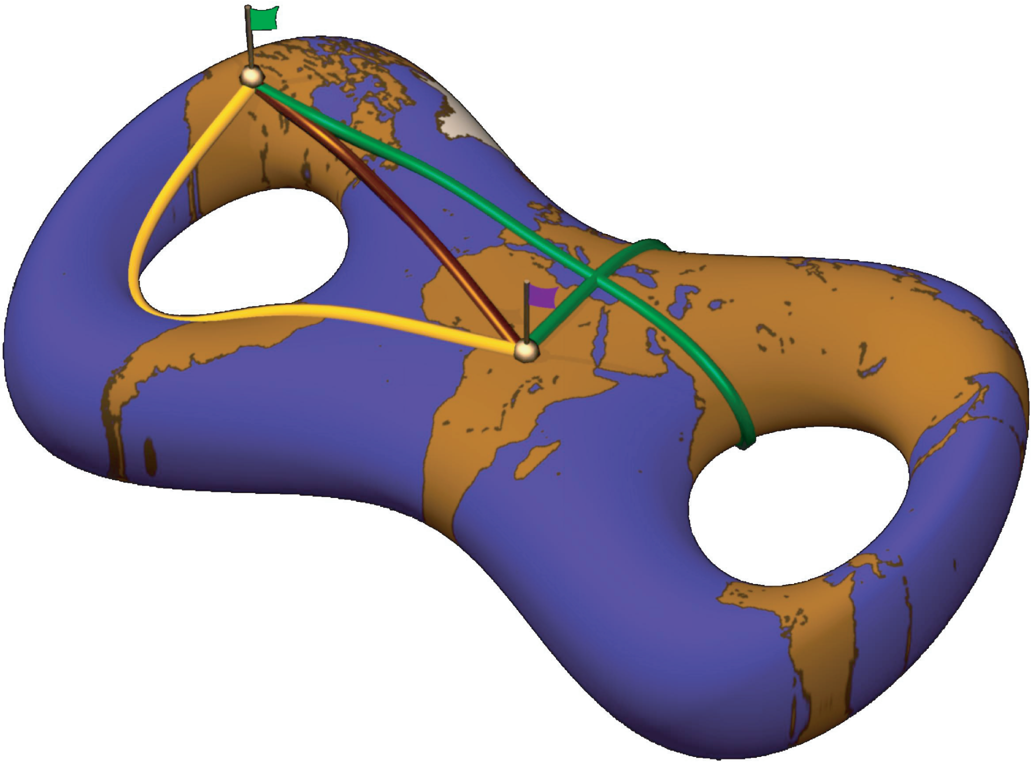
European Women in Mathematics
Proceedings of the 9th General Meeting

Loccum, Germany, 30 August–4 September 1999



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This volume contains the proceedings of the ninth general meeting of the European Women in Mathematics held at The Conference Centre at Loccum, Germany, under the auspices of the Deutsche Mathematiker-Vereinigung (DMV), it was also financially supported by the Gesellschaft für Angewandte Mathematik und Mechanik (GAMM), UNESCO, and the EU. This book was printed with support from the London Mathematical Society and the Danish Mathematical Society.

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PREFACE

During the EWM meeting in Trieste 1997, it was decided that the 9th general meeting of EWM should be held in Germany in 1999. Though there had been a few EWM meetings on the national level in Germany, this was going to be the first international EWM meeting in the country, reflecting the growing number of women mathematicians interested in EWM activities.

The meeting was held under the auspices of the Deutsche Mathematiker-Vereinigung (DMV), it was also financially supported by the Gesellschaft für Angewandte Mathematik und Mechanik (GAMM), UNESCO, and the EU. All these organizations shared our concern in making participation as widely available as possible, in particular for mathematicians from countries where salaries, especially for women, are low, such as Eastern Europe. Their support is gratefully acknowledged; it made a big difference to the success of the meeting.

Looking for a place to hold the meeting, the Conference Center at Loccum was suggested, it is located about 50 km from Hannover in a remote setting in the countryside. Both before and during the meeting the staff members at Loccum were very helpful and cooperative. Also, the conference center itself turned out to be a good choice as, besides its nice rooms and well-equipped lecture halls, it had many different places to sit and talk, creating a warm atmosphere and inviting informal discussions.

Information about the meeting was distributed via the usual channels of communication of EWM, that is, the information was spread through the e-mail network and in particular by the regional coordinators; announcements were also published in the newsletters of some mathematical societies like the DMV. As a main new medium the World Wide Web was used to provide extensive and up-to-date information on many details of the programme and practical matters such as information on the location and travel hints via a special conference website. Also, after the meeting the website was kept open with information on the conference.

In the preregistration time before the meeting a great number of people expressed their interest in participating in the meeting. Mostly due to financial problems and sometimes also due to visa problems, in the end many could not attend the meeting; in particular, these problems prevented many women mathematicians from Eastern European countries and from countries outside Europe from taking part in the conference. Finally, the meeting was attended by 50 participants from 13 European countries.

It is by now a tradition of EWM meetings to have a focus on three mathematical areas, at least one being of an interdisciplinary nature. The mathematical sessions at Loccum were on *Hilbert problems* (organised by Ina Kersten), on

Mathematical Modelling in theoretical physics, geophysics, and biology (organised by Tsou Sheung Tsun), and on *Discrete Mathematics and its Applications* (organised by Christine Bessenrodt and Tsou Sheung Tsun). As at earlier meetings, renowned women mathematicians had been invited beforehand to give talks at the sessions, and these lectures were complemented by short talks delivered by participants in Loccum.

A broad spectrum of research interests was covered by the participants. This was demonstrated in a particularly impressive way by the poster session (chaired by Polina Agranovich), which was an important part of the meeting. Following the ideas laid out by Laura Fainsilber for the poster session at the Trieste meeting in 1997, many participants presented very creative and original posters which inspired lively and fruitful discussions (and sometimes hands-on activities!).

Besides the mathematical sessions, thought-provoking talks were given at the session on *The Ideal University* (organised by Irene Pieper-Seier and Christine Bessenrodt) which also included heated debates. Also, the EWM video "Women and Mathematics across Cultures" was shown by Marjatta Näätänen, illustrating statistics on the participation of women in mathematics in different European countries and giving personal views across countries by four women mathematicians. In planning future activities, we exchanged ideas about ways of encouraging communication on all levels and about a questionnaire project.

Thinking back on the conference, there are a few things which make EWM meetings different. Organizing an international EWM meeting seems a more special enterprise than organizing other conferences. Seeing so many enthusiastic women mathematicians gathered in one place is an experience many of us rarely have in the workplace; it is also very special to have mathematicians from so many countries, age levels, and interests come together. It was very rewarding to see the interactions between people with so many different backgrounds during the week of the meeting. Altogether, the fine lectures and posters and the lively discussions on many topics were part of the success of the conference, and it is a pleasure to thank everyone for their contribution to it. In particular, I would like to thank the members of the organizing committee, some of whom have already been mentioned above; especially, for the "local" part of the organization, it was a cherished experience for me to collaborate with Irene Pieper-Seier.

Christine Bessenrodt

The following women made up the organizing commitee of the 9th General Meeting of EWM: Polina Agranovich (Ukraine), Christine Bessenrodt (Germany), Ina Kersten (Germany), Olga Kounakovaskaya (Russia), Irene Pieper-Seier (Germany), Ufuk Taneri (Turkey), and Tsou Sheung Tsu (UK).

These proceedings contain reports of the talks held at this meeting as well as articles about other aspects of the meeting: A discussion on age-limits, the poster session, the e-mail list, the general assembly and its findings, and also some reports on the life of the organization between meetings. The proceedings are available in the electronic form at

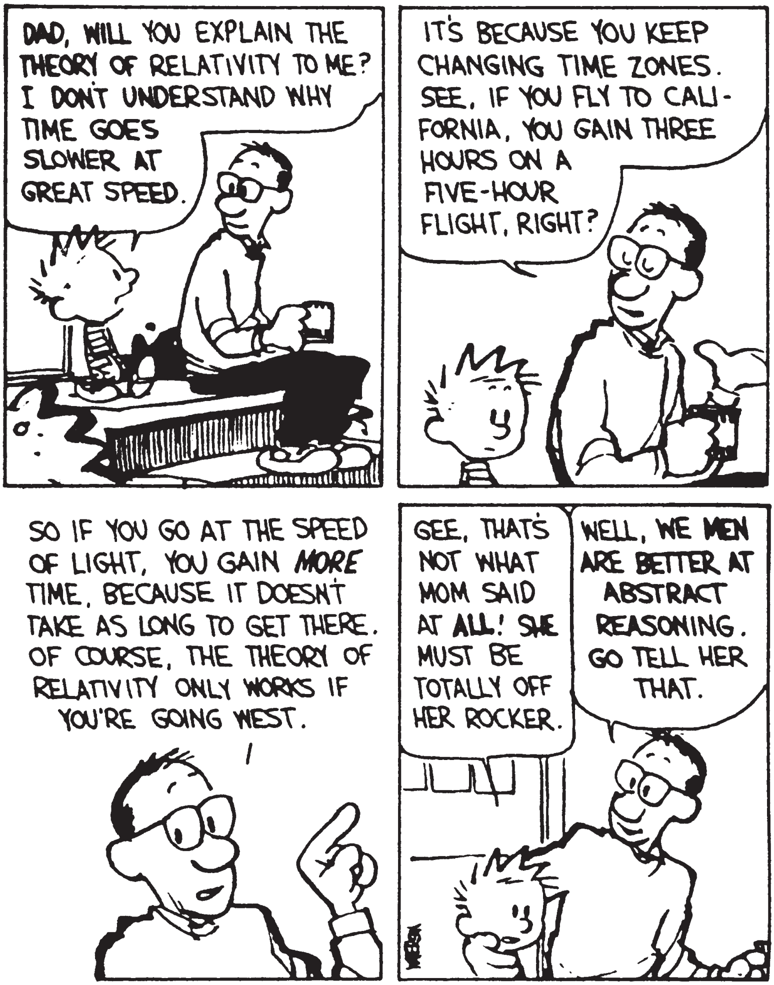
http://www.math.helsinki.fi/EWM

or of Hindawi Publishing Corporation

http://books.hindawi.com/977594502X/.

We thank the London Mathematical Society and the Danish Mathematical Society for financially supporting the printing of the proceedings. Also, we wish to thank the organizers of the meeting and the people who contributed to the proceedings, in particular Nadja Kutz who designed the cover. The design is based on a poster of hers which won the third place in a competition held by the European Mathematical Society. We must also mention Laura Fainsilber and Cathy Hobbs who edited the proceedings of the 8th General Meeting of EWM and who have been very helpful with many aspects of the completion of these proceedings. Finally, Lisbeth Grubbe Nielsen who is a secretary at the Department of Mathematical Sciences in Aalborg has been a great help and we thank her for that and also the department for letting her take time to help us.

The editors, Rachel Camina and Lisbeth Fajstrup


DAD, WILL YOU EXPLAIN THE THEORY OF RELATIVITY TO ME? I DON'T UNDERSTAND WHY TIME GOES SLOWER AT GREAT SPEED.
IT'S BECAUSE YOU KEEP CHANGING TIME ZONES. SEE, IF YOU FLY TO CALIFORNIA, YOU GAIN THREE HOURS ON A FIVE-HOUR FLIGHT, RIGHT?
SO IF YOU GO AT THE SPEED OF LIGHT, YOU GAIN MORE TIME, BECAUSE IT DOESN'T TAKE AS LONG TO GET THERE. OF COURSE, THE THEORY OF RELATIVITY ONLY WORKS IF YOU'RE GOING WEST.
GEE, THAT'S NOT WHAT MOM SAID AT ALL! SHE MUST BE TOTALLY OFF HER ROCKER.
WELL, WE MEN ARE BETTER AT ABSTRACT REASONING. GO TELL HER THAT.

EWM and Discussion from the 9th General Meeting

EUROPEAN WOMEN IN MATHEMATICS

EWM is an affiliation of women bound by a common interest in the position of women in mathematics. Our purposes are as follows:

- Encourage women to take up and continue their studies in mathematics.
- Support women with or desiring careers in research in mathematics or mathematics related fields.
- Provide a meeting place for these women.
- Foster international scientific communication among women and men in the mathematical community.
- Cooperate with groups and organizations, in Europe and elsewhere, with similar goals.

Our organization was conceived at the International Congress of Mathematicians in Berkeley, August 1986, as a result of a panel discussion organized by the Association for Women in Mathematics, in which several European women mathematicians took part. There have since been nine European meetings: In Paris (1986), in Copenhagen (1987), in Warwick (England) (1988), in Lisbon (1990), in Marseilles (1991), in Warsaw (1993), in Madrid (1995), in Trieste (1997), in Loccum, Germany (1999). The next meeting will be in Malta (2001).

At the time of writing, there are participating members in the following countries: Austria, Bulgaria, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Italy, Latvia, Lithuania, Malta, the Netherlands, Norway, Poland, Portugal, Romania, Russia, Spain, Sweden, Switzerland, Turkey, Ukraine, and the United Kingdom; contacts in Albania, Brazil, Chile, Egypt, India, Iran, Khirghistan, Nepal, Tunisia, Uzbekistan, and the West Bank. Activities and publicity within each country are organized by regional coordinators. Each country or region is free to form its own regional or national organization, taking whatever organizational or legal form is appropriate to the local circumstances. Such an organization, Femmes et Mathematiques, already exists in France. Other members are encouraged to consider the possibility of forming such local, regional or national groups themselves.

There is also an e-mail network and a web page:

http://www.math.helsinki.fi/EWM

where you will find this report as well as the proceedings of the previous general meetings in Madrid in 95, and in Trieste in 97, the yearly Newsletters, access to a bibliography on women mathematicians, and more. To subscribe to the *ewm-all* e-mail network send the following command (typing your own personal names instead of firstname(s) and lastname): *Join ewm-all firstname(s) lastname* as the only text in the body of a message addressed to:

mailbase@mailbase.ac.uk

You will then receive confirmation of your subscription.
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August, 2000

REPORT ON SOME ACTIVITIES OF EWM BETWEEN GENERAL MEETINGS

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Since the 9th general meeting of EWM there have been various EWM regional meetings, some of which are listed below. These meetings have not only concentrated on the interesting ongoing mathematical research but also on the presence (or lack) of women in the mathematical world.

• Helen Robinson and Cathy Hobbs organized a meeting at the International Centre for Mathematical Sciences in Edinburgh, Scotland in September 1999. Talks were given on medical statistics, explosions, LEGO blocks and graphs among others. There was also an animated debate on the grades obtained by women at Cambridge University.

• There was a joint meeting with Femmes et Mathematiques in June 1999.

• Irene Sciriha and Cathy Hobbs set up a stand and showed the EWM video at the Brussels EU meeting Women and Science: Networking the Networks which was held in July 1999. Various networks, AWISE (UK) in particular, showed interest in collaborating with us.

• In February 2000, there was a work shop in Malta aimed to arouse awareness of EWM. Mathematics students and lectures discussed new proofs and posed new problems.

• In April 2000, Irene Sciriha participated in a much larger follow-up meeting called Women and Science: Making Change Happen. A stand was set up to highlight EWM activities. A powerpoint presentation on EWM activities was made available and a leaflet on EWM's interests was disseminated.

• In June 2000, there was a joint EWM-AWM session at the AMS Scandinavian meeting in Odense, Denmark.

DISCUSSION OF THE EWM E-MAIL NETWORK

The following is a summary of the discussion of the EWM e-mail network. The EWM e-mail network consists at the moment of three lists, namely ewm-all, ewm-discuss, and ewm-uk, which is a national sublist for Great Britain.

> **Ewm-discuss.** Is a list intended for discussing issues that are of interest to the members of the network. It is a SUBLIST of ewm-all. The naming may be misleading, but for technical reasons it has to be called a sublist. It means that the members of the ewm-discuss list form a SUBSET of the members of ewm-all, that is, *everybody who is on the ewm-discuss list also gets all e-mails sent to ewm-all.*
>
> **Ewm-all.** Is now essentially for information, for example, for job offers conference announcements, short requests, etc. It should also give regular updates on topics discussed on ewm-discuss. The idea being that women can join and leave ewm-discuss as and when the current discussion interests/does not interest them.

