European Women in Mathematics Workshop on Moduli Spaces in Mathematics and Physics

Oxford, July 1998

Oval Sculpture (No. 2) 1943 by Barbara Hepworth

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EUROPEAN WOMEN IN MATHEMATICS
WORKSHOP ON
MODULI SPACES IN MATHEMATICS AND PHYSICS

Oxford, 2 and 3 July 1998

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FOREWORD

These are the proceedings of the European Women in Mathematics workshop on Moduli Spaces which was held at Oxford in July 1998.

The aim of this interdisciplinary workshop was to explain to nonspecialists different uses of moduli spaces in various areas of mathematics and physics such as differential and algebraic geometry, dynamical systems, Yang-Mills theory and conformal field theory, and to facilitate the exchange of ideas between workers in these fields.

The workshop followed the very successful meeting on "Renormalisation in Mathemtaics and Physics" organised jointly by *femmes et mathématiques* and European Women in Mathematics in Paris in June 1996. It is hoped that these two meetings will be the start of a biennial series of interdisciplinary workshops.

The workshop was a small scale two days meeting, organised around seven talks giving different points of view on the concept and uses of moduli spaces. Special efforts were made by the speakers to present their topics in a form accessible to nonspecialists. The different perspectives presented contributed to the richness of the meeting which was attended by about forty participants (including a good number of graduate students), about half of whom were from continental Europe and half from British universities.

The organisers would like to express their sincere thanks to the London Mathematical Society, to Algebraic Geometry in Europe and to the Mathematical Institute, Oxford University for their support. We would also like to thank the staff of the Mathematical Institute and the staff of Balliol College, where the participants stayed, for their help, and to thank all the participants for making the meeting such an enjoyable and lively one.

Frances Kirwan
Sylvie Paycha
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September 1999

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MODULI SPACES IN ALGEBRAIC GEOMETRY

FRANCES KIRWAN

The word "moduli" is due to Riemann [18] in 1857, who observed that an isomorphism class of compact Riemann surfaces of genus g "hängt ... von $3g-3$ stetig veränderlichen Grössen ab, welche die Moduln dieser Klasse genannt werden sollen". For the next century the concept of moduli as parameters in some sense measuring or describing the variation of geometric objects was used in algebraic geometry, but it was not until the 1960s that Mumford [14] gave precise definitions of moduli spaces and methods for constructing them. Since then there has been an enormous amount of work on and using moduli spaces from very many different points of view.

The aim of this article is to describe some of the basic ideas in the theory of moduli spaces in algebraic geometry, and thus to serve in part as an introduction to the other articles in this proceeding.

1. Classification problems in algebraic geometry

Moduli spaces arise naturally in classification problems in algebraic geometry [14], [15], [16], [19], [20], [23]. A typical such problem, for example the classification of nonsingular complex projective curves up to isomorphism (or equivalently compact Riemann surfaces up to biholomorphism), can be resolved into two basic steps.

Step 1 is to find as many discrete invariants as possible (in the case of nonsingular complex projective curves the only discrete invariant is the genus).

Step 2 is to fix the values of all the discrete invariants and try to construct a "moduli space"; that is, a complex manifold (or an algebraic variety) whose points correspond in a natural way to the equivalence classes of the objects to be classified.

What is meant by "natural" here can be made precise given suitable notions of families of objects parametrised by base spaces and of equivalence of families. A *fine moduli space* is then a base space for a universal family of the objects to be classified (any family is equivalent to the pullback of the universal family along a unique map into the moduli space). If no universal family exists there may still be a *coarse moduli space*

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satisfying slightly weaker conditions, which are nonetheless strong enough to ensure that if a moduli space exists it will be unique up to canonical isomorphism.

It is often the case that not even a coarse moduli space will exist. Typically, particularly "bad" objects must be left out of the classification in order for a moduli space to exist. For example, a coarse moduli space of *nonsingular* complex projective curves exists (although to have a fine moduli space we must give the curves some extra structure, such as a level structure), but if we want to include singular curves (often important so that we can understand how nonsingular curves can degenerate to singular ones) we must leave out the so-called "unstable curves" to get a moduli space. However all nonsingular curves are stable, so the moduli space of stable curves of genus g is then a compactification of the moduli space of nonsingular projective curves of genus g.

There are several different methods available for constructing moduli spaces, involving very different techniques. Among these are the following:

- orbit spaces for group actions (using geometric invariant theory [14] or more recently ideas due to Kollar [8] and to Mori and Keel [6]; geometric theoretic quotients can also often be described naturally as symplectic reductions, and it is in this guise that many moduli spaces in physics appear [21]);
- period maps, Torelli theorems and variations of Hodge structures, initiated by Griffiths et al [4] and described by Claire Voisin in [24];
- Teichmüller theory (for Riemann surfaces; see, e.g., [10] and also [17] by Mary Rees).

As we shall see in Section 6, all three methods can be used for Riemann surfaces, to give alternative descriptions of their moduli spaces.

Remark. Recall that a compact Riemann surface (i.e., a compact complex manifold of complex dimension 1) can be thought of as a nonsingular complex projective curve, in the sense that every compact Riemann surface can be embedded in some complex projective space

$$\mathbb{P}_n = \mathbb{C}^{n+1} - \{0\}/(\text{multiplication by nonzero complex scalars})$$

as the solution space of a set of homogeneous polynomial equations. Moreover, two nonsingular complex projective curves are biholomorphic if and only if they are algebraically isomorphic. So there is a natural identification between the moduli space of compact Riemann surfaces of genus g up to biholomorphism and the moduli space of nonsingular complex projective curves up to isomorphism.

There are other situations where an "algebraic" moduli space can be naturally identified with the corresponding "complex analytic" moduli space, but this is not always the case. For example, if we consider K3 surfaces (compact complex manifolds of complex dimension 2 with first Betti number and first Chern class both zero), we find that the moduli space of all K3 surfaces has complex dimension 20, whereas the moduli spaces of algebraic K3 surfaces (which have one more discrete invariant, the degree, to be fixed) are 19-dimensional.

When $n > 1$ the question of classifying n-folds (i.e., compact complex manifolds—or in the algebraic category nonsingular complex projective varieties—of dimension n) becomes much harder than in the case $n = 1$ (which is the case of compact Riemann surfaces or nonsingular complex projective curves). The problems include the following:

(i) we need to worry about algebraic moduli spaces versus nonalgebraic ones (cf. K3 surfaces);
(ii) families of n-folds can be "blown up" along families of subvarieties to produce even more complicated families.

Remark. Recall that we blow up a complex manifold X along a closed complex submanifold Y by removing the submanifold Y from X and glueing in the projective normal bundle of Y in its place. We get a complex manifold $\tilde{X}$ with a holomorphic surjection $\pi : \tilde{X} \to X$ such that π is an isomorphism over $X - Y$ and if $y \in Y$, then $\pi^{-1}(y)$ is the complex projective space associated to the normal space T_yX/T_yY to Y in X at y. If $X = \mathbb{C}^{n+1}$ and $Y = \{0\}$ and we identify $\mathbb{P}_n$ with the set of one-dimensional linear subspaces of $\mathbb{C}^{n+1}$, then

$$\tilde{X} = \left\{(v, w) \in \mathbb{C}^{n+1} \times \mathbb{P}_n : v \in w\right\}$$

with $\pi(v, w) = v$.

We have already seen that the first problem (i) does not arise when $n = 1$. The second problem does not arise either when $n = 1$, because blowing up a 1-fold makes no difference unless the 1-fold has singularities (in which case blowing up may help to "resolve" the singularities; for example when we blow up the origin $\{0\}$ in $\mathbb{C}^2$ then the singular curve C in $\mathbb{C}^2$ defined by $y^2 = x^3 + x^2$ is transformed into a nonsingular curve $\tilde{C}$ with the origin in C replaced by two points, corresponding to the two complex "tangent directions" in C at 0).

Because of this second difficulty, the classification of n-folds when $n > 1$ requires a preliminary step before there is any hope of carrying out the two steps described above.

Step 0 (the "minimal model programme" of Mori et al [12]): Instead of all the objects to be classified, consider only specially "good" objects, such that every object is obtained from one of these specially good objects by a sequence of blow-ups.

How to carry out Mori's minimal model programme is well understood for algebraic surfaces and 3-folds, but in higher dimensions is incomplete as yet [9].

We shall ignore both Step 0 and Step 1 from now on, and concentrate on Step 2, the construction of moduli spaces.

2. The ingredients of a moduli problem

The formal ingredients of a moduli problem are:

(1) a set A of *objects* to be classified;
(2) an *equivalence relation* $\sim$ on A;
(3) the concept of a *family* of objects in A with *base space* S (or *parametrised by* S); and sometimes

(4) the concept of *equivalence of families*.

These ingredients must satisfy:

(i) a family parametrised by a single point $\{p\}$ is just an object in A (and equivalence of objects is equivalence of families over $\{p\}$);

(ii) given a family X parametrised by S and a map[1] $\phi : \tilde{S} \to S$, there is a family $\phi^* X$ parametrised by $\tilde{S}$ (the "pullback of X along ϕ"), with pullback being functorial and preserving equivalence.

In particular, for any family X parametrised by S and any $s \in S$ there is an object X_s given by pulling back X along the inclusion of $\{s\}$ in S. We think of X_s as the object in the family X whose parameter is the point s in the base space S.

3. Examples of families and deformations

Example 1. A family of compact Riemann surfaces parametrised by a complex manifold S is a surjective holomorphic map,

$$\pi : T \longrightarrow S,$$

from a complex manifold T of (complex) dimension $\dim(T) = \dim(S) + 1$ to S, such that π is proper (i.e., the inverse image $\pi^{-1}(C)$ of any compact subset C of S under π is compact) and has maximal rank (i.e., its derivative is everywhere surjective). Then $\pi^{-1}(s)$ is a compact Riemann surface for each $s \in S$, and is the object in the family with parameter s.

The family defined by π is an algebraic family if π is a morphism of nonsingular projective varieties.

Example 2. A family of nonsingular complex projective varieties parametrised by a nonsingular complex variety S is a proper surjective morphism,

$$\pi : T \longrightarrow S,$$

with T nonsingular and π having maximal rank. We can also allow T and S to be singular, but then we require an extra technical condition (that π must be flat with reduced fibres).

In this example equivalence of families $\pi_1 : T_1 \to S_1$ and $\pi_2 : T_2 \to S_2$ is given by isomorphisms $f : T_1 \to T_2$ and $g : S_1 \to S_2$ such that $g \circ \pi_1 = \pi_2 \circ f$. Equivalence of families in the first example is similar.

Definition. A *deformation* of a nonsingular projective variety or compact complex manifold M is a family $\pi : T \to S$ together with an isomorphism

$$\pi^{-1}(s_0) \cong M$$

for some $s_0 \in S$ (or it is the germ at s_0 of such a π).

[1]Here 'map' means 'morphism' in algebraic geometry, and 'complex analytic map' in complex analytic geometry.

Example 3. A *family of holomorphic (or algebraic) vector bundles* over a compact Riemann surface (or nonsingular complex projective curve) Σ is a vector bundle over $\Sigma \times S$ where S is the base space (see, e.g., [22]).

A *deformation of a vector bundle* E_0 over Σ is a vector bundle E over a product $\Sigma \times S$ together with an isomorphism

$$E|_{\Sigma \times \{s_0\}} \cong E_0$$

for some $s_0 \in S$ (or the germ at s_0 of such a family of vector bundles).

4. Fine and coarse moduli spaces

For definiteness, let us consider moduli problems in algebraic geometry rather than complex analytic geometry until it is specified otherwise.

Definition. A *fine moduli space* for a given (algebraic) moduli problem is an algebraic variety M with a family U parametrised by M having the following (universal) property:

- for every family X parametrised by a base space S, there exists a unique map $\phi : S \to M$ such that

$$X \sim \phi^* U.$$

Then, U is called a *universal family* for the given moduli problem.

Many moduli problems have no fine moduli space, but nonetheless there may be a moduli space satisfying slightly weaker conditions, called a coarse moduli space. If a fine moduli space does exist, it will automatically satisfy the conditions to be a coarse moduli space. Both fine and coarse moduli spaces, when they exist, are unique up to canonical isomorphism.

Definition. A *coarse moduli space* for a given moduli problem is an algebraic variety M with a bijection

$$\alpha : A/\sim \longrightarrow M$$

(where A is the set of objects to be classified up to the equivalence relation $\sim$) from the set $A/\sim$ of equivalence classes in A to M such that

(i) for every family X with base space S, the composition of the given bijection $\alpha : A/\sim \to M$ with the function

$$\nu_X : S \longrightarrow A/\sim,$$

which sends $s \in S$ to the equivalence class $[X_s]$ of the object X_s with parameter s in the family X, is a morphism;

(ii) when N is any other variety with $\beta : A/\sim \to N$ such that the composition $\beta \circ \nu_X : S \to N$ is a morphism for each family X parametrised by a base space S, then

$$\beta \circ \alpha^{-1} : M \longrightarrow N$$

is a morphism.

5. The jump phenomenon

For some moduli problems, a family X with base space S which is connected and of dimension strictly greater than zero may exist such that for some $s_0 \in S$ we have

(i) $X_s \sim X_t$ for all $s, t \in S - \{s_0\}$;
(ii) $X_s \not\sim X_{s_0}$ for all $s \in S - \{s_0\}$.

Then we cannot construct a moduli space including the equivalence class of the object X_{s_0}. Typically, "unstable" objects must be left out because of this jump phenomenon (e.g., when trying to construct moduli spaces of complex projective curves—including singular curves—or moduli spaces of vector bundles).

6. Classical examples of moduli spaces

• The *Jacobian* $J(\Sigma)$ of a compact Riemann surface Σ is a fine moduli space for holomorphic line bundles (i.e., vector bundles of rank 1) of fixed degree over Σ up to isomorphism. As a complex manifold

$$J(\Sigma) \cong \mathbb{C}^g/Ł,$$

where g is the genus of Σ and Ł is a lattice of maximal rank in $\mathbb{C}^g$ (i.e., $J(\Sigma)$ is a *complex torus*). Since $J(\Sigma)$ is also a complex projective variety, it is an *abelian variety*.

More precisely, $J(\Sigma)$ is the quotient of the complex vector space $H^0(\Sigma, K_\Sigma)$ of dimension g by the lattice $H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Here K_Σ is the complex cotangent bundle of Σ and $H^0(\Sigma, K_\Sigma)$ is the space of its holomorphic sections, i.e., the space of holomorphic differentials on Σ. If we choose a basis $\omega_1, \dots, \omega_g$ of holomorphic differentials and a standard basis $\gamma_1, \dots, \gamma_{2g}$ for $H_1(\Sigma, \mathbb{Z})$ such that

$$\gamma_i \cdot \gamma_{i+g} = 1 = -\gamma_{i+g} \cdot \gamma_i$$

if $1 \le i \le g$ and all other intersection pairings $\gamma_i \cdot \gamma_j$ are zero, then we can associate to Σ the $g \times 2g$ *period matrix* $P(\Sigma)$ given by integrating the holomorphic differentials ω_i around the 1-cycles γ_j. The Jacobian $J(\Sigma)$ can then be identified with the quotient of $\mathbb{C}^g$ by the lattice spanned by the columns of this period matrix.

We can in fact always choose the basis $\omega_1, \dots, \omega_g$ of holomorphic differentials so that the period matrix $P(\Sigma)$ is of the form

$$\begin{pmatrix} I_g & Z \end{pmatrix},$$

where I_g is the $g \times g$ identity matrix. This period matrix is called a *normalised period matrix*. The Riemann bilinear relations tell us that Z is symmetric and its imaginary part is positive definite.

• The moduli space $\mathcal{A}_g$ of all abelian varieties of dimension g was one of the first moduli spaces to be constructed. We have

$$\mathcal{A}_g \cong \mathcal{H}_g / \mathrm{Sp}(2g; \mathbb{Z}),$$

where $\mathcal{H}_g$ is Siegel's upper half space, which consists of the symmetric $g \times g$ complex matrices with positive definite imaginary part.

One way to construct and study the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g is via the *Torelli map*

$$\tau : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

given by

$$\Sigma \longmapsto J(\Sigma).$$

Torelli's theorem tells us that τ is injective (cf. [4], [24]). Describing the image of $\mathcal{M}_g$ in $\mathcal{A}_g$ is known as the Schottky problem.

• For the Teichmüller approach (cf. [17]) to $\mathcal{M}_g$ we consider the space of all pairs consisting of a compact Riemann surface of genus g and a basis $\gamma_1, \dots, \gamma_{2g}$ for $H_1(\Sigma, \mathbb{Z})$ as above such that

$$\gamma_i \cdot \gamma_{i+g} = 1 = -\gamma_{i+g} \cdot \gamma_i$$

if $1 \leq i \leq g$ and all other intersection pairings $\gamma_i \cdot \gamma_j$ are zero. If $g \geq 2$ this space (called Teichmüller space) is naturally homeomorphic to an open ball in $\mathbb{C}^{3g-3}$ (by a theorem of Bers). The mapping class group Γ_g (which consists of the diffeomorphisms of the surface modulo isotopy) acts discretely on Teichmüller space, and the quotient can be identified with the moduli space $\mathcal{M}_g$. This gives us a description of $\mathcal{M}_g$ as a complex analytic space, but not as an algebraic variety.

• To construct the moduli space $\mathcal{M}_g$ as an algebraic variety using geometric invariant theory, we use the fact that every compact Riemann surface of genus g can be embedded canonically as a curve of degree $6(g-1)$ in a projective space of dimension $5g-6$. The use of the word "canonical" here is a pun; it refers both to the canonical line bundle (although here "tri-canonical" would be more accurate) and to the fact that no choices are involved, except that a choice of basis is needed to identify the projective space with the standard one $\mathbb{P}_{5g-6}$. This enables us to identify $\mathcal{M}_g$ with the quotient of an algebraic variety by the group $\mathrm{PGL}(n+1;\mathbb{C})$. Here however we do not have a discrete group action, and to construct the quotient we must use Mumford's geometric invariant theory (see Section 8), which was developed in the 1960s in order to provide algebraic constructions of this moduli space and others.

For a very recent guide to many different aspects of the moduli spaces $\mathcal{M}_g$ see [5].

7. Moduli spaces as orbit spaces

Example 4. As a simple example, let us consider the moduli space of hyperelliptic curves of genus g. By a hyperelliptic curve of genus g we mean a nonsingular complex projective curve C with a double cover $f : C \to \mathbb{P}_1$ branched over $2g+2$ points in the complex projective line $\mathbb{P}_1$.

Let S be the set of unordered sequences of $2g+2$ distinct points in $\mathbb{P}_1$, which we can identify with an open subset of the complex projective space $\mathbb{P}_{2g+2}$ by associating to an unordered sequence $a_1, \dots, a_{2g+2}$ of points in $\mathbb{P}_1$ the coefficients of the polynomial whose roots are $a_1, \dots, a_{2g+2}$. Then it is not hard to construct a family $\mathcal{X}$ of hyperelliptic curves of genus g with base space S such that the curve parametrised by $a_1, \dots, a_{2g+2}$

is a double cover of $\mathbb{P}_1$ branched over $a_1, \ldots, a_{2g+2}$. This family is not quite a universal family, but it does have the following two properties.

(i) The hyperelliptic curves $\mathcal{X}_s$ and $\mathcal{X}_t$ parametrised by elements s and t of the base space S are isomorphic if and only if s and t lie in the same orbit of the natural action of $G = \mathrm{SL}(2;\mathbb{C})$ on S.

(ii) (Local universal property.) Any family of hyperelliptic curves of genus g is locally equivalent to the pullback of $\mathcal{X}$ along a morphism to S.

These properties, (i) and (ii), imply that a (coarse) moduli space M exists if and only if there is an *orbit space* for the action of G on S [15]. Here as in [15] by an orbit space we mean a G-invariant morphism $\phi : S \to M$ such that every other G-invariant morphism $\psi : S \to M$ factors uniquely through ϕ and $\phi^{-1}(m)$ is a single G-orbit for each $m \in M$. (We can think of an orbit space as the set of G-orbits endowed in a natural way with the structure of an algebraic variety).

This sort of situation arises quite often in moduli problems, and the construction of a moduli space is then reduced to the construction of an orbit space. Unfortunately such orbit spaces do *not* in general exist. The main problem (which is closely related to the jump phenomenon discussed above) is that there may be orbits contained in the closures of other orbits, which means that the natural topology on the set of all orbits is not Hausdorff, so this set cannot be endowed naturally with the structure of a variety. This is the situation with which Mumford's geometric invariant theory [14] attempts to deal with, telling us how to throw out certain "unstable" orbits in order to be able to construct an orbit space [2].

Example 5. Let $G = \mathrm{SL}(2;\mathbb{C})$ act on $(\mathbb{P}_1)^4$ via Möbius transformations on the Riemann sphere

$$\mathbb{P}_1 = \mathbb{C} \cup \{\infty\}.$$

Then

$$\left\{(x_1, x_2, x_3, x_4) \in (\mathbb{P}_1)^4 : x_1 = x_2 = x_3 = x_4\right\}$$

is a single orbit which is contained in the closure of every other orbit. On the other hand, the open subset

$$\left\{(x_1, x_2, x_3, x_4) \in (\mathbb{P}_1)^4 : x_1, x_2, x_3, x_4 \text{ distinct}\right\}$$

of $(\mathbb{P}_1)^4$ has an orbit space which can be identified with

$$\mathbb{P}_1 - \{0, 1, \infty\}$$

via the cross ratio (cf. [17]).

[2] See also [6], [8] for more general constructions of orbit spaces which can be used for moduli problems where geometric invariant theory seems not to be of use.

8. Mumford's geometric invariant theory

Let X be a complex projective variety (i.e., a subset of a complex projective space defined by the vanishing of homogeneous polynomial equations), and let G be a complex reductive group acting on X. To apply geometric invariant theory we require a *linearisation* of the action; that is, an ample line bundle L on X and a lift of the action of G to L. We lose very little generality by assuming that, for some projective embedding

$$X \subseteq \mathbb{P}_n,$$

the action of G on X extends to an action on $\mathbb{P}_n$ given by a representation

$$\rho : G \longrightarrow \mathrm{GL}(n+1),$$

and taking for L the hyperplane line bundle on $\mathbb{P}_n$. Algebraic geometry associates to $X \subseteq \mathbb{P}_n$ its homogeneous coordinate ring

$$A(X) = \bigoplus_{k \geq 0} H^0\big(X, L^{\otimes k}\big),$$

which is the quotient of the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ in $n+1$ variables by the ideal generated by the homogeneous polynomials vanishing on X. Since the action of G on X is given by a representation $\rho : G \to \mathrm{GL}(n+1)$, we get an induced action of G on $\mathbb{C}[x_0, \dots, x_n]$ and on $A(X)$, and we can therefore consider the subring $A(X)^G$ of $A(X)$ consisting of the elements of $A(X)$ left invariant by G. This subring $A(X)^G$ is a graded complex algebra, and because G is reductive it is finitely generated [14]. To any finitely generated graded complex algebra we can associate a complex projective variety, and so we can define $X//G$ to be the variety associated to the ring of invariants $A(X)^G$. The inclusion of $A(X)^G$ in $A(X)$ defines a *rational* map ϕ from X to $X//G$, but because there may be points of $X \subseteq \mathbb{P}_n$ where every G-invariant polynomial vanishes this map will not in general be well defined everywhere on X (i.e., it will not be a morphism).

We define the set X^{ss} of *semistable* points in X to be the set of those $x \in X$ for which there exists some $f \in A(X)^G$ not vanishing at x. Then the rational map ϕ restricts to a surjective G-invariant morphism from the open subset X^{ss} of X to the quotient variety $X//G$. However $\phi : X^{ss} \to X//G$ is still not in general an orbit space: when x and y are semistable points of X we have $\phi(x) = \phi(y)$ if and only if the closures $\overline{O_G(x)}$ and $\overline{O_G(y)}$ of the G-orbits of x and y meet in X^{ss}. Topologically $X//G$ is the quotient of X^{ss} by the equivalence relation for which x and y in X^{ss} are equivalent if and only if $\overline{O_G(x)}$ and $\overline{O_G(y)}$ meet in X^{ss}.

We define a *stable* point of X to be a point x of X^{ss} with a neighbourhood in X^{ss} such that every G-orbit meeting this neighbourhood is closed in X^{ss}, and is of maximal dimension equal to the dimension of G. If U is any G-invariant open subset of the set X^s of stable points of X, then $\phi(U)$ is an open subset of $X//G$ and the restriction $\phi|_U : U \to \phi(U)$ of ϕ to U is an orbit space for the action of G on U in the sense described above, so that it makes sense to write U/G for $\phi(U)$. In particular there is an orbit space X^s/G for the action of G on X^s, and $X//G$ can be thought of as a compactification of this orbit space.

$$\begin{array}{ccccc} X^s & \underset{\text{open}}{\subseteq} & X^{ss} & \underset{\text{open}}{\subseteq} & X \\ \downarrow & & \downarrow & & \\ X^s/G & \underset{\text{open}}{\subseteq} & X^{ss}/\sim & = & X//G. \end{array}$$

Example 6. Let us return to hyperelliptic curves of genus g. We have seen that the construction of a moduli space reduces to the construction of an orbit space for the action of $G = \mathrm{SL}(2;\mathbb{C})$ on an open subset S of $\mathbb{P}_{2g+2}$. If we identify $\mathbb{P}_{2g+2}$ with the space of unordered sequences of $2g+2$ points in $\mathbb{P}_1$, then S is the subset consisting of unordered sequences of distinct points. When the action of G on $\mathbb{P}_{2g+2}$ is linearised in the obvious way then an unordered sequence of $2g+2$ points in $\mathbb{P}_1$ is semistable if and only if at most $g+1$ of the points coincide anywhere on $\mathbb{P}_1$, and is stable if and only if at most g of the points coincide anywhere on $\mathbb{P}_1$ (see, e.g., [7, Chapter 16]). Thus S is an open subset of $\mathbb{P}^s_{2g+2}$, so an orbit space S/G exists with compactification the projective variety $\mathbb{P}_{2g+2}//G$. This orbit space is then the moduli space of hyperelliptic curves of genus g.

Other moduli spaces (such as moduli spaces of curves and of vector bundles; see, e.g., [2], [3], [14], [13], [15]) can be constructed as orbit spaces via geometric invariant theory in a similar way. For an example of one of many infinite dimensional versions, see [1].

9. Symplectic reduction

Geometric invariant theoretic quotients are closely related to the process of reduction in symplectic geometry, and thus many moduli spaces can be described as symplectic reductions.

Suppose that a compact, connected Lie group K with Lie algebra $\mathbf{k}$ acts smoothly on a symplectic manifold X and preserves the symplectic form ω. Let us denote the vector field on X defined by the infinitesimal action of $a \in \mathbf{k}$ by

$$x \longmapsto a_x.$$

By a moment map for the action of K on X we mean a smooth map

$$\mu : X \longrightarrow \mathbf{k}^*$$

that satisfies

$$d\mu(x)(\xi)\cdot a = \omega_x\big(\xi, a_x\big)$$

for all $x \in X$, $\xi \in T_x X$, and $a \in \mathbf{k}$. In other words, if $\mu_a : X \to \mathbb{R}$ denotes the component of μ along $a \in \mathbf{k}$ defined for all $x \in X$ by the pairing

$$\mu_a(x) = \mu(x)\cdot a$$

between $\mu(x) \in \mathbf{k}^*$ and $a \in \mathbf{k}$, then μ_a is a Hamiltonian function for the vector field on X induced by a. We shall assume that all our moment maps are equivariant moment maps; that is, $\mu : X \to \mathbf{k}^*$ is K-equivariant with respect to the given action of K on X and the coadjoint action of K on $\mathbf{k}^*$.

It follows directly from the definition of a moment map $\mu : X \to \mathbf{k}^*$ that if the stabiliser K_ζ of any $\zeta \in \mathbf{k}^*$ acts freely on $\mu^{-1}(\zeta)$, then $\mu^{-1}(\zeta)$ is a submanifold of X and the symplectic form ω induces a symplectic structure on the quotient $\mu^{-1}(\zeta)/K_\zeta$. With this symplectic structure the quotient $\mu^{-1}(\zeta)/K_\zeta$ is called the Marsden-Weinstein reduction, or symplectic quotient, at ζ of the action of K on X. We can also consider the quotient $\mu^{-1}(\zeta)/K_\zeta$ when the action of K_ζ on $\mu^{-1}(\zeta)$ is not free, but in this case it is likely to have singularities.

Example 7. Consider the cotangent bundle T^*Y of any n-dimensional manifold Y with its canonical symplectic form ω which is given by the standard symplectic form

$$\omega = \sum_{j=1}^{n} dp_j \wedge dq_j \tag{9.1}$$

with respect to any local coordinates $(q_1, \ldots, q_n)$ on Y and the induced coordinates $(p_1, \ldots, p_n)$ on its cotangent spaces. If Y is the configuration space of a classical mechanical system then T^*Y is the phase space of the system and the coordinates $p = (p_1, \ldots, p_n) \in T_q^*Y$ are traditionally called the momenta of the system.

If Y is acted on by a Lie group K, the induced action on T^*Y preserves ω and there is a moment map $\mu : T^*Y \to \mathbf{k}^*$ whose components μ_a along $a \in \mathbf{k}$ are given by pairing the moment coordinates p with the vector fields on X induced by the infinitesimal action of K; that is,

$$\mu_a(p, q) = p \cdot a_q$$

for all $q \in Y$ and for all $p \in T_qY$. When $K = \mathrm{SO}(3)$ acts by rotations on $Y = \mathbb{R}^3$ then μ is the angular momentum, or moment of momentum, about the origin.

