ds + \frac{1}{2}\int_{s>0} s\coth\left(\frac{\pi s}{2}\right)\mathrm{Tr}\mathcal{P}_s^-(\varphi)\, ds.$$

This formula is obtained by several methods ([13] or [12]).

The first proof of the Plancherel formula for semisimple connected Lie groups was given by Harish-Chandra ([10]). M. Duflo and M. Vergne later gave a new proof using the orbits method [3].

A. Bouaziz proved, using the orbit method, the inversion formula for the orbital integrals on connected semisimple Lie groups [2] and this method can be adapted to find a similar formula on symmetric spaces $G_{\mathbb{C}}/G$, where $G_{\mathbb{C}}$ is a complex semisimple Lie groups and G a real form of $G_{\mathbb{C}}$ ([4] and [5]). The Plancherel formula can be deduced from the inversion of orbital integrals using the limit formula of Harish-Chandra.

An interesting open problem consists of studying how the orbit method can be applied to prove the Plancherel formula on general symmetric spaces G/H where G is a semisimple Lie group with an involution σ and H is an open subgroup of the group G^{σ} of the elements fixed by σ.

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REPRESENTATIONS OF THE SYMMETRIC GROUPS

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1. Introduction. Since early on in the representation theory of finite groups the representations of the symmetric groups S_n have played an important rôle. The irreducible complex characters of S_n were classified by Frobenius 1900; already from these very beginnings of the complex representation theory of the symmetric groups, the connections with symmetric functions have been of particular importance (see [21]). Since partitions of n label the irreducible representations in a natural way, there has always been an intimate relation between algebraic and representation theoretic properties and combinatorial questions. A recurring theme is the determination of representation theoretical data by combinatorial algorithms on the partition labels. Via partitions, there is also a link to number theory.

Of particular interest are the dimensions of the S_n-representations, their branching behaviour with respect to restriction to the subgroup S_{n-1}, and the result of tensoring with the sign representation. To all these questions there are well-known answers available if the representations are defined over a field of characteristic 0; nice combinatorial descriptions are given via branching and conjugation of the partitions of n labelling these representations. An important problem that is still open even for representations at characteristic 0 is the computation of general tensor products.

It turned out that for p-modular representations (i.e. those defined over a field of characteristic p) the problems mentioned above are much harder. The interest in such questions has increased in recent years as there are strong connections between the symmetric groups and their representations and related groups such as the alternating groups or the covering groups of these groups, and also strong relations to representations of the general linear groups and Hecke algebras and their quantum analogues; all these topics are developing very fast (see [13, 22] and the literature cited there). One particular reason for looking into the representation theory of the alternating groups comes from a general strategy in the representation theory of general finite groups: reduce a conjecture to the case of finite simple groups and then, using the classification of finite simple groups, check it for all these groups. Of course, in particular with this strategy in mind, one often starts developing or testing conjectures for the infinite families of symmetric and alternating groups.

We will describe below some of the recent results on the restriction of irreducible S_n-representations and the tensor product with the sign representation at characteristic p. These results have been the basis for progress on the p-modular representations of the alternating groups.

2. Representations at characteristic 0. First we will introduce representations and some of their basic properties.

Let G be a finite group, and let A be a commutative ring (with 1); in this article, the ring A will usually be a field or $A = \mathbb{Z}$. Then a *(linear) representation* of G on a finitely generated free A-module V (of rank m) is a homomorphism

$$G \to \mathrm{GL}(V) \text{ resp. } G \to \mathrm{GL}_m(A)$$

from G to the group of invertible transformations on V. Taking traces gives the associated character $\chi_V : G \to A$. Note that $\chi_V(1) = m$ is the rank of V; also, χ_V is a *class function*, i.e. constant on conjugacy classes of G. With respect to this G-action V is a module for the *group algebra* AG, which is the algebra of formal sums $\sum_{g\in G} a_g g$ with coefficients in A, central multiplication by scalars in A and componentwise addition and multiplication induced from the multiplication in G (linearly extended). Thus the terms AG-module and (A-)representation of G may be used interchangeably.

The AG-module V (resp. the corresponding representation) is *irreducible* if it contains only the two (trivial) AG-submodules $\{0\}$ and V; the corresponding character χ_V is then also called irreducible.

Let us look at some examples for the group $G = S_n$, and take $A = \mathbb{Q}$. The two easiest representations of S_n are the *trivial representation*

$$\mathbb{1} : S_n \to \mathbb{Q}^*, \qquad \sigma \mapsto 1$$

and the *sign representation*

$$\mathrm{sgn} : S_n \to \mathbb{Q}^*, \qquad \sigma \mapsto \mathrm{sgn}\,\sigma.$$

Like any one-dimensional representation, they are obviously irreducible and they coincide with the corresponding characters.

The *natural representation* of S_n is given on an n-dimensional $\mathbb{Q}$-vector space V with basis $\{b_1, \ldots, b_n\}$ by

$$\sigma(b_i) = b_{\sigma(i)} \quad \text{for all } \sigma \in S_n,\ i = 1, \ldots, n.$$

For $n \geq 2$, this representation is not irreducible since V has the S_n-invariant subspaces

$$U = \mathbb{Q}\left(\sum_{i=1}^{n} b_i\right), \qquad W = \left\{\sum_{i=1}^{n} c_i b_i \in V \mid \sum_{i=1}^{n} c_i = 0\right\}.$$

In fact, the $\mathbb{Q}S_n$-module V decomposes into a direct sum of these modules, i.e. $V = U \oplus W$ (as $\mathbb{Q}S_n$-modules). The module U is just the trivial representation again, so in particular U is irreducible; in fact, also W is irreducible.

In explicit matrix terms, for $n = 3$ the natural representation is given by the following matrices for generators of S_3

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So the corresponding character χ_V has the values $\chi_V(1) = 3, \chi_V((12)) = 1, \chi_V((123)) = 0$ (note that this gives the values on all conjugacy classes of S_3). Changing the basis to one adapted to the submodules U and W, i.e. taking as a new basis $b'_1 = \sum_i b_i$, $b'_2 = b_1 - b_3$, $b'_3 = b_2 - b_3$, we obtain the matrix representation

$$(12) \mapsto \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{0} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad (123) \mapsto \begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{-1} & \mathbf{-1} \\ 0 & \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

So all representation matrices have the same structure with diagonal block matrices of size 1×1 resp. 2×2 (indicated above in boldface). From the lower diagonal 2×2 block matrices we immediately obtain the character belonging to W as given by the values $\chi_W(1) = 2$, $\chi_W((12)) = 0$, $\chi_W((123)) = -1$.

In the example above, we have been in the situation of *ordinary representation theory*, i.e. the ring A is a field K of characteristic 0, which is "sufficiently large" for G, e.g. the field of complex numbers will always do (for $G = S_n$ the field of rationals is already large enough). Some of the most important basic properties of ordinary representations of G are the following (see [7, 9]):

Basic facts in ordinary representation theory.

(a) (Maschke) KG is semisimple, so any K-representation of G is *completely reducible*, i.e. any KG-module V can be written as

$$V = V_1 \oplus \cdots \oplus V_k$$

with irreducible KG-submodules $V_1, \ldots, V_k$.

(b) The K-representations of G are determined (up to isomorphism) by their characters.

(c) The number of irreducible K-representations of G (up to isomorphism) equals the number $k(G)$ of conjugacy classes of G.

(d) Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G over K; then

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2.$$

By the properties above, the irreducible representations are the basic building blocks for all representations, so the basic problem of ordinary representation theory is to determine these resp. their characters.

In the example $G = S_3$, $K = \mathbb{Q}$ considered above, we had already determined three irreducible representations resp. characters, namely the trivial representation $\mathbb{1}$, the sign representation and the 2-dimensional $\mathbb{Q}S_n$-module W with its character χ_W. By the general properties stated above, these are *all* the irreducible representations resp. characters of S_3.

For the symmetric groups S_n, the classification of their irreducible characters has been achieved early in the history of representation theory by Frobenius. Important at all stages of the development of the representation theory of S_n was to find the right combinatorial notions. In the case of ordinary representation theory, the fundamental associated combinatorial objects are partitions (which naturally label the conjugacy classes of S_n!) and tableaux.

A *partition* $\lambda = (\lambda_1, \dots, \lambda_l)$ of a natural number n is a weakly decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_l > 0$ of integers with $\sum_{i=1}^{l} \lambda_i = n$, for short we write: $\lambda \vdash n$. The integer $l = l(\lambda)$ is the *length* of λ, the numbers λ_i are the *parts* of λ. We also write the partition exponentially as $\lambda = (l_1^{a_1}, \dots, l_m^{a_m})$, $l_1 > \cdots > l_m > 0$. Counting the partitions of a fixed number n gives the partition function

$$p(n) = |\{\lambda \mid \lambda \vdash n\}|;$$

this has been studied in depth since Euler in combinatorics as well as in number theory [1].

Not only are the conjugacy classes of S_n and their irreducible complex characters equinumerous, but more importantly Frobenius obtained in 1900 the following result:

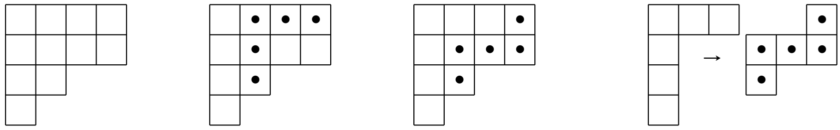
Classification of the complex irreducible S_n-characters. The irreducible complex characters of S_n are *naturally* labelled by partitions of n.

Originally, the character values of the character labelled by λ were determined via the expansion of the *Schur functions* s_λ in terms of the power sum functions. This link between the character theory of the symmetric groups and the theory of symmetric functions has been of great importance to both areas (see [21]). Fortunately, there is an easier way to compute the character values; we will describe the precise connection between the partition label and the actual character values by a recursion formula below. We denote the complex irreducible character labelled by the partition λ by $[\lambda]$, so $\mathrm{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$.

Example. The characters of the trivial representation and the sign representation of S_n correspond to the partitions (n) and (1^n), respectively, i.e. $\mathbb{1} = [n]$, $\mathrm{sgn} = [1^n]$. For $n \geq 2$, the character of the natural representation of S_n is the sum $[n] + [n-1, 1]$.

In particular for the purpose of computing the character values, it has been extremely fruitful to represent a partition graphically as follows. For $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, its *Young diagram* $Y(\lambda)$ has λ_i boxes in row i, for $i = 1, \dots, n$ (see the example below for an illustration of the notions defined here!). A particular rôle, e.g. for induction arguments, is played by the so-called *hooks* in λ. The (i,j)*-hook* $H_{i,j}$ in λ consists of the box at position (i,j) (using matrix notation) together with all boxes in $Y(\lambda)$ to the right and below. The *hooklength* $h_{i,j}$ counts the number of boxes in $H_{i,j}$. An *l-hook* of λ is a hook of length l in λ. The *leg length* $L(H_{i,j})$ is the number of boxes below the (i,j)-box in $Y(\lambda)$. Corresponding to an (i,j)-hook $H_{i,j}$, λ contains an (i,j)*-rim hook* $R_{i,j}$, which connects the end box of the ith row with the end box of the jth column along the rim of the diagram. Removal of $H_{i,j}$ from λ then means the removal of $R_{i,j}$ from λ; the resulting partition is denoted $\lambda \setminus H_{i,j}$. We illustrate all these notions now by an example.

Example. For $\lambda = (4^2, 2, 1)$, its Young diagram $Y(\lambda)$ is shown to the left, then the Young diagram with the (1,2)-hook $H_{1,2}$ indicated; here, $h_{1,2} = 5$ and $L(H_{1,2}) = 2$. Next, the corresponding rim hook $R_{1,2}$ is indicated, and the removal process to obtain $\lambda \setminus H_{1,2} = (3, 1^3)$.



While the computation of the character values via Schur functions is rather cumbersome, the following result provides an easy combinatorial recursion formula for computing the character values (see [14, 25]):

Murnaghan-Nakayama formula. Let $\lambda \vdash n$, $\sigma_\alpha \in S_n$ of cycle type $\alpha \vdash n$, e a part of α, and let $\alpha - e$ denote the partition where the part e has been removed from α. Let $\sigma_{\alpha-e} \in S_{n-e}$ be an element of cycle type $\alpha - e$. Then

$$[\lambda](\sigma_\alpha) = \sum_{H\, e-\text{hook in } \lambda} (-1)^{L(H)} [\lambda \setminus H](\sigma_{\alpha-e}).$$

An important special case is the restriction to the subgroup S_{n-1}; a 1-hook in λ is called a *removable box* in λ.

Branching theorem.

$$[\lambda]|_{S_{n-1}} = \sum_{A \text{ removable box in } \lambda} [\lambda \setminus A].$$

Example. For $\lambda = (4^2, 2, 1)$, the restriction of the corresponding character to S_{10} is

$$[4^2, 2, 1]|_{S_{10}} = [4, 3, 2, 1] + [4^2, 1^2] + [4^2, 2].$$

For studying the representations themselves rather than only their characters we have to introduce the notion of tableaux, which has seen many occurrences also in other contexts.

For a partition $\lambda \vdash n$, a λ-*tableau* t is a filling of the boxes of the Young diagram $Y(\lambda)$ with the numbers $1, \ldots, n$. A λ-tableau is *standard* if its entries increase along rows to the right and down the columns. It describes an inductive construction of λ, starting from the empty partition and adding the box with entry i at step i, where at each intermediate step we have the Young diagram of a partition. Phrased differently, a standard tableau corresponds to a path in the *Young graph* which is the infinite graph having all partitions as its vertices, and where two vertices are joined if the corresponding partitions $\lambda \vdash n$ and $\mu \vdash n+1$ differ only by adjoining a box to λ to obtain μ.

Example. Here are two $(4^2, 2, 1)$-tableaux of which only the second is standard

$$\begin{array}{cccc} 10 & 2 & 5 & 8 \\ 3 & 1 & 9 & 6 \\ 7 & 11 & & \\ 4 & & & \end{array} \qquad\qquad \begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 6 & 10 & 11 \\ 5 & 9 & & \\ 8 & & & \end{array}$$

As we have noticed before, the character value at 1 is the dimension of the corresponding representation. There are several ways of computing this dimension for an ordinary irreducible representation of S_n [14]:

Dimension formulae. Let $\lambda \vdash n$ be a partition. Then

(a) $[\lambda](1) = \frac{n!}{\prod \text{hooklengths in } \lambda}$ (*Hook formula*)

(b) $[\lambda](1) = f^\lambda := |\{t \mid t \text{ standard } \lambda\text{-tableau}\}|$.

Note that the equality

$$f^\lambda = \frac{n!}{\prod \text{hooklengths in } \lambda}$$

is a purely combinatorial statement; for a nice "probabilistic" proof of this due to Greene, Nijenhuis and Wilf see [25].

From the basic facts in ordinary representation theory we also deduce the following combinatorial identity

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2.$$

A "bijective proof" of this assertion (mapping permutations in S_n to pairs of standard tableaux of the same shape) is given by the Robinson-Schensted-Knuth algorithm (see [25]) which has many generalizations and variations.

Knowing the irreducible characters does not imply that one can easily write down the matrix representations to which they correspond; for S_n, such explicit matrix representations have been given by Young; in fact, he constructed the so-called seminormal, orthogonal and natural representations (see [14, 15, 25]). An explicit (but complicated) combinatorial description of the modules is given via the so-called *Specht modules* S^λ, which are defined over $\mathbb{Z}$ with the help of tableaux, and which are irreducible over $\mathbb{C}$. They are important also in the next section, when we discuss representations at positive characteristic.

An important problem for representations of finite groups is the computation of tensor products. Given two AG-modules V and W, their tensor product $V \otimes_A W$ is again an AG-module, with the group G acting diagonally. The matrices of the matrix representation corresponding to the tensor product are then the Kronecker products of the matrix representations corresponding to V and W. In general, it is very hard to compute such tensor products and only little information is known. For $A = K$ a field of characteristic 0, it suffices to compute the character of the tensor product of two representations, which is just the pointwise product $\chi_V \cdot \chi_W$ of the two corresponding characters, sometimes also called *Kronecker product*.

So in the case of S_n, given two irreducible characters $[\lambda]$ and $[\mu]$, one would like to know the coefficients $d^\nu_{\lambda,\mu} \in \mathbb{N}_0$ in the expansion

$$[\lambda] \cdot [\mu] = \sum_{\nu \vdash n} d^\nu_{\lambda,\mu} [\nu].$$

For the trivial character $[n]$ we have, of course, just

$$[\lambda] \cdot [n] = [\lambda].$$

Let us now consider the easiest non-trivial case: tensor products with the sign representation. Here we have for an arbitrary $\lambda \vdash n$

$$[\lambda] \cdot \mathrm{sgn} = [\lambda] \cdot [1^n] = [\lambda'],$$

where the *conjugate partition* λ' is obtained from λ by reflecting its Young diagram in the main diagonal.

Example. For $\lambda = (4^2, 2, 1)$, the conjugate partition is $\lambda' = (4, 3, 2^2)$.

The computation of general Kronecker products for S_n is one of the big open problems in the ordinary representation theory of the symmetric groups! There are many partial results, e.g. products for special partitions or information on particular constituents, but no satisfying combinatorial algorithm is known. Only recently, the slightly vague phrase "In general, Kronecker products are reducible." has been made precise

Theorem 1 [2]. *Let λ and μ be partitions of n. Then the Kronecker product of the corresponding irreducible characters is homogeneous, i.e.*

$$[\lambda] \cdot [\mu] = c[\nu]$$

for some partition ν of n and some $c \in \mathbb{N}$, if and only if one of the partitions λ, μ is (n) or (1^n) (and in this case the multiplicity c is 1).

So such Kronecker products are irreducible *only* in the two easy cases discussed above! In the case of the alternating groups, whose representation theory is closely related with the symmetric groups, this is no longer true: here there are tensor products of representations of dimension > 1 which are irreducible. The corresponding situations are classified; for this, Kronecker products of characters of S_n are studied which have very few different constituents [2].

3. Representations at characteristic p. We now turn to *p-modular representation theory*, i.e. to the situation where $A = F$ is a field of characteristic $p > 0$, p dividing the group order $|G|$, and F is again chosen to be "sufficiently large" for the group G (for $G = S_n$ the prime field $F = \mathbb{Z}_p$ is already large enough). If p is a prime not dividing the group order, the representation theory is similar to the one at characteristic 0. In many respects, p-modular representation theory is more complicated than ordinary representation theory (see below). One reason for studying modular representations is similar as in number theory: guided by a local-global principle one studies representation theory at different primes p to understand the global situation of *integral representations*, e.g. in the case of $G = S_n$ representations over the ring $\mathbb{Z}$ of integers.

Let F and G be as above; here are some of the

Basic facts in p-modular representation theory.

(a) (Maschke) The group algebra FG is not semisimple.

(b) The composition factors of an F-representation of G are determined by its *Brauer character* (which is a p-analogue of the "ordinary" character, but not just the trace of the F-matrices).

(c) The number $\ell(G)$ of irreducible F-representations of G (up to isomorphism) equals the number of *p-regular* conjugacy classes of G (which are the ones containing only elements of order not divisible by p).

Again, the main task is to determine all the irreducible FG-representations since they are the main building blocks of all FG-representations; because the irreducible representations may be "glued" together in different ways as composition factors our knowledge of all FG-representations is not as complete as in the case of ordinary representation theory.

Example. The 2-dimensional representation W of S_3 discussed before is even a representation over $\mathbb{Z}$. Let us look at the corresponding matrices with respect to the basis $\{w_1, w_2\}$ (say)

$$(12) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad (123) \mapsto \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and reduce the entries modulo p to obtain the representation $\overline{W} = F \otimes_{\mathbb{Z}} W$ in a characteristic p dividing the group order. For $p = 2$, it is easily checked that $\overline{W}$ is irreducible as a $\mathbb{Z}_2 S_3$-module. For $p = 3$, the sum $w_1 + w_2$ of the two basis vectors w_1 and w_2 is fixed, so $\overline{W}$ is reducible. But $\overline{W}$ has no other proper submodule, so it does not decompose into a direct sum of submodules; note also that at characteristic 3 also the natural module $\overline{V}$ does not decompose into the direct sum of $\overline{U}$ and $\overline{W}$, since in this case $\overline{U} = F(\sum_i b_i) \subset \overline{W} = \{\sum_i c_i b_i \mid \sum_i c_i = 0\}$. Modules with this property are called *indecomposable*. Unfortunately, for most finite groups G there are infinitely many indecomposable FG-modules and their classification is a so-called *wild* problem. Coming back to our module $\overline{W}$, its matrix representation over $\mathbb{Z}_3$ with respect to the basis $w_1 + w_2, w_2$ is

$$(12) \mapsto \begin{pmatrix} \mathbf{1} & 1 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad (123) \mapsto \begin{pmatrix} \mathbf{1} & -1 \\ 0 & \mathbf{1} \end{pmatrix}.$$

From the diagonal 1×1-blocks (marked in boldface) one immediately reads off the composition factors of the $\mathbb{Z}_3 S_3$-module $\overline{W}$: it has the trivial representation as a submodule (see above!) and the sign representation as a quotient module.

In the previous section, we have mentioned the Specht modules S^λ which are defined over $\mathbb{Z}$ and give the irreducible complex S_n-representations. Via reduction modulo p, the p-modular irreducible S_n-representations can also be obtained from the Specht modules. By the above, we know that the number $\ell(S_n)$ of such representations equals the number of p-regular conjugacy classes of S_n, so

$$\ell(S_n) = |\{\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n \mid p \nmid \lambda_i \text{ for all } i\}|.$$

By an old result of Glaisher, the set of partitions on the right hand side is equinumerous with the set of *p-regular partitions of n*, which are those partitions where no part is repeated p (or more) times. The p-analogue of our previous classification theorem is the following:

Classification of the p-modular irreducible S_n-representations. For a p-regular partition λ of n, the Specht module $S_F^\lambda = F \otimes_{\mathbb{Z}} S^\lambda$ has a unique irreducible quotient module, denoted by D^λ. The modules D^λ, where λ runs through the p-regular partitions of n, form a complete system of representatives for the (isomorphism classes of) irreducible FS_n-modules.

Unfortunately, this description of the p-modular irreducible representations is not very explicit, and our knowledge is far from being as detailed as at characteristic 0.

Example. Take again $n = 3$ and $p = 3$. We have only two 3-regular conjugacy classes, with representatives (1) and (12), so we only have two 3-modular irreducible representations. Since the partition (1^3) is not a 3-regular partition, it does not appear as a label of a 3-modular representation. The partition (n) labels the trivial representation at any characteristic. But the sign representation has different partition labels depending on the characteristic; observe that at characteristic 2 the sign representation equals the trivial representation! Our $\mathbb{Z}_3S_3$-module $\overline{W}$ is the Specht module $S^{(2,1)}$, and we have seen before that over $\mathbb{Z}_3$, $\overline{W}$ has the sign representation as a quotient. Hence we have at characteristic 3: $\mathrm{sgn} = D^{(2,1)}$.

In recent years, important progress on modular S_n-representations has been achieved in particular with Kleshchev's Branching theorems. For his modular branching results, Kleshchev has introduced the important new combinatorial concepts of good and normal boxes of a partition. The properties "good" and "normal" (or more precisely: p-good and p-normal) single out special removable boxes of a partition with respect to the prime p. These properties are purely combinatorial; for the somewhat involved definition see [17] or [5]. Corresponding to the Young graph mentioned in the preceding section, the *p-good Young graph* has all p-regular partitions as its vertices, and two vertices $\lambda \vdash n$ and $\mu \vdash n+1$ are joined by an edge if they differ only by adding a box to λ to obtain μ such that the box is p-good in μ. As noted by Lascoux, Leclerc and Thibon, the p-good Young graph coincides with the crystal graph occurring in the work of physicists on quantum affine algebras (see [20] and the references quoted there). We collect some of Kleshchev's results in the following theorem.

p-modular Branching theorem [17]. *Let λ be a p-regular partition of n, $n \in \mathbb{N}$, $n \geq 2$. Then the following holds:*

(i)

$$\operatorname{soc}\left(D^\lambda|_{S_{n-1}}\right) \simeq \bigoplus_{A \text{ good}} D^{\lambda \setminus A},$$

where soc M *denotes the* socle *of the module M, i.e. its largest completely reducible submodule.*

(ii) $D^\lambda|_{S_{n-1}}$ *is completely reducible if and only if all normal boxes in λ are good.*

Moreover, Kleshchev [19] also showed that only normal removable boxes A of λ give rise to composition factors corresponding to partitions of the form $\lambda \setminus A$, and he provided an explicit combinatorial description for the multiplicity of such composition factors $D^{\lambda\setminus A}$ in $D^\lambda|_{S_{n-1}}$.

As a consequence, the results by Kleshchev provide lower bounds for the dimension of the p-modular irreducible representations: the dimension of the representation D^λ is at least the number of p-good standard tableaux of λ, which are those tableaux corresponding to the adjoining of only good boxes at each step. The additional information on the multiplicities mentioned above improves this bound further. Unfortunately, an exact dimension formula comparable to the ones for ordinary representations is still not in sight; this is a central open question on irreducible p-modular S_n-representations.

From the description of the restriction of irreducible complex characters we immediately deduce that an ordinary irreducible S_n-representation restricts to an irreducible S_{n-1}-representation if and only if the Young diagram of its partition label has rectangular shape, since only in this case the partition has only one removable box. From Kleshchev's Branching theorem, we can deduce the corresponding answer in the modular case: the restriction $D^\lambda|_{S_{n-1}}$ is irreducible if and only if λ has exactly one normal node (which is then the only good node in λ). These partitions are called *JS-partitions*, since Jantzen and Seitz had conjectured the criterion for such irreducible restrictions in [16]. In fact, they described these partitions via a condition on their parts: a p-regular partition $\lambda = (l_1^{a_1}, \ldots, l_t^{a_t})$ is a JS-partition if and only if

$$l_i - l_{i+1} + a_i + a_{i+1} \equiv 0 \mod p \quad \text{for } 1 \le i < t.$$

So the p-analogues of rectangles are quite complicated! The JS-partitions have recently also appeared in different contexts, e.g. they play a special rôle in the study of certain exactly solvable models in statistical mechanics called the *RSOS-models* (for: restricted-solid-on-solid) [10], as well as in work on restrictions of representations from $\mathrm{GL}(n)$ to $\mathrm{GL}(n-1)$ [6].

In the previous section, we have discussed tensor products of complex irreducible S_n-representations; while there was no good answer for general such tensor products, at least tensoring with the sign representation was easy. at characteristic p, even computing the tensor product with the sign representation was a hard problem. In 1979, Mullineux [23] defined a rather complicated p-analogue of conjugation for p-regular partitions and conjectured that this gave the combinatorial answer to the question on the tensor product with the sign representation for p-modular irreducible S_n-representations; so for a p-regular partition λ the Mullineux map describes the p-regular partition λ^M defined by

$$D^\lambda \otimes \mathrm{sgn} \simeq D^{\lambda^M}.$$

The branching results have been applied successfully for the affirmative solution of the long-standing Mullineux Conjecture. Kleshchev had reduced this conjecture to a purely combinatorial conjecture which was subsequently proved by him and Ford in a long paper; a short proof of this combinatorial conjecture providing further insights was given in [5].

The Mullineux map has motivated the definition of residue symbols, which may be viewed as a p-analogue of the well-known Frobenius symbols for partitions. As a first application of the residue symbols it was shown that these behave well with respect to

p-branching and p-conjugation (i.e. the Mullineux map) simultaneously; they served as the main tool in the short proof of the combinatorial conjecture mentioned above. The residue symbols have also been applied in the investigation of the JS-partitions; in particular, their p-cores (which are special p-regular partitions associated to partitions) have been determined, and it turned out that these are partitions of rectangular shape [4].

The better understanding of the Mullineux map, in particular via residue symbols, has opened up the road to studying the modular irreducible representations of the alternating groups A_n and their branching behaviour [3].

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FINITE DIMENSIONAL ALGEBRAS AND SINGULARITY THEORY

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The purpose of this lecture is to give some impression of the type of work which has been done in the area of algebra called representation theory of finite dimensional algebras. Since around 1970 there have been various interesting developments, including establishing connections with other parts of algebra. We will not attempt to give a general survey, but instead concentrate on one particular topic, where there are connections with commutative ring theory/singularity theory. We just give the minimal background necessary, and rather than giving formal definitions, we illustrate concepts through examples. We give no specific references for the developments we discuss, but list relevant references, together with a guide to the literature, at the end.

1. Background. Let k be an algebraically closed field. A finite *quiver* Γ is a finite set of vertices and a finite set of arrows between vertices. For example,

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

$$\begin{tikzcd} & & \bullet \\ \bullet \arrow[r] & \bullet \arrow[ur] \arrow[dr] & \\ & & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet \end{tikzcd}$$

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \end{tikzcd}$$

are quivers.

Associated with a quiver Γ and the field k is the *path algebra* $k\Gamma$: When Γ is the quiver

$$\bullet_1 \xrightarrow{\alpha} \bullet_2 \xrightarrow{\beta} \bullet_3$$

then $k\Gamma$ has as k-basis the paths of Γ, that is $e_1, e_2, e_3, \alpha, \beta, \beta\alpha$, where the e_i are the trivial paths corresponding to the vertices. As for multiplication, the product of two paths is given by composition when possible, and is defined to be 0 otherwise. When Γ has no oriented cycles, the path algebra $k\Gamma$ is finite dimensional over k. Up to Morita equivalence (that is, equivalence of module categories) every finite dimensional k-algebra is isomorphic to a factor algebra $k\Gamma/I$, where Γ is a finite quiver and I is an ideal in $k\Gamma$.

We give some examples of finite dimensional k-algebras appearing in other forms. One class of examples is provided by group algebras over k, where G is a finite group G. If k has characteristic two and G has two elements, we have $kG \simeq k\Gamma/\langle \alpha^2 \rangle$, where Γ is the quiver

$$\begin{tikzcd} \bullet \arrow[loop right, "\alpha"] \end{tikzcd}$$

and $\langle \alpha^2 \rangle$ is the ideal in $k\Gamma$ generated by α. If G is the Klein 4-group, we have $kG \simeq k\Gamma/\langle \alpha^2, \beta^2, \beta\alpha - \alpha\beta \rangle$, where Γ is the quiver

$$\begin{tikzcd} \bullet \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd} .$$

Other examples are provided by various matrix algebras, like $\begin{pmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{pmatrix}$. Actually this algebra is isomorphic to the path algebra of the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet .$$

Appropriate factors of commutative rings, for example $k[X,Y]/(X,Y)^2$, give other examples of finite dimensional k-algebras.

The representation theory of a finite dimensional k-algebra Λ deals with the study of $\operatorname{mod}\Lambda$, the finitely generated (left) Λ-modules. One problem has been to decide when Λ is of finite representation type, that is, has only a finite number of indecomposable modules up to isomorphism. Recall that a module M is indecomposable if $M = N \oplus L$ (direct sum) implies $N = 0$ or $L = 0$. Another problem is to describe all indecomposable modules over an algebra of finite representation type, and for other classes of algebras. Various techniques have been developed to deal with these and other questions. We illustrate with the following examples. If $\Lambda = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ there are three indecomposable modules: $\begin{pmatrix} k \\ k \end{pmatrix}, \begin{pmatrix} 0 \\ k \end{pmatrix}, \begin{pmatrix} k \\ k \end{pmatrix} \Big/ \begin{pmatrix} 0 \\ k \end{pmatrix}$. When $\Lambda = kG$ where G is a finite group whose order is divisible by the characteristic of k, then kG is of finite representation type if and only if the p-Sylow subgroups of G are cyclic.

2. Preprojective algebras. Let Γ be a finite connected quiver with no oriented cycles, so that the path algebra $k\Gamma$ is finite dimensional over k. We associate with Γ a new quiver $\bar{\Gamma}$, where for each arrow α in Γ we add an arrow $\alpha^\star$ in the opposite direction. For example if Γ is the quiver

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet ,$$

then $\bar{\Gamma}$ is

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\alpha"] & \bullet \arrow[l, shift left, "\alpha^\star"] \arrow[r, shift left, "\beta"] & \bullet \arrow[l, shift left, "\beta^\star"] \end{tikzcd} .$$

Let I be the ideal of $k\bar{\Gamma}$ generated by $\alpha^\star\alpha, \alpha\alpha^\star + \beta^\star\beta, \beta\beta^\star$ (one element corresponding to each vertex). The factor algebra $k\bar{\Gamma}/\langle\alpha^\star\alpha, \alpha\alpha^\star + \beta^\star\beta, \beta\beta^\star\rangle = \Pi(\Gamma)$ is the *preprojective algebra* of Γ. (The principle is the same for any Γ; one just has to be careful about whether the coefficients of the paths are $+1$ or -1.)

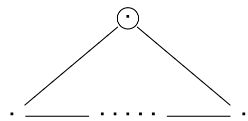
There is a class of indecomposable $k\Gamma$-modules called preprojective modules, which up to isomorphism are all indecomposable modules exactly when $k\Gamma$ is of finite representation type. We mention that the indecomposable summands of $k\Gamma$ are always preprojective. Also we have $k\Gamma \subset \Pi(\Gamma)$, and actually as a $k\Gamma$-module $\Pi(\Gamma)$ is the direct sum of the indecomposable preprojective $k\Gamma$-modules.

The preprojective algebras which are finite dimensional are of special interest. But some of the others lead to connections with commutative ring theory/singularity theory. In particular we are interested in those which are noetherian. The following is therefore of interest, for a finite connected quiver Γ.