Archives of both lists are always available from the website at

http://www.mailbase.ac.uk

so it is easy to follow a discussion even if one has not received the messages.

A great majority of the women at the discussion agreed to keep the lists as they currently are. This should not be viewed as a rigid law, but as a flexible reaction to necessities. A discussion about a different structure of the ewm-network could take place on ewm-discuss.

The *disadvantages* of having these two lists instead of just one are:

• It may sometimes be hard to decide whether something is information or a topic for discussion.

• Ewm-discuss is a sublist that means that fewer (or as many) members of the network are taking (at least passively) part in discussions.

• Under the assumption that there are subjects that people should feel morally obliged to discuss, having two lists means that they can circumvent this obligation.

It was agreed that people should be regularly informed of how to switch lists in order to avoid people missing discussions they may wish to contribute.

The *advantages* of having these two lists are:

• The function of information list is preserved and reaches many without overloading mailboxes.

• People may feel more encouraged to discuss on a list which is intended for discussion, rather than on an all-list.

- People have a choice whether they want to discuss an issue or not.

The number of people quitting the network during the Kosovo war discussion and their comments showed that there was a need for the option to choose.

There was the suggestion to open up a *newsgroup*, that is, a list which can be attended via a world wide web (www) browser. A newsgroup is even less intrusive than a discussion list. However, the experience of people at the discussion with news groups seemed to be not so good. The main argument was that there were too many comments sent to the newsgroup, since nearly everybody in the world could attend a discussion. Consequently people would stop reading the lists after a while due to an information overload. One could moderate this effect by restricting access via, for example, a password, but this was considered too complicated. In addition, people in general tend to forget to regularly check the newsgroup.

There was the question whether one should restrict access to the ewm-list archives at mailbase (for details, see http://www.mailbase.ac.uk), where all e-mails sent to the ewm-lists are being kept and which are accessible to everybody via www. Most of the people at the discussion thought it better to keep them open, as there may be other people who are interested in the discussion and would like to browse the archives but who would not want to be on the e-mail list. One could also think about opening up something like an ewm-private list, which has a restricted archive access, which means that only members of this list are allowed to read the archives. Some people at the discussion found that a webpage with password access leaves a bad impression. In general, EWM tries to be open and encourage as many people as possible to participate in activities regardless of whether or not they are members. Restricting access to some of the means of communication could blur that attitude. However, it was agreed that the list of members of ewm-all shall be accessible only to list members, which is the case by now.

Another issue raised at the discussion was the question of netiquette rules (those rules are quoted later in this proceedings). There have been some incidents where basic netiquette rules were very much on the edge of being broken. The question was whether one should moderate the list in order to avoid such incidents. However, moderation would be very time consuming and it was concluded that hopefully a regularly sent out reminder of Netiquette rules would be sufficient for a decent use of the lists. In cases where Netiquette rules are broken in a very severe way, one has to react anyway, since the ewm-lists have to conform to the mailbase rules (see Mailbase Acceptable Use Policy at http://www.mailbase.ac.uk/docs/aup.html). The woman in charge for our list is Sarah Rees.

It was remarked that there was lately not much discussion on ewm-discuss. One suggestion was to appoint someone, who—if there is silence on the net—sends out an e-mail for initiating a discussion. It was asserted that a network is as good and effective as its members and that everybody should feel responsible for contribute. This assertion should be part of the regularily sent out Netiquette rules.

There seem to be a lot of EWM members not on the ewm-lists. It was said that even some regional coordinators do not take part in the ewm electronic network. Hence every member of the list should encourage others to join the lists!

End of discussion.

Nadja Kutz

DISCUSSION ON AGE LIMITS

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One evening during the EWM meeting in Loccum was dedicated to a discussion on age limits stipulated in many announcements for grants. The wish to have such a discussion was prompted by a recent series of letters on the ewm-net, which expressed concern at imposing age limits when grants are advertised. As women have often had gaps in their careers it is felt that these age limits are particularly harmful to women.

Laura Tedeschini-Lalli, who moderated this discussion in Loccum, reminded us of the first time the issue was taken up by the EWM. At the meeting in Warwick in 1988, the topic of cultural differences came up during an informal exchange of opinions, and from there the topic of mathematicians and age followed. It seemed that the stereotypical image of a creative mathematician as being young and male was common. Someone suggested that a study be made, to collect examples of mathematicians who continued to do good research late into life.

The aim of the discussion in Loccum was to try to clarify the impact on women's mathematical careers of the many age limits present in job adverts and grant descriptions. One of the participants wondered whether these age limits put an extra strain on women's careers. It is well known that women are generally the ones who take care of the children, and in some countries also of the elderly in the family. As a result of these duties at home, women are more likely to have a late start in their education as well as a slower career path. The effect of imposing age limits might then simply be that women are cut off from pursuing their careers.

During the discussion two concrete suggestions where made in order to enlighten the question of how age limits affect women's careers: First, all participants in the discussion briefly presented themselves, explained why they chose to do mathematics, and outlined the path of their careers; second, we agreed that it would be useful to gather concrete data on a large number of mathematicians, both female and male, in order to find out how their research work and job situation were conditioned by age. Towards this last point we drew up a questionnaire, included at the end of this report, which is intended to be distributed. When presented to organizations that award grants, such as EU educational programmes, the output of the questionnaire will hopefully illustrate the problems of age limits.

Several interesting aspects were revealed by the personal presentations. True and deep interest in math was the main reason for ending up being a mathematician, but the encouragement or lack of the same from the ambient environment also proved crucial. Several of the participants in the discussion told that they did not consider doing a Ph.D. in the first place, as they had not believed they were good enough for doing research, but that such a move was suggested to them. It turned out that having somebody in the family, typically the parents, doing math was an important support, and lessened doubt of the sort “Am I good enough to do math?” On the other hand, some participants recalled that at different stages during their education they were warned against doing mathematics, as being too hard. Explanations for why it is difficult to decide on doing math were presented, ideas included: Women worry about succeeding in what they do, the social pressure could be important, if things go wrong women tend to blame themselves. Attending a girls school proved positive towards making the decision to do math, but it was not the only motivation.

The career patterns of the participants covered a wide spectrum. The age when the math education or research work started varied greatly. Some started research work late in life, and discovered then that it was what they wanted to do. Only a few had a more or less “uninterupted” math career, the majority had experienced gaps, typically due to the family situation. Sometimes the gaps were caused by painful events, and getting over them was not easy and immediate. It was mentioned that if one was lucky to come back to math after a temporary interruption, the desire to do research was certainly stronger. The reason our discussion focused on career gaps was that any gap brings a delay, and therefore one can get closer to not fulfilling the age limit requests.

The overall impression after the presentation was that math careers often start later for women, that there is no standard career path for women mathematicians, that confidence in doing research work increases in time, and that a lot of good research is done later in life.

Questionnaire

(1) Are you male or female?
(2) How old are you?
(3) What is your nationality?
(4) What is your mother’s job?
    What is your father’s job?
(5) How many children do you have?
(6) At what age did you complete your Ph.D.?
(7) How many countries have you studied/worked in?
(8) How many temporary mathematical jobs have you had?
(9) Do you have a permanent job?

(10) How many years after your Ph.D. did you obtain your first permanent job?

(11) At what age did you write the paper of which you are most proud?

(12) Have you had any gaps in your mathematical production?
If so, how long were these gaps?
In your opinion what were the reasons for these gaps?

DO MATHEMATICIANS' CAREERS FOLLOW A COMMON CAREER PATH?—ANALYSIS OF A QUESTIONNAIRE

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1. Introduction. Frequently when a mathematician reads a job advert or the conditions for a grant she is confronted with an age-limit. Although these adverts often include a statement along the lines of "Applicants over 40 may be considered under very exceptional circumstances" (see advert for the Royal Society's University Research Fellowships), the implication is that any mathematician worthy of the given job/grant should have already proved themselves by the stated age. As this age is often reasonably young, 30 or 35, say, it was felt by many members of EWM that such age-limits were restrictive and biased against mathematicians who, for one reason or another, had decided to take a few years out from their mathematical careers or decided to work part-time. It was also felt that the majority of such mathematicians would be female and therefore that these age-limits were particularly harmful to women. By the time of the 9th General Meeting of EWM a discussion along these lines had been running for some time on ewm-discuss (the e-mail list of EWM devoted to such debates). It was therefore decided that a further discussion should be held on this topic at the General Meeting and that a questionnaire should be compiled to see whether these suspicions were based on reality.

The discussion was interesting, lively and wide-ranging (see Nadia Larsen's article for a detailed report) and the questionnaire put together at the time reflected the many topics covered during the discussion. However, when we were faced with distributing the questionnaire and analysing the results, we decided it would be preferable to reduce the scope of the questionnaire and to focus on what we thought was a critical assumption lying behind the idea of age-limits: That the career paths of mathematicians followed a common path and therefore by comparing two mathematicians of, for example, age 30, you could confidently compare the future success of the two mathematicians.

How many times as a mathematician have you heard members of the general public stating, as fact(!), that mathematicians burn-out young and that mathematicians do their best work by the time they are 30? Although there are classic cases where this is true, many participants in the discussion felt that this was really a myth and a harmful myth at that; harmful to mathematicians of both sexes, not just women mathematicians. This gave us additional motivation in trying to find out how mathematicians' careers really do evolve and whether such general statements can be made.

The questionnaire that follows is an adaptation of the one drawn up at the discussion and aims to tackle the particular issue of how mathematicians' careers develop. We were interested in all mathematicians, not only female, and we were not looking for particular problems faced by women mathematicians. The questionnaire was distributed via ewm-all (the e-mail list of EWM) and women were asked to pass the questionnaire on to male mathematicians of similar standing within their departments. Copies were also distributed at the British Mathematical Colloquium in Leeds, April 2000.

2. The questionnaire

(1) Are you female or male?
(2) How old are you?
(3) (a) How many years is it since you completed your Ph.D.?
(b) Where did you complete your Ph.D.?
(4) When did you publish your first mathematical paper?
(5) What is your current position? Is it temporary or permanent, part-time or full-time?
(6) (a) How many children do you have? If none go to question 7.
(b) Did you take maternity/paternity leave for each child?
(c) If so, for how long?
(d) At what age did you have your first child?
(7) (a) Have you worked part-time?
(b) If so for how long?
(c) When was this?
(8) (a) At what age did you write the published (accepted) paper of which you are most proud?
(b) To date, when has been your most productive mathematical period?
(c) Did the work on paper (a) occur during period (b)?
(9) (a) Have you had any gaps in your publishing mathematical career?
(b) If so, how long were these gaps and when were they?
(c) In your opinion, what were the reasons for these gaps?
(10) Comments.

Remarks

(i) It was thought that the country where a mathematician completed their Ph.D. would have a greater influence on their mathematical career than their country of origin, hence question 3(b).

(ii) Although question 3(a) asked for number of years since completion of Ph.D. this was then combined with question 2 to find the age at which the respondent completed their Ph.D., thus question 3(a) was probably the wrong question.

(iii) Although some respondents complained that question 8(a) was too difficult to answer we felt it was an important indicator as to when a mathematician's career peaks.

3. Results

3.1. Gender and Ph.D. origin

TABLE 3.1. Country where respondent completed their Ph.D. (response to questions 1 and 3(b)).

Country	Female	Male	Total
Argentina	1	0	1
Australia	0	1	1
Austria	1	0	1
Beligium	1	0	1
Bulgaria	1	0	1
Czech Republic	1	0	1
Denmark	2	2	4
France	4	5	9
Germany	16	12	28
Italy	1	0	1
Netherlands	1	1	2
Russia	1	0	1
Sweden	1	0	1
Switzerland	3	0	3
UK	11	10	21
USA	2	1	3
Grand total	47	32	79

3.2. Age

TABLE 3.2. Ages of respondents (response to question 2).

Age	27–29	30–39	40–49	50–59	60–69	70–71
Frequency	5	31	16	22	3	2

Mean age of respondents: 43	Female: 41	Male: 46
Median age of respondents: 41		
Mean age when completed Ph.D.: 28	Female: 29	Male: 27
Mean age when published first paper: 28	Female: 28	Male: 27
Mean age when wrote best paper: 35	Female: 35	Male: 36

3.3. Children

Of the 52 respondents (29 female and 23 male) with children, 24 of the women had taken maternity leave and 2 of the men had taken paternity leave.

TABLE 3.3. Number of children of respondents (response to question 6(a)).

No. of children	Female	Male	Total
0	18	9	27
1	9	8	17
2	17	7	24
3	1	6	7
4	2	2	4

3.4. Jobs

TABLE 3.4. Type of jobs of respondents (response to questions 1 and 5).

Job type	Female	Male	Total
Full-time and permanent	27	20	47
Full-time and temporary	10	6	16
Part-time and permanent	1	0	1
Part-time and temporary	3	1	4
Retired	1	2	3
Between jobs	1	0	1
Incomplete answer	4	3	7

TABLE 3.5. Whether respondent had worked part-time (response to question 7(a)).

Worked part-time	Female	Male	Total
Yes	21	5	26
No	26	27	53

3.5. Productive periods

TABLE 3.6. Whether respondent's most productive mathematical period coincided with the writing of the paper of which they are most proud (response to question 8).

Coincided	Female	Male	Total
Yes	35	15	50
No	10	11	21
No answer	2	6	8

3.6. Gaps in publishing career

42 respondents (27 female and 15 male) said they had gaps in their publishing career.

TABLE 3.7. Frequency of reasons given by respondents for gaps in their publishing career (response to question 9(c)).

Reason	Female	Male	Total
Children/Family	16	5	21
Teaching	7	3	10
Administration/Managerial	4	5	9
New research area	5	2	7
Personal problems	4	3	7
Unfriendly environment	3	2	5
Disruptive moves	2	0	2
Writing books	0	2	2
No progress on difficult problem	0	2	2

4. Discussion. Over half of the respondents completed their Ph.D.'s in either Germany or the UK. However since these two countries have very different academic systems, a typical British student completing their Ph.D. several years before the average German student (in our sample the median age for a British student completing their Ph.D. is 25 in comparison with 28 for a German student and 29 for the rest), the sample should not be biased too heavily by a particular mathematical culture.

In our sample the average age of a female respondent is 5 years younger than that of the male respondent. To investigate whether this difference was significant, a 2-sample t-test assuming unequal variances was performed which did not quite reach a 5% significance level ($p = 0.051$). However, we claim from personal experience (departments we have worked in, conferences attended, etc.), that the average female mathematician is younger than the average male mathematician, but it is not possible to check this assumption without collecting large scale data. Whether the ages of our respondents accurately reflects that of the mathematical population is unclear to us, but it seems likely that our sample is on the young side.

Further to the calculation of the mean age of the respondent when they wrote their best paper, we checked to see if this age was related to the age of the respondent. This turned out to be the case. However, even after allowing for the current age of the respondent the age when they wrote their best paper was still very similar for men and women and still around 35. We can maybe conclude that men and women's mathematical careers peak at similar times, and that this is relatively young, although maybe not as young as is generally believed. This is also supported by the response to question 8(c). Most respondents said that their most productive period coincided with the writing of their best paper and for those when this was not the case the two times were generally pretty close together. This also implies that mathematicians' best work is the culmination of an intense period of research as opposed to a "flash in the dark"—maybe common knowledge to the mathematician, if not to the general public.

That 16% of the male respondents had worked part-time in comparison to 45% of the women (a statistically significant difference, $p = 0.004$) confirmed the idea that women tend to have a much more disrupted career path than men. This was also supported by the fact that of the 29 women and 23 men with children, 24 women took maternity leave in comparison with only 2 of the men taking paternity leave.