The connection with GIT arises as follows. Let X be a nonsingular complex projective variety embedded in complex projective space $\mathbb{P}_n$, and let G be a complex Lie group acting on X via a complex linear representation $\rho : G \to \mathrm{GL}(n+1; \mathbb{C})$. A necessary and sufficient condition for G to be reductive is that it is the complexification of a maximal compact subgroup K (e.g., $G = \mathrm{GL}(m; \mathbb{C})$ is the complexification of the unitary group $U(m)$). By an appropriate choice of coordinates on $\mathbb{P}_n$ we may assume that ρ maps K into the unitary group $U(n+1)$. Then the action of K preserves the Fubini-Study form ω on $\mathbb{P}_n$, which restricts to a symplectic form on X. There is a moment map $\mu : X \to \mathbf{k}^*$ defined (up to multiplication by a constant scalar factor depending on differences in convention on the normalisation of the Fubini-Study form) by

$$\mu(x) \cdot a = \frac{\overline{\hat{x}}^t \rho_*(a) \hat{x}}{2\pi i \|\hat{x}\|^2} \tag{9.2}$$

for all $a \in \mathbf{k}$, where $\hat{x} \in \mathbb{C}^{n+1} - \{0\}$ is a representative vector for $x \in \mathbb{P}_n$ and the representation $\rho : K \to U(n+1)$ induces $\rho_* : \mathbf{k} \to \mathbf{u}(n+1)$ and dually $\rho^* : \mathbf{u}(n+1)^* \to \mathbf{k}^*$.

In this situation we have two possible quotient constructions, giving us the GIT quotient $X//G$, if we want to work in algebraic geometry and the symplectic reduction $\mu^{-1}(0)/K$, if we want to work in symplectic geometry. In fact these give us the same quotient space, at least up to homeomorphism (and diffeomorphism away from the singularities). More precisely, any $x \in X$ is semistable if and only if the closure of its G-orbit meets $\mu^{-1}(0)$, and the inclusion of $\mu^{-1}(0)$ into X^{ss} induces a homeomorphism

$$\mu^{-1}(0)/K \to X//G.$$

There are other quotient constructions closely related to symplectic reduction and geometric invariant theory, which are useful when working with Kähler or hyperkähler manifolds.

10. Moduli spaces of vector bundles

In physics, moduli spaces are often described as symplectic reductions of infinite-dimensional symplectic manifolds by infinite-dimensional groups (although the moduli spaces themselves are usually finite-dimensional). One example is given by moduli spaces of holomorphic vector bundles (cf. [11]), which can also be described using Yang-Mills theory (cf. [21]).

The Yang-Mills equations arose in physics as generalisations of Maxwell's equations. They have become important in differential and algebraic geometry. Yang-Mills equations are formulated over arbitrary compact oriented Riemannian manifolds, and in particular over compact Riemann surfaces and higher dimensional Kähler manifolds. The fundamental theorem of Donaldson, Uhlenbeck and Yau—a holomorphic bundle over a compact Kähler manifold admits an irreducible Hermitian Yang-Mills connection if and only if it is stable—can be thought of as an infinite-dimensional illustration of the link between symplectic reduction and geometric invariant theory.

Let M be a compact oriented Riemannian manifold and let E be a fixed complex vector bundle over M with a Hermitian metric. Recall that a connection A on E (or equivalently on its frame bundle) can be defined by a covariant derivative $d_A : \Omega^p_M(E) \to \Omega^{p+1}_M(E)$, where $\Omega^p_M(E)$ denotes the space of C^∞-sections of $\bigwedge^p T^*M \otimes E$ (i.e., the space of p-forms on M with values in E). This covariant derivative satisfies the extended Leibniz rule

$$d_A(\alpha \wedge \beta) = \big(d_A\alpha\big) \wedge \beta + (-1)^p \alpha \wedge d_A\beta$$

for $\alpha \in \Omega^p_M(E)$, $\beta \in \Omega^q_M(E)$, and therefore is determined by its restriction $d_A : \Omega^0_M(E) \to \Omega^1_M(E)$. The Leibniz rule implies that the difference of two connections is given by an $E \otimes E^*$-valued 1-form on M, and hence that the space of all connections on E is an infinite-dimensional affine space $\mathscr{A}$ based on the vector space $\Omega^1_M(E \otimes E^*)$. Similarly, the space of all *unitary* connections on E (i.e., connections compatible with the Hermitian metric on E) is an infinite-dimensional affine space based on the space of

1-forms with values in the bundle $\mathbf{g}_E$ of skew-adjoint endomorphisms of E. The Leibniz rule also implies that the composition $d_A \circ d_A : \Omega^0_M(E) \to \Omega^2_M(E)$ commutes with multiplication by smooth functions, and thus we have

$$d_A \circ d_A(s) = F_A s$$

for all C^∞ sections s of E, where $F_A \in \Omega^2_M(\mathbf{g}_E)$ is defined to be the curvature of the unitary connection A. The Yang-Mills functional on the space $\mathscr{A}$ of all unitary connections on E is defined as the L^2-norm square of the curvature, given by the integral over M of the product of the function $\|F_A\|^2$ and the volume form on M defined by the Riemannian metric and the orientation. The Yang-Mills equations are the Euler-Lagrange equations for this functional, given by

$$d_A * F_A = 0,$$

(see [1, Proposition 4.6]) where d_A has been extended in a natural way to $\Omega^*_M(\mathbf{g}_E)$. The gauge group $\mathscr{G}$, that is, the group of unitary automorphisms of E, acts by preserving the Yang-Mills functional and the Yang-Mills equations.

If M is a complex manifold we can identify the space $\mathscr{A}^{(1,1)}$ of unitary connections on E with curvature of type $(1,1)$ with the space of holomorphic structures on E, by associating to a holomorphic structure $\mathscr{E}$ the unitary connection whose $(0,1)$-component is given by the $\bar{\partial}$-operator defined by $\mathscr{E}$. This space $\mathscr{A}^{(1,1)}$ is an infinite-dimensional complex subvariety of the infinite-dimensional complex affine space $\mathscr{A}$, acted on by the complexified gauge group $\mathscr{G}_c$ (the group of complex C^∞ automorphisms of E), and two holomorphic structures are isomorphic if and only if they lie in the same $\mathscr{G}_c$-orbit.

When (M,ω) is a compact Kähler manifold there is a $\mathscr{G}$-invariant Kähler form Ω on $\mathscr{A}$ defined by

$$\Omega(\alpha,\beta) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(\alpha\wedge\beta)\wedge\omega^{n-1},$$

where n is the complex dimension of M. The Lie algebra of $\mathscr{G}$ is the space $\Omega^0_M(\mathbf{g}_E)$ of sections of $\mathbf{g}_E$, and there is a moment map $\mu : \mathscr{A} \to (\Omega^0_M(\mathbf{g}_E))^*$ for the action of $\mathscr{G}$ on $\mathscr{A}$ given by the composition of

$$A \longmapsto \frac{1}{8\pi^2} F_A \wedge \omega^{n-1} \in \Omega^{2n}_M(\mathbf{g}_E)$$

with integration over M. On $\mathscr{A}^{(1,1)}$ the norm square of this moment map agrees up to a constant factor with the Yang-Mills functional, which is minimised by the Hermitian Yang-Mills connections.

As in the finite-dimensional situation, for a suitable definition of stability the moduli space of stable holomorphic bundles of topological type E over M (which plays the role of the GIT quotient) can be identified with the moduli space of (irreducible) Hermitian Yang-Mills connections on E (which plays the role of the symplectic reduction). This was proved in general for vector bundles over compact Kähler manifolds Uhlenbeck and Yau with a different proof for nonsingular complex projective varieties given by Donaldson.

Over a compact Riemann surface M the situation is relatively simple, as all connections on E have curvature of type $(1,1)$ and so the infinite-dimensional complex affine space $\mathscr{A}$ can be identified with the space $\mathscr{C}$ of holomorphic structures on E. A moment map for the action of the gauge group on $\mathscr{A}$ is given by assigning to a connection $A \in \mathscr{A}$ its curvature $F_A \in \Omega^2_M(\mathbf{g}_E)$, and after a suitable central constant has been added the Hermitian Yang-Mills connections are exactly the zeros of the moment map.

A holomorphic bundle $\mathscr{E}$ over a Riemann surface M is stable (resp., semistable) if $\mu(\mathscr{F}) < \mu(\mathscr{E})$ (resp., $\mu(\mathscr{F}) \leq \mu(\mathscr{E})$) for every proper subbundle $\mathscr{F}$ of $\mathscr{E}$, where

$$\mu(\mathscr{F}) = \deg(\mathscr{F})/\mathrm{rank}(\mathscr{F}).$$

When the theory of stability of holomorphic vector bundles was first introduced, Narasimhan and Seshadri proved that a holomorphic vector bundle over M is stable if and only if it arises from an irreducible representation of a certain central extension of the fundamental group $\pi_1(M)$. Atiyah and Bott [1] translated this in terms of connections to show that a holomorphic vector bundle over M is stable if and only if it admits a unitary connection with constant central curvature. They deduced from this the existence of a homeomorphism between the moduli space $\mathscr{M}(n,d)$ of stable bundles of rank n and degree d over M and the moduli space of irreducible connections with constant central curvature on a fixed C^∞ bundle E of rank n and degree d over M.

11. The Kodaira-Spencer infinitesimal deformation map

If we are working in complex analytic geometry, rather than algebraic geometry, then there are nice methods for studying deformations and thus the local structure of moduli spaces.

Let $\pi : X \to S$ be a deformation of a compact complex manifold $M = \pi^{-1}(s_0)$, where $s_0 \in S$. We can cover M (thought of as a subset of X) with open subsets W_i of X such that there exist isomorphisms

$$h_i : W_i \longrightarrow U_i \times V_i,$$

where $V_i = \pi(W_i)$ is open in S and $U_i = M \cap W_i$ is open in $M = \pi^{-1}(s_0)$ and the projection of h_i onto V_i is just $\pi : W_i \to V_i$.

For each $i \neq j$ we then get a holomorphic vector field θ_{ij} on $U_i \cap U_j$ by differentiating $h_i \circ h_j^{-1}$ in the direction of any tangent vector $v \in T_{s_0}S$. These holomorphic vector fields define a 1-cocycle in the tangent sheaf Θ of M. This gives us the *Kodaira-Spencer map*

$$\rho_\pi : T_{s_0}S \longrightarrow H^1(M,\Theta).$$

THEOREM (Kuranishi). *If M is a compact complex manifold, then it has a deformation $\pi : X \to S$ with $\pi^{-1}(s_0) = M$ such that*

(i) *the Kodaira-Spencer map $\rho_\pi : T_{s_0}S \to H^1(M,\Theta)$ is an isomorphism;*
(ii) *π has the local universal property for deformations (i.e., any deformation of M is locally the pullback of π along a map f into S);*
(iii) *if $H^0(M,\Theta) = 0$, then the map f in (ii) is unique;*

(iv) *if* $H^2(M,\Theta)=0$*, then* S *is nonsingular at* s_0 *and so* $\dim S=\dim H^1(M,\Theta)$.

This deformation π is called the *Kuranishi deformation* of M (its germ at s_0 is unique up to isomorphism), and S is called the *Kuranishi space* of M.

Suppose there exists a fine moduli space of complex manifolds diffeomorphic to M. Then the moduli space is locally isomorphic near $[M]$ to the Kuranishi space near s_0. More often there is only a coarse moduli space, and the moduli space is locally isomorphic near $[M]$ to the quotient of the Kuranishi space by the action of the group of automorphisms of M. However this nice behaviour only happens if the dimension of $H^0(M,\Theta)$ is locally constant; otherwise jumping phenomena tend to arise so that no moduli space can exist.

Application

The dimension of the moduli space $\mathcal{M}_g$ of curves of genus g can be calculated using Kuranishi theory:

$$\dim \mathcal{M}_g = 3g-3.$$

12. Conclusion

Even within algebraic geometry, moduli spaces appear in many varied and important rôles. As the other articles in these proceedings indicate, they also appear in many other areas of mathematics and also physics.

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HODGE THEORY AND DEFORMATIONS OF COMPLEX STRUCTURES

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1. Introduction

These notes are intended for a reader with a certain knowledge of algebraic geometry. We have however provided each section with a few lines of introduction in order to give an idea of the content for the other readers.

The purpose of these notes is to provide an introduction to the theory and applications of the variations of Hodge structure (VHS), that is the way the Hodge decomposition on the cohomology groups of a projective or Kähler variety varies with its complex structure.

Associating to a complex structure on X the Hodge decomposition on its cohomology groups allows one to define the period map from the moduli space of X to a period domain. The latter are to a certain extent well-understood, since they are homogeneous spaces, so we can use the period map to prove local or global properties of the moduli space of X. For example, we can prove by curvature computation the positivity of the Hodge bundles: an essential point here is the transversality property that we will explain in detail, which says essentially that the image of the period map is tangent to a certain distribution defined on the period domain, and called the horizontal distribution. This property is used in many ways, as we will show in these notes, but it is also responsible for the fact that in general the period map cannot be surjective so that we cannot use the period map to uniformize the moduli spaces, except in a few well-understood cases, such as abelian varieties and K3 surfaces (or more generally hyperkähler varieties).

In any case, in order to use the period map for the study of the moduli space, an essential problem to solve is the so-called Torelli problem, which asks whether the period map is injective, that is whether the complex structure on X is determined by the Hodge structures on its cohomology groups. Of course there are many counterexamples to this general statement: for example, given X, one may consider the family of varieties, parametrized by X, consisting of the blow-ups of X at one of its points; these varieties are not isomorphic in general, but they have isomorphic Hodge structures. However, we shall focus on the positive results on this problem and we shall turn especially to the use of

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infinitesimal computations to attack it. The point is that, proving a Torelli theorem means being able to determine a complex structure on X from the data of its periods, which are a mixture of algebro-geometric data (the Hodge filtration) and of transcendental data (the integral (co)homology). Now going to the corresponding infinitesimal objects, that is differentiating the period map, we can forget about the transcendental part, and stay inside algebraic geometry.

We shall first explain how to compute the differential of the period map, and we shall give a few examples where the period map is known to be an immersion. In a more subtle vein, we shall next explain an argument due to Donagi [4] which eventually allows us to prove a generic Torelli theorem (i.e., to prove that a period map is of degree one on its image) by purely algebro-geometric means: the point is that the differential of the period map at a given point has itself a moduli point, i.e., an isomorphism equivalence class, and if it is true that the map to which X associates the moduli point of the differential of the period map at X is generically injective, the same is true of the period map itself.

We shall conclude these notes with a few applications of infinitesimal variations of Hodge structures to the study of the Hodge theory and algebraic cycles of the general fiber of a family. In fact, despite its transcendental character, Hodge theory is conjectured to be related in a very precise way to the Chow groups of a variety. For example, it is conjectured that if a variety has only trivial Hodge structures, that is, no odd dimensional rational cohomology and only type (p, p) rational cohomology classes of degree $2p$ for every p, then its rational Chow ring is isomorphic to its cohomology ring by the cycle class map. In the other direction it is known that if the cycle class map is injective, the Hodge structures are trivial, a statement first proved by Mumford [10] for surfaces and generalized in [12], [13]. Here, to give an idea of how the variations of Hodge structure can be useful, we shall describe the Abel-Jacobi map, and explain a few triviality [6] or non-triviality [16] results for the Abel-Jacobi map of the general member of certain families.

Finally, we shall sketch the argument of the most beautiful result in the theory of variations of Hodge structure, Nori's theorem [11], of which the triviality result mentioned above appears now as a corollary. Nori proves that the relative rational cohomology groups $H^k(X \times T, Z)$ vanish when $k \leq 2n$ and $Z \subset X \times T$ is any complete family of sufficiently ample complete intersections in X of dimension n. This is to be put in contrast with Lefschetz theorem, which would give this vanishing only for $k \leq n$. The main point here is to show that certain complexes built from the infinitesimal variation of Hodge structure of the family $Z \to T$ become exact for sufficiently ample Z_t's. Finally, a simple but important point in Nori's result is the fact that the infinitesimal invariants extracted from infinitesimal variations of Hodge structure are related to geometric object such as Dolbeault cohomology classes on the total space of the universal family, via a spectral sequence.

2. The period map and the Torelli problem

In this section, we define the Hodge filtration of a complex Kähler variety and the period map. It associates to a complex structure on an underlying differentiable variety X the corresponding Hodge filtration on the (fixed) cohomology groups of X. The *Torelli* *problem* addresses the question whether the period map is injective, that is, whether the complex structure is determined up to isomorphism by the Hodge structure. There are other related questions such as the *infinitesimal Torelli problem* which asks whether the local period map is an immersion. To report on this last aspect, we first describe the derivative of the period map, which involves the *Kodaira-Spencer map*, that is, the classifying map for first-order deformations. We turn to Donagi's approach to the *generic Torelli problem*, which uses in certain cases the derivative of the period map to show that the period map is generically one-to-one on its image.

2.1. Hodge structure. Let X be a projective complex variety or a Kähler compact variety; the tangent space of X at each point has a complex structure, which decomposes the space $A^n(X)$ of global complex C^∞ differential forms on X as

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X),$$

where $A^{p,q}(X)$ is the space of differential forms everywhere of type (p,q), that is, locally in the space generated by the $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge \overline{dz}_{j_1} \wedge \cdots \wedge \overline{dz}_{j_q}$, where z_k, $k = 1, \dots, \dim X$ are holomorphic coordinates.

Now consider the cohomology groups

$$H^n(X, \mathbb{C}) = H^n(X, \mathbb{Z}) \bigotimes \mathbb{C} = \{\text{closed forms in } A^n(X)\}/dA^{n-1}(X).$$

Hodge theory tells us that the decomposition of forms into types passes to the cohomology, that is

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the set of classes representable by a closed form in $A^{p,q}(X)$, and is isomorphic to the Dolbeault cohomology group $H^q(X, \Omega_X^p)$. The decomposition above is called the Hodge decomposition. One defines the Hodge filtration $F^\bullet H^n(X)$ as follows:

$$F^p H^n(X) = \bigoplus_{k \geq p} H^{k,n-k}(X).$$

The Hodge structure on $H^n(X)$ is given by the position of the spaces $H^{p,q}$ with respect to the integral lattice $H^n(X, \mathbb{Z})$.

2.2. The period map. Suppose that $\pi : \mathcal{X} \to B$ is a family of complex projective or Kähler varieties deforming X, that is, $\mathcal{X}$ and B are smooth connected complex varieties, π is a holomorphic proper submersion and $X = \pi^{-1}(0)$ for some point $0 \in B$. We shall denote by X_b the fiber $\pi^{-1}(b)$, for $b \in B$. If B is contractible, which is locally the case and which we shall assume for the moment, there exists a diffeomorphism over B

$$\mathcal{X} \cong X_0 \times B,$$

so that we have canonical isomorphisms

$$H^n\big(X_b, \mathbb{Z}\big) \cong H^n(\mathcal{X}, \mathbb{Z}) \cong H^n(X, \mathbb{Z}).$$

The period map associates to $b \in B$ the Hodge filtration $F^\bullet H^n(X_b)$ on the space $H^n(X_b, \mathbb{C})$ considered as a fixed space via the canonical isomorphism $H^n(X_b, \mathbb{C}) \cong H^n(X, \mathbb{C})$. So this is a map with value in the variety of flags of given ranks on $H^n(X, \mathbb{C})$. As first discovered by Griffiths [7], [8] it has two essential properties:

• First, it is holomorphic, which means that if $\mathcal{H}^n = H^n(X, \mathbb{C}) \otimes \mathcal{O}_B$ is the (trivial) holomorphic vector bundle on B with fiber $H^n(X_b, \mathbb{C})$ (more intrinsically $\mathcal{H}^n = R^n\pi_*\mathbb{C} \otimes \mathcal{O}_B$), there are holomorphic subbundles $F^p\mathcal{H}^n \subset \mathcal{H}^n$ such that $F^p\mathcal{H}^n_b = F^pH^n(X_b) \subset H^n(X_b, \mathbb{C})$.

• Second, it satisfies the *transversality property*: let $\nabla : \mathcal{H}^n \to \mathcal{H}^n \otimes \Omega_B$ be the Gauss-Manin connection, which is simply the usual differentiation in the local natural trivializations of $\mathcal{H}^n$ used above over a contractible B. Then the transversality property means

$$\nabla F^p\mathcal{H}^n \subset F^{p-1}\mathcal{H}^n \bigotimes \Omega_B. \tag{2.1}$$

In other words, under infinitesimal deformations of the complex structure, the Hodge filtration is only shifted by 1.

2.3. Derivative of the period map. It is well known that if $W \subset V$ is a vector subspace of a vector space, the tangent space to the Grassmannian at the point W is canonically isomorphic to $\mathrm{Hom}(W, V/W)$. So the derivative of the map which to $b \in B$ associates the subspace $F^pH^n(X_b) \subset H^n(X_b, \mathbb{C}) \cong H^n(X, \mathbb{C})$ has to be a map from $T_{B,b}$ to $\mathrm{Hom}(F^pH^n(X_b), H^n(X_b, \mathbb{C})/F^pH^n(X_b))$. It is now an immediate consequence of transversality that this map takes in fact its values in

$$\mathrm{Hom}\left(F^p/F^{p+1}H^n(X_b), F^{p-1}/F^pH^n(X_b)\right)$$

which is naturally contained in

$$\mathrm{Hom}\left(F^pH^n(X_b), H^n(X_b, \mathbb{C})/F^pH^n(X_b)\right).$$

In fact, as one can show retracing through the identifications made, the map

$$T_{B,b} \to \mathrm{Hom}\left(F^p/F^{p+1}H^n(X_b), F^{p-1}/F^pH^n(X_b)\right) \tag{2.2}$$

comes by dualisation and restriction at the point b from the map

$$\overline{\nabla} : F^p/F^{p+1}\mathcal{H}^n \longrightarrow F^{p-1}/F^p\mathcal{H}^n \otimes \Omega_B, \tag{2.3}$$

which fits in the following diagram:

$$\begin{array}{ccc}
\nabla : F^{p+1}\mathcal{H}^n & \longrightarrow & F^p\mathcal{H}^n \otimes \Omega_B \\
\downarrow & & \downarrow \\
\nabla : F^p\mathcal{H}^n & \longrightarrow & F^{p-1}\mathcal{H}^n \otimes \Omega_B \\
\downarrow & & \downarrow \\
\overline{\nabla} : F^p/F^{p+1}\mathcal{H}^n & \longrightarrow & F^{p-1}/F^p\mathcal{H}^n \otimes \Omega_B.
\end{array}$$

Next note that we have the identifications

$$F^p/F^{p+1}H^n\big(X_b\big) \cong H^q\big(\Omega^p_{X_b}\big), \quad p+q=n.$$

The important result is the following Proposition.

PROPOSITION 1. *The map*

$$T_{B,b} \longrightarrow \operatorname{Hom}\Big(H^q\big(\Omega^p_{X_b}\big), H^{q+1}\big(\Omega^{p-1}_{X_b}\big)\Big)$$

of (2.2) is the composition of the Kodaira-Spencer map

$$\kappa : T_{B,b} \longrightarrow H^1\big(T_{X_b}\big)$$

and of the map given by the cup-product in cohomology and the interior product

$$H^1\big(T_{X_b}\big) \bigotimes H^q\big(\Omega^p_{X_b}\big) \longrightarrow H^{q+1}\big(\Omega^{p-1}_{X_b}\big).$$

Here the Kodaira-Spencer map κ is the classifying map for the first-order deformation of the complex structure of X_b parametrized by $T_{B,b}$. Indeed, one can show that $H^1(T_{X_b})$ parametrizes exactly the isomorphism classes of first order deformations

$$X_{b,\epsilon} \longrightarrow \Delta_\epsilon,$$

where $\Delta_\epsilon = \operatorname{Spec}\mathbb{C}[\epsilon]/\epsilon^2$ is the infinitesimal disk consisting of one point together with one tangent vector.

Concretely, κ is induced by the long exact sequence of cohomology associated to the short exact sequence of holomorphic bundles on X_b

$$0 \longrightarrow T_{X_b} \longrightarrow T_{\mathscr{X}|X_b} \longrightarrow \pi^* T_{B|X_b} \longrightarrow 0,$$

using the fact that $H^0(\pi^* T_{B|X_b}) = T_{B,b}$.

2.4. The Torelli problem. Usually, if B is not simply connected, there is a nontrivial monodromy action

$$\pi_1(B,0) \longrightarrow \operatorname{Aut}\big(H^n(X,\mathbb{Z})\big)$$

obtained as the local trivializations used above along paths. Because of this there is no canonical identification $H^n(X_b,\mathbb{Z}) \cong H^n(X,\mathbb{Z})$. However we are always allowed to choose one. Furthermore, one works usually with polarized variations of Hodge structure, which means that there is an intersection form $\langle\,,\rangle$ on $H^n(X_b,\mathbb{Z})$, which is locally constant, that is, compatible with the local identifications $H^n(X_b,\mathbb{Z}) \cong H^n(X_{b'},\mathbb{Z})$ for any contractible open subset of B containing b' and b. This form has to be nondegenerate, skew if n is odd, symmetric if n is even, and the Hodge filtration has to satisfy with respect to $\langle\,,\rangle$:

$$F^pH^n\big(X_b\big)^\perp = F^{n-p+1}H^n\big(X_b\big). \tag{2.4}$$

There are in fact also sign conditions given by the Hodge index theorem which are satisfied by the Hodge filtration but we shall not consider them.

A typical example of polarization is the one given by the intersection pairing, when $n = \dim_{\mathbb{C}} X$.

In any case, it follows from the existence of the locally constant intersection pairing $\langle\,,\rangle$ that the monodromy group will be contained in $\Gamma = \mathrm{Aut}(H^n(X,\mathbb{Z}), \langle\,,\rangle)$ so that we can choose near each b a locally constant isomorphism

$$\big(H^n(X_b,\mathbb{Z}), \langle\,,\rangle\big) \cong \big(H^n(X,\mathbb{Z}), \langle\,,\rangle\big)$$

well defined up to Γ.

This allows us to define the period map $\mathcal{P}$ from the moduli space of X, to $\mathcal{D}/\Gamma$, where $\mathcal{D}$ is the local period domain consisting of all filtrations $F^\bullet H^n$ on $H^n(X,\mathbb{C})$ of given ranks, satisfying the condition:

$$H^n(X,\mathbb{C}) = F^p H^n \bigoplus \overline{F^{n-p+1} H^n},$$

where the bar is complex conjugation, and the polarization condition (2.4). The period map $\mathcal{P}$ associates to b the Hodge filtration on $H^n(X_b,\mathbb{C})$, that is, on $H^n(X,\mathbb{C})$ via any choice of an isomorphism

$$\big(H^n(X_b,\mathbb{Z}), \langle\,,\rangle\big) \cong \big(H^n(X,\mathbb{Z}), \langle\,,\rangle\big),$$

as above.

Here the moduli space of X parametrizes all possible Kähler complex structures on X up to isomorphism and is built from a variety B parametrizing a family $\mathcal{X} \to B$ as above by identifying points with isomorphic fibers.

The Torelli problem asks for the injectivity of $\mathcal{P}$. That is, we ask whether the existence of an isomorphism $(H^n(X_b,\mathbb{Z}), \langle\,,\rangle) \cong (H^n(X_{b'},\mathbb{Z}), \langle\,,\rangle)$ inducing an isomorphism $H^n(X_b,\mathbb{C}) \cong H^n(X_{b'},\mathbb{C})$ compatible with the Hodge filtrations implies that X_b is isomorphic to $X_{b'}$. It is known to hold for curves, K3 surfaces, cubic threefolds, cubic fourfolds.

The generic Torelli problem asks whether the period map is of degree 1 on its image. We shall first of all turn to the infinitesimal Torelli problem, which asks whether the local period map is an immersion. By the description of the derivative of the period map given in Proposition 1, this will be the case if the map induced by the cup-product

$$\bigoplus d\mathcal{P}^{p,q} : H^1(T_X) \longrightarrow \bigoplus \mathrm{Hom}\Big(H^q\big(\Omega_X^p\big), H^{q+1}\big(\Omega_X^{p-1}\big)\Big)$$

is injective. This is known to be true for X a nonhyperelliptic curve, by Noether's theorem, and also for smooth hypersurfaces in projective space, by Carlson-Griffiths description of their IVHS [1], with the exception of cubic surfaces. Finally it is also true for any Calabi-Yau variety, that is a Kähler variety with trivial canonical bundle. Indeed, let η be a nonzero generator of $H^0(\Omega_X^n) = H^0(K_X) = \mathbb{C}$, $n = \dim X$; then η corresponds to a nowhere vanishing holomorphic n-form, hence induces by interior product an isomorphism

$$T_X \cong \Omega_X^{n-1},$$

which induces an isomorphism in cohomology

$$H^1(T_X) \cong H^1\big(\Omega_X^{n-1}\big).$$

Since this last map is obviously equal to $d\mathcal{P}^{n,0}(\eta)$, it follows that $d\mathcal{P}^{n,0}$ is injective.

We shall conclude this section with Donagi's approach [4] to the generic Torelli problem for certain families. One starts with the observation that the image $\operatorname{Im} \oplus d\mathcal{P}^{p,q}$ of $H^1(T_X)$ in $\oplus \operatorname{Hom}(H^q(\Omega_X^p), H^{q+1}(\Omega_X^{p-1}))$ has a moduli point in the quotient $\mathcal{Q}$ of the Grassmannian

$$\operatorname{Grass}\Big(N, \bigoplus \operatorname{Hom}\Big(H^q\big(\Omega_X^p\big), H^{q+1}\big(\Omega_X^{p-1}\big)\Big)\Big)$$

under the group $\Pi_{p,q} \operatorname{Aut}(H^q(\Omega_X^p))$, where $N = \dim H^1(T_X)$. Notice that this quotient does not depend on the complex structure of X, assuming $H^1(T_X)$ of constant dimension; for clarity it would be better to replace here $H^q(\Omega_X^p)$ by abstract spaces $V^{p,q}$. So we have a natural map $d\mathcal{P}$ from the moduli space of X to $\mathcal{Q}$, which to X associates the orbit of $\operatorname{Im} \oplus d\mathcal{P}^{p,q}(H^1(T_X))$ under the group $\Pi_{p,q} \operatorname{Aut}(H^q(\Omega_X^p))$.

Now to prove the generic Torelli theorem for the family of deformations of X, we have to show that if we have two open subsets U_1 and U_2 of the moduli space of X and an isomorphism $j : U_1 \cong U_2$ such that $\mathcal{P} \circ j = \mathcal{P}$, then $U_1 = U_2$ and $j = id$. But it is immediate to conclude under these assumptions that we also have

$$d\mathcal{P} \circ j = d\mathcal{P}.$$

The conclusion is that if the map $d\mathcal{P}$ is of degree 1 on its image, the same is true of $\mathcal{P}$.