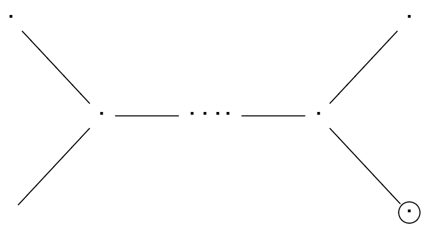
- $k\Gamma$ is finite dimensional $\Leftrightarrow$ The underlying graph $|\Gamma|$ is Dynkin $(A_n, D_n, E_6, E_7, E_8)$
- $k\Gamma$ is noetherian $\Leftrightarrow$ The underlying graph $|\Gamma|$ is extended Dynkin (and not finite dimensional) $(\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8)$

Recall that the extended Dynkin diagrams are

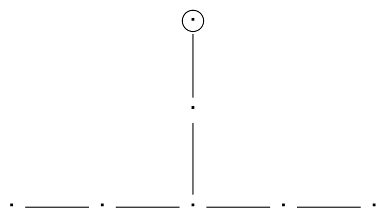
$\tilde{A}_n$



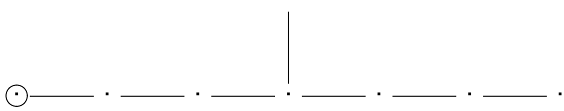
$\tilde{D}_n$



$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$

and we obtain the Dynkin diagrams by dropping the encircled vertices and the corresponding edges.

3. Invariant rings. Assume that the characteristic of k is zero, and let G be a finite subgroup of $SL(2,k)$. Then G acts naturally on the power series ring $S = k[[X,Y]]$. For example if $G = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$, the generator $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ sends X to $-X$ and Y to $-Y$. Hence X^2, Y^2, XY are left fixed under the action of G, and the invariant ring $R = S^G$, which consists of the elements of S which are left fixed under the action of G, turns out to be $k[[X^2,Y^2,XY]]$.

Let $\operatorname{Ref} R$ denote the finitely generated R-modules which are *reflexive*, that is, if $M^* = \operatorname{Hom}_R(M,R)$, the natural map $M \to M^{**}$ is an isomorphism. For example R is a reflexive R-module. Then one can investigate questions of finite representation type with respect to this class of modules. For some classes of commutative rings, including the invariant rings $R = S^G$, there are similar methods as for finite dimensional algebras to deal with such questions. In this language the above rings are all of finite representation type.

An important ring closely related with $R = S^G$ is the *skew group ring* SG. The elements, and the addition, are as for the ordinary group ring. For the multiplication we have $(sg)(s'g') = sg(s')gg'$ for s, s' in S and g, g' in G. Then we have the following.

- $\operatorname{Ref} SG$ is equivalent to $\operatorname{Ref} R$, and the indecomposable reflexive SG-modules are exactly the indecomposable summands of SG.

Let M be the direct sum of one copy of each indecomposable R-module in $\operatorname{Ref} R$, and consider $\Sigma = \operatorname{End}_R(M)^{\mathrm{op}}$. Then Σ and SG are known to be Morita equivalent. In particular $\operatorname{Ref}\Sigma$ and $\operatorname{Ref} SG$ are equivalent categories.

4. Associating quivers/graphs with invariant rings. Let as in section three $G \subset SL(2,k)$ and $R = k[[X,Y]]^G$. Then we have associated quivers or graphs from the following three different points of view.

(i) *Modules*: Define a quiver where the vertices correspond to the nonisomorphic indecomposable modules in $\operatorname{Ref} R$. The arrows correspond to the existence of a certain type of maps, called *irreducible* maps. If $G = \langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rangle$, and hence $R = k[[X^2,Y^2,XY]]$, there are two indecomposable modules in $\operatorname{Ref} R$: R and $R(X,Y)$. There are two "basic maps" from R to $R(X,Y)$, given by sending 1 to X or 1 to Y. From $R(X,Y)$ to R we get two "basic maps" by multiplication with X or with Y. The associated quiver, called Auslander-Reiten quiver or AR-quiver, will be

$$\begin{tikzcd} R\bullet \arrow[r, shift left=3] \arrow[r, shift left=1] & \bullet R(X,Y) \arrow[l, shift left=1] \arrow[l, shift left=3] \end{tikzcd} .$$

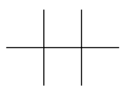
Whereas this quiver is more complicated for the other invariant rings, it is a general feature that the arrows occur in pairs

$$\bullet \rightleftarrows \bullet$$

(ii) *Group representations*: The inclusion $G \subset SL(2,k)$ determines a two-dimensional kG-module V in a natural way. Let $V_1 = k, \ldots, V_n$ be the simple kG-modules (that is, the irreducible representations of G over k). We define a quiver where the vertices correspond to the V_i. For each V_i consider $V \otimes_k V_i$, which is a kG-module. Write $V \otimes_k V_i = \oplus_{j=1}^{n} V_j^{r_j}$. Then there are r_j arrows from V_j to V_i. This quiver is called the McKay quiver. If $G = \langle g \rangle$ where $g = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we have two irreducible representations: k and k_-, where g acts trivially on k and $g \cdot a = -a$ for a in k. Then $V = k_- \oplus k_-$, and $V \otimes_k k = k_- \oplus k_-, V \oplus_k k_- = k \oplus k$. Hence we get the McKay quiver

$$k\bullet \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\end{smallmatrix} \bullet K$$

(iii) *Geometry*: Let $k = \mathbb{C}$. G acts on $\mathbb{C}^2$, where $\mathbb{C}$ denotes the complex numbers, so we have a quotient $\mathbb{C}^2/G$. This surface, which is a hypersurface, has a singular point at the origin. This singularity can be "resolved." In the resolution there is a finite number of curves above the singular point, some of which intersect. This gives rise to the *resolution graph*, where the vertices correspond to the curves, and there is an edge between two vertices exactly when the corresponding curves intersect. For example the intersection pattern



gives rise to the graph

$$\bullet \text{———} \bullet \text{———} \bullet\ .$$

When replacing

$$\bullet \rightleftarrows \bullet \qquad \text{by} \qquad \bullet \text{———} \bullet$$

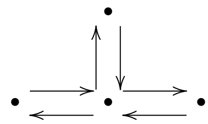
it turns out that the graphs occuring in (i) and (ii) are the extended Dynkin diagrams, and the same ones in the two cases. Also all extended Dynkin diagrams occur. If the vertex corresponding to R in case (i) and to k in case (ii) is dropped, we get the corresponding Dynkin diagram, which is the same as the diagram occurring in (iii).

Historically things were discovered in the opposite order. For (ii), it was first the question of an observation by McKay that there was the nice relationship with (iii). Then (i) and (ii) were proved to give the same quivers by Auslander. Now each statement can be proved directly, and the connections led to new relationships.

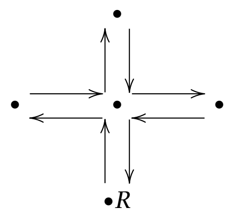
5. Connections. Note that it follows from the above discussion that the AR-quiver associated with $R = k[[X,Y]]^G$ is the quiver for the preprojective algebra of an extended Dynkin quiver. In view of our previous comments on equivalences of categories of reflexive modules, it is easy to see that same quiver is also the AR-quiver of the skew group ring $k[[X,Y]]G$ and of $\mathrm{End}_R(M)^{\mathrm{op}} = \Sigma$. And actually, if $\underline{r}$ denotes the radical of Σ and $gr\Sigma = \Sigma/\underline{r} + \underline{r}/\underline{r}^2 + \cdots$ the associated graded ring, we have the following:

- $\Pi(\Gamma) \simeq gr\Sigma$ (Note that $grk[[X,Y]]G = k[X,Y]G$). Σ can be constructed from $gr\Sigma$ via completion, so we can formulate the connection as follows:
- $\widehat{\Pi(\Gamma)} \simeq \Sigma$ (Here $\hat{}$ denotes completion with respect to the graded radical of $\Pi(\Gamma)$).

6. How to use the connections. Consider the preprojective algebras which are factor algebras of the path algebras of the quivers

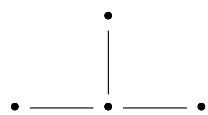


and

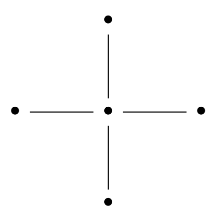


over k.

The first one has associated diagram



which is Dynkin, and is hence a finite dimensional algebra. The second one has associated diagram



which is extended Dynkin, and hence corresponds to some $R = k[[X,Y]]^G$. If we drop the vertex of R from the second quiver, we obtain the first quiver. Dropping this vertex has a nice module theoretic interpretation. Let M be a direct sum of one copy of each indecomposable reflexive R-module, let $\mathrm{End}_R(M)$ denote the endomorphism ring of M, and $\underline{\mathrm{End}_R}(M)$ the factor ring modulo the ideal $P(M,M)$ consisting of the R-homomorphisms $f\colon M \to M$ which factor through some finite direct sum of copies of R. Then we have the following, where Γ is a quiver, for example

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[r] & \bullet \arrow[u] \arrow[r] & \bullet
\end{tikzcd}$$

with underlying graph

$$\begin{tikzcd}
& \bullet & \\
\bullet \arrow[r, dash] & \bullet \arrow[u, dash] \arrow[r, dash] & \bullet
\end{tikzcd}$$

- $\underline{\mathrm{End}}_R(M)^{\mathrm{op}} \simeq \Pi(\Gamma)$.

(The same holds for the other Dynkin/extended Dynkin diagrams).

In addition to this connection being interesting in itself, an important point is that we can use information on modules over R to study $\Pi(\Gamma)$-modules. For example we have a functor $\Omega_R^1 : \underline{\mathrm{Ref}\, R} \to \underline{\mathrm{Ref}\, R}$ called the syzygy functor, and it is known that $\Omega_R^2 = \Omega_R^1 \Omega_R^1$ is isomorphic to the identity functor. Here $\underline{\mathrm{Ref}\, R}$ denotes the category whose objects are those of $\mathrm{Ref}\, R$ and where the morphism groups are the ordinary groups of R-homomorphisms, modulo the maps which factor through a finite direct sum of copies of R. This can be used to prove that $\Omega^6_{\Pi(\Gamma)}$ is isomorphic to the identity. This fact can again be used as a basis for further information on the module theory for the preprojective algebra $\Pi(\Gamma)$.

In recent years deformation theory and Hochschild cohomology have been investigated for preprojective algebras.

7. Guide to the literature. For the material in section 1, as well as for a more general introduction to the representation theory of finite dimensional algebras, we refer to the book [5]. Preprojective algebras are investigated in for example [11, 16, 8, 6, 4, 7, 17]. For material related to sections 3 and 4 we refer to [9, 12, 1, 2, 3, 18]. For section 5 we refer to [15] and for section 6 to [4, 10]. Also the articles [13] and [14] deal with material related to the topic of this lecture.

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SYMMETRIES

An interdisciplinary workshop organized by
INA KERSTEN AND SYLVIE PAYCHA

As the previous interdisciplinary session that took place in the Madrid E.W.M. meeting, this session on Symmetries was organized in a different way from the two other mathematical sessions of this meeting. Only two of the talks had been planned in advance,

- Symmetries of the Painlevé equations and connection with projective differential geometry by Ljudmila Bordag
- Symmetry and symmetry breaking in particle physics by Sheung Tsun Tsou.

The other talks (in chronological order)

- Symmetric attractors and symmetric fractals by Emilia Petrisor
- On infinitesimal symmetries of the self dual Yang-Mills equations by Tatiana Ivanova
- Some properties of Hamiltonian symmetries by Inna Yemelyanova
- q-dimensional formulas for the cyclic polyene Hubbard model by Ufuk Taneri
- Hamiltonians and Fock spaces associated to root systems by Valentina Golubeva

were arranged during the meeting and prepared together with all the speakers of the session. It was an exciting experience for the participants of the session to work out together the details of the talk one of us was about to give. We think they would agree to say we all learned a lot from this confrontation to other topics related to symmetry and other mathematical approaches to this concept. The spirit of the interdisciplinary session being that of bringing together spontaneous contributions and thus giving participants the opportunity to report on their work in a written form, the contributions to this session have not been refereed. Their contents are therefore left to the entire responsibility of the authors.

The concept of symmetry has been running through mathematics and physics more or less since these were born; the singular form *symmetry* is reductive as can been seen from the variety of titles, each of them giving a different way of approaching this concept, so that its plural form *symmetries* seems indeed more appropriate for this session. Symmetries arise here as a tool to solve equations (Painlevé equations), as properties of some dynamical systems (Hamiltonian symmetries, symmetric attractors). They can be local or non-local (as in Yang-Mills theory), they can be conserved or instead break (symmetry breaking in particle physics), they can be related to symmetries in the underlying algebraic structures (Fock spaces associated to root systems). This is only a small insight into the richness of the concept of symmetry and we hope the reader will get some of the pleasure we experienced listening to these lively talks.

INA KERSTEN and SYLVIE PAYCHA

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SYMMETRIES OF THE PAINLEVÉ EQUATIONS AND THE CONNECTION WITH PROJECTIVE DIFFERENTIAL GEOMETRY

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Abstract. The well known six Painlevé equations are characterized by unmovable critical points and they are useful for the investigation of the integrability of nonlinear partial differential equations.

As a new approach we investigate a series of invariants under general point transformations for the Painlevé equations. The general theory of invariants for equations of the type $y'' = f(x, y, y')$ under transformations $x = \phi(u, v)$, $y = \psi(u, v)$ had been developed in the works of R. Liouville, S. Lie and A. Tresse. Later, E. Cartan introduced the idea of projective connections and showed that the vanishing of invariants implies some geometric properties.

Applying these ideas we show that the geometrical images of the Painlevé equations have a very special structure and can be embedded as some set of surfaces in $\mathbb{R}\mathbf{P}^3$. We expect that this allows the derivation of new tests of the integrability of nonlinear partial differential equations as well as of nonlinear dynamical systems.

1. Introduction. In the middle of the 19th century one of the most important problems in analysis was the investigation and classification of ordinary differential equations. The methods of investigation were suggested by L. Fuchs and H. Poincaré and an essential part of them was the analysis of the singularities of the general solution. In the case of linear equations we can predict all singularities of the general solution by studying the coefficients of the equation. A quite different situation occurs if we are concerned with *nonlinear* differential equations. Neither we can describe the position of the singularities of the solutions nor the kind of the singularities from the coefficients of the equation because they may (and in fact they do so in many cases) depend on the initial data.

In order to make any progress it was necessary to restrict the investigation to such equations for which the singularities of their general solution depend only on the coefficients of the equation as it was in the linear case. Let us give some definitions in order to distinguish the classes of equations and the singularities of their solutions.

Definition. Points where the solutions have singularities are called *movable* (resp. *unmovable*) if their position depends (resp. does not depend) on the initial values of the solution.

Definition. The differential equation

$$\Omega\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0 \tag{1}$$

defined in a domain G is called *algebraic*, if $\Omega = \Omega(x, y_0, y_1, \dots, y_n)$ is a polynomial in y_i $(i = 0, 1, \dots, n)$ with coefficients which are meromorphic functions of x. The differential equation (1) is called *rational* if it is algebraic and of degree one with respect to y_n. The differential equation (1) is called *linear* if it is algebraic and if it is linear in y_i, $(i = 0, 1, \dots, n)$.

Examples

(1) The equation

$$k y' y^{k-1} = 1, \quad k \in \mathbb{N}$$

is a nonlinear rational equation of the first order. The general solution is

$$y(x) = (x - c)^{1/k}, \quad c \in \mathbb{C}$$

and it has an algebraic branch point for $x = c$. The position of this singular point depends on the integration constant c. Therefore it is a movable singularity.

(2) The equation

$$y'' + (y')^2 = 0$$

is a rational equation of the second order. Its general solution is

$$y(x) = \log(x - c_1) + c_2, \quad c_1, c_2 \in \mathbb{C}$$

and has a logarithmic branch point at c_1. The position of this singular point depends on the initial data, i.e. we have a movable singular point.

(3) The equation

$$y y'' + (y')^2 \left(\frac{2y}{y'} - 1\right) = 0$$

is a rational equation of the second order with the general solution

$$y(x) = c_1 \exp\left(-\frac{1}{(x - c_2)}\right), \quad c_1, c_2 \in \mathbb{C}.$$

This solution has an essential singular point at $x = c_2$, i.e. the equation has a movable singular point too.

Definition. A point $x \in \mathbb{C}$ is called a *critical* point if the general solution of a differential equation is not unique in any of its neighbourhoods.

A critical point may be isolated or not. For instance, it can be a branch (or ramification) point or belong to a line of branch points. Now we can reformulate our problem as follows:

Describe all possible nonlinear differential equations without movable critical points.

Despite of the restriction we have made the above problem remains very complicated and up to now it is not completely solved.

In order to simplify it further restrictions are introduced. We can consider for example only first order differential equations. The general solution of a first order differential equation can have as critical points only branch points. The general solution may have also other singularities, poles or essential singularities, but singularities of such type can not destruct the uniqueness of the solution and in the case of a first order equation all essential singularities are fixed (unmovable). Now we can classify all nonlinear differential equations of the first order without movable critical points. In this context classification means the possibility to describe all possible classes of equivalence under a given group of transformations of the variables x and y. L. Fuchs, H. Poincaré, and later P. Painlevé looked for equivalence classes under the following groups of transformations

$$\tilde{x} = \varphi(x), \qquad \tilde{y} = \frac{\alpha(x)y + \beta(x)}{\gamma(x)y + \delta(x)}, \tag{2}$$

where $\alpha, \beta, \gamma, \delta, \varphi$ are holomorphic functions. The problem of the classification of all algebraic ordinary differential equations of the first order was solved by L. Fuchs and H. Poincaré and the solution is very elegant.

Theorem 1 (Classification). *Each algebraic ordinary differential equation of the first order without movable critical points is equivalent to one of the following two equations:*

(1) *The equation for the Weierstrass $\wp$-function*

$$\left(\frac{dy}{dx}\right)^2 = 4y^3 - g_2 y - g_3, \quad g_2, g_3 \in \mathbb{C}. \tag{3}$$

(2) *The Riccati equation*

$$\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x), \tag{4}$$

where a, b, c are holomorphic functions.

Nearly all known special functions are defined by nonlinear differential equations of the first order without movable critical points. Well known examples are the functions called after Bessel, Hankel, Weber, Legendre, Struve and so on. It is well known that all these functions have many applications in mathematical physics which shows that the solutions of the equations with unmovable critical points proved to be very useful transcendental functions. This was the reason for the enthusiasm for the classification of the next, more complicated, class of equations—the equations of the second order. However E. Picard noticed that for such equations the essential singularities in general can be movable and concluded that such a classification never can be done. In fact, many attempts to solve this problem were unsuccessful. Later, P. Painlevé restricted the problem further and proposed to consider only *rational* differential equations of the type

$$\frac{d^2y}{dx^2} = \mathfrak{R}(x, y, y')$$

with fixed critical points. After a huge amount of calculations P. Painlevé found 50 (fifty) equations [34]. In fact he found 50 equivalence classes under the group of transformations (2). After that he investigated these equations and found that most of them can be solved explicitly or can be reduced to linear equations. But there still remained 3 equations that turned out to describe new transcendental functions. Painlevé's calculations were improved by his student B. O. Gambier [17] who added another 3 equations to this list. So there are 6 equations, called now *Painlevé equations* PI,…,PVI for which general solutions are principally new transcendental functions

$$y'' = 6y^2 + x, \tag{5}$$

$$y'' = 2y^3 + xy + \alpha, \tag{6}$$

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y}, \tag{7}$$

$$y'' = \frac{y'^2}{2y} + \frac{3y^3}{2} + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{8}$$

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{x} + \frac{\delta y(y+1)}{y-1}, \tag{9}$$

$$\begin{aligned} y'' = {} & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha - \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \left(\frac{1}{2} - \delta\right)\frac{x(x-1)}{(y-x)^2}\right). \end{aligned} \tag{10}$$

All of this equations have poles only as movable singularities, the critical points and essential singularities are fixed. The equations PI, PII, PIV have essential singularities in the point ∞, the equations PIII and PV have critical points $(0, \infty)$ and the last PVI equation has three critical points $(0, 1, \infty)$.

P. Painlevé believed that these equations can not have solutions in terms of known special functions. However it turned out that he was not right. Much later in [7] and subsequently in [2] there were found and classified rational and algebraic solutions of these equations for special choices of the parameters as well as further special solutions. For example, for particular values of the parameters PII possess Airy functions as special solutions, PIII Bessel functions, PIV Hermite-Weber functions, PV confluent hypergeometric functions and PVI hypergeometric functions. An exceptional case is the first equation PI (5) for which every solution is a new transcendental function.

In honour of the investigations of P. Painlevé an ordinary differential equations is today called to have the *Painlevé property* if it does not have movable critical points.

All attempts to generalize Painlevé's results despite of considerable efforts did not have success. In this connection we shall mention the results about the equations of the third order obtained by J. Chazy [13, 14], R. Garnier [22]. These investigations which use classical methods have been continued and summarized by F. J. Bureau [8].

Thus the problem was considered as too difficult and forgotten for a long time.

New life came into this problem from the development of the inverse scattering method. The method allowed to discover many nonlinear integrable partial differential equations with remarkable properties. In some sense these equations can be viewed as the first step from linearity to nonlinearity and they have many applications in modern physics, biology and other areas. These equations possess large classes of explicit solutions with truly nonlinear properties and are usually called soliton equations because of one of the classes of their solutions is characterized by its the particle-like properties. A essential feature of the soliton equations is that the proper scaling solutions (or similarity solutions) are described by some ordinary differential equation with fixed critical points, i.e. with the Painlevé property.

Examples

(1) The Korteweg-deVries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

where the subscript denotes the corresponding derivatives, has scaling solutions that are described by the first PI (5) or the second PII (6) Painlevé equations. For example, after the substitution

$$u(x,t) = \frac{w(z)}{(3t)^{2/3}}, \qquad z = \frac{x}{(3t)^{1/3}}$$

we get

$$w''' - 6w^2w' - (zw)' = 0,$$

or

$$w'' = 2w^3 + zw + \alpha.$$

(2) The nonlinear Schrödinger equation (NLS)

$$iu_t = u_{xx} \pm 2|u|^2u$$

leads to the second PII (6) and fourth PIV (8) Painlevé equations.

(3) The Boussinesq equation

$$u_{tt} = u_{xx} + (u^2)_{xx} \pm u_{xxxx}.$$

If we make the substitution

$$u(x,t) = w(x - ct) = w(z)$$

we get the equation

$$(1-c^2)w'' + \frac{1}{2}(w^2)'' + \frac{1}{4}w'''' = 0,$$

which after integration leads to one of the equations

$$w'' + 2w^2 + \alpha = 0, \qquad w'' + 2w^2 + z = 0,$$

that it leads to PI (5). It can be shown that other substitutions lead to PII (6) and PIV (8).

(4) The Sine-Gordon equation

$$u_{xt} = \sin u$$

after the substitution

$$u(x,t) = f(z), \quad z = xt, \quad \to w(z) = \exp(if)$$

leads to

$$ww'' = (w')^2 - \frac{ww'}{z} + \frac{w^3 - w}{z},$$

that is to the equation PIV.

From the above examples we see that the Painlevé equations can arise from integrable partial differential equations after dimensional reduction, that is the Painlevé equations arise taking the scaling symmetries into account.

We are going to look for symmetries for the Painlevé equations themselves. In order to do this, it would be useful to say some words about the notion of symmetry which is very broad and used in quite different meanings. For example, in physics we call a physical system symmetric if its observable quantities are the same seen from different observers which are connected by transformations of the space-time. In mathematics we investigate the invariance properties of a object under a given group of transformations and we say that the object have given symmetry if it is invariant under the action of this group of transformations.

Now we must specify what kind of symmetries we will investigate in connection with differential equations. Usually we look for some types of transformations that map solutions of the given equation into other solutions of the same equation (possibly with different parameters). There are different types of symmetries of differential equations, for example, master symmetries, isomonodromic symmetries, nonlocal symmetries etc., but their investigation is out of scope of this talk. We shall concentrate now on the Painlevé equations. They possess a rich family of symmetries. Some of them are well known, other are not. We intend to present some new results and for this we must explain the notations in more detail.

How can we use symmetries to solve differential equations? It turns out that we must look for invariant values of given equation. Let us define what this means. We call a function τ_n a *semi invariant* of weight n for the equation (1) if after some group transformation we have

$$\tilde{\tau}_n = \Delta^n \tau_n,$$

where Δ is the Jacobian of the transformation. We call the invariant τ for $n = 0$ an *absolute invariant.* It is evident that a vanishing semi invariant is an absolute invariant.

The expression for a semi invariant τ_n is constructed out of the coefficients of the equation and their derivatives. Differential invariants are not the only possible ones. There also may exist integral invariants for a given equation. Once a semi invariant is found it can be used as a new variable and this change may result in a simpler form of the equation written in terms of the new variables.

The theory of Lie point symmetries of partial and ordinary differential equations is a well investigated topic and most of the steps involved are algorithmic but unfortunately many important equations do not possess any Lie point symmetries. It can be shown that Painlevé equations do not allow nontrivial Lie point symmetries. On the other hand we have seen that the Painlevé equations have a lot of remarkable properties. Therefore we must look for some more general symmetry structures. So we consider the general smooth point transformations

$$\tilde{x} = \varphi(x,y), \qquad \tilde{y} = \psi(x,y).$$

We do not restrict ourself here to the case when they form a finite Lie group. For example, we can consider a transformation

$$\tilde{x} = x+1, \qquad \tilde{y} = y+x.$$

as it has the above form.

The set of all such transformations is called the set of smooth local diffeomorphisms of $\mathbb{R}^2$ and is denoted by $(\mathrm{diff})^\infty$. The general point transformations forms a pseudo group. The investigation of invariance properties with respect to the pseudo group $(\mathrm{diff})^\infty$ is incomparably more difficult than in the case of a Lie point symmetry. The reason for this is the fact that the elements of a pseudo group are not parametrised and there is no corresponding finite dimensional algebra of operators.

What kind of results can we expect from such investigations? In the case of Lie point symmetries we had the possibility to choose convenient variables using the differential invariants and simplify the equations. It turns out that in the case of pseudo groups there is a similar procedure. We look for invariants under the pseudo group and try to simplify the differential equation or to solve it. However, on this way there are some difficulties. First of all, the theory of differential invariants is more or less well developed only for ordinary differential equations of the first and second order, but not for arbitrary differential equation. Let us describe the history of this theory and introduce some new notations.

Let us consider the case of explicit differential equations of the second order

$$y'' = \Phi(x,y,y') \tag{11}$$

and look for differential expressions which are invariant under the pseudo group of point transformations

$$\tilde{x} = \varphi(x,y), \qquad \tilde{y} = \psi(x,y). \tag{12}$$

It is evident that the simplest second order differential equation has the form

$$y'' = 0.$$

After a general transformation (12) this equation takes the form

$$\frac{d^2\tilde{y}}{rd\tilde{x}^2} = a_1(\tilde{x},\tilde{y})\left(\frac{d\tilde{y}}{d\tilde{x}}\right)^3 + 3a_2(\tilde{x},\tilde{y})\left(\frac{d\tilde{y}}{d\tilde{x}}\right)^2 + 3a_3(\tilde{x},\tilde{y})\left(\frac{d\tilde{y}}{d\tilde{x}}\right) + a_4(\tilde{x},\tilde{y}). \tag{13}$$

Now, if we apply another point transformation of the kind (12) to the equation (13), we can easily see that its form will not change. This observation can be reformulated in the following way. The expression

$$\frac{\partial^4\Phi}{\partial y'^4} = 0$$

is the first (and simplest) differential invariant of ordinary differential equations of the second order. It means that the equations of the type (13) belong to the first and simplest class of ordinary differential equations of the second order. The next, more complicated class of equations has the form

$$(y'')^2 = P_5(y';x,y),$$

where P_5 is some polynomial of fifth order in the variable y'.

The six Painlevé equations belong to the simplest type (13). Now we have to consider the differential invariants for this type of equations, i.e. we have to construct some differential expressions out of the coefficients $a_1,\dots,a_4$ that are invariant under the pseudo group $(\mathrm{diff})^\infty$.

The first investigation of invariants of the equation (13) was done by R. Liouville [27]. He found some series of absolute and semi invariants and discovered a procedure to construct other invariants of higher weights if the initial semi invariant for this series does not vanish. A major role in his investigation are playing two quantities L_1, L_2. These quantities are also very important in our work. Let us introduce

$$\Pi^0_{11} = 2(a_3^2 - a_2a_4) + a_{3x} - a_{4y}, \tag{14}$$

$$\Pi^0_{22} = 2(a_2^2 - a_1a_3) + a_{1x} - a_{2y}, \tag{15}$$

$$\Pi^0_{12} = \Pi^0_{21} = a_2a_3 - a_1a_4 + a_{2x} - a_{3y}, \tag{16}$$

where by the subscripts x, y we denote the corresponding partial derivatives. Then L_1, L_2 are defined by

$$L_1 = -\frac{\partial \Pi^0_{11}}{\partial y} + \frac{\partial \Pi^0_{12}}{\partial x} - a_2\Pi^0_{11} - a_4\Pi^0_{22} + 2a_3\Pi^0_{12}, \tag{17}$$

$$L_2 = -\frac{\partial \Pi^0_{12}}{\partial y} + \frac{\partial \Pi^0_{22}}{\partial x} - a_1\Pi^0_{11} - a_3\Pi^0_{22} + 2a_2\Pi^0_{12}. \tag{18}$$

In the case when both L_1 and L_2 are equal to zero, the equation (13) is equivalent to $y'' = 0$ [27]. The value ν_5, the most important semi invariant discovered by R. Liouville, is

$$\nu_5 = L_2(L_1L_{2x} - L_2L_{1x}) + L_1(L_2L_{1y} - L_1L_{2y}) - a_1L_1^3 + 3a_2L_1^2L_2 - 3a_3L_1L_2^2 + a_4L_2^3. \tag{19}$$

If we have

$$\nu_5 = 0 \tag{20}$$

then ν_5 is an absolute invariant. It cannot be used for the construction of any other invariants of higher weights because it is equal to zero. R. Liouville discovered another initial semi invariant of weight 1 for the equations for which (20) holds, it is

$$w_1 = \frac{1}{L_1^4}\Big[-L_1^3\big(\Pi^0_{12}L_1 - \Pi^0_{11}L_2\big) - R_1(L_1^2)_x - L_1^2R_{1x} + L_1R_1(a_3L_1 - a_4L_2)\Big], \tag{21}$$

where

$$R_1 = L_1L_{2x} - L_2L_{1x} + a_2L_1^2 - 2a_3L_1L_2 + a_4L_2^2$$

and $L_1 \neq 0$. For the case $L_2 \neq 0$ we have a similar formula. The first non vanishing semi invariant in case $w_1 = 0$ found by R. Liouville has the weight 2 and is expressed as follows

$$i_2 = \frac{3R_1}{L_1} + \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial y}. \tag{22}$$

It can be used for the construction of the last series of invariants investigated by R. Liouville.

R. Liouville had some problems with the equations for which invariants v_5, w_1, i_2 vanish. He stated that all such equations are equivalent to $y'' = 0$ under the general point transformation (12). This incorrectness was remarked later by P. Painlevé and was a subject of some controversial discussion between P. Painlevé [33] and R. Liouville [26]. P. Painlevé had heavy doubts about the method of R. Liouville and as a result this voluminous and up to few pages correct work was neither cited nor used (to our knowledge).

A further consideration of the same problem from the point of view of groups of infinitesimal transformations was introduced by S. Lie [24, 25] and completed by his student A. Tresse. In his first work [39] A. Tresse looked for complete series of invariants for the equations of type (13). Later he investigated the most general second order differential equation (11) with arbitrary smooth function $\Phi(x, y, y')$ in [40]. A. Tresse got an award for this work because he found the complete set of invariants for the equation (11) under general point transformations (12). The work of A. Tresse was almost forgotten perhaps due to his inconvenient notations and we unsuccessfully tried to find an application of his work. Note that the results of A. Tresse are inapplicable to the case $v_5 = w_1 = 0$.

At the beginning of our century, G. Thomsen [38] found the series of invariants for the equation (13) under the point transformation (12) using methods of differential geometry. His invariants were quite the same as were found by R. Liouville. He excluded the case $v_5 = w_1 = 0$ from the consideration too.

Let us compute the invariants v_5, w_1 and i_2 for the six Painlevé equations. For all equations we get $v_5 = 0$ and $w_1 = 0$. Therefore the Painlevé equations must be a quite narrow subclass of the equations obeying (13). There must exist some transformation of the variables x and y after which the equation takes the form

$$y'' = f(x, y). \tag{23}$$

In fact, we found this transformation in paper [6] and are in a position that we can rewrite all Painlevé equations in the form (23).

Theorem 2. *The PI-PVI equations can be reduced to the canonical form (23).*

The first two equations, PI and PII already have this form. PIII equation takes the canonical form

$$y'' = \alpha e^{x+y} + \beta e^{x-y} + \gamma e^{2(x+y)} + \delta e^{2(x-y)}. \tag{24}$$

PIV equation is now

$$y'' = \frac{3}{4}y^5 + 2xy^3 + (x^2 - \alpha)y + \frac{\beta}{2y^3}. \tag{25}$$

PV equation takes the form

$$y'' = 4\alpha e^{y}\frac{1+e^{y}}{(1-e^{y})^3} + 4\beta e^{y}\frac{1-e^{y}}{(1+e^{y})^3} + \frac{\gamma}{4}\left(e^{x-y}-e^{x+y}\right) + \frac{\delta}{8}\left(e^{2(x-y)}-e^{2(x+y)}\right). \quad (26)$$

Finally the PVI equation takes the canonical form [5]

$$4\pi^2\frac{d^2y}{dx^2} = \beta\wp'(y \mid 1/2, ix/2) + \gamma\wp_1' + \alpha\wp_2' + \delta\wp_3'. \quad (27)$$

Here $\wp'(y \mid 1/2, ix/2)$ is the derivative of the Weierstrass $\wp$-function with the periods 1 and ix;$\wp_1'$, $\wp_2'$, $\wp_3'$ are its shifts on the half-periods $1/2$, $ix/2$ and $(1+ix)/2$ correspondingly. This completes the list.