Family and, in particular, the care of young children came up as the main reason for gaps in respondents publishing careers. Perhaps not surprisingly, this reason affected a higher proportion of female respondents than male (female: 16/47 and male: 5/32). That duties other than research, such as teaching and managerial responsibilities, can often create gaps in people's publishing careers is also not surprising. In fact, whether research is our primary aim was also questioned, one (male) respondent commented "there are more ways to contribute to mathematics than writing lemmas." Gaps can also be due to unhappy circumstances. It appears that mathematicians need to be in a friendly environment and have a happy personal life to succeed in research. But the reasons for gaps are not all negative: People switch research areas, take time out to write books and concentrate on difficult problems which do not always lend themselves to solutions. So maybe gaps themselves should not always be viewed in a negative light and instead thought of as a necessary time-out in people's careers.

It is often claimed that, due to the additional pressures having children puts on their careers, female academics have on average fewer children than their male counterparts. In our sample the female respondents had a mean number of children of 1.1 in comparison to 1.5 for that of the male respondents. However, this could be due to the fact that the majority of our female respondents are not past child-bearing age and at an average age of 41 years are 5 years younger than our average male respondent.

5. Conclusions. We should treat the conclusions we draw from this survey with caution. First, we note that our sample size of only 79 is too small to draw any strong conclusions. Second, we note that our distribution of the questionnaire is very likely to introduce a bias as we relied on people volunteering to fill it in. We have no way of monitoring those who chose not to participate. In fact, our sample is probably too female to be truly representative of the mathematical population. However, we will make some additional comments.

Although it seems clear from our results that the career path of a female mathematician is much more disrupted than that of her male colleague this does not seem to have a significant impact on when she judges herself to have peaked as a mathematician. However, it might help to explain why she has a slightly higher tendency to have gaps in her career (27/47 women claimed to have gaps in comparison with 15/32 men). Further, although we seem drawn to conclude that there is a similarity in mathematicians' career paths, we should also note that our sample consists of mathematicians who have remained and therefore succeeded in academia (although one respondent worked in industry) and therefore conformed to our own stereotype of a successful mathematician.

With regard to the issue of sample size we note that the questionnaire is due to appear in the September 2000 issue of the Association for Women in Mathematics (AWM) newsletter and a similar questionnaire has appeared in the Notiziario of the Italian Mathematical Society. We look forward to analysing this additional data which will enhance our sample size as well as providing interesting comparisons. Thus, we hope in the future to be able to draw more reliable conclusions about the career paths of mathematicians.

INTRODUCTION TO THE SESSION ON THE IDEAL UNIVERSITY

There were two talks and a discussion devoted to the topic "The Ideal University." Renate Tobies and Britta Schinzel gave talks about the history of women in mathematics in Germany, the history of universities, the effect of globalization on universities and on the situation of women, and minorities in universities. These talks gave facts, personal experiences, and personal opinions. The talks raised many questions, which gave rise to a very lively debate on Saturday afternoon.

In these proceedings, we have written contributions from both Renate Tobies and Britta Schinzel, a report of the discussion and a comment from Tsou Sheung Tsun, in particular pointing out where she disaggrees with Britta Schinzel. All in all, this gives a good impression of the situation in Loccum: There are very different opinions about these subjects.

Lisbeth Fajstrup

IN SPITE OF MALE CULTURE: WOMEN IN MATHEMATICS

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I will present a survey of conditions for women in mathematical research at German universities since the last third of the 19th century. Conditions which need to be considered, are laws for women studies, circumstances at different universities, the role of the parent's home, the influence of the doctor father, the importance of the chosen mathematical research field, as well as the image of women mathematicians in public.

The title of my talk uses a quotation from Marie Vaerting (1880–1964), who, in 1910, gained a doctorate in mathematics. It is taken from her autobiographical novel "Haßkampfs Anna" and the full quotation is: "*We dedicate ourselves to mathematics and natural sciences 'in spite of male culture, female bodies and ballet*'." Marie Vaerting is not just any doctor of mathematics; she and her sisters are representative of those women who just after the turn of the century in Germany set out to conquer areas which are considered to be male domains even today.

The proportion of female students of mathematics in Germany today is around 33%, further mathematics is not one of the ten courses of the studies most frequently chosen by women. In the case of male students, however, mathematics is currently in 8th place. These figures make it difficult to imagine that mathematics could once have been one of the most popular courses of studies for women. Yet this was the conclusion revealed by a project we carried out in Kaiserslautern.

The mathematician Gerhard Kowalewski (1876–1950) wrote about his time in Bonn in the first decade of this century as follows (see [7]).

> *"That was when the first female students started to show up in Bonn. At other universities they were still being brusquely rejected by respected professors. In Bonn there was a whole string of competent women mathematicians, and many of them came to me to take their state examinations ...."*
>
> *Marie Vaerting, who later became a famous novelist and whose first novel "Haßkampfs Anna" was penned during her time as a student in Bonn, was also a student of mine, as was her sister Mathilde ...,*

> *Marie Vaerting worked in Bonn under my supervision on a doctorate on a quite difficult subject.*

Marie Vaerting came from the Catholic region of Emsland and had seven sisters and two brothers. At least four of the girls studied mathematics. The most famous of them was Mathilde Vaerting (1884–1977), Marie's younger sister by four years, who, like Marie, studied mathematics, physics, and philosophy and who in 1923 managed to become an ordinary professor (as one of only two women in the whole of the 1920s). The professorship, at Jena University in Thuringia, was not one of mathematics but of education. Here she did much to promote the teaching of girls in mathematics and natural sciences [10]. The Vaerting sisters were part of a whole group of women who broke into male domains.

Where did the considerable affinity of women for mathematics at this time come from? This affinity was not only the desire for improved general education but above all arose from a new professional opportunity. Prussia had passed a ministerial decree on 18th August 1908, which not only allowed the enrollment of women at universities but at the same time introduced a new order in state-run secondary schooling for girls. The new element was the fact that for the first time in a German state the subjects of mathematics and natural sciences were to be taught at state-run girls' schools. A result of the new regulation was that more teachers were needed. This led to a new professional opening for women, shown clearly by the numbers choosing mathematics as the subject of their studies (ranked 3rd among the subjects chosen by women and staying in that position from 1909 to 1919, by which time the schools were overflowing with teachers). Although the decree stipulated that men should make up at least a third of the teachers at these girls schools (of course there was no such quota on female teachers in boys' schools), it also presented an opportunity for women who were interested in mathematics and natural sciences. It should be noted that at that time there were no fixed regulations on a course of studies leading to a diploma in mathematics. These were only introduced in 1942, and professional openings in industry and commerce, insurance and statistics offices only developed gradually. The principal achievement of the new statutory regulations, as far as women were concerned, was that they could take up the profession of teacher of mathematics and natural sciences. At the same time, however, the newly accessible course of studies for this profession created a basis for women to find their way into research. This path to research was governed by factors which are useful to know about even today.

Parents had to support their daughters' inclinations and be able to afford to finance their studies. The analysis of the lives of a group of women with doctorates showed that most of the fathers were either teachers, medium-level civil servants, tradesmen, or businessmen. Even if it had been made possible to study up to doctorate level, excellent academic performance was not enough to pursue a university career if financial backing was not guaranteed.

Emmy Noether (1882–1935), who is probably still the most significant female mathematician even today, led a meagre existence on an inheritance and a teaching contract. Käte Hey (1904–1990), a student of Emil Artin's (1898–1962) in Hamburg, had to give up her academic leanings on financial grounds. Her doctoral thesis on numerical theory won a competition in 1927, was graded "excellent" and gained her a prize of 300 marks. In his report on her thesis Artin wrote:

> *"Transferring the functional equation of the zeta function from ideal classes in algebraic number fields to ideal classes in extremely complex number systems is an important and difficult problem in the arithmetic of these systems ...."*

The results of Käte Hey's doctoral thesis are quoted at length in Max Deuring's (1907–1884) book "Algebren" (1968) and have, therefore, been preserved for posterity. As I found out only recently, Emmy Noether brought Hey's results to the attention of her student Ernst Witt (1911–1991), who took them up in his doctoral thesis (1933). Käte Hey, however, was unable to continue with her research interests. Her father, a medium-level civil servant working in customs, wanted to finance courses of studies for other children too and called a halt. This is reflected in a poem in the family chronicle:

> *Ihre Leistungen sind beachtlich*
> *und nach der Doktorprüfung fragt sich,*
> *ob sie nun ganz in Forschung und Lehre*
> *ihre Wissenschaft verehre.*
> *Jetzt aber sagt der Vater: "Stop!*
> *Es geht nicht nur nach deinem Kopp,*
> *ich kann dich nicht ewig speisen und kleiden.*
> *Sind auch die Ansprüche noch so bescheiden*
> *so kostet dies doch alles Geld,*
> *das mir nicht reichlich in den Schoß fällt.*
> *Du mußt das Studium beenden*
> *dich einem Brotberuf zuwenden,*
> *damit du dich mit dem Verdienst*
> *fürder selbst durchs Leben bringst."*

Käte Hey took and passed the state examinations in the subjects of pure and applied mathematics and physics. She became a teacher, married, and in the end was forced to give up her career. She had four children—two sons and two daughters (see [8]).

Combining family and work is difficult for a woman with an academic career even today—especially in Germany. Up until 1919, there were laws in Germany which prohibited married women from working in civil service. They had to choose between following a career or having a family. Even after 1919, new paragraphs were passed which permitted the dismissal of a woman working in civil service if she married (the argument given being double income).

Up until recently, the conclusions on the role of the parents have been based on comparisons within a relatively small group of subjects. In the course of the investigations being carried out in the project "Women in Mathematics. Career Development in Mathematics under a Gender-comparative Perspective," supported by the German Volkswagen Foundation, we aim to consider a larger number of graduates in mathematics in order to arrive at significant conclusions. The project is interdisciplinary, combining methods of investigation used in historiography and social psychology. Personnel questionnaires of all the teachers, male and female, who passed state examinations in Prussia with mathematics as their major subject are being analysed so as to show the development of professional careers in the period from the beginning of this century to 1942. The different types of families the subjects came from is one of the factors which will be clarified. A project for a doctoral thesis in social psychology is investigating corresponding data among subjects who graduated in mathematics in 1998. Together we will examine which factors, constant or variable, governed and still govern the careers of men and women in mathematics.

A look at the historical development reveals that Germany lagged behind other countries, only allowing the enrollment of women at a later stage. It should be noted, however, that the universities did make exceptions. The first woman ever to gain a doctorate with a mathematical thesis, for example, achieved this at a German university. I imagine you all know the example of the Russian mathematician Sofja Kowalewskaja (1850–1891), who was awarded a doctorate by the University of Göttingen in 1874 and finally, in 1884, a professorship in mathematics in Stockholm. There were other foreigners who paved the way for women at German universities by being allowed to listen to lectures and even undertake doctoral studies.

Being allowed to enroll at a university and to take doctoral examinations did not imply that a university career was guaranteed. In Germany there was and still is the hurdle of the habilitation. In 1907 a law excluding women from habilitation was passed. A few years later Emmy Noether arrived on the scene with an excellent academic record. The mathematicians at Göttingen encouraged her to apply for a habilitation. This application, made in 1915, and another in 1917 failed because no exceptions to this law were permitted. It was only in 1919, after the collapse of the German empire, that Emmy Noether passed her habilitation in mathematics, the first woman in Germany to do so. And it was only after this exception that an official decree was provided which repealed the law of 1907. On 21st February 1920, it was ruled that the sex of a person was no longer a determining factor of whether they were allowed to take the habilitation examination. Habilitation was the precondition for a professorship at a German university. If we consider the time before 1945, then Emmy Noether was the only woman to be awarded the title professor, although in fact it was only an extraordinary professorship without the status (and advantages) of a civil servant, since this was the highest position a woman could be given in Prussia at that time. By 1945, only six women had obtained a habilitation in mathematics.

Year of habilitation	Women mathematician	University
1919	Emmy Noether (1882–1935)	Göttingen
1927	Hilda Geiringer (1893–1973)	Berlin
1936	Ruth Moufang (1905–1977)	Frankfurt a.M.
1941	Helene Braun (1914–1986)	Göttingen
1942	Maria-Pia Geppert (1907–1998)	Gießen
1945	Erna Weber (1897–1988)	Jena

Their careers were further hindered by political developments in the 1930s. Emmy Noether and Hilda Geiringer emigrated to the USA because they were Jews. Ruth Moufang was hindered purely because she was a woman, she had successfully defended her habilitation thesis in 1936, but her appointment as a university lecturer was refused. The Ministry informed her: “A lecturer in the Third Reich is expected not only to pursue academic activities but also to perform duties which are mainly educational and require leadership qualities, and as the body of students consists almost exclusively of men, any female professor lacks the first essential for the profitable fulfilment of these functions.” (See [5].) And so the application was rejected.

In the cases of Maria-Pia Geppert and Erna Weber, we must point out that they conformed politically and became members of the Nazi party [9]. All four of the women who had obtained a habilitation and who had stayed in Germany received professorships after 1945. The few examples of women in mathematics being able to enter the higher echelons seem to support the general conclusion that it was more likely to be possible in newly arising areas or scientific border areas. Emmy Noether created a new field of mathematics with abstract algebra. Hilda Geiringer was the first woman to obtain a lectureship (albeit unpaid) in applied mathematics. Geppert and Weber were able to make use of their interdisciplinary training and finally—after the World War II—received professorships in the respective fields of mathematical statistics and biostatistics, fields which were still little established and recognised.

Those female professors were exceptions, and the situation has changed little to this day. Women in mathematics deviate from the usual image of a woman. I would like to continue by illustrating how this image has developed, as this is the only way to gain a proper understanding of many problems existing today.

To practise mathematics was considered unfeminine, contrary to the nature of women. This idea has a long tradition going back to antique times—even Aristoteles concluded that physiological differences led to differing character traits (see [1]). There may have been changes in the traits attributed to the female sex in the course of history, and it has not been altogether denied that women possess intelligence. But women have been attributed *another type* of intelligence. In the 19th century discussions centred on the opposition of concrete, female, and abstract, male characteristics. Abstract mathematical and scientific thinking was therefore completely contrary to the image of a *normal* woman. Wilhelm von Humboldt (1767–1835) considered women had "... an admirable strength in that part of the investigation of truth which requires lively, flexible sensitiveness and quick, effortless comprehension and association, however a no less conspicuous weakness in and almost greater aversion towards that part which is founded on autonomy and discriminatory powers ...." (See [3].) And so it would be *contrary to nature* to train women for exact sciences—a consideration which resulted in mathematics and natural sciences not being taught at state-run girls' schools in Germany. The widely held opinion is expressed in the frequently quoted book "On aptitude for mathematics" (1900) by the Leipzig neurologist Paul Moebius (1853–1907):

> *"It may therefore be said that a mathematical woman is contrary to nature, a kind of hybrid. Scholarly and artistic women are the results of degeneration. It is only through deviation from the species, through pathological changes, that a woman can acquire talents other than those qualifying her to be a sweetheart and mother."*

Although—on the basis of tests—male and female intelligence have been declared equal, from a statistical point of view, by the American psychologist Lewis Terman (1877–1956), there are still educated people who adhere unflaggingly to the old theories. I would like to quote some comments made recently by a 46 year-old physics professor, a statement that is not an isolated example: "Nature made women less capable mentally than men. A woman with a particular talent for natural sciences is obviously possible, but it is exceptional .... It is a scientifically proven fact that men are better scientists. If women are to be given positions then at most less important ones such as that of a laboratory assistant .... A woman's personality is bound to be changed considerably by working in the sciences, since it is contrary to her nature. ... a woman's natural destiny is to serve and be passive."

I must vindicate the honour of mathematicians in comparison with physicists by saying that they were quicker to support women. The great mathematician Carl Friedrich Gauss (1777–1855) would very much have liked to see an honorary diploma awarded to the French woman mathematician Sophie Germain (1776–1831), with whom he maintained correspondence. The Berlin mathematician Karl Weierstrass (1815–1897) led the first woman in mathematics, the mentioned Sofja Kowalewskaja, to a doctorate. And in the 1890s mathematicians in Göttingen advocated women's being allowed to study even before official enrollment had been regulated. There is a book with collected contemporary statements. Among others it features the physicist Max Planck (1858–1947), quoting the "contrary to nature" argument and sending women back to the kitchen.