This reasoning has been applied successfully by Donagi [4] to get a generic Torelli theorem for hypersurfaces of degree d in $\mathbb{P}^n$, with a few series of exceptions, the most significant series of exceptions being the cases where d divides $n+1$, which includes the Calabi-Yau hypersurfaces ($d = n+1$). It has been also used by M. Green in [5] to prove the generic Torelli theorem for the family of sufficiently ample hypersurfaces of any given variety with very ample canonical bundle. Finally, I used it recently [14] to prove the generic Torelli theorem for the quintic threefolds (essentially the first family of exceptions to Donagi's theorem).

Remark 1. The quintic threefold is the typical example of a Calabi-Yau threefold, and, in relation to mirror symmetry [15], a Torelli theorem might be especially interesting for these varieties. It is also interesting to note that for Calabi-Yau threefolds the orbit of $\operatorname{Im} \oplus d\mathcal{P}^{p,q}(H^1(T_X))$ under the group $\Pi_{p,q} \operatorname{Aut}(H^q(\Omega_X^p))$ determines and is determined by the so-called Yukawa coupling, which is a cubic form on $H^1(T_X)$. By mirror symmetry this cubic form is conjectured to correspond to the quantum product of elements of $H^2(Y)$, Y being the mirror variety.

3. Application of VHS to algebraic cycles and Nori's theorem

In this section we give several applications of the variations of Hodge structures to the study of algebraic cycles. We first explain the cycle class and the Deligne cycle class, defined on the codimension p cycles of a Kähler complex variety. Here the image of the cycle class and the continuous part in the image of the Deligne cycle class are conjecturally described by the Hodge conjecture. However the discrete part in the image of the Deligne cycle class is quite mysterious. We give two results concerning it, at least for the general member of certain families. We finally conclude these notes with a sketch of Nori's connectivity theorem. It shows that the usual Lefschetz theorems, which concern the relative cohomology of a pair (X, Z), where Z is an ample complete intersection in X, can be enormously improved if one works with a universal pair $(X \times T, Z_T)$, where T is a big enough parameter space for such complete intersections. Again the machinery of infinitesimal variations of Hodge structures plays the major role.

3.1. Cycle class and Abel-Jacobi maps. If $Z \subset X$ is a codimension p cycle, that is, an integral combination of codimension p analytic (not necessarily smooth) subvarieties, $Z = \sum_i n_i Z_i$, we can define the class $[Z] \in H^{2p}(X, \mathbb{Z})$ of Z as

$$[Z] = \sum_i n_i [Z_i],$$

where $[Z_i]$ is Poincaré dual of the fundamental homology class of Z_i. In fact $[Z]$ is a Hodge class, that is, viewed as an element of $H^{2p}(X, \mathbb{C})$, it lies in $H^{p,p}(X)$. The rational Hodge conjecture predicts that all classes in $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ are classes of codimension p cycles with rational coefficients.

There is also a more refined invariant attached to a codimension p cycle, which is its Deligne cycle class. On cycles Z homologous to zero, i.e., such that $[Z] = 0 \in H^{2p}(X, \mathbb{Z})$, the Deligne cycle class map is equal to the Abel-Jacobi map, with values in the intermediate jacobian $J^{2p-1}(X)$ defined by Griffiths [7], [8]. The intermediate jacobian $J^{2p-1}(X)$ is the complex torus defined as

$$J^{2p-1}(X) = H^{2p-1}(X, \mathbb{C}) / \big(F^p H^{2p-1}(X) \oplus H^{2p-1}(X, \mathbb{Z})\big).$$

Using Poincaré duality, it is also isomorphic to

$$F^{n-p+1} H^{2n-2p+1}(X)^* / H_{2n-2p+1}(X, \mathbb{Z}),$$

where $n = \dim X$ and the map $H_{2n-2p+1}(X, \mathbb{Z}) \to F^{n-p+1} H^{2n-2p+1}(X)^*$ is given by integration of forms over homology classes. If Z is a codimension p cycle homologous to zero, the image $\Phi_X^{2p-1}(Z)$ of Z by the Abel-Jacobi map Φ_X^{2p-1} is then constructed as follows: since Z is homologous to zero, one can write $Z = \partial \Gamma$ for some real $2n-2p+1$-chain in X. Then using a little Hodge theory one can show that, although Γ is not closed, the integration $\int_\Gamma$ is well defined on $F^{n-p+1} H^{2n-2p+1}(X)$, so that we have a well-defined element $\int_\Gamma \in F^{n-p+1} H^{2n-2p+1}(X)^*$. Finally, any other choice of Γ will be of the form $\Gamma' = \Gamma + T$, where T is a cycle, and it follows that

$$\int_{\Gamma'} - \int_\Gamma = \int_T \in H_{2n-2p+1}(X, \mathbb{Z}).$$

So $\Phi_X^{2p-1}(Z) = \int_\Gamma \bmod H_{2n-2p+1}(X, \mathbb{Z})$ is a well-defined element of $J^{2p-1}(X)$.

Now a natural question is: what is the image of Φ_X^{2p-1}? First of all, one can show that the continuous part of this image is a complex subtorus of $J^{2p-1}(X)$ which satisfies the property that its tangent space is contained in

$$H^{p-1,p}(X) \subset H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X) = TJ^{2p-1}(X).$$

Conversely, the rational Hodge conjecture predicts that the continuous part in the image is equal to the maximum subtorus of $J^{2p-1}(X)$ satisfying this property.

Next, what about the discrete part? It is known to be a countable group, but it is not known how to describe or characterize it in general. We shall now state two theorems which show that in the presence of parameters one can answer this question at least for the general complex structure: the first one is due to M. Green [6] and myself independently.

THEOREM 1. *Let X be a general hypersurface of degree at least* 6 *in $\mathbb{P}^4$; then the Abel-Jacobi map Φ_X^3 is trivial modulo torsion.*

Note that by Lefschetz theorem only Φ_X^3 can be nonzero for such variety. Also in any degree, one can construct hypersurfaces in $\mathbb{P}^4$ for which the discrete part in the image of Φ_X^3 is nontrivial (e.g., generic hypersurfaces containing two lines).

A similar result [6] holds in fact for higher dimensional hypersurfaces.

To give an idea of how the possibility of deforming X is used in the proof, let us denote by U the parameter space for smooth hypersurfaces of degree d in $\mathbb{P}^4$. Over U we have the family of intermediate jacobians, with fiber $J^3(X_u)$ over u. This family has a natural holomorphic structure, the sheaf of holomorphic sections being equal to

$$\mathcal{J}^3 = \mathcal{H}^3/\big(F^2\mathcal{H}^3 \oplus H^3_{\mathbb{Z}}\big).$$

Here, we use the notation of Section 2.2, and $H^3_{\mathbb{Z}}$ is the local system with fiber $H^3(X_u,\mathbb{Z})$.

Now suppose that over some generically finite cover $V \to U$ we have a holomorphic family of one-cycles homologous to zero: $v \mapsto Z_v \subset X_v$. Then there is a corresponding section ν_Z of the family of intermediate jacobians, given by

$$\nu_Z(v) = \Phi^3_{X_v}\big(Z_v\big) \in J\big(X_v\big).$$

Griffiths [9] proves that ν_Z is holomorphic and that it satisfies a differential equation analogous to the transversality condition (2.1).

Then the essential point in the proof of the theorem above is to note that for $d \geq 6$ this differential equation is satisfied only by those sections of $\mathcal{J}^3$ which are locally the projection of a flat (with respect to the Gauss-Manin connection) section of $\mathcal{H}^3$. A monodromy argument then allows one to conclude that ν_Z is in fact a torsion section, which finishes the proof.

The second one [16] is a generalization of a result first proved by Clemens [2] in the case of the quintic threefold. It shows, in contrast to the previous theorem, that the discrete part of the image of the Abel-Jacobi map Φ_X^3 may be very big even for the general complex structure on X.

THEOREM 2 (Voisin). *Let X be a nonrigid Calabi-Yau threefold. Then for a general deformation X_t of X, the image of the Abel-Jacobi map of X_t is a countable subgroup of $J(X_t)$ which, tensored with $\mathbb{Q}$, is not a finite dimensional $\mathbb{Q}$-vector space.*

Here a Calabi-Yau threefold is a Kähler variety with trivial canonical bundle and $h^{2,0}$ number equal to zero. These varieties are in particular algebraic, by Kodaira embedding theorem. The nonrigidity condition means that one can deform the complex structure on X.

Notice that there is no theoretical contradiction between Theorems 1 and 2. The point is that the differential equation used in the proof of Theorem 1 is weak in the case of Calabi-Yau threefolds, because then $h^{3,0} = 1$. (When $h^{3,0} = 0$ there is no differential equation at all.)

The variations of Hodge structure are used in two ways in the proof of Theorem 2; first of all we study the variation of Hodge structure of the family of sufficiently ample hypersurfaces in X; this study allows us to show that there are at least countably many smooth surfaces $S \overset{j}{\hookrightarrow} X$ such that there is a nonzero integral class

$$\lambda \in H^2(S,\mathbb{Z})_0 \cap H^{1,1}(S),$$

where

$$H^2(S,\mathbb{Z})_0 = \operatorname{Ker}\big(j_* : H^2(S,\mathbb{Z}) \longrightarrow H^4(X,\mathbb{Z})\big).$$

By the Lefschetz theorem for $(1,1)$ classes, that is the Hodge conjecture for degree 2 classes, such λ is then Poincaré dual to the homology class of some integral combination Z of curves in S, and Z is homologous to zero in X, since $\lambda \in \operatorname{Ker} j_*$. These one-cycles homologous to zero are the ones used to show that the image of the Abel-Jacobi map is nonfinitely generated, at least if the complex structure of X is general.

As for the nonfinite generation, the argument is quite technical, but again it uses in an essential way the fact that the moduli space B of deformations of X is positive dimensional: in fact, putting the construction above in family over B, we get countably many families of one-cycles $Z_b \subset X_b$, hence as in the proof of Theorem 1, countably many associated sections ν_Z of the jacobian bundle

$$\mathcal{J}^3 = \mathcal{H}^3/\big(F^2\mathcal{H}^3 \oplus H^3_{\mathbb{Z}}\big)$$

on B. Now we can choose local liftings $\tilde{\nu}_Z$ of the ν_Z's to sections of the holomorphic vector bundle $\mathcal{H}^3/F^2\mathcal{H}^3$ on B, and what we show in fact is:

The sections $\tilde{\nu}_Z$ generate an infinite dimensional complex vector space of sections of $\mathcal{H}^3/F^2\mathcal{H}^3$ on B.

This is easily seen to imply Theorem 2. (The fact that there is no continuous part in the image of the Abel-Jacobi map of X_t for general t is quite standard.)

3.2. Nori's theorem. Let X be a projective variety, and let $L_1, \dots, L_r$ be holomorphic line bundles on X which are very ample: this means that L_i has enough global holomorphic sections to embed X in a projective space. Let $Z \subset X$ be the smooth complete intersection of the hypersurfaces defined by $\sigma_i \in H^0(L_i)$, that is, these hypersurfaces intersect transversally along Z, so that in particular $\dim X = n+r$, with $n = \dim Z$. We know by Lefschetz theorem that the restriction map

$$H^k(X,\mathbb{Z}) \longrightarrow H^k(Z,\mathbb{Z})$$

is an isomorphism for $k < n$ and is injective for $k = n$, which is equivalent to the following vanishing theorem for the relative cohomology groups:

$$H^k(X,Z,\mathbb{Z}) = 0, \quad \text{for } k \leq n. \tag{3.1}$$

Now let $S = \Pi_i H^0(L_i)$, which is a parameter space for the set of all such complete intersections. For any morphism $\phi : T \to S$, one has the universal complete intersection parametrized by T

$$Z_T \subset X \times T$$

such that the fiber of Z_T over $t \in T$ is the intersection of the hypersurfaces defined by $\sigma_i \in H^0(L_i)$, where $\phi(t) = (\sigma_1, \dots, \sigma_r)$.

Now the vanishing of (3.1) easily implies that for any $\phi : T \to S^0$, where S^0 denotes the open set of S consisting of r-tuples $(\sigma_1, \dots, \sigma_r)$ such that the hypersurfaces defined by σ_i intersect transversally, one has

$$H^k(X \times T, Z_T, \mathbb{Z}) = 0, \quad \text{for } k \leq n. \tag{3.2}$$

Indeed, this follows from the fact that the Leray spectral sequences for $pr_2 : X \times T \to T$ and $pr_2 : Z_T \to T$, which are known to degenerate at E_2 by Deligne [3], will coincide at E_2 in degrees $< n$ by (3.1). The argument works in fact as well for degree n.

Nori's theorem [11] improves (3.2) as follows.

THEOREM 3. *If the L_i's are sufficiently ample, for any submersive morphism $\phi : T \to S$, one has*

$$H^k(X \times T, Z_T, \mathbb{Z}, \mathbb{Q}) = 0, \quad \textit{for } k \leq 2n.$$

It is an interesting question to decide whether a topological proof of Nori's theorem can be given. As it stands, it is completely algebraic. Assuming ϕ takes value in S^0, the essential argument is as follows: we want to understand the cohomology groups $H^k(Z_T, \mathbb{C})$ for $k \leq 2n$. They are computed by the (degenerating) Leray spectral sequence for $pr_2 : Z_T \to T$ as $\oplus_{p+q=k} H^q(R^p pr_{2*}\mathbb{C})$. Now the local constant system $H^p_{\mathbb{C}} := R^p pr_{2*}\mathbb{C}$ identifies with the space of ∇-flat sections of the Hodge bundle $\mathcal{H}^p$ (notation as in Section 2.2) hence, since ∇ is integrable, it admits a resolution by the de Rham complex

$$0 \longrightarrow H^p_{\mathbb{C}} \to \mathcal{H}^p \xrightarrow{\nabla} \mathcal{H}^p \bigotimes \Omega_T \longrightarrow \mathcal{H}^p \bigotimes \Omega^2_T \cdots.$$

So we have to compute the cohomology of the de Rham complex $DR(\mathcal{H}^p)$,

$$0 \longrightarrow \mathcal{H}^p \xrightarrow{\nabla} \mathcal{H}^p \bigotimes \Omega_T \longrightarrow \mathcal{H}^p \bigotimes \Omega^2_T \cdots$$

and this can be done by putting on it the Hodge filtration, which is well defined thanks to the transversality property (2.1)

$$F^l DR(\mathcal{H}^p) : 0 \longrightarrow F^l\mathcal{H}^p \xrightarrow{\nabla} F^{l-1}\mathcal{H}^p \bigotimes \Omega_T \longrightarrow F^{l-2}\mathcal{H}^p \bigotimes \Omega_T^2 \cdots .$$

Now the graded pieces of the de Rham complex for this filtration are the complexes that we started to define in (2.3)

$$0 \longrightarrow \mathcal{H}^{l,p-l} \xrightarrow{\overline{\nabla}} \mathcal{H}^{l-1,p-l+1} \bigotimes \Omega_T \xrightarrow{\overline{\nabla}} \mathcal{H}^{l-2,p-l+2} \bigotimes \Omega_T \cdots ,$$

where $\mathcal{H}^{l,p-l} = F^l\mathcal{H}^p/F^{l+1}\mathcal{H}^p$. The main technical point is then the fact that if the L_i's are sufficiently ample, enough of the cohomology sheaves of these complexes are governed by the cohomology of X, to imply the desired isomorphisms $H^p(X \times T, \mathbb{C}) \cong H^p(Z_T, \mathbb{C})$, $p < 2n$.

The actual proof of Nori looks technically different, but the essential point lies here, and its important meaning is the fact that infinitesimal invariants, lying in the cohomology sheaves of the complexes (3.2) are related to Dolbeault cohomology classes of the total family Z_T by the spectral sequence associated to the filtration above on the complexes $DR(\mathcal{H}^p)$.

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MODULI SPACES OF VECTOR BUNDLES ON ALGEBRAIC VARIETIES

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These notes are intended to support our cross disciplinary discussion on moduli spaces. In no case do I claim it is a survey on moduli spaces of vector bundles on algebraic projective varieties. Many people have made important contributions without even being mentioned here and I apologize to those whose work I may have failed to cite properly.

1. Introduction

Moduli spaces are one of the fundamental constructions of Algebraic Geometry and they arise in connection with classification problems. Roughly speaking a moduli space for a collection of objects A and an equivalence relation $\sim$ is a classification space, i.e., a space (in some sense of the word) such that each point corresponds to one, and only one, equivalence class of objects. Therefore, as a set, we define the moduli space as equivalence classes of objects $A/\sim$. In our setting the objects are algebraic objects, and because of this we want an algebraic structure on our classification set. Finally, we want our moduli space to be unique (up to isomorphism).

General facts on moduli spaces can be found, for instance, in [25], [26] or [27] (see also [22]). In this paper, we shall restrict our attention to moduli spaces of stable vector bundles on smooth, algebraic, projective varieties. We have attempted to give an informal presentation of the main results, addressed to a general audience.

A moduli space of stable vector bundles on an algebraic, projective variety X is a scheme whose points are in "natural bijection" to isomorphic classes of stable vector bundles on X. The phrase "natural bijection" can be given a rigorous meaning in terms of representable functors. Using Geometric Invariant Theory the moduli space can be constructed as a certain Quot-scheme by a natural group action.

Once the existence of the moduli space is established, the question arises as what can be said about its local and global structure. More precisely, what does the moduli space look like, as an algebraic variety? Is it, for example, connected, irreducible, rational or smooth? What does it look as a topological space? What is its geometry? Until now, there is no general answer to these questions.

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The goal of this paper is to review some of the known results which nicely reflect the general philosophy that moduli spaces inherit a lot of properties of the underlying variety; essentially when the underlying variety is a surface. In Section 2, we recall some generalities on moduli spaces of vector bundles, all of which are well known to experts on this field. Section 3 deals with vector bundles on algebraic surfaces. Quite a lot is known in this case and we will review some of the main results. Section 4 is devoted to moduli spaces of vector bundles on higher-dimensional varieties. Very few results are known. As we shall stress, the situation drastically differs and results like the smoothness and irreducibility of moduli spaces of stable vector bundles on algebraic surfaces turn out to be untrue for moduli spaces of stable vector bundles on higher-dimensional algebraic varieties. We could not resist discussing some details that perhaps only the experts will care about, but hopefully will also introduce the nonexpert reader to a subtle subject. To this end, we present new results on moduli spaces of stable vector bundles on rational normal scrolls of arbitrary dimension (see [3]) with the hope of finding a clue which could facilitate the study of moduli spaces of stable vector bundles on arbitrary n-dimensional varieties.

Notation. Let $(X, \mathcal{O}_X(1))$ be a polarized irreducible smooth projective scheme over an algebraically closed field k of characteristic zero. Recall that the Euler characteristic of a locally free sheaf E is

$$\chi(E) := \sum_i (-1)^i h^i(X, E),$$

where $h^i(X, E) = \dim_k H^i(X, E)$. The Hilbert polynomial $P_E(m)$ is given by

$$m \longrightarrow \chi\big(E \otimes \mathcal{O}_X(m)\big)/rk(E).$$

2. The moduli functor; fine and coarse moduli spaces

The first step in the classification of vector bundles is to determine which cohomology classes on a projective variety can be realized as Chern classes of vector bundles. On curves the answer is known. On surfaces the existence of vector bundles was settled by Schwarzenberger; and it remains open on higher-dimensional varieties. The next step aims at a deeper understanding of the set of all vector bundles with a fixed rank and Chern classes. This naturally leads to the concept of moduli spaces which I shall shortly recall.

Let $(X, \mathcal{O}_X(1))$ be a polarized projective scheme over an algebraically closed field k. For a fixed polynomial $P \in \mathbb{Q}[z]$, we consider the contravariant functor

$$\mathcal{M}_X(P)(-) : (\mathrm{Sch}/k) \longrightarrow (\mathrm{Sets}), \qquad S \longmapsto \mathcal{M}_X(P)(S),$$

where $\mathcal{M}_X(P)(S) = \{S$-flat families $\mathcal{F} \to X \times S$ of vector bundles on X all of whose fibers have Hilbert polynomial $P\}/\sim$, with $\mathcal{F} \sim \mathcal{F}'$ if and only if, $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$ for some $L \in \mathrm{Pic}(S)$, $p : S \times X \to S$ being the natural projection. And if $f : S' \to S$ is a morphism in (Sch/k), let $\mathcal{M}_X(P)(f)(-)$ be the map obtained by pulling back sheaves via $f_X = f \times \mathrm{id}_X$:

$$\mathcal{M}_X(P)(f)(-) : \mathcal{M}_X(P)(S) \longrightarrow \mathcal{M}_X(P)\big(S'\big), \quad [F] \longrightarrow \big[f_X^* F\big].$$

Definition 2.1. A fine moduli space of vector bundles on X with Hilbert polynomial $P \in \mathbb{Q}[z]$ is a scheme $M_X(P)$ together with a family (Poincaré bundle) of vector bundles $\mathcal{U}$ on $M_X(P) \times X$ such that the contravariant functor $\mathcal{M}_X(P)(-)$ is represented by $(M_X(P), \mathcal{U})$

If $M_X(P)$ exists, it is unique up to isomorphism. Nevertheless, in general, the functor $\mathcal{M}_X(P)(-)$ is not representable. In fact, there are very few classification problems for which a fine moduli space exists. To get, at least, a coarse moduli space (see, e.g., [27] or [25] for a precise definition) we must somehow restrict the class of vector bundles that we consider. What kind of vector bundles should we taken? In [17] and [18], M. Maruyama found an answer to this question: stable vector bundles.

Definition 2.2. Let $(X, \mathcal{O}_X(1))$ be a polarized projective scheme of dimension d. For a torsion-free sheaf F on X one sets

$$\mu_H(F) := \frac{c_1(F)H^{d-1}}{rk(F)}, \qquad P_F(m) := \frac{\chi(F \otimes \mathcal{O}_X(mH))}{rk(F)}$$

with $H = \mathcal{O}_X(1)$. The sheaf F is μ-semistable (resp., semistable) with respect to the polarization H if and only if

$$\mu_H(E) \leq \mu_H(F) \qquad (\text{resp., } P_E(m) \leq P_F(m) \text{ for } m \gg 0)$$

for all nonzero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then F is μ-stable (resp., stable) with respect to $H = \mathcal{O}_X(1)$.

One easily checks the implications

$$\mu\text{-stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu\text{-semistable}.$$

Remark 2.3. The definition of stability depends on the choice of the polarization H. The changes of the moduli space that occur when the polarization H varies have been studied by several people in greater detail often with respect to their relation to Gauge theory and the computation of Donaldson polynomials (see, e.g., [7], [9], [30], and [29]).

Definition 2.4. Let $(X, H = \mathcal{O}_X(1))$ be a polarized projective scheme over an algebraically closed field k. For a fixed polynomial $P \in \mathbb{Q}[z]$, we consider the contravariant subfunctor $\mathcal{M}^s_X(H, P)(-)$ of the functor $\mathcal{M}_X(P)(-)$:

$$\mathcal{M}^s_X(H, P)(-) : (\text{Sch}/k) \longrightarrow (\text{Sets}), \qquad S \longmapsto \mathcal{M}^s_X(H, P)(S),$$

where $\mathcal{M}^s_X(H, P)(S) = \{S\text{-flat families} \mathcal{F} \to X \times S$ of vector bundles on X all of whose fibers are stable with respect to H and have Hilbert polynomial $P\}/\sim$, with $\mathcal{F} \sim \mathcal{F}'$ if and only if $\mathcal{F} \cong \mathcal{F}' \otimes p^*L$ for some $L \in \text{Pic}(S)$, $p : S \times X \to S$ being the natural projection.

THEOREM 2.5. *The functor $\mathcal{M}^s_X(H, P)(-)$ has a coarse moduli scheme $M^s_X(H, P)$ which is a separated scheme and locally of finite type over k. This means*

(1) *There is a natural transformation*

$$\Psi : \mathcal{M}^s_X(H,P)(-) \longrightarrow \mathrm{Hom}\big(-, M^s_X(H,P)\big),$$

which is bijective for any reduced point x_0.

(2) *For every scheme* N *and every natural transformation* $\Phi : \mathcal{M}^s_X(H,P)(-) \to \mathrm{Hom}(-,N)$ *there is a unique morphism* $\varphi : M^s_X(H,P) \to N$ *for which the following diagram commutes.*

$$\begin{tikzcd} \mathcal{M}^s_X(H,P)(-) \arrow[rr, "\Psi"] \arrow[dr, "\Phi"'] & & \mathrm{Hom}\big(-, M^s_X(H,P)\big) \arrow[dl, "\varphi_*"] \\ & \mathrm{Hom}(-,N) & \end{tikzcd}$$

Proof. See [17, Theorem 5.6]. □

Remark 2.6. (1) If a coarse moduli space exists for a given classification problem, then it is unique (up to isomorphism).

(2) A fine moduli space for a given classification problem is always a coarse moduli space for this problem but, in general, not vice versa. In fact, there is no a priori reason why the map

$$\Psi(S) : \mathcal{M}^s_X(H,P)(S) \longrightarrow \mathrm{Hom}\big(S, M^s_X(H,P)\big)$$

should be bijective for varieties S other than $\{pt\}$.

We refer to [13, Section 4.5] for general facts on the infinitesimal structure of the moduli space $M^s = M^s_X(H,P)$. Let me just recall that if E is a stable vector bundle on X with Hilbert polynomial P, represented by a point $[E] \in M^s$, then the Zariski tangent space of M^s at $[E]$ is canonically given by $T_{[E]}M^s \cong \mathrm{Ext}^1(E,E)$. If $\mathrm{Ext}^2(E,E) = 0$, then M^s is smooth at $[E]$. In general, we have the following bounds:

$$\dim_k \mathrm{Ext}^1(E,E) \geq \dim_{[E]} M^s \geq \dim_k \mathrm{Ext}^1(E,E) - \dim_k \mathrm{Ext}^2(E,E).$$

Remark 2.7. In spite of the great progress made during the last decades in the problem of moduli spaces of stable vector bundles on smooth projective varieties (essentially in the framework of the Geometric Invariant Theory by Mumford) many problems remain open, and for varieties of arbitrary dimension, very little is known about their local and global structure.

See [27], [25] or [26] for the definition of categorical quotient of a variety by the action of a group and its connection with moduli problems.

3. Moduli spaces of vector bundles on algebraic surfaces

Throughout this section X will be a smooth, irreducible, algebraic surface over the complex field and we will denote by $M_{X,H}(r;L,n)$ (resp., $\overline{M}_{X,H}(r;L,n)$) the moduli space of rank r, vector bundles (resp., torsion free sheaves) E on X, μ-stable (resp., semistable) with respect to a polarization H with $\det(E) = L \in \mathrm{Pic}(X)$ and $c_2(E) = n \in \mathbb{Z}$. Moduli spaces for stable vector bundles on smooth algebraic surfaces were constructed in the 1970's and quite a lot is known about them. In the 1980's, Donaldson proved that the moduli space $M_{X,H}(2; 0, n)$ is generically smooth of the expected dimension provided n is large enough (see [5]). As a consequence, he obtained some spectacular new results on the classification of C^∞ four-manifolds. Since then, many authors have studied the structure of the moduli space $M_{X,H}(r; L, n)$ from the point of view of algebraic geometry, of topology and of differential geometry; giving very pleasant connections between these areas.

Many interesting results have been proved, and before reminding you of some of them, let me just give one example to show how the geometry of the surface is reflected in the geometry of the moduli space.

Example 3.1 (Mukai: [23] and [24]). Let X be a K3 surface. Then, the moduli space $M_{X,H}(r; L, n)$ is a smooth, quasi-projective variety of dimension $2rn + (1-r)L^2 - 2(r^2-1)$ with a symplectic structure. In addition, if $M_{X,H}(r; L, n)$ is 2-dimensional and compact, then it is isomorphic to a K3 surface isogenous to X.

A more precise example could be the following one.

Example 3.2. Let $X \subset \mathbb{P}^3$ be a general quartic hypersurface. X is a K3 surface and its Picard group is generated by the restriction, $\mathcal{O}_X(1)$, of the tautological line bundle on $\mathbb{P}^3$ to X. We have an isomorphism

$$\rho : X \cong M_{X,\mathcal{O}_X(1)}\big(2; \mathcal{O}_X(-1), 3\big)$$

which on closed points $y \in X$ is defined by $\rho(y) := F_y$, F_y being the kernel of the epimorphism $H^0(X, \mathcal{I}_y(1)) \otimes \mathcal{O}_X \to \mathcal{I}_y(1)$.

From now on, we will assume that the discriminant

$$\Delta(r; L, n) := 2rn - (r-1)L^2 \gg 0.$$

The moduli space is empty if $\Delta(r; L, n) < 0$, by Bogomolov's inequality and, on the other hand, it is nonempty if $\Delta(r; L, n) \gg 0$. (See, e.g., [18] and [10].) For small values of the discriminant $\Delta(r; L, n)$ the moduli space $M_{X,H}(r; L, n)$ of vector bundles on an algebraic surface X can look rather wild; there are many examples of moduli spaces which are not of the expected dimension, and which are not irreducible nor reduced (see, e.g., [10], [28] or [20]). This changes when the discriminant increases and we have the following result which is one of the most important results in the theory of vector bundles on an algebraic surface X.

THEOREM 3.3. *Let H be an ample divisor on X. If $\Delta(r; L, n) \gg 0$, then the moduli space $M_{X,H}(r; L, n)$ is a normal, generically smooth, irreducible, quasi-projective variety of dimension $2rn - (r-1)L^2 - (r^2-1)\chi(\mathcal{O}_X)$.*

Proof. Generic smoothness was first proved by Donaldson in [5] for rank 2 vector bundles with trivial determinant, and by Zuo in [31] for general determinants. Asymptotic irreducibility was proved for the rank 2 case by Gieseker and Li in [11], and for arbitrary ranks by Gieseker and Li in [12] and by O'Grady in [28]. Finally, asymptotic normality was proved by Li in [14]. □

Another remark should be made. As we pointed out in Remark 2.3 the definition of stability depends on the choice of the polarization and the following natural question arises: let H and H' be two different polarizations, what is the difference between the moduli spaces $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$? It turns out that the ample cone of X has a chamber structure such that $M_{X,H}(r; L, n)$ only depends on the chamber of H and, in general, $M_{X,H}(r; L, n)$ changes when H crosses a wall between two chambers (see, e.g., [7], [9], [30] and [29]). However, we have (see [13, Theorem 4.C.7]) the following theorem.