This new form of the Painlevé equations is not only easy to recognize, but also very convenient for many investigations. This result yields an easy way to prove whether an equation belongs to the same equivalence class under $(\text{diff})^\infty$ or not. One has simply to check whether the invariants v_5 and w_1 for this equation vanish or not. After that we are concerned only with equations of the form (23).

Using the new form of the Painlevé equations we can better understand another type of symmetries—the symmetries resulting in the transformation of parameters. The first Painlevé equations don't have any parameter, but the other equations have up to four parameters. Now let us look for all possible transformations, point transformations, Lie-Bäcklund transformations, canonical transformations and so on, which transform a given solution with a prescribed set of parameters into some other solution of the same equation with possibly another set of parameters. The classification of such transformations for the PIII and PV equations as well as the classification for those sets of parameters for which these equations possess rational and algebraic solutions was done in [7]. Later K. Okamoto in [31] used the canonical and isomonodromic transformations together with the Hamiltonian structure of the Painlevé equations to describe the parameter symmetries of the Painlevé equations.

For some Painlevé equations such symmetries are quite evident. For example, we can see that a solution of the equation PII with some parameter α can be transformed by a point transformation into another solution with the parameter $-\alpha$ or by a canonical transformation into a solution with the parameter $1-\alpha$. The most complicated from the point of view of the parameter transformations is the sixth equation, PVI. Its investigation requires some unusual methods [31]. If we take now the sixth Painlevé equation in the new form, we can reproduce the result quite elementary. Let us formulate and prove the following theorem.

Theorem 3. *The group of parameter transformations of the equation PVI is isomorphic to $\mathcal{S}^4$.*

Proof. Consider a two dimensional lattice Λ with primitive periods $W_1, W_2 \in \mathbb{C}$, such that $\Im W_2/W_1 > 0$. We get 3 new lattices Λ_1, Λ_2, Λ_3 if we shift our lattice by half periods $W_1/2, W_2/2, (W_1+W_2)/2$. The right hand side of the sixth Painlevé equation is the sum of four $\wp'$-functions multiplied each with the corresponding parameter. Each of the $\wp'$-functions has a double pole respectively at the points $\{0, W_1/2, W_2/2, (W_1+W_2)/2\}$ with coefficients $\alpha, \beta, \gamma, \delta$. This means that we have some composition of 4 lattices with vertices marked by the parameters $\alpha,\beta,\gamma,\delta$. Now we prove that for an arbitrary permutation $s \in S^4$ of the parameters $\alpha,\beta,\gamma,\delta$ we get a solution of the equation PVI. In fact, the $\wp'$-function is double periodic and, consequently, we can start with an arbitrary weighted vertex. On the other hand we can introduce some new primitive periods

$$(\tilde{W}_1,\tilde{W}_2) = (W_1,W_2)\,A, \quad A \in SL(2,\mathbb{C})$$

due to the modular transformation A. We take into account the homogenity of $\wp'(y \mid W_1, W_2(x))$ with respect to the transformation of the variable x which is induced by A. The function $\wp'$ is invariant under such modular transformation A. So we can reproduce all permutations $s \in S^4$ using these two properties. □

2. The equivalence classes generated by the Painlevé equations. In this section we consider the cases where some different Painlevé equations can be transformed to one another by some general point transformation.

The canonical form (23) is very special. The only point transformations preserving it are linear in y. As a result, the equivalence problem for the equations in the canonical form become trivial.

Let us denote by $\{\mathrm{P}j(\alpha,\beta,\gamma,\delta)\}$ the set of equations equivalent to the j-th Painlevé equation with parameters $\alpha,\beta,\gamma,\delta$.

Consider the sets

$$\begin{aligned}
&\{\mathrm{PI}\},\\
&\{\mathrm{PII}\} = \bigcup_{\alpha}\{\mathrm{PII}(\alpha)\},\\
&\{\mathrm{PIII}\} = \bigcup_{\alpha,\beta,\gamma,\delta}\{\mathrm{PIII}(\alpha,\beta,\gamma,\delta)\},\\
&\{\mathrm{PIV}\} = \bigcup_{\alpha,\beta}\{\mathrm{PIV}(\alpha,\beta)\},\\
&\{\mathrm{PV}\} = \bigcup_{\alpha,\beta,\gamma,\delta}\{\mathrm{PV}(\alpha,\beta,\gamma,\delta)\},\\
&\{\mathrm{PVI}\} = \bigcup_{\alpha,\beta,\gamma,\delta}\{\mathrm{PVI}(\alpha,\beta,\gamma,\delta)\}.
\end{aligned}$$

From the canonical forms of the Painlevé equations the following theorems result

Theorem 4.

$$\{\mathrm{P}j\} \cap \{\mathrm{P}k\} = \phi, \quad k < j,\ k,j \in \{\mathrm{I},\mathrm{II},\mathrm{III},\mathrm{IV},\mathrm{V},\mathrm{VI}\},$$

except for the case $k = \mathrm{III}$, $j = \mathrm{V}$

$$\{\mathrm{PIII}\} \cap \{\mathrm{PV}\} = \bigcup_{\gamma,\delta}\{\mathrm{PIII}(-\gamma,\gamma,-\delta,\delta)\} = \bigcup_{\gamma,\delta}\{\mathrm{PV}(0,0,\gamma,\delta)\},$$

$$\{\mathrm{PIII}(-\gamma,\gamma,-\delta,\delta)\} = \{\mathrm{PV}(0,0,4\gamma,8\delta)\}.$$

Inside of the classes corresponding to each of the six Painlevé equations it is possible to give much more detailed description. For example for PII the following relations hold.

Theorem 5. *For the second Painlevé equation* (PII)

$$\{\mathrm{PII}(\alpha)\}\cap\{\mathrm{PII}(\tilde{\alpha})\}=\phi,\quad \text{if } \alpha^2\neq\tilde{\alpha}^2,$$
$$\{\mathrm{PII}(\alpha)\}=\{\mathrm{PII}(-\alpha)\}.$$

3. The geometry of the Painlevé equations. The adequate geometrical theory of the equations (13) was created by E. Cartan [11, 12]. He introduced the concept of the space of the projective connection (SPC). The equation (13) can be considered as the equation on the geodesics in this space. E. Cartan found some special classes of SPC so called spaces of the normal projective connection (SNPC) that are in one-to-one correspondence with the equivalence classes of (13) under the general point transformations (12).

We can imagine SNPC as follows. As a base we take a two-dimensional manifold $\mathscr{X}$ and in every point $X\in\mathscr{X}$ attach a typical fiber $\mathbb{R}\mathbf{P}^2$, i.e. we consider a product bundle with the structure group $\mathrm{PGL}(2,\mathbb{R})$: $\mathscr{X}\times\mathbb{R}\mathbf{P}^2\xrightarrow{p}\mathscr{X}$. In every fiber $\mathbb{R}\mathbf{P}^2$ we fix a frame $\mathbf{P}(X)=(P_0,P_1,P_2)^T(X)$. We take a point $P=\vec{z}\mathbf{P}(X)$ with coordinates $\vec{z}$. We move along an infinitesimal path from the point X to the point X', for $X,X'\in\mathscr{X}$; then the image of the point P will have the coordinates $\vec{z}(\mathbf{I}-\omega)$ in the frame $\mathbf{P}(X')$, where ω is the matrix of 1-forms.

The normal projective connection corresponding to (13) can be assigned by the following matrix of 1-forms ω

$$\begin{aligned}
&\omega_0^0=0, &&\omega_0^1=dx, &&\omega_0^2=dy,\\
&\omega_1^0=\Pi_{11}^0dx+\Pi_{12}^0dy, &&\omega_1^1=-a_3dx-a_2dy, &&\omega_1^2=a_4dx+a_3dy,\\
&\omega_2^0=\Pi_{21}^0dx+\Pi_{22}^0dy, &&\omega_2^1=-a_2dx-a_1dy, &&\omega_2^2=-\omega_1^1
\end{aligned}\tag{28}$$

where $\Pi_{11}^0,\Pi_{22}^0,\Pi_{12}^0,\Pi_{21}^0$ are defined by (14–16).

E. Cartan [12] and S. S. Chern [16] proved that each n-dimensional SNPC $\mathscr{X}^n$ can be immersed into the projective space $\mathbb{R}\mathbf{P}^N$. S. S. Chern [16] found that $N=n(n+1)/2+[n/2]$, i.e. for $n=2$ we have $N=4$. If the invariant $\nu_5=0$ then it is possible to immerse $\mathscr{X}^2$ into $\mathbb{R}\mathbf{P}^3$ and the image of $\mathscr{X}^2$ is a developable surface. For instance, the surface corresponding to the equation $y''=0$ is the projective plane (see E. Cartan [12] and V. Prokofjev [37]). In all other cases the immersion is possible into $\mathbb{R}\mathbf{P}^4$ only.

Let us now look for an image of SNPC for the Painlevé equations. For all of these equations we have $\nu_5=0$, $w_1=0$, i.e. we can describe the same normal projective connection if we take a set of surfaces in $\mathbb{R}\mathbf{P}^3$. In $\mathbb{R}\mathbf{P}^3$ we fix a frame $\mathbf{P}(X)=(P_0,P_1,P_2,P_3,)^T(X)$ and choose different points $P_i(X)$ as follows. Let the point $P_0(X)$ lie on an arbitrary cone with a quadric directrix and with fixed vertex $P_2(X)=\mathrm{const}$, let $P_1(X)$ lie on a tangent plane to this cone at the point $P_0(X)$. We fix one of the rulings $\mathscr{F}$ on this cone and take into account a plane through the ruling $\mathscr{F}$ and the point $P_0(X)$. On this plane we take an arbitrary point $P_3(X)(P_3(X)\neq P_0(X),P_2(X),\ P_3(X)\notin\mathscr{F})$. If we now move along a curve $y=y(x,C)$ (corresponding to ω), then the point $P_0(X)$ for fixed x will move along a ruling of the cone and at the same time the point $P_3(X)$ will describe a flat curve, touching the cone at the point $P_2(X)$. For every fixed x we have its own flat curve, all of the curves are forming the surface $\mathcal{P}_3 \in \mathbb{R}\mathbf{P}^3$. Such a surface is a geometrical characteristic of the equation (23). In case of the PI equation the flat curve is an ellipse (or another cone section) and the surface $\mathcal{P}_3$ looks like an bowed together croissant. For the PII equation it is the flat cubic curve and so on. The much more complicated case is the PVI equation. The flat curve in this case looks like a deformed spring.

E. Cartan defined and investigated the holonomy group of SNPC corresponding to (13). He proved that there is only one special case when the holonomy group is nontrivial. It has a fixed point on $\mathbb{R}\mathbf{P}^2$ and it can be shown that in this case the equation is equivalent to $y'' = f(x,y)$ for some $f(x,y)$. In other cases the holonomy group is either the projective group $\mathrm{PGL}(2,\mathbb{R})$ or it is trivial (for equations equivalent to $y'' = 0$). We reduced all Painlevé equations to the form (23) and proved that they have a nontrivial holonomy group. Therefore we found new symmetry properties of the Painlevé equations.

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SYMMETRY AND SYMMETRY BREAKING IN PARTICLE PHYSICS

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ABSTRACT. Symmetry, in particular gauge symmetry, is a fundamental principle in theoretical physics. It is intimately connected to the geometry of fibre bundles. A refinement to the gauge principle, known as "spontaneous symmetry breaking," leads to one of the most successful theories in modern particle physics. In this short talk, I shall try to give a taste of this beautiful and exciting concept.

1. Introduction. The concept of symmetry is one of the very few on which mathematicians and physicists agree, namely that

$$\text{SYMMETRY} \equiv \text{GROUPS}.$$

Hence we shall use these terms interchangeably.

In particle physics, there are two main uses of groups:

(1) as transformation groups under which a theory is *invariant*;

(2) as group representations for classifying the many particles we see.

In a sense, the first is all important, just like the main characters of a play. The second is more like the supporting cast, without which the theory, although it can stand on its own, is much less interesting and also much less realistic.

The next question is: which groups does one use or need? Generally speaking, finite-dimensional compact semi-simple Lie groups. In this talk, in order to simplify the presentation but without losing the essentials, I shall consider almost exclusively only the following: for abelian groups $U(1)$, and for nonabelian groups the unitary groups $U(N)$ and $SU(N)$. At the end I shall mention an example where a discrete group figures.

2. The particles: a lightning view. Particles used to be called *elementary* particles, which made good sense when we knew only the electron, the proton and the neutron, and they were adequate for forming *all* the elements in the Periodic Table. Then Einstein proved the existence of the photon as a particle. Also Dirac postulated the existence of anti-particles, which was well borne out by later experiments. … All in all, there are now more than 150 of them listed, and the number keeps on increasing! It would be highly unsatisfactory if we had to put them all in one or more representations or 'multiplets' without a good theoretical guidance.

Fortunately, we do now have a theoretical basis, the gauge principle, which we shall study in the next section. In the light of the gauge principle, particles can be classified under three headings:

• Vector bosons: γ (the photon), W^+, W^-, Z^0.
• Leptons: $e, \nu_e; \mu, \nu_\mu; \tau, \nu_\tau$. (In words, the electron, the electron neutrino, etc.)
• Quarks: these are not observable themselves, but they form most of the other particles by combining two or three together. Each quark q is in the 3-dimensional or fundamental representation, and directly observable particles occur in the 1-dimensional or singlet representation as follows

$$qqq : \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \cdots$$
$$q\bar{q} : \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \cdots .$$

Note that only singlets can be observed as free particles, as will be explained later.

3. The gauge principle. We said at the beginning that the invariance of a theory under certain group transformations is the most important aspect of symmetry. Let us study it now in greater detail.

Recall classical electromagnetism. The skew rank 2 field tensor $F_{\mu\nu}$ $(\mu, \nu = 0,1,2,3)$ has as its components the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}.$$

These are directly measurable quantities and hence do not transform under any symmetries. However, one can and does introduce a vector potential A_μ, related to $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu,$$

so that there is a freedom in changing A_μ without affecting $F_{\mu\nu}$

$$A_\mu \mapsto A_\mu + ie\partial_\mu \Lambda,$$

where $\Lambda(x)$ is a scalar field, and e is a 'coupling' constant representing the strength of interaction. In classical theory, there is no need to consider the potential A_μ. However, in quantum theory, it was demonstrated that $F_{\mu\nu}$ is not enough to describe the physics and one *needs* A_μ. This is the famous Bohm-Aharonov experiment.

The 'gauge freedom' in A_μ is in fact linked to the arbitrary phase of the electron wave function

$$\psi \mapsto e^{ie\Lambda}\psi.$$

Hence the relevant group for the symmetry of electromagnetism is

$$G = U(1).$$

In 1954, Yang and Mills extended this gauge principle to a nonabelian group G

$$A_\mu \mapsto SA_\mu S^{-1} - \frac{i}{g}(\partial_\mu S)S^{-1},$$
$$\psi \mapsto S\psi,$$

Physics	*Mathematics*
Special Relativity	Flat Space-time
General Relativity	Riemannian Geometry
Quantum Mechanics	Hilbert Space
Electromagnetism and Yang-Mills Gauge Theory	Fibre Bundles

TABLE 1. Mathematics and physical theories.

where $S \in G$.

This is the famous Yang-Mills theory. In the last 20 years or so, it has been generally accepted that Yang-Mills theory is the basis of *all* of particle physics

YANG-MILLS THEORY = BASIS OF ALL PARTICLE PHYSICS.

A refinement of gauge symmetry is called *symmetry breaking*, where the whole theory (including equations of motion) is invariant under a group G but a particular solution (or 'vacuum') is invariant only under a subgroup $H \subset G$. This will be important for later applications.

4. The geometry of gauge theory. Although it was not realized at the time, gauge theory is intimately linked with geometry. In fact it is as geometric a theory as Einstein's general relativity. Table 1, borrowed from a paper by Yang, underlines this fact.

Recall the definition of a principal fibre bundle, as illustrated in the accompanying sketch (Figure 1).

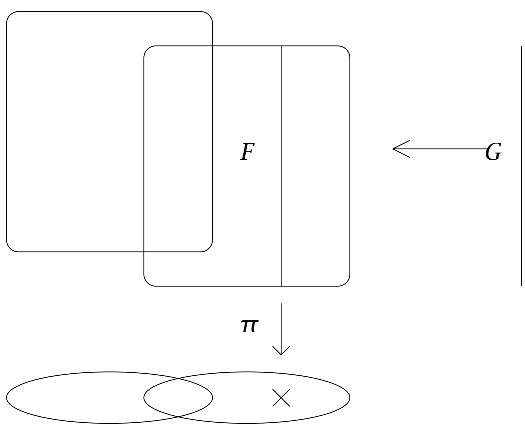

FIGURE 1. Sketch of a principal bundle.

Thus a principal fibre bundle consists of a manifold P (*total space*), a manifold X (*base space* or *spacetime*), a projection π and a group G (*structure* or *gauge group*).

Above any point $x \in X$ the inverse image $\pi^{-1}(x) \subset P$ is called the typical fibre F, and is homeomorphic to G. Above an open set U_α of X, the inverse image $\pi^{-1}(U_\alpha) \subset P$ is homeomorphic to the product $U_\alpha \times F$

$$\phi_\alpha \colon U_\alpha \times F \to \pi^{-1}(U_\alpha).$$

Thus in a sense, the manifold P is a 'twisted' product of G and X, the twisting being done by the action of the group

$$\phi_{\alpha,x} \colon F \to \pi^{-1}(x), \qquad y \mapsto \phi_\alpha(x,y),$$

with

$$\phi_{\beta,x}^{-1}\phi_{\alpha,x} \colon F \to F$$

giving the relevant action of the group G.

A trivial bundle is then just the product $X \times G$. The most well-known example of a nontrivial bundle is the Möbius band, where twisting is done by the 2-element group $\mathbf{Z}_2$. An example which is useful in physics is the *magnetic monopole*, which can be represented topologically by S^3, which in turn is a nontrivial S^1 bundle over S^2 (the Hopf bundle, of Chern class 1, for the experts). Here spacetime is thought of as $S^2 \times \mathbf{R}^2$, where the second factor is just a vector space with no topology, and can thus be ignored for the present purpose. Ordinary electromagnetism without magnetic monopoles is given topologically by the trivial bundle bundle $\mathbf{R}^4 \times S^1$. In both cases, the typical fibre is the circle S^1, which is homeomorphic to the group $U(1)$.

To proceed further we need to introduce a *connection* on the principal bundle P. This is a 1-form A on P with values in the Lie algebra $\mathbf{g}$ of G, satisfying certain properties and giving a prescription for differentiating vectors and tensors. Locally it combines with the usual partial derivative to give the *covariant derivative*

$$D_\mu = \partial_\mu - ig[A_\mu, \cdot\,].$$

In differential geometry and in gauge theory one has to replace the partial derivative by the covariant derivative so as to preserve the invariance or symmetry of the system.

From the connection one can define the *curvature*

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\mu, A_\nu].$$

One recognizes immediately that these are respectively the gauge potential and the gauge field introduced in the last section, where the extra commutators (in the Lie algebra) take into account that now the group is in general nonabelian.

With this language, the mechanism of symmetry breaking can be stated as the case when the twisting of the bundle are by elements of a subgroup H of G (and when the connection 1-form takes values in the corresponding Lie subalgebra, one says then that the bundle with connection is reducible to the subgroup H). An important example is the 't Hooft-Polyakov magnetic monopole, which is a nontrivial $U(1)$ reduction of a trivial $SU(2)$ bundle, given by the exact sequence (for those who are fond of such things)

$$\cdots \to \pi_2(SU(2)) \to \pi_2(SU(2)/U(1)) \to \pi_1(U(1)) \to \pi_1(SU(2)) \to \cdots.$$

FORCE	GROUP	GAUGE BOSONS	MATTER
Strong (QCD)	$SU(3)$	[Gluons]	[Quarks]
Electroweak (Weinberg-Salam)	$U(2)$	$\gamma, W^{\pm}, Z^0$	Leptons [Higgs]

TABLE 2. Forces and Fields in the Standard Model.

The first and last terms being zero, one gets the isomorphism

$$\pi_2(SU(2)/U(1)) \cong \pi_1(U(1)).$$

5. Briefest summary of the Standard Model. Following the gauge principle, we can now try to fit the three types of particles of Section 2 into a more systematic pattern, the better to exhibit their symmetry properties.

The vector bosons, also known as gauge bosons, are the potential $A_\mu(x)$ when considered as fields. Note that in the language of quantum field theory, the concept of "particles" and "fields" are interchangeable: particles interact by influencing the space-time in their neighbourhood and thus giving rise to fields, that is, functions of space-time with a definite tensor property (whether scalar, vector, rank 2 skew tensor, etc.). According to the interaction, we have a specific symmetry or gauge group. The other two types of particles are usually thought of as "matter fields" belonging to representations of the corresponding groups.

We now recognize that, other than gravitation, there are two fundamental forces of Nature: the strong and the electroweak. The electroweak theory is an example of a gauge theory with symmetry breaking. The idea, called the Weinberg-Salam model, is that at high energies when the Universe was much younger the symmetry was not broken, but as the Universe cooled down the $U(2)$ gauge group broke down to the $U(1)$ subgroup which is the electromagnetism of today. The rest of the $U(2)$ interaction manifests itself in the present-day weak interaction, of which radioactivity is the most commonly known aspect. The breaking also leaves some remnant fields called the Higgs fields which are yet to be discovered.

As mentioned already, each quark is in a 3-dimensional representation of $SU(3)$. Hence a quark has in fact three states, fancifully called *colour*. This "colour" is not directly observable, as only states in the singlet representation can exist free. We say that the $SU(3)$ symmetry is *exact* and *confined*.

Table 2 summarizes these ingredients of the so-called Standard Model of particle physics. The particles in square brackets are not (or have not been) directly observed, but they are part of the theory.

The standard model can in fact be schematically represented as

$$(\text{QCD} + \text{Weinberg-Salam}) \times 3$$

the gauge group being $SU(3) \times SU(2) \times U(1)/\mathbf{Z}_6$. Most physicists neglect the six-fold identification, but it is important for identifying the correct particle representations.

The multiplication by 3 above is necessary to model another aspect of the particle spectrum known as *generations*. Take the charged leptons as an example. There are

Quarks			Leptons	
$\begin{pmatrix} u \\ d \end{pmatrix}_L$	u_R	d_R	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	e_R
$\begin{pmatrix} c \\ s \end{pmatrix}_L$	c_R	s_R	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	μ_R
$\begin{pmatrix} t \\ b \end{pmatrix}_L$	t_R	b_R	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	τ_R

TABLE 3. Generations of Quarks and Leptons

3 of them: the electron e, the muon μ and the tauon τ. Except for their very different masses, they behave in extremely similar fashion. The same pattern is repeated for their neutral 'partners' the neutrinos ν_e, ν_μ, ν_τ. The quarks also come in three generations: the 'up' and 'down' as the lightest generation, the 'charm' and 'strange' as the next in mass, and the 'top' and 'bottom' as the heaviest. Table 3 arranges the 3 generations as 3 rows. The subscripts L and R refer to the left-handed and right-handed field components, a refinement we shall not have time to go into.

The role of the Higgs fields in the standard model is crucial. They break the $U(2)$ symmetry, give masses to the gauge bosons W, Z and also give masses to the quarks and charged leptons. Without them, all particles would be massless. Notice that the neutrinos are supposed to be massless, although some recent experiments in particle physics and astrophysics indicate that they may have extremely small masses.

Even with this briefest of summaries of the Standard Model we can already see how symmetry plays a crucial organizing role in our understanding of particle physics. And in this *gauge* symmetry is of prime importance.

6. Electric-magnetic duality: example of a discrete symmetry. It is well-known that electromagnetism has a discrete $\mathbf{Z}_2$ symmetry, that is, the equations are invariant under the change from 'electric' to 'magnetic' and vice versa. Let us look at this in a little more detail.

As described in Section 3, we can start with the potential A_μ and define the field tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu.$$

Further introduce the Hodge star operator, which in this case goes from 2-forms to 2-forms

$$^*F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}.$$

This operation interchanges electric fields and magnetic fields. We then have the identity

$$\partial_\mu {}^*F^{\mu\nu} = 0,$$

which always holds for $F_{\mu\nu}$ defined as above in terms of an A_μ. On the other hand, by Gauss' theorem, this 'divergen-free' condition is equivalent to the absence of magnetic monopoles, because ${}^*F_{\mu\nu}$ gives the magnetic flux out of such an object if present. This very significant link between a geometric statement and a physical statement can be schematically represented as

$$\underbrace{A_\mu \text{ exists}}_{\text{geometry}} \overset{\text{Poincaré}}{\iff} \partial_\mu {}^*F^{\mu\nu} = 0 \overset{\text{Gauss}}{\iff} \underbrace{\text{no magnetic monopoles}}_{\text{physics}}.$$

In the language of differential forms, the geometric statement is no other than

$$F \text{ exact} \overset{\text{locally}}{\iff} F \text{ closed}.$$

Now in the absence of electric charges (remember: only the main characters and no supporting cast!), we have

$$\partial_\mu F^{\mu\nu} = 0,$$

just as for the case of magnetic monopoles above, only this time we have $F^{\mu\nu}$ instead of ${}^*F^{\mu\nu}$. So we have the 'dual' of the scheme above

$$\underbrace{\tilde{A}_\mu \text{ exists}}_{\text{geometry}} \overset{\text{Poincaré}}{\iff} \partial_\mu F^{\mu\nu} = 0 \overset{\text{Gauss}}{\iff} \underbrace{\text{no electric charges}}_{\text{physics}}.$$

We see that the electric-magnetic discrete symmetry indeed holds.

It can further be shown that electromagnetism is dual symmetric in the above sense even in the presence of charges.

What is even more interesting—and this is what I am currently working on—is that Yang-Mills theory (or nonabelian gauge theory) is also dual symmetric, but the proof is not all that straightforward. One has to use techniques involving infinite-dimensional loop variables and the dual transform is no longer just the Hodge star but a loop space generalization of it. What is interesting, and intriguing, is that this *discrete* symmetry is clearly linked to the *continuous* gauge symmetry. One consequence is that the gauge symmetry is now doubled

$$G \times \tilde{G},$$

where as groups the two factors are identical, only the physical aspects they refer to are not identical but dual to each other. Now 't Hooft proved a theorem which can be stated as follows: the G symmetry is exact and confined if and only if the $\tilde{G}$ symmetry is broken and massive. Compare this to the actual symmetries of the Standard Model

$SU(3)$	exact and confined
$U(2)$	broken and massive

Applying 't Hooft's theorem to these symmetries lead to very interesting consequences which I do not have time to talk about.

7. Conclusions. Let me summarize the salient points about symmetry in particle physics that I have mentioned:

(1) Symmetry is all important in physics. For lack of time (and expertise) I have omitted to treat many symmetries, such as Lorentz symmetry, diffeomorphism symmetry, supersymmetry, ...

(2) There are two main uses of groups:
   (a) in the gauge principle as invariance, and
   (b) for particle classification using representations.

(3) The Standard Model is a triumph of the gauge principle.

(4) Electric-magnetic duality (a discrete $\mathbf{Z}_2$ symmetry), when generalized to Yang-Mills theory, leads to very interesting results.

If, however, you wish to take away with you just one point, then I recommend

$$\text{SYMMETRY} \equiv \text{GROUPS}.$$

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There are many excellent textbooks and semi-popular books on modern particle physics which emphasize its symmetry properties. There are also excellent articles in *Scientific American* which are most suitable to give a taste of the beauty of the subject. Below is a random selection of such, the first being a more general appreciation of symmetry in physics by the originator of Yang-Mills theory: [1, 2, 3, 4, 5, 6]

For the reader who might want to know more about the last part of this lecture, here are a few of my recent articles (the last with an amusing application from the Serret-Frenet formulae for space curves): [7, 8, 9, 10]

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SYMMETRIC ATTRACTORS AND SYMMETRIC FRACTALS

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1. Introduction. A *continuous time dynamical system* is a pair (M, Φ_t), where M is a smooth manifold as the state space, and Φ_t is the flow of a complete C^r-vector field F $(r \geq 1)$ defined on M, as the evolution law. $\Phi_t x_0$ represents the position at the moment t during the evolution of the state x_0. We refer here to the case $M = \mathbf{R}^n$. The orbits or trajectories in the system are the solutions of the system of differential equations

$$\dot{x} = F(x). \tag{1}$$

A *discrete dynamical system* is defined by a pair $(\mathbb{R}^n, f)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map, a homeomorphism or a diffeomorphism. f defines the law of evolution. The orbit of a state $x_0 \in \mathbb{R}^n$ is $\mathbb{O}(x_0) = \{f^m(x_0) \mid m \in \mathbf{N}\}$ if f is only continuous and $\mathbb{O}(x_0) = \{f^m(x_0) \mid m \in \mathbf{Z}\}$ if f is a homeomorphism or a diffeomorphism. Hence the evolution of a state x_0 is not watched continuously in time, but at regular intervals of time.

The concept of symmetry in a dynamical system is well known in physics. For the system of differential equations (1) symmetries are transformations T of the state space that leave the equations of motion invariant, i.e. along with $x(t)$, $Tx(t)$ is also a solution of the system. This happens iff $F \circ T = T \circ F$.

It is straightforward to show that if T is a symmetry of the dynamical system (1) and T is invertible, then T^{-1} is also a symmetry. So the symmetries of a dynamical system form a group, called group of symmetries.

Analogously, a transformation T of the state space of a discrete dynamical system is a symmetry if $T \circ f = f \circ T$. The study of symmetric discrete systems on $\mathbb{R}^n$ was motivated by symmetric patterns observed in experimental fluid dynamics [3].

When a system is symmetric, i.e. it has a nontrivial group of symmetries, one expects that the system has symmetric orbits, symmetric fixed points and periodic orbits, symmetric attractors or repellers. Also, symmetric steady states can generate symmetric patterns in the state space of the system.

Recent results in dynamical systems theory [3, 5] have shown the coexistence of chaos and symmetry. This coexistence seems to be a paradox because symmetry represents order and regularity, while chaos-disorder and impredictability. Next we show that this coexistence is possible and not contradictory.

2. Symmetric attractors in equivariant discrete dynamical systems. In order to understand the structure of symmetric attractors we fix the context in which they are generated and studied.

Consider the simplest case of discrete dynamical systems, namely the systems of the form $(\mathbb{R}^n, f)$, where f is a continuous map. Let Γ be a subgroup of the orthogonal group $O(n)$, acting linearly on $\mathbb{R}^n$, i.e. there is a continuous mapping (the action)

$$\Gamma \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(\gamma, x) \longmapsto \gamma x$$

such that the following conditions are satisfied

(i) For each $\gamma \in \Gamma$ the mapping $\rho_\gamma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by $\rho_\gamma(x) = \gamma x$ is linear;

(ii) $\gamma_1(\gamma_2 x) = (\gamma_1\gamma_2)x \ \forall \gamma_1, \gamma_2 \in \Gamma$ and $x \in \mathbb{R}^n$.

Definition. The system $(\mathbb{R}^n, f)$ is Γ-symmetric or Γ-equivariant if

$$f(\gamma x) = \gamma f(x) \quad \forall \gamma \in \Gamma, \ \forall x \in \mathbb{R}^n.$$

Some elements that characterize the dynamics of a discrete system $(\mathbb{R}^n, f)$ are: fixed points $(f(x_0) = x_0)$, q-periodic points $(f^q(x_0) = x_0, q > 1)$, ω–limit set of an orbit: $\omega(x) = \{y \in \mathbb{R}^n \mid \exists k_i \to \infty$ such that $f^{k_i}(y) \to x\}$, attractors.

There exist many definitions of an attractor in the literature concerning dynamical systems. Golubitsky and coworkers [7, 1] consider a fairly general one.

Definition. Let $(\mathbb{R}^n, f)$ be a dynamical system with f continuous. An f-invariant set Λ is called a stable set if for any open neighbourhood U of Λ there is a smaller open neighbourhood V of Λ such that $f^n(V) \subset U \ \forall n \in \mathbf{N}$.

Definition. An *attractor* of the dynamical system $(\mathbb{R}^n, f)$ is a stable ω-limit set (or in other words an attractor is a Lyapunov stable ω-limit set).

All above mentioned elements (in fact subsets in the state space) that characterize the dynamics of a system $(\mathbb{R}^n, f)$ are f-invariant subsets. For an f-invariant subset Λ of the state space of a Γ-equivariant system it is important to know the "amount" of symmetry exhibited by Λ. This "amount" is measured by the symmetry group:

$$\Sigma_\Lambda = \{\gamma \in \Gamma \mid \gamma\Lambda = \Lambda\}.$$

Most results on attractors in Γ-symmetric dynamical systems are known in the case of *planar systems* $(\mathbb{R}^2, f)$, with Γ a it finite group ([3, 1, 7, 2]). The only finite groups acting linearly on $\mathbb{R}^2$ are the *cyclic group* $\mathbf{Z}_m$ of order m, and the *dihedral group* $\mathbf{D}_m$ of order $2m$. We identify $\mathbf{Z}_m$ with the group of linear transformations generated by the planar rotation of angle $2\pi/m$. The dihedral group $\mathbf{D}_m$ is generated by $\mathbf{Z}_m$, togheter with an element of order two that does not commute with $\mathbf{Z}_m$. So we identify $\mathbf{D}_m$ with the group of linear transformations generated by planar rotation $\mathbb{R}_{2\pi/m}$ and an involution I, $I \circ I = \mathrm{id}$.