The mathematicians' statements, on the other hand, were, without exception, positive. The famous mathematician Felix Klein (1849–1925), who by that time had already led two women to doctorates, emphasized the following (see [7, page 135]).

> *"I am glad to answer the question, because the continuing opinion in Germany that female mathematical studies should be practically inaccessible is probably a major obstacle to all efforts aimed at developing higher female education. I am not referring to exceptional cases, which as such do not prove much, but to our average experiences here in Göttingen. Not wishing to go into great detail, I should merely like to mention that in this semester, for example, no less than six ladies have attended our higher mathematics lectures and tutorials and in the process have continuously proved themselves to be equal to their male colleagues in every respect. The fact of the matter at present is that these ladies are all from foreign countries: Two Americans, one Briton, three Russians—but no one will seriously want to claim that other nations could by nature have a specific ability which we lack, in other words, that our German ladies could not accomplish the same if they received the appropriate previous training."*

The mathematicians' efforts on behalf of women may not have been entirely altruistic, however, since the number of students had fallen substantially at the beginning of the 1890s and professors depended on greater numbers for pecuniary reasons. But it must be said for the mathematicians that they still continued to support women even after the curve had risen again.

When the mathematician David Hilbert (1862–1943) wanted to lead the first woman, an American, to a doctorate in 1899, he prepared himself for the decisive faculty meeting in writing. Ann Lucy Bosworth (born 1868), Hilbert's first female doctoral student, had started with Hilbert's "Grundlagen der Geometrie" (Foundations of geometry) 1899, but she had chosen her field of research independently, this was certainly exceptional. Hilbert wrote a detailed text with the title "Über Frauenstudium" (On women's studies), this handwritten paper is kept in Hilbert's "Nachlass" (left papers) in the Göttingen library. Here are the notes he made on the subject of "Women's studies."

"… At the same time I was examining the dissertations of two men. I believe, no examiner would find that these men's papers were of a higher level. In one point, the woman's treatise is even a little better. It is a point, which everybody, including me, usually considers to be a weakness of female candidates, namely, in this case, she was more independent."

Hilbert already announced that his second female doctoral student, the Russian Ljubowa Sapolskaja (born 1871) would submit her thesis (Über die Theorie der relativ-cubischen Abelschen Zahlkörper, 1900/1902.). He wrote about her lecture in his seminar: "If you were present in our seminar yesterday, you would have been surprised to see with what fervour and enthusiasm a woman could speak about mathematics. It was a Russian, who lectured."

In another document, Hilbert explained the level of difficulty of Sapolskaja's field of research: "The subject belongs to the field of pure number theory. I had repeatedly offered this subject to other students; but they rejected it on the grounds that it was too abstract. The problem required a special measure of energy, diligence and time, and the intellectual ability not to give up, even when one is forced to think in purely abstract terms. She had this ability in plenty. Nothing was too abstract for her."

Although the dissertations of these women were of a high level, it was not very easy to convince the other members of the philosophical faculty to permit these women to submit their theses. We should keep in mind that the philosophical faculty at the Göttingen university consisted of professors of philology, philosophy, and history as well as mathematicians and scientists. Although some women had got their doctorates at the Göttingen university in the 1890s, there were still some opponents around 1900, who were against women's studies. Hilbert put this opposition into words, when he wrote his above-mentioned paper about women's studies in 1899. While preparing for the admission of his first female doctoral candidate, Hilbert wrote: "A number of you, gentlemen, are not well-disposed towards women's studies. But, I ask you for the sake of mathematics to refrain from the urge to do something averse." Current investigations also show that the supervisor's influence is critical for the candidate's future career. Up until 1933, fifty-seven mathematicians, one of whom was female, supervised 98 doctoral theses (of which 9 had been written by foreign women), also the women taking doctorates in mathematics came from 23 German universities. In the second decade of this century there were 53 chairs in mathematics, to which six more were added in 1919 (when the universities of Hamburg and Cologne were founded). The most significant universities for doctoral students were Göttingen and Bonn. The majority of foreign women took their doctorates in Göttingen, as this was considered the international centre of mathematics. Bonn, on the other hand, was favoured by German women, as described by Gerhard Kowalewski who I quoted at the beginning of this talk. Up until 1933, fourteen women were conferred doctorates in Bonn. A more recent bibliography documents the doctoral theses in mathematics from 1961 to 1970. During this period, Bonn ranked first among all the universities in the Federal Republic of Germany, with 92 doctoral theses, although only one of these was written by a woman.

If we look back to these earlier times, we see that only seldom did the supervisors continue to support their female doctoral students in a scientific career after they had received their doctorates. Emmy Noether—the only female supervisor—inspired 14 doctoral theses, of which one was written by a woman in Göttingen and another by a woman in the USA. Noether's position as an extraordinary professor without the status of a civil servant did not allow her to take on assistants herself. Her scientifically talented student Margarete Hermann (1901–1984) [2] ended up taking up a post as assistant to Leonard Nelson (1882–1927), a philosopher at the Göttingen university. Even when women did receive a post as scientific assistant to a professor of mathematics—to Gerhard Kowalewski in Dresden, August Gutzmer (1860–1924) in Halle, Lothar Heffter (1862–1962) in Freiburg, for example, they were usually restricted to support work. They seldom managed to produce scientific work in their own right once they had received their doctorates.

The analysis of the contents of the doctoral theses written by women before 1933 shows a dominance of geometrical work. Now we know this was also the case with doctoral theses written by men in Germany. It is one of the aspects we are currently analysing within the framework of our Volkswagen Foundation project. It must be pointed out that a gender analysis has already been produced in the USA. Judy Green and Jeanne LaDuke discovered that there, too, the first doctoral theses were mainly in the field of geometry, regardless of sex. But there was a change of direction at the beginning of the 1930s. Men dedicated themselves more strongly to the field of analysis, whereas women stuck to geometry as their primary field of research. Our investigation shows no change in this respect, although we have yet to give detailed reasons for this. Maybe we have to pay attention to the emigration of many famous mathematicians in the 1930s (see [6]).

Whether a person is capable of making scientific achievements in the field of geometry probably has less to do with the frequently discussed problem of spatial imagination. International tests prove over and again that women produce worse results. A test we carried out among first-year mathematics students at the beginning of the winter semester 1998/99 brought us to the same conclusion. The number of subjects was too small for significant conclusions to be made, but a trend was confirmed. However, it is still completely unclear theoretically whether this has any kind of relation to mathematical abilities.

I would like to close with some comments on support for women in Kaiserslautern.

The university was only founded in 1970 and is oriented towards mathematics and natural sciences. In the winter semester of 1997/98, there were 25.2% women among the students. Various ideas were developed in order to increase this proportion. One of them is the Ada Lovelace Programme, named after the first female programmer, who had contacts with Charles Babbage. This programme sends female students into schools as mentors, to arouse schoolgirls' interest in studying these subjects and to include them in project work. Engineering and technology days and mathematics days for local schoolgirls are held at the university regularly. The schoolgirls are introduced to scientific and technical experiments, participate in mathematics competitions, can take part in projects in various ways and generally gather information. In some departments professors supervise special groups of female students. The field of experimental physics has been quite successful. Seminars and lectures on the analysis of gender research results for the teaching of mathematics and for the field of architecture and urban and environmental planning were very popular with the students but were not held regularly. This is mainly due to the inadequate formulation of the examination regulations, which only make such courses optional or, as an exception, include them in the Didactics of Mathematics syllabus. At the entire university there are only two ordinary professors who are women—in architecture and in biology. Besides this there is the Sofja Kowalewskaja visiting professorship for women mathematicians, named after the first woman to gain a doctorate in mathematics. This visiting professorship is awarded to one woman per semester, the visitors come from various countries. This professorship was set up in 1991, for the most part due to the initiative of the mathematics professor Helmut Neunzert, who thus followed in the footsteps of the mathematician Felix Klein with his aim of promoting not only the applications of mathematics but also women in mathematics. The Institute of Techno- and Economic Mathematics, set up by Neunzert in 1995 and run under Fraunhofer management, also promotes women scientists. I would particularly like to emphasise that previous interdisciplinary training is especially useful for projects to be worked on at the Institute. It was also Neunzert who initiated the Volkswagen Foundation project I am currently working on, together with women social psychologists in Erlangen. The opening ceremony took place in July 1998, on the invitation was a picture of a woman holding the Boltzmann equation in her hand. In 1991, Neunzert published a popular science book in cooperation with Bernd Rosenberger, its title was "Schlüssel zur Mathematik" or "The Key to Mathematics." Among other things, it says [4]:

> *"… people must be made much more aware of the fact that mathematics is not an exclusively male discipline. In Germany, too, we can only bring about a change if we change the dry, unworldly, unfeminine image of mathematics. It is worth emancipating oneself from one's own and from others' prejudices about the relation of women to mathematics."*

It is good when men stand up for women to be allowed to take a foothold in areas which even today are still considered unfeminine by some professors. Let us therefore end with the quotation from the title of my talk: We work in mathematics "in spite of male culture" and in doing so try to make the squaring of the circle, the impossible, as near possible as we can.

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CHALLENGES FACING AN IDEAL UNIVERSITY

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1. Introduction. What do we need at universities both in the near future and in the long run? The challenges of the near future might, for example, be reacting to the new needs of the job market, to the financial situation of universities, or to the new possibilities supplied by the new digital media. But the university must also be prepared to confront the long-term future needs of preserving our world, its nature and environment as well as mankind. The cooperation of all disciplines is vital to carry out necessary change. Not only scientific and technical cooperation but also the cooperation needed to provide the political, social, and cultural environment for change to take place. The challenge is to set up new measures and principles and to develop theories, strategies, knowledge, and aids for a new civilization capable of dealing with all these problems. There have been considerable changes in all societies of the world, in particular the effects of globalization, national governance loses its power in favour of European or global regulations. As is well known, the specialization and particularization of the sciences and even within the sciences is such that understanding between the disciplines is hardly possible, hence less scientific cooperation. Therefore, the challenges of a new university should not only lie in the further development of sciences for their own sake, but also aim to shift the objectives of sciences away from the partition of life and towards a dedication to developments which lead to the reduction of civilization costs. Undoubtedly, every civilization is bound to change, both scientifically and technologically. But differently from before, the finiteness of the world, the circulation of materials, pollution, ozone-holes, the global effects of local actions come into our view. All efforts, both realised and potential, are necessary to care for future social developments without damage to the environment. Women and minorities should be integrated more than before into the design of such changes in order to use all possible potential of qualifications, ideas, and knowledge.

To meet all these challenges, one has to reflect on the role of the university:

- Is the integration of science and teaching still a suitable model?
- Should a university education also qualify people for the job market, or only for the reproduction of science?

- Should the university be open to other courses of education, such as adult education or continuation of education?
- Is the Humboldt model of a genius scientist working in solitude and freedom on self-defined problems, gaining objective results, still a realistic one?
- What is “objective” science? Does it exist and can it exist, especially for interdisciplinary research? And if not, by which epistemological claims can or should it be replaced? For example, reproducibility of results, post-modern coexistence of everything, discourse?
- Is it still adequate to leave sciences to their isolated developments, to follow intrinsic self-defined goals and subjects, or should they be (or are they already) open to influences of whatever type? For example, by interchange with other sciences, by a kind of educational market, by political goals and there again, by which influences, for example, by directing state money flows?
- Is the commercialization of education a necessary answer to the change in social perception of intellectual activity into intellectual capital and hence intellectual property?
- To what extent can technology help adaptation to the modern professional needs, such as flexibility and orientation?

In the following, we will discuss some very different answers. These answers will be concerned with science, educational and university structures, technological means, and an experiment enabling the integration of women and minorities.

2. Historical remarks. In Europe, science of the Middle Ages was above all theology, with all other sciences contributing to it. Arts and crafts, on the other hand, were the profane skills. Mathematics contributed to both science and profane skills. Science was universalistic in the sense that all subjects could be combined into a consistent architecture and all contradictions were ruled out by theological governing and dogma. It was also universalistic in the other sense that it was possible for skilled scientists to have an overview of the whole structure of science. The monasteries held large libraries where the current scientific knowledge was kept and, to some extent, was also distributed. Also, the role of encyclopedias, such as the “speculum maius” by Vinzenz of Beauvais in the 13th century, or the one by Diderot and d’Alembert established in the mid 18th century, for allowing universal access to knowledge and universalising this knowledge cannot be underrated.

The term and the institution “university” was first created in Italy during the 11th century. The first university of the world was the university of Salerno (1050) followed by a series of nearly 30 other universities throughout Italy: Bologna (1119), Ferrara, Siena, and others. Teaching existed at Oxford in 1096 (there is no clear date of foundation of the university), Paris was founded in 1150, and later Prague in 1348. The first German-speaking universities were Vienna (1365), Heidelberg (1386), and Cologne (1388). The (male) students were the creators of these democratic universities: They paid their teaching personnel for teaching them law, medicine, and so forth and could also dismiss them. Further, the students could determine the subjects taught.

Renaissance brought a new definition of science: The rationalistic science, which was coupled with technological usage from the beginning. The Royal Society in London with Francis Bacon as an outstanding contributor and the Académie Francaise in Paris with René Descartes on the French side implemented this new science, now explicitly defined as "male" in opposition to a new notion of "femaleness," which was identified with nature, feelings and a sexuality worth being feared. The consequences of this orientation have been widely discussed in women's and gender studies (cf. Fox Keller [6]). In Germany, the foundings of the university of Halle (1694) and of Göttingen (1734) first represented the inclusion of sciences and the new rationalistic methods.

The 19th century again introduced new cultures of science and humanities, to be connoted with the foundations of the universities of Berlin, Breslau, and Bonn and with the names of, for example, Virchow, Helmholtz, or Linné on the science side, and in contrast on the side of the humanities with, for example, Mommsen and Weber, showing a polarization between disciplinary orientation. This divergence is exemplary with the scientific directions of the brothers Alexander and Wilhelm von Humboldt. Wilhelm established the leading ideals for the culture of universities in Germany. The so-called "Humboldt Universität" still forms the current university: The duty to search for "the truth" for both science and humanities, the union of research and teaching, the serving of research for its own sake only, for nothing and nobody else, emancipated from feudal rules: Research "in solitude and freedom." Of course, life, body, and subjectivity were eliminated when the scientist followed his objective epistemology [3]. Alexander, on the other hand, favoured a scientific restoration for the sake of a progressive civilization, and he was also successful, especially in an economic respect. He wanted to implement the école polytechnique—founded by Napoleon in Paris—in Germany. In the late 19th century, this project was finally successful. The new foundations of technical universities were "cultureless" and had to fight for their reputation for a long time. But the engineers gained resources and also stood for the industrial civilization, not military engineering. As for the Humboldt university, the engineering culture was separated from body and life, but it took a turn to reality with the orientation to technical processes. Engineering, moreover, became a habitual model for male identity.

Today, all these divisions seem to be no longer adequate, nor do solitude and absolute freedom of research. Technical education is more and more integrated with science and humanities and technical research has won the same validity as the one of the classical sciences. This integration is of urgent need, because social processes should no longer follow the rapid changes of technological ones, but conversely. However, the problems to integrate communicative, social, and language qualifications into engineering education still show the strong cultural characteristic features of technology.