THEOREM 3.4. *Let H and H' be ample divisors on X. If $\Delta(r; L, n) \gg 0$, then the moduli spaces $M_{X,H}(r; L, n)$ and $M_{X,H'}(r; L, n)$ are birational.*

The last result implies that for many purposes we can fix the polarization H; and this is what we do for studying the birational geometry of the moduli spaces $M_{X,H}(r; L, n)$. For example, we can reduce the study of the rationality of the moduli space $M_{X,H}(r; L, n)$ for any ample divisor H to the study of the rationality of $M_{X,H}(r; L, n)$ for a suitable ample divisor H.

In the last part of this section we turn our attention to the study of the rationality of the moduli space $M_{X,H}(r; L, n)$ and the computation of the Kodaira dimension of $\overline{M}_{X,H}(r; L, n)$. For $X = \mathbb{P}^2$, Maruyama (resp., Ellingsrud and Stromme) proved that if $c_1^2 - 4c_2 \not\equiv 0 (\mathrm{mod}\, 8)$, then the moduli space $M_{\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ of rank 2, $\mathcal{O}_{\mathbb{P}^2}(1)$-stable vector bundles on $\mathbb{P}^2$ with Chern classes c_1 and c_2 is rational (see [19] and [8]). Later on, Maeda proved that the rationality of $M_{\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ holds for all $(c_1, c_2) \in \mathbb{Z}^2$ provided $M_{\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ is nonempty [15]. In particular, $\overline{M}_{\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)}(2; c_1, c_2)$ has Kodaira dimension $-\infty$. For some ruled surfaces X, Qin also showed that $\overline{M}_{X,H}(r; L, n)$ has Kodaira dimension $-\infty$. As for K3 surfaces X, a consequence of Mukai's work [23] shows that $\overline{M}_{X,H}(r; L, n)$ has Kodaira dimension zero. More recently, Li has proved that if X is a minimal surface of general type with reduced canonical divisor, then $\overline{M}_{X,H}(r; L, n)$ is of general type (see [14]). All these indicate that the Kodaira dimension of $\overline{M}_{X,H}(r; L, n)$ is closely related to the Kodaira dimension of X and moduli spaces associated to rational surfaces should be rational. In fact, we have the following result.

THEOREM 3.5. *Let X be a smooth rational surface, $L \in \mathrm{Pic}(X)$ and $n \in \mathbb{Z}$. Assume that $\Delta(2; L, n) \gg 0$. Then, there exists an ample divisor H on X such that the moduli space $M_{X,H}(2; L, n)$ is rational.*

Proof. This was proved in [4, Theorem A] . □

Unless for the discussion of the Kodaira dimension and of the rationality, all stated results hold for arbitrary surfaces and we have not considered the branch of beautiful results that works for special surfaces like K3 surfaces, elliptic surfaces or ruled surfaces. Many interesting results are also missing; for instance, Picard group of moduli spaces, Fourier-Mukai transformations, symplectic structures, Gauge theoretical aspects of moduli spaces, and so forth.

4. Moduli spaces of vector bundles on high-dimensional varieties

Let X be a smooth, projective, n-dimensional variety over the complex field and let $M_{X,L}(r; c_1, \dots, c_{\min\{r,n\}})$ denote the moduli space of rank r, vector bundles E on X, μ-stable with respect to a polarization L with fixed Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$.

A major result, in the theory of vector bundles on an algebraic surface S, is the proof that the moduli space of rank r, with stable (with respect to a polarization L) vector bundles on S for fixed c_1 and fixed polarization L, is irreducible and smooth for large c_2. The result is not true for higher-dimensional varieties and it is rather common to have the existence of moduli spaces of stable vector bundles on X which are not irreducible nor smooth. Indeed, in [6] (resp., [1]), Ein (resp., Ancona and Ottaviani) proved that the minimal number of irreducible components of the moduli space of rank 2 (resp., rank 3) stable vector bundles on $\mathbb{P}^3$ (resp., $\mathbb{P}^5$) with fixed c_1 and c_2 going to infinity grows to ∞. Inspired by Ein's result we have proved the following theorem.

THEOREM 4.1. *Let X be a smooth projective 3-fold, $c_1, H \in \mathrm{Pic}(X)$ with H ample and $d \in \mathbb{Z}$. Assume that there exist integers $a \neq 0$ and b such that $ac_1 \equiv bH$. Let $M_{X,H}(c_1, d)$ be the moduli space of rank 2, μ-stable vector bundles (with respect to H) E on X, with $\det(E) = c_1$ and $c_2(E)H = d$ and let $m(d)$ be the number of irreducible components of $M_{X,H}(c_1, d)$. Then $\liminf_{d\to\infty} m(d) = +\infty$.*

Proof. See [2, Theorem 0.1] . □

See [21] for examples of singular moduli spaces of vector bundles on $\mathbb{P}^{2n+1}$ with $c_2 \gg 0$.

Nevertheless, we will see that for a $(d+1)$-dimensional, rational, normal scroll X and for suitable choice of $c_i \in H^{2i}(X, \mathbb{Z})$, $i = 1, 2$, and a fixed $L = L(c_1, c_2)$ the moduli space $M_{X,L}(2; c_1, c_2)$ is a smooth, irreducible, rational, projective variety. To prove this, we need to fix some more notation.

Take $\mathscr{E} := \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $0 = a_0 \le a_1 \le \cdots \le a_d$ and $a_d > 0$. Let

$$X := \mathbb{P}(\mathscr{E}) = \mathrm{Proj}\big(\mathrm{Sym}(\mathscr{E})\big) \xrightarrow{\pi} \mathbb{P}^1$$

be the projectived vector bundle and let $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ be the tautological line bundle. $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ defines a birational map

$$X := \mathbb{P}(\mathscr{E}) \xrightarrow{f} \mathbb{P}^N,$$

where $N = d + \sum_{i=0}^{d} a_i$. The image of f is a variety of dimension $d+1$ and minimal degree called *rational normal scroll*. By abuse of language, we also call X rational normal scroll.

Let H be the class in $\mathrm{Pic}(X)$ associated to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on X and let F be the fiber of π. We have

$$\mathrm{Pic}(X) \cong \mathbb{Z}^2 \cong \langle H, F\rangle \quad \text{with } H^{d+1} = \sum_{i=0}^{d} a_i; \qquad H^d F = 1; \qquad F^2 = 0.$$

Let E be a rank 2 vector bundle on a $(d+1)$-dimensional rational normal scroll X. Since $H^2(X, \mathbb{Z})$ is generated by the classes H and F, and $H^4(X, \mathbb{Z})$ is generated by the classes HF and H^2; the Chern classes $c_i(E) \in H^{2i}(X, \mathbb{Z})$, $i = 1, 2$ of E may be written as $c_1(E) = aH + bF$ and $c_2(E) = xH^2 + yHF$ with $a, b, x, y \in \mathbb{Z}$. Moreover, since a rank 2 vector bundle E on X is μ-stable with respect to a polarization L if and only if $E \otimes \mathcal{O}_X(D)$ is μ-stable with respect to L for any divisor $D \in \mathrm{Pic}(X)$, we may assume, without loss of generality, that $c_1(E)$ is one of the following: 0, H, F or $H+F$.

From now on, X will be a $(d+1)$-dimensional, rational, normal scroll. We compute the dimension and prove the irreducibility, smoothness and rationality of the moduli spaces $M_L(2; c_1, c_2)$ of rank 2 vector bundles E on X with certain Chern classes c_1 and c_2; and μ-stable with respect to a polarization L closely related to c_2. We want to stress that the polarization L that we choose strongly depends on c_2, our results turn out to be untrue if we fix c_1, L and c_2L^{d-1} goes to infinity. Indeed, for $d = 2$ and fixed L, the minimal number of irreducible components of the moduli space $M_L(2; c_1, c_2)$ of rank 2, μ-stable vector bundles with respect to L with fixed c_1 and c_2L going to infinity grows to infinity (it follows from Theorem 4.1).

Our approach will be to write μ-stable with respect to L, rank 2 vector bundles E on X, as an extension of two line bundles. A well-known result for vector bundles on curves is that any vector bundle of rank $r \geq 2$ can be written as an extension of lower rank vector bundles. For higher-dimensional varieties we may not be able to get such a nice result. (For instance, it is not true for vector bundles on $X = \mathbb{P}^n$.) However, it turns to be true for certain μ-stable with respect to L, rank 2 vector bundles E on rational normal scrolls.

THEOREM 4.2. *Let X be a $(d+1)$-dimensional, rational, normal scroll and c_2 an integer such that $c_2 > (H^{d+1} + d + 2)/2$. We fix the ample divisor $L = dH + bF$ on X with $b = 2c_2 - H^{d+1} - (1-\epsilon)$ and $\epsilon = 0, 1$. Then $M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d+2)$.*

Sketch of the Proof. We divide the proof into several steps.

Step 1: We prove that any vector bundle $E \in M_L(2; H + \epsilon F, (c_2 + \epsilon)HF)$ sits in an exact sequence of the following type:

$$0 \longrightarrow \mathcal{O}_X\big(H - c_2F\big) \longrightarrow E \longrightarrow \mathcal{O}_X\big((c_2 + \epsilon)F\big) \longrightarrow 0.$$

The key point for proving the first step is the fact that for any rank 2 vector bundle $E \in M_L(2; H+\epsilon F, (c_2+\epsilon)HF)$, $E(-H+c_2F)$ has a section whose scheme of zeros has codimension greater than or equal to 2 (see [3, Proposition 2.8]).

Step 2: For any vector bundle $E \in M_L(2; H+\epsilon F, (c_2+\epsilon)HF)$, we compute the Zariski tangent space of $M_L(2; H+\epsilon F, (c_2+\epsilon)HF)$ at $[E]$ and we get

$$\dim T_{[E]} M_L\big(2; H+\epsilon F, (c_2+\epsilon)HF\big) = 2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d+2).$$

Step 3: We prove that

$$M_L\big(2; H+\epsilon F, (c_2+\epsilon)HF\big) \cong \mathbb{P}\big(\mathrm{Ext}^1\big(\mathcal{O}_X((c_2+\epsilon)F), \mathcal{O}_X(H-c_2F)\big)\big).$$

It follows from the last step that the moduli space $M_L(2; H+\epsilon F, (c_2+\epsilon)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(d+1)c_2 - H^{d+1} + \epsilon(d+1) - (d+2)$. This completes the proof. □

Analogously we prove the following theorem.

THEOREM 4.3. *Let X be a $(d+1)$-dimensional, rational, normal scroll and c_2 an integer such that $c_2 > H^{d+1} + d + 1$. We fix the ample divisor $L = dH + bF$ on X with $b = c_2 - H^{d+1} - (1-\epsilon)$ and $\epsilon = 0, 1$. Then $M_L(2; \epsilon F, (2c_2+\epsilon)HF)$ is a smooth, irreducible, rational, projective variety of dimension $2(\epsilon d+1)c_2 - H^{d+1} + 2(\epsilon-1) - \epsilon(d+2)H^{d+1}$.*

Theorems 4.2 and 4.3 reflect nicely the general philosophy that, at least for suitable choice of the Chern classes and the polarization, the geometry of the underlying variety and of the moduli spaces are intimately related. We hope that phenomena of this sort will be true for other high-dimensional varieties.

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SOME USES OF MODULI SPACES IN PARTICLE AND FIELD THEORY

TSOU SHEUNG TSUN

In this talk I shall try to give an elementary introduction to certain areas of mathematical physics where the idea of moduli space is used to help solve problems or to further our understanding. In the wide area of gauge theory, I shall mention instantons, monopoles and duality. Then, under the general heading of string theory, I shall indicate briefly the use of moduli space in conformal field theory and M-theory.

1. Introduction

Physicists seldom define their terms. So although I know roughly what a moduli space is, and the sort of thing one does with it in physics, I was not really very sure of what exactly it is. So I asked Frances (Kirwan), just as the porters at Balliol College (where participants were lodged) did when they also wanted to know what a moduli space was. I have always taken it to be some sort of useful parameter space, convenient in the sense that mathematicians have already worked out all its properties (at least in the classical cases). But Frances told me something much more significant—she describes it as a parameter space in the *nicest possible way*.

So in the next 55 minutes or so, I shall try to give you a rough picture of how physicists have made use of this nice concept of a parameter space. We should note, however, that it is far from a one-way traffic. Much of the tremendous progress in 4-manifold theory, and a large part of it is done here, came about by studying certain moduli spaces occurring in mathematical physics.

A few notes of warning, however, are in place. For a hard-nosed or pragmatic physicist, (A) spacetime X has 4 dimensions, 3 space and 1 time, with an indefinite metric. By an indefinite metric I mean that the quadratic form giving the metric is not positive definite, so that two distinct points in spacetime can be null-separated. In fact, distances along light-paths are always zero. For him (or her) also (B) spacetime is by and large like $\mathbb{R}^4$, that is, (i) flat, (ii) looking more or less the same in all directions, (iii) real, and (iv) more or less infinite in all its four directions and hence noncompact.

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On the other hand, algebraic geometry is more about Riemannian manifolds and the best results are almost always obtained for the compact case. In order to make contact, the concept of spacetime has to be modified in several significant ways.

(1) One considers definite metrics, a process known as *euclideanization*. Then many nice things happen. In particular, the *wave operator*

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

which is hyperbolic, becomes the 4-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

which is elliptic, and for elliptic operators there are all sorts of good results like the index theorems. Euclideanization is done in the following: self-dual Yang-Mills theory, instantons, monopoles, Seiberg-Witten theory, strings,….

(2) Alternatively, one *complexifies* spacetime, and then the question of definite or indefinite metric disappears. In this case, one can use powerful complex manifold techniques including twistor theory. This is also where supersymmetry comes in mathematically. Moreover, by a change of point of view (see later), Riemann surfaces also play an important role. Complexification is done in superstrings, supersymmetric Yang-Mills theory, M-theory,….

(3) One also changes the topology of spacetime by *compactifying* some or all of its directions. In some cases, this is only a mild change, amounting to imposing certain decay properties at infinity (see later). In other cases, this gives rise to important symmetries of the theory. Compactification is done in instantons, superstrings, M-theory,….

(4) One either changes the number of spacetime dimensions or re-interprets some of them as other degrees of freedom. This dimensional change is done in strings, superstrings, monopoles, M-theory,….

At first sight, these modifications look drastic. The hope is that they somehow reflect important properties of the real physical world, and that the nice results we have do not disappear once we know how to *undo* the modifications. Surprisingly, the (largely unknown) mathematics underlying real 4-dimensional spacetime looks at present quite intractable!

2. Yang-Mills theory (Gauge theory)

Unlike most of the other theories I shall mention, Yang-Mills theory is an experimentally 'proven' theory. In fact, it is generally believed, even by hard-nosed or pragmatic physicists, that Yang-Mills theory is the basis of *all* of particle physics. From the physics point of view, Yang-Mills theory is the correct framework to encode the invariance of particle theory under the action of a symmetry group—the gauge group G—at each spacetime point. For example, let $\psi(x)$ be the wave-function of a quantum particle. Then the physical system is invariant under the action of the group

$$\psi(x) \longmapsto \Lambda(x)\psi(x), \quad \Lambda(x) \in G.$$

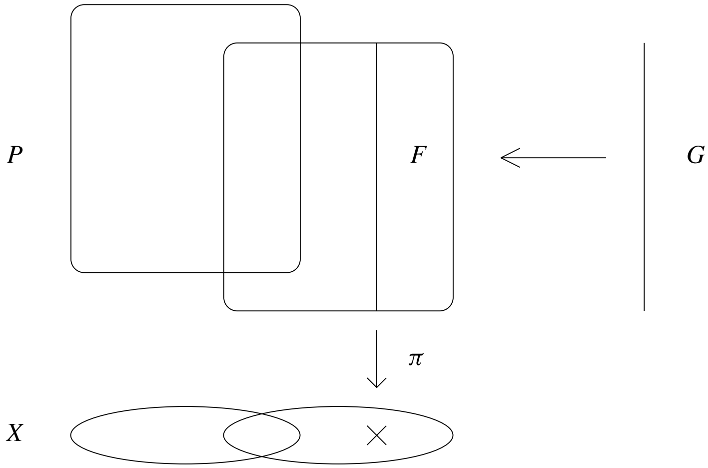

Figure 1. Sketch of a principal bundle

This invariance is known as *gauge invariance*. Now the groups that are most relevant to particle physics are $\mathrm{U}(1)$, $\mathrm{SU}(2)$, $\mathrm{SU}(3)$. However, we shall come across other groups as well. But for simplicity, we shall take $G = \mathrm{SU}(2)$, unless otherwise stated.

There is an additional ingredient in many favoured gauge theories, namely *supersymmetry*. This is a symmetry relating two kinds of particles: bosons (e.g., a photon) with integral spin and fermions (e.g., an electron) with half-integral spin. Spin is a kind of internal angular momentum which is inherently quantum mechanical. Since bosons and fermions, in general, behave quite differently (e.g., they obey different statistics), this symmetry is not observed in nature. However, one can imagine this symmetry holding for example at ultra-high energies. What makes this symmetry theoretically interesting is that many theories simplify and often become complex analytic with this extra symmetry, making much of the underlying mathematics accessible. Also the complex analyticity links such theories with most studies of moduli spaces.

Mathematically, Yang-Mills theory can be modelled (in the simplest case) by a principal bundle P (see Figure 1) together with a connection on it. I remind you that, roughly speaking, a *principal bundle* is a manifold P with a projection π onto a *base space* X, and a right action by the *structure group* G. In general, the base space can be any smooth manifold, but here we consider only the case of spacetime X. Above each point $x \in X$, the inverse image (called the *fibre*) $\pi^{-1}(x)$ is homeomorphic to G. The *total space* P is locally a product, in the sense that X is covered by open sets U_α and $\pi^{-1}(U_\alpha)$ is homeomorphic to $U_\alpha \times G$. A *connection* A is a 1-form on P with values in the Lie algebra $\mathfrak{g}$ of G, satisfying certain conditions and giving a prescription for differentiating vectors and tensors on X. It combines with the usual exterior derivative d to give the covariant exterior derivative d_A:

$$d_A = d + A$$

in such a way as to preserve gauge invariance.

Next, we need the curvature 2-form:

$$F_A = dA + AA \quad \left(F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu + ig[A_\mu, A_\nu]\right).$$

The second formula (in brackets) is the same as the first one, but written in local coordinates, or 'with indices', where $\mu = 0, 1, 2, 3$.

Since $\dim X = 4$ (for the moment, anyway), we have the Hodge star operator which takes 2-forms to 2-forms:

$$\begin{aligned} *\colon \Omega^2 &\longrightarrow \Omega^2 \\ F_A &\longmapsto {}^*F_A. \end{aligned}$$

In local coordinates, this can be written as

$${}^*F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

where $\epsilon_{\mu\nu\rho\sigma}$ is a completely skew symbol defined by $\epsilon_{0123} = 1$. Notice that

$$\begin{aligned} (*)^2 &= +1 \quad \text{in euclidean metric,} \\ (*)^2 &= -1 \quad \text{in Minkowskian metric.} \end{aligned}$$

Yang-Mills theory is given by the Yang-Mills action or functional

$$S(A) = \frac{1}{8\pi^2} \int_X \operatorname{tr}\left(F_A^* F_A\right) = \frac{1}{8\pi^2} \left\| F_A \right\|^2.$$

The curvature satisfies:

$$\begin{aligned} d_A F_A &= 0 \quad \text{(Bianchi identity),} \\ d_A {}^*F_A &= 0 \quad \text{(Yang-Mills equation).} \end{aligned}$$

These are the *classical* equations for Yang-Mills theory. Notice that the first one is an identity from differential geometry, and the second one comes from the first variation of the action.

The space of connections $\mathcal{A}$ is an affine space, but we are really interested in connections modulo gauge equivalence. Two connections A, A' are gauge equivalent if they are "gauge transforms" of each other:

$$A' = \Lambda^{-1} A \Lambda + \Lambda^{-1} d\Lambda.$$

In other words, $\Lambda(x) \in G$, Λ is a fibre-preserving automorphism of P invariant under the action of G. We shall use the symbol $\mathcal{G}$ for the group of gauge transformations Λ.

So we come to our first, most basic, *moduli space*

$$\bar{\mathcal{M}} = \mathcal{A}/\mathcal{G}.$$

It is in general infinite-dimensional with complicated topology.

We shall be interested in various subspaces or refinements of $\bar{\mathcal{M}}$.

One theoretical use of $\bar{\mathcal{M}}$ itself is in (the euclidean formulation of) quantum field theory, where with the Feynman path integral approach, one has to consider the integral of the exponential of the Yang-Mills action over $\bar{\mathcal{M}}$:

$$\int_{\bar{\mathcal{M}}} e^{-S(A)}.$$

But this integral is very difficult to define in general!

The moduli space $\bar{\mathcal{M}}$ has a singular set which represents the *reducible connections*, which are connections with holonomy group $H \subset G$ such that the centralizer of H properly contains the centre of G. We say then that the connection *reduces* to H. The complement $\mathcal{M}$ of this singular set is dense in $\bar{\mathcal{M}}$, and represents the irreducible connections. For $G = \mathrm{SU}(2)$, near an irreducible connection $\bar{\mathcal{M}}$ is smooth, but reducible connections lead to cone-like singularities in $\bar{\mathcal{M}}$.

2.1. Instantons. Recall that $G = \mathrm{SU}(2)$. Bundles P over X are classified by the second Chern class of the associated rank 2 vector bundle E (cf. Rosa-Maria Miró-Roig's talk):

$$k = c_2(E)[X] = \frac{1}{8\pi^2}\int_X \operatorname{tr} F_A^2 \in \mathbb{Z}.$$

We say that a connection A is self-dual (or anti-self-dual) if its curvature F_A satisfies

$$F_A = {}^*F_A \quad \big(\text{resp., } F_A = -{}^*F_A\big).$$

Then given any connection A, we can decompose the corresponding curvature F_A into its self-dual and anti-self-dual parts:

$$F_A = F_A^+ + F_A^-.$$

In the context of Yang-Mills theory a self-dual connection is called an *instanton*[1]:

$$F_A = {}^*F_A \Leftrightarrow F_A^- = 0.$$

In this case,

$$\text{Bianchi identity} \cong \text{Yang-Mills equation}.$$

In other words, a self-dual connection is *automatically* a classical solution.

Now we have

$$S(A) = \frac{1}{8\pi^2}\int_X |F_A^+|^2 + |F_A^-|^2,$$
$$k = \frac{1}{8\pi^2}\int_X |F_A^+|^2 - |F_A^-|^2.$$

Hence one has immediately

$$S(A) \geq k,$$

[1]It is a matter of convention whether one so defines a self-dual or anti-self-dual connection.

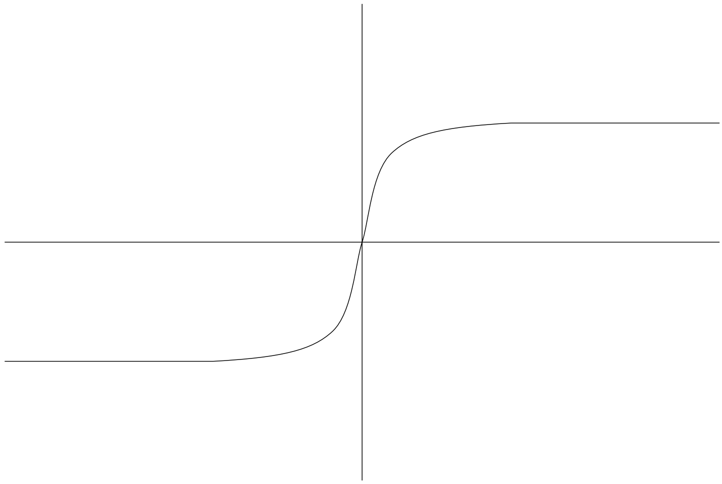

Figure 2. Sketch of a kink connecting two different states.

and

$$S(A) = k \Leftrightarrow F_A^- = 0.$$

So a self-dual connection gives an absolute minimum for the action. The integer k is known as the *instanton number*.

Warning: Nontrivial self-dual connections exist only when X is either euclidean or complex.

The *mathematical* magic of instantons is that instead of solving the second order Yang-Mills equations we have only the first-order self-duality equation to deal with. These connections can actually be constructed using euclidean *twistor methods* without explicitly solving any equations (cf. Tatiana Ivanova's talk).

Physically, the presence of instanton contribution in the path integral allows tunnelling between different vacua (i.e., lowest energy states) of the relevant Yang-Mills theory (namely, quantum chromodynamics for strong interactions or QCD). This role of the instantons can be compared to lower-dimensional objects such as "solitons" or topological defects called "kinks" which connect up two different states at infinity (see Figure 2). The two phenomena are quite similar, since "tunnelling" means a quantum particle can penetrate a potential barrier which a classical particle cannot go through, thus connecting two classically separate states. The effect of instantons is "nonperturbative" in the sense that such an effect cannot be obtained as a term in a power series expansion of g the coupling constant (which is measure of the "strength" of the interaction under consideration, and which appears for example in the nonlinear term of the curvature form $F_{\mu\nu}$). This is a direct manifestation of the fact that instantons are topological in nature and cannot be obtained by any "local" considerations such as power series expansions.

Since in euclidean space the Yang-Mills equations are elliptic, and concentrating on irreducible connections gets rid of zero eigenvalues, one can use the index theorem to count the "formal dimension" of *instanton moduli space*. Typically the smooth part of the moduli space will have this formal dimension as its actual dimension. For example,

$$X = S^4, \quad \dim_{\mathbb{C}}\left(\mathcal{M}_{I,k}\right) = 8k-3.$$

Uhlenbeck has given a unique compactification of $\mathcal{M}_I$, the union for all k. For more details about instanton moduli spaces, I again refer you to Tatiana Ivanova's talk.

2.2. Monopoles. Recall $G = \mathrm{SU}(2)$. Consider a Yang-Mills theory with a scalar field (called *Higgs field*) ϕ, together with a potential term $V(\phi)$ which is added to the Yang-Mills action. Suppose further that

$$V(\phi_0) = \text{ minimum for } |\phi_0| \neq 0,$$

and that $V(\phi)$ is invariant under a subgroup $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Then for those connections of P which are reducible to this $\mathrm{U}(1)$ subgroup, we can for certain purposes concentrate on this "residual gauge symmetry" and have a $\mathrm{U}(1)$ gauge theory. If we interpret this $\mathrm{U}(1)$ as Maxwell's theory of electromagnetism, then a nontrivial reduction of P can be regarded as a *magnetic monopole*. The *magnetic charge* k is given by the first Chern class of the reduced bundle. In fact we have the following exact sequence which gives us an isomorphism:

$$\begin{tikzcd}
\pi_2\big(\mathrm{SU}(2)\big) \arrow[r] \arrow[d, equal] & \pi_2\big(\mathrm{SU}(2)/\mathrm{U}(1)\big) \arrow[r, "\sim"] & \pi_1\big(\mathrm{U}(1)\big) \arrow[r] & \pi_1\big(\mathrm{SU}(2)\big) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

Unlike the original magnetic monopole considered by Dirac, these 't Hooft-Polyakov monopoles have finite energy and are the soliton solutions of the field equations corresponding to the action:

$$S(A,\phi) = S(A) + \|D\phi\|^2 + \lambda\big(1-|\phi|^2\big)^2,$$

where the last term is the usual form of the potential $V(\phi)$. From this we get the Yang-Mills-Higgs equations (YMH):

$$\begin{aligned}
D_A F &= 0, \\
D_A{}^*F &= -\big[\phi, D_A\phi\big], \\
D_A{}^*D_A\phi &= 2\lambda\phi\big(|\phi|^2-1\big).
\end{aligned}$$

Now we specialise to a certain limit, the Prasad-Somerfield limit: $V(\phi) = 0$, but $|\phi| \to 1$ at infinity. Then the Yang-Mills-Higgs system becomes:

$$\begin{aligned}
D_A F &= 0, \\
D_A{}^*F &= -\big[\phi, D_A\phi\big], \\
D_A{}^*D_A\phi &= 0.
\end{aligned}$$

Consider next a Yang-Mills theory in euclidean $\mathbb{R}^4$, invariant under x_4-translations. Then we can write

$$A = A_1\,dx_1 + A_2\,dx_2 + A_3\,dx_3 + \phi\,dx_4,$$

where A_1, A_2, A_3, ϕ are Lie algebra-valued functions on $\mathbb{R}^3$. The action can be written as

$$S(A) = \left\| F_A \right\|^2 = \|F\|^2 + \|D\phi\|^2,$$

where now F is the curvature of the connections in three dimensions:

$$A' = A_1\,dx_1 + A_2\,dx_2 + A_3\,dx_3,$$

and D is the corresponding 3-dimensional covariant derivative. In this way, we can make the following identification since the actions for the two theories are identical:

$$\text{YMH on } \mathbb{R}^3 \cong \text{dimensionally reduced YM on } \mathbb{R}^4.$$

In this case,

$$F_A = {}^*F_A \Rightarrow \text{first 2 YMH}.$$

Hence a solution to the Bogomolny equation

$$F = {}^*D_A\phi$$

gives a solution of YMH. These are known as "static monopoles".

The moduli spaces $\mathcal{M}_k$ corresponding to a given charge k are well studied, at least for $k = 1, 2$. The translation group $\mathbb{R}^3$ acts freely on $\mathcal{M}_k$, so does an overall phase factor S^1. Dividing these out we get the reduced monopole moduli spaces $\mathcal{M}_k^0$, $\dim_{\mathbb{C}} = 4k - 4$. Taking the k-fold covers, one obtains:

$$\tilde{\mathcal{M}}_k \cong \mathbb{R}^3 \times S^1 \times \tilde{\mathcal{M}}_k^0.$$

The special case of $k = 2$ has been studied by Atiyah and Hitchin as an entirely novel way of obtaining the scattering properties of two monopoles, using a metric on $\mathcal{M}_2^0$ they discovered, and assuming (with Manton) that geodesic motion on it describes adiabatic motion of the two monopoles. This is the most direct use that I know of moduli space for deriving something akin to dynamics!

2.3. Topological field theory. I wish just to mention a class of quantum field theories called *topological quantum field theories* (TQFT), where the observables (correlation functions) depend only on the global features of the space on which these theories are defined, and are independent of the metric (which, however, may appear in the classical theory). Atiyah gave an axiomatic approach to these, but there are so many local experts here that I do not feel justified in expanding on that!