In other words we have defined here a representation of the group Γ $(\Gamma = \mathbf{Z}_m, \mathbf{D}_m)$ on $\mathbb{R}^2$, i.e. a map $\rho : \Gamma \longrightarrow \mathrm{O}(2, \mathbb{R})$, $\rho(\gamma) = \rho_\gamma \in \mathrm{O}(2, \mathbb{R})$.

In order to analyse the behaviour of planar $\mathbf{Z}_m$ or $\mathbf{D}_m$-symmetric systems we will work in complex coordinates. Hence consider dynamical systems of the form $(\mathbf{C}, f)$, where f is a polynomial function, $f : \mathbf{C} \longrightarrow \mathbf{C}$.

The general form of a $\mathbf{D}_m$-symmetric polynomial function is [3]

$$f(z) = p(z\overline{z}, \mathrm{Re}(z^m))z + q(z\overline{z}, \mathrm{Re}(z^m))\overline{z}^{m-1},$$

where p and q are real-valued polynomial functions uniquely determined by f.

For computer simulations of the dynamics of such systems one has used the truncation

$$f(z) = (\lambda + \alpha z\overline{z} + \beta \mathrm{Re}(z^m))z + \gamma\overline{z}^{m-1}, \quad \lambda, \alpha, \beta, \gamma \in \mathbb{R}.$$

A polynomial mapping g with $\mathbf{Z}_m$-symmetry is obtained adding the term $i\omega z$

$$g(z) = (\lambda + i\omega + \alpha z\overline{z} + \beta \mathrm{Re}(z^m))z + \gamma\overline{z}^{m-1}.$$

These symmetric dynamical systems have symmetric attractors. So it is natural to address the following questions:

(1) Can every subgroup $\Sigma \subset \mathbf{D}_m$ (or $\mathbf{Z}_m$) be the symmetry group of an attractor of a planar $\mathbf{D}_m$ (or $\mathbf{Z}_m$)-equivariant map?
(2) In a family f_λ of planar $\mathbf{D}_m$-equivariant maps how do symmetry subgroups of attractors A_λ change as the parameter λ increases?

The first question is one concerning the *admissibility of a subgroup* $\Sigma \subset \mathbf{D}_m$ as symmetry group of attractors for a polynomial dynamical system $(\mathbb{R}^2, f)$. (Recall that the subgroups of $\mathbf{D}_m$ are $\mathbf{D}_k$ and $\mathbf{Z}_k$, $k \geq 1$, and k divides m).

Definition. A subgroup $\Sigma \subset \mathbf{D}_m$ is admissible if there is a continuous $\mathbf{D}_m$-equivariant map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ having an attractor Λ with symmetry group $\Sigma_\Lambda = \Sigma$.

The answer to the first question is negative in the case of $\mathbf{D}_m$. There are restrictions on the symmetry groups as follows [7, 1]:

If f is a planar $\mathbf{D}_m$-equivariant map then are admissible the following groups: $1, \mathbf{D}_1$, $\mathbf{D}_m$, $\mathbf{Z}_k, k > 1$, k divides m, and $\mathbf{D}_2$ when m is even. The subgroups $\mathbf{D}_k, 2 < k < m$ (k divides m) are inadmissible.

But if f is a planar $\mathbf{Z}_m$-equivariant map, then all subgroups $\mathbf{Z}_k$, where k divides m, are admissible.

It was shown that group elements which act as reflections play a crucial role in determining admissibility.

In the case when f is a homeomorphism there are greater restrictions on admissibility [6].

The change of the symmetry group Σ_λ of an attractor A_λ of a planar $\mathbf{D}_m$-equivariant map f_λ as λ varies, observed in computer simulations [3] and proved in theoretical approaches is a *symmetry-increasing bifurcation*. Namely f_λ has a symmetry-increasing bifurcation at $\lambda = \lambda_0$ if:

- $\Sigma_\lambda = \Sigma$ for $\lambda < \lambda_0$;
- $\Sigma_\lambda = \Sigma'$ for $\lambda > \lambda_0$; and $\Sigma \subset \Sigma'$.

In order to explain (at least heuristically) this type of bifurcation we give some properties of attractors in Γ-equivariant maps.

If A is an attractor of $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e. there is an $x \in \mathbb{R}^n$ such that $A = \omega(x)$ and A is stable, then for every $\rho \in \Gamma$, $\rho(A)$ is also an attractor for f because of the equivariance of f. $\rho(A)$ is called a conjugate attractor.

The basic result explaining the symmetry-increasing bifurcation is:

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous mapping commuting with the linear transformation ρ of $\mathbb{R}^n$. If A is an attractor of f and*

$$A \cap \rho(A) \neq \emptyset$$

then $\rho(A) = A$.

Proof. In order to show the equality $\rho(A) = A$ we have to prove the inclusions $\rho(A) \subset A$, $\rho(A) \supset A$. We show only the first one, the second being proved using similar arguments.

Since A is a stable set, for every open neighbourhood U of A there is an open neighbourhood V of A such that $f^k(V) \subset U$ for all $k \in \mathbf{N}$. Moreover by equivariance property of f, $\rho(A)$ is also an attractor for f, i.e. there exists an $x \in \mathbb{R}^n$ such that $\rho(A) = \omega(x)$. Take $y \in A \cap \rho(A)$. Then $y \in V$. V being an open set, there is a $j \in \mathbf{N}$ such that $f^j(x)$ is close to y. Hence $f^j(x) \in V$. Therefore $\rho(A) = \omega(x) = \omega(f^j(x)) \subset U$. It is well known that an ω–limit set is a closed subset in the space of the states of the system. So $\rho(A)$ is also closed, and as a result $\rho(A) \subset A$. □

Now we are able to explain the scenario of symmetry-increasing bifurcation. As the parameter λ increases the conjugate attractors A_λ, $\rho(A_\lambda)$, $\rho \in \Gamma \setminus \Sigma_\lambda$, collide and merge at λ_0 into a single attractor with symmetry group Σ including the group generated by Σ_λ and ρ.

In fig. 1 and 2 is shown the symmetry increasing bifurcation in a $\mathbf{D}_3$-equivariant family corresponding to parameters $\alpha = -1, \beta = 0.1, \gamma = -0.8$. fig. 1 represents for $\lambda = 1.5$ three conjugated attractors having $\mathbf{D_1}$ symmetry. As λ increases the three attractors collide and give rise to a single attractor having $\mathbf{D_3}$–symmetry. In fig. 2, the attractor coresponds to $\lambda = 1.55$.

Numerical simulations and theoretical approach of the dynamics of planar $\mathbf{D}_m$-equivariant maps lead to the conclusion of coexistence of chaos and symmetry. Here we call the system $(\mathbb{R}^n, f)$ chaotic if it exhibits some kind of sensitive dependence on initial conditions, i.e. in every neighbourhood of any state x_0 there exists a state y whose orbit diverges in time from that of x_0.

The attractors of Γ-symmetric discrete dynamical systems seem to be chaotic aperiodic attractors.

Indeed if an orbit (not an ω–limit set) of such a system had a symmetry group Σ, then that orbit would be periodic, because the symmetric property of the orbit means that there exists an $m \in \mathbf{N}^\star$ and $\gamma \in \Sigma$ such that $f^m(x) = \gamma x$. Then by the equivariance property we have that $f^{km}(x) = \gamma^k x, \forall k \in \mathbf{N}^\star$. Since the symmetry group is finite there is a k_0 such that $\gamma^{k_0} = \mathrm{id}$, and so x is a periodic point of period $k_0 n$.

Moreover a $\mathbf{D}_m$–symmetric attractor of a $\mathbf{D}_m$-equivariant planar mapping, $m \geq 2$, exhibits certain kind of sensitive dependence on initial conditions [7]. Therefore dynamics on an attractor having full symmetry is chaotic. The symmetry imposes an order in chaos (see figs. 3 and4).

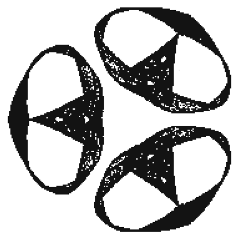

FIGURE 1. Conjugate attractors of a $\mathbf{D}_3$-symmetric map.

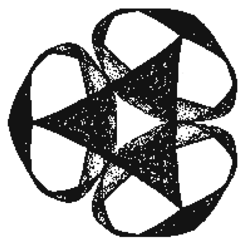

FIGURE 2. $\mathbf{D}_3$-symmetric attractor generated by collision of conjugated attractors.

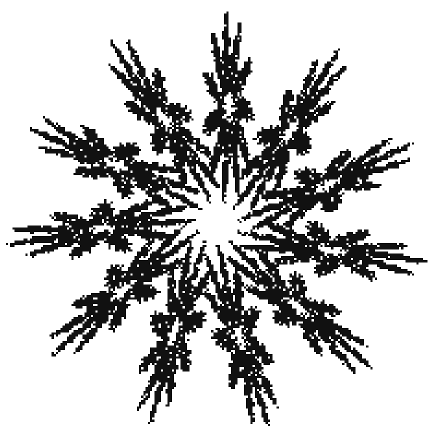

FIGURE 3. $\mathbf{Z}_{11}$-symmetric fractal.

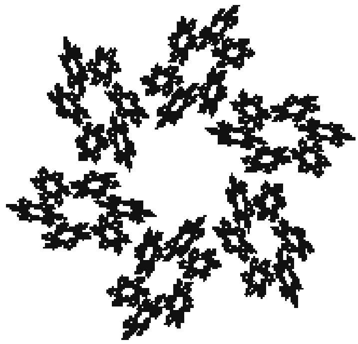

FIGURE 4. $\mathbf{Z}_6$-symmetric fractal defined by an IFS satisfying open set condition.

3. Symmetric fractals. The above results on chaotic symmetric attractors suggested the idea of generating Γ-symmetric fractal sets in $\mathbb{R}^2$, where Γ is the dihedral group $\mathbf{D}_m$ or the cyclic group $\mathbf{Z}_m$ [8].

Given an affine contraction $C : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and a finite group Γ acting linearly in $\mathbb{R}^2$ ($\Gamma = \mathbf{D}_m$ or $\mathbf{Z}_m$), $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_N\}$ consider the IFS (Iterated Function System) $\{C_j = \gamma_j \circ C\}_{j=1,\dots,N}$ i.e. a finite family of affine contractions.

Associate to this family a mapping $\mathscr{C}$ defined on the class $\mathscr{K}(\mathbb{R}^2)$ of nonempty compact subsets in $\mathbb{R}^2$ endowed with Hausdorff metric ρ [4].

$$\mathscr{C}(K) = \cup_{i=1}^{N} C_j(K), \tag{2}$$

$\mathscr{C}$ is a contraction on the complete metric space $(\mathscr{K}(\mathbb{R}^2), \rho)$ and its fixed point F is Γ-invariant.

Indeed, $F = \cup_{j=1}^{N} \gamma_j C(F)$, and $\gamma F = \cup_{j=1}^{N} \gamma\gamma_j C(F) = \cup_{i=1}^{N} \gamma_i C(F) = F$, where $\gamma_i = \gamma\gamma_j$ (when j runs over $\{1, 2, \dots, N\}$, i also runs over the same set).

Question: Every pair (Γ, C) defines a fractal set F, i.e. F has Hausdorff dimension [4] less than two? The answer is given by:

PROPOSITION 2. *The fixed point set F associated to the pair (Γ, C) is a fractal set if the order of the group Γ, denoted $|\Gamma|$, satisfies: $|\Gamma| < \alpha_1^{-1}\alpha_2^{-1}$, where $\alpha_1 \geq \alpha_2$, are the singular values of the linear part T of the affine contraction $C = T + a$.*

Moreover, the Hausdorff dimension of the set F, $\dim_H(F)$, also depends on the order of the group Γ and singular values α_1, α_2:

(1) If $|\Gamma| \leq \alpha_2^{-1}$ then

$$\frac{\ln|\Gamma|}{\ln \alpha_2^{-1}} \leq \dim_H(F) < \min\left(2, \frac{\ln|\Gamma|}{\ln \alpha_1^{-1}}\right)$$

and if $|\Gamma| \leq \alpha_1^{-1}$ then $\dim_H(F) \leq 1$.

(2) If $\alpha_2^{-1} < |\Gamma| < \alpha_1^{-1}\alpha_2^{-1}$, then

$$\frac{\ln|\Gamma|}{\ln\alpha_2^{-1}} \le \dim_H(F) < \frac{\ln\left(|\Gamma|\alpha_1\alpha_2^{-1}\right)}{\ln\alpha_2^{-1}}.$$

An IFS $(C_j)_{j=0,1,\dots,N}$ is said to satisfy the open set condition if there is a bounded open set $G \subset \mathbb{R}^2$ such that $\mathscr{C}(G) = \cup_{j=0}^{N} C_j(G) \subset G$, with this union disjoint.

If an IFS (C_j) satisfies the open set condition then the components of the associated invariant set F, that is the subsets $F_j = C_j(F) \subset F$, $j = 1,2,\dots,N$, are disjoint subsets.

If a Γ-invariant subset F associatted to the IFS defined by a pair (Γ, C) has disjoint components, then it may be defined as the repeller of a piecewise affine Γ-invariant dynamical system $(\mathbb{R}^2, f)$, that is any point near the fractal set F evolves away from F under the action of f.

As a conclusion, dynamics of $\mathbf{D}_m$—or $\mathbf{Z}_m$—equivariant planar maps, as the simplest symmetric discrete systems, shows on the one hand the existence of an order in chaos, and on the other hand provides an explanation for patterned turbulence in hydrodynamics [3].

Symmetric fractals in turn appear to have a structural order more subtle than traditional symmetric patterns because of their geometry.

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TWISTOR CORRESPONDENCE AND SYMMETRIES OF THE SELF-DUAL YANG-MILLS EQUATIONS

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1. Introduction. Yang-Mills theory is a non-Abelian generalization of the Maxwell theory of electromagnetism. The electromagnetic interactions are described by the gauge fields of the Abelian group $U(1)$. In the pioneer paper of Yang and Mills [26] it was suggested to consider gauge fields of a non-Abelian group G. Now it is well-known that in particle physics the strong, weak and electromagnetic interactions are described by gauge fields of the group $SU(3)\times SU(2)\times U(1)/Z_6$. So, the Yang-Mills theory forms a basis of the Standard Model describing these interactions and accumulating our modern knowledge in particle physics. The dynamics of the non-Abelian gauge fields is described by the Yang-Mills (YM) equations, and the study of the space of solutions to the YM equations is of particular interest. Later, in 1975, the equations giving a very important subclass of solutions to the YM equations were introduced [3]. These equations were called the self-dual Yang-Mills (SDYM) equations; their solutions provide absolute minima for the Yang-Mills functional in Euclidean 4-space.

The correct mathematical language to deal with various aspects of the classical gauge field theories is the language of fibre bundles. By the 1950's the theory of fibre bundles, based on ideas of Cartan and Weyl who introduced connections and curvature in the early part of this century, was a well-established part of differential geometry (see, e.g., [15]). In the 1970's this area of mathematics has again received close attention by both mathematicians and physicists in the form of the Yang-Mills theory (see [17, 9, 25] and references therein). In a geometric language, the gauge potentials A_μ are components of the connection 1-form in a principal fibre bundle, the gauge fields $F_{\mu\nu}$ are components of the curvature 2-form, etc. There exists a large literature on the geometric meaning of the SDYM equations (see, e.g., [1, 17, 25, 18]).

Our aim is to investigate infinitesimal symmetries of the SDYM equations. Under a symmetry we understand a transformation which maps solutions of the SDYM equations into solutions of these equations. In other words, symmetry transformations preserve the solution space. It is known that all local symmetries of the SDYM equations, which are also called manifest symmetries, are given by gauge transformations and conformal transformations. Since 1979, in a number of papers [22], it was shown that the SDYM equations have nonlocal, so-called 'hidden' symmetries which are related to global gauge transformations. More general gauge-type symmetries were described in [23, 8, 14]. In [21], an affine extension of conformal symmetries was introduced. The twistor interpretation of this algebra was discussed in [14]. But the problem of describing all possible (local and nonlocal) symmetries is not yet solved.

The paper is organized as follows: in Sections 2 and 3 we recall the main definitions; in Section 4 we describe the Penrose-Ward twistor correspondence [24, 2, 19], which helps us to reduce the problem of investigating nonlocal symmetries of the SDYM equations to the problem of describing local symmetries of holomorphic bundles over a twistor space; and, finally, in Sections 5 and 6 we give the cohomological description of the above-mentioned symmetries.

2. Definitions and notation

2.1. Principal fibre bundle. We assume that the notion of an n-dimensional *differentiable manifold* is known [15]. Briefly, it is a topological space M with an open *cover* $\{U_\alpha\}, \alpha \in I$, and *local coordinates* $x_\alpha : U_\alpha \to \mathbb{R}^n$ such that $M = \cup_{\alpha \in I} U_\alpha$, and *transition functions* $x_\alpha \circ x_\beta^{-1}$ are smooth.

Let P be a manifold and G a Lie group. A *differentiable principal fibre bundle* over M with the *structure group* G consists of a manifold P and an action of G on P satisfying the following conditions:

(i) G acts freely on P on the right: $(p,a) \mapsto pa$ and $pa = p \Leftrightarrow a = e$, where $(p,a) \in P \times G$, $pa \in P$, $a \in G$ and e is the identity in G;

(ii) M is a quotient space of P by an equivalence relation ($p \sim pa$) induced by G: $M = P/G$, and the *canonical projection* $\pi : P \to M$ is differentiable;

(iii) P is *locally trivial*, that is, every point $x \in M$ has a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic with $U \times G$.

A principal fibre bundle, or G-bundle over M, is denoted by $P(M,G)$. P is called the *bundle space*, M is the *base space*, G the *structure group* and π the *projection*. For each $x \in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called the *fibre* over x. Every fibre is diffeomorphic to G.

2.2. Associated fibre bundle. Let N be a manifold on which G acts on the left: $(a,\xi) \mapsto a\xi$, $(a,\xi) \in G \times N$, $a\xi \in N$. On $P \times N$ we define the action of $a \in G$ by $(p,\xi) \mapsto (pa, a^{-1}\xi)$. This action defines an equivalence relation between points of $P \times N$: $(p,\xi) \sim (pa, a^{-1}\xi)$. Let us introduce a projection π_E: π_E (equivalence class of (p,ξ))$= \pi(p)$. By definition a *fibre bundle associated to P with fibre N* is a space $E(M,G,N,P) \equiv P \times_G N = (P \times N)/G$ with the projection $\pi_E : E \to M$ and the following differentiable structure:

(i) for any open $U \subset M$, $\pi_E^{-1}(U)$ is an open submanifold of E,

(ii) for each $x \in M$, there exists an open neighbourhood U of x, $x \in U \subset M$, such that $\pi_E^{-1}(U) \simeq U \times N$ (a local triviality property).

2.3. Pull-back bundle. Let $\varphi : K \to M$ be a smooth map, and let $\pi : P \to M$ be a principal G-bundle. The *pull-back bundle* $\varphi^* P \to K$ is the principal G-bundle over K defined by

$$\begin{array}{ccc} \varphi^* P & \xrightarrow{\hat{\varphi}} & P \\ \downarrow & & \downarrow \\ K & \xrightarrow{\varphi} & M \end{array}$$

where $\hat{\varphi}$ is a natural map $\varphi^*P \to P$ which covers φ: $\hat{\varphi}(\kappa, a) = (\varphi(\kappa), a)$ for $\kappa \in K$, $a \in G$ and $\hat{\varphi}(pa) = \hat{\varphi}(p)a$.

2.4. Sections of a fibre bundle and sheaf of sections. A *local section* over U of the principal G-bundle P over M is a map $\sigma_U : U \to P$ such that $\pi(\sigma_U(x)) = x$, $\forall x \in U \subset M$. A *global section* of the principal fibre bundle $P(M,G)$ is a map σ from the base space M to the bundle space P, satisfying $\pi(\sigma(x)) = x$, $\forall x \in M$. A local section over U of the associated fibre bundle $E(M,G,N,P)$ is a map $s_U : U \to E$ such that $\pi_E(s_U(x)) = x$, $\forall x \in U \subset M$, and a global section is a map $s : M \to E$ such that $\pi_E \circ s$ is the identity map of the base space M.

Consider a vector bundle $E(M,G,V,P)$ with a vector space V as a typical fibre. Let s_U and s'_U be any sections over $U \subset M$ of the bundle E. These sections are called equivalent at the point $x \in U$, if there exists an open neighbourhood $W \subset U$ of the point x such that $s_U|_W = s'_U|_W$. The equivalence class of such sections is called a *germ* $\mathbf{s}_x$ of sections at the point x. Let us denote by $\mathcal{S}_x$ a set of germs at the point x of all sections of the vector bundle $E(M,G,V,P)$. Then, the topological space

$$\mathcal{S} = \bigcup_{x \in M} \mathcal{S}_x$$

with the canonical projection $\mathcal{S} \ni (x, s_x) \mapsto x \in M$ is called a *sheaf* of germs of sections of the vector bundle E.

2.5. Transition matrices. The local triviality of $P(M,G)$ (see Section 2.1(iii)) means that there exists a diffeomorphism $\psi : \pi^{-1}(U) \to U \times G, U \subset M$ such that $\psi(p) = (\pi(p), \phi(p))$, where ϕ is a map of $\pi^{-1}(U)$ into G satisfying $\phi(pa) = \phi(p)a$, $\forall p \in \pi^{-1}(U)$ and $a \in G$.

Let $\{U_\alpha\}, \alpha \in I$, be an open cover of M. For any given point $x \in M$, there always exists U_α such that $x \in U_\alpha$. Choose a point p in $\pi^{-1}(x)$ and define

$$\sigma_\alpha(x) = p\phi_\alpha^{-1}(p), \tag{1}$$

where $\phi_\alpha : \pi^{-1}(U_\alpha) \to G$, and $\phi_\alpha^{-1}(p) \in G$ is inverse to $\phi_\alpha(p) \in G$: $\phi_\alpha(p)\phi_\alpha^{-1}(p) = e$. Recall that for any point $p' \in \pi^{-1}(x)$ $\exists a \in G$ such that $p' = pa$. Then, using the property of $\phi_\alpha : \phi_\alpha(pa) = \phi_\alpha(p)a$, we have

$$p'\phi_\alpha^{-1}(p') = p\phi_\alpha^{-1}(p),$$

i.e. $\sigma_\alpha(x)$ is independent of the choice of the point p in the fibre $\pi^{-1}(x)$. Moreover, $\pi(\sigma_\alpha(x)) = x$ and $\psi_\alpha(\sigma_\alpha(x)) = (x, e)$, where e is the identity in G.

Suppose $x \in U_\alpha \cap U_\beta$. Then

$$\sigma_\beta(x) = p\phi_\beta^{-1}(p) = \sigma_\alpha(x)\phi_\alpha(p)\phi_\beta^{-1}(p).$$

Since the action of G on P is free ($pa = p \Leftrightarrow a = e$), and since σ_α and σ_β depend only on x, we can define

$$f_{\alpha\beta}(x) = \phi_\alpha(p)\phi_\beta^{-1}(p). \tag{2}$$

The maps $f_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are called *transition matrices*. For any $x \in U_\alpha \cap U_\beta \cap U_\gamma$, the cocycle condition

$$f_{\alpha\beta}(x) f_{\beta\gamma}(x) f_{\gamma\alpha}(x) = e \tag{3}$$

takes place. The transition matrices (2) are differentiable as G-valued functions of x.

In the pull-back bundle transition matrices are defined as maps $\varphi^* f_{\alpha\beta} : \varphi^{-1}(U_\alpha) \cap \varphi^{-1}(U_\beta) \to G$. Let V be a vector space on which G acts via a representation ρ. Then $\rho(f_{\alpha\beta})$ are transition matrices in the associated vector bundle $E(M, G, V, P)$.

2.6. Trivial principal fibre bundle. A principal fibre bundle $P(M, G)$ is *trivial* if, using the group action, we can construct another set of local sections and transition matrices such that all transition matrices $f_{\alpha\beta}$ are equal to the identity ($f_{\alpha\beta}(x) = e; \alpha, \beta \in I, \forall x \in M$). Then, $P \simeq M \times G$, and it admits a global section.

2.7. Gauge transformation. A *gauge transformation* on P is a bundle automorphism $\mathfrak{f} : P \to P$ satisfying the following conditions

(i) for $\forall p \in P$, $\exists\, g(p) \in G$ such that $\mathfrak{f}(p) = p\, g(p)$,

(ii) $g(pa) = a^{-1} g(p)\, a$, $\forall p \in P$, $\forall a \in G$.

It is easy to see that $\mathfrak{f}(pa) = \mathfrak{f}(p)a$. A set of such automorphisms of P is a group $\mathfrak{G}$ which can be given the structure of a smooth infinite-dimensional Lie group. This group is called the *group of gauge transformations*. Note that $\mathfrak{G}$ is the set of sections of the associated bundle of groups $P \times_{\mathrm{AdG}} G$. A Lie algebra $\mathfrak{g}$ of the Lie group $\mathfrak{G}$ is the space of sections of the associated vector bundle $P \times_{\mathrm{AdG}} \mathcal{G}$, where $\mathcal{G}$ is a Lie algebra for the structure group G.

2.8. The connection form. If $v \in \mathcal{G}$, then v defines a *fundamental vector field* $\mathcal{T}(v)$ on P as follows

$$(\mathcal{T}(v) f)(p) \equiv \frac{d}{dt} f(p \exp(tv))|_{t=0},$$

where $f : P \to \mathbb{R}$ is a function on P, $\exp : \mathcal{G} \to G$ is the exponential map. Note that $\pi_* \mathcal{T}_p = 0$, hence $\mathcal{T}_p \equiv \mathcal{T}(v)_p$ is a *vertical* vector. It is a vector tangent to the fibre through p, at p.

Let us consider a tangent space T_p for the manifold P at the point $p \in P$. It can be split in the following way

$$T_p = V_p \oplus H_p,$$

where V_p is the subspace of vectors tangent to the fibre at p, H_p is the supplementary linear subspace in T_p to V_p.

A *connection* A in P is a choice of H_p satisfying

(i) $H_{pa} = (R_a)_* H_p$,

(ii) H_p depends differentiably on p,

where $(R_a)_*$ is a linear map $H_p \to H_{pa}$, induced by the right action of G on P. H_p is called the *horizontal subspace* at p and has the same dimension as M. V_p is called the *vertical subspace*.

The connection A in P can be realized as a *connection 1-form* such that for $\forall X \in T_p(P)$, $A(X)$ coincides with the vertical component for X. X is horizontal $\Leftrightarrow$ $A(X) = 0$.

Let us define a local connection 1-form on $U_\alpha \subset M$ as $A_\alpha = \sigma_\alpha^* A$, where $\sigma_\alpha : U_\alpha \to P$ is a local section of P. If $x_\alpha = \{x^\mu_{(\alpha)}\}, \mu = 1, \dots, n$, are local coordinates in U_α, then A_α can be written as

$$A_\alpha = \sum_\mu A_\mu^{(\alpha)}(x_\alpha) dx^\mu_{(\alpha)}, \tag{4a}$$

where the components $A_\mu^{(\alpha)}(x_\alpha)$ are functions of $x_\alpha \in U_\alpha$ with values in $\mathcal{G}$.

On $U_\alpha \cap U_\beta$ we have the compatibility condition

$$A_\beta = f_{\alpha\beta}^{-1} A_\alpha f_{\alpha\beta} + f_{\alpha\beta}^{-1} df_{\alpha\beta}, \tag{4b}$$

where $f_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are transition matrices, and d is the exterior derivative on M.

By the help of gauge transformations we can construct another set of local sections $\sigma'_\alpha : U_\alpha \to P$, related to (1) by

$$\sigma'_\alpha(x_\alpha) = \sigma_\alpha(x_\alpha) g_\alpha(x_\alpha), \tag{5}$$

where $x_\alpha \in U_\alpha \subset M$, $g_\alpha(x_\alpha) \in \mathfrak{G}$ is a section over U_α of the bundle $P \times_{\mathrm{AdG}} G$. Then, the gauge transformations of the local connection form are the following

$$A'_\alpha = g_\alpha^{-1} A_\alpha g_\alpha + g_\alpha^{-1} dg_\alpha, \tag{6}$$

where $g_\alpha(x_\alpha)$ is understood as a G-valued function of x_α.

2.9. Covariant derivative. Given a connection A in P and local coordinates $\{x^\mu_{(\alpha)}\}$ on an open set $U_\alpha \subset M$, we can construct a lift $D_\mu^{(\alpha)}$ of the vector field $\partial_\mu^{(\alpha)} = \partial/\partial x^\mu_{(\alpha)}$ on U_α to $\pi^{-1}(U_\alpha)$. Suppose σ_α is a section over U_α, then

$$A_\alpha\left(\partial_\mu^{(\alpha)}\right) = A\left(\sigma_{\alpha *}\partial_\mu^{(\alpha)}\right) = A_\mu^{(\alpha)} = A\left(\mathcal{T}(A_\mu^{(\alpha)})\right),$$

where $\mathcal{T}(A_\mu^{(\alpha)})$ denotes the fundamental vector field associated to $A_\mu^{(\alpha)}$. Therefore,

$$A\left(\sigma_{\alpha *}\partial_\mu^{(\alpha)} - \mathcal{T}(A_\mu^{(\alpha)})\right) = 0,$$

and $\sigma_{\alpha *}\partial_\mu^{(\alpha)} - \mathcal{T}(A_\mu^{(\alpha)})$ is the horizontal vector field such that

$$\pi_*\left(\sigma_{\alpha *}\partial_\mu^{(\alpha)} - \mathcal{T}(A_\mu^{(\alpha)})\right) = \partial_\mu^{(\alpha)}.$$

Since $\pi^{-1}(U_\alpha) \simeq U_\alpha \times G$, we can identify $\partial_\mu^{(\alpha)}$ with $\sigma_{\alpha *}\partial_\mu^{(\alpha)}$.

Thus, the *covariant derivative*

$$D_\mu^{(\alpha)} = \partial_\mu^{(\alpha)} - \mathcal{T}\left(A_\mu^{(\alpha)}\right) \tag{7}$$

is the horizontal lift of $\partial_\mu^{(\alpha)}$ at the point $\sigma_\alpha(x_\alpha)$.

2.10. The curvature form. Let us consider the bundle $P \times_{\mathrm{AdG}} \mathcal{G}$, associated to P, with the typical fibre $\mathcal{G}$, on which G acts by the automorphisms $\mathrm{Ad}_a : v \mapsto a^{-1} v a$, $v \in \mathcal{G}, a \in G$. We can associate with any connection A on P a 2-form on M with values in the set of sections of $P \times_{\mathrm{AdG}} \mathcal{G}$.

It can be shown that the action of the covariant derivative (7) on fields in the adjoint representation of G coincides with the action of the operator $D_\mu^{(\alpha)} = \partial_\mu^{(\alpha)} + [A_\mu^{(\alpha)}, \cdot\ ]$. Let us calculate the commutator

$$F_{\mu\nu}^{(\alpha)} \equiv \left[D_\mu^{(\alpha)}, D_\nu^{(\alpha)}\right] = \partial_\mu^{(\alpha)} A_\nu^{(\alpha)} - \partial_\nu^{(\alpha)} A_\mu^{(\alpha)} + \left[A_\mu^{(\alpha)}, A_\nu^{(\alpha)}\right]. \tag{8}$$

Note that $F_{\mu\nu}^{(\alpha)}$ depends on $x_\alpha \in U_\alpha \subset M$ and takes values in the Lie algebra $\mathcal{G}$ of the structure group G.

Let us define a set of $\mathcal{G}$-valued 2-forms F_α

$$F_\alpha = \frac{1}{2} \sum_{\mu,\nu} F_{\mu\nu}^{(\alpha)} dx_{(\alpha)}^\mu \wedge dx_{(\alpha)}^\nu. \tag{9a}$$

On $U_\alpha \cap U_\beta$ we have

$$F_\beta = f_{\alpha\beta}^{-1} F_\alpha f_{\alpha\beta}, \tag{9b}$$

where $f_{\alpha\beta}$ are the transition matrices of the bundle P, and, therefore, F_α can be understood as a section over U_α of the bundle $P \times_{\mathrm{AdG}} \mathcal{G}$.

Recall that any point $p \in \pi^{-1}(U_\alpha)$ verifies $p = \sigma_\alpha(\pi(p))\phi_\alpha(p)$ (see (1)). We can use ϕ_α to construct F in $\pi^{-1}(U_\alpha)$

$$F = \mathrm{Ad}_{\phi_\alpha^{-1}} (\pi^* F_\alpha),$$

which is called the *curvature 2-form* in the bundle space.

One can directly check that

$$DF = 0, \tag{10a}$$

where D is the *covariant differential*. In components $D = \sum_\mu D_\mu^{(\alpha)} dx_{(\alpha)}^\mu$ on U_α, and we have

$$D_{[\mu}^{(\alpha)} F_{\rho\sigma]}^{(\alpha)} = 0. \tag{10b}$$

The identities (10) are called the *Bianchi identities*.