3. Long-term challenges. The major challenge of society and therefore also of sciences is to cope with the problems of the future: Ecology, migration, globalization, balance between rich and poor, women and men, minorities and majorities, etc. For these aims, all resources, powers, and technological possibilities have to be used, since the challenge is nothing less than building a new civilization capable of accepting and solving these problems. Special emphasis lies on research to find reliable solutions in every respect. To cope with the meshed economic, social, juridical, and cultural aspects of the problems, any solution will not only include parts from natural science and technology, but also all the other sciences. This requires all disciplines to cooperate on scientific themes stemming from the demands of building such a new civilization. Today, it is imperative that the different disciplines not only follow their disciplinary goals and questions—this, of course, as well—but they also have to pursue research themes with external goals, goals which subordinate disciplinary development under research questions contributing to solutions for the urgent needs of the present society and ecology. Of course, this does not mean that basic research within disciplines should be abandoned, nor does it mean that a basic training in the different disciplines can be abandoned. On the contrary, disciplinary training must be very good to be able to work in interdisciplinary contexts. But it means that also the basic researchers should cooperate with those working on problems of the future, in order to know what the problems are and what those might be able to contribute. It also means that with the new goals, Humboldt's ideals of freedom and solitude must fall. The freedom to choose one's subject of interest should obey an ethical self-restriction to the urgent goals mentioned above. That is, not everything that can potentially be done, needs to be done, nor should it be. Intrinsic goals of single sciences should be weighed against social and ecological goals. Problems of this sort can only be solved with the efforts of all sciences.

With the current faculty structures of today's university, such a cooperation is hardly possible. The disciplinary ideals, goals, ethics, languages, and cultures heavily withstand the necessary integrating features. But there is an urgent need to lead the university away from the strong partition into subjects and sub-subjects, where crossing the border is hardly possible, to a new university, where an understanding and cooperation is possible, without leaving one's disciplinary grounds and competence. Universality today cannot consist in integrating all scientific knowledge of the world in single minds, it can only be brought into life by cooperation between researchers from different subjects. This implies the willingness and the capability to follow common scientific aims and efforts. This kind of new universality consists in the possibility of gaining mental access to all scientific results of all subjects relevant to the problems chosen. Gaining mental access that is as far as necessary for defining the interfaces and working together. Moreover, it requires to renounce quite a lot of one's own professional values, and also of hermetic scientific languages. One is required to make oneself understood, to communicate and to cooperate and to follow the common goals.

New competence to work in trans-, multi-, and inter-disciplinary ways on the subjects mentioned must be developed. This requires one to be able to speak a common language between scientists or at least to understand other scientific languages and to moderate between them.

Note. The universal language of mathematics is not capable of serving for this cooperation. Of course it is good for those parts which are able to be formalised and quantifiable, but this is the minor amount. For the whole range of the meaning of life, the language of mathematics is not apt to serve for such a cooperation. There are many reasons: The necessity of prior rational reconstruction of reality, which would imply that all secrets of the world would have been recognised before formalization; the integration of the different mathematical models for single parts of reality is mostly impossible, etc. Gödel's results hint on such problems as well. With formalization, uniqueness and stratifying concepts are introduced, meanings by definition, not open to interpretation and discourse or moving targets.

More qualifications are required for this cooperation than before. The lonely genius working alone is not the type required anymore, but people with wide interests, competent in their science, being able to explain their own and their own sciences' contributions, capable of discourse, leaving pluralism alive, and still finding ways to solutions.

4. Short-term challenges. There are a lot of necessities for change at universities in the near future. The majority of universities are confronted with reduced resources, and the students are faced with new qualificational demands, with lifelong learning, flexibility, and the adaptation to a global work market. The role or the help of the new technologies in coping with the challenges mentioned is yet unclear. Is it necessary for every course of study to have ubiquitous MBONE-access, to use authoring systems, to have network and multimedia I/O, hardware and support for every student?

Globalization of work gives education and learning a new role. The most important goals of education are no longer memorising subject contents, but being capable of learning on demand, of finding information where it is available, and of remaining flexible. On the teaching side, it requires a shift from support of the acquisition of knowledge to strategies for the acquisition of knowledge. The fixed knowledge bases and curricula have to be made flexible. The capability to acquire short-time external knowledge effectively becomes more important than a huge amount of internalised (learnt) knowledge. The creation of a new knowledge is becoming a globally coordinated activity of many people. Presentation and design becomes more important and even an educational goal, both for teaching and research. The necessity to communicate and to cooperate raises the importance of social, language, and team competence.

This gives universities a new role and forces them to find a new standpoint. First of all, new forms of study have to be found which can cope with the necessities of a new working society. A fixed division between learning phase and working phase does not exist anymore. A continuous phase of working in a fixed career track will no longer be a general model. Lifelong learning also requires new courses and forms of study from the universities. They have to offer part-time, short-time, and remote courses. Moreover, they have to enable movement between different kinds of educational institutions, between practice and research. Using webbed digital media can help to design the necessary redefinition of universities.

By loosening and marginalizing the traditional models of courses of study, room is opened for new constellations, such as overlapping and interdisciplinary courses, taking elements from different subjects and courses and combining them unconventionally. Gender Studies can serve as models—even if it is true that these prototypes arose from the lack of resources—or courses in media theory. Other courses directly oriented towards certain professions are possible models as well.

All methods and new technologies should be used to meet the needs of tomorrow's university, nothing should be excluded. Multimedia is setting the pace for innovations in education, university and school. Virtual universities, virtual courses of study, schools, self-organised learning, tele-learning, education on demand, and the new knowledge society are the buzzwords of exposition. There is a wide range between euphoria and scepticism, and estimations of costs, such as those of potential savings, differ very much. On one hand, the technical possibilities are by far overestimated—they will solve all our problems of future education, on the other hand, their fundamental importance is also mostly undervalued. A third position is that there is no way to avoid the use of the new digital media for education, for good reasons of increasing means and possibilities of learning, but to use them critically, that is, to select and to integrate them into classical teaching. It should be noted that technology only helps to solve technological problems, that didactic and pedagogical questions have to be solved with didactic and pedagogical means. But by technological means, medial access is channelled and all thinking and learning relies heavily on media.

Dennis Tsichritzis, the director of the German GMD and professor at the university of Geneva writes in his article "Reengineering the University" [12]: "Today's university is at a turning point, and turn it must. The time has come to recognise that education is a business and students are the customers.

Pressure for such a change comes from the public, the media, and the political groups, which become aware of the new technological means and therefore demand new learning environments." He claims a radical restructuring of university and research, which abandons Humboldt's ideals of a university integrating research and education. He claims that the new teaching environments are virtual classrooms, with the possibility of visiting digital libraries. Students have a wide selection from the world's best offerings, they can specialise in arbitrary directions, because every existing course is also available to them. Teaching personnel saves teaching time and wins it for research by using authoring on the fly-technology, and institutions have a valuable instrument of validation by just evaluating the market of course choices.

A similar vision is given by the "Expertenkreis Hochschulentwicklung durch neue Medien" by Encarnacao et al. [2]. The scenario outlines a global market of education, corporate universities, networks and international consortia, virtual universities, educational brokers, etc., all mostly now centred in or around firms. Only a very small classical university is left with the task of carrying out fundamental research and educating its own personnel. According to the authors, the very use of the new media creates transparency and, in consequence, quality of education, and the evolution of education according to the needs of the market, etc. Similar visions, if not developments, exist in most countries of the world.

It is necessary to confront these visions with reality. David Noble from the York University of Toronto summarizes experiences from the UCLA and from the York University of Toronto on the automation of higher education [9]. The development of courseware had been put into commercial hands. A two month strike of the students of York with slogans like "the classroom versus the boardroom" stopped the unlimited exploitation of online education by private firms. Beneath the technological change and camouflaged by it, as he realises, lies the commercialization of education. A change in social perception has resulted in the systematic conversion of intellectual activity into intellectual capital and hence intellectual property. After a phase of commercialization of scientific and engineering knowledge via patents and exclusive licenses, where industrialists invented ways to socialise the risks and the costs of creating this knowledge while privatising their benefits, the change we are now facing is the turn of the activity of instruction itself into commercially viable proprietary products that can be owned and sold on the market: Copyrighted videos, courseware, CD-ROMs and web sites. Experience there demonstrates that computer-based teaching, with its unlimited demands upon instructor time and vastly expanded overhead requirements—equipment, upgrades, maintenance, technical and administrative support staff—costs much more than traditional education. The use of technology entails an inevitable extension of working time by the university staff, to stay on top of the technology, to respond, via chat rooms, virtual office and e-mail to both students and administrators to whom they have now become instantly and continuously accessible. Once faculty and courses go online, administration gains much greater direct control over faculty performance and course content. The technology allows for more careful administrative monitoring of faculty availability, activities, and responsiveness. Once faculties put their course material online, the knowledge and course design skill embodied in the material is taken out of their possession. The administration is now in a position to hire less skilled and cheaper and short-time workers to deliver the course material. It also can peddle the course elsewhere without the original designers' involvement or even knowledge. The buyers of the packaged courses are able to outsource the work of their own employees, etc. The students, if ever asked, have rejected the initiatives, for example, at UCLA and York. They fear the costs and plea for face-to-face education. But they also realise that the course material is only thinly veiled trials for product and market development, that while they are studying their courses, the courses are studying them. All online courses are monitored, automatically locked and archived in the system for use by the vendor. David Noble fears that higher quality education as a whole will become the exclusive preserve of the privileged, thus swinging the democratic achievements in equality of access to education to the other extreme.

I will not further refer to the economic aspects and assume that education will not be privatised. Then the question arises: Which specific effects of support can be obtained by interactive computer systems (including achievements such as thinking, learning, or processing of information)? The possibilities of the new computer networks with multimedia often mentioned are rationalization (by money-giving institutions such as the ministry of education) and quality improvements, especially in the process of appropriation (by the money-taking institutions, such as universities). The question cannot be decided because the technology is not yet developed enough, reliable TA studies do not exist, and also because the technical potentials cannot be separated from other factors, such as better didactical concepts or better preparation of the teaching materials. Moreover, the implementation of the new technologies is usually supported by additional resources from projects, which are finally making comparisons impossible.

Reinhard Keil-Slavik [5] puts the questions into a more theoretical frame by differentiating between primary, secondary, and tertiary functions of media. The primary functions of media are to: Make phenomena realisable, arrange artefacts so that they can simultaneously be observed and their content relations are mapped by the layout relations, and combine connected artefacts. The potential of rationalization by multimedia, in his opinion, lies in the possibilities to process these primary functions. The new media enable a pictorial turn, which reaches our mind more directly than the written and even the spoken word. By this, the above functions gain even more relevance: Arrangement, layout, and room visions. The secondary functions, such as selection of contents and instructional processes, can be processed by learning software. The function of multimedia here is to improve the quality of learning materials. This cannot be separated from competence in the subject's contents and from didactic competence. The tertiary media functions lie in the implementation of systems that learn, used in teaching, for example, for filtering or in the form of know-bots for searching, selecting, combining, and processing knowledge according to a person's profile of interest. The higher the layer, the more complex not only the media functions, but also the intertwining with other competence and efforts.

A lot of questions and requirements are still left open with these scenarios.

(1) To begin with, the very use of global archives and global teaching offers requires media competence in teaching and learning in all subjects. It also requires the shift already mentioned from the acquisition of knowledge to *multimedia strategies* for the acquisition of knowledge. Navigation, filtering, and effective search within the huge amount of mostly nonvalidated knowledge offered requires technological and strategic skill and ability to validate. That is, competent use of the mediated knowledge requires new *"meta"-skills and -knowledge.*

(2) It requires *technical equipment*, not only for scientists and teaching personnel, but also for every student—a very expensive task.

(3) Along with the decentralised furtherance, a *future university ordering of knowledge* also requires a new ordering of the traditional centralised carriers of technical media: The integration of library, computing centre and media centre in a university support unity: *"Digital media" centre.* And, with the new institutions, also modified and new professions are required: Cybrarians who are able to integrate the skills of librarians, information brokering, and technical capabilities for running the computer centre and the multimedia equipment.

(4) It will require a lot of installation and running costs. For students, this also means running telephone costs. The model of locally free telephone calls would also help to get more people on the net in Germany.

(5) Also the readiness of students in self-engagement in using the digital media has to be supported by removing hindrances, such as costs, access, teaching media competence, and also by the *quality of the offer.*

(6) The preparation costs are extremely high. For example, the Freiburg high-tech project *"authoring on the fly"* computes 100 to 500 hours of preparation of one hour's teaching unit on the computer. The preparation of synchronous on-line teaching in several places using MBONE technology and shared white-board has similar requirements.

(7) The teaching methods change radically by the use of digital aids for virtualised teaching. A traditional lecture with chalk and blackboard may be prepared in the evening before the lesson. A technically mediated demo, the use of a software package for simulation or animation during the lecture must be carefully prepared and well in advance.

(8) One important aspect of technology for the *usability* and with it for the rationalization of teaching and learning is that systems must be suitable for everyday use. With the current state of the high-tech multimedia possibilities, much more effort has to be put into making the offered learning units run, rather than into the learning itself.

(9) Another important aspect is that isolated learning alone does not suffice. Learning is also a social process, where it is necessary to discuss and compare one's performance with others.

(10) High-tech learning software has to be actualised, that is, it does not suffice to build authoring on the fly-units once and for all, they have to be supplied with possibilities for change, improvement, and addition.

(11) Usability and the possibility to combine different technical means is important. Reusability and the availability of materials adapted to different learning types and qualities, that is, everywhere (e.g., in the home as well as in the class room) and anytime, are necessary prerequisites to make the multimedial program work.

If the new technologies are used in a way that they contribute to all these goals, and not with a technology centred habitat, but with a critical one which only seeks to use it for efficiency to serve for more effective learning, this will be helpful to serve the new requirements of the job market: Flexibility for lifelong learning. The following list gives some claims or hints on using them successfully.

(1) Digital media have to take second place: The use of digital media is an aid to establish the new overlapping courses of study, but not a goal itself. Virtual teaching and learning has to follow nontechnological needs, not conversely. But it is helpful, both in the form of stand-alone remote universities, and as one means to an end among others.

Still, also within classical courses of study, a heavier use of webbed digital media is highly recommendable.

(2) Teaching is a decentralised task and should remain so in general. But it can be extended by central provisions within the single courses of study.

Universities should support such activities by giving resources and taking such efforts adequately into account. All subjects should be stimulated to participate in this reform and to use the existing digital networks as the university intranet.

(3) A side effect of such a networking will be the intended bridging within the university, from strengthening of the cooperation between faculties, better knowledge of one's colleagues' activities by consulting the WWW, and even interdisciplinary work.

(4) Another side-effect might be the quality of the teaching offered: Multimedia teaching is a lot of work, which usually also flows into the preparation of the contents and didactics. The competition between different offers might lead to better quality as well.

(5) High-tech variants should be handled with care. Professional TV-shows cannot become a model for university teaching. But low-tech use of the new digital media, such as e-mail, newsgroups, mailing lists, web-archives, MUDs, MOOs, and IRCs require considerably less time and effort and they can be integrated into classical teaching much easier as well.

(6) The potential of communication possibilities by using the new digital media has not yet been investigated thoroughly—but this must be done.

(7) Technical visions of teaching and learning have to be replaced by *social visions of teaching and learning. To learn how to learn* will be the most important challenge for our students, and with it *to teach how to learn* for ourselves.

The new order of knowledge. Another challenge comes from the *globalization of knowledge* and *globalization of the production of knowledge* which creates new knowledge orders (e.g., an internet governance of knowledge), again implying the use of the new digital media. Open global computer networks, by allowing new forms of communication, information, saving, and archiving, also heavily participate in building a new mediated order of knowledge [10]. This, in the sense of Michel Foucault, that is, not only with epistemological goals, but also goals of ruling in technology and society. The new instrumental media will be used to search, select, process information and to mediate it to users in adequate form. Users, in turn, have to be involved with the media to be in a position to reconstruct relevant knowledge for themselves. If universities do not want to leave the evolving of a new knowledge order to chance and other powers, they must participate in the structuring and design process of the order of knowledge. If, according to Spinner, knowledge can be structured according to zones of quality, shelter, and distribution, science also has to take responsibility for them.

Michael Nentwich describes in his working paper “the future of science” [8] not so much the future of science, but the actual use of digital media within the scientific communities. The changes are manifested in new forms of publishing and in the ways scientists work and communicate. Communication via e-mail, but also online in video- and online-conferences, makes cooperation easier and allows the scientists to gain knowledge of greater actuality. Texts are receiving a more dynamic component with publication they need not remain fixed. The so-called open peer commentary, as well as the online-referencing, can lead to ongoing modification of publications. The role of authorship is thereby changed. Articles are becoming the products of groups of scientists with different roles in the production of texts. Rating of these articles can be done by automatic follow up of reading.