Instead, I shall just indicate the role of moduli space in Witten's approach. Starting with a moduli space $\mathcal{M}$ one can get fields, equations and symmetries of the theory. Witten postulates the existence of certain operators $\mathbb{O}_i$ corresponding to cohomology classes η_i of $\mathcal{M}$ such that

$$\langle \mathbb{O}_1 \cdots \mathbb{O}_n \rangle = \int_{\mathcal{M}} \eta_1 \cdots \eta_n,$$

where $\langle \cdots \rangle$ denotes the correlation function of the operators. Hence he obtains these correlation functions as intersection numbers of $\mathcal{M}$, using Donaldson theory. So in a sense the TQFT is entirely defined by $\mathcal{M}$.

The observables called correlation functions can best be understood in the case of, for example, a 2-point function in statistical mechanics. This is the probability, given particle 1, of finding particle 2 at another fixed location.

To go into any further details about TQFT would require more detailed knowledge both of quantum field theory and supersymmetry. These would lead us unfortunately too far from the context of this workshop.

2.4. Seiberg-Witten theory. Recall that a spin structure on X is a lift of the structure group of the tangent bundle of X from SO(4) to its double cover $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$. Because of this isomorphism, one can represent a spin structure more concretely as a pair of complex 2-plane bundles $S^+, S^- \to X$, each with structure group SU(2). A slightly more general concept is a spin^c structure over X, which is given by a pair of vector bundles W^+, W^- over X with an isomorphism for the second exterior powers

$$\Lambda^2 W^+ = \Lambda^2 W^- = L, \text{ say},$$

such that one has locally

$$W^{\pm} = S^1 \otimes L^{1/2},$$

where $L^{1/2}$ is a local square root of $L : L^{1/2} \otimes L^{1/2} = L$.

Given a spin^c manifold X, the Seiberg-Witten equations (SW) are written for a system consisting of (1) a unitary connection A on $L = \Lambda^2 W^{\pm}$, and (2) ψ a section of W^+. Then these equations are:

$$\begin{aligned} D_A \psi &= 0, \\ F_A^+ &= -\tau(\psi, \psi), \end{aligned}$$

where τ is a sesquilinear map $\tau\colon W^+ \times W^+ \to \Lambda^+ \otimes \mathbb{C}$.

The Seiberg-Witten equations (SW) can be obtained from varying the following functional:

$$E(A, \psi) = \int_X \left|D_A \psi\right|^2 + \left|F_A^+ + \tau(\psi, \psi)\right|^2 + \frac{R^2}{8} + 2\pi^2 c_1(L)^2,$$

where R is the scalar curvature of X and $c_1(L)$ is the first Chern class of L. Notice that the last two terms depend only on X and L, so that solutions of SW are absolute minima of E on the given bundle L.

The relevant moduli space here is the space $\mathcal{M}$ of all irreducible solution pairs (A, ψ), modulo gauge transformations. The Seiberg-Witten invariants are then homology classes of $\mathcal{M}$, independent of the metric on X. These invariants prove very useful in 4-manifold theory. In particular, Seiberg and Witten give a "physicist's proof" that the instanton invariants of certain 4-manifolds (namely with $b^+ > 1$, where b^+ is the dimension of the space of self-dual harmonic forms) can be expressed in terms of the Seiberg-Witten invariants.

From the quantum field theory point of view, the importance of Seiberg-Witten theory lies in the concept of *duality*. In a modified version of Yang-Mills theory, called $N = 2$ supersymmetric Yang-Mills theory, the quantum field theory is described by a scale parameter t and a complex parameter u (here supersymmetry is essential). In the limit $t \to \infty$, the theory is described by an analytic function τ of u. If $b^+(X) > 1$, then τ is *modular* (in the classical sense) with respect to the action of $\mathrm{SL}(2, \mathbb{Z})$. This means, in particular, that a theory with parameter u is related to a theory with parameter u^{-1} in a definite and known way. The transformation $u \mapsto u^{-1}$ corresponds to changing the coupling constant to its inverse. Hence for the magnetic monopoles of the theory this represents a *duality* transformation: from *electric* with coupling e to *magnetic* with coupling $\tilde{e}$ and vice versa, since Dirac's quantization condition states that $e\tilde{e} = 1$ in suitable units. By relating a "strongly coupled" theory to a "weakly coupled" theory, one can hope to obtain results on the former by performing perturbative calculations (which are meaningless when coupling is strong) in the latter. By inspecting their moduli spaces one is often able to identify pairs of dually related theories.

3. String and related theories

I shall be extremely brief about these theories. The reason is, apart from my own obvious ignorance, that they are considerably more complicated than gauge theories and require much more knowledge not only of quantum physics but also of algebraic geometry than can reasonably be dealt with in this workshop. My aim here is just to give a taste of some immensely active areas of research in mathematical physics in recent years where moduli spaces play an important role.

The gist of string theory is that the fundamental objects under study are not point-like particles as in gauge field theories but 1-dimensional extended strings. These strings are really the microscopic quantum analogues of violin strings: they move in space and they also vibrate. The equation of motion of a free string can be obtained from an action which is similar to that for a massless free particle. In the latter case we have

$$S_0 = \int d\tau \, \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

which is just the *length* of the "worldline" in spacetime X traced out by the particle as it travels through space. Here $\eta_{\mu\nu}$ is the metric on X and x^μ are the coordinates of the particle. For the string the free action is the *area* of the "worldsheet" (with coordinates σ, τ) traced out by the 1-dimensional string in spacetime X:

$$S_1 = \int d\sigma \, d\tau \, \eta^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu,$$

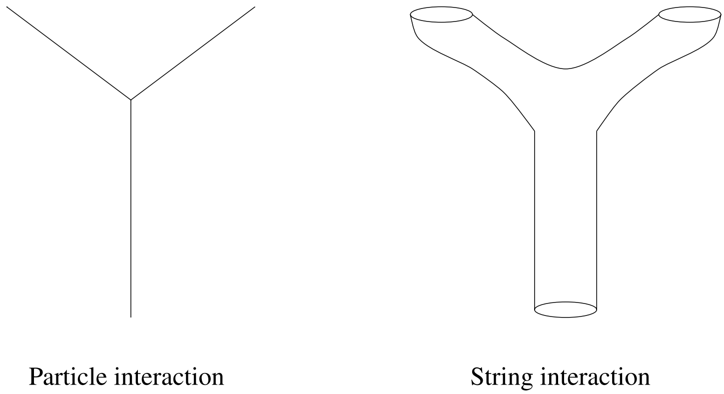

Figure 3. Schematic representation of particle and string interactions.

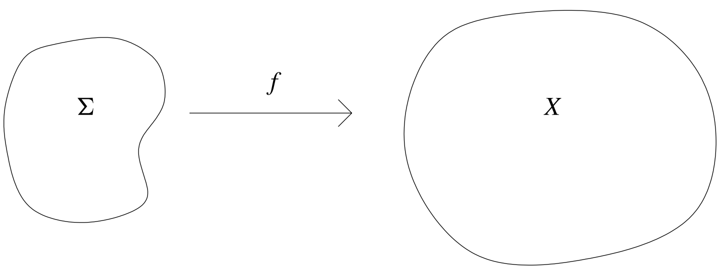

Figure 4. Embedding worldsheet into spacetime.

where the indices $\alpha, \beta = 0, 1$ refer to the worldsheet. Varying S_1 with respect to x gives simply the 2-dimensional wave equation:

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)x^\mu = 0.$$

We see that, in this context, spacetime coordinates can be regarded as fields on the 2-dimensional surface which is the worldsheet.

Interaction between strings are given by the joining and splitting of strings so that the resultant worldsheet can be visualized, on euclideanization, as a Riemann surface Σ with a given genus (see Figure 3). For example, a hole in Σ can be obtained by one closed string splitting into two and then joining together again. In fact, a useful way of looking at string theory is to think of it as being given by an embedding f of a Riemann surface Σ into spacetime X (Figure 4).

3.1. Conformal field theory. We have written the action S_1 for a free string in terms of a particular parametrization of Σ, but obviously the physics ought to be invariant under reparametrizations. The group of reparametrizations on Σ is the infinite-dimensional conformal group, and that is the symmetry group of string theory.

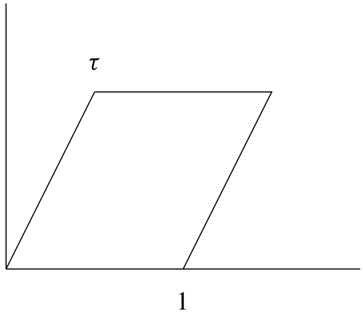

Figure 5. A 2-torus represented on the complex plane.

On the other hand, on a given Riemann surface Σ, one can consider certain field theories which have this invariance. These are called conformal field theories (CFT) and play important roles in statistical mechanics and critical phenomena (e.g., phase change), when the theories become independent of the length scale (so that quantities are defined only up to conformal transformations).

The concept of moduli plays an important role in CFT. In fact, the original idea of modulus is defined for Riemann surfaces (see talk by Frances Kirwan). So a torus T^2 has one modulus τ (see Figure 5). The conformal structure of T^2 is invariant under the action of the *modular group* $\mathrm{SL}(2, \mathbb{Z})$ on τ.

CFT are often studied for their own sake, but as far as string theories are concerned their use lies in the fact that they are the terms in a first-quantized, perturbative formulation of string theory. Schematically, one can think of string theory as the "sum over g" of CFT on Riemann surfaces of genus g. Unfortunately, this "summation" has never yet been given a precise meaning. What provides some hope that the problem may be tractable is the fact that the infinite-dimensional integral $\int e^{-S_1(x)}$ occurring in the path integral formalism can be reduced to one on the moduli space of the Riemann surface, which is finite-dimensional.

3.2. Various string theories. Up to now I have been carefully vague about the nature of spacetime X in string theory. It turns out that to get a consistent, first-quantized theory, one needs X to have twenty-six dimensions! If we modify the theory by adding supersymmetry to produce a *superstring* theory, then $\dim X = 10$. However, this potentially disastrous requirement has been turned to good use to produce interesting theories in four dimensions, as we now briefly sketch.

We shall concentrate on the supersymmetric version as being the more favoured by string theorists, in that we now assume $\dim X = 10$. Imagine that one can compactify six of these ten dimensions so that

$$X \cong K \times \mathbb{R}^4$$

with K a compact 6-dimensional space, and moreover that the size of K is small. Since the length is an inverse measure of energy, this means that to observers of low-energy (such as us) spacetime will just look 4-dimensional and the other six dimensions are curled up so tight we cannot see them. The often-quoted example is that a water pipe looks like a thin line from a distance.

Not only that, the symmetries of X can be factored into that of $\mathbb{R}^4$ (the usual ones) and that of K. The latter can then be interpreted as the internal symmetries of Yang-Mills theory. In fact, the choice of K is dictated by which gauge symmetry one wants.

There are in all five string theories. A string can be open (homeomorphic to an interval) or closed (homeomorphic to a circle). An open string theory is called Type I. For closed strings, depending in the boundary conditions one imposes, one has Type IIA or Type IIB. If one combines both the usual and the supersymmetric versions one obtains the heterotic string, with gauge group (after suitable compactification) either $E_8 \times E_8$ or SO(32). The $E_8 \times E_8$ heterotic string is particularly favoured as being able to include various Yang-Mills theories which are important in particle physics.

3.3. *M*-Theory. One can generalize the 1-dimensional strings to higher-dimensional objects called "membranes"; similarly superstrings to "supermembranes". The study of these last objects have become particularly fashionable, especially after the introduction of something called M-theory.

Now supersymmetry can also be made into a local gauge theory which is then called *supergravity*. It was shown some time ago that in supergravity, $\dim X \leq 11$, so 11-dimensional supergravity was studied as being in some sense a unique theory.

M-theory is perceived as an 11-dimensional supergravity theory, where the 11-dimensional manifold X can be variously compactified to give different superstring theories. Moreover, solitonic solutions are found which are supermembranes. By examining the moduli of these solutions one can connect pairs of underlying string theories. For example, reminiscent of the Seiberg-Witten duality and using the modular transformations on the modulus τ of the torus (in one of the compactifications of X), one can connect the two different versions of the heterotic string. In fact, by using both compactification and duality one finds that M-theory can give rise to all the five superstring theories mentioned above. So in some sense, all the five theories are equivalent and one can imagine that they are just different perturbative expansions of the same underlying M-theory.

Most recently, Maldecena suggested that M-theory on compactification on a particular 5-dimensional manifold (called anti-de Sitter space), including all its gravitational interactions, may be described by a (nongravitational) Yang-Mills theory on the boundary of X which happens to be 4-dimensional Minkowski space (i.e., flat spacetime). This opens up some new vistas in the field.

Although progress is made in an almost day-to-day basis, we are still waiting for a fuller description, perhaps even a definition, of M-theory. Meanwhile, it has generated a lot of interest and especially intense study into the various moduli spaces that occur.

4. Conclusion

I have endeavoured to describe a few physical theories in which moduli space plays an important role. However, I must say that the success in the reverse direction is more spectacular—using Yang-Mills moduli spaces (in different specializations) to understand

4-manifolds, following Donaldson, Kronheimer and many others. At the beginning I have explained why the success in physics is more restricted. Nevertheless, there are many high points:

(1) Self-dual Yang-Mills $\rightsquigarrow$ instantons $\rightsquigarrow$ vacuum structure of QCD.

(2) Monopole moduli spaces $\rightsquigarrow$ identification of pairs of dual theories in Seiberg-Witten scheme $\rightsquigarrow$ hope for possibility of practical computations in quantum field theory.

(3) Classification of conformal field theories $\rightsquigarrow$ application of theoretical statistical mechanics.

(4) Identifying moduli spaces to connect up the different string theories $\rightsquigarrow$ leading to a unification in eleven dimensions?

But for lack of time and expertise, I have omitted many other areas of mathematical physics being actively pursued at present in which moduli spaces play significant roles.

Acknowledgement

I thank Sylvie Paycha for taking time to read and comment on a first draft of this paper.

The following is only a small selection of articles that I have used in preparing this talk. They are in no way even representative.

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ON THE USE OF PARAMETER AND MODULI SPACES IN CURVE COUNTING

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In order to solve problems in enumerative algebraic geometry, one works with various kinds of parameter or moduli spaces: Chow varieties, Hilbert schemes, Kontsevich spaces. In this note we give examples of such spaces. In particular we consider the case where the objects to be parametrized are algebraic curves lying on a given variety. The classical problem of enumerating curves of a given type and satisfying certain given conditions has recently received new attention in connection with string theories in theoretical physics. This interest has led to much new work—on the one hand, within the framework of more traditional algebraic geometry, on the other hand, with rather surprising results, using new methods and ideas, such as the theory of quantum cohomology and generating functions.

1. Introduction

Enumerative geometry has a long history. Apollonius of Perga (262–200 B.C.) considered and solved problems like the following: construct all circles tangent to three given circles in the plane. The enumerative part of this problem is to determine the *number* of solutions: there is one such circle containing (or circumscribing) the three circles, three containing precisely two, three containing only one, and one containing none, hence the answer is eight.

Similar questions can be asked for arbitrary conics (curves of degree 2) in the complex projective plane $\mathbb{P}^2 := \mathbb{P}^2_{\mathbb{C}}$—e.g., how many conics are tangent to five given conics. A conic in $\mathbb{P}^2$ is given by the six coefficients of its defining equation (up to multiplication by a nonzero scalar), hence the parameter space of conics can be identified with $\mathbb{P}^5$. The points corresponding to *degenerate* conics (pairs of lines) form a hypersurface in $\mathbb{P}^5$, and the points corresponding to "double" lines form a 2-dimensional subvariety V in this hypersurface. For a given conic, the points corresponding to conics tangent to that conic, form a hypersurface of degree 6. Hence one might be led to think (as Jakob Steiner did in 1848), that there are $6^5 = 7776$ conics tangent to five given conics—in other words, that

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http://books.hindawi.com/9775945011/

the points corresponding to these conics are the points of intersection of five hypersurfaces, each of degree 6. This argument is wrong, however, because the parameter space is not "complete" with respect to the given problem in the following sense: any such "tangency condition" hypersurface contains the subvariety V of double lines, hence their intersection is never finite. The correct solution to the problem is 3264, as was found by de Jonquières (1859) and Chasles (1864). Essentially, what Chasles did, was to replace the parameter space $\mathbb{P}^5$ with a space B, whose points correspond to pairs consisting of a conic and its dual conic, and all limits of such pairs. The space B is the blow-up of $\mathbb{P}^5$ along V, and on B the intersection of the "tangency conditions" is finite. (See [11] for the history and the details.) More generally, one can ask to determine the *characteristic numbers* $N_{a,b}$ of a given family of plane curves; here $N_{a,b}$ is, by definition, the number of curves passing through a given points and tangent to b given lines, where $a+b$ is equal to the dimension of the family. Classically, this problem was solved—by Schubert and Zeuthen—for curves of degree at most 4, and these numbers have been verified by modern, rigorous methods.

To solve problems like the ones above, for example, to count curves lying on a given variety and satisfying certain conditions, a natural procedure is to represent the curves as points in some space, and then to represent the conditions as cycles on this parameter space. If the intersection theory of the parameter space is known, then the solution to a given enumerative problem can be obtained as the intersection number of the cycles corresponding to the given conditions (at least up to multiplicities of the solutions)—provided the cycles intersect properly.

A particular problem of this kind, which goes back to Severi and Zariski, is the following: given an r-dimensional family of curves on a surface, determine the number of curves in that family having r nodes (a *node* is an ordinary double point, that is, a singular point formed by two branches of the curve meeting transversally). The family can, for example, be a subsystem of a complete linear system, given by imposing the curves to pass through a certain number of points on the surface. There has been a lot of work on this problem in the last few years—here are just a very few sample references: [22], [3], [8], [2], [23], [6], [12], [21], [1].

The more nodes (or other singularities) a curve has, the smaller geometric genus it has. Therefore, one can also consider enumerative problems where instead of fixing the number (and type) of singularities, one fixes the geometric genus of the curves. For example, if one considers *rational* curves (i.e., curves of geometric genus zero) in the projective plane, then such an irreducible nodal curve of degree d must have $(d-1)(d-2)/2$ nodes. The question of enumerating rational curves is the one that first came up in the context of string theory in theoretical physics, and it is also one that has been central to many problems in algebraic and symplectic geometry.

In Section 2, we give a very brief presentation of three parameter spaces: the *Chow variety*, the *Hilbert scheme*, and the *Kontsevich moduli space of stable maps*. We elaborate a little on the last, which is the newest of the three, and we also define Gromov-Witten invariants in certain cases. Section 3 gives a simple example of a situation where the three above spaces lead to different compactifications of the same space, namely that of twisted cubic curves. In Section 4, we give a short introduction to quantum cohomology and show how the associativity of the quantum product can be used to deduce a recursive formula for the number of rational plane curves of given degree d, passing through $3d-1$ general points.

This note is only meant as a tiny introduction to what has recently become a very lively area of research. No proofs are given, and not all statements are completely true the way they are written. I refer to the papers in the bibliography for precise statements and proofs, more material, and further references. In particular, parts of Sections 2 and 4 draw heavily on [5] and [10].

Notation. I use standard notation from algebraic geometry (as in [7]). Note that a complex, projective, nonsingular *variety* can be considered as a complex analytic manifold, and more generally, a *projective scheme* can be considered as a complex analytic space (see [7, p. 438]). A *curve* (resp. a *surface*) is a variety of (complex) dimension 1 (resp. 2).

2. Parameter and moduli spaces

Let $X \subset \mathbb{P}^n$ be a complex, projective, nonsingular variety. There are at least three approaches to representing the set of curves $C \subset X$:

(1) Chow variety: its points correspond to 1-dimensional cycles on X.

(2) Hilbert scheme: its points correspond to 1-dimensional subschemes of X.

(3) Kontsevich space: its points correspond to stable maps from a curve to X.

The first approach is the oldest; it goes back to Cayley, but was developed by Chow (see [14, p. 40]). The idea is to parametrize effective 1-dimensional cycles $C = \sum n_i C_i$ (the C_i are reduced and irreducible curves and $n_i \geq 0$) on X by associating to each such C a hypersurface $\Phi(C)$ in $\mathbb{P}^{n*} \times \mathbb{P}^{n*}$ (where $\mathbb{P}^{n*}$ is the *dual* projective space whose points are hyperplanes in $\mathbb{P}^n$): intuitively, $\Phi(C)$ is the set of pairs of hyperplanes (H, H') such that $C \cap H \cap H' \neq \emptyset$. For each C, the coefficients of $\Phi(C)$ determine a point in an appropriate projective space, and the union of these points, as C varies, is $\mathrm{Chow}_1(X)$. In order to get something of reasonable size, we restrict the set of curves we consider by fixing the degree, say d, of the cycle. The corresponding parameter space is denoted $\mathrm{Chow}_{1,d}(X)$.

The second approach is due to Grothendieck. It gives a projective scheme, $\mathrm{Hilb}(X)$, which parametrizes all closed subschemes of a given projective variety X. The advantage with this approach is that there exists a universal flat family of subschemes having the Hilbert scheme as a base. In fact, to give a morphism from a scheme T to the Hilbert scheme, is equivalent to giving a flat family of schemes over T, where each fiber is a subscheme of X. Since the Hilbert polynomial is constant in a flat family, the Hilbert scheme splits into (not necessarily irreducible) components $\mathrm{Hilb}^{P(t)}(X)$ according to the Hilbert polynomial $P(t)$. For a projective curve, the Hilbert polynomial is of the form $P(t) = dt + 1 - g_a$, where d is the degree and g_a the arithmetic genus of the curve.

The last approach is relatively new and is part of the "revolution" in enumerative geometry due to the appearance of the physicists on the scene. The Kontsevich moduli space of pointed morphisms can be defined as follows (see [15], [5]). Fix an element $\beta \in A_1X := H_2(X;\mathbb{Z})$ and consider the set of isomorphism classes of pointed morphisms:

$$M_{g,n}(X,\beta) = \big\{\big(\mu : C \longrightarrow X; p_1,\ldots,p_n\big) \mid \mu_*([C]) = \beta\big\}/\backsim,$$

where C is an irreducible smooth curve of genus g, the p_i's are distinct points on C, and

$$\left(\mu : C \longrightarrow X; p_1, \ldots, p_n\right) \backsim \left(\mu' : C' \longrightarrow X; p'_1, \ldots, p'_n\right)$$

if there exists an isomorphism $\nu : C \to C'$ with $\mu' \circ \nu = \mu$ and $\nu(p_i) = p'_i$, for $i = 1, \ldots, n$. By adding the so-called *stable* pointed morphisms from not necessarily irreducible curves, one obtains a compactification $\overline{M}_{g,n}(X, \beta)$ of this space, which is a *coarse moduli space* (see [9] and [18]). In particular, if X is a point (so that $\beta = 0$), then $\overline{M}_{g,n}(\{\text{point}\}, 0) = \overline{M}_{g,n}$ is the usual Deligne-Mumford moduli space of stable, n-pointed curves of genus g.

In what follows, we shall only consider the case where $g = 0$ (so that $C = \mathbb{P}^1$) and X is convex (e.g., X is a projective space, a Grassmannian, a flag variety, ...). Then one can show that $M_{0,n}(X, \beta)$ is a normal projective variety of dimension

$$\dim X + \int_\beta c_1\left(T_X\right) + n - 3,$$

where $c_1(T_X)$ denotes the first Chern class of the tangent bundle of X, and $\int_\beta \alpha$ is the degree of the zero cycle $\alpha \cap \beta$. Set $\overline{M} = \overline{M}_{0,n}(X, \beta)$, and let

$$\rho_i : \overline{M} \longrightarrow X$$

denote the ith evaluation map, given by

$$\rho_i\left(\mu : C \longrightarrow X; p_1, \ldots, p_n\right) = \mu\left(p_i\right).$$

If $\gamma_1, \ldots, \gamma_n \in A^*X := H^*(X; \mathbb{Z})$ are cohomology classes, we define *Gromov–Witten invariants* as follows:

$$I_\beta\left(\gamma_1, \ldots, \gamma_n\right) = \int_{\overline{M}} \rho_1^* \gamma_1 \cup \cdots \cup \rho_n^* \gamma_n.$$

Assume each γ_i is effective, that is, γ_i is equal to the class $[\Gamma_i]$ of some subvariety Γ_i of X. Assume moreover that $\sum_i \operatorname{codim} \Gamma_i = \dim \overline{M}$. If the Γ_i are in "general position", the Gromov-Witten invariants have enumerative significance:

$$I_\beta\left(\gamma_1, \ldots, \gamma_n\right) = \deg \rho_1^{-1}\left(\Gamma_1\right) \cap \cdots \cap \rho_n^{-1}\left(\Gamma_n\right)$$

is the number of pointed maps $(\mu : C \to X; p_1, \ldots, p_n)$ such that $\mu_*([C]) = \beta$ and $\mu(p_i) \in \Gamma_i$. This is the same as the number of rational curves in X of class β and meeting all the subvarieties Γ_i.

Example. Let $X = \mathbb{P}^2$ and $\beta = d$ [line]. We shall write $M_{0,n}(\mathbb{P}^2, d)$ instead of $M_{0,n}(\mathbb{P}^2, d$ [line]). This space has dimension $2 + 3d + n - 2 = 3d - 1 + n$, since $\int_\beta c_1(T_X) = 3d$. Consider the case $n = 3d - 1$. Take points $x_1, \ldots, x_{3d-1} \in X$ in general position, and set $\Gamma_i = \{x_i\}$. Then

$$\sum_i \operatorname{codim} \Gamma_i = 2(3d - 1) = \dim M_{0,n}\left(\mathbb{P}^2, d\right)$$

holds, and

$$I_\beta\big([\Gamma_1], \ldots, [\Gamma_{3d-1}]\big) = N_d$$

is the number of rational plane curves of degree d passing through the $3d-1$ points $x_1, \ldots, x_{3d-1}$.

There exist linear relations between the boundary components of the compactification $\overline{M}_{0,n}(\mathbb{P}^2, d)$ of $M_{0,n}(\mathbb{P}^2, d)$, and these can be used to find a recursive formula for the numbers N_d in terms of N_{d_i} with $d_i < d$ (see [5, 0.6]). In Section 4, we shall indicate how this formula also can be derived from quantum cohomology.

Consider the case of plane conics, that is, take $d = 2$. In this case, both $\mathrm{Chow}_{1,2}(\mathbb{P}^2)$ and $\mathrm{Hilb}^{2t+1}(\mathbb{P}^2)$ are equal to the space $\mathbb{P}^5$ of plane conics. Hence, as we have seen, neither is good for enumerative problems involving tangency conditions. Classically, one considered the variety B of *complete* conics: $B \subset \mathbb{P}^5 \times \mathbb{P}^{5*}$ is the set of pairs of a conic and its dual conic (the conic in the dual projective plane whose points are the tangent lines of the original conic) and limits of such pairs; one shows that B is equal to the blow-up of $\mathbb{P}^5$ in the locus V corresponding to double lines. The limits of a pair consisting of a conic and its dual conic can be identified with the following three types of configurations: a pair of lines (the limit of the dual conic in this case is the "double line" consisting of all lines through the point of intersection of the line pair), a line with two marked points (the limit of the dual is the union of the sets of lines through each of these points), and a line with one marked point (the dual is the set of lines through the point, considered as a "double line" in the dual plane).

The space $M_{0,0}(\mathbb{P}^2, 2)$ is the set of isomorphism classes of maps $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ such that the image cycle $\mu_*(\mathbb{P}^1)$ has degree 2. The class of a map μ which is one-to-one is determined by its image, $\mu(\mathbb{P}^1)$, which is a nonsingular conic. A map which is two-to-one is a degree 2 map from $\mathbb{P}^1$ to some line in $\mathbb{P}^2$. Its isomorphism class is determined by that line together with two distinct points on it (the two branch points of the map). The maps corresponding to points on the boundary of the compactification $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ are maps from the union of two $\mathbb{P}^1$'s intersecting in a point; if the map is an immersion, its isomorphism class is determined by its image, the union of two lines—otherwise, it maps the two $\mathbb{P}^1$'s onto the same line, and its class is determined by that line together with the point which is the image of the intersection point of the two $\mathbb{P}^1$'s. Hence we can indeed identify $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ with the space B of complete conics (see [5, 0.4]).

The example above is typical for hypersurfaces, in the sense that the (relevant components of the) Chow variety and Hilbert scheme are equal for hypersurfaces, e.g., for curves on surfaces. We shall see in the next section that this does not hold when we consider curves on higher dimensional varieties.

3. Twisted cubic curves

A *twisted cubic* is a nonsingular, rational curve of degree 3 in $\mathbb{P}^3$. The set $\mathcal{T}$ of twisted cubics has a natural structure as a homogeneous space of dimension 12: since any twisted cubic is projectively equivalent to the image of the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$, given by sending a point with homogeneous coordinates (s,t) to the point (s^3, s^2t, st^2, t^3), we get $\mathcal{T} = \mathrm{SL}(4;\mathbb{C})/\mathrm{SL}(2;\mathbb{C})$. Consider the following three compactifications of the variety $\mathcal{T}$.

(1) A twisted cubic is a 1-cycle of degree 3 in $\mathbb{P}^3$, so $\mathcal{T} \subset \mathrm{Chow}_{1,3}(\mathbb{P}^3)$. Let $\mathcal{C}$ denote the irreducible component containing $\mathcal{T}$.

(2) A twisted cubic is a curve of degree 3 and arithmetic genus zero in $\mathbb{P}^3$, hence $\mathcal{T} \subset \mathrm{Hilb}^{3t+1}(\mathbb{P}^3)$. Let $\mathcal{H}$ denote the irreducible component containing $\mathcal{T}$.

(3) A twisted cubic is the image of a map $\mathbb{P}^1 \to \mathbb{P}^3$, hence $\mathcal{T} \subset \overline{M}_{0,0}(\mathbb{P}^3, 3)$. Let $\mathcal{M}$ denote the irreducible component of the Kontsevich space containing $\mathcal{T}$.