3. Yang-Mills model in $\mathbb{R}^4$

3.1. The Yang-Mills action. Let us consider a principal fibre bundle $P = P(\mathbb{R}^4, G)$ over the Euclidean space $\mathbb{R}^4$ with structure group G. Since our base space is $\mathbb{R}^4$ with coordinates $\{x^\mu\}$, $\mu, \nu, \ldots = 1, \ldots, 4$, the bundle P is a trivial principal fibre bundle: $P = \mathbb{R}^4 \times G$. Then components $A_\mu(x)$ of the connection 1-form A in the bundle P are defined globally on $\mathbb{R}^4$ ($x \in \mathbb{R}^4$), and components

$$F_{\mu\nu}(x) \equiv [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{11}$$

of the curvature 2-form F in P will not have an additional index α (cf. (8)). Fields $A_\mu(x)$ and $F_{\mu\nu}(x)$ are defined on $\mathbb{R}^4$ and take values in the Lie algebra $\mathcal{G}$.

The *Yang-Mills model* in $\mathbb{R}^4$ with the structure group G is the model with the following action

$$S[A] = \int_{\mathbb{R}^4} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\, d^4x. \tag{12}$$

Here and in what follows summation over repeated indices is understood.

The *Lagrangian* $L[A] = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ is invariant under the gauge transformations

$$A_\mu \mapsto A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g, \tag{13}$$

where $g(x) \in \mathfrak{G}$ can be understood as a G-valued function of $x \in \mathbb{R}^4$.

3.2. The self-dual Yang-Mills equations. From (12) one can easily derive the equations of motion for the YM model. They are called the *Yang-Mills equations* and have the following form

$$D_\mu F_{\mu\nu} = 0. \tag{14}$$

In $\mathbb{R}^4$ we have the completely antisymmetric tensor $\varepsilon_{\mu\nu\mathcal{R}\sigma}$ such that $\varepsilon_{1234} = 1$. Then the Bianchi identities (10b) can be written as

$$D_\mu(\varepsilon_{\mu\nu\mathcal{R}\sigma}F_{\rho\sigma}) = 0.$$

The *self-dual Yang-Mills equations* have the form

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\mathcal{R}\sigma}F_{\rho\sigma}. \tag{15}$$

We see that if $F_{\mu\nu}$ satisfies equations (15), the Bianchi identities ensure that the YM equations (14) are satisfied.

3.3. Manifest symmetries of the SDYM equations. As mentioned in Section 1, all local (manifest) symmetries of the SDYM equations are given by gauge transformations (13) and conformal transformations of $\mathbb{R}^4$. Let us write them down in the infinitesimal form: $A_\mu \mapsto A'_\mu = A_\mu + \delta A_\mu + \cdots$.

The action of the algebra $\mathfrak{g}$ of gauge transformations on A_μ is given by

$$\delta_\vartheta A_\mu = \partial_\mu \vartheta + [A_\mu, \vartheta], \tag{16}$$

where $\vartheta(x) \in \mathfrak{g}, x \in \mathbb{R}^4$.

It is well-known that for $n > 2$ the group of conformal transformations of $\mathbb{R}^n$ is locally isomorphic to the group $SO(n+1,1)$ [12]. Infinitesimal transformations of A_μ under the action of the group of conformal transformations of $\mathbb{R}^4$, locally isomorphic to $SO(5,1)$, have the form

$$\delta_V A_\mu = V^\nu \partial_\nu A_\mu + A_\nu \partial_\mu V^\nu, \tag{17}$$

where a vector field $V = V^\nu \partial_\nu$ is any generator

$$X_a = \delta_{ab}\eta^b_{\mu\nu}x_\mu\partial_\nu, \qquad Y_a = \delta_{ab}\bar{\eta}^b_{\mu\nu}x_\mu\partial_\nu,$$
$$P_\mu = \partial_\mu, \qquad K_\mu = \frac{1}{2}x_\sigma x_\sigma\partial_\mu - x_\mu B,$$
$$B = x_\sigma\partial_\sigma, \qquad a,b,\ldots = 1,2,3,$$

of the 15-parameter conformal group. Here $\{X_a\}$ and $\{Y_a\}$ generate two commuting $SO(3)$ subgroups in $SO(4)$, P_μ are the translation generators, K_μ are the generators of special conformal transformations and B is the dilatation generator; $\eta^a_{\mu\nu} = \{\varepsilon^a_{bc}, \mu = b, \nu = c;\ \delta^a_\mu, \nu = 4;\ -\delta^a_\nu, \mu = 4\}$ and $\bar{\eta}^a_{\mu\nu} = \{\varepsilon^a_{bc}, \mu = b, \nu = c;\ -\delta^a_\mu, \nu = 4;\ \delta^a_\nu, \mu = 4\}$ are the 't Hooft tensors satisfying

$$\frac{1}{2}\varepsilon_{\mu\nu\mathcal{R}\sigma}\eta^a_{\rho\sigma} = \eta^a_{\mu\nu},$$
$$\frac{1}{2}\varepsilon_{\mu\nu\mathcal{R}\sigma}\bar{\eta}^a_{\rho\sigma} = -\bar{\eta}^a_{\mu\nu},$$

i.e. $\eta^a_{\mu\nu}$ are the self-dual tensors and $\bar{\eta}^a_{\mu\nu}$ are the anti-self-dual tensors.

4. The Penrose-Ward correspondence. Our aim is to describe an infinite-dimensional algebra of all infinitesimal symmetries of the SDYM equations. It can be done with the help of the Penrose-Ward correspondence which we shall briefly discuss.

4.1. Complex structure on $\mathbb{R}^4$. A *complex structure* on $\mathbb{R}^4$ is a tensor J^ν_μ such that $J^\nu_\mu J^\sigma_\nu = -\delta^\sigma_\mu$. The most general constant complex structure $J = (J^\nu_\mu)$ on $\mathbb{R}^4$ has the form

$$J^\nu_\mu = s_a\bar{\eta}^a_{\mu\sigma}\delta^{\sigma\nu}, \tag{18}$$

where real numbers s_a parametrize a two-sphere S^2, $s_as_a = 1$, $\bar{\eta}^a_{\mu\sigma}$ are the anti-self-dual 't Hooft tensors. By using J, one can introduce $(0,1)$ vector fields $\bar{V}_{\bar{1}}$, $\bar{V}_{\bar{2}}$ ($J^\nu_\mu\bar{V}^\mu = -i\bar{V}^\nu$) in the following way

$$\bar{V}_{\bar{1}} = \partial_{\bar{y}} - \lambda\partial_z, \qquad \bar{V}_{\bar{2}} = \partial_{\bar{z}} + \lambda\partial_y, \tag{19}$$

where $y = x_1 + ix_2,\ z = x_3 - ix_4,\ \bar{y} = x_1 - ix_2,\ \bar{z} = x_3 + ix_4$ are complex coordinates on $\mathbb{R}^4 \simeq \mathbb{C}^2$, and $\lambda = (s_1 + is_2)/(1 + s_3)$ is a local complex coordinate on $S^2 \simeq \mathbb{C}P^1$.

4.2. Twistor space for $\mathbb{R}^4$. Let $C_+ := \{\lambda \in \mathbb{C}P^1 : |\lambda| \le 1 + \alpha\}$, where $0 < \alpha < 1$ is a positive real number, $C_- := \{\lambda \in \mathbb{C}P^1 : |\lambda| \ge 1 - \alpha\}$ (including $\lambda = \infty$). Then C_+ and C_- form a two-set open cover of the Riemann sphere $\mathbb{C}P^1$ with the intersection $C_\alpha = C_+ \cap C_- = \{\lambda : 1 - \alpha \le |\lambda| \le 1 + \alpha\}$. The vector field $\partial_{\bar{\lambda}} := \partial/\partial_{\bar{\lambda}}$ is antiholomorphic $(0,1)$ vector field with respect to the standard complex structure $\varepsilon = i\, d\lambda \otimes \partial_\lambda - i\, d\bar{\lambda} \otimes \partial_{\bar{\lambda}}$ on $\mathbb{C}P^1$ (λ, $\bar{\lambda}$ are complex coordinates on $\mathbb{C}P^1$).

Twistor space $\mathcal{Z}$ of $\mathbb{R}^4$ is the bundle $\pi : \mathcal{Z} \to \mathbb{R}^4$ of complex structures on $\mathbb{R}^4$ associated with the principal $SO(4)$-bundle of orthogonal frames of $\mathbb{R}^4$ [1]. It means that the fibre $\pi^{-1}(x)$ of $\mathcal{Z} \to \mathbb{R}^4$ over a point $x \in \mathbb{R}^4$ coincides with the space $\mathbb{C}P^1$ of complex structures on $\mathbb{R}^4$ defined in Section 4.1. The space $\mathcal{Z}$ is the trivial bundle over $\mathbb{R}^4$ with fibre $\mathbb{C}P^1$, hence $\mathcal{Z} = \mathbb{R}^4 \times \mathbb{C}P^1$ is a manifold which can be covered by two coordinate patches $\mathcal{Z} = U_+ \cup U_-$

$$U_+ := \{x \in \mathbb{R}^4, \lambda \in C_+\}, \qquad U_- := \{x \in \mathbb{R}^4, \lambda \in C_-\} \tag{20a}$$

with the intersection

$$U := U_+ \cap U_- = \{x \in \mathbb{R}^4, \lambda \in C_\alpha = C_+ \cap C_-\}. \tag{20b}$$

Let us denote the cover (20) by $\mathfrak{U}$.

The twistor space $\mathcal{Z}$ is a complex manifold with complex structure $\mathcal{J} = (J, \varepsilon)$ on $\mathcal{Z}$. Vector fields $\bar{V}_{\bar{1}}, \bar{V}_{\bar{2}}$ from (19) and $\bar{V}_{\bar{3}} = \partial_{\bar{\lambda}}$ are the vector fields on $\mathcal{Z}$ of type $(0,1)$ with respect to the complex structure $\mathcal{J}$.

4.3. Complex vector bundle $\tilde{E}$ over the twistor space. Let us consider a trivial principal fibre bundle $P = P(\mathbb{R}^4, SU(n))$ over $\mathbb{R}^4$ with the structure group $SU(n)$. Then, A_μ and $F_{\mu\nu}$ take values in the Lie algebra $su(n)$.

Let $E = P \times_{SU(n)} \mathbb{C}^n$ be a complex vector bundle associated to P. Sections of this bundle are $\mathbb{C}^n$-valued vector-functions depending on $x \in \mathbb{R}^4$. By using the projection $\pi : \mathcal{Z} \to \mathbb{R}^4$, we can pull back the bundle E with the connection $D = D_\mu dx^\mu$ to the bundle $\tilde{E} := \pi^* E$ over $\mathcal{Z} = \mathbb{R}^4 \times \mathbb{C}P^1$

$$\begin{tikzcd} \pi^* E \arrow[r] \arrow[d] & E \arrow[d] \\ \mathcal{Z} \arrow[r, "\pi"] & \mathbb{R}^4 \end{tikzcd}$$

The trivial smooth complex vector bundle $\tilde{E}$ over the twistor space $\mathcal{Z}$ can be considered as a bundle associated to the principal fibre bundle $\tilde{P} = \tilde{P}(\mathcal{Z}, SL(n,\mathbb{C}))$, i.e., $\tilde{E} = \tilde{P} \times_{SL(n,\mathbb{C})} \mathbb{C}^n$. By definition of the pull back, the pulled back connection $\tilde{D} := \pi^* D$ in $\tilde{E}$ will be flat along the fibres $\mathbb{C}P^1_x$ of the bundle $\mathcal{Z} \to \mathbb{R}^4$, and, therefore, the components of $\tilde{A} := \pi^* A$ along the vector fields ∂_λ, $\partial_{\bar{\lambda}}$ in $\mathbb{C}P^1_x$ can be set equal to zero. Then we have $\tilde{D} = D + d\lambda \partial_\lambda + d\bar{\lambda}\partial_{\bar{\lambda}}$.

4.4. Linear system for the SDYM equations and holomorphic bundles. Let $\tilde{D}^{(0,1)}_{\bar{a}}$ ($a = 1,2,3$) be components of $\tilde{D}$ along the $(0,1)$ vector fields $\bar{V}_{\bar{a}}$ on $\mathcal{Z}$. A section ξ of the bundle $\tilde{E}$ is called a *local holomorphic section* if it is a local solution of the equations $\tilde{D}^{(0,1)}_{\bar{a}} \xi = 0$ or, in local coordinates on $\mathcal{Z}$,

$$(D_{\bar{y}} - \lambda D_z)\xi(x, \lambda, \bar{\lambda}) = 0, \tag{21a}$$

$$(D_{\bar{z}} + \lambda D_y)\xi(x, \lambda, \bar{\lambda}) = 0, \tag{21b}$$

$$\partial_{\bar{\lambda}} \xi(x, \lambda, \bar{\lambda}) = 0. \tag{22}$$

The equations $\tilde{D}^{(0,1)}_{\bar{a}} \xi = 0$ on sections ξ of the complex vector bundle $\tilde{E}$ define a *holomorphic structure* in $\tilde{E}$. Accordingly, the bundle $\tilde{E}$ is said to be *holomorphic* if equations (21), (22) are compatible, i.e., the (0,2) components of the curvature of the bundle $\tilde{E}$ are equal to zero.

The solution of equation (22) is $\xi(x,\lambda)$. Equations (21) on $\xi(x,\lambda)$ are called the *linear system* for the SDYM equations [4, 24]. It is easy to see that the compatibility conditions of the linear system (21) coincide with the SDYM equations written in the complex coordinates $y,z,\bar{y},\bar{z}$ on $\mathbb{R}^4 \simeq \mathbb{C}^2$.

Equations (21) have local solutions $\xi_\pm(x,\lambda)$ over $U_\pm \subset \mathcal{Z}$, and $\xi_+ = \xi_-$ on $U = U_+ \cap U_-$ (for definitions of $U_\pm, U$ see (20)). We can always represent $\xi_\pm$ in the form $\xi_\pm = \psi_\pm \chi_\pm$, where $\psi_\pm$ are matrices of fundamental solutions of (21) defining a trivialization of $\tilde{E}$ over $U_\pm$, and $\chi_\pm \in \mathbb{C}^n$ are Čech fibre coordinates satisfying $\bar{V}_{\bar{a}}\chi_\pm = 0$ and $\chi_- = \mathcal{F}\chi_+$ on $U = U_+ \cap U_- \subset \mathcal{Z}$. The matrix $\mathcal{F} = \psi_-^{-1}\psi_+$ is the transition matrix in the bundle $\tilde{E}$, i.e., holomorphic $SL(n,C)$-valued function on U with non-vanishing determinant satisfying the conditions on transition matrices [13].

4.5. Ward's theorem. So, starting from the complex vector bundle E over $\mathbb{R}^4$ with the self-dual connection D, we can construct the holomorphic vector bundle $\tilde{E}$ over $\mathcal{Z}$ with transition matrix $\mathcal{F} = \psi_-^{-1}\psi_+$ defined on $U \subset \mathcal{Z}$.

Conversely, if we are given the holomorphic vector bundle $\tilde{E} = \tilde{P}(\mathcal{Z}, SL(n,\mathbb{C})) \times_{SL(N,\mathbb{C})} \mathbb{C}^n$ associated to the principal fibre bundle $\tilde{P}$ over $\mathcal{Z}$, which is holomorphically trivial on each fibre $\mathbb{C}P^1_x$: $\tilde{E}\,|_{\mathbb{C}P^1_x} \simeq \mathbb{C}P^1_x \times \mathbb{C}^n$ (Ward's twistor construction [24]), then on $\mathbb{C}P^1_x$ the transition matrix $\mathcal{F}$ can be factorized in the form (Birkhoff's theorem)

$$\mathcal{F} = \psi_-^{-1}(x,\lambda)\psi_+(x,\lambda), \tag{23}$$

where $\psi_\pm(x,\lambda)$ are $SL(n,\mathbb{C})$-valued functions holomorphic in $\lambda^{\pm 1}$ on $C_\pm$.

From the holomorphicity of $\mathcal{F}$ on U ($\bar{V}_{\bar{a}}\mathcal{F} = 0$) it follows that $(\bar{V}_{\bar{a}}\psi_+)\psi_+^{-1} = (\bar{V}_{\bar{a}}\psi_-)\psi_-^{-1}$ and, therefore,

$$\begin{aligned}(\partial_{\bar{y}}\psi_+ - \lambda\partial_z\psi_+)\psi_+^{-1} &= (\partial_{\bar{y}}\psi_- - \lambda\partial_z\psi_-)\psi_-^{-1} \\ &= -(A_{\bar{y}}(x) - \lambda A_z(x)),\end{aligned} \tag{24a}$$

$$\begin{aligned}(\partial_{\bar{z}}\psi_+ + \lambda\partial_y\psi_+)\psi_+^{-1} &= (\partial_{\bar{z}}\psi_- + \lambda\partial_y\psi_-)\psi_-^{-1} \\ &= -(A_{\bar{z}}(x) + \lambda A_y(x)),\end{aligned} \tag{24b}$$

and the potentials $\{A_\mu\}$ defined by (24) satisfy the SDYM equations and do not change after transformations: $\psi_\pm \mapsto \psi_\pm h_\pm$, where $h_\pm$ are regular holomorphic matrix-valued functions on $U_\pm$. This means that the bundles with transition matrices $h_-^{-1}\mathcal{F}h_+$ and $\mathcal{F}$ are holomorphically equivalent.

We summarize the facts about the Penrose-Ward correspondence in the theorem [2, 1]:

Theorem. *There is a one-to-one correspondence between gauge equivalence classes of solutions to the SDYM equations in the Euclidean 4-space and equivalence classes of holomorphic vector bundles $\tilde{E}$ over the twistor space $\mathcal{Z}$, that are holomorphically trivial over each real projective line $\mathbb{C}P^1_x$ in $\mathcal{Z}$.*

5. Infinitesimal gauge-type symmetries

5.1. The algebras $C^0(\mathfrak{U},\mathcal{H})$ and $C^1(\mathfrak{U},\mathcal{H})$. We consider the principal fibre bundle $\tilde{P} = \tilde{P}(\mathfrak{T}, SL(N,\mathbb{C}))$ over the twistor space $\mathfrak{T}$ and the associated bundle $\operatorname{Ad}\tilde{P} = \tilde{P}\times_{\operatorname{Ad} SL(n,\mathbb{C})} sl(n,\mathbb{C})$ with the adjoint action of the group $SL(n,\mathbb{C})$ on the algebra $sl(n,\mathbb{C})$: $\xi \mapsto \operatorname{Ad}_g \xi = g\xi g^{-1}$, $g \in SL(n,\mathbb{C})$, $\xi \in sl(n,\mathbb{C})$. Let $\mathcal{H}$ be a sheaf of germs of holomorphic sections of the bundle $\operatorname{Ad}\tilde{P}$ (see Section 2.4), $\Gamma(\mathcal{U},\mathcal{H})$ be a set of all sections of the sheaf $\mathcal{H}$ over an open set $\mathcal{U} \subset \mathfrak{T}$.

A collection $\{\varphi_+, \varphi_-\}$ of sections of $\mathcal{H}$ over the open sets U_+ and U_- from (20a) is called a *0-cochain over* $\mathfrak{T}$, subordinate to the cover $\mathfrak{U} = \{U_+, U_-\}$. Thus, a 0-cochain is an element of the space

$$C^0(\mathfrak{U},\mathcal{H}) := \Gamma(U_+,\mathcal{H}) \oplus \Gamma(U_-,\mathcal{H}).$$

The space of *1-cochains* with values in $\mathcal{H}$

$$C^1(\mathfrak{U},\mathcal{H}) := \Gamma(U,\mathcal{H})$$

is a set of sections φ of the sheaf $\mathcal{H}$ over $U = U_+ \cap U_-$. Notice that $C^0(\mathfrak{U},\mathcal{H})$ and $C^1(\mathfrak{U},\mathcal{H})$ are Lie algebras of holomorphic maps: $U_\pm \to sl(n,\mathbb{C})$ and $U \to sl(n,\mathbb{C})$ respectively with pointwise commutator.

5.2. Action of $C^1(\mathfrak{U},\mathcal{H})$ on transition matrices. The standard action of the algebra $C^0(\mathfrak{U},\mathcal{H})$ on the space of holomorphic transition matrices $\mathcal{F}$

$$\delta\mathcal{F} = \varphi_-\mathcal{F} - \mathcal{F}\varphi_+$$

gives us holomorphically equivalent bundles. Hence, these transformations are trivial. But we shall consider the action of the algebra $C^1(\mathfrak{U},\mathcal{H})$ on $\mathcal{F}$

$$\delta_\varphi\mathcal{F} = \varphi(\lambda)\mathcal{F} - \mathcal{F}\varphi^\dagger\left(-\frac{1}{\bar{\lambda}}\right), \tag{25}$$

where $\varphi \in C^1(\mathfrak{U},\mathcal{H})$, $\varphi = \varphi(\lambda) \equiv \varphi(y - \lambda\bar{z}, z + \lambda\bar{y}, \lambda)$, $\varphi(-1/\bar{\lambda}) \equiv \varphi(y + \bar{z}/\bar{\lambda}, z - \bar{y}/\bar{\lambda}, -1/\bar{\lambda})$, and $\dagger$ denotes Hermitian conjugation.

Transformations (25) preserve the holomorphicity of $\mathcal{F}$ and preserve the hermiticity of the bundle E; they are local infinitesimal transformations of the transition matrix.

5.3. Infinitesimal gauge-type transformations of self-dual connections. Let us introduce the $sl(n,\mathbb{C})$-valued function ϕ on U

$$\phi := \psi_-(\delta_\varphi\mathcal{F})\psi_+^{-1} = \psi_-\varphi(\lambda)\psi_-^{-1} + \psi_+\varphi^\dagger\left(-\frac{1}{\bar{\lambda}}\right)\psi_+^{-1},$$

which is holomorphic in $\lambda \in C_\alpha$ and can be expanded in Laurent series

$$\phi = \sum_{n=-\infty}^{\infty} \lambda^n\phi_n(x) = \phi_- - \phi_+,$$

$$\phi_+ := \tilde{\phi}_0(x) - \sum_{n=1}^{\infty}\lambda^n\phi_n(x), \qquad \phi_- := \hat{\phi}_0(x) + \sum_{n=-\infty}^{-1}\lambda^n\phi_n(x),$$

$$\hat{\phi}_0(x) - \tilde{\phi}_0(x) = \phi_0(x).$$

The splitting $\phi = \phi_- - \phi_+$ is a solution of the infinitesimal variant of the Riemann-Hilbert problem, and functions $\phi_\pm \in sl(n,\mathbb{C})$ are holomorphic in $\lambda \in C_\pm$. It follows from (24) that $\tilde{D}^{(0,1)}_{\bar{a}}\phi = 0$, therefore,

$$(D_{\bar{y}} - \lambda D_z)\phi_+ = (D_{\bar{y}} - \lambda D_z)\phi_-, \tag{26a}$$

$$(D_{\bar{z}} + \lambda D_y)\phi_+ = (D_{\bar{z}} + \lambda D_y)\phi_-. \tag{26b}$$

The action of the algebra $C^1(\mathfrak{U},\mathcal{H})$ on $SL(n,\mathbb{C})$-valued functions $\psi_\pm$ and on gauge potentials $\{A_\mu\}$ is given by formulae

$$\delta_\varphi\psi_+ = -\phi_+\psi_+, \qquad \delta_\varphi\psi_- = -\phi_-\psi_-, \tag{27}$$

$$\delta_\varphi A_{\bar{y}} - \lambda\delta_\varphi A_z = D_{\bar{y}}\phi_+ - \lambda D_z\phi_+ = D_{\bar{y}}\phi_- - \lambda D_z\phi_-, \tag{28a}$$

$$\delta_\varphi A_{\bar{z}} + \lambda\delta_\varphi A_y = D_{\bar{z}}\phi_+ + \lambda D_y\phi_+ = D_{\bar{z}}\phi_- + \lambda D_y\phi_-. \tag{28b}$$

It follows from (28) that

$$\begin{aligned}
\delta_\varphi A_y &= \oint_{S^1}\frac{d\lambda}{2\pi i\lambda^2}(D_{\bar{z}}\phi_+ + \lambda D_y\phi_+),\\
\delta_\varphi A_z &= -\oint_{S^1}\frac{d\lambda}{2\pi i\lambda^2}(D_{\bar{y}}\phi_+ - \lambda D_z\phi_+),\\
\delta_\varphi A_{\bar{y}} &= \oint_{S^1}\frac{d\lambda}{2\pi i\lambda}(D_{\bar{y}}\phi_+ - \lambda D_z\phi_+),\\
\delta_\varphi A_{\bar{z}} &= \oint_{S^1}\frac{d\lambda}{2\pi i\lambda}(D_{\bar{z}}\phi_+ + \lambda D_y\phi_+),
\end{aligned} \tag{29}$$

where $S^1 = \{\lambda \in CP^1 : |\lambda| = 1\}$. Thus, we have described the action of $C^1(\mathfrak{U},\mathcal{H})$ on the space of solutions of SDYM equations.

Example 1. For $\varphi = 0$ we have $\phi = 0$. Choose $\phi_+ = \phi_- = \vartheta(x)$, $x \in \mathbb{R}^4$, then formulae (29) give us manifest gauge symmetries (16).

Example 2. If we choose $\varphi = \varphi(\lambda)$ (i.e. $\partial_\mu\varphi(x,\lambda) = 0$), then obtain the action of the algebra $su(n)\otimes C[\lambda,\lambda^{-1}]$ on the space of solutions of SDYM equations [22].

6. Infinitesimal diffeomorphism-type symmetries

6.1. The algebra $C^0(\mathfrak{U},\mathcal{V})$. Let us consider a complexified tangent bundle $T^{\mathbb{C}}(\mathcal{Z}) = T^{(1,0)}(\mathcal{Z}) \oplus T^{(0,1)}(\mathcal{Z})$ of the twistor space $\mathcal{Z}$ and the sheaf $\mathcal{V}$ of germs of holomorphic sections of the bundle $T^{(1,0)}(\mathcal{Z})$. The set of all sections of the sheaf $\mathcal{V}$ over an open set $\mathcal{U} \in \mathcal{Z}$ is denoted by $\Gamma(\mathcal{U},\mathcal{V})$. If we take sections of $\mathcal{V}$ over each of the open sets U_+ and U_- from the cover $\mathfrak{U}$, then the resulting collection of holomorphic vector fields is called a 0-cochain over $\mathcal{Z}$, subordinate to the cover $\mathfrak{U}$. Thus, a 0-cochain $\{\eta_+,\eta_-\}$ is an element of the space

$$C^0(\mathfrak{U},\mathcal{V}) := \Gamma(U_+,\mathcal{V}) \oplus \Gamma(U_-,\mathcal{V}).$$

The space of 1-cochains is defined as follows: $C^1(\mathfrak{U},\mathcal{V}) := \Gamma(U,\mathcal{V})$, where $U = U_+ \cap U_-$. Thus, elements of $C^1(\mathfrak{U},\mathcal{V})$ are holomorphic vector fields η_{+-} defined on U.

6.2. Action of $C^0(\mathfrak{U},\mathcal{V})$ on transition matrices. The vector space $C^0(\mathfrak{U},\mathcal{V})$ can be described as the Lie algebra of holomorphic vector fields with pointwise commutator, defined on U_+ and U_-. For any $\eta = \{\eta_+,\eta_-\} \in C^0(\mathfrak{U},\mathcal{V})$ we define two actions of $C^0(\mathfrak{U},\mathcal{V})$ on the transition matrix $\mathcal{F}$

$$\delta_\eta^\pm \mathcal{F} = \eta_\pm(\mathcal{F}), \tag{30}$$

i.e. as a derivative of $\mathcal{F}$ along the vector fields $\eta_\pm \in C^0(\mathfrak{U},\mathcal{V})$.

One can also consider a combination of these actions

$$\delta_\eta \mathcal{F} = \delta_\eta^- \mathcal{F} - \delta_\eta^+ \mathcal{F}.$$

It is easy to see that the algebra $C^0(\mathfrak{U},\mathcal{V})$ acts on the algebra $C^1(\mathfrak{U},\mathcal{H})$ by derivations, and we can consider a semidirect sum $C^0(\mathfrak{U},\mathcal{V}) \dotplus C^1(\mathfrak{U},\mathcal{H})$ of these algebras.

6.3. Action of $C^0(\mathfrak{U},\mathcal{V})$ on self-dual connections. Let us introduce the $sl(n,C)$-valued functions $\theta^\pm$ on U

$$\theta^\pm := \psi_-(\delta_\eta^\pm \mathcal{F})\psi_+^{-1},$$

which are holomorphic in $\lambda \in C_\alpha$

$$\theta^\pm = \sum_{n=-\infty}^{\infty} \lambda^n \theta_n^\pm(x) = \theta_-^\pm - \theta_+^\pm,$$

where

$$\begin{aligned}
\theta_+^\pm &:= \tilde{\theta}_0^\pm(x) - \sum_{n=1}^{\infty} \lambda^n \theta_n^\pm(x),\\
\theta_-^\pm &:= \hat{\theta}_0^\pm(x) + \sum_{n=-\infty}^{-1} \lambda^n \theta_n^\pm(x),\\
\hat{\theta}_0^\pm(x) &- \tilde{\theta}_0^\pm(x) = \theta_0^\pm(x).
\end{aligned}$$

Thus, the functions $\theta_\pm^\pm(x,\lambda) \in sl(n,C)$ are holomorphic in $\lambda^{\pm 1} \in C_\pm \subset \mathbb{C}P^1$.

For $\theta_-^\pm$ and $\theta_+^\pm$ we have

$$(D_{\bar{y}} - \lambda D_z)\theta_+^\pm = (D_{\bar{y}} - \lambda D_z)\theta_-^\pm, \tag{31a}$$

$$(D_{\bar{z}} + \lambda D_y)\theta_+^\pm = (D_{\bar{z}} + \lambda D_y)\theta_-^\pm. \tag{31b}$$

The action of $C^0(\mathfrak{U},\mathcal{V})$ on matrix-valued functions $\psi_\pm \in SL(n,C)$ and on gauge potentials $\{A_\mu\}$ is given by formulae

$$\delta_\eta^\pm \psi_+ := -\theta_+^\pm \psi_+, \qquad \delta_\eta^\pm \psi_- := -\theta_-^\pm \psi_-, \tag{32}$$

$$\delta_\eta^\pm A_{\bar{y}} - \lambda \delta_\eta^\pm A_z : = D_{\bar{y}}\theta_+^\pm - \lambda D_z \theta_+^\pm = D_{\bar{y}}\theta_-^\pm - \lambda D_z \theta_-^\pm, \tag{33a}$$

$$\delta_\eta^\pm A_{\bar{z}} + \lambda \delta_\eta^\pm A_y : = D_{\bar{z}}\theta_+^\pm + \lambda D_y \theta_+^\pm = D_{\bar{z}}\theta_-^\pm + \lambda D_y \theta_-^\pm. \tag{33b}$$

It follows from 33 that

$$\begin{aligned}
\delta_\eta^\pm A_y &= \oint_{S^1} \frac{d\lambda}{2\pi i\lambda^2}\left(D_{\bar z}\theta_+^\pm + \lambda D_y\theta_+^\pm\right),\\
\delta_\eta^\pm A_z &= -\oint_{S^1} \frac{d\lambda}{2\pi i\lambda^2}\left(D_{\bar y}\theta_+^\pm - \lambda D_z\theta_+^\pm\right),\\
\delta_\eta^\pm A_{\bar y} &= \oint_{S^1} \frac{d\lambda}{2\pi i\lambda}\left(D_{\bar y}\theta_+^\pm - \lambda D_z\theta_+^\pm\right),\\
\delta_\eta^\pm A_{\bar z} &= \oint_{S^1} \frac{d\lambda}{2\pi i\lambda}\left(D_{\bar z}\theta_+^\pm + \lambda D_y\theta_+^\pm\right),
\end{aligned}\tag{34}$$

where $S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\}$.

Example 3. Let us consider the holomorphic vector fields $\eta = \lambda^{-n}\tilde N$, $n = 0, \pm 1, \pm 2, \ldots$, where $\tilde N$ are vector fields on $\mathcal{Z}$ realizing the action of $so(5,1)$ on $\mathcal{Z}$, which preserves the holomorphicity of the bundle $\tilde E \to \mathcal{Z}$. Such lift $N \to \tilde N$ of vector fields from $\mathbb{R}^4$ to $\mathcal{Z}$ was described in [16]. As it has been shown in [14], symmetries (34) for $\eta = \lambda^{-n}\tilde N$, $n = 0, \pm 1, \pm 2, \ldots$ with $n \geq 0$ are in one-to-one correspondence with the symmetries from [21].

7. Conclusion. To sum up, using the one-to-one correspondence between the classes of holomorphically equivalent transition matrices $\mathcal{F}$ and the gauge equivalent classes of self-dual connections, to any infinitesimal transformations (25) and (30) of transition matrices we have associated the infinitesimal transformations (29) and (34) of solutions $\{A_\mu\}$ of the SDYM equations. There are no other infinitesimal automorphisms of the bundle $\tilde E$ over $\mathcal{Z}$ besides those generated by the algebras $C^0(\mathfrak{U}, \mathcal{V})$ and $C^1(\mathfrak{U}, \mathcal{H})$. Thus, an infinite-dimensional algebra of all infinitesimal transformations of solutions of the SDYM equations has the form $C^0(\mathfrak{U}, \mathcal{V}) \dotplus C^1(\mathfrak{U}, \mathcal{H})$.

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SOME PROPERTIES OF HAMILTONIAN SYMMETRIES

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Hamiltonian systems with a finite number of degrees of freedom are under consideration. Different conditions of symmetry are compared. The role of Emmy Noether's results for analytical mechanics is discussed.