Of course, the changes also depend on the scientific subject. In areas where the acquisition of data is necessary, this can be done from remote places, exchange of data is eased, the access to large data bases gives new empirical quality to research. Also, extended research groups can be established working in virtual laboratories, so-called collaboratories. They cooperate and coproduce in a modularised way, more independently of place. Of course, there remains research which is bound to place, such as field research, interviewing, etc.

This kind of remote cooperation also has its deficits, well known as decontextualization of information, as the lacking of bodily information channels such as mimics and gesture. On the other hand, the forms of discourse enabled by electronic communication are capable of infinity [11]. And it also needs new forms of written moderation, which do not yet exist. Generally, there is the possibility to move from local and national discourses (which are more typical for social sciences, arts, and humanities, less for science, mathematics, and technology) to global communities and a democratization of science, a break-up of hierarchies, and on the other hand, the well-known specialization and particularization of knowledge and expertise.

5. The technical university of women in Europe—a necessary experiment. There have been a lot of names to try to characterise new types of society: Post-industrial society, services society, consumer's society, communication society, information society, knowledge society, risk society, virtual society. The huge variety of classifications shows that old structures are dissolving, but a (one) new formation has not arisen. Yet the visible changes indicate the break-up into a new civilization. Following Janshen [3], a new civilization for the new millennium cannot be defined, even less implemented from scratch. It must and will use the structures defined within the last centuries. Therefore, it also bears the burden of a male culture of progress: Engineers, enterprise holders, and colonizers. James Watt's steam engine marks the beginning of a new epoch. The industrial use of fossil energies to run the machines led to the development of big industries in steel, machine construction, electric industry, mining, and chemical industry. Productivity rose and with it the production and use of goods. Ships, railways, and telegraphic aids had the effect of shrinking time and space for human activity. Rationalization, not only by machines, but also by bureaucracy and the formalization of social processes, made human activity more effective, it also enabled the potential of power by science and technology: Taylorism, Fordism, military weapons, and mass-destructive technology, but also precipitated the decline of agriculture. Today, rationalization is not only restricted to public and institutional sectors, but increasingly also to the private one.

As is well known, technology also structured society in the times of the first industrialization, the polarization between the possessors and the workers, and also the emergence of the town (where men worked) as a place for public action as opposed to the private home (where women worked) without possibilities for empowerment. Education and professionalization are gaining impetus compared with family and inherited position. The standard of public health is rising as is women's power. On the other hand, there is also an awareness of the threats of global damage. Such damage affects everyone, not just flora and fauna, and thus, the responsibility for ecological balance, pollution of air, earth, water, and nutrition cannot be attributed to single persons. It is imperative that women also gain control over technological processes. The tendency for key social qualifications only to be available from technical universities is still effective. And it relies on structures and norms stemming from military rationality and order. Of course, it is speculation to claim that our civilization would have very different characteristics with a more active participation of women. But today, the breaking up of old structures and the yet open future is a challenge, especially for women. New basic technologies are opening new tracks. A new scientific practice should combine technical innovation with the social one. Signs of a new civilization are visible with the changing of norms, institutions, and jobs. With post-modern plurality, a greater tolerance towards all kinds of social, cultural, and epistemic differences and values has arisen. The postmodern discourse about construction of social attributes and values lets bindings seem deconstructable, giving dimensions of new freedom. The fact that there is a lot of openness for structures and values, calls for design, especially designs by women. Universities can no longer be the shelter for socially isolated individuals, they must be embedded into variable social processes. Intrinsic motivation of subjects should move in favour of taking responsibility for scientific actions, seen as part of a duty for building-up a new civilization and taking responsibility for the natural environment, more or less constructed or left to itself. Instrumental technology, constructed for different aims, is moving in favour of integrated technologies, processing relations and communication, but without explicit goals. The development and implementation of technologies for intrinsic reasons should be abandoned in favour of development for sustainable and social goals. This requires the cooperation of all sciences.

I would like to present concepts for a type of university that meets both the imaginations of a group of women and the claims for education and research for the needs of "tomorrow." Most of them have been discussed in the context of the development of the "Technical University of Women in Europe" (TUFE) led by Doris Janshen.

One starting point is the observation that the male power of definition in the process of civilization rules out or marginalizes female participation in this process, and this especially through the development of technology and technological skills. It is therefore necessary for women to define their own "university." A second point considers the social and economic costs of today's richness through industrialization, which is based on scientific and technological knowledge. The question arises of how future civilization can be developed, also implying social and ecological responsibility, and how science and technology can integrate this in the very processes of teaching and research.

Feminist research is favouring specific epistemological interests. That is, approaches which use the power of knowledge to support life, peace, and the development of civil cultures are favoured against such which are used to gain power over nature, people, and nations, such as destructive and military technology. Consequently, the focus lies on long-lasting concepts and social and ecological aims instead of research which is readily usable for economic and political aims. A further feminist desideratum is to reintegrate epistemological questions and foundations into science in order to support self-reflection into scientific discourses. Critical questioning of the sciences about their aims and their knowledge produced is a necessary condition for regaining an orientation of sciences towards social aims, away from Humboldt's ideals of objectivity, (absolute) freedom (of research aims) and solitude. Moreover, feminist scientists fight for a culture of communication and discourse. They do not take the post-modern trend of "anything goes." Instead of seeking affirmation, more wisdom is won by working on differences and exploring their reasons.

Feminist critique of science and technology is well known, it attacks the "rational method" of recursive partition of problems into smaller parts up to atomic level, solving the parts and composing them to a general solution. Further, the objectivist distance and view of scientific work and many other partitioning features, like the partition of science and life, of theory and practice, of subject and object, etc. A constructive turn is seldom performed and it always bears the danger of essentialist positions. Here, they are avoided by just listing aims and desires for change and by trying to set up conditions for their fulfilment. One basic claim is to use the potential of women, and to use it not only as isolated women in science, but to bring women together in a technical university and to integrate different scientific views of women in one place, where teaching and research, practice and science is performed.

As is already mentioned, there is an urgent need for interdisciplinary, multidisciplinary, and transdisciplinary work and research. For this, it is necessary to dissolve the classical faculty structures for the sake of setting up new structures which help to investigate in cross-section problems both in teaching and in research. To know the "real problems," it is necessary to integrate science and practice, that is, to also involve people from business and firms and from different countries. Also, there should be a commitment to taking responsibility for ecological problem areas and problems of the third world as well as socially and economically usable research results. Women should have the power of definition in this university, but also cooperative male teachers and researchers may be integrated. This implies that possibilities to integrate work and even career and family life must be guaranteed, also for single mothers. Also the integration of students and practitioners in the research processes has to be enabled, among other ways, by deconstruction of hierarchies, also between scientific cultures and disciplines. This also implies a social integration for the well-being of students and teaching and research personnel. All this involves experiments with new structures for science and teaching, focussing on the integration of women and the orientation towards problems of the present. This has consequences, not only on scientific subjects and forms of study, but also on the whole culture of science and its mediation. Room for a different knowledge must be opened, for research that transgresses the borders of the disciplines and which takes its motivation from social practice of today and tomorrow. Of course, this also implies competence in, and development of, basic research within disciplines.

The target groups of the Technical University of Europe are researchers, graduating students, fellow students, professionals, the areas of working, the whole society. Fellowships should enable women and men to study abroad, to make possible further development for undergraduates, as for graduates with continued education or postdoctoral studies. Also, employed people should be able to study at the technical university of women, especially women after the family phase should have the possibility of reintegrating into the job market by receiving educational offers.

People of the new civilization not only have to be more flexible, they also have to keep stability and strength within the flow of meanings, as well as to preserve social competences. Barbara Mettler Meibom, founder of the institute for communicational ecology, speaks of the necessity to develop a new ecology of communication [7]. Only those who treat others and oneself plus the new technologies ecologically with respect to communication will be capable of communication. The TUFE therefore develops learning modules for "social training in technology." The goals of technological development by the TUFE will not be innovation for the sake of being the first, but to find meaningful applications for ecologically neutral or positive processes and to develop technology for them afterwards. At the TUFE, innovative processes will be embedded into networks of applications from the outset.

The TUFE wishes to develop courses of study integrating different disciplinary methods and areas of knowledge—oriented towards a specific field of interest. It wishes to reflect the disciplinary language habits and to develop a didactics of interdisciplinarity. For this disciplinary training, at least a two years' basic course is necessary, but on the main study level the different fields of actual interest should drive the multidisciplinary contributions. (There already exist such courses of study, for example, gender studies, media studies, etc.)

The forms of organization should be left open in a first phase, in order to be able to probe new forms. But there should not exist a disciplinary order, faculties will be bound together by areas of research and courses of study rather than by disciplines. On the other hand, each discipline existing within a faculty should be represented by two researchers at least. Studying is oriented towards communication, discussion and projects, less to lectures. Research careers should have longer-lasting perspectives than is the case within German universities today.

The hope is that the new qualifications will be exchanged throughout the whole labour world and society via influences of the research pools, further education, practice and industry, state and institutions. The effect could be to build up concepts for a new civilization and to let them emerge into society.

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DISCUSSION ON THE IDEAL UNIVERSITY

Report by TSOU SHEUNG TSUN

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Britta Schinzel's talk "Challenges facing an Ideal University" raises many interesting and controversial points, as expected from the nature of such a topic.

While agreeing that many of her points are valid, I would like to register my comments and disagreements here for the sake of balance, and as a record for the discussion session afterwards. Also, this seems a good opportunity to counter some of the very unfair things said by the general public (*not* by Britta Schinzel) against scientists.

(1) To build an ideal university is a very difficult, and to my mind impossible and superhuman, objective. In spite of globalization, we are still very different depending on our civilizations and our geographical positions. What suits Germany would in all probability not suit India, for instance, at the same epoch. Also, to build it using modern technology ("MBONE" was cited more than once), last-minute though it may be, is somewhat short-sighted, given the time scale required to realize such a project.

(2) The need to reinvent a totally different type of university is based significantly on the "Humboldt model" being no longer "realistic." The very phrase "a genius scientist working in solitude and freedom on self-defined problems" begs many questions. First of all, universities are not meant for "geniuses," scientist or not. Then, with very very few exceptions, scientists, genius or not, have not been working "in solitude" for at least a century. A good example is the human genome project published last week (June 2000). As to "self-defined problems," even Archimedes, genius though he was, worked out his principle of flotation because a king mistrusted his goldsmith. So I think the "Humboldt model" has never been realistic, but this does not prevent actual existent universities from flourishing, nor significant science being done.

(3) Schinzel asks: "What is "objective science"? Does it exist? … replaced by reproducibility of results …." All sciences are objective by definition and if results are not reproducible, then they are *not* scientific results. These criteria held at least since the end of the 19th century, if not since Descartes.

(4) Under "Long-term challenges," Schinzel says "The freedom to choose one's subject of interest should obey an ethical self-restriction … to renounce quite a lot of one's own professional values." Such and similar remarks suggest

that in the modern world considerable external constraints should be imposed on scientists doing research, including basic research, such as questions of relevance as perceived by those funding the research. I would like to stress that basic research is a creative activity, which those engaged in the arts subjects seem not to recognize. Just as the nonpainting public do not restrict the painter to, say, use certain colours only, nor does the nonmusician ask the composer to think of the environment when composing, so those who do not do basic research should not *direct* what a basic researcher does. In fact, basic researchers very often cannot tell in advance what directions they are going, except in a general way. There are innumerable historical examples where some very abstract discoveries were put to very practical uses. One oft-quoted one is the discovery of electricity. When asked by the Chancellor what use was electricity, the scientist (was it Faraday?) said, “One day you will be able to tax it, sir.” How true it is!

(5) A common language is urged among scientists, who should not use “hermetic scientific language.” Mathematics is dismissed as being “not apt to serve for such cooperation,” because of Gödel’s incompleteness theorem (which, if I understand correctly, implies that no arithmetic system can be entirely consistent, and this implies—but I may be very wrong—the same is true of any sensible language system). If, on the other hand, “hermetic scientific languages” means something that can be understood only after some years of study, then I do not think there is any shortcut. Science constitutes part of our total knowledge accumulated over thousands of years and over the five continents. No nonnative English (or German) speaker, for instance, would demand Shakespeare-(or Goethe-) on-tap! He or she knows that to appreciate either it is imperative to learn English (or German) to considerable depth, and that requires also years. Scientists are often blamed for using obscure language, but nobody seems to abuse Shakespeare for the same!

(6) I agree heartily with Schinzel that education should not be privatized. She quotes Tsichritzis’ version of “students … can specialize in arbitrary directions,” and a scenario imagined by Encarnacao, Leidhold, and Reuter with “global market of education, … educational brokers … centered round firms.” I am glad that the students of UCLA and Toronto reject them! Somehow it is implied that in the ideal university, students should be able to design their own courses. I do not know how that can be achieved. Students come to the university, or college, or whatever, to learn, and before they start learning how could they know what exactly to learn? Just think how difficult and complicated it is to design the undergraduate syllabus for any course and you will see my point. Besides, which students would be far-sighted enough to realize that it is not always the easiest courses that would get you the knowledge you want? For example, a medical student must go through a lot of arduous training, including memorizing many details of anatomy for instance, before she/he is qualified to treat patients. Given the choice and no prior knowledge of what

is necessary, would the medical student be able and willing to go through five years of arduous study?

(7) Concerning a women's university, it is stated that one of the more fundamental problem is that women feel secure together. Perhaps, but since we are concerned with ideals, we should think of ways of changing social attitudes which make women insecure, rather than trying to perpetuate such attitudes by segregating women. Also, I think Marylis Delest made an important point, namely that women's universities would automatically have less able teachers, whether we like it or not.

29 June 2000

DISCUSSION ON THE IDEAL UNIVERSITY

Report by WIBKE JÜRGENSEN

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The talks of Renate Tobies and Britta Schinzel, which covered historical facts, personal experiences and future aspects of ideal universities, provided plenty of information and topics for a discussion. So, on Saturday afternoon a lively debate took place. Various opinions, impressions, and experiences were expressed, especially about the question as to whether or not universities solely for women are a good way to support women. In the following, we sketch some of the main topics of the session in the hope of inspiring further discussions.

Restrictions on basic research. The general feeling of the discussion was that as a creative activity, basic research should not be limited by public restrictions. Often it is not clear at the beginning of a project where the research will lead, in particular whether abstract ideas can be put into practice. Therefore, it is important to be in a position to be able to explore ideas based on intuition and vague ideas.

Although there are at present no obvious restrictions on research, totally independent research rarely takes place. Often the filling of positions, social structures and—last but not least—industry funded scientific research restrict the freedom for pure research.

But there are also some desirable restrictions. For example, every researcher should reflect on her work by participating in discourse. After all, academics are paid by the public sector and should not forget this.

The demand for a discourse between researchers and the public leads to a further point of the discussion.

Discipline and interdisciplinary work. Interdisciplinary projects often offer the opportunity to connect research with applications. Such projects require contacts between researchers from different disciplines and the readiness to participate in discourse. However, one has to be competent in ones discipline before working in an interdisciplinary team.

Experiences in Denmark and Germany show that project oriented teaching also works well since students, responsible for their projects, are highly motivated and enjoy working on a major task. Nevertheless, the given task has to be carefully chosen to ensure a learning effect, neither too difficult nor too easy.

Womens universities: Expectations and experiences. One of the main discussion points was about womens universities, that is, universities solely for women. Various impressions were expressed, among others, fears regarding the quality of the academic staff at a womens university, would succesful academics opt for a position at a womens university? Also would women learn to fight if studying in the protected world of female groups?

However, womens colleges in the USA show that their graduates are normally strong, self-confident and good at managing. The colleges have gained a reputation comparable to the best coeducational colleges.

Single sex courses within coeducational establishments were also discussed. Some participants of the discussion reported of contradictory experiences. For example, a professor made a fool of himself when suggesting single sex groups for the exercise classes; the following year women asked for female groups. Grouping students proportionally fifty-fifty seems to work well, helping to establish a good atmosphere and enabling women to play an equal role in the group.