These three spaces, $\mathcal{C}$, $\mathcal{H}$, and $\mathcal{M}$, are birationally equivalent, but they are not equal. We have a map $\phi : \mathcal{H} \to \mathcal{C}$, which "forgets" the scheme structure except for the multiplicities of the components—e.g., any scheme structure of multiplicity 3 on a line L in $\mathbb{P}^3$ maps to the same point $3L \in \mathcal{C}$. Similarly, there is a map $\psi : \mathcal{M} \to \mathcal{C}$, but no obvious maps between $\mathcal{H}$ and $\mathcal{M}$.

As an example, consider the point $2L + L' \in C$, where L, L' are lines in $\mathbb{P}^3$ intersecting in a point. Points in $\phi^{-1}(2L + L')$ correspond to double structures of genus -1 on the line L, and one can show that there is a 2-dimensional family of such structures. Points in $\psi^{-1}(2L + L')$ correspond to stable maps from a union of $\mathbb{P}^1$'s onto $L \cup L'$, of degree 2 on L and 1 on L', and where some $\mathbb{P}^1$'s may map to points. The set of isomorphism classes of such maps also contains a 2-dimensional set, but there seems to be no natural relation between the fibres $\phi^{-1}(2L + L')$ and $\psi^{-1}(2L + L')$ (cf. [17]).

There are also other natural compactifications of $\mathcal{T}$. The ideal of a twisted cubic in the homogeneous coordinate ring is generated by three quadrics, and one can show that $\mathcal{T}$ has a "minimal" compactification $\overline{\mathcal{T}}$ equal to the moduli space of nets (i.e., 2-dimensional linear systems) of quadrics. In fact, one can show that $\mathcal{H}$ is the blow up of $\overline{\mathcal{T}}$ along the boundary $\overline{\mathcal{T}} - \mathcal{T}$; points in the boundary correspond to degenerate nets, i.e., nets with a plane as fixed component (see [4]).

For enumerative problems, one is led to consider a space of "complete" twisted cubics, similarly to the case of plane curves, by taking triples consisting of the curve, its tangent developable surface, and its strict dual curve, and limits of such triples. Depending on whether one takes these limits in the Hilbert schemes or in the Chow varieties, one gets different spaces, and they also differ from the ones considered above (see [19] and [20]).

4. Quantum cohomology and rational curves

The quantum cohomology ring of a projective variety can be thought of as a deformation of the ordinary cohomology ring, where the deformation parameters—or "quantum variables"—are "dual" to a basis of the cohomology groups (viewed as a complex vector space.) To get a ring structure, one deforms the ordinary cup product to get a "quantum product." The structure of this ring has quite surprising implications in enumerative geometry. In particular we shall see how the recursive formula for the number N_d can be deduced from the associativity of the quantum product.

Let $T_0, \ldots, T_m$ be a basis for A^*X such that $T_0 = 1$, $T_1, \ldots, T_p$ is a basis for A^1X, and $T_{p+1}, \ldots, T_m$ is a basis for the sum of the other cohomology groups. Consider the "universal element"

$$\gamma = \sum_{i=0}^{m} y_i T_i,$$

where the coefficients y_i are the "quantum variables". The main idea from physics is to form a generating function (or "potential" or "free energy" function) for the Gromov-Witten invariants as follows:

$$\Phi(y_0, \ldots, y_m) = \sum_{n_0+\cdots+n_m \geq 3} \sum_{\beta \in A_1 X} I_\beta(T_0^{n_0}, \ldots, T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}.$$

One can show that for each β there are only finitely many nonzero Gromov-Witten invariants, hence $\Phi(y_0, \ldots, y_m) \in \mathbb{Q}[[y_0, \ldots, y_m]]$ is a power series ring.

The part of Φ corresponding to $\beta = 0$ (corresponding to maps $\mathbb{P}^1 \to X$ with image a point) is the "classical" part—the rest is the "quantum" part:

$$\Phi = \Phi_{\mathrm{cl}} + \Gamma.$$

Define numbers g_{ij} by

$$g_{ij} = \int_X T_i \cup T_j$$

and let (g^{ij}) denote the inverse matrix $(g_{ij})^{-1}$. Then we define the quantum product:

$$T_i * T_j = \sum_{k,l} \Phi_{ijk} g^{kl} T_l,$$

where $\Phi_{ijk} = \delta^3\Phi/\delta y_i \delta y_j \delta y_k$. By extending this product $\mathbb{Q}[[y_0, \ldots, y_m]]$-linearly to the $\mathbb{Q}[[y_0, \ldots, y_m]]$-module $A^*X \otimes \mathbb{Q}[[y_0, \ldots, y_m]]$ we obtain a $\mathbb{Q}[[y_0, \ldots, y_m]]$-algebra that we denote by QA^*X—this is our "quantum cohomology" ring. Obviously, the above product is commutative, and one sees easily that T_0 is a unit. On the contrary, it takes a lot more effort to prove that the product is associative! In view of the consequences of associativity, this is not so surprising (see [5], [10]).

The case $X = \mathbb{P}^2$. In this case, $p = 1$ and $m = 2$: $T_0 = [X]$, $T_1 = [\text{line}]$, $T_2 = [\text{point}]$, and $\beta = dT_1$, for $d \in \mathbb{Z}$. We have $(g_{ij}) = 1$ if $i + j = 2$ and $g_{ij} = 0$ otherwise, so that

$$g_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (g^{ij}).$$

Hence we get

$$T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_0.$$

The classical part of Φ becomes

$$\Phi_{\mathrm{cl}} = \sum_{n_0+n_1+n_2=3} \left(\int_{\mathbb{P}^2} T_0^{n_0} \cup T_1^{n_1} \cup T_2^{n_2} \right) \frac{y_0^{n_0}}{n_0!} \cdot \frac{y_1^{n_1}}{n_1!} \cdot \frac{y_2^{n_2}}{n_2!} = \frac{1}{2} y_0 y_1^2 + \frac{1}{2} y_0^2 y_2.$$

Hence

$$\left(\Phi_{\mathrm{cl}}\right)_{ijk} = 1$$

if (i,j,k) is (a permutation of) $(0,1,1)$ or $(0,0,2)$, and zero otherwise.

From the definition of the Gromov-Witten invariants it follows that the quantum part Γ does not contain the variable y_0. In fact, we get

$$\begin{aligned}
\Gamma\left(y_0, y_1, y_2\right) &= \sum_{n_1+n_2\geq 3}\ \sum_{d>0} I_{dT_1}\left(T_1^{n_1}, T_2^{n_2}\right)\frac{y_1^{n_1}}{n_1!}\cdot\frac{y_2^{n_2}}{n_2!} \\
&= \sum_{n_1}\sum_{d>0}\left(\int_{dT_1} T_1\right)^{n_1} I_{dT_1}\left(T_2^{3d-1}\right)\frac{y_1^{n_1}}{n_1!}\cdot\frac{y_2^{3d-1}}{(3d-1)!} \\
&= \sum_{d>0} N_d e^{dy_1}\cdot\frac{y_2^{3d-1}}{(3d-1)!},
\end{aligned}$$

where $(\int_{dT_1} T_1)^{n_1} = d^{n_1}$, $\sum_{n_1} d^{n_1}\frac{y_1^{n_1}}{n_1!} = e^{dy_1}$, and $N_d := I_{dT_1}(T_2^{3d-1})$ is, as observed earlier, the number of rational plane curves of degree d passing through $3d-1$ general points.

We now deduce a recursive relation for the numbers N_d. Since the quantum part Γ of Φ does not contain the variable y_0, we have $\Phi_{ij0} = (\Phi_{\mathrm{cl}})_{ij0}$. We can therefore compute

$$\begin{aligned}
T_1 * T_1 &= T_2 + \Gamma_{111}T_1 + \Gamma_{112}T_0, \\
T_1 * T_2 &= \Gamma_{121}T_1 + \Gamma_{122}T_0, \\
T_2 * T_2 &= \Gamma_{221}T_1 + \Gamma_{222}T_0.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\left(T_1 * T_1\right) * T_2 &= \left(\Gamma_{221}T_1 + \Gamma_{222}T_0\right) + \Gamma_{111}\left(\Gamma_{121}T_1 + \Gamma_{122}T_0\right) + \Gamma_{112}T_2, \\
T_1 * \left(T_1 * T_2\right) &= \Gamma_{121}\left(T_2 + \Gamma_{111}T_1 + \Gamma_{112}T_0\right) + \Gamma_{122}T_1.
\end{aligned}$$

The associativity of the product now gives the following differential equation for the function Γ:

$$\Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111}\Gamma_{122}.$$

We now plug in the power series expression for the function Γ in this differential equation and solve for N_d:

$$N_d = \sum_{d_1+d_2=d} N_{d_1}N_{d_2}\left[d_1^2 d_2^2\binom{3d-4}{3d_1-2} - d_1^3 d_2\binom{3d-4}{3d_1-1}\right].$$

The initial condition is $N_1 = 1$—there is exactly one line through two given points in the plane—and so one can compute all N_i recursively—here are the first 8:

$$\begin{aligned}
N_1 &= 1,\\
N_2 &= 1,\\
N_3 &= 12,\\
N_4 &= 620,\\
N_5 &= 87\,304,\\
N_6 &= 26\,312\,976,\\
N_7 &= 14\,616\,808\,192,\\
N_8 &= 13\,525\,751\,027\,392.
\end{aligned}$$

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TEICHMÜLLER DISTANCE, MODULI SPACES OF SURFACES, AND COMPLEX DYNAMICAL SYSTEMS

MARY REES

1. Introduction

This paper arises, strange as it may seem, out of a project in complex dynamics. Complex dynamics is the study of the dynamics of rational maps of the Riemann sphere: that is, given a rational map f of $\hat{\mathbf{C}}$, one studies the behaviour of the sequence $\{f^n(z)\}$ for varying z. (Alternatively, people study the dynamics of entire or meromorphic functions of the complex plane.) As one varies a rational map through some natural parameter space—such as the space of all maps of some fixed degree $d > 1$—dynamical behaviour varies widely—even wildly. An important aspect of study in complex dynamics is to understand variation of dynamics within parameter spaces. I shall attempt to give some indication of how this led me to develop a calculus of Teichmüller distance, in the hope of striking one or two chords. Two types of moduli spaces (at least) are involved here: the parameter spaces of rational maps (the original objects of study); the classical moduli space arising as the quotient of Teichmüller space by the mapping class group also plays a rôle. Teichmüller distance arises naturally in the classical problem of isotopy classification of surfaces homeomorphisms, as will be described later. The way in which Teichmüller distance arises in complex dynamics is related to this.

1.1. Periodic points. In any area of dynamics, periodic points are important, and this is especially true in complex dynamics. A point z is *periodic* under a map f, of period n, if $f^n(z) = z$, and n is the least integer greater than zero for which this is true. Periodic points come in various types. A periodic point z of period n is *attractive* if the derivative $(f^n)'(z)$ is less than 1 in modulus. Here, I am restricting to holomorphic maps. This definition is independent of choice of local coordinates, and of the point in the forward orbit of z, that is, for a periodic point z, z is attractive if and only if $f(z)$ is. Attractive periodic points influence the behaviour of nearby points: for any point w sufficiently near an attractive periodic point z, the sequence $f^m(w)$ converges to the periodic orbit of z. Further, attractive periodic points influence the behaviour of nearby maps: if z is attractive periodic under f, then for g near f, g has an attractive periodic point

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near z. This is a simple consequence of the Implicit Function Theorem, and, although elementary, is an important tool in dynamics in general.

1.2. Hyperbolic maps. It can happen, for a rational map f, that for almost all points w (in a topological or measure-theoretic sense) the sequence $f^m(w)$ converges to one of finitely many attractive periodic orbits. This happens, for example, if such convergence holds for all *critical* points w. A point w is *critical* for a (holomorphic) map f if $f'(w) = 0$. This definition is independent of the choice of local coordinates. Since there are only finitely many critical points (in fact, at most $2d-2$ for a rational map of degree d), it is possible—at least in theory, and often in practice too—to check whether their orbits converge to attractive periodic orbits. If this does happen for all critical points of a map f, then f is *hyperbolic*. This is a very important concept in dynamical systems. The formal definition of hyperbolic is somewhat different from the above in general, but the general consequences are much the same. For hyperbolic maps, dynamics can be analysed: as stated above, for a hyperbolic rational map, a generic orbit converges to an attractive periodic orbit. This is considered rather dull behaviour, and the set of points with orbits not converging to some attractive periodic orbit is called the *Julia set*. Dynamics on this set can be analysed thoroughly, using standard techniques available in dynamical systems.

1.3. Hyperbolicity and stability. In summary, there are two (dynamical) kinds of rational maps: hyperbolic and nonhyperbolic. The hyperbolic maps can be analysed. They are *J-stable*—stable is generally a good word in mathematics, but apparently with many different meanings. In dynamics, a map f is stable if all g near f are conjugate to f, that is, of the form $\phi \circ f \circ \phi^{-1}$ for some homeomorphism ϕ. For rational maps, J-stable means that such a conjugacy holds in a neighbourhood of the Julia set. Thus, each hyperbolic map is in an open connected set of maps which are all hyperbolic and have the same dynamics, at least on their Julia sets (and with very minor variations elsewhere). It is conjectured that hyperbolic maps are generic—open and dense—in natural parameter spaces of rational maps, for example the space of all maps of some fixed degree $d > 1$. It is known that stable maps are dense [7]. So the conjecture is that all stable maps are hyperbolic: a result which is true in some other categories of dynamical systems. (However, in other categories, stable systems are not usually dense.) In contrast to some other situations in mathematics, there are infinitely many hyperbolic (and hence stable) components in any reasonable parameter space of rational maps, so even if the conjecture holds and hyperbolic maps are dense, the variation of dynamcis in the parameter space will necessarily be complicated.

1.4. Versions of homotopy equivalence for maps. A rational map is obviously hyperbolic if every critical point w is periodic, because then w is attractive with $(f^n)'(w) = 0$ (by the chain rule). Many hyperbolic components will contain such a map, which is called *critically finite*. The total dynamics of such a map is determined by its critical orbits, or, more precisely, by the appropriate type of homotopy class of the map with respect to these orbits. This type of homotopy equivalence for critically finite branched coverings is known as *Thurston equivalence*. If f is a branched covering, then the *postcritical set* $P(f)$ is the set $\{f^n(c) : n > 0,\ c \text{ critical}\}$. Then critically finite branched coverings f_0

and f_1 are *Thurston equivalent* if there is a homotopy f_t through critically finite branched coverings from f_0 to f_1 such that $P(f_t)$ varies isotopically. Thus Thurston equivalence is stronger than homotopy equivalence and can be regarded as the appropriate analogue of isotopy for homeomorphisms: we recall that two homeomorphisms are isotopic if they are homotopic through homeomorphisms.

1.5. The importance of homotopy-type information. Another common theme in dynamics is that isotopy, or sometimes even homotopy, determines a map up to some kind of semiconjugacy, for certain homotopy (or isotopy) classes. There are many results of this type. For example [5], any continuous map f of a torus (of any dimension) which is homotopic to a *hyperbolic* toral automorphism g is semiconjugate to g, that is, there is a continuous map φ such that $\varphi \circ f = g \circ \varphi$. There is a result of this type [6] for other surface homeomophisms (with a somewhat weaker semiconjugacy statement), and another such result for critically finite branched coverings (see [8, 4.1]). This is one rough reason for dynamicists to be interested in the problems of: classification of surface homeomorphisms up to isotopy; classification of critically finite branched coverings up to Thurston equivalence; and so on.

1.6. Classical problems. The problem of isotopy classification of surface homeomorphisms has been well worked over during the past seventy years or so. There have been varying approaches to the problem by (among others) Nielsen, Thurston [4], and (following Thurston to some extent) Bers [1]. It is Bers' approach that I shall be highlighting shortly. First, I want to try and indicate certain aspects of all the proofs which seem to give important clues as to how investigations into dynamical structure should develop. All proofs point towards a "best map", or at least, a small class of best maps, within an isotopy, or Thurston equivalence, class. These best maps have strong geometric structure and strong dynamical properties. For example, the best map in a Thurston equivalence class is often a rational map, usually unique up to Möbius conjugacy. The family of best maps in an isotopy class of surface homemorphism is, for some choices of isotopy class (the *pseudo-Anosov* classes), a family of maps each of which preserves two transverse measured foliations (with singularities) on the surface, expanding leaves of one foliation and contracting leaves of the other. The other important point about these proofs is that they turn out to give information about the topology of the isotopy class of homeomorphisms (or Thurston equivalence class of critically finite branched coverings) itself. Let us consider the case of isotopy classes of surface homeomorphisms for the moment. Any isotopy class itself (at least for a surface of negative Euler characteristic) is easily seen to be contractible. However, the proof of isotopy classification gives, further, the topology of the connected components of the space of pairs (f, S), where S is homeomorphic to a fixed compact surface S_0 and f is a homeomorphism of S. A component of this space corresponding to a pseudo-Anosov isotopy class is homotopy equivalent to the circle. Now let us consider critically finite branched coverings. Thurston [10], [2] showed that certain Thurston equivalence classes of critically finite branched coverings (satisfying a certain condition concerning sets of disjoint simple closed loops) contained a unique rational map up to Möbius conjugation. The proof, almost incidentally, also showed that such Thurston equivalence classes (after quotienting by Möbius conjugation) were contractible—in this case, a nontrivial fact.

1.7. Generalizing the classical problems. I shall shortly come to some concrete definitions, statements, and even proofs. However, the point of the work I shall attempt to describe is not to reprove the results stated above but to carry out other investigations which were suggested by the above. We have seen that the dynamics of a rational map is heavily influenced by what happens to the critical orbits—and, in fact, critically finite maps are relatively easy to analyse. It is therefore natural to consider slices in parameters space in which various critical points are constrained to have finite forward orbits. Such slices will intersect many hyperbolic components, for example, and will contain many critically finite maps. By constraining all but one of the critical points, one obtains slices of complex dimension 1, which are good for initial consideration. Such spaces of rational maps are natural subspaces of spaces of branched coverings with specific dynamics on certain finite sets, which include some critical points. These can be regarded as "one dimension up" from, for example, the isotopy class of a surface homeomorphism, or of a homeomorphism of $(\hat{\mathbf{C}}, A)$, where $A \subset \hat{\mathbf{C}}$ is a finite set. I have tried to show the evidence that investigation of dynamical structure should go hand-in-hand with a study with the space of maps itself. It seems likely that the structure of a moduli space often reflects structure of the objects in the moduli space in some way. Also, (perhaps less clear in the above) although one might want to study a relatively small parameter space, in order to study it comprehensively, it might be necessary to consider a somewhat larger space. This is because of the fact that dynamical structure is often implied by very simple homotopy-type information. It is also because information about varying dynamics within a parameter space is inextricably linked to information about the structure—especially topological and geometric structure—of the parameters space itself, including the topological structure of inclusions into certain much larger spaces (but which are nevertheless quite simply defined), topologically. The larger spaces can also be regarded as moduli spaces, although not in the strict mathematical sense described in Frances Kirwan's talk (of this meeting), because they are not in general algebraic varieties. I am afraid it is not possible to justify this completely here. However, there might possibly be an analogy with other situations in mathematics—in particular, involving moduli spaces—such as blow-ups of singularities. Some of the enlargement (but not all) of rational map parameter spaces mentioned above is indeed connected with singularities in these spaces.

2. Calculus

2.1. The programme from now on. For the rest of this article, I shall restrict to developing some calculus of Teichmüller distance, and applying it to an adaptation of Bers' proof of isotopy classification of surface homeomorphisms. It is not possible to give much detail—for further detail (see [9, Chapters 8–16]). I think this calculus is of interest in its own right. As we shall see, I have developed it only in a restricted case—for marked spheres rather than even general finite type surfaces. I do believe that it can be developed for general finite type surfaces though, a belief that was strengthened by some discussion at the EWM meeting, and subsequently with a colleague. The calculus is not that easy to use—nor formidably difficult. It is certainly totally unnecessary to burden Bers' elegant proof with it. However, the isotopy classification is useful as an illustration, and the calculus seems to be essential to the problem for which it was developed.

This specific problem (for which the calculus was developed) is part of the general project of understanding variation of dynamics in parameter spaces of rational maps. The specific problem would take a very long time to state, but is, in fact, quite closely related to the problem of isotopy classification of surface homeomorphisms. For further details, see [9].

2.2. Teichmüller space. Let S_0 be a compact surface (of two real dimensions) and $Y_0 \subset S_0$ finite. Then homeomorphisms $\varphi_0, \varphi_1 : (S_0, Y_0) \to (S, Y)$ are *isotopic* if there is a continuous family φ_t of homeomorphisms for $t \in [0, 1]$.

Now let S be a Riemann surface (or a 2-dimensional hyperbolic manifold). Then $\phi : (S_0, Y_0) \to (S, Y)$ and $\varphi' : (S_0, Y_0) \to (S', Y')$ are *equivalent* (as elements of Teichmüller space) if there is a biholomorphic map (or hyperbolic isometry) $\tau : (S, Y) \to (S', Y')$ such that φ' and $\tau \circ \varphi$ are isotopic. If $[\varphi]$ denotes the equivalence class of φ, then the *Teichmüller space* of (S_0, Y_0), which we denote by $\mathcal{T}(S_0, Y_0)$, is

$$\big\{[\phi] : \varphi : (S_0, Y_0) \to (S, Y) \text{ an orientation-preserving homeomorphism}\big\}.$$

Roughly speaking, $\mathcal{T}(S_0, Y_0)$ is the space of framed geometric structures homeomorphic to (S_0, Y_0).

2.3. Example. Let $S_0 = \hat{\mathbf{C}}$. Then $\mathcal{T}(S_0, Y_0)$ is a point if $\#(Y_0) \leq 3$, because given any two sets of three points in $\hat{\mathbf{C}}$, there is a Möbius transformation taking the first set to the second. If $\#(Y_0) = n \geq 3$, then applying a Möbius transformation, we can assume that $0, 1, \infty \in Y_0$. Then, for any $[\varphi] \in \mathcal{T}(S_0, Y_0)$, we can choose φ in its equivalence class to fix $0, 1, \infty$, and we do this from now on. From now on, we write

$$Y_0 = \{0, 1, \infty\} \cup \{y_i : 1 \leq i \leq n-3\}.$$

Then the map

$$[\phi] \longmapsto \big(\phi(y_i)\big) \in \mathbf{C}^{n-3}$$

is a local homeomorphism, and gives $\mathcal{T}(S_0, Y_0)$ the structure of a complex manifold. In fact, it can be shown that $\mathcal{T}(S_0, Y_0)$ is homeomorphic to $\mathbf{R}^{2n-6}$. If $\#(Y_0) = 4$, then $\mathcal{T}(S_0, Y_0)$ is biholomorphic to the unit disc.

2.4. The mapping class group. Teichmüller space was used by Thurston (and Bers) to analyse isotopy classes of homeomorphisms $\psi : (S_0, Y_0) \to (S_0, Y_0)$. The *mapping class group* $M(S_0, Y_0)$ is the group of orientation-preserving homeomorphisms modulo isotopy. The mapping class group acts on $\mathcal{T}(S_0, Y_0)$ on the right by

$$[\varphi] \cdot [\psi] = [\varphi \circ \psi].$$

The quotient of Teichmüller space by the mapping class group is often called the *moduli space of the surface*.

2.5. Teichmüller distance. Let $\chi : U \to \mathbf{C}$ be a local diffeomorphism, where $U \subset \mathbf{C}$ is open. Then one can define the (pointwise) *distortion* $K(\chi)(z)$ at a point $z \in U$ by

$$K(\chi)(z) = \sqrt{\frac{|\lambda_1|}{|\lambda_2|}},$$

where $\lambda_1 \geq \lambda_2 > 0$ are the eigenvalues of $D\chi_z^t D\chi_z$, where $D\chi_z$ denotes the derivative of χ at z (considered as a 2×2 matrix). Note that $K(\chi)(z) = 1$ if χ is holomorphic near z. We can define $K(\chi)(z)$ similarly for $z \in S_1$ if $\chi : S_1 \to S_2$ is a local diffeomorphism and S_1, S_2 are any Riemann surfaces: the definition is independent of the choice of local coordinates. Then we can define

$$\|\chi\|_{qc} = \|K(\chi)\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the L_∞ norm. This norm is finite if χ is *quasi-conformal*, which is true, for example, if χ is a C^1 diffeomorphism between compact spaces. It is also finite more generally: since we are taking the L_∞ norm, $\|\chi\|_{qc}$ might be finite even if χ is nondifferentiable on a set of zero measure.

The *Teichmüller distance*

$$d_{\mathcal{T}} : \mathcal{T} \times \mathcal{T} \to (0, \infty)$$

is defined by

$$d_{\mathcal{T}}\big([\varphi_1], [\varphi_2]\big) = \frac{1}{2} \inf \big\{ \log \|\chi\|_{qc} : [\chi \circ \varphi_1] = [\varphi_2] \big\}.$$

The Teichmüller distance coincides with half the Poincaré distance if $\mathcal{T} = \mathcal{T}(\hat{\mathbf{C}}, Y_0)$ and $\#(Y_0) = 4$, so that $\mathcal{T}(\hat{\mathbf{C}}, Y_0)$ is the unit disc. The Teichmüller distance is a genuine metric: if χ is biholomorphic, then $\log \|\chi\|_{qc} = 0$ for all z. Moreover, the action of $M(S_0, Y_0)$ on $\mathcal{T}(S_0, Y_0)$ preserves Teichmüller distance, that is,

$$d_{\mathcal{T}}\big([\varphi_1] \cdot [\psi], [\varphi_2] \cdot [\psi]\big) = d_{\mathcal{T}}\big([\varphi_1], [\varphi_2]\big)$$

for all $[\varphi_1]$, $[\varphi_2] \in \mathcal{T}(S_0, Y_0)$ and for all $[\psi] \in M(S_0, Y_0)$. This ensures that the Teichmüller distance metric descends to a metric on moduli space. The infimum in the definition of $d_{\mathcal{T}}$ is attained for a unique χ. This χ is defined by the following properties. Except at finitely many points on $\varphi_1(S_0)$ and $\varphi_2(S_0)$, there are local coordinates $x + iy$ on $\varphi_1(S_0)$ and $\varphi_2(S_0)$ with respect to which χ has the formula

$$\chi(x + iy) = \lambda x + i \frac{1}{\lambda} y$$

for some $\lambda \geq 1$. We then have

$$\log \lambda = \frac{1}{2} \log \|\chi\|_{qc} = d_{\mathcal{T}}\big([\varphi_1], [\varphi_2]\big).$$

Moreover, the local coordinates (with singularities) are given by *quadratic differentials* on $\varphi_1(S_0)$, $\varphi_2(S_0)$. A quadratic differential on $(\varphi_j(S_0), \varphi_j(Y_0))$ has the following property with respect to (any system of nonsingular) charts (U, ξ_U), (V, ξ_V) on $\varphi_j(S_0)$, that is, with $U, V \subset \varphi_j(S_0)$. Suppose that $U \cap V \neq \emptyset$ and that the quadratic differential is given in $\xi_U(U)$, $\xi_V(V)$ by $q_U(z)dz^2$, $q_V(z)dz^2$. Then q_U, q_V are meromorphic with at worst simple poles, with these occurring only (at most) at points of $\xi_U \circ \varphi_j(Y_0)$, $\xi_V \circ \varphi_j(Y_0)$, and on $\xi_U(U \cap V)$ we have

$$q_U(z) = \Big(\big(\xi_V \circ \xi_U^{-1}\big)'(z)\Big)^2 \cdot q_V \circ \xi_V \circ \xi_U^{-1}(z).$$

The singular local coordinates with respect to which χ has its special form are then given on U by

$$(x+iy)\big(\xi_U^{-1}(z)\big) = \int_{z_0}^{z} \sqrt{q_U(\zeta)}d\zeta.$$

Up to addition of a constant and plus or minus sign, these local coordinates are independent of choice of chart U: the way the quadratic differential transforms under change of chart, and the change of variable formula for integrals, ensure that.

Now let $S_0 = \hat{\mathbf{C}} = \varphi_j(S_0)$ and consider the chart $\mathbf{C}$. Then a quadratic differential on $(\hat{\mathbf{C}}, \varphi(Y_0))$ is given in this chart by $q(z)dz^2$, where q is a rational function with at worst simple poles, occurring at most at points of $\varphi(Y_0)$, and with at least three more poles than zeros. This last condition assumes that ∞ is a point of $\varphi(Y_0)$: otherwise we need at least four more poles than zeros. It arises from considering the local coordinate $1/z$ at ∞: we need $q(1/z)z^{-4}$ to have at most a simple pole at zero.

2.6. Bers' approach to isotopy classification of surface homeomorphisms. Let an isotopy class $[\varphi]$ in $M(S_0, Y_0)$ be given. Then consider the map

$$F : \mathcal{T}(S_0, Y_0) \longrightarrow \mathbf{R}_+$$

given by

$$F\big([\varphi]\big) = d_{\mathcal{T}}\big([\varphi] \cdot [\psi], [\varphi]\big).$$

Consider where the infimum of this function is attained. There are three possibilities, corresponding to three different types of isotopy classes.

(1) The minimum value "zero" is attained at a point in $\mathcal{T}$. In that case, it is attained at a unique point in $\mathcal{T}$ at a point $[\varphi]$, and $\varphi \circ \psi \circ \varphi^{-1}$ is isotopic to a biholomorphism (or hyperbolic isometry) of $(\varphi(S_0), \varphi(Y_0))$, which is necessarily of finite order.

(2) The infimum is "zero", but this is not attained at any point of $\mathcal{T}$. This is the *reducible* case. In this case, it is possible to show that the infimum is only attained by going to infinity in $\mathcal{T}$ in specific directions. We omit the details.

(3) The infimum is strictly positive, and is attained uniquely on a curve of one real dimension in $\mathcal{T}$ which is invariant under the action of $[\psi]$: it has to be, since right action by $[\psi]$ preserves Teichmüller distance. This curve is known as a *geodesic* (and distance between any two points on the geodesic is indeed attained uniquely along paths in the geodesic). Such a minimizing geodesic is uniquely determined by any fixed point $[\varphi]$ on it, and a unique quadratic differential (up to multiplication by a nonzero real number) for $(\varphi(S_0), \varphi(Y_0))$. If we take the singular local coordinates $x+iy$ on $\varphi(S_0)$ given by this

quadratic differential, then all other points on the geodesic are $[\chi_\lambda \circ \varphi]$ $(\lambda > 0)$, where, if $S'_\lambda = \chi_\lambda \circ \varphi(S_0)$, then $\chi_\lambda : S'_0 = \varphi(S_0) \to S'_\lambda$ is given in singular local coordinates by

$$\chi_\lambda(x + iy) = \lambda x + i\frac{y}{\lambda}.$$

This is known as the *pseudo-Anosov* case. If $[\varphi]$ is a point on the geodesic, then so is $[\varphi \circ \psi]$, and it is apparent from the characterization above of points on the geodesic that $\varphi \circ \psi \circ \varphi^{-1}$ has very special properties. In fact, the singular local coordinates $x + iy$ give the two transverse measured foliations (with singularities) mentioned earlier in this article. Leaves of these foliations are given locally by the curves $y =$ constant and $x =$ constant.