A term "group analysis of differential equations" was introduced by the Russian academician Lev Ovsiannikov [17] for continuous Lie symmetry analysis of differential equations. The main ideas and the first fundamental results in the sphere were developed and detailed by the outstanding Norwegian mathematician Sophus Lie [12, 13]. A new approach to the search for analytical solutions of both ordinary and partial differential equations was proposed. Local continuous symmetry transformations of differential equations became the basis for the analysis.

Linear algebra of symmetry vector fields corresponds to nonlinear differential equations in the general case or to a system of differential equations. It gives us a chance for investigating a sufficiently simple object to come to a conclusion on the more complicated one. Algebra of vector fields is accessible to detailed analysis with modern algebraic means.

Some changes of variables naturally spring up under group analysis of differential equations. It gives the most adequate way to use symmetry properties for simplification of the original problem.

Russia can be proud of the Syberian scientific school of group analysis and its applications with Lev Ovsiannikov at the head. The school gave a new vital power to the classical results of Sophus Lie. Mathematical physics was found to be a fruitful topic for Lie theory applications. Many considerable broadenings of the theory in hydrodynamics, nonlinear elasticity and plasticity theory, nonlinear acoustics, magnetic hydrodynamics, nonlinear field theory and in some problems of chemistry, biology and economics were obtained.

On the threshold of the XXI-th century group analysis of differential equations became a fully formed and actively developing scientific method. There is a close contact among the scientists of the world community in group analysis now. Lev Ovsiannikov's talented successor Nail Ibragimov became one of the most prominent specialists in the sphere [1, 7, 2, 8]. N. Ibragimov became the editor-in-chief of the international journal "Lie groups and their applications." He regularly organizes international conferences on group analysis of differential equations. Essentially new results were obtained by Peter Olver [16] who is a successor of the famous mathematician George Birkhoff. Peter Olver was the first to apply Lie theory in hydromechanics. Moreover P. Olver did a lot in popularizing and developing group analysis culture. Lie-Ovsyannikov theory in quantum field theory was applied by Dmitry Shirkov [4]. A wide bibliography on the group analysis of differential equations can be found in [1, 7, 2].

As a rule the traditional university course on "Differential equations" is a set of empirical regulations for solution finding. The following sacramental phrase is used: "We shall find the solution in this way ... " When a student asks: "Why shall we find it this way?" a typical answer is: "It is the only way needed. It's impossible to find the solution in another way. You must trust mathematical experience." Group analysis answers the questions: "What is the method of solution finding? What changes of variables should be done? What is the order of changing? How can some trivial solutions be used for the family of nontrivial solutions or even for common solution finding? Is this equation integrable or not at all?" It is surprising that at the very end of the XX-th century most universities, at least in Russia, have the syllabus of the "Differential equations" course that was established in the middle of the last century!

Let us start with a brief presentation of the main group analysis of differential equations ideas.

Transformation

$$q' = f(q,a), \quad q \in R^n,\ a \in A^r \tag{1}$$

(n-variable space dimension, r-arithmetical space of parameters dimension) is called an *r-parametric local Lie group*, iff the following three axioms are satisfied:

1. Closure axiom. Two sequential transformations (1) $q' = f(q,a)$ and $q'' = f(q',b)$ are equivalent to transformation (1) $q'' = f(q,c)$, $(a,b,c \in A^r)$. The law of parameters transformation $c = \varphi(a,b)$ does not contain q.

2. Identity axiom. There is a set of parameters a^0, that transforms a space of the variables q in itself: $q = f(q,a^0)$, $a^0 \in A^r$.

3. Inversion axiom. There is a set of parameters a^{-1}, that transforms a point q' to the initial point q: $q = f(q',a^{-1})$, $a^{-1} \in A^r$.

A tangent vector field of transformation (1) concept plays the most significant part in the Lie theory. The Taylor series in the nearest neighborhood of the identity transformation is the following

$$q' = q + \xi(q)\epsilon + o(\epsilon). \tag{2}$$

$\epsilon = a - a^0$—a small parameter increment. Without any restriction $a^0 = 0$, $\epsilon = a$ can be chosen. Taylor series (2) is completely defined by its two first terms. They characterize infinitely small (infinitesimal) transformation that corresponds to (1). A function

$$\xi(q) = \frac{\partial f}{\partial a}\,|_{a=0} \tag{3}$$

is called *infinitesimal* or *vector field* of transformation (1).

A derivation operator corresponding to vector field (3)

$$X = \xi(q)\frac{\partial}{\partial q} = \xi(q)\partial_q, \tag{4}$$

is the first order linear operator.

The main properties of operator X (4) are defined by the following Lie theorems.

Theorem 1. *Function $f(q,a)$ satisfying axioms 1–3, is a solution of Cauchy problem*

$$\frac{df}{da} = \xi(f) \qquad f(q,0) = q. \tag{5}$$

Equations (5) are called *Lie equations*. Theorem 1 is formulated for the simplest case of one-parametric transformation (1). As a rule it's enough for applications.

There is a useful binary operation for operators (4). It is called *commutator*

$$[X_i, X_j] \equiv X_i X_j - X_j X_i.$$

Theorem 2. *The commutator for the two operators X_i and X_j is a linear combination of operators X*

$$[X_i, X_j] = c_{ij}^k X_k, \quad (i,j,k = 1,2,\dots,r).$$

where c_{ij}^k is the structural constants tensor. It is not a function of parameters or variables.

Thus nonlinear object-r-parametric transformation (1) has a linear correspondence. That is a linear algebra formed by operators $X_1, X_2, \dots, X_r$.

Theorem 3. *Structural constants c_{ij}^k satisfy the following conditions*

$$c_{ij}^k = -c_{ji}^k; \qquad c_{ij}^l c_{lk}^m + c_{ki}^l c_{lj}^m + c_{jk}^l c_{li}^m \equiv 0.$$

A concept of invariance plays a central role in the Lie theory. According to the definition function $F(q)$ is an *invariant of Lie transformation (1)*, iff $F(q') = F(q)$ is satisfied. The condition is equivalent to $X(F) = 0$.

Differential equations in the Lie theory are considered as algebraic ones in the functional space of independent and differential variables as well as of all the derivatives that are included in the equation. The "continuation of transformation (1)" technique for the derivatives is introduced.

The rule of the first derivation of an operator

$$X = \xi(t,q)\partial_q + \eta(t,q)\partial_t,$$

(t-independent variable, $q - n$-dimensional vector of differential variables) is the following

$$X^1 = X + (\dot{\xi} - \dot{\eta}\dot{q})\partial_{\dot{q}} = X + \zeta(t,q,\dot{q})\partial_{\dot{q}}.$$

($d\xi/dt = \dot{\xi}$). Symmetry condition for an ordinary differential equation of the first order (ODE-1) $F(t,q,\dot{q}) = 0$ is the following

$$X^1 F|_{F=0} = 0.$$

The second prolongation of X is

$$X^2 = X^1 + (\dot{\zeta} - \dot{\eta}\ddot{q})\partial_{\ddot{q}} = X^1 + \theta(t,q,\dot{q},\ddot{q})\partial_{\ddot{q}}$$

and a condition of symmetry for ODE-2 $F(t,q,\dot{q},\ddot{q}) = 0$ is

$$X^2F|_{F=0} = 0.$$

For ODE-n we have

$$X^nF|_{F=0} = 0.$$

Group analysis of differential equations offers some prescriptions of symmetry utilization for reducing an order of ODE. Sophus Lie proved that ODE-n is fully integrable if it assumes solvable n-parametric Lie symmetry algebra L_n. L_n is a solvable algebra iff its derivative algebra of a certain order reduces to null [17, 18]. Cartan introduces the following condition of solvability

$$c_{il}^m c_{jm}^p c_{kp}^l = c_{im}^l c_{jp}^m c_{kl}^p.$$

The same scheme can be used for finding symmetry vector fields of partial differential equations.

Example 1. [8, p. 185], [23, p. 63]. One-dimensional heat conductivity equation

$$u_t = u_{xx} \tag{6}$$

admits Lie symmetries with operators

$$\begin{aligned}
X_1 &= \partial_t;\\
X_2 &= \partial_x;\\
X_3 &= u\partial_u;\\
X_4 &= 2t\partial_t + x\partial_x;\\
X_5 &= 2t\partial_x - xu\partial_u;\\
X_6 &= 4t^2\partial_t + 4tx\partial_x - (x^2+2t)u\partial_u.
\end{aligned}$$

Integration of Lie equations (5) for operator X_6

$$\begin{aligned}
\frac{dt'}{da} &= 4t'^2;\\
\frac{dx'}{da} &= 4t'x';\\
\frac{du'}{da} &= -(x'^2+2t')u',
\end{aligned} \tag{7}$$

with initial conditions: $a = 0$, $t' = t$; $x' = x$; $u' = u$, leads to

$$\begin{aligned}
t' &= t/(1-4at);\\
x' &= x/(1-4at);\\
u' &= u\sqrt{(1-4at)}\exp\left(ax^2/(4at-1)\right).
\end{aligned}$$

A trivial solution of the heat conductivity equation $u = c = \text{const}$ is transformed to the fundamental solution of the equation (the Green function of the Cauchy problem) when applying the transformation with X_6

$$u = \frac{c}{(1+4at)^{1/2}} \exp\left(\frac{-ax^2}{1+4at}\right)$$

if a point $t_0 = -1/4a$, $x_0 = 0$ is chosen as an initial one.

Under the fixed normalization $c = \sqrt{a/\pi}$ we have

$$u = \frac{1}{(4\pi(t-t_0))^{1/2}} \exp\left(\frac{-x^2}{4(t-t_0)}\right).$$

This example demonstrates that traditional analysis of a differential equation can be successfully replaced by its group analysis.

Canonical equations

$$\dot{q} = H_p, \qquad \dot{p} = -H_q \tag{8}$$

admit *dynamical symmetry* (DS), if the following condition is satisfied:

$$[X, \Gamma] = -\Gamma(T)\Gamma \tag{9}$$

($H_p = \partial H/\partial p$, $H_q = \partial H/\partial q$, summation symbol $i = \overline{1, n}$ is omitted).

$$[X, \Gamma] = X\Gamma - \Gamma X$$

is a commutator of a symmetry vector field

$$X = Q\partial_q + P\partial_p + T\partial_t \tag{10}$$

and of a Hamiltonian stream

$$\Gamma = H_p\partial_q - H_q\partial_p + \partial_t. \tag{11}$$

All $2n+1$ components of vector field (10) Q, P, T are supposed to be functions of extended phase space q, p, t.

Example 2. For $H = qp(1 - 2\sqrt{q/p})$ the equations (8)

$$\dot{q} = q\left(1 - \sqrt{q/p}\right), \qquad \dot{p} = p\left(3\sqrt{q/p} - 1\right).$$

admit DS (9) with operator (10)

$$X = \frac{1}{2\sqrt{q^3 p}}\left(-\frac{1}{p}\partial_q + \frac{3}{q}\partial_p + \partial_t\right)\exp t.$$

An integral of conservation $I = 1/(\sqrt{q^3 p})\exp t$ with the help of the found symmetry can be constructed.

As a rule partial cases of DS are formulated in Lagrange form [20, 19, 26].

Cartan symmetry (CS) condition is

$$\mathscr{L}_X \Theta(L) = d\varphi(q, \dot{q}, t), \tag{12}$$

where $\mathcal{L}_X$ is the Lie derivative of Cartan differential 1-form

$$\Theta = L_{\dot{q}}\,dq - (L_{\dot{q}}\dot{q} - L)\,dt \tag{13}$$

and φ is a gauge function. The Lie derivative of 1-form (13) with respect to a vector field

$$X = Q\partial_q + P\partial_p + T\partial_t \tag{14}$$

is given by

$$\mathcal{L}_X\Theta = d(X \rfloor \Theta) + X \rfloor d\Theta,$$

where Q, P, T in (14) are supposed to be functions of $q, \dot{q}, t$; $\rfloor$ denotes the contraction of vectors and forms. Contrary to (10) P's in (14) are contravariant vector field components.

When (12) is true, Euler-Lagrange equations

$$\frac{d}{dt}L_{\dot{q}} - L_q = 0 \tag{15}$$

admit *the Noether law of conservation* [5, 9, 14, 20, 19, 26]

$$I = \langle X, \Theta \rangle - \varphi \tag{16}$$

(the brackets mean natural pairing function). It means that the function I in (16) is conserved along the classical trajectories of (14), i.e. $\Gamma(I) = 0$, where Lagrangian stream has the form

$$\Gamma = \dot{q}\partial_q + \lambda\partial_{\dot{q}} + \partial_t \tag{17}$$

instead of (11); λ in (17) satisfies the Euler–Lagrange equations (15). It is supposed that the Lagrangian $L(q, \dot{q}, t)$ is regular.

Lie symmetry (LS) is the same as DS (9), but with

$$\begin{aligned} X &= Q(q,t)\partial_q + (\dot{Q} - \dot{T}\dot{q})\partial_{\dot{q}} + T(q,t)\partial_t. \\ X^{(0)} &= Q\partial_q + T\partial_t \end{aligned} \tag{18}$$

is a point symmetry vector field and X (18) is its first prolongation.

Noether symmetry (NS) is the same as CS (12) but with (18) and with a gauge function $\varphi = \varphi(q, t)$.

The question is what the operation of "prolongation" means in Hamiltonian terms? It is proved in [26] that it can be constructed in the following way. Let the Cartan symmetry condition (12) be expressed in Hamiltonian form

$$\mathcal{L}_X\Theta(H) = d\varphi(q, p, t), \tag{19}$$

where X is (10) and $\Theta(H)$ is the Cartan form

$$\Theta(H) = p\,dq - H\,dt. \tag{20}$$

Then the law of conservation (16) can be presented as

$$I = Qp - TH - \varphi \tag{21}$$

with P in (10) having the following form

$$P = -Q_q P + T_q H + \varphi_q. \tag{22}$$

When (12) has a simple form

$$\mathcal{L}_X \Theta(H) = 0 \tag{23}$$

without a gauge function φ, then (22) becomes

$$P = -Q_q p + T_q H. \tag{24}$$

The above-mentioned condition may be considered as an operation of "prolongation" as in Lagrangian case (18) takes place. In particular for the case of NS the expression (24) becomes simpler

$$P = -Q_q(q,t)p + T_q(q,t)H.$$

Example 3. It is easy to check (23) in Example 2 and find that it is not fulfilled. The supposition that $\varphi(q,p,t) \neq 0$ does not save the situation. Really the gauge function $\varphi(q,p,t)$ ought to satisfy the following conditions

$$\varphi_q = P + Q_q p - T_q H; \qquad \varphi_p = Q_p p - T_p H. \tag{25}$$

The conditions (25) are found with the help of (21), (22). Some simple calculations show that in Example 3 conditions (25) come into conflict. As gauge function $\varphi(q,p,t)$ does not exist for the considered symmetry vector field so it is true that DS and CS are not the same for Hamiltonian systems.

Hamiltonian symmetry (HS). According to the definition a Hamiltonian system possesses HS, if there is a vector field (10) for which the condition

$$XH(q,p,t) = 0 \tag{26}$$

takes place.

Example 4. Dynamics of a rigid body with fixed point problem has the partial case of Goryachev–Chaplygin [21, 26]. By Hamiltonian description in Euler coordinates the function H is [26]

$$H = \frac{1}{2}\left(p_\theta^2 + \left(4 + \operatorname{ctg}^2\theta\right)p_\varphi^2\right) - \sin\theta\sin\varphi$$

(θ, φ are the Euler angles; p_θ, p_φ are the coincident general impulses).

The well known law of conservation

$$I = p_\varphi\left(p_\theta^2 + \operatorname{ctg}^2\theta p_\varphi^2\right) + (p_\theta\cos\varphi - p_\varphi\operatorname{ctg}\theta\sin\varphi)\cos\theta$$

forms a lot of HS's with

$$X = F(\varphi, \theta, p_\varphi, p_\theta)(I_p \partial_q - I_q \partial_p).$$

Under the condition $F = 1$ the symmetries with X are both HS and DS. Is it correct for all cases?

Hamiltonian symmetry that permits to build a law of conservation $I(q,p,t) = \text{const}$ of canonical equations (8) is not a dynamical symmetry in general case (argument see [27]).

Example 5. Let's return to Example 4. If $F = 1/p_\theta$ is chosen, it is easy to find that the HS is not DS.

Example 6. We return once more to Example 2. Is DS in the example the same as HS? By calculating XH we find

$$XH = \frac{1}{\sqrt{q^3 p}} \neq 0.$$

The symmetry is not a Hamiltonian one. Our conclusion is that DS (9) and CS (12) are not the same for Hamiltonian systems.

Our next conclusion is that DS (9) and HS (26) are not the same. There are Hamiltonian symmetries that are not dynamical ones and vice versa. If DS is not found for a Hamiltonian system we have another opportunity to construct some laws of conservation by using Hamiltonian symmetries.

A well-known Emmy Noether theorem that was formulated and proved by her in 1918 [14] established a connection between symmetries and laws of conservation. In the general case this connection is not easy to realize. For example DS symmetry (9) (or in a partial case LS) vector field (10) needs to be straightened for applying. It is not a simple procedure at all. A system of partial differential equations appears

$$XI = 0; \qquad XI_1 = 1. \tag{27}$$

The system (27) has $n-1$ solutions for invariants I and one solution for I_1. The last function belongs to an invariant family. If we could change the initial variables so that functions I and I_1 take the position of some new variables then the symmetry vector field X would be straightened along the new variable I_1. In any case we need to solve the system (27) for simplifying the initial problem.

The Emmy Noether result gives us a surprising opportunity to build laws of conservation constructively without such difficulties. It shows the procedure for finding the law (16) without a necessity to solve a system of partial differential equations like (27). That is why Emmy Noether's result is so popular in applications. It should be noted that the Emmy Noether result applying to analytical mechanics with finite freedom degrees had a long and twisted history. The point is that Emmy Noether herself proved the very general theorem for not only the Cartan form (20) but for a functional

$$A(u(x)) = \int L(x, u(x), u_{,j}(x))\, d^n x.$$

$x = (x^1, x^2, \dots, x^n)$ are independent variables; $u(x) = u^1, u^2, \dots, u^N$ are functions with domain of definition $D \subset R^n$; $u_{,j} = \partial u/\partial x^j$ are their partial derivatives; L is a function (Lagrange function).

The Euler-Lagrange equations

$$d\left(\partial L/\partial u^a_{,j}\right)/dx^j - \partial L/\partial u^a = 0$$

admit laws of conservation

$$\xi^j L + \left(\eta^a - u^a_{,i}\xi^i\right)\left(\partial L/\partial u^a_{,j}\right) dx^1 \wedge \cdots \wedge dx^j \wedge \cdots dx^n = I = \text{const}$$

for infinitesimal symmetry with the operators

$$X_k = \xi^i{}_k(x)\partial/\partial x^j + \eta^a_k(x,u)\partial/\partial u^a,$$

($k = 1, \dots, R$; R-dimension of linear Lie symmetry algebra) with $A(u(x))$ as an invariant.

The partial case $N = 1$ was not selected by Emmy Noether explicitly and the assimilation of E. Noether's result to analytical mechanics of finite freedom degrees has more then a half a century of history. There is a detailed review of the history in our monograph [26] with the bibliography containing 182 titles. It should be remarked that even now "Cartan symmetry" (12) as a rule is considered to be a generalization of the so called "Noether symmetry." The latter is regarded as a partial case of NS without the gauge function φ and without the infinitesimal T (see for example [3, 10])

$$\mathcal{L}_X \Theta(L) = d\varphi(q,t); \qquad X = Q(q)\partial_q \tag{28}$$

with the law of conservation

$$I = Q^i(q)p_i, \quad i = 1, \dots, n.$$

In reality Emmy Noether's result overlaps both CS and NS and all their partial cases including (28). The overwhelming number of "generalizations" of the Noether theorem is only partial cases of above mentioned genuine Emmy Noether theorem (see, for example, [6, 11, 15, 22, 24, 25, 28]).

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q-DIMENSIONAL FORMULAS FOR THE CYCLIC POLYENE HUBBARD MODEL

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ABSTRACT. Through rather tedious algebraic manipulations involving ratios of polynomials with fractional powers the generating polynomials result from q-dimensional formulas. $U(n) \downarrow O(m,m) \downarrow C_n$ branching rules are given to account firstly for the Quasi-Spin invariance $(U(n) \downarrow O(m,m))$ and then the Spatial Symmetry invariance $(O(m,m) \downarrow C_n)$ due to C_n. The resulting expressions may be efficiently handled using the symbolic computation language MAPLE and the dimensional information for an arbitrary spin, isospin and quasimomentum obtained.

1. Introduction. It is well known today that characteristics of symmetry groups are extensively exploited in quantum mechanics. When deriving or calculating the characters of symmetric group irreducible representations (S_N irreps) one often relies on suitable generating polynomials; for example, the Frobenius Theorem of the symmetric group representation theory [7] yields directly all the simple characters of the irrep of S_N as the coefficients of the Schur polynomial. As Cayley's Theorem states, any finite group G of order N is isomorphic to a suitable subgroup of the symmetric or permutation group S_N. Furthermore, there is a close relationship between the representation theory of S_N and the representation theory of compact Lie groups. Also the unitary groups $U(n)$ play the same role for the compact Lie groups as do permutation groups for finite groups. This interrelationship is reflected in various group theoretical approaches to the N-electron correlation problem employing n-orbital models, where the close relationship

$$\begin{tikzcd} S_N \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & U(n) \arrow[d, leftrightarrow] \arrow[dl, shift right] \\ SU(2) \arrow[r, leftrightarrow] \arrow[ur, shift right] & U(2) \end{tikzcd}$$

among the respective theories is well known [22, 19, 7]. We are interested in the solution of the Schrödinger Equation

$$\mathbf{H}\Psi = \mathbf{E}\Psi,$$

where $\mathbf{H}$ is the spin-independent Hamiltonian, Ψ is the wave function and the hamiltonian operator acting on the wave function gives the same wave function multiplied by the energy eigenvalue $\mathbf{E}$. (see e.g. [14, 11, 17, 18, 20]). Ψ is expressed as a linear combination of configuration state functions (CSFs) $\psi_{\mathbf{i}}$

$$\Psi = \sum_{\mathbf{i}} \mathbf{c_i}\psi_{\mathbf{i}}.$$

The total number of $\psi_{\mathbf{i}}$, κ say, is given by Weyl's Dimension formula [14, 12, 13, 8, 11, 17, 9] and can be a huge number leading to a Hamiltonian matrix of size $\kappa \times \kappa$. (e.g., calculations of the order of $10^9 \times 10^9$ have been performed). As a special case of character theory, one often requires only the characters of the identity providing the appropriate dimensions. For example, Weyl's Dimension Formula

$$D_n(a,b,c) = \frac{b+1}{n+1}\binom{n+1}{a}\binom{n+1}{c}$$

gives the number of spin adapted configurations $\psi_{\mathbf{i}}$, characterized by the total spin quantum number S, for the n-orbital model of an N-electron system which is described by a spin-independent Hamiltonian $\mathbf{H}$. Here a, b, c, also called *Paldus* labels, label the two column $U(n)$ irrep$\langle 2^a, 1^b, 0^c\rangle$ with $a = (1/2)N - S$, the number of doubly occupied orbitals, $b = 2S$, the number of singly occupied orbitals, $c = n - (a+b)$ and $\binom{n}{m} = n!/(n-m)!m!$, the number of unoccupied orbitals, is the usual binomial coefficient. Alternatively, the dimension can be computed from the Young diagram YD, (or Weyl Tableau WT) using the "n-graph" and the "hook-graph." Considering a matrix form, the YD / WT is represented by a rows of double boxes followed by b rows of single boxes in the first column. Then, the n-graph is constructed by placing integers n along the diagonal of the WT and then placing integers which increase (decrease) in unit steps to the right (left) of the diagonal. For the hook-graph, on the other hand, each box of the WT is assigned a "hook" that consists of the box itself together with the box to its right and all boxes in the same column below it. The "hook length" is then defined as the number of boxes making such a hook and the boxes of the WT are labeled by the hook lengths. One then has the dimension given by

$$\dim_{\mathrm{WT}} = \frac{\text{product of integers on } n\text{-graph}}{\text{product of integers on hook graph}}.$$

For the $U(4)$ irrep $\langle 2^1, 1^2, 0^1\rangle$ with $n = 4 =$n, $a = 1$, $b = 2$, then

$$D_n(1,2,1) = \frac{3}{5}\binom{5}{1}\binom{5}{1} = 15,$$

$$\dim_{\mathrm{WT}} = \frac{\begin{array}{ll} 4 & 5 \\ 3 & \\ 2 & \end{array}}{\begin{array}{ll} 4 & 1 \\ 2 & \\ 1 & \end{array}} = \frac{2\cdot 3\cdot 4\cdot 5}{1\cdot 2\cdot 4\cdot 1} = 15.$$

An equivalent form of this and related dimension formulas may often be derived using simple combinatorial means, independently of group theory [15]. Even then it is often useful to employ the technique of generating polynomials. The direct exploitation of generating polynomials may prove rather demanding; however, in view of recent developments in symbolic computation, it is of interest to explore the possibilities offered by these powerful tools in handling directly the generating polynomials and extracting the desired information. Here, we shall employ MAPLE [1] to extract the dimensional information from recently obtained generating polynomials [2] for the number of symmetry adapted configurations of the Hubbard cyclic polyene model with conserved spin, quasispin, and quasi-momentum. These generating polynomials are obtained with the help of the $U(n)$-q-dimension formula [25] that originates from the theory of quantum groups. Before we give the formulation in the next section, let us begin our discussion by providing some motivation for the introduction of Unitary Groups into quantum chemistry with a simple discussion of the CI-Configuration Interaction matrix element problem. The ψ_i can be chosen in a variety of ways; however, in general they are chosen to be antisymmetric and to be eigenfunctions of S^2 and S_z. Denoting ψ_i by $|K\rangle$, $|L\rangle$ in Dirac notation, the central practical problem is the evaluation of $\langle K|H|L\rangle$, [11, 17, 18],

$$\langle K|H|L\rangle = \sum_{ij}\langle i|h|j\rangle A_{ij}^{KL} + \frac{1}{2}\sum_{ijkl}[ij \mid kl]\, B_{ijkl}^{KL},$$

where $\langle i|h\,|\,j\rangle, [ij \mid kl]$ are the usual one- and two-electron repulsion integrals in charge cloud notation; the $A_{ij}^{KL}, B_{ijkl}^{KL}$ are numerical "vector coupling" coefficients that depend on the nature of $|K\rangle, |L\rangle$. In second quantization formalism, these coupling coefficients emerge as matrix elements of creation and annihilation operators $\mathrm{X}_{i\sigma}^{+}$ and $\mathrm{X}_{j\sigma}$. The operator $\mathrm{X}_{i\sigma}^{+}$ creates an electron in orthonormal spin-orbital $|i\sigma\rangle$ where $|i\sigma\rangle = |i\rangle\,|\sigma\rangle$, and $\sigma = \alpha$ or β. Similarly, the operator $\mathrm{X}_{j\sigma}$ destroys an electron in orthonormal spin-orbital $|i\sigma\rangle$. In quantum chemistry problems in which the number of particles is conserved, the creation and annihilation operators will always occur in pairs. Then, for the spin independent Hamiltonian, we obtain

$$\langle K|H|L\rangle = \sum_{ij}\langle i \mid h \mid j\rangle \left\langle K \,\middle|\, \sum_{\sigma}\mathrm{X}_{i\sigma}^{+}\mathrm{X}_{j\sigma} \,\middle|\, L\right\rangle + \frac{1}{2}\sum_{ijkl}[ij \mid kl] \left\langle K \,\middle|\, \sum_{\sigma\gamma}\mathrm{X}_{i\sigma}^{+}\mathrm{X}_{k\gamma}^{+}\mathrm{X}_{l\gamma}\mathrm{X}_{j\sigma} \,\middle|\, L\right\rangle.$$

Since the creation and annihilation operators always occur in pairs, we define a "generator"

$$E_{j\gamma}^{i\sigma} = \mathrm{X}_{i\sigma}^{+}\mathrm{X}_{j\gamma},$$

and because of the summations over spin σ

$$E_j^i = \sum_{\sigma}\mathrm{X}_{i\sigma}^{+}\mathrm{X}_{j\sigma}.$$

The vector coupling coefficients now take the form

$$A_{ij}^{KL} = \left\langle K \,\middle|\, E_j^i \,\middle|\, L\right\rangle,$$
$$B_{ijkl}^{KL} = \left\langle K \,\middle|\, E_j^i E_l^k - \delta_{jk}E_l^i \,\middle|\, L\right\rangle.$$

Furthermore, the B_{ijkl}^{KL} can be written as

$$\left\langle K \left| E_j^i E_l^k - \delta_{jk} E_l^i \right| L \right\rangle = \sum_M \left\{ \left\langle K \left| E_j^i \right| M \right\rangle \left\langle M \left| E_l^k \right| L \right\rangle \right\} - \delta_{jk} \left\langle K \left| E_l^i \right| L \right\rangle,$$

where the summation over M must include the complete range of the E_j^i. Thus $\langle K| E_j^i |L\rangle$ is all that is required for CI matrix element evaluation [11]. To summarize, the utility of the Unitary Groups stems from the remarkable result that the Hamiltonian becomes expressible as a linear and bilinear form in the generators of $U(n)$

$$H = \sum_{ij} \langle i| h \mid j\rangle E_j^i + \frac{1}{2} \sum_{ijkl} [ij \mid kl] \left(E_j^i E_l^k - \delta_{jk} E_l^i \right),$$

and the computation of A_{ij}^{KL}, B_{ijkl}^{KL} becomes an exercise in the theory of Unitary Groups. The E_j^i, E_l^k satisfy a commutation relation

$$\left[E_j^i, E_l^k \right] = E_j^i E_l^k - E_l^k E_j^i = \delta_{jk} E_l^i - \delta_{li} E_j^k$$

and form the Lie Algebra of $U(n)$. The $\{|K\rangle\}$ must also carry a representation and thus the generators $\{\langle K| E_j^i |L\rangle\}$ will also satisfy the above commutation relation. Hamiltonian matrix is Hermitian; i.e. totally symmetric about the diagonal and thus it suffices to calculate upper/or lower diagonal elements only along with the diagonals [11]. The basic theory of symmetric and linear (or unitary) groups is perhaps best documented in the classic books by Weyl [22], Robinson [19], or Hammermesh [7] and the reader is referred to those texts for details.