New technologies for teaching. At many universities efforts are made to use new technologies for teaching as well as for distance learning. Nevertheless, going multimedia is not as easy as sometimes supposed. Experiences show that up to now, technology offered by computer science is not yet usable, for example, the network is cut during lessons or the rates of transfer are too low.

Technological evolution is certainly important. However, instead of being indiscriminately innovative, it would be better to improve teaching within the universities and reflect the use of technologies.

Suggestions to improve the atmosphere within a university. To end the discussion, we focused on which aspects of a university made it a good learning/research environment. For example, an important point for female researchers seems to be a place in the department where people meet in an informal way and where discussions can take place, a coffee room, for example.

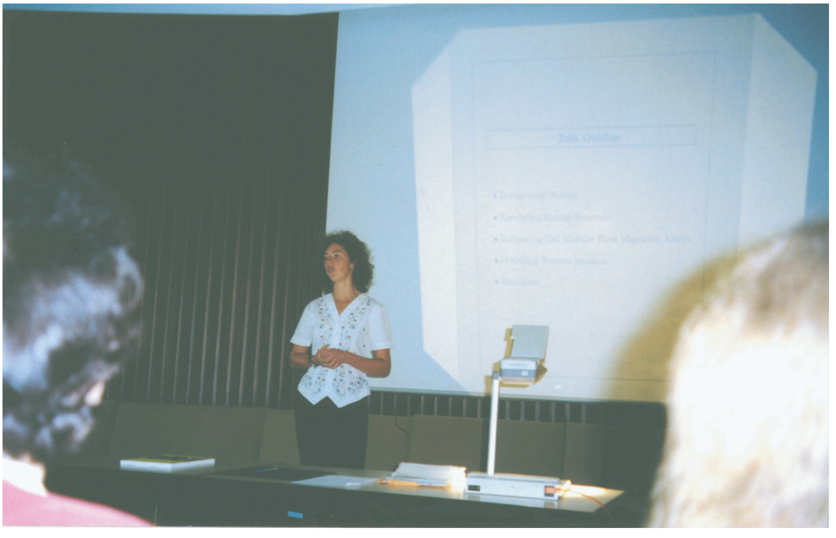
To attract more female students different teaching systems are worth mentioning. For example, as opposed to the traditional exercise class where someone presents solutions at the blackboard, classes before the solutions are submitted where students can ask for help seem to be popular. Other ways of support can consist of extra courses for female newcomers to a university, offering lessons, exercises, and discussions where they can put any questions to experienced students.

For women who want to work at a university after their Ph.D. having a long-term perspective is extremely important. More permanent jobs at universities, changing structures of employment as well as more financial support for female researchers could offer such a perspective.

PHOTO GALLERY

MATHEMATICAL MODELLING

Cecilia Jarlskog



Helen Byrne

Rosa Maria Spitaleri

Laura Tedeschini Lalli

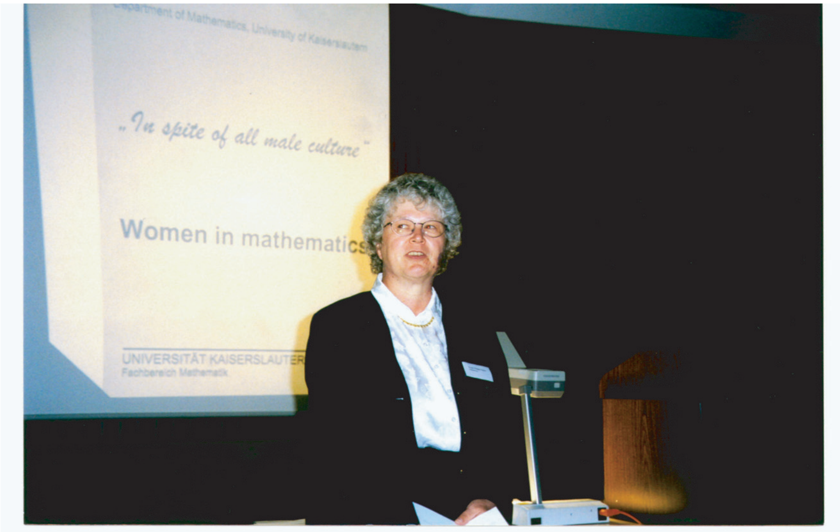
HILBERT PROBLEMS

Marie Francoise Coste Roy

Ruth Kellerhals

Ina Kersten

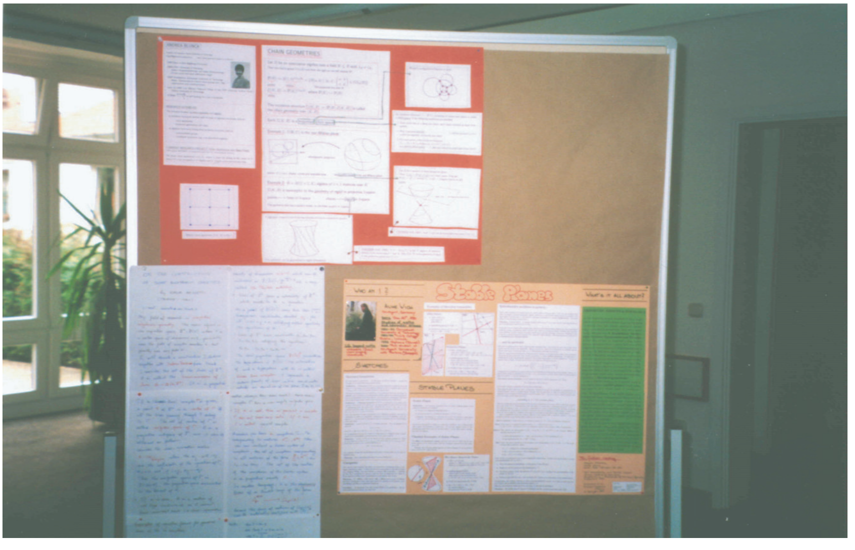
Discrete mathematics, Ulrike Tilmann



Irene Pieper Seier

Britta Schinzel

Renate Tobies



Postersession

Postersession

Wibke Jürgensen and her Poster

Amel Chaabouni and her Poster

Lisbeth Fajstrup and Tsou Sheung Tsun

A boattrip on Wednesday afternoon

Nadia S. Larsen, Emilia Mezzetti and Rosa Maria Spitaleri

Rachel Camina, Maylis Delest and Nadja Kutz

Maren Riewenschneider and Cathy Hobbs

Two participants brought their children.
The staff at Kloster Loccum found daycare for them

Group picture

THE MATHEMATICAL PART

INTRODUCTION TO THE POSTER SESSION

The Poster Session played a central role in the 9th International Meeting of EWM. Many participants arrived with carefully prepared visual explanations of their work. In all, there were 29 posters covering a wide spectrum of Mathematics, the width and depth of the interests of women-mathematicians was apparent. The posters also illustrated the type of problems that are investigated by the mathematical schools of different European countries. Due to the success of the Poster Session it was decided that poster abstracts should be published for inspection by the wider mathematical community.

Of course, the success of this Poster Session was built on the positive experiences gained from the Poster Session of the 8th General Meeting. I anticipate that the Poster Session will be an evermore interesting and informative feature of subsequent meetings.

Finally, I would like to mention a particular poster which, as a collection of puzzles, did not lend itself to an abstract. I quote from the poster of Dörfner, Preissler, and Wich "A mathematician is a machine to turn tea into … or a short course in the manufacturing of mathematical objects." This poster was always surrounded by participants—everybody wanted to play with the puzzles.

Good luck to everybody for the future

Polina Agranovich

POSTER PRESENTATIONS

Organized by POLINA AGRANOVICH

Integral representations of generalized Legendre functions

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Joint work with L. Chibrikova

In our paper, we consider the equation

$$(1-z)^2 w'' - 2zw' + \left[\nu(\nu+1) - \frac{\mu^2}{2(1-z)} - \frac{\lambda^2}{2(1+z)}\right] w = 0. \tag{1}$$

This equation is known as a generalized associated Legendre differential equation.

We construct fundamental pair of solutions $P_\nu^{(\mu,\lambda)}(z)$ and $Q_\nu^{(\mu,\lambda)}(z)$ of differential equation (1) for arbitrary ν, μ, λ using method of contour integrals with Oiler kernel. Functions $P_\nu^{(\mu,\lambda)}(z)$ and $Q_\nu^{(\mu,\lambda)}(z)$, when $\lambda = \pm\mu$, are represented through associated Legendre functions of the first and the second kind.

Exceptional sets for entire functions

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Joint work with Vladimir Logvinenko

The Levin-Pflüger theory of completely regular growth (one-term asymptotic representation) of entire functions is used intensively in different parts of mathematics and physics. One of the extension of this theory is the consideration of n-term asymptotics of entire functions.

In this work, the differences between the Levin-Pflüger theory of completely regular growth and the case of n-term asymptotics are studied. The main of these differences is the existence of points of elevation in the exceptional sets for the n-term asymptotics.

Weierstrass functions in boundary-value problems for the analytic functions

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D. Hilbert in Goettingen was the first to give a solution of linear homogeneous Riemann problem $\Phi^+(t) = G(t)\Phi^-(t)$ by using integral equations.

F. D. Gakhov in Kazan gave a constructive solution of homogeneous and inhomogeneous Riemann boundary-value problems by using the integral of Cauchy type.

The representatives of Gakhov scientific school, which the author of the report belong to, work intensively in Kazan: Three Ph.D. dissertations have recently been defended by I. A. Bikchantaev, F. N. Garifyanov, and Yu. V. Obnosov.

In a review, the summaries of these dissertations as well as the authors solutions of nonlinear boundary-value problem $[\Phi^+(t)]^\alpha = G(t)[\Phi^-(t)]^\beta$, where Weierstrass functions $\zeta(z), \sigma(z), \wp(z)$ are given. Then, the connection between mathematical schools of Goettingen and Berlin (Germany) and of Kazan (Russia) is shown.

Chain geometries

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Let R be an associative algebra with 1 over a field K. The incidence structure $\Sigma(K,R) = (\mathbb{P}(R), \mathscr{C}(K,R))$, whose point set $\mathbb{P}(R)$ is the projective line over R, and whose *chain* set $\mathscr{C}(K,R)$ consists of the K-sublines of $\mathbb{P}(R)$, is called the *chain geometry* over (K,R). It satisfies the axioms of an abstract *chain space*.

The *real Möbius plane* is the chain geometry $\Sigma(\mathbb{R},\mathbb{C})$. It is the geometry of points and plane sections of an ellipsoid in real projective 3-space. Analogously, one can introduce the chain space $\Sigma(\mathfrak{Q})$ on a quadric $\mathfrak{Q}$ in an arbitrary projective space. Under certain conditions, $\Sigma(\mathfrak{Q})$ can be embedded into a suitable chain geometry $\Sigma(K,R)$.

The chain geometry $\Sigma(K,R)$ over the K-algebra $R = M(2\times 2, K)$ of 2×2 matrices is isomorphic to the geometry $(\mathscr{L},\mathfrak{R})$ of all lines and all *reguli* in the projective 3-space over K. A similar result holds in projective spaces of arbitrary, not necessarily finite, dimension. Then a regulus consists of subspaces that possess an isomorphic complement, and R is the ring of linear endomorphisms of such a subspace.

A geometrical approach to the Emden-Fowler equation

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This paper is devoted to the investigation of Emden-Fowler equations of the type

$$y''(x) + e(x)\, y^p(x) = 0, \quad x \in \mathbb{R}.$$

The motivation to study such equations systematically came from astro-, atomic-physics, and Riemannian geometry. We study the properties of Emden-Fowler equation, which are invariant under the general point transformations.

The underlying geometrical theory of second order ordinary equations was developed by E. Cartan. He introduced the concept of the space of normal projective connection. The Emden-Fowler equation can then be considered as the equation of the geodesics in such space. In case of Emden-Fowler equation the corresponding space of normal projective connection can be immersed in the flat projective space $\mathbb{R}P^3$. The description of related surfaces is given.

The local density-functional method as a tool of mathematical modeling in solid states physics

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We report in our work *ab initio* calculations of the electronic structure and total energy of pure Mg with *hcp* structure as well as *fcc* structure for comparision. We have done these computations also for pure Al with *fcc* lattice as well for typical alloys $MgAl_3$ and $MgAl_6$. We have varied the size of the unit cell in each case to obtain the minimum of total energy in relation to the lattice parameters $a, c/a$. To calculate the total energies we take an accurate method using density functional theory within the LDA using the implementation of this method by Dr. M. Methfessel. The present calculations for Mg in the *hcp* structure agree extremely well with the data obtained from the experiments.

Conjugacy class sizes—some implications for finite groups

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Let G be a finite group and $x \in G$. Then $x^G = \{g^{-1}xg : g \in G\}$ defines the conjugacy class of x in G. Note that $|x^G| = |G : C_G(x)|$, where $C_G(x) = \{g \in G : g^{-1}xg = x\}$ is the centraliser of x in G. The *conjugate type vector* of G is the r-tuple $\{n_1, n_2, \ldots, n_r\}$, where $n_1 > n_2 > \cdots > n_r = 1$ are all the numbers that occur as sizes of conjugacy classes of G. This definition leads to the following general question.

Question. Given the conjugate type vector of a finite group G, what can you say about the structure of G?

For example, if $\{n_1, \ldots, n_r\} \times \{m_1, \ldots, m_s\}$ is defined to be $\{n_i m_j \mid 1 \le i \le r,\ 1 \le j \le s\}$, then it can be proved that G is nilpotent if its conjugate type vector is of the form

$$\{p_1, 1\} \times \cdots \times \{p_r, 1\},$$

where $p_1, \ldots, p_r$ are distinct primes.

For a further discussion on this type of question (and references) see the article of the same title in this proceedings.

Some properties of singular invariant distributions

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Let P be the relative invariant of an irreducible regular prehomogeneous space defined over $\mathbb{R}$ with V as real vector space of dimension n. Then P is a real polynomial of degree d. Let $V_1, \dots, V_\ell$ be the connected components of $V \setminus P^{-1}(0)$.

Problem. Find an explicit "topological" condition in order that the coefficients of the Laurent expansion at the negative poles of the distributions $|P|^s_{|V_i}$ be nonzero.

In the Bull. Sci. Math. (2) 1989, I expressed all the linear relations (there are $d\ell/2$ of them) between the above-mentioned coefficients, which are invariant distributions, using the matrices $\partial^\alpha U(c_j)$; here the c_j are the roots of the b-function of P, $0 \le \alpha < d$ and $U(s)$ is the analytic matrix which appears in the Fourier transform of the vector-valued distribution $(|P|^s_{|V_i})_{1 \le i \le \ell}$.

In 1999, D. Barlet gave a condition on $V_i \cap F_{\mathbb{R}}$, where $F_{\mathbb{R}}$ is the real Milnor fiber of P, which is a solution in the case where P is a germ of an analytic function at zero whose complexification has an isolated singularity at zero. P is not necessarily a relative invariant. However, most relative invariants are not in this case.

History of the travelling salesman problem

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Introduction. The travelling salesman problem is one of the discrete optimization problems. However, there is no polynomial algorithm for this problem. From the historical point of view, the development of the problem formulations and solutions can be traced.

Mathematicians use basically two definitions of the problem: They are looking for a path which passes through each node either *exactly* once, or *at least* once. The solution of the former produces a Hamiltonian circuit, whereas the one of the latter need not.

Origins. It is said that the travelling salesman problem was formulated by Hassler Whitney in 1934. However, he himself denies this. In 1934, another mathematician, Merrill Flood, was trying to find the cheapest route for a school bus. His colleague Alan Tucker pointed out to him that this problem has something to do with *Hamiltonian circuits*.

After the Second World War, the problem became increasinlgy interesting. The solutions were more and more formalised with the use of programming languages.