Bers [1] was able to do his analysis of the function F without any reference to its derivative, which means without any reference to the derivative of $d_{\mathcal{T}}$. (Note that, with respect to the natural local coordinates on $\mathcal{T}(\hat{\mathbf{C}}, Y_0)$ that we described earlier, the right action of $[\psi]$ is given by the identity map.) However, at the same time that Bers was carrying out his analysis, Earle [3] was developing a formula for the derivative of $d_{\mathcal{T}}$. I shall not give the precise formula. It is not difficult, but involves the introduction of Beltrami differentials associated to elements of Teichmüller space. Anyway, the nub of the formula is that the derivative of $d_{\mathcal{T}}([\varphi_1], [\varphi_2])$ at $([\varphi_1], [\varphi_2])$ with $[\varphi_1] \neq [\varphi_2]$ is given by the pair of quadratic differentials at $(\varphi_1(S_0), \varphi_1(Y_0))$, $(\varphi_2(S_0), \varphi_2(Y_0))$ used to define the map χ with $[\chi \circ \varphi_1] = [\varphi_2]$ which minimises distortion. I shall give a formula below in the special case of $\mathcal{T}(\hat{\mathbf{C}}, Y_0)$. Of course, when one is thinking of infima of functions—which often turn out to be minima—then one thinks of the second derivative. If F is twice differentiable at any minimum value, the first derivative must be zero at such a point, and the second derivative must be positive at such a point. This is indeed the case, at least in the cases for which I have been able to compute second derivative of distance so far, that is, for marked spheres.

2.7. The derivative formula for $\mathcal{T}(\hat{\mathbf{C}}, Y_0)$. This particularly simple formula (which is a special case of Earle's) is possible because of the simple local coordinates on $\mathcal{T}(\hat{\mathbf{C}}, Y_0)$. Let $\#(Y_0) = n$, $n \geq 3$. Let $[\varphi]$, $[\varphi'] \in \mathcal{T}(\hat{\mathbf{C}}, Y_0)$, so that φ, φ' are orientation-preserving homeomorphisms of $\hat{\mathbf{C}}$, and assume, without loss of generality, that they fix 0, 1, ∞. Then if $h = (h_i) \in \mathbf{C}^{n-3}$ is small enough given $[\varphi]$, the element $[\varphi] + h$ of $\mathcal{T}(\hat{\mathbf{C}}, Y_0)$ is well-defined by taking this to be the isotopy class of the homeomorphism near φ which fixes 0, 1, ∞ and sends y_i to $\varphi(y_i) + h_i$. Let $q(z)dz^2$ and $-p(z)dz^2$ be the quadratic differentials at $[\varphi]$, $[\varphi']$ for $d_{\mathcal{T}}([\varphi], [\varphi'])$. Thus, $p(z)dz^2$ is the *stretch* of $q(z)dz^2$, that is, with respect to the local coordinates given by $q(z)dz^2$, $p(z)dz^2$, the homeomorphism minimizing distortion is in the form

$$x + iy \longmapsto x + i\frac{y}{\lambda},$$

where $\log\lambda = d_{\mathcal{T}}([\varphi], [\varphi'])$. We recall from the definition of quadratic differentials that q and p have at most simple poles. Then the formula for the first derivative in this special case (see [9, Chapter 8]) is given by

$$\begin{aligned}&d_{\mathcal{T}}\big([\varphi]+h,[\varphi']+h'\big)\\&=d_{\mathcal{T}}\big([\varphi],[\varphi']\big)+2\pi\sum_{i=1}^{n-3}\mathrm{Re}\left(\mathrm{Res}\big(q,\varphi(y_i)\big)h_i-\mathrm{Res}\big(p,\varphi'(y_i)h_i'\big)\right)+o(h)+o\big(h'\big).\end{aligned}$$

2.8. Quadratic differentials and hyperelliptic curves. We continue with the conventions on Y_0, φ established above. We recall that if $q(z)dz^2$ is a quadratic differential at $(\hat{\mathbf{C}},\varphi(Y_0))$, then q is a rational function with at most n simple poles, at most occurring at the points $\varphi(Y_0)$, and at most 3 less zeros than poles, up to multiplicity. We suppose for the moment that all zeros are simple, that all points of $\varphi(y_i)$ are simple poles of q, and that q has exactly three more poles than zeros. (This is essentially the definition of ∞ being a simple pole of q.) We consider the Riemann surface

$$S_q=\big\{(z,w)\in\mathbf{C}^2:q(z)=w^2\bar{\big\}}.$$

Here, we use $\bar{\}}$ to denote the following possibly nonstandard closure: without closure, the set described above is biholomorphically a compact surface minus finitely many points, corresponding to taking z or $w=\infty$. We define the closure to be the union with these finitely many points, thus giving a compact Riemann surface S_q, such that $\pi:S_q\to\hat{\mathbf{C}}$ given by $\pi(z,w)=z$ is a branched double cover, branched over the zeros and poles of q—including ∞. Thus, S_q is a compact surface of genus $n-3$ which I shall call a *hyperelliptic curve*. (This is the standard definition, modulo possible quibbles about the closure operation.)

Then $wdz=\pi^*(\sqrt{q(z)}dz)$ is a holomorphic 1-form on S_q. Some checking in local coordinates near points where $w=0$ or ∞, or where $z=\infty$, is necessary to confirm this, but it is so. The standard theory tells us that the holomorphic 1-forms on S_q form a vector space of complex dimension $n-3$. The general formula for a holomorphic 1-form on S_q is

$$r(z)w\,dz=\pi^*\big(r(z)\sqrt{q(z)}dz\big),$$

where r is a rational function whose denominator is the numerator of q and the numerator is a polynomial of degree less than or equal to $n-3$. The real and imaginary parts of holomorphic 1-forms are *harmonic 1-forms*. The real and imaginary parts of a complex valued form given locally by $(a+ib)(dx+i\,dy)$ (where a and b are functions) are $a\,dx-b\,dy$ and $b\,dy+a\,dx$. A harmonic 1-form is *closed*—i.e., $\partial a/\partial y=\partial b/\partial x$—and the *harmonic* condition is $\partial a/\partial x+\partial b/\partial y=0$. Let ω be a holomorphic or harmonic 1-form on S_q, and γ a loop on S_q. Then

$$\int_\gamma\omega$$

depends only on the homology class of γ. Thus each holomorphic 1-form defines an element of $H^1(S_q,\mathbf{C})$ and each harmonic 1-form defines an element of $H^1(S_q,\mathbf{R})$. In fact, the space of harmonic 1-forms is isomorphic to $H^1(S_q,\mathbf{R})$.

2.9. Connection with the second derivative of Teichmüller distance. In this section, I shall indicate very briefly how to use harmonic 1-forms on S_q, S_p to obtain the second derivative of the Teichmüller distance function. See [9] for further details, and also on the function F. (For reasons of space, I shall not attempt to indicate, here, the precise form of $D^2 d_{\mathcal{T}}$, nor why certain terms in this are positive, nor why $D^2 F$ is positive.)

We recall that if $\log\lambda = d_{\mathcal{T}}([\varphi],[\varphi'])$, then λ is the distortion of the best map χ : $(\hat{\mathbf{C}},\varphi(Y_0)) \to (\hat{\mathbf{C}},\varphi'(Y_0))$ with $[\chi\circ\varphi] = [\varphi']$. The key point is that λ can be characterised in terms of a linear map between $H^1(S_q,\mathbf{R})$ and $H^1(S_p,\mathbf{R})$, as follows. Note that the map $\chi : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ lifts to a map $\tilde{\chi} : S_q \to S_p$. Then $\tilde{\chi}$ induces a map from $H_1(S_q,\mathbf{Z})$ to $H_1(S_p,\mathbf{Z})$. Then the following holds, where we identify complex numbers with 2-dimensional real column vectors, so that left multiplication by real 2×2 matrices makes sense. For all $\gamma \in H_1(S_q,\mathbf{Z})$,

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\int_\gamma \pi_q^*\big(\sqrt{q(z)}dz\big) = \int_{\tilde{\chi}(\gamma)} \pi_p^*\big(\sqrt{p(z)}dz\big).$$

That is, in terms of $H^1(S_q,\mathbf{R})$, $H^1(S_p,\mathbf{R})$,

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\pi_q^*\big(\sqrt{q(z)}dz\big) = \tilde{\chi}^*\pi_p^*\big(\sqrt{p(z)}\big)dz. \tag{2.1}$$

Now suppose that $[\varphi]$ and $[\varphi']$ are changed to $[\varphi]+h$ and $[\varphi']+h'$, respectively, and we want to find the perturbations q_1 and p_1 which satisfy the equation corresponding to (2.1). Write

$$\varphi\big(y_i\big) = b_i, \qquad \text{and} \qquad a_i = \mathrm{Res}\big(q,b_i\big).$$

Then q_1 has possible poles at 0, 1, ∞ and b_i+h_i, $1\le i\le n-3$, where the residues are k_i, $1\le i\le n-3$. Then

$$q(z) = \sum_{i=1}^{n-3}\frac{b_j(b_j-1)a_j}{z(z-1)(z-b_j)}.$$

Then expanding, $\sqrt{q_1}$ this gives

$$\sqrt{q_1(z)}dz = \sqrt{q(z)}dz + \frac{1}{2}\sum_{j=1}^{n-3}\frac{b_j(b_j-1)k_j}{z(z-1)(z-b_j)\sqrt{q}}dz$$
$$+\sum_{j=1}^{n-3}\frac{b_j(b_j-1)a_jh_j}{z(z-1)(z-b_j)^2\sqrt{q(z)}}dz + o(h) + o(k).$$

Then we note that

$$\pi_q^*\left(\frac{b_j(b_j-1)}{z(z-1)z-b_j\sqrt{q(z)}}dz\right) \tag{2.2}$$

form a basis of holomorphic 1-forms on $H^1(S_q,\mathbf{C})$. The real and imaginary parts then form a basis of harmonic 1-forms in $H^1(S_q,\mathbf{R})$. Meanwhile, the

$$\pi_q^*\left(\frac{b_j(b_j-1)a_j}{z(z-1)(z-b_j)^2\sqrt{q(z)}}dz\right)$$

are meromorphic 1-forms of the *first kind*, that is, the residues at singularities are zero. Thus these forms, also, define elements of $H^1(S_q, \mathbf{C})$. In fact, each of these 1-forms has exactly one singularity, which is a double pole. It is possible to write these forms in terms of the basis of harmonic 1-forms given by (2.2). The coefficient matrix is a sum of products of matrices of improper intergrals and inverses of such. This looks like a very classical calculation, apart from the nonstandard form of hyperelliptic curve (i.e., using q rather than a polynomial). Nevertheless, details can be found in [9, Chapters 10–11].

In exactly the same way, an expresson can be found for the perturbation $\pi_p^*(\sqrt{p_1(z)}dz)$ of $\pi_p^*(\sqrt{p(z)}dz)$. Then the perturbation of the equation (2.1) can be written down to first order, and the vectors k and k', and the perturbation of the distance, can be computed to first order in terms of h and h'. This, then, gives the formula for the second derivative of Teichmüller distance, at least generically. The formula extends continuously at other points. For further details see [9, Chapters 10–13].

The formula for the second derivative of $d_{\mathcal{T}}$ then gives a formula for the second derivative of the function F of Section 2.6. It is possible to show that, as expected, D^2F is positive, although not, in general, positive definite. The whole procedure can be regarded as an exercise in Morse theory, that is, using level and critical sets of a function to analyze the topology of a space.

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MODULI SPACE OF SELF-DUAL GAUGE FIELDS, HOLOMORPHIC BUNDLES, AND COHOMOLOGY SETS

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We discuss the twistor correspondence between complex vector bundles over a self-dual 4-dimensional manifold and holomorphic bundles over its twistor space and describe the moduli space of self-dual Yang-Mills fields in terms of Čech and Dolbeault cohomology sets. The cohomological description provides the geometric interpretation of symmetries of the self-dual Yang-Mills equations.

1. Introduction

The purpose of this paper is to describe the moduli space of self-dual Yang-Mills fields and a symmetry algebra acting on the solution space of the self-dual Yang-Mills equations. The description of the moduli space of self-dual Yang-Mills fields is based on the twistor construction [13], [16], [1].

Let us briefly outline the differential-geometric background. We take M to be an oriented Riemannian 4-manifold, G a semisimple Lie group, $P(M,G)$ a principal fibre bundle over M with the structure group G, A a connection 1-form on P, F_A its curvature 2-form and D a covariant differential on P. A connection 1-form A on P is called *self-dual* if its curvature F_A is *self-dual*, that is,

$$*F_A = F_A, \tag{1.1}$$

where $*$ is the Hodge star operator acting on 2-forms on M. We call equations (1.1) the *self-dual Yang-Mills* (SDYM) equations. By virtue of the Bianchi identity $DF_A = 0$, solutions of the SDYM equations automatically satisfy the Yang-Mills equations

$$D\big(*F_A\big) = 0. \tag{1.2}$$

Notice that solutions to equations (1.2) are of considerable physical importance (see the talk in this volume by Tsou S.T.). Physicists use Yang-Mills theory (by which we mean any non-Abelian gauge theory) to describe the strong and electroweak interactions (see, e.g., [3]). They call the connection 1-form A the *gauge potential* and the curvature

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2-form F_A the *gauge* or *Yang-Mills field*. The SDYM equations (1.1) describe a subclass of solutions to the Yang-Mills equations (1.2). A choice of different boundary conditions for self-dual gauge fields gives such important solutions of the Yang-Mills equations as instantons, monopoles and vortices.

It is well known that the SDYM equations are manifestly invariant under the gauge transformations of the gauge potential A and gauge field F_A and under the rescaling of a metric $\mathbf{g}$ on $M : \mathbf{g} \mapsto e^{\varphi}\mathbf{g}$ (Weyl transformation), where φ is an arbitrary smooth function on M. The gauge transformations have the form (cf. talk by Tsou S.T.)

$$A \longmapsto A^g = g^{-1}Ag + g^{-1}dg, \tag{1.3a}$$

$$F_A \longmapsto F_A^g = g^{-1}F_A g, \tag{1.3b}$$

where g is a global section of the associated bundle of groups $\operatorname{Int} P = P \times_G G$ (G acts on itself by internal automorphisms: $h_1 \mapsto h_2^{-1}h_1h_2,\ h_1, h_2 \in G$), that is, $g \in \Gamma(M, \operatorname{Int} P)$. We denote the infinite-dimensional Lie group $\Gamma(M, \operatorname{Int} P)$ by $\mathfrak{G}_M$ and call it the *gauge group*.

We denote by $\mathcal{A}_M$ the space of smooth global solutions to (1.1). The *moduli space* $\mathcal{M}$ of self-dual gauge fields is the space of gauge nonequivalent self-dual gauge potentials on M,

$$\mathcal{M} := \mathcal{A}_M / \mathfrak{G}_M. \tag{1.4}$$

Let $U \subset M$ be such an open ball that the bundle P is trivializable over U. We consider smooth self-dual connection 1-forms A on U, that is, *local solutions* of the SDYM equations. Denote by $\mathcal{A}_U$ the space of all smooth solutions to (1.1) on U and by $\mathcal{M}_U$ the *moduli space* of smooth self-dual gauge potentials A on U,

$$\mathcal{M}_U := \mathcal{A}_U / \mathfrak{G}_U, \tag{1.5}$$

where $\mathfrak{G}_U := \Gamma(U, \operatorname{Int} P) = C^{\infty}(U, G)$ is an infinite-dimensional group of local gauge transformations.

The use of the moduli spaces (1.4) and (1.5) in physics is discussed in the talk by Tsou. An important example of their use in mathematics is given by Donaldson's discovery of exotic smooth structures on 4-manifolds, which is based on topological properties of the moduli space of self-dual gauge fields over the manifolds in question [4], [5].

The paper is organized as follows: in Section 2 we recall the twistor description of self-dual manifolds and self-dual gauge fields, in Section 3 we discuss the cohomological description of the moduli space of self-dual gauge fields mainly following [15], and in Section 4 we describe the infinitesimal symmetries of the SDYM equations from the cohomological point of view (see also [7], [8]).

2. An important tool: Twistors

Twistors were introduced by Penrose in order to translate the massless free-field equations in space-time into holomorphic structures on a related complex manifold known as a twistor space. The twistor theory is based on an integro-geometric transformation which transforms complex-analytic data on the twistor space to solutions of massless field equations. Suggested originally for the description of linear conformally invariant equations, the twistor method has proved very fruitful for solving nonlinear equations of general relativity and Yang-Mills theories. Namely, the Penrose nonlinear graviton construction [13] gives the general local solution of the self-dual conformal gravity equations, and the Ward twistor interpretation of self-dual gauge fields [16] gives the general local solution of the SDYM equations on self-dual 4-manifolds M.

2.1. Twistor spaces. For each oriented Riemannian 4-manifold M one can introduce the manifold

$$\mathcal{Z} := P\big(M, \mathrm{SO}(4)\big)/\mathrm{U}(2) \simeq P\big(M, \mathrm{SO}(4)\big) \times_{\mathrm{SO}(4)} S^2,$$

where $P(M, \mathrm{SO}(4))$ is the principal SO(4)-bundle of oriented orthogonal frames on M. So, the space $\mathcal{Z}$ is a bundle associated to $P(M, \mathrm{SO}(4))$ with typical fibre $\mathbb{C}P^1 \simeq S^2$ and canonical projection $\pi : \mathcal{Z} \to M$. The manifold $\mathcal{Z}$ is called the *twistor space* of M.

A Riemannian metric $\mathbf{g}$ is self-dual if the anti-self dual part of the Weyl tensor vanishes [13], [1], [17]. Manifolds M with self-dual metrics are called *self-dual*. In [13], [1] it was shown that the twistor space $\mathcal{Z}$ for such M is a complex 3-manifold. In what follows, we shall consider a self-dual manifold M and the twistor space $\mathcal{Z}$ of M.

The Levi-Civita connection on M generates the splitting of the tangent bundle $T(\mathcal{Z})$ into a direct sum

$$T(\mathcal{Z}) = V \oplus H \tag{2.1}$$

of the vertical $V = \mathrm{Ker}\,\pi_*$ and horizontal H distributions. The complexified tangent bundle of $\mathcal{Z}$ can be split into a direct sum

$$T^{\mathbb{C}}(\mathcal{Z}) = V^{\mathbb{C}} \oplus H^{\mathbb{C}} = T^{1,0} \oplus T^{0,1} \tag{2.2}$$

of subbundles of type (1,0) and (0,1). Analogously, one can split the complexified cotangent bundle of $\mathcal{Z}$ into a direct sum of subbundles $T_{1,0}$ and $T_{0,1}$. Using the standard complex structure on $S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathcal{Z}$, one obtains

$$T^{\mathbb{C}}(\mathcal{Z}) = \big(V^{1,0} \oplus H^{1,0}\big) \oplus \big(V^{0,1} \oplus H^{0,1}\big). \tag{2.3}$$

The distribution $V^{0,1}$ is integrable.

Denote by $\{V_a\}$, $\{\bar{V}_a\}$, $\{\theta^a\}$ and $\{\bar{\theta}^a\}$ ($a = 1, 2, 3$) local frames for the bundles $T^{1,0}$, $T^{0,1}$, $T_{1,0}$ and $T_{0,1}$, respectively. Because of (2.3), each of the local frames is spanned by horizontal (when $a = 1, 2$) and vertical (when $a = 3$) parts. The derivative operator d on $\mathcal{Z}$ splits as follows:

$$d = \partial + \bar{\partial}, \qquad \partial^2 = \bar{\partial}^2 = 0, \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0, \tag{2.4}$$

where locally $\partial = \theta^a V_a$, $\bar{\partial} = \bar{\theta}^a \bar{V}_a$.

We consider a sufficiently small open ball $U \subset M$ such that $\mathcal{Z}\,|_U$ is a direct product $\mathcal{P} \equiv \mathcal{Z}\,|_U \simeq U \times S^2$ as a smooth real 6-manifold. The space $\mathcal{P} \subset \mathcal{Z}$ is called the *twistor space* of U. This space is covered by two coordinate patches $\mathcal{U}_1$ and $\mathcal{U}_2$,

$$\mathcal{U}_1 := U \times \Omega_1, \qquad \mathcal{U}_2 := U \times \Omega_2,$$

where $\Omega_1 = \{\lambda \in \mathbb{C} : |\lambda| < \infty\}$, $\Omega_2 = \{\zeta \in \mathbb{C} : |\zeta| < \infty\}$ form the covering $\Omega = \{\Omega_1, \Omega_2\}$ of the complex projective line $\mathbb{C}P^1$ and $\lambda = \zeta^{-1}$ on $\Omega_1 \cap \Omega_2$. On $\mathcal{U}_1$ and $\mathcal{U}_2$ we have the local coordinates $\{x^\mu, \lambda, \bar{\lambda}\}$ and $\{x^\mu, \zeta, \bar{\zeta}\}$, respectively. We denote by $\mathfrak{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ the two-set open covering of $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ and by $\mathcal{U}_{12}$ the intersection $\mathcal{U}_1 \cap \mathcal{U}_2 = U \times (\Omega_1 \cap \Omega_2)$.

Recall that for any self-dual manifold its twistor space is a complex manifold. So, on $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{P}$ one can introduce holomorphic coordinates $\{z_1^a\}$, $\{z_2^a\}$, $a = 1, 2, 3$. On the intersection $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$ these coordinates are related by a holomorphic transition function $f_{12} : z_1^a = f_{12}^a(z_2^b)$. For local frames $\{\bar{V}_a^{(1)}\}$ and $\{\bar{V}_a^{(2)}\}$ of the bundle $T^{0,1}$ over $\mathcal{U}_1$ and $\mathcal{U}_2$ one has $\bar{V}_a^{(1)} z_1^b = 0$ on $\mathcal{U}_1$ and $\bar{V}_a^{(2)} z_2^b = 0$ on $\mathcal{U}_2$. Notice that as local frames of $T^{0,1}$ over $\mathcal{U}_1$, $\mathcal{U}_2$ one can take the antiholomorphic vector fields $\{\partial/\partial \bar{z}_1^a\}$ on $\mathcal{U}_1$ and $\{\partial/\partial \bar{z}_2^a\}$ on $\mathcal{U}_2$.

2.2. Twistor correspondence. Let M be a self-dual 4-manifold with the twistor space $\mathcal{Z}$. There is a bijective correspondence [16], [2], [1] between complex vector bundles $E \to M$ on M with self-dual connections and holomorphic vector bundles $\tilde{E} \to \mathcal{Z}$ on $\mathcal{Z}$ which are trivial on fibres $\mathbb{C}P^1$ of the bundle $\pi : \mathcal{Z} \to M$ (see also [14], [18], [9] and references therein).

We briefly describe the twistor correspondence for the case of a vector bundle $\mathcal{E}$ over an open set $U \subset M$ with a self-dual connection 1-form A. Such a bundle $(\mathcal{E}, A)$ can be lifted to a bundle $(\pi^*\mathcal{E}, \pi^*A)$ over the twistor space $\mathcal{P}$ of U. By definition of the pullback, the pulled back connection 1-form π^*A on $\pi^*\mathcal{E}$ is flat along the fibres $\mathbb{C}P^1$ of the bundle $\pi : \mathcal{P} \to U$. Therefore, the components of the connection 1-form π^*A on the bundle $\tilde{\mathcal{E}}_0 := \pi^*\mathcal{E}$ along the distribution V can be set equal to zero. Moreover, the bundle $\tilde{\mathcal{E}}_0$ is a trivial complex vector bundle $\tilde{\mathcal{E}}_0 = \mathcal{P} \times \mathbb{C}^n$ with the transition matrix $\mathcal{F}_{12}^0 = 1$ on $\mathcal{U}_1 \cap \mathcal{U}_2$. As it was demonstrated in [16], [2], [1], the SDYM equations (1.1) on a connection 1-form A on $\mathcal{E}$ is the condition for the connection 1-form π^*A to define a *holomorphic structure* on the bundle $\tilde{\mathcal{E}}_0$. Namely, the 1-form π^*A can be split into a direct sum of $(1,0)$- and $(0,1)$-parts, and the operator $\bar{\partial}$ can be lifted from $\mathcal{P}$ to $\tilde{\mathcal{E}}_0$,

$$\bar{\partial}_{\bar{B}} = \bar{\partial} + \bar{B}, \tag{2.5}$$

where $\bar{B}$ is the $(0,1)$-part of π^*A satisfying the equations

$$\bar{\partial}_{\bar{B}}^2 \equiv \bar{\partial}\bar{B} + \bar{B} \wedge \bar{B} = 0. \tag{2.6}$$

In the local frame $\{\bar{\theta}^a\}$, $a = 1, 2, 3$, we have $\bar{B} = \bar{B}_a \bar{\theta}^a$ and $\bar{B}_3 = 0$. We denote the correspondence described above by $(\mathcal{E}, A) \sim (\tilde{\mathcal{E}}_0, \bar{B})$. From (2.6) it follows that the trivial holomorphic vector bundle $\tilde{\mathcal{E}}_0$ with the flat $(0,1)$-connection $\bar{B}$ is diffeomorphic to a *holomorphic vector bundle* $\tilde{\mathcal{E}}$ with a holomorphic transition matrix $\mathcal{F}_{12}$, that is, $(\tilde{\mathcal{E}}_0, \bar{B}) \sim (\tilde{\mathcal{E}}, \mathcal{F}_{12})$. Therefore, there exist smooth G-valued functions ψ_1 on $\mathcal{U}_1$ and ψ_2 on $\mathcal{U}_2$ such that $\bar{B}_a^{(1)} = -(\bar{V}_a^{(1)}\psi_1)\psi_1^{-1}$, $\bar{B}_a^{(2)} = -(\bar{V}_a^{(2)}\psi_2)\psi_2^{-1}$ and $\mathcal{F}_{12} = \psi_1^{-1}\mathcal{F}_{12}^0\psi_2 = \psi_1^{-1}\psi_2$, where $\mathcal{F}_{12}^0 = 1$ is the transition matrix in the bundle $\tilde{\mathcal{E}}_0$. Since $\bar{B}$ is zero along the distribution $V^{0,1}$, we have $\bar{V}_3^{(1)}\psi_1 = 0$ on $\mathcal{U}_1$ and $\bar{V}_3^{(2)}\psi_2 = 0$ on $\mathcal{U}_2$, which means that $\tilde{\mathcal{E}}$ is holomorphically trivial after the restriction to any projective line $\mathbb{C}P_x^1 \hookrightarrow \mathcal{P}$, $x \in U$.

To sum up, we have a one-to-one correspondence between the complex vector bundle $\mathcal{E}$ over $U \subset M$ with a self-dual connection 1-form A and the trivial complex vector bundle $\tilde{\mathcal{E}}_0$ over $\mathcal{P}$ with the flat (0, 1)-connection $\bar{B}$ on $\tilde{\mathcal{E}}_0$ having zero component along the distribution $V^{0,1}$. In its turn, there is a diffeomorphism between the bundle $(\tilde{\mathcal{E}}_0, \bar{B})$ and the holomorphic vector bundle $\tilde{\mathcal{E}}$ over $\mathcal{P}$ that is trivializable as a smooth bundle over $\mathcal{P}$ and is holomorphically trivializable after restricting to $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}$, $x \in U$. Thus we have the following equivalence of data:

$$(\mathcal{E}, A) \sim \big(\tilde{\mathcal{E}}_0, \bar{B}\big) \sim \big(\tilde{\mathcal{E}}, \mathcal{F}_{12}\big),$$

which is called the twistor correspondence between the bundles $(\mathcal{E}, A)$, $(\tilde{\mathcal{E}}_0, \bar{B})$ and $(\tilde{\mathcal{E}}, \mathcal{F}_{12})$.

3. Čech and Dolbeault descriptions of holomorphic bundles

In the Čech approach holomorphic bundles are described by holomorphic transition matrices, and in the Dolbeault approach they are described by flat (0,1)-connections. In this section, we recall definitions of cohomology sets of manifolds with values in sheaves of groups and reformulate the equivalence of the Čech and Dolbeault descriptions of holomorphic bundles in cohomology terms. Funally, using the twistor correspondence, we obtain two cohomological descriptions of the moduli space $\mathcal{M}_U$ of self-dual gauge fields.

3.1. Sheaves and cohomology sets. We recall some definitions [6], [10], [11], [12]. We consider a complex manifold X, smooth maps from X into a non-Abelian group G and a sheaf $\mathfrak{S}$ of such G-valued functions. Let $\mathfrak{U} = \{\mathcal{U}_\alpha\}, \alpha \in I$, be an open covering of the manifold X. A *q-cochain* of the covering $\mathfrak{U}$ with values in $\mathfrak{S}$ is a collection $\psi = \{\psi_{\alpha_0 \cdots \alpha_q}\}$ of sections of the sheaf $\mathfrak{S}$ over nonempty intersections $\mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_q}$. A set of q-cochains is denoted by $C^q(\mathfrak{U}, \mathfrak{S})$; it is a group under the pointwise multiplication.

Subsets of *cocycles* $Z^q(\mathfrak{U}, \mathfrak{S}) \subset C^q(\mathfrak{U}, \mathfrak{S})$ for $q = 0, 1$ are defined as follows:

$$Z^0(\mathfrak{U}, \mathfrak{S}) := \big\{\psi \in C^0(\mathfrak{U}, \mathfrak{S}) : \psi_\alpha \psi_\beta^{-1} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing\big\}, \tag{3.1a}$$

$$\begin{aligned} Z^1(\mathfrak{U}, \mathfrak{S}) := \big\{\psi \in C^1(\mathfrak{U}, \mathfrak{S}) : \psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1} \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing;& \\ \psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = 1 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \varnothing\big\}.& \end{aligned} \tag{3.1b}$$

It follows from (3.1a) that $Z^0(\mathfrak{U}, \mathfrak{S})$ coincides with the group $H^0(X, \mathfrak{S}) := \mathfrak{S}(X) \equiv \Gamma(X, \mathfrak{S})$ of global sections of the sheaf $\mathfrak{S}$. The set $Z^1(\mathfrak{U}, \mathfrak{S})$ is not in general a subgroup of the group $C^1(\mathfrak{U}, \mathfrak{S})$.