2. q-dimensional formulas for the cyclic polyene hubbard model. The Hubbard Hamiltonian [6, 23, 24, 16, 3, 4, 5] for the cyclic chain with n equidistant and equivalent sites, satisfying the Born-von-Kármán cyclic boundary conditions, is as

$$H^{\mathbf{H}} = \mathsf{U} \sum_{i=1}^{n} \left(n_{i,1} - \frac{1}{2} \right) \left(n_{i,-1} - \frac{1}{2} \right) - \sum_{i=1}^{n} \left(E_{i+1}^i + E_i^{i+1} \right),$$

where U is the one-center on-site Coulomb integral; associated with the i-th site spin orbital with azimuthal spin σ, $\sigma = \pm 1$, the $n_{i,\sigma}$ are the corresponding spin orbital occupation number operator

$$n_{i,\sigma} = X_{i,\sigma}^+ X_{i,\sigma};$$

and we have employed the $U(n)$ generators. The total electron number operator is

$$\hat{N} = \sum_{i=1}^{n} n_i, \quad n_i = n_{i,1} + n_{i,-1};$$

and the spin-independent Hubbard Hamiltonian, entirely in terms of $U(n)$-generators, is thus

$$H^{\mathbf{H}} = \frac{1}{2} \mathsf{U} \sum_{i=1}^{n} \left(E_i^i \right)^2 - \sum_{i=1}^{n} \left(E_{i+1}^i + E_i^{i+1} \right) + \frac{1}{2} \mathsf{U} \left(\frac{1}{2} n - 2\hat{N} \right).$$

We can thus restrict ourselves to a fixed irrep of $U(n)$. To reflect the cyclic boundary conditions all the indices are taken modulo n, $n+1 \equiv 1(\text{mod } n)$. $H^{\mathbf{H}}$ commutes with the $SU(2)$ operators

$$\begin{aligned}
S_z &= \frac{1}{2}(\epsilon_{1,1} - \epsilon_{-1,-1}), \\
S_+ &= \epsilon_{1,-1}, \\
S_- &= \epsilon_{-1,1}, \\
S^2 &= [S_z(S_z+1)] + [(S_-)(S_+)]
\end{aligned}$$

with the $SU(2)$ generators

$$\epsilon_{\sigma,\gamma} = \sum_{i=1}^{n} X_{i,\sigma}^{+} X_{i,\gamma}, \quad {\scriptstyle (\sigma,\gamma=\pm 1)}.$$

The electron number operator $\hat{N}$ represents the first order invariant of both $U(n)$ and $U(2)$,

$$\hat{N} = \sum_{i=1}^{n} E_i^i = \sum_{\sigma=\pm 1} \epsilon_{\sigma,\sigma}.$$

$H^{\mathbf{H}}$ with its apparent $SU(2)$ invariance also possesses a quasi-spin $SU(2)$ invariance; the quasi-spin operator Q is defined with its components

$$\begin{aligned}
Q_z &= \frac{1}{2}\left(\hat{N} - n\right), \\
Q_+ &= \sum_{i=1}^{n} (-1)^i X_{i,-1}^{+} X_{i,1}^{+}, \\
Q_- &= Q_+^{\dagger}.
\end{aligned}$$

We are considering lattices with even number of sites; defining

$$n = 2m,$$

the pseudo-orthogonal group $O(m,m)$ is introduced so as to account for this quasi-spin invariance; the infinitesimal generators are

$$\alpha_{ij} = (-1)^i E_j^i - (-1)^j E_i^j = -\alpha_{ji},$$

satisfying the commutation relations

$$[\alpha_{ij}, \alpha_{kl}] = g_{jk}\alpha_{il} + g_{il}\alpha_{jk} - g_{ik}\alpha_{jl} - g_{jl}\alpha_{ik},$$

with the orthogonal group metric

$$g_{ij} = (-1)^i \delta_{ij};$$

and the Hermiticity condition is

$$\alpha_{ij}^{\dagger} = (-1)^{i+j}\alpha_{ji} = (-1)^{i+j+1}\alpha_{ij}.$$

With the $O(m,m)$ as well, at most two column irreps occur [6]; designating the $O(m, m)$ irreps with $\langle a_0, b_0, c_0\rangle, a_0 + b_0 + c_0 = m$, the dimension of the $O(m,m)$ irrep is given by

$$D^0_m(a_0,b_0,c_0) = \frac{(b_0+1)(n-2a_0-b_0+1)}{(n+1)(n+2)} \binom{n+2}{a_0} \binom{n+2}{a_0+b_0+1}.$$

With the above background, considering firstly the $U(n) \downarrow O(m,m)$ branching rules, the $U(n)$ irrep$\langle 2^a, 1^b, 0^c \rangle$, also denoted as $\Gamma(a,b,c)$, decomposes as

$$\Gamma(a,b,c,) = \bigoplus_{a_0=0}^{a\wedge c} \Gamma^0(a_0,b_0,c_0),$$

with

$$b_0 = b \wedge (n-2a_0-b),$$

$$x \wedge y = \min\{x,y\},$$

and $\Gamma^0(a_0,b_0,c_0)$ is the $O(m,m) irrep$ with the highest weight (lexicographic labeling). All the states in $U(n)$ irrep$\Gamma(a,b,c)$ have the azimuthal quasi-spin quantum number

$$Q_z = \frac{1}{2}(N-n),$$
$$N = 2a+b,$$

and the $O(m,m)$ irrep$\Gamma^0(a_0,b_0,c_0)$ is characterized by the quasi-spin

$$Q = \frac{1}{2} \mid n-N \mid +c\wedge a - a_0 = \frac{1}{2}(n-b) - a_0.$$

The allowed values of quasi-spin are thus

$$\frac{1}{2} \mid n-N \mid \leq Q \leq \frac{1}{2}(n-b).$$

Secondly we consider the $O(m,m) \downarrow C_n$ branching rules, so as to account for the spatial symmetry invariance characterized by the cyclic group C_n. C_n irreps are labeled by the quasi-momentum quantum number k. The q-character formalism [25] provides the generating polynomials [21] yielding the desired dimensional information or multiplicities of the states characterized by quantum numbers $\{N,S,Q,k\}$ and n. To summarize, with a,b,c the $U(n)$ irrep$\Gamma(a,b,c)$, with

$$a_0 = \frac{1}{2} \mid n-N \mid +c\wedge a - Q,$$
$$b_0 = b \wedge [n-2(c\wedge a)-b],$$
$$c_0 = m-(a_0+b_0),$$

the $O(m,m)$ irrep $\Gamma^0(a_0,b_0,c_0)$, and with the quasi-momentum $k, 0 \leq k < n$, the relevant C_n irrep are defined. The generating polynomial $F_{(a_0,b_0,c_0)}(q)$ [21] is given by

$$F_{(a_0,b_0,c_0)}(q) = q^\rho D^0_q[a_0,b_0],$$

with

$$\rho = \frac{1}{2}(n+1)N - (m+1)(a-a_0),$$

and the $O(m,m)$ q-dimension

$$D_q^0[a_0,b_0] = \frac{[b_0+1]}{[n+1]}\begin{bmatrix} n+2 \\ a_0 \end{bmatrix}\begin{bmatrix} n+2 \\ a_0+b_0+1 \end{bmatrix} \times\left(\frac{[n+2-a_0][n+1-a_0-b_0]-q^{-m-1}[a_0+b_0+1][a_0]}{[n+2]^2}\right),$$

where

$$[v] = q^{(1/2)v} - q^{(-1/2)v}$$

designates the polynomials in q and the binomial coefficients are defined in analogy to their standard meaning by

$$\begin{bmatrix} v \\ w \end{bmatrix} = \frac{[v]!}{[v-w]![w]!}, \qquad 0 \le w \le v$$

$$[v]! = \begin{cases} [v]\cdot[v-1]\cdot\cdots\cdot[1], v \ge 1 \\ 1, v = 0. \end{cases}$$

The generating polynomial, once obtained, is transformed to the standard form

$$F_{(a_0,b_0,c_0)}(q) = \sum_{k=1}^{n-1} m_k^0(a_0,b_0)q^k,$$

by reducing its exponent modulo n. The coefficients $m_k^0(a_0,b_0)$ are the desired dimensions or C_n-multiplicities.

3. MAPLE implementation of the q-dimensional formalism. Considering the simplest cyclic polyene with non-degenerate ground state, the π-*electron* model of benzene, will summarize the algebra involved in the construction of generating polynomials [23, 24, 16], yielding the desired C_n-multiplicities. For this model, $n = 6 = N$, $m = 3$, all states have azimuthal quasi-spin $Q_z = 0$, and the possible spin quantum numbers are $S = 0,1,2$, and 3. Consequently, the $U(n)$ irreps involved are

$$(a,b,c) = \begin{cases} (3,0,3) \\ (2,2,2) \\ (1,4,1) \\ (0,6,0) \end{cases}.$$

Considering the first two cases that correspond to the most important cases, (with an indication of the quasi-spin character of the $O(m,m)$ irrep), i.e. firstly the singlet configurations,

$$\begin{aligned}\Gamma(3,0,3) &= \bigoplus_{a_0=0}^{3} \Gamma_Q^0(a_0,b_0,c_0)\\ &= \Gamma_3^0(0,0,3) \bigoplus \Gamma_2^0(1,0,2) \bigoplus \Gamma_1^0(2,0,1) \bigoplus \Gamma_0^0(3,0,0),\end{aligned}$$

and then the triplet configurations,

$$\begin{aligned}\Gamma(2,2,2) &= \bigoplus_{a_0=0}^{2} \Gamma_Q^0(a_0,b_0,c_0)\\ &= \Gamma_2^0(0,2,1) \bigoplus \Gamma_1^0(1,2,0) \bigoplus \Gamma_0^0(2,0,1).\end{aligned}$$

The generating polynomial for, e.g. , $\Gamma_1^0(2,0,1)$ is then

$$F_{(2,0,1)}(q) = q^{17} D_q^0[2,0],$$

where

$$\begin{aligned}D_q^0[2,0] &= \frac{[1]}{[7]} \begin{bmatrix} 8 \\ 2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} \frac{[6][5] - q^{-4}[3][2]}{[8]^2}\\ &= \frac{[7][6]([6][5] - q^{-4}[3][2])}{[3][2]^2[1]}.\end{aligned}$$

With the decomposition

$$\begin{aligned}[r] &\equiv [2\rho + 1]\\ &= q^{r/2} - q^{-r/2}\\ &= [1]\left(q^{\rho} + q^{\rho-1} + \cdots + q + 1 + q^{-1} + \cdots + q^{-\rho}\right),\end{aligned}$$

with $\rho = 1,2,3$, and

$$\left(q^2 + 1 + q^{-2}\right)\left(q^2 + q + 1 + q^{-1} + q^{-2}\right) = \left(q + 1 + q^{-1}\right)\left(q^3 + q + 1 + q^{-1} + q^{-3}\right),$$

$$\begin{aligned}D_q^0[2,0] = \left(q^3 + q^2 + q + 1 + q^{-1} + q^{-2} + q^{-3}\right)\left(q^2 + 1 + q^{-2}\right)\\ \times\left(q^3 + q + 1 + q^{-1} + q^{-3} - q^{-4}\right).\end{aligned}$$

Finally, with mod 6

$$D_q^0[2,0] = 16 + 12q + 16q^2 + 12q^3 + 16q^4 + 12q^5,$$

gives the C_n-multiplicities

$$m_k^0(2,0) = \{16,12,16,12,16,12\} \quad \text{for } k = 0,1,\ldots,5, \text{ in order.}$$

It is observed that the D_{nh} symmetry of the model as well as the particle-hole symmetry for the half-filled shell case are reflected in the results and $m_k^0(a_0,b_0) = m_{n-k}^0(a_0,b_0)$. We can now talk about the exploitation of MAPLE for the computations; the bottleneck of the calculation is the factorization of the $O(m,m)$-q-dimension followed by the expansion and normalization of the resulting generating polynomial. Denoting the $O(m,m)$-q-dimension by Dq, ρ by qhat, the generating polynomial $F_{(a_0,b_0,c_0)}$ by Dq 3, the required code section is as follows:

```
≈
Dq2:= factor(Dq);
Dq3:= q^ qhat*Dq2;
Dq4:= expand(Dq3);
≈
reducer:= proc(t,x,mm)     local d;
    d:=degree(t,x);
    subs(x^ d=x^ (d mod mm),t);
end;
≈
res:=map(reducer,Dq4,q,n);
resfct:=sort(res);
Fq[ic,jc]:=resfct;
≈
```

clearly, following the factorization, the resulting generating polynomial Dq 3 is expanded and reduced to the standard form with the help of a procedure called "reducer." Except for this procedure, all the other operations (factor, expand, map and sort) are standard functions in MAPLE. The computed C_n-multiplicities for all the singlet and triplet states of the cyclic polyene with 6 sites are summarized below: letting quasi-momentum k label vertices of the benzene molecule

$$\begin{tikzcd} & 0 \arrow[dr] & \\ 5 \arrow[ur] & & 1 \arrow[d] \\ 4 \arrow[u] & & 2 \arrow[dl] \\ & 3 \arrow[ul] & \end{tikzcd}$$

we have

(a) for the singlets:

k	(a_0,b_0)	$\longrightarrow$	$(3,0)$	$(2,0)$	$(1,0)$	$((0,0)$	
$\downarrow$	Q	$\longrightarrow$	0	1	2	3	
			–	–	–	–	
0			16	16	4	0	
1, 5			8	12	3	0	
2, 4			14	16	4	0	
3			10	12	2	1	
		$\sum \hookleftarrow$					
		$\dim\Gamma(a_0,b_0)$	70	84	20	1	$\to \sum = 175 = \dim\Gamma(a,b,c)$

(b) for the triplets:

k	(a_0, b_0)	$\longrightarrow$	$(2,0)$	$(1,2)$	$(0,2)$			
$\downarrow$	Q	$\longrightarrow$	0	1	2			
			–	–	–			
0			12	16	2			
1,5			16	14	3			
2,4			12	16	2			
3			16	14	3			
		$\sum \hookleftarrow$						
			84	90	15	$\longrightarrow$	$\sum =$	$\dim(a,b,c) = 189$

The timing data of some quasi-spin MAPLE implementations (in seconds and on four processor Silicon Graphics, Challenge L) are listed below so as to give some idea of the computing time; factorization scheme-timing data are distinguished by giving them in parentheses

$n = N$ $\rightarrow$	4	6	10	14	18	22
Singlet + Triplet	2.55 (0.95)	5.32 (2.40)	12.08 (7.55)	28.26 (20.62)	51.53 (40.15)	110.62 (93.65)

4. Discussion. Symbolic manipulation language MAPLE proves an efficient handling of rather complex algebraic expressions. This result is an implication of the usefulness of any such languages. Although the computational time increases appreciably with the increasing polyenic size, a good deal of useful information can be extracted by directly exploiting various generating functions. Symbolic computation can also be exploited in other group theoretical problems of quantum chemical calculations; the relationship with Gaussian Polynomial based combinatorial approaches is addressed in Taneri and Paldus's paper [21].

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HAMILTONIANS OF THE CALOGERO-SUTHERLAND TYPE MODELS ASSOCIATED TO THE ROOT SYSTEMS AND CORRESPONDING FOCK SPACES

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The object of this paper is to construct the principal operators for the Calogero-Sutherland type models associated to arbitrary root system and to use them to generate the analogues of the quantum mechanical Fock spaces and the Heisenberg-Weyl algebras. First, a short review of the classical theory of the Calogero-Sutherland models corresponding to the root system A_n, B_n, C_n is given (for detailed exposition see S. Kakei [1, 2]). Then using some constructions from Lie algebra theory we give a generalization of these models to the case of arbitrary root systems. We construct the generalizations of the momentum, Laplace and Dunkl operators, and establish the commutation relations between these operators. Further, generalizations of Fock spaces and Heisenberg-Weyl algebras will be given. In conclusion a conjecture concerning isomorphisms of these algebras and spaces for Calogero and Sutherland models will be given.

Let us consider two one-dimensional quantum integrable models:

(a) The Calogero rational model of a harmonic oscillator on the line with Hamiltonian

$$H_C = \frac{1}{2}\sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + x_j^2\right) + \sum_{j<k}\frac{\beta(\beta-1)}{(x_j - x_k)^2};$$

(b) The Sutherland trigonometric model of a harmonic oscillator on the circle (θ_j are angles of particles on the circle)

$$H_S = -\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta_i^2} + \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}\frac{\omega^2\beta(\beta-1)}{\sin[\omega(\theta_i-\theta_j)/2]}.$$

Integrability of these models signifies that there exists a family of commuting differential operators (so called conserved quantities in physics) which the Hamiltonian belong to. Let

$$x_j = e^{i\theta_j}, \quad j = 1,\dots,n.$$

be the change of variables in H_S. Then we have

$$H_S = \sum_{i=1}^{n}(x_j\partial_j)^2 - \beta(\beta-1)\sum_{j<k}\frac{2x_jx_k}{(x_j-x_k)^2}.$$

Let s_{ij}, $1 \le i < j \le n$, be the elements of the symmetric group S_n. The action of the operator s_{ij} on the function $f(x_1,\dots,x_n)$ is given by permutation of the variables x_i and x_j.

Denote by ∇_j an operator of the form

$$\nabla_j = \frac{\partial}{\partial x_j} - \beta \sum_{\substack{k \\ k\neq j}} \frac{s_{jk}-1}{x_j - x_k}.$$

It is called the Dunkl operator of rational type.

Let us define the algebra $\mathfrak{U}_S$ (so called Heisenberg-Weyl algebra) generated by elements ∇_j, x_j, s_{ij}, $i,j = 1,\dots,n$. These elements satisfy the following commutation relations:

(1) $[\nabla_i, \nabla_j] = 0, \quad i,j = 1,2,\dots,n,$
(2) $s_{ij}\nabla_j = \nabla_i s_{ij},$
(3) $s_{ij}\nabla_k = \nabla_k s_{ij}, k \neq i,j,$
(4) $[\nabla_i, x_j] = \delta_{ij}(1+\beta\sum_{k\neq i} s_{ik}) - (1-\delta_{ij})\beta s_{ij}.$

The $\mathfrak{U}_S$-module generated by the vacuum vector v_0=1 is called the Fock space and will be denoted by $\mathcal{F}_S$. The operator ∇_j annihilates v_0 and the s_{ij} conserve it.

We also introduce the operators ∇^-, ∇^+

$$\nabla_j^- = \frac{1}{\sqrt{2}}\left(-\nabla_j + x_j\right),$$
$$\nabla_j^+ = \frac{1}{\sqrt{2}}\left(\nabla_j + x_j\right).$$

Define the algebra $\mathfrak{U}_C$ generated by the elements ∇^-, ∇^+ and s_{ij} (the so-called Heisenberg-Weyl algebra). They satisfy the following commutation relations

(1) $[\nabla_i^\varepsilon, \nabla_i^\varepsilon] = 0,\ \varepsilon = \pm,\ i,j = 1,\dots,n,$
(2) $s_{ij}\nabla_j^\varepsilon = \nabla_i^\varepsilon s_{ij};\ s_{ij}\cdot\nabla_k^\varepsilon = \nabla_k^\varepsilon \cdot s_{ij},\ \varepsilon = \pm 1,$
(3) $[\nabla_i^\varepsilon, x_j] = \frac{\varepsilon}{\sqrt{2}}[\delta_{ij}(1+\beta\sum_{k\neq j} s_{jk}) - (1-\delta_{ij})\beta s_{ij}],$
(4) $[\nabla_i^-, \nabla_i^+] = [\nabla_i, x_j].$

We denote by $\mathcal{F}_C$ the $\mathfrak{U}_C$-module (Calogero module) generated by the vacuum vector $v_0 = e^{-\sum_1^n x_j^2}/2.$

Define a homomorphism $\rho : \mathfrak{U}_S \to \mathfrak{U}_C$ by the following rules

$$\rho\left(\nabla_j\right) = \nabla_j^+, \qquad \rho\left(x_j\right) = \nabla_j^-, \qquad \rho\left(s_{ij}\right) = s_{ij}.$$

ρ is an isomomorphism. The isomorphism $\rho : \mathcal{F}_S \to \mathcal{F}_C$ can be defined similarly. The set of commuting operators for Calogero model is generated by the set of coefficients of powers of variables u^k in the polynomial

$$\Delta(u) = \sum_j \left(u + \hat{h}_j\right),$$

where

$$\hat{h}_j = \nabla^+\nabla^- + \beta \sum_{k<j} s_{ik}.$$

1. Laplacians and other operators associated to the root system. Now we extend the theory described above to the case of arbitrary root system. For this purpose we define some set of operators, called generalized operators of coordinates, momenta and Laplacians.

Let V be a finite-dimensional vector space, R a root system in V (see App. 1), and W its Weyl group.

Let us define the map

$$V \to \mathbb{C}^N$$

given by the formulas

$$x = (x_1, \dots, x_n) \to u = (u_\alpha, u_\alpha, \dots), \quad u_\alpha = F(\alpha, x), \dots .$$

Define the action of the Weyl group $W(R)$ in $\mathbb{C}^N$ by the rule

$$wu_\alpha = u_{w\alpha}, \quad \alpha \in R, w \in W(R).$$

Define the action of $W(R)$ on the space of complex-valued functions on $\mathbb{C}^N$

$${}^w f(u) = f(wu) = f(u_{w\alpha}), \quad \alpha \in R.$$

For an operator of generalized coordinates

$$L_\gamma(u) = \sum_{\alpha \in R} F(\gamma, \beta) u_\alpha$$

we have

$${}^w L_\gamma(u) = L_{w\gamma}(u).$$

Indeed,

$$\begin{aligned} {}^w L_\gamma(u) &= \sum_{\alpha \in R} F(\gamma, \alpha) u_{w\alpha} \\ &= \sum_{\alpha' \in R} F(\gamma, w^{-1}\alpha') u_{a'} \\ &= \sum_{\alpha' \in R} F(w\gamma, \alpha') u_{\alpha'} = L_{w\gamma}(u). \end{aligned}$$

We now introduce the Laplacians and define some maps for the rational, trigonometric and elliptic cases.

Denote $\partial_\alpha = \partial / \partial u_\alpha$. Consider the differential operator of the first order (momentum operator)

$$D_\gamma = \sum_{\alpha \in R} F(\gamma, \alpha) \partial_\alpha.$$

Using the Lemma of App. 1 and commutativity of ∂_α and ∂_β for the differential operator of second order

$$\Delta_1 = \sum_{\alpha, \beta \in R} F(\alpha, \beta) \partial_\alpha \partial_\beta,$$

we obtain

$$\hat{\Delta}_1 = \sum_{\gamma \in R} D_\gamma^2.$$

This operator is naturally called the Laplacian for the rational case.

Similarly for the trigonometric case we have the momentum operator

$$\tilde{D}_\gamma = \sum_{\alpha \in R} F(\gamma, \alpha) u_\alpha \partial_\alpha.$$

and for the differential operator of the second order $\hat{\Delta}_2$ we obtain

$$\hat{\Delta}_2 = \sum_{\alpha, \beta \in R} F(\alpha, \beta)(u_\alpha \partial_\alpha)(u_\beta \partial_\beta) = \sum_{\gamma \in R} \tilde{D}_\gamma^2.$$

This operator is naturally called the Laplacian for the trigonometric case.

For the elliptic case we have the momentum operator

$$\widetilde{\tilde{D}_\gamma} = \sum_{\alpha \in R} F(\gamma, \beta) u_\alpha u_{-\alpha} \partial_\alpha,$$

and the Laplacian

$$\hat{\Delta}_3 = \sum_{\alpha, \beta \in R} F(\alpha, \beta)(u_\alpha u_{-\alpha} \partial_\alpha)(u_\beta u_{-\beta} \partial_\beta) = \sum_{\gamma \in R} \widetilde{\tilde{D}_\gamma}^2.$$

The following assertion is an easy consequence of the $W(R)$-invariancy of $F(\alpha, \beta)$ and the equation

$$w \circ \partial_\gamma = \partial_{w\gamma} \circ w.$$

Proposition 1. *The families of operators $\partial_\gamma, D_\gamma$ and $\tilde{D}_\gamma$ are equivariant with respect to the action of the Weyl group $W(R)$, that is*

$$w \partial_\gamma = \partial_{w\gamma} w,$$
$$w D_\gamma = D_{w\gamma} w,$$
$$w \tilde{D}_\gamma = \tilde{D}_{w\gamma} w,$$

and the operators Δ_1 and Δ_2 are invariant with respect to $W(R)$

$$w \circ \Delta_1 = \Delta_1 \circ w,$$
$$w \circ \Delta_2 = \Delta_2 \circ w.$$

Proof. The proof of the first group of equations follows easily from the fact that $F(\alpha, \beta)$ is $W(R)$-invariant. It is sufficient to do the proof of the second group of equations for the generators s_α, $\alpha \in R$ of $W(R)$. Indeed, if $s_\alpha \gamma = \delta$, then $s_\alpha \delta = \gamma$ (since

$s_\alpha^2 = 1$), and we have $s_\alpha \circ \partial_\gamma = \partial_\delta \circ s_\alpha$.

$$\begin{aligned}
w \circ \Delta_1 &= w \circ \sum_{\alpha,\beta\in R} F(\alpha,\beta)\partial_\alpha\partial_\beta \\
&= \sum_{\alpha,\beta\in R} F(\alpha,\beta) w \circ \partial_\alpha\partial_\beta \\
&= \sum_{\alpha,\beta\in R} F(\alpha,\beta)\partial_{w\alpha} \circ w \circ \partial_\beta \\
&= \sum_{\alpha,\beta\in R} F(\alpha,\beta)\partial_{w\alpha} \circ \partial_{w\beta} \circ w \\
&= \left(\sum_{\alpha',\beta'\in R} F(\alpha',\beta')\partial_{\alpha'}\partial_{\beta'} \right) \circ w \\
&= \Delta_1 \circ w.
\end{aligned}$$

□

Let $h : M \to N$ be the map of smooth manifolds and f_M and F_N the spaces of functions on M and N respectively. Let further D be some differential operator on F_N.

Definition. The differential operator on F_M, that makes the following diagram

$$\begin{tikzcd}
F_M \arrow[d, "h^*D"'] & F_N \arrow[l, "h^*"'] \arrow[d, "D"] \\
F_M & F_N \arrow[l, "h^*"']
\end{tikzcd}$$

commutative is called the inverse image h^*D of the differential operator D.

Define the following maps from $V = V \otimes \mathbb{C}$ to $\mathbb{C}^N$

$$\begin{aligned}
U : V \to \mathbb{C}^N, &\qquad x \mapsto U(x) = \{u_\alpha(x) = F(\alpha, x), \alpha \in R\}, \\
E : V \to \mathbb{C}^N, &\qquad x \mapsto E(x) = \{u_\alpha(x) = \exp F(\alpha, x), \alpha \in R\}.
\end{aligned}$$

Proposition 2. *The Laplace operator $\Delta = \sum_{i=1}^n \partial_i^2$ on V is the inverse image of the operators Δ_1 and Δ_2 by the maps U and E respectively.*

Proof. We have

$$\begin{aligned}
\partial_i f(U(x)) &= \sum_{\alpha\in R} \partial_\alpha f \frac{\partial U_\alpha(x)}{\partial x_i} \\
&= \sum_{\alpha\in R} F(\alpha, e_i)\partial_\alpha f, \\
\partial_i^2 f(U(x)) &= \partial_i \left(\sum_{\alpha\in R} F(\alpha, e_i)\partial_\alpha f \right) \\
&= \sum_{\alpha\in R} F(\alpha, e_i) \sum_{\beta\in R} F(\beta, e_i)\partial_\alpha\partial_\beta f \\
&= \sum_{\alpha,\beta\in R} F(\alpha, e_i)F(\beta, e_i)\partial_\alpha\partial_\beta f(U(x)),
\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{n} \partial_i^2 f(U(x)) &= \sum_{\alpha,\beta\in R}\sum_{i=1}^{n} F(\alpha,e_i)F(\beta,e_i)\partial_\alpha\partial_\beta f(U(x))\\ &= \sum_{\alpha,\beta\in R} F(\alpha,\beta)\partial_\alpha\partial_\beta f(U(x))\\ &= \Delta_1 f(U(x)).\end{aligned}$$

Similarly the equation $\Delta = E^*\Delta_2$ can be proved.

Note: use the equations

$$\begin{aligned}\frac{\partial U_\alpha(x)}{\partial x_i} &= \frac{\partial \exp F(\alpha,x)}{\partial x_i}\\ &= F(\alpha,e_i)\exp F(\alpha,x)\\ &= F(\alpha,e_i)u_\alpha(x).\end{aligned}$$

□

2. Universal Dunkl operators and Hamiltonians

The Bethe-Dunkl varieties. Let $\alpha \to k_\alpha, \alpha \in R$, be the $W(R)$-invariant function on R and let A_γ and B_γ, $\gamma \in R$, be the operators of the form

$$A_\gamma = \sum_{\alpha\in R_+} \frac{F(\gamma,\alpha)k_\alpha}{u_\alpha - u_{-\alpha}} s_\alpha, \qquad B_\gamma = \sum_{\alpha\in R_+} F(\gamma,\alpha)k_\alpha \frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}} s_\alpha.$$

These operators generate equivariant families of operators, that is for A_γ we have

$$w \circ A_\gamma = A_{w\gamma} \circ w.$$

Indeed,

$$\begin{aligned} wA_\gamma &= w \circ \sum_{\alpha\in R_+} \frac{F(\gamma,\alpha)k_\alpha}{u_\alpha - u_{-\alpha}} s_\alpha\\ &= \sum_{\alpha\in R_+} \left(\frac{F(\gamma,\alpha)k_\alpha}{u_{w\alpha} - u_{-w\alpha}} s_{w\alpha}\right) w\\ &= \sum_{\alpha'\in R_+} \left(\frac{F(\gamma,w^{-1}\alpha')k_{w^{-1}\alpha'}}{u_{\alpha'} - u_{-\alpha'}} s_{\alpha'}\right) w\\ &= \sum_{\alpha'\in R_+} \left(\frac{F(w\gamma,\alpha')k_{\alpha'}}{u_{\alpha'} - u_{-\alpha'}} s_{\alpha'}\right) w\\ &= A_{w\gamma} w.\end{aligned}$$

The case of B_γ is considered similarly.

Introduce the "universal" Dunkl operators

$$\begin{aligned}\nabla_\gamma &= -D_\gamma + A_\gamma,\\ \tilde{\nabla}_\gamma &= -\tilde{D}_\gamma + B_\gamma,\\ \nabla_\gamma^\pm &= \nabla_\gamma \pm L_\gamma(u)\end{aligned}$$

for $\gamma \in R$. Each of the families of introduced operators is equivariant.

Proposition 3. *The following commutation relations hold*

$$[\nabla_\gamma,\nabla_\delta] = \sum_{w\in W(R)}\left\{\sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta=w}} k_\alpha k_\beta \frac{(F(\gamma,\alpha)F(\delta,\beta)-F(\gamma,\beta)F(\delta,\alpha))}{(u_\alpha-u_{-\alpha})\,(u_\beta-u_{-\beta})}\right\}w,$$

$$[\tilde\nabla_\gamma,\tilde\nabla_\delta] = \sum_{w\in W(R)}\left\{\sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta=w}} k_\alpha k_\beta (F(\gamma,\alpha)F(\delta,\beta)-F(\gamma,\beta)F(\delta,\alpha)) \cdot\frac{u_\alpha+u_{-\alpha}}{u_\alpha-u_{-\alpha}}\cdot\frac{u_\beta+u_{-\beta}}{u_\beta-u_{-\beta}}\right\}w,$$

$$[D_\gamma,L_\delta] = F(\gamma,\delta),$$

$$[\nabla_\gamma,L_\delta] = -F(\gamma,\delta)-2\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha L_\alpha(u)s_\alpha}{F(\alpha,\alpha)(u_\alpha-u_{-\alpha})},$$

$$[L_\gamma\nabla_\gamma,L_\delta\nabla_\delta] = -K(\gamma,\delta)\Big(L_\gamma\nabla_\delta-L_\delta\nabla_\gamma\Big) -4\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)L_\alpha^2(u)s_\alpha}{F^2(\alpha,\alpha)(u_\alpha-u_{-\alpha})}\Big(F(\gamma,\alpha)\nabla_\delta-F(\delta,\alpha)\nabla_\gamma\Big),$$

where

$$K(\gamma,\delta) = F(\gamma,\delta)+2\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha L_\alpha(u)s_\alpha}{F(\alpha,\alpha)(u_\alpha-u_{-\alpha})}.$$

In the equations above the terms related to $w\in W(R)$ that cannot be represented in the form of a product $s_\alpha s_\beta$ are assumed to be equal to zero. A proof of the first commutative relation is given in App.2. The other relations are proved similarly.

We now give some definitions.

Definition. Let $\gamma,\delta\in R$ and $\alpha,\beta\in R_+$ are such that $s_\alpha s_\beta = w\in W$. Then for all $w\in W$ we can define algebraic varieties by the equations

$$M_D(R) = \left\{(u_\alpha\in C^N \mid \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta\\ s_\alpha s_\beta=w}} k_\alpha k_\beta\left\{\frac{F(\gamma,\alpha)F(\delta,\beta)-F(\gamma,\beta)F(\delta,\alpha)}{(u_\alpha-u_{-\alpha})(u_\beta-u_{-\beta})}\right\}=0\right\}.$$

This variety will be called the Dunkl variety for Calogero model.

Proposition 3 for the operator ∇ gives the following

Proposition 4. *On the Dunkl variety we have*

$$[\nabla_\gamma,\nabla_\delta] = 0.$$

We now introduce the "universal" Hamiltonians of Calogero-Sutherland type

$$H_C = -\Delta_1+\sum_{\alpha\in R_+}\frac{F(\alpha,\alpha)({k_\alpha}^2-2k_\alpha s_\alpha)}{(u_\alpha-u_{-\alpha})^2},$$

$$H_S = -\Delta_2+\sum_{\alpha\in R_+}\frac{F(\alpha,\alpha)4u_\alpha u_{-\alpha}({k_\alpha}^2-k_\alpha s_\alpha)}{(u_\alpha-u_{-\alpha})^2},$$

$$H_C^h = H_C+Q(u).$$

It is easy to verify that these Hamiltonians are W-invariant, i.e.

$$wH_C = H_C w, \qquad wH_C^h = H_C^h w \quad \forall w \in W(R).$$

Proposition 5. *The following equations hold*

$$\sum_{\gamma\in R} \nabla_\gamma^2 = -H_C - \sum_{w\in W(R)} \left\{ \sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta = w}} k_\alpha k_\beta \frac{F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} \right\} w,$$

$$\sum_{\gamma\in R} \tilde{\nabla}_\gamma^2 = -H_S - \sum_{w\in W(R)} \left\{ \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta\\ s_\alpha s_\beta = w}} k_\alpha k_\beta F(\alpha,\beta) \cdot \frac{u_\alpha + u_{-\alpha}}{u_\alpha - u_{-\alpha}} \cdot \frac{u_\beta + u_{-\beta}}{u_\beta - u_{-\beta}} \right\} w$$
$$- \sum_{\alpha\in R_+} k_\alpha^2 F(\alpha,\alpha),$$

$$\sum_{\gamma\in R} \nabla_\gamma^- \nabla_\gamma^+ = -H_C^h - \left\{ \sum_{\gamma\in R} F(\gamma,\gamma) + 2 \sum_{\alpha\in R_+} \frac{k_\alpha L_\alpha(u) s_\alpha}{u_a - u_{-\alpha}} \right\}.$$

The proof of the first of these equations is given in App. 3. The others equations are proved similarly.

We now give two definitions.

Definition. The algebraic subvariety in $\mathbb{C}^{|R|}$ with equations

$$\sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta\\ s_\alpha s_\beta = w}} k_\alpha k_\beta \frac{F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} = 0$$

is called the Bethe variety for the Calogero model.

We have the following Theorem.

Theorem 6. *On the Bethe variety we have*

$$H_C = -\sum_{\gamma\in R} \nabla_\gamma^2.$$

Definition. The intersection of the Dunkl variety and the Bethe variety will be called the Bethe-Dunkl variety.

Theorem 7. *On the Bethe-Dunkl variety the set of algebraically independent integrals of Calogero problem is given by the formulas*

$$I_k = \sum_{\gamma\in R} \nabla_\gamma^k, \qquad k = 2, 3, \ldots.$$

Evidently, we have

$$H_C = I_2.$$

3. Weyl-Heisenberg algebras and Fock spaces. The Weyl-Heisenberg algebras for the Sutherland and Calogero models respectively are

$$A_S = \mathbb{C}[\nabla_\gamma, u_\gamma, s_\gamma], \qquad \gamma \in R,$$
$$A_C = \mathbb{C}[\nabla_\gamma^+, \nabla_\gamma^-, s_\gamma],$$

where

$$\nabla_\gamma^+ = \nabla_\gamma + L_\gamma(u),$$
$$\nabla_\gamma^- = \nabla_\gamma - L_\gamma(u).$$

Define now the Fock spaces for both models:

F_S is A_S-module genegated by the vacuum vector $v_0 = e^{-Q(u)/2}$;

F_C is the A_C-module generated by a vector $v_0 = 1$.

We have the following assertion.

Assertion: The Hamiltonian H_C and the integrals I_k belong to A_C.

Analogous assertion for A_S can be stated. Define the homomorphisms

$$\rho_A : A_S \to A_C,$$
$$\rho_F : F_S \to F_C.$$

We can state the following conjecture.

Conjecture 1. *The homomorphism ρ is an isomorphism.*

The analogous statement was announced for the classical root systems A, B, C, D in short publications of S. Kakei [1, 2].

4. Appendix 1: Definition of a root system

Canonical bilinear form. Let V be a finite-dimensional vector space, R is a finite subset generating V. For any $\alpha \in R$, $\alpha \neq 0$, there exists at most one reflection s of V such that $s(\alpha) = -\alpha$ and $s(R) = R$. Let G be the group of automorphisms that leaves R stable. Since R generates V, G is isomorphic to a subgroup of the symmetric group of R. Let s, s' be two reflections. Then we have $t = ss' \in G$ and $s(\alpha) = -\alpha$.

Definition. The subset R of V is a root system if

(1) R is finite, 0 is not in R and R generates V;

(2) $\forall\, \alpha \in R$ there exists an element $\alpha^\vee \in V^*$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and $s_{\alpha,\alpha^\vee}$ leaves R invariant, i.e.

$$s^2_{\alpha,\alpha^\vee}(x) = x + (\langle \alpha, \alpha^\vee \rangle - 2)\langle x, \alpha^\vee \rangle \alpha \qquad (s_{\alpha,\alpha^\vee} = -\alpha).$$

Note. $s_{\alpha,\alpha^\vee}$ is a reflection iff $\langle \alpha, \alpha^\vee \rangle = 2$.

Introducing the notation $s_{\alpha,\alpha^\vee} = s_\alpha$ we can write

$$s_\alpha(x) = x - \langle \alpha^\vee, x \rangle, \quad \forall x \in V.$$

The elements of R are called the roots.

The automorphisms of V leaving R stable are automorphisms of R. They form a group $A(R)$. The subgroup of $A(R)$ generated by the s_α's is called the Weyl group of R and is denoted by $W(R)$ or simply W.

Let $V = \mathbb{C}^n(\mathbb{R}^n)$. Let R be a reduced and irreducible root system in a n-dimensional real vector space V. Let R_+ be the set of positive roots, $R_0 = \{\alpha_1, \dots, \alpha_m\}$ the set of simple roots, $R_0 \subset R_+$, $W(R)$ the Weyl group of R generated by reflections. For a given root system the unique non-degenerated positive symmetric bilinear form $F_R(x,y)$ on V invariant under $W(R)$ can be constructed. This form satisfies the following condition

$$F_R(x,y) = \sum_{\alpha\in R} F_R(x,\alpha)F_R(\alpha,y).$$

From now on we will denote by the same characters $V, F, s_\alpha \in W(R)$ the complexifications $V\otimes\mathbb{C}$, and the natural extensions of F, s_α to $V\otimes\mathbb{C}$.

Let $e_1, \dots, e_n$ be an orthonormal basis in V with respect to F. Any vector $x \in V$ can be represented in the following form

$$x = \sum_{i=1}^{n} F(x,e_i)e_i.$$

Then, by the bilinearity of F, we have

$$F(x,y) = \sum_{i=1}^{n} F(x,e_i)F(e_i,y).$$

Let $|R| = N$ be the number of roots in the a root system R, $\mathbb{C}^N$ the complex space associated to R. Let $\{u_\alpha, \alpha \in R\}$ be coordinates in $\mathbb{C}^N$ ordered by some order chosen on R. For example, $\alpha \succ \beta$ if $\alpha - \beta \in P^+$ where P^+ is the positive part of the root lattice. Define on $\mathbb{C}^N$ the quadratic form

$$Q(u) = \sum_{\alpha,\beta\in R} F(\alpha,\beta)u_\alpha u_\beta$$

and its polar symmetric bilinear form

$$Q(u,v) = \sum_{\alpha,\beta\in R} F(\alpha,\beta)u_\alpha v_\beta.$$

The following lemma holds.

Lemma 8. *Let $L_\gamma(u) = \sum_{\alpha\in R} F(\gamma,\beta)u_\alpha$. Then we have*

$$Q(u) = \sum_{\gamma\in R} \left(L_\gamma(u)\right)^2,$$

$$Q(u,v) = \sum_{\gamma\in R} L_\gamma(u)L_\gamma(v).$$

Corollary 9. *The restriction of Q to $\mathbb{R}^N \subset \mathbb{C}^N$ is a non-negative form.*

The proof is immediate.

5. Appendix 2: Proof of the first equation of Proposition 3. We have

$$\begin{aligned}[\nabla_\gamma, \nabla_\delta] &= [-D_\gamma + A_\gamma, -D_\delta + A_\delta] \\ &= [D_\gamma, D_\delta] - ([D_\gamma, A_\delta] - [D_\delta, A_\gamma]) + [A_\gamma, A_\delta].\end{aligned}$$

Evidently, the first bracket is equal to zero because the operators D_δ and D_γ are the sums of commuting differential operators ∂_α and hence commute also. Further, we have

$$\begin{aligned}[D_\gamma, A_\delta] &= D_\gamma \circ A_\delta - A_\delta \circ D_\gamma \\ &= D_\gamma(A_\delta) + \sum_{\alpha\in R_+} \frac{k_\alpha F(\delta,\alpha) s_\alpha}{(u_\alpha - u_{-\alpha})} D_{s_\alpha \gamma} - A_\delta D_\gamma,\end{aligned}$$

Using the equation

$$s_\alpha \gamma = \gamma - 2\frac{F(\gamma,\alpha)}{F(\alpha,\alpha)}\alpha$$

we obtain

$$\begin{aligned}[D_\gamma, A_\delta] &= D_\gamma(A_\delta) + \sum_{\alpha\in R_+}\left(\frac{k_\alpha F(\delta,\alpha)s_\alpha}{u_\alpha - u_{-\alpha}}\right) D_\gamma - 2\sum_{\alpha\in R_+}\frac{k_\alpha F(\gamma,\alpha)F(\delta,\alpha)}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})} D_\alpha - A_\delta D_\gamma \\ &= D_\gamma(A_\delta) + A_\delta D_\gamma - 2\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha D_\alpha}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})} - A_\delta D_\gamma \\ &= -2\sum_{\alpha\in R_+}\frac{k_\alpha F(\gamma,\alpha)F(\delta,\alpha)s_\alpha}{(u_\alpha - u_{-\alpha})^2} - 2\sum_{\alpha\in R_+}\frac{F(\gamma,\alpha)F(\delta,\alpha)k_\alpha D_\alpha}{F(\alpha,\alpha)(u_\alpha - u_{-\alpha})}.\end{aligned}$$

It is easy to see that this expression is symmetrical with respect to γ and δ. For this reason, the commutator $[D_\delta, A_\gamma]$ is equal to the same expressiom and we obtain that

$$[\nabla_\gamma, \nabla_\delta] = [A_\gamma, A_\delta].$$

The computation of the last commutator gives

$$[A_\gamma, A_\delta] = \sum_{\alpha,\beta\in R_+}\frac{k_\alpha k_\beta F(\gamma,\alpha)F(\delta,\beta)}{(u_\alpha - u_{-\alpha})(u_{s_\alpha\beta} - u_{-s_\alpha\beta})} s_\alpha s_\beta.$$

Let $\beta' = \varepsilon_\alpha(s_\beta)s_\alpha\beta = \pm s_\alpha\beta$, $\beta' \in R_+$. We obtain further

$$\begin{aligned}[A_\gamma, A_\delta] = &\sum_{\alpha,\beta\in R_+}\frac{k_\alpha k_{\beta'} F(\gamma,\alpha)F(s_\alpha\delta,\beta)}{(u_\alpha - u_{-\alpha})\,(u_{s_\alpha\beta} - u_{-s_\alpha\beta})} s_\alpha s_\beta \\ &- \sum_{\alpha,\beta'\in R_+}\frac{k_\alpha k_{\beta'} F(s_\alpha\gamma,\beta)F(\delta,\alpha)}{(u_\alpha - u_{-\alpha})\,(u_{\beta'} - u_{-\beta'})} s_\alpha s_{\beta'} \\ = &\sum_{w\in W(R)}\sum_{\substack{\alpha,\beta\in R_+\\ s_\alpha s_\beta = w}}\frac{k_\alpha k_\beta (F(\gamma,\alpha)F(\delta,\beta) - F(\gamma,\beta)F(\delta,\alpha))}{(u_\alpha - u_{-\alpha})\,(u_\beta - u_{-\beta})} w.\end{aligned}$$

The first equation is proved.

6. Appendix 3:Proof of Proposition 5. We prove the first equation. Indeed, we have

$$\begin{aligned}\sum_{\gamma\in R}\nabla_\gamma^2 &= \sum_{\gamma\in R}(-D_\gamma + A_\gamma)^2 \\ &= \sum_{\gamma\in R}\left({D_\gamma}^2 - (D_\gamma\circ A_\gamma + A_\gamma D_\gamma) + {A_\gamma}^2\right) \\ &= \sum_{\gamma\in R} D_\gamma^2 - \sum_{\gamma\in R}(D_\gamma\circ A_\gamma + A_\gamma D_\gamma) + \sum_{\gamma\in R} A_\gamma^2.\end{aligned}$$

Earlier it was shown that $\Delta_1 = \sum_{\gamma\in R} D_\gamma^2$. Furter we compute $\sum_{\gamma\in R} D_\gamma \circ A_\gamma$. We have

$$\begin{aligned}
\sum_{\gamma\in R} D_\gamma \circ A_\gamma &= \sum_{\gamma\in R}\sum_{\alpha\in R_+} D_\gamma(A_\gamma) + \sum_{\gamma\in R} \frac{k_\alpha F(\gamma,\alpha) s_\alpha D_{s_\alpha\gamma}}{u_\alpha - u_{-\alpha}} \\
&= -\sum_{\gamma\in R}\sum_{\alpha\in R_+} \frac{2k_\alpha F(\gamma,\alpha)F(\gamma,\alpha)s_\alpha}{(u_\alpha-u_{-\alpha})^2} + \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} \sum_{\gamma\in R} F(\gamma,\alpha) D_{s_\alpha\gamma} \\
&= -\sum_{\alpha\in R_+} 2\frac{F(\alpha,\alpha)k_\alpha s_\alpha}{(u_\alpha-u_{-\alpha})^2} + \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha D_{s_\alpha\alpha}}{u_\alpha - u_{-\alpha}} \\
&= -\sum_{\alpha\in R_+} 2\frac{F(\alpha,\alpha)k_\alpha s_\alpha}{(u_\alpha-u_{-\alpha})^2} - \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha D_\alpha}{u_\alpha - u_{-\alpha}}.
\end{aligned}$$

Further

$$\begin{aligned}
\sum_{\gamma\in R} A_\gamma D_\gamma &= \sum_{\gamma\in R}\sum_{\alpha\in R_+} \frac{F(\gamma,\alpha)k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} D_\gamma \\
&= \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} \sum_{\gamma\in R} F(\alpha,\gamma) D_\gamma \\
&= \sum_{\alpha\in R_+} \frac{k_\alpha s_\alpha}{u_\alpha - u_{-\alpha}} D_\alpha.
\end{aligned}$$

We see that

$$\sum_{\gamma\in R} D_\gamma \circ A_\gamma + \sum_{\gamma\in R} A_\gamma D_\gamma = -\sum_{\alpha\in R_+} 2\frac{F(\alpha,\alpha)k_\alpha s_\alpha}{(u_\alpha-u_{-\alpha})^2}.$$

Compute now $\sum_{\gamma\in R} A_\gamma^2$. We have

$$\begin{aligned}
\sum_{\gamma\in R} A_\gamma^2 &= \sum_{\gamma\in R}\sum_{\alpha\in R_+} \frac{k_\alpha F(\gamma,\alpha)s_\alpha}{u_\alpha - u_{-\alpha}} \sum_{\beta\in R_+} \frac{k_\beta F(\gamma,\beta)s_\beta}{u_\beta - u_{-\beta}} \\
&= \sum_{\alpha\in R_+} \frac{k_\alpha^2 s_\alpha^2}{(u_\alpha - u_{-\alpha})(u_{-\alpha} - u_\alpha)} \sum_{\gamma\in R} F(\gamma,\alpha)F(\gamma,\alpha) \\
&\quad + \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} \frac{k_\alpha k_\beta s_\alpha s_\beta}{(u_\alpha - u_{-\alpha})(u_{s_\alpha\beta} - u_{-s_\alpha\beta})} \sum_{\gamma\in R} F(\alpha,\gamma)F(\beta,\delta) \\
&= -\sum_{\alpha\in R_+} \frac{k_\alpha^2 F(\alpha,\alpha)}{(u_\alpha-u_{-\alpha})^2} + \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta'}} \frac{k_\alpha k'_\beta F(\alpha, s_\alpha\beta')}{(u_\alpha - u_{-\alpha})(u'_\beta - u_{-\beta'})} s'_\beta s_\alpha \\
&= -\sum_{\alpha\in R_+} \frac{k_\alpha^2 F(\alpha,\alpha)}{(u_\alpha-u_{-\alpha})^2} - \sum_{\substack{\alpha,\beta\in R_+\\ \alpha\neq\beta}} \frac{k_\alpha k_\beta F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} s_\alpha s_\beta \\
&= -\sum_{\alpha\in R_+} \frac{k_\alpha^2 F(\alpha,\alpha)}{(u_\alpha-u_{-\alpha})^2} - \sum_{w\in W} \left\{ \sum_{\alpha,\beta\in R_+} \frac{k_\alpha k_\beta F(\alpha,\beta)}{(u_\alpha - u_{-\alpha})(u_\beta - u_{-\beta})} \right\} w.
\end{aligned}$$

The proof of the first assertion of Proposition 5 is finished.

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Appendices

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Appendix B

Report of General Assembly and EWM Statutes

DECISIONS TAKEN DURING THE GENERAL ASSEMBLY

Following the statutes of EWM a general assembly was held during this meeting. The decisions taken are valid until the next general assembly which should take place during the next general meeting of EWM in Hannover in 1999.

Time: December 14, 1997, 14.15–18.10 o'clock.
Place: ICTP Trieste.
Present: 45 members and 4 non-members from more than five countries.

1. Opening of the General Assembly. Bodil Branner from Denmark and Ragni Piene from Norway wished everyone present welcome to the general assembly. The general assembly was announced in the EWM Newsletter in February 1997 and in separate announcements of the Trieste meeting. Thus the requirements for the announcement of the general assembly were met and the meeting was valid.

Bodil Branner and Ragni Piene were chosen to chair the assembly. They presented an agenda which was approved as the working procedure of the meeting.

2. Appoint two people to take the minutes, two people to check the minutes and two people to count the votes. Karin Baur from Switzerland and Ufuk Taneri from Turkey/North Cyprus were appointed to take minutes together with Riitta Ulmanen, Finland. Sylvie Paycha, France, and Marie Demlova, Czech Republic, were appointed to check the minutes and count votes.

3. Approving of new members. The general assembly approved to membership of EWM those who had sent their application forms to a regional coordinator since the previous general assembly held in Madrid in 1995. Those who had given their application forms during the meeting in Trieste were also approved. Fees for these newly joined were waived for 1997.

4. Approving of the minutes of the previous general assembly. Minutes of the general assembly held in Madrid 1995 were published in the Proceedings of the meeting. The minutes were also distributed the present general assembly. The minutes were approved.

5. Electing two auditors and a deputy. The previous auditors, Seija Kämäri and Kirsi Peltonen from Finland were elected as auditors for 1998–99. No deputy was elected.

6. Confirming the financial statement and discharging those responsible of liabilities. Marjatta Näätänen from Finland explained briefly the financial situation of EWM: she presented the accounts. The accounts were confirmed by the general assembly. Those responsible of liabilities were discharged.

7. Deciding fees. It was decided to keep the fees as they were: 1 ECU (low), 20 ECU (standard), and 50 ECU (high).

The question of how to collect the fees and how to send it to EWM was raised. Riitta Ulmanen explained that every regional coordinator collects the fees whichever way is most convenient for her. She may open an account for that purpose if necessary. After making deductions necessary for local use she then sends the rest to the EWM account in Finland either in her own currency or in Finnish currency, if the amount is reasonable. It was also suggested that coordinators could bring the fees they have collected to the general assembly and give them to the EWM secretary there. These procedures were approved.

8. Electing Standing Committee and convenor and deputy convenor for 1998–99. According to the statutes the Standing Committee consists of 8–12 members. The term of a member is four years. Half of the terms will expire at the general assembly meeting and half will continue. In the by-laws it is defined that the standing committee will propose names for the new standing committee. Any woman either at the general meeting or otherwise involved in EWM can also make propositions. Standing committee proposed candidates presented in Appendix 3.

As old members the following were elected

- Polyna Agranovich, Ukraine.
- Capi Corrales Rodriganez, Spain.
- Marie Demlova, Czech Republic.
- Laura Fainsilber, France/Sweden.
- Emilia Mezzetti, Italy.
- Ragni Piene, Norway.

[Note: Capi Corrales Rodriganez subsequently resigned from her position on the Standing Committee.]

As new members:

- Christine Bessenrodt, Germany.
- Catherine Hobbs, UK.
- Irene Sciriha, Malta.
- Betül Tanbay, Turkey.
- Tsou Sheung Tsun, UK.
- Inna Yemelyanova, Russian Federation.

Laura Fainsilber was elected as convenor and Irene Sciriha and Inna Yemelyanova as deputy convenors.

9. Electing international coordinators. The following persons were elected as international coordinators

- East: Marie Demlova, Czech Republic and Tatiana Vasilieva, Russian Federation.
- Central/North: Marja Kankaanrinta, Finland.
- West: Capi Corrales Rodriganez, Spain.

[Note: Capi Corrales Rodriganez subsequently resigned from her position as

International Co-ordinator (West) and has been replaced by Rosa Maria Miro-Roig, Spain.]

10. Confirming regional coordinators. Regional coordinators were confirmed (see page 235). Sweden will choose a coordinator later.

It was also decided to have contact persons in non-European countries which have shown interest in EWM. The standing committee should contact all the regions.

11. Choosing a time and place for the next meeting and electing the organizing committee. It was preliminary decided in Madrid that the next meeting should take place in Germany. Christine Bessenrodt had made preparations for the next meeting.

It was decided to have the next meeting of EWM in September 1999 in Germany, near Hanover. Exact dates were not set yet.

The possibility of Russia organizing the meeting in 2001 was discussed. Inna Yemelyanova said that the Russians would be proud to organize the meeting in 2001.

After discussion following persons were elected to the organizing committee for 1999:

- Christine Bessenrodt, Germany.
- Polyna Agranovich, Ukraine.
- Irene Pieper-Seier, Germany.
- Ina Kersten, Germany.
- Olga Kounakovskaia, Russian Federation.
- Tsou Sheung Tsun, UK.

The organizing committee can complete itself when necessary with the consent of the standing committee.

12. Setting commissions for specific issues. It was decided on the following:

- Link with Association for Women in Mathematics (AWM): Christine Bessenrodt.
- Link with European Mathematical Society (EMS): Bodil Branner.
- Editing the EWM Newsletter: Eva Bergqvist, Sweden, Nadia Larsen, Denmark, Nina Rudälv, Sweden, and Ufuk Taneri, Turkey.
- Bulletin news: International coordinators.
- Web page: Olga Caprotti, Italy, and Hilda Irene van der Veen, Netherlands.
- E-mail EWM network: Sarah Rees, UK.

13. Publication of the proceedings of the present meeting. Catherine Hobbs and Laura Fainsilber had volunteered to do the proceedings.

Catherine Hobbs and Laura Fainsilber had studied different possibilities to publish the proceedings. Several options were presented. E.g. the proceedings might be published as a hard copy only, as a hard copy and a CD-ROM disk—and be put on the Web pages. The publisher could be some outside publisher or EWM could be the publisher.

All the possibilities were carefully considered. The decision of whom to use was left to Catherine Hobbs and Laura Fainsilber to make.

Also the contents of the proceedings was discussed. This subject raised a vivid discussion and several suggestions were made. As alternatives the following were presented: talks only, talks and posters, posters only.

The conclusion was that the proceedings should consist of talks and of some posters which relate to the talks presented at the meeting. The editors would select the posters they saw fit and ask for permission to print them.

14. The question of funding. This subject was postponed to be held discussed in the evening.

15. The life of EWM in between general meetings. There was a vivid discussion on this subject. No general opinion was reached. The question, who can use the name of EWM and on which occasions can it be used, was raised. It was decided that the name of EWM can be used on occasions that are not directly linked to EWM, only with the consent of the standing committee.

16. The ICM '98 and EMS 2000 activities. The general assembly decided to have a round table in ICM '98 meeting. The subject should be close to women in science in countries undergoing political changes.

The subject of the round table discussion to be held in EMS 2000 meeting should change from "Women and mathematics" which has been the topic twice now. "How are research institutes of mathematics organized in different countries" was proposed as a possible topic.

The question of interdisciplinary meetings: Call of proposals of on subjects of interdisciplinary meetings was given. Proposals should be given to the standing committee by July 1998.

17. Open discussion on any other subject. Open discussion was postponed.

18. Closing the General Assembly. Bodil Branner and Ragni Piene declared the general assembly closed for this part.

EUROPEAN WOMEN IN MATHEMATICS: STATUTES

Name and Location

Article 1.

1. European Women in Mathematics, informally EWM, is an association established in accordance with the laws of Finland.
2. Its seat is in Helsinki, Finland.

Purpose and Nature of Activities

Article 2.

1. The purposes of EWM are:
   - To encourage women to take up and continue their studies in mathematics and to promote mathematics among women.
   - To support women with or desiring careers in research in mathematics or mathematics related fields.
   - To provide a meeting place for these women.
   - To foster international scientific communications among women within and across fields in mathematics.
   - To promote equal opportunity and equal treatment of women and men in the mathematical community.
   - To cooperate with groups and organizations with similar goals.
2. To achieve its aims EWM may organize meetings, conferences, courses and seminars, arrange negotiations, disseminate a newsletter and other material related to its aims, operate as a publisher, prepare proposals and motions, make statements, award grants and prizes, and represent its membership.
3. The organization may, according to the situation, act directly, co-operate with individuals or bodies having similar aims, and set up subordinate bodies for special tasks.
4. The organization is non-profit making.

Membership

Article 3.

1. A member of the organization may be any woman, who supports the purposes of the organization.
2. The number of non-Finnish members may exceed one third of the total.
3. Members are approved and dismissed by the general assembly.
4. Members pay membership dues as determined by the general assembly.
5. A member can be removed for non payment of fees for more than two years.
6. Members may terminate their membership by giving a written notice to the convenor or to a member of the standing committee or by announcing the

termination at the meeting of the general assembly to be recorded in the minutes.

7. The organization can have honorary members. An honorary member has the right to vote and does not pay registration or membership fees.
8. The organization can have either women or men as well as organizations as supporting members.
9. Supporting members pay dues and receive relevant information. They do not have the right to vote.

Organs

Article 4.

1. The organ of the organization with decision making power is the general assembly.
2. The standing committee is the main executive organ.
3. Other executive organs are the international and regional coordinators.

Decision Making

Article 5.

- Decisions are made by simple majority vote of the general assembly unless the statutes require a qualified majority. The requirement of qualified majority is at least 3/4 of the votes cast.

The General Meeting

Article 6.

- The main tool for implementing the organization's statutory goals is the general meeting.
- EWM will aim to have a general meeting in Europe at least once every two years. The aim is to arrange these meetings in years to alternate with the European and International mathematical congresses, and in addition to have some activity at European congresses and International congresses.

The General Assembly

Article 7.

- The general assembly is held every second year during the general meeting. It is the responsibility of the standing committee to announce the general meeting. The announcement is done with the assistance of the coordinators at least six months in advance in a letter sent to each member or by e-mail or by an announcement in an appropriate newsletter.
- The General Assembly
  1. elects the standing committee, the convenor from the members of the standing committee, and deputies. The convenor may be non-Finnish
  2. elects a team of at least three international coordinators
  3. confirms the choice of regional coordinators

4. elects two auditors and a deputy
5. approves new members and elects honorary members
6. decides on removal of members by qualified majority
7. decides on registration and annual dues
8. accepts the minutes of the previous general assembly
9. receives the auditors' reports
10. confirms the financial statements and discharges those concerned from liability
11. chooses the time and place for the next general meeting from possibilities proposed as specified by the by-laws, and a local person who will choose a group to be responsible for the practical and financial arrangements
12. sets up commissions for specific issues
13. decides on changes of the statutes by qualified majority
14. decides on by-laws, the changing of which requires a qualified majority

The Extraordinary General Assembly

Article 8.

- An extraordinary general assembly can be called by giving six weeks notice by e-mail or in writing to all members. The reason for such a meeting must be clearly specified in writing.

The Standing Committee

Article 9.

1. The standing committee consists of 8–12 members and their deputies. The term of office of committee members will be four years. Half of the terms will expire at the general assembly meeting and half will continue. The first terms to expire will be drawn by lots. The members of the standing committee must be members of the organization.
2. The standing committee will be called together by the convenor when necessery or when a member of the committee so requests.
3. The standing committee will assist in organizing the forthcoming general meeting as specified by the by-laws.
4. The standing committee proposes the budget to the general assembly, receives the accounts to be presented to the general assembly and approves donations from outside organizations.
5. The standing committee shall appoint and dismiss the staff, define their duties and confirm their renumeration.

Coordinators

Article 10.

- The coordinators are chosen as specified in Article 7 and the by-laws. The job of a coordinator is to gather and pass on information.
    1. The team of international coordinators will also assist the standing committee in taking care of other business such as links with other organizations,

other types of meetings, emergency situations, etc.

2. As far as possible, there should be at least one regional coordinator in each country or region of Europe and also in non-European countries in which there is sufficient interest in EWM.

Signing for the Organization

Article 11.

- The organization may be signed for either by the convenor together with another member of the standing committee or by any two members of the standing committee.

Finances

Article 12.

1. EWM may receive gifts, grants, bequests and legacies. The association may raise funds for purposes connected to its aims by selling mathematical or similar material and it can own property and shares.
2. The general assembly appoints for each fiscal year two auditors and one deputy who are not members of the standing committee. These auditors may at all times require that the books and all relevant documents be presented to them, and they may examine the cash and financial situation.

   The fiscal year shall be one calendar year.

   The accounts shall be submitted to the auditors by the end of March.

   The auditors' report shall be submitted to the standing committee by the end of April.

Amendments

Article 13.

1. Amendment of the statutes or dissolving or merging the organization shall be mentioned in a notice to all members of the organization before the meeting of the general assembly.
2. Amendment of the statutes must be endorsed by a qualified majority of the general assembly; dissolving or merging the organization must be endorsed by a qualified majority of the meeting of the general assembly or an extraordinary general assembly.

Dissolving of the Organization

Article 14.

- In the event of the organization being dissolved or abolished, any assets remaining after discharge of all debts shall be transferred to a legal body having aims similar to those of the organization.

By-Laws

Article 15.

- These statutes are followed by a collection of by-laws approved by the general assembly.

By-Laws

1. Membership.

- An individual may become a member by contacting a coordinator or a member of the standing committee. The membership will be temporary until it becomes confirmed by the following general assembly. No election of members shall be effective until the relevant fees have been paid. Either the general assembly or the standing committee may waive fees in particular cases. Other than for non-payment of fees, a member can only be removed on the basis of written reasons, and after she has had the opportunity to let her case be heard, by a qualified vote of the general assembly.

  Members may terminate their membership by giving a written or e-mailed notice to a regional or international coordinator, or to a member of the standing committee.

  Newly elected members should be informed and receive the relevant documents.

  All matters of doubt or difficulty relating to membership shall be decided by the standing committee subject to confirmation by the general assembly.

2. The General Assembly.

- A meeting of the general assembly must be held during each general meeting of EWM. The general assembly is open to members and guests.

  Decisions are made if possible by consensus. If no members present object, routine decisions may be made by simple majority vote. However, 10% of members present may ask that a particular decision only be made subject to a qualified majority.

  In this case discussion continues until a decision can be reached by qualified majority.

  The requirement of qualified majority is at least 3/4 of the votes cast, also it is necessary that the the following requirements are met: at least 20 persons from at least 5 countries represented in the organization.

  The opinion of the membership may be solicited by a mail vote at any time.

  Unless otherwise specified by the general assembly, decisions on matters other than changes of the statutes or by-laws or those specified by Article 7 are delegated by the general assembly to the standing committee.

3. Standing Committee

Selection of the Standing Committee

- The standing committee will propose names for the new standing committee. Any woman either at the general meeting or otherwise involved in EWM can also propose members. The standing committee will arrange the proposed names as for fields in mathematics and geographic location in order to get a broad representation across subjects and countries. The written list of suggestions should be available before the general assembly. The final choice will be made by the general assembly, if necessary by vote.

The Standing Committee Includes:

(a) A woman responsible for the forthcoming general meeting

(b) Someone who was centrally involved in the last general meeting.
(c) The convenor and one or two deputy convenors (these may well be women (a) or (b)).
(d) If possible, a person at another institution in the country where the meeting is to be held.

The Main Responsibilities of the Standing Committee Are:

1. To be responsible for advising and assisting with the forthcoming general meeting. To draw up a detailed programme for that meeting, to try to raise funding for that meeting, and especially to think about organizational matters and prepare issues for discussion.
2. To announce the general meeting with the assistance of the team of coordinators. The general meeting is announced as early and widely as possible (in particular in EMS and AWM newsletters).
3. To draw up the agenda for the meeting of the general assembly and if possible to announce major issues to be considered in advance. To ensure that a report of the general meeting is prepared.
4. To take care of other issues which may arise between general meetings.
5. To ensure there is always a functioning team of international coordinators.
6. To consider the possibility of other types of meetings, and to delegate responsibilities appropriately.
7. To take care of any emergency situation which may arise.
8. To receive the financial report, and to propose the budget. The report and budget should be publicized in a EWM report or newsletter.
9. To select the treasurer.
10. To confirm the choice of treasurers for regional or national groups.

4. The Convenors.

- The job of the convenor and her deputies is to ensure that appropriate actions are taken at appropriate time, thus activating the standing committee as necessary.

5. Committees.

- Committees for specific purposes, for example finances or local organization, may be set up as and when necessary, either by the general assembly or by the standing committee.

6. Deputies.

- In so far as possible all people with responsible jobs should have deputies. In particular, there must be a deputy convenor for the standing committee.

7. The Coordinators.

- The job of a coordinator is to gather and pass on information among EWM members and to answer enquiries, send information to interested people etc..
  - There will always be at least three international coordinators, from at least two countries, elected by the general assembly, by simple majority vote.
  - As far as possible there should be at least one and preferably two regional coordinators in each country or region in Europe and also in non-European

countries in which there is sufficient interest in EWM. There will also be coordinators for links with organizations with related purposes, for example AWM and EMS. As far as possible, coordinators should change every two to four years.

- International Coordinators.
  - The job of the international coordinator is to:
    - Maintain an up-to-date list of regional coordinators.
    - Seek out replacements for regional coordinators as necessary, making sure if possible that all regional coordinator are active.
    - Seek out people who might act as regional coordinators in countries or regions which are not yet represented.
    - Supply the list of regional coordinators to the standing committee as requested and also to EMS.
    - Send out mailings to the regional coordinators for distribution in their region as requested by the standing committee or when otherwise appropriate.
    - Liaise with the standing committee, especially about publicity for the forthcoming general meeting.
    - Answer general enquiries or pass them on to the appropriate regional coordinator or member of the standing committee.
    - Keep copies of important correspondence such as past applications for money to the EEC.
    - Take care of other business such as links with other organizations, other types of meetings, emergency situations, etc.
- Regional Coordinators
  - Selection of regional coordinators.
    - EWM members in a region should agree among themselves the best method of choice suited to their region. In case of serious disagreement, the matter should be referred to the standing committee and the international coordinators and if necessary put to a vote in the general assembly.
    - The choice of all regional coordinators should be confirmed by the general assembly.
- Duties of the Regional Coordinators.
  - The job of a regional coordinator is to:
    - maintain some form of address list of people interested in, and members of, EWM
    - arrange for collection of membership dues
    - mail out information as requested by the international coordinators, and other information as appropriate
    - advertise EWM meetings in the newsletter of her region or country's mathematical society and elsewhere in her area as appropriate
    - act as a liason for anyone wanting to contact women mathematicians or to get information about women mathematicians in her country or region
    - give her name to national, regional and local mathematical societies as a person to contact in matters relating to women mathematicians in that country or region

- if possible, either collect or arrange to get collected information about numbers of women mathematicians in her country, and about factors relating to their status, programmes to assist them, etc.

8. The Extraordinary General Assembly.

- A request for such a meeting must be endorsed either by at least five members of among the standing committee and the international coordinators, or by at least 25 EWM members. The reason for such a meeting must be clearly specified in writing. The meeting must be held at an easily accessible place in Europe. The responsibility for organizing such a meeting is that of the persons calling for the extraordinary assembly.