Cocycles $\hat{f}, f \in Z^1(\mathfrak{U}, \mathfrak{S})$ are called *equivalent* $\hat{f} \sim f$ if $\hat{f}_{\alpha\beta} = \psi_\alpha f_{\alpha\beta} \psi_\beta^{-1}$ for some $\psi \in C^0(\mathfrak{U}, \mathfrak{S})$, $\alpha, \beta \in I$. The cocycle f equivalent to $\hat{f} = 1$ is called *trivial* and for such cocycles $f = \{f_{\alpha\beta}\}$ we have $f_{\alpha\beta} = \psi_\alpha^{-1}\psi_\beta$. A set of equivalence classes of 1-cocycles is called the *1-cohomology set* and is denoted by $H^1(\mathfrak{U}, \mathfrak{S})$. After taking the direct limit of the sets $H^1(\mathfrak{U}, \mathfrak{S})$ over successive refinements of the covering $\mathfrak{U}$ of X, one obtains the *Čech 1-cohomology set* $H^1(X, \mathfrak{S})$ of X with coefficients in $\mathfrak{S}$. In the case when $\mathfrak{U}_\alpha$ are Stein manifolds, $H^1(\mathfrak{U}, \mathfrak{S}) = H^1(X, \mathfrak{S})$.

We shall also consider a sheaf $\dot{\mathfrak{S}}$ of smooth functions on X with values in an abelian group. Then the subgroups of cocycles $Z^q(\mathfrak{U}, \dot{\mathfrak{S}}) \subset C^q(\mathfrak{U}, \dot{\mathfrak{S}})$ for $q = 0, 1$ are defined as follows:

$$Z^0(\mathfrak{U}, \dot{\mathfrak{S}}) := \{\theta \in C^0(\mathfrak{U}, \dot{\mathfrak{S}}) : \theta_\alpha - \theta_\beta = 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing\}, \tag{3.2a}$$

$$\begin{aligned} Z^1(\mathfrak{U}, \dot{\mathfrak{S}}) := \{\theta \in C^1(\mathfrak{U}, \dot{\mathfrak{S}}) : \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \varnothing; \\ \theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = 0 \text{ on } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma \neq \varnothing\}, \end{aligned} \tag{3.2b}$$

that is, everywhere in the definitions, the multiplication is replaced by addition. Trivial cocycles (coboundaries) are given by the formula $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$, where $\{\theta_{\alpha\beta}\} \in Z^1(\mathfrak{U}, \dot{\mathfrak{S}})$, $\{\theta_\alpha\} \in C^0(\mathfrak{U}, \dot{\mathfrak{S}})$. Quotient spaces (cocycles/coboundaries) are the cohomology spaces $H^i(\mathfrak{U}, \dot{\mathfrak{S}}), i = 1, 2, \ldots$.

Now we consider the twistor space $\mathcal{P}$ and the two-set open covering $\mathfrak{U} = \{\mathcal{U}_1, \mathcal{U}_2\}$ of $\mathcal{P}$. Then the space of cocycles $Z^1(\mathfrak{U}, \mathfrak{S})$ with coefficients in a sheaf $\mathfrak{S}$ of non-Abelian groups over $\mathcal{P}$ is a special case of formula (3.1b),

$$Z^1(\mathfrak{U}, \mathfrak{S}) := \{f \in C^1(\mathfrak{U}, \mathfrak{S}) : f_{21} = f_{12}^{-1} \text{ on } \mathcal{U}_1 \cap \mathcal{U}_2\}. \tag{3.3}$$

Any cocycle $f = \{f_{12}, f_{21}\} \in Z^1(\mathfrak{U}, \mathfrak{S})$ defines a unique complex vector bundle $\tilde{\mathcal{E}}$ over $\mathcal{P} = \mathcal{U}_1 \cup \mathcal{U}_2$ by glueing the direct products $\mathcal{U}_1 \times \mathbb{C}^n$ and $\mathcal{U}_2 \times \mathbb{C}^n$ with the help of G-valued transition matrix f_{12} on $\mathcal{U}_{12}$. Equivalent cocycles define isomorphic complex vector bundles over $\mathcal{P}$ and smooth complex vector bundles are parametrized by the set $H^1(\mathcal{P}, \mathfrak{S})$.

We introduce the sheaf $\mathcal{H}$ of all holomorphic sections of the trivial bundle $\mathcal{P} \times G$, where G is a Lie group. Then holomorphic vector bundles over $\mathcal{P}$ are parametrized by the set $H^1(\mathcal{P}, \mathcal{H})$.

3.2. Exact sequences of sheaves and cohomology sets. We consider the sheaf $\mathfrak{S}$ of smooth sections of the bundle $\mathcal{P} \times G$ and the subsheaf $\mathcal{S} \subset \mathfrak{S}$ of such smooth sections that are annihilated by the distribution $V^{0,1}$ on $\mathcal{P}$, that is, locally $\bar{V}_3 \psi = 0$ on $\mathcal{U} \subset \mathcal{P}$. So we have $\mathcal{H} \subset \mathcal{S} \subset \mathfrak{S}$ and there is the canonical embedding $\mathrm{i} : \mathcal{H} \to \mathcal{S}$.

We also consider the sheaf $\mathcal{B}^{0,1}$ of such smooth $(0, 1)$-forms $\bar{B}$ on $\mathcal{P}$ with values in the Lie algebra $\mathcal{G}$ of G that have zero components along the distribution $V^{0,1}$. Define a map $\bar{\delta}^0 : \mathcal{S} \to \mathcal{B}^{0,1}$ given for any open set $\mathcal{U} \subset \mathcal{P}$ by the formula

$$\bar{\delta}^0 \psi = -(\bar{\partial}\psi)\psi^{-1}, \tag{3.4}$$

where $\psi \in \mathcal{S}(\mathcal{U})$, $\bar{\delta}^0\psi \in \mathcal{B}^{0,1}(\mathcal{U})$, $d = \partial + \bar{\partial}$. One can also consider the sheaf $\mathfrak{B}^{0,2}$ of smooth $\mathcal{G}$-valued $(0, 2)$-forms on $\mathcal{P}$ and introduce an operator $\bar{\delta}^1 : \mathcal{B}^{0,1} \to \mathfrak{B}^{0,2}$ defined for any open set $\mathcal{U} \subset \mathcal{P}$ by the formula

$$\bar{\delta}^1 \bar{B} = \bar{\partial}\bar{B} + \bar{B} \wedge \bar{B}, \tag{3.5}$$

where $\bar{B} \in \mathcal{B}^{0,1}(\mathcal{U})$, $\bar{\delta}^1\bar{B} \in \mathfrak{B}^{0,2}(\mathcal{U})$.

Denote by $\mathcal{B}$ the subsheaf in $\mathcal{B}^{0,1}$ of such $\bar{B}$ that $\bar{\partial}\bar{B} + \bar{B} \wedge \bar{B} = 0$, that is, $\mathcal{B} = \mathrm{Ker}\,\bar{\delta}^1$. The sheaf $\mathcal{S}$ acts on the sheaf $\mathcal{B}$ by means of the adjoint representation:

$$\bar{B} \longmapsto \mathrm{Ad}_\psi \bar{B} = \psi^{-1}\bar{B}\psi + \psi^{-1}\bar{\partial}\psi.$$

It can be checked that the sequence of sheaves

$$\mathbf{1} \longrightarrow \mathcal{H} \xrightarrow{\mathfrak{i}} \mathcal{S} \xrightarrow{\bar\delta^0} \mathcal{B} \xrightarrow{\bar\delta^1} 0 \tag{3.6}$$

is exact that is, $\mathcal{B} \simeq \mathcal{S}/\mathcal{H}$. The exact sequence of sheaves induces the following exact sequence of cohomology sets [10], [11], [12], [15]:

$$e \longrightarrow H^0(\mathcal{P},\mathcal{H}) \xrightarrow{\mathfrak{i}_*} H^0(\mathcal{P},\mathcal{S}) \xrightarrow{\bar\delta^0_*} H^0(\mathcal{P},\mathcal{B}) \xrightarrow{\bar\delta^1_*} H^1(\mathcal{P},\mathcal{H}) \xrightarrow{\mathfrak{f}} H^1(\mathcal{P},\mathcal{S}), \tag{3.7}$$

where e is a marked element of these sets and $\mathfrak{f}$ is an embedding induced by the map $\mathfrak{i}$.

The sets $H^0(\mathcal{P},\mathcal{H})$, $H^0(\mathcal{P},\mathcal{S})$ and $H^0(\mathcal{P},\mathcal{B})$ are the spaces of global sections of the sheaves $\mathcal{H},\mathcal{S}$ and $\mathcal{B}$. The set $H^1(\mathcal{P},\mathcal{H})$ is the moduli space of holomorphic vector bundles over $\mathcal{P}$, and the set $H^1(\mathcal{P},\mathcal{S})$ is the moduli space of smooth complex vector bundles over $\mathcal{P}$ that are holomorphic on any projective line $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}$, $x \in U$.

3.3. Cohomological description of the moduli space $\mathcal{M}_U$. By definition the moduli space $\mathcal{M}_U$ of local solutions to the SDYM equations is the space of gauge nonequivalent self-dual connections A on U (see equation (1.5)). The space $H^0(\mathcal{P},\mathcal{B})$ is the space of smooth $\mathcal{G}$-valued global $(0,1)$-forms $\bar B$ on $\mathcal{P}$ satisfying equation (2.6) and having zero component along the distribution $V^{0,1}$. By virtue of the twistor correspondence $(\mathcal{E},A) \sim (\tilde{\mathcal{E}}_0,\bar B)$, the space $H^0(\mathcal{P},\mathcal{B})$ coincides with the space $\mathcal{A}_U$ of local solutions to the SDYM equations, $H^0(\mathcal{P},\mathcal{B}) \simeq \mathcal{A}_U$. The group $H^0(\mathcal{P},\mathcal{S})$ is isomorphic to the group $\mathfrak{G}_U$ of local gauge transformations, because G-valued smooth functions ψ defined globally on $\mathcal{P} = U \times \mathbb{C}P^1$ and holomorphic on $\mathbb{C}P^1$ do not depend on local complex coordinates of $\mathbb{C}P^1$, that is, $\psi \equiv g(x) \in \mathfrak{G}_U$, $x \in U$. Therefore we have the bijection

$$\mathcal{M}_U \simeq H^0(\mathcal{P},\mathcal{B})/H^0(\mathcal{P},\mathcal{S}), \tag{3.8}$$

that follows from the definition (1.5) of the moduli space $\mathcal{M}_U$ and the twistor correspondence briefly described in Section 2.2. The description of $\mathcal{M}_U$ in terms of $\mathcal{G}$-valued (0,1)-forms $\bar B$ on $\mathcal{P}$ is called the *Dolbeault description* of $\mathcal{M}_U$.

Now we consider the set $\operatorname{Ker}\mathfrak{f} = \mathfrak{f}^{-1}(e)$, $e \in H^1(\mathcal{P},\mathcal{S})$. It consists of such elements from $H^1(\mathcal{P},\mathcal{H})$ that are mapped into the class $e \in H^1(\mathcal{P},\mathcal{S})$ of smoothly trivial complex vector bundles over $\mathcal{P}$ that are holomorphically trivial on any projective line $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}, x \in U$. Therefore, the set $\operatorname{Ker}\mathfrak{f}$ is the moduli space of holomorphic vector bundles $\tilde{\mathcal{E}}$ that are diffeomorphic to the bundle $\tilde{\mathcal{E}}_0$ from the class $e \in H^1(\mathcal{P},\mathcal{S})$. For any representative $\mathcal{F} = \{\mathcal{F}_{12},\mathcal{F}_{12}^{-1}\} \in Z^1(\mathfrak{U},\mathcal{H}) \subset Z^1(\mathfrak{U},\mathcal{S})$ of the set $\operatorname{Ker}\mathfrak{f}$ one can find a decomposition

$$\mathcal{F}_{12} = \psi_1^{-1}(x,\lambda)\psi_2\big(x,\lambda^{-1}\big), \tag{3.9}$$

where ψ_1,ψ_2 are smooth G-valued functions on $\mathcal{U}_1,\mathcal{U}_2$ that are holomorphic on $\mathbb{C}P^1_x \hookrightarrow \mathcal{P}, x \in U$. Note that $\psi = \{\psi_1,\psi_2\} \in C^0(\mathfrak{U},\mathcal{S})$.

It follows from the exact sequence (3.7) that

$$\operatorname{Ker}\mathfrak{f} \simeq H^0(\mathcal{P},\mathcal{B})/H^0(\mathcal{P},\mathcal{S}). \tag{3.10}$$

Therefore we have the bijection

$$\mathcal{M}_U \simeq \operatorname{Ker}\mathfrak{f}, \tag{3.11}$$

and the description of $\mathcal{M}_U$ in terms of transition matrices $\mathcal{F} \in \operatorname{Ker}\mathfrak{f}$ is called the *Čech description* of the moduli space $\mathcal{M}_U$.

Let us collect the bijections (3.8), (3.10), and (3.11) in the following table:

Dolbeault description		moduli space of s-d gauge fields		Čech description
$H^{0,1}_{\bar\partial_{\hat B}}(\mathcal{P}) \supset H^0(\mathcal{P},\mathcal{B})/H^0(\mathcal{P},\mathcal{S})$	$\simeq$	$\mathcal{M}_U$	$\simeq$	$\operatorname{Ker}\mathfrak{f} \subset H^1(\mathcal{P},\mathcal{H})$,

where $H^{0,1}_{\bar\partial_{\hat B}}(\mathcal{P})$ is a Dolbeault 1-cohomology set defined as a set of orbits of the group $H^0(\mathcal{P},\mathfrak{S})$ in the set $H^0(\mathcal{P},\mathfrak{B})$ and $\mathfrak{B}$ is the sheaf of $\mathcal{G}$-valued (0, 1)-forms $\hat B$ on $\mathcal{P}$ such that $\bar\partial^2_{\hat B} = 0$.

4. Infinitesimal symmetries of the SDYM equations

We can now use the results of the previous sections to study symmetries of the SDYM equations. Cohomological description of the moduli space of self-dual gauge fields simplifies the problem of finding symmetries of the SDYM equations and clarifies the geometric meaning of these symmetries. Namely, in the Čech approach, to solutions of the SDYM equations there correspond holomorphic G-valued functions $\mathcal{F}_{12}$ (1-cocycles) on the overlap $\mathcal{U}_{12}$ of the open sets $\mathcal{U}_1, \mathcal{U}_2$ covering the twistor space $\mathcal{P}$. Therefore any holomorphic perturbation of $\mathcal{F}_{12}$ determines a tangent vector on the solution space of the SDYM equations. In Section 4.2 we define these infinitesimal holomorphic transformations of $\mathcal{F}_{12}$ by multiplying $\mathcal{F}_{12}$ on holomorphic $\mathcal{G}$-valued matrices θ_{12}, θ_{21} defined on $\mathcal{U}_{12}$. Then, using a solution of the infinitesimal variant of the Riemann-Hilbert problem from Section 4.3, we proceed in Section 4.4 to the Dolbeault description and define a transformation of the flat (0,1)-connection $\bar B$. Finally, we introduce the algebra $C^1(\mathfrak{U},\dot{\mathcal{H}})$ of 1-cochains of $\mathcal{P}$ with values in the sheaf $\dot{\mathcal{H}}$ of $\mathcal{G}$-valued holomorphic functions on $\mathcal{P}$ and, using the Penrose-Ward correspondence, we describe in Section 4.5 the action of the algebra $C^1(\mathfrak{U},\dot{\mathcal{H}})$ on self-dual gauge potentials.

4.1. Action of the group $C^1(\mathfrak{U},\mathcal{H})$ on the space $Z^1(\mathfrak{U},\mathcal{H})$. The group $C^1(\mathfrak{U},\mathcal{H})$ and the space $Z^1(\mathfrak{U},\mathcal{H})$ have been described in Section 3.1. Let us define the action ρ of $C^1(\mathfrak{U},\mathcal{H})$ on $Z^1(\mathfrak{U},\mathcal{H})$ by the formula

$$(\rho_h f)_{12} = h_{12} f_{12} h_{21}^{-1}, \tag{4.1}$$

where $h = \{h_{12}, h_{21}\} \in C^1(\mathfrak{U},\mathcal{H})$, $f = \{f_{12}, f_{12}^{-1}\} \in Z^1(\mathfrak{U},\mathcal{H})$. It is clear that for an arbitrary cocycle $f = \{f_{12}, f_{21}\} \in Z^1(\mathfrak{U},\mathcal{H})$, one can always find a cochain $\{h_{12}, h_{21}\} \in C^1(\mathfrak{U},\mathcal{H})$ such that $f_{12} = h_{12}h_{21}^{-1}$, $f_{21} = h_{21}h_{12}^{-1}$, that is, the group $C^1(\mathfrak{U},\mathcal{H})$ acts transitively on $Z^1(\mathfrak{U},\mathcal{H})$. The stability subgroup of the trivial cocycle $f^0 = 1$ is

$$C_\triangle(\mathfrak{U},\mathcal{H}) = \big\{\{h_{12}, h_{21}\} \in C^1(\mathfrak{U},\mathcal{H}) : h_{12} = h_{21}\big\}.$$

Therefore, $Z^1(\mathfrak{U},\mathcal{H})$ is a homogeneous space,

$$Z^1(\mathfrak{U},\mathcal{H}) = C^1(\mathfrak{U},\mathcal{H})/C_\triangle(\mathfrak{U},\mathcal{H}).$$

4.2. Action of the algebra $C^1(\mathfrak{U},\dot{\mathcal{H}})$ on the space $Z^1(\mathfrak{U},\mathcal{H})$. Denote by $\dot{\mathcal{H}}$ the sheaf of holomorphic sections of the trivial bundle $\mathcal{P}\times\mathcal{G}$, where $\mathcal{G}$ is the Lie algebra of a Lie group G. Denote by $\dot{\mathcal{S}}$ the sheaf of smooth partially holomorphic sections of the bundle $\mathcal{P}\times\mathcal{G}$, that is, such smooth maps $\phi:\mathcal{P}\to\mathcal{G}$ that $\partial_{\bar\lambda}\phi = 0$ in the local coordinates $\{x^\mu,\lambda,\bar\lambda\}$ on $\mathcal{P}$.

We consider the infinitesimal form of the action (4.1). Substituting $h_{12}=\exp(\theta_{12})\simeq 1+\theta_{12}$, $h_{21}=\exp(\theta_{21})\simeq 1+\theta_{21}$, we have

$$\delta_\theta\mathcal{F}_{12} = \theta_{12}\mathcal{F}_{12} - \mathcal{F}_{12}\theta_{21}, \tag{4.2}$$

where $\theta=\{\theta_{12},\theta_{21}\}\in C^1(\mathfrak{U},\dot{\mathcal{H}})$, $\mathcal{F}=\{\mathcal{F}_{12},\mathcal{F}_{12}^{-1}\}\in Z^1(\mathfrak{U},\mathcal{H})$. Here and in what follows as $\mathcal{F}=\{\mathcal{F}_{12},\mathcal{F}_{12}^{-1}\}$ we take representatives of the space $\operatorname{Ker}\mathfrak{f}$ (see Section 3.3), that is, such cocycles $\mathcal{F}_{12}$ that admits the decomposition (3.9).

4.3. The map $\phi : C^1(\mathfrak{U},\dot{\mathcal{H}})\to C^0(\mathfrak{U},\dot{\mathcal{S}})$. Now we construct the following $\mathcal{G}$-valued function:

$$\Phi_{12}(\theta) = \psi_1\big(\delta_\theta\mathcal{F}_{12}\big)\psi_2^{-1}, \tag{4.3}$$

where $\{\psi_1,\psi_2\}\in C^0(\mathfrak{U},\mathcal{S})$ and $\mathcal{F}_{12}=\psi_1^{-1}\psi_2$. Then one can check that

$$\Phi_{21} = -\Phi_{12}$$

and Φ_{12} is a smooth $\mathcal{G}$-valued function on $\mathcal{U}_{12}$ such that $\partial_{\bar\lambda}\Phi_{12}=0$ in the local coordinates $\{x^\mu,\lambda,\bar\lambda\}$ on $\mathcal{U}_{12}$. Therefore, $\Phi=\{\Phi_{12},\Phi_{21}\}\in Z^1(\mathfrak{U},\dot{\mathcal{S}})$.

It can be shown that $H^1(\mathcal{P},\dot{\mathcal{S}})=0$, since $\dot{\mathcal{S}}$ is the sheaf of smooth $\mathcal{G}$-valued functions on $\mathcal{P}$ that are holomorphic on $\mathbb{C}P^1\hookrightarrow\mathcal{P}$. Therefore, each 1-cocycle with values in $\dot{\mathcal{S}}$ is a 1-coboundary, and we have

$$\Phi_{12}(\theta) = \phi_1(\theta) - \phi_2(\theta), \tag{4.4}$$

where $\phi(\theta)=\{\phi_1(\theta),\phi_2(\theta)\}\in C^0(\mathfrak{U},\dot{\mathcal{S}})$.

Notice that the splitting (4.4) defined for any $\theta\in C^1(\mathfrak{U},\dot{\mathcal{H}})$ is not unique. Namely, as a 0-cochain from $C^0(\mathfrak{U},\dot{\mathcal{S}})$ instead of $\phi(\theta)$ one can also take

$$\tilde\phi(\theta) = \big\{\phi_1(\theta)+\varphi_1,\phi_2(\theta)+\varphi_2\big\},$$

where $\varphi_1=\varphi_2$ on $\mathcal{U}_{12}$, that is, $\varphi=\{\varphi_1,\varphi_2\}\in H^0(\mathcal{P},\dot{\mathcal{S}})$. Fix $\varphi\in H^0(\mathcal{P},\dot{\mathcal{S}})$ for each $\theta\in C^1(\mathfrak{U},\dot{\mathcal{H}})$, then the splitting (4.4) defines a subspace $\phi(C^1(\mathfrak{U},\dot{\mathcal{H}}))$ in $C^0(\mathfrak{U},\dot{\mathcal{S}})$. It can be checked that

$$\phi\big([\theta,\tilde\theta]\big) = \big[\phi(\theta),\phi(\tilde\theta)\big] = \big\{\big[\phi_1(\theta),\phi_1(\tilde\theta)\big],\big[\phi_2(\theta),\phi_2(\tilde\theta)\big]\big\}\in C^0\big(\mathfrak{U},\dot{\mathcal{S}}\big)$$

for any $\theta,\tilde\theta\in C^1(\mathfrak{U},\dot{\mathcal{H}})$. Therefore, the map $\phi: C^1(\mathfrak{U},\dot{\mathcal{H}})\to C^0(\mathfrak{U},\dot{\mathcal{S}})$ is a homomorphism.

4.4. Action of the algebra $C^1(\mathfrak{U},\dot{\mathscr{H}})$ on the space $H^0(\mathcal{P},\mathcal{B})$. Using the action (4.2) and the homomorphism ϕ, we obtain an action

$$\delta_\theta\psi_1 = -\phi_1(\theta)\psi_1, \qquad \delta_\theta\psi_2 = -\phi_2(\theta)\psi_2, \tag{4.5}$$

of the algebra $C^1(\mathfrak{U},\dot{\mathscr{H}})$ on a 0-cochain $\{\psi_1,\psi_2\}\in C^0(\mathfrak{U},\mathcal{S})$ such that $\mathcal{F}_{12}=\psi_1^{-1}\psi_2$.

By definition, for $\bar{B}=\{\bar{B}^{(1)},\bar{B}^{(2)}\}\in H^0(\mathcal{P},\mathcal{B})$ we have

$$\begin{aligned}
\bar{B}^{(1)} &= -(\bar\partial\psi_1)\psi_1^{-1} \quad \text{on } \mathcal{U}_1,\\
\bar{B}^{(2)} &= -(\bar\partial\psi_2)\psi_2^{-1} \quad \text{on } \mathcal{U}_2,\\
\bar{B}^{(1)} &= \bar{B}^{(2)} \quad \text{on } \mathcal{U}_{12}=\mathcal{U}_1\cap\mathcal{U}_2.
\end{aligned} \tag{4.6}$$

Therefore, the action of $C^1(\mathfrak{U},\dot{\mathscr{H}})$ on $H^0(\mathcal{P},\mathcal{B})$ has the form

$$\delta_\theta\bar{B}^{(1)} = \bar\partial\phi_1(\theta)+\big[\bar{B}^{(1)},\phi_1(\theta)\big], \tag{4.7a}$$

$$\delta_\theta\bar{B}^{(2)} = \bar\partial\phi_2(\theta)+\big[\bar{B}^{(2)},\phi_2(\theta)\big]. \tag{4.7b}$$

The transformations (4.7) look like infinitesimal gauge transformations

$$\delta_\varphi\bar{B} = \bar\partial\varphi+\big[\bar{B},\varphi\big], \tag{4.8}$$

where φ is an element of the Lie algebra $H^0(\mathcal{P},\dot{\mathcal{S}})\simeq\mathfrak{g}_U$ of the gauge group $H^0(\mathcal{P},\mathcal{S})\simeq\mathfrak{G}_U$. But for $\phi(\theta)=\{\phi_1(\theta),\phi_2(\theta)\}\in C^0(\mathfrak{U},\dot{\mathcal{S}})$ we have $\phi_1(\theta)\neq\phi_2(\theta)$ on $\mathcal{U}_{12}$, and the transformations (4.7) differ from (4.8).

4.5. Action of the algebra $C^1(\mathfrak{U},\dot{\mathscr{H}})$ on the space $\mathcal{A}_U$. Recall that we consider a self-dual 4-manifold M, the twistor space $\mathcal{Z}$ of which is a complex 3-manifold, and the SDYM equations (1.1) on M. To describe infinitesimal symmetries of the SDYM equations, we take an open ball $U\subset M$ and the twistor space $\mathcal{P}$ of U that is covered by two coordinate patches $\mathcal{U}_1$ and $\mathcal{U}_2$ (see Section 2.2).

The twistor correspondence gives us the following relation between a self-dual connection $A=A_\mu dx^\mu$ on the complex vector bundle $\mathcal{E}$ over U and a flat (0, 1)-connection $\bar{B}=\{\bar{B}^{(1)},B^{(2)}\}$ on the bundle $\tilde{\mathcal{E}}_0=\pi^*\mathcal{E}$:

$$\bar{B}_1^{(1)} = A_{\bar y}-\lambda A_z, \qquad \bar{B}_2^{(1)}= A_{\bar z}+\lambda A_y, \qquad \bar{B}_3^{(1)}=0 \quad \text{on } \mathcal{U}_1, \tag{4.9a}$$

$$\bar{B}_1^{(2)} = \zeta A_{\bar y}-A_z, \qquad \bar{B}_2^{(2)}= \zeta A_{\bar z}+A_y, \qquad \bar{B}_3^{(2)}=0 \quad \text{on } \mathcal{U}_2, \tag{4.9b}$$

where $y=x^1+ix^2,\ z=x^3-ix^4, \bar y=x^1-ix^2,\ \bar z=x^3+ix^4$ are complex coordinates on U.

One can always choose such local frames $\{\bar V_a^{(1)}\}$, $\{\bar V_a^{(2)}\}$ of the bundle $T^{0,1}$ over $\mathcal{U}_1$, $\mathcal{U}_2$, respectively, that $[\bar V_a^{(1)},\bar V_b^{(1)}]=0,\ [\bar V_a^{(2)},\bar V_b^{(2)}]=0,\ \bar V_3^{(1)}=\partial_{\bar\lambda},\ \bar V_3^{(2)}=\partial_{\bar\zeta}$ and on the intersection $\mathcal{U}_{12}=\mathcal{U}_1\cap\mathcal{U}_2$ the local frames are connected by the formulae [13], [1], [17]

$$\bar V_1^{(1)}=\lambda\bar V_1^{(2)}, \qquad \bar V_2^{(1)}=\lambda\bar V_2^{(2)}, \qquad \bar V_3^{(1)}=-\bar\lambda^2\bar V_3^{(2)}.$$

From (4.5), (4.7) we obtain the following action of the algebra $C^1(\mathfrak{U}, \dot{\mathcal{H}})$ on the space $\mathcal{A}_U$ of solutions to the SDYM equations on U:

$$\delta_\theta A_y = \oint_{S^1} \frac{d\lambda}{2\pi i\lambda}\big(\bar{V}_2^{(2)} + \bar{B}_2^{(2)}\big)\phi_2(\theta), \qquad \delta_\theta A_z = -\oint_{S^1} \frac{d\lambda}{2\pi i\lambda}\big(\bar{V}_1^{(2)} + \bar{B}_1^{(2)}\big)\phi_2(\theta),$$

$$\delta_\theta A_{\bar{y}} = \oint_{S^1} \frac{d\lambda}{2\pi i\lambda}\big(\bar{V}_1^{(1)} + \bar{B}_1^{(1)}\big)\phi_1(\theta), \qquad \delta_\theta A_{\bar{z}} = \oint_{S^1} \frac{d\lambda}{2\pi i\lambda}\big(\bar{V}_2^{(1)} + \bar{B}_2^{(1)}\big)\phi_1(\theta), \tag{4.10}$$

where $S^1 = \{\lambda \in \mathbb{C}P^1 : |\lambda| = 1\}$.

5. Conclusion

The space of local solutions to the SDYM equations on a self-dual 4-manifold M has been considered. Choosing the concrete self-dual 4-manifold (e.g., S^4, T^4,...) or imposing some boundary conditions on gauge fields, one can obtain instantons, monopoles or other special solutions of the SDYM equations, the moduli spaces of which are discussed in the talk by Tsou. Our purpose was to describe the moduli space and symmetries of *local* solutions to the SDYM equations. The use of twistor correspondence and cohomologies reveals the geometric meaning of symmetries of the SDYM equations, which may help in quantizing the SDYM model.

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