Heuristics. The published heuristics can be divided into three basic types: tour-to-tour improvement, tour building, and subtour elimination. A short description of these methods can be found, for example, in [1]. The methods are usually aimed at getting approximate results. What is then interesting from historical point of view is the way mathematicians reflect on their and the others' results; whereas in early 1960s it was usually referred to a specific computer and the time as measured by stopwatch, at the end of 1960, first complexity characteristics appear.

Instead of a conclusion, here is a question:

Is mathematics going to change its style from Theorem-Proof *to the* Algorithm-Analysis *method, or will there be "two kinds of mathematics"?*

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Quadratic and hermitian forms over rings

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The upper third of the poster consists of pictures of various places I have lived and worked in mathematics: Paris, Berkeley, Besançon, and Göteborg.

I am interested in *quadratic and hermitian forms* whose coefficients are taken in rings such as the p-adic integers, the rational integers, matrix rings, and group rings $\mathbb{Z}[G]$, $\mathbb{Z}_p[G]$.

The study of quadratic and hermitian forms is related to number theory, via the trace form, to algebraic geometry via the study of the variety of zeroes of a form, to algebraic groups via the group of symmetries, to algebra when we study for which rings R the family of all forms has certain properties.

An application to gas kinetics. In Discrete Velocity Models for the Boltzmann equation, one tries to approximate the collision operator (a multiple integral over $\mathbb{R}^n\times$sphere) using summations over points in a lattice. One needs to know which lattice points lie exactly on the surface of spheres of given radius, that is, which points with integer coordinates x_i satisfy $\sum x_i^2 = n$. In high dimensions, it is easy to find enough points to get good approximations. In dimension 2, many circles have no points, or very few, and it is difficult to see whether the points are equidistributed.

Geometry and topology in concurrent computing

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Coordination of concurrent processes is a central problem within distributed computing. Modern multiprocessor systems are inherently asynchronous, so this is an important but difficult subject. It is not obvious how to model concurrent systems, and computer scientists do not agree on which model to choose. In recent years, a variety of mathematical methods have been proposed to model concurrency situations. Among those are techniques borrowed from algebraic and geometric topology: Simplicial techniques have led to new theoretical bounds for coordination problems [2]. Higher dimensional automata have been modelled as cubical complexes with a partial order reflecting the time flows, and their homotopy properties allow to reason about a systems global behaviour. At the workshop Geometric and Topological Methods in Concurrency Theory,

http://www.math.auc.dk/raussen/admin/workshop/workshop.html

computer scientists and mathematicians discussed and presented various aspects of this new common ground between mathematics and computer science. The poster presents some of the results of this workshop together with earlier results in this area including the deadlock detection algorithm described in [1].

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Local state space reduction of multiscale systems

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Joint work with Klaus Schneider

Modelling reaction kinetics in a homogeneous medium usually leads to nonlinear systems of ordinary differential equations the dimension of which can be very large. A well-known approach to reduce the dimension of such systems is the quasi-steady state assumption (QSSA): The derivative of fast variables is assumed to be zero. This procedure requires some knowledge of the underlying chemistry, moreover the corresponding differential system must be explicitely given. We have described and justified a procedure for a local reduction of the dimension of state space which does not require chemical insight as well as an explicit knowledge of the system in a singularly perturbed form. The mathematical justification is based on the theory of invariant manifolds. Numerical examples show that the method works well [1].

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Uniform asymptotics of oscillating integrals

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Joint work with N. P. Kirk, J. W. Bruce, and J. N. L. Connor

The uniform asymptotic evaluation of oscillating integrals has numerous applications to short wavelength phenomena in chemistry, physics, and mathematics. In particular, it is important for the study of collisions of atoms, molecules, and nuclear heavy ions.

The technique we have developed in order to uniformly evaluate certain types of oscillating integrals to which the standard asymptotic techniques do not apply is to employ suitable integral representations to convert the original one-dimensional integrals into multidimensional ones. The phases of these integrals can then be analysed using Singularity Theory techniques (i.e., by computing invariants such as corank and codimension), and then transformed into their polynomial normal forms using diffeomorphic changes of coordinates. We can use numerical techniques [1] to evaluate the integrals containing these normal forms, and use these values to derive the uniform asymptotic expansions of the original integrals.

Acknowledgment. This project was funded by the Engineering and Physical Sciences Research Council, UK.

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Automaticity of solutions of Mahler equations

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Automatic sequences are intermediate between periodic and random sequences. Considering sequences with values in a finite field $\mathbb{F}_{p^\alpha}$, it was shown in 1979 that the coefficients of a formal power series $F \in \mathbb{F}_{p^\alpha}[[X]]$ are p-automatic if and only if F satisfies a Mahler equation $\sum_{j=0}^{d} P_j(X)F(X^{p^j}) = Q(X)$ with polynomials $P_j, Q \in \mathbb{F}_{p^\alpha}[X]$ $(j = 0, \dots, d)$.

We give sufficient criteria for the automaticity of the coefficients of solutions of the more general Mahler equations

$$\sum_{j=0}^{d} P_j(X)F(X^{m^j}) = Q(X)$$

for arbitrary $m \in \mathbb{N}$. In that case, the automaticity depends on the characteristic of the finite field $\mathbb{F}_{p^\alpha}$ as well as on the order of the polynomial Q.

A characterization of orthogonal and symplectic groups

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How can we characterize a subgroup of an orthogonal or symplectic group as a subgroup of the general linear group $\mathrm{GL}(n,K)$ with certain properties? We give an example for the orthogonal case in this summarization.

Definition 1. Let V be a K-vector-space of dimension $n \in \mathbb{N}$ over a commutative field K. Let $\pi \in \mathrm{GL}(V)$. Then

$$\mathbf{F}(\pi) = \{v \in V \mid v\pi = v\} \text{ is the fixed space of } \pi,$$
$$\mathbf{B}(\pi) = \{v\pi - v \mid v \in V\} \text{ the path of } \pi,$$
$$\pi \text{ is } \textit{simple} :\Longleftrightarrow \dim B(\pi) = 1;$$
$$\mathrm{id} \neq \pi \text{ is an } \textit{involution} :\Longleftrightarrow \pi^2 = \mathrm{id}.$$

Let $f : V \times V \to K$ be a *symmetric bilinearform*, that is, f is linear in both components and holds $f(v,w) = f(w,v)$ for every $v,w \in V$. We define the *orthogonal group*, respectively, f as

$$O(V,f) := \{\pi \in \mathrm{GL}(V) : f(v\pi, w\pi) = f(v,w) \, \forall v,w \in V\},$$

and denote $v \perp w$ if $f(v,w) = 0$, $v,w \in V$, and $V^\perp := \{v \in V : v \perp w \;\; \forall w \in V\}$.

Definition 2. Let $\sigma \in O(V,f)$. We say, σ is a *symmetry* if and only if σ is simple and holds $B(\sigma) \nsubseteq V^\perp$.

Theorem 1 (Characterization theorem)**.** *Let K be a field with* $\mathrm{char} K \neq 2$ *and V be an n-dimensional K-vector-space. Further, let S be a set of simple involutions in* $\mathrm{GL}(V)$ *and there are $\sigma_1, \ldots, \sigma_n \in S$ with $\sum_{i=1}^n B(\sigma_i) = V$ and $\bigcap_{i=1}^n F(\sigma_i) = \bigcap_{\sigma \in S} F(\sigma)$ and $G := \langle S \rangle$. Then the following statements are equivalent:*

(1) *$G \leq O(V,f)$ for a symmetric bilinearform $f : V \times V \to K$. Especially S is a set of symmetries.*

(2) *G holds the following properties:*

(K1) $\tau\sigma\tau \in S$ *for all* $\tau, \sigma \in S$.

(K2) *For* $\tau, \sigma \in S$ *and* $B(\sigma) = B(\tau)$, *we have* $\tau = \sigma$.
(K3) *For each* $\sigma_1, \sigma_2, \sigma_3 \in S$, *we have* $F(-\sigma_1\sigma_2\sigma_3) \neq \{0\}$.

We refer to [2, 3] for the proof of this result and the symplectic case.

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Factorization dynamics and Coxeter Toda lattices

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The posters present a joint work of Tim Hoffmann, Johannes Kellendonk, Nicolai Reshetikhin and the author of the posters. Roughly speaking, the posters explain how a certain matrix factorization can be used to construct a discrete integrable dynamical system. This system can be regarded as a discretization of the well-known Toda flow. More precisely, it was shown that the factorization relation on simple Lie groups with standard Poisson Lie structure restricted to Coxeter symplectic leaves gives an integrable system.

Semigroup crossed products

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Crossed products of C^*-algebras by groups of automorphisms have been extensively studied over the past few decades and are now a well-established area of operator algebras.

In recent years, there has been growing interest in a similar notion, involving semigroups of endomorphisms rather than automorphisms, as it turned out that the resulting objects were very fit to model C^*-algebras generated by families of isometries, the so-called Toeplitz algebras. Uniqueness results were, for instance, immediate from characterisations of faithful representations of appropriate semigroup crossed products (Adji-Nilsen-Laca-Raburn, Laca-Raeburn, Fowler-Raeburn). In a different direction, semigroup crossed products and their techniques were employed in giving an alternate description of the Bost-Connes Hecke C^*-algebra arising in number theory and generalisations hereon (Laca-Raeburn, Arledge-Laca-Raeburn, Laca).

Two simple questions have arisen in work on the Bost-Connes C^*-algebra and Toeplitz algebras: Do semigroup crossed products behave well under tensor products and short exact sequences? Here we give an affirmative answer for a large class of semigroup crossed products.

On the construction of some Buchsbaum varieties

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The set of the lines of a projective space $\mathbf{P}^n = \mathbf{P}(V)$, denoted by $\mathbf{G}(1,n)$, is a projective algebraic variety, called the Grassmannian. It is naturally embedded in $\mathbf{P}(\Lambda^2 V)$. The hyperplane sections of the Grassmannian parametrize some families of lines, called *linear line complexes*: They correspond bijectively to elements in $\mathbf{P}(\Lambda^2 V)^*$, or else to sections of the twisted cotangent bundle $\Omega_{\mathbf{P}^n}(2)$ (using the Euler exact sequence).

Hence, fixed an integer number $m \leq \binom{n+1}{2}$, giving a bundle map $\varphi : \mathcal{O}_{\mathbf{P}^n}^{\oplus m} \to \Omega_{\mathbf{P}^n}(2)$ is the same as giving a linear system Λ of linear line complexes of projective dimension $m-1$. The degeneracy locus of the map φ is a projective scheme X, which can be interpreted as the set of centres of the complexes of Λ, where, by definition, a point P is a centre of a complex Γ if Γ contains all lines through P.

In a joint paper with Dolores Bazan, we have studied such degeneracy loci, determining in particular their degree, the locally free resolution of their homogeneous ideal, and proving that they are arithmetically Buchsbaum. In particular, if $n = 5$, for $m = 2X$ is a triple of skew lines, for $m = 3X$ is an elliptic normal scroll surface. As an application of the general theory, we obtain an explicit description of the Hilbert scheme of such elliptic scrolls in $\mathbf{P}^5$.

Solving evolution equations with approximate approximations

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The new method of approximate approximations was introduced by V. Maz'ya. One represents a smooth function u by

$$\mathcal{M}_h u(\mathbf{x}) = \frac{1}{\sqrt{\mathcal{D}}^d} \sum_{\mathbf{m}\in\mathbb{Z}^d} u(h\mathbf{m})\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right),$$

where the function η is continuous and fulfills a decay condition. The larger class of such basis functions (compared with, e.g., splines) is payed with an additional so-called saturation error in the approximation process. This error can be controlled by $\mathcal{D}$ in such a way that it is smaller than, for example, the machine precision, although it does not tend to zero for h tending to zero. This makes the method useful for practical applications. One application is the solution of nonlinear partial differential equations, for example, evolution equations like

$$u_t - \nu u_{xx} = \partial_x f(x,t,u), \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = \varphi(x).$$

Algorithms for this equation based on approximate approximations were developed by Maz'ya and Karlin. We proved error estimates for these algorithms. The error estimates are of the form

$$||u(\cdot,n\tau) - \bar{u}(\cdot,n\tau)||_{L_2(\mathbb{R})} \le (\mathbb{O}(h^N + \tau) + \text{saturation error} + \|\varphi - \bar{\varphi}\|_{L_2(\mathbb{R})}) \exp(Cn\tau),$$

where $\bar{u}(\cdot,n\tau)$ is the approximate solution at time $n\tau$, $\bar{\varphi}$ is the approximation of the initial value and N is the approximation order of $\mathcal{M}_h$ depending on the choice of η. The method was also used to solve nonlinear nonlocal evolution equations. Similar error estimates were derived.

A counterexample to the infinite-dimensional version of the matrix Hunt-Muckenhoupt-Wheeden theorem: Operator BMO and factorizations of $H^1(c^1)$

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Joint work with F. Nazarov, A. Gillespie, S. Treil, A. Volberg, and J. Wilson

The classical Hunt-Muckenhoupt-Wheeden theorem in its version for L^2 states that the Hilbert transform H, which assigns to each L^2 function on the unit circle $\mathbb{T}$ its harmonic conjugate $Hf = \tilde{f}$, extends to a bounded linear operator on the weighted L^2 space $L^2_w(\mathbb{T})$ if and only if the weight function $w : \mathbb{T} \to \mathbb{R}^+$ satisfies the Muckenhoupt A_2 condition

$$\sup_{I \subset \mathbb{T} \text{ interval}} (m_I w)(m_I w^{-1}) < \infty,$$

where m_I denotes averaging over the interval I. In 1996, S. Treil and A. Volberg proved a matrix version of this theorem: They showed that the Hilbert transform is bounded on the space of vector-valued functions

$$L^2_W(\mathbb{T},\mathbb{C}^d) = \left\{ f : \mathbb{T} \longrightarrow \mathbb{C}^d : \int_{\mathbb{T}} \langle W(t)f(t), f(t)\rangle dt < \infty \right\}$$

for the matrix weight function $W : \mathbb{T} \to \mathrm{Mat}(d \times d)^+$ if and only if W satisfies the so-called matrix Muckenhoupt A_2 condition

$$\sup_{I \subset \mathbb{T} \text{ interval}} \rho((m_I W)(m_I W^{-1})) < \infty.$$

Here ρ denotes the spectral radius.

We show that the infinite-dimensional version of this theorem is false. Our main tool is an exponential-like connection between matrix Muckenhoupt A_2 weights and matrix BMO, which allows us to reformulate the question of the validity of an infinite-dimensional matrix Hunt-Muckenhoupt-Wheeden theorem for a certain class of weights as the question of equality for two spaces of operator-valued BMO functions. Using Sarason's Factorization theorem, this in turn can be reformulated as a factorization problem for $H^1(c^1)$. From an earlier result of F. Lust-Piquard, pointed out to us by G. Pisier, it follows that this factorization of $H^1(c^1)$ does not hold.

Willmore hypercyclides

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The poster deals with Möbius invariant hypersurfaces in euclidean space E^n which are both Willmore hypersurfaces and hypercyclides.

In the twenties, Blaschke characterized Willmore surfaces in E^3, using their central spheres. This result leads to the definition of Willmore hypersurfaces for higher dimensions $n \geq 3$.

Willmore hypersurfaces are those hypersurfaces $\mathcal{F}$ where their central hyperspheres Z have a special second envelope $\hat{\mathcal{F}}$ (apart from $\mathcal{F}$). More precisely, if $\mathcal{F}$ is a Willmore hypersurface, each central hypersphere Z of $\mathcal{F}$ has to be a central hypersphere for this second envelope $\hat{\mathcal{F}}$, too. A central hypersphere of $\mathcal{F}$ is a hypersphere which touches $\mathcal{F}$ and has the same mean curvature as $\mathcal{F}$ in its point of contact with $\mathcal{F}$.

Hypercyclides are generalizations of the classical Dupin cyclides for $n = 3$, and they were classified by Cecil in a Möbius invariant way in 1992. This classification shows that for a Möbius invariant view on hypercyclides it suffices to consider hypertori of revolution $S^k \times S^{n-k-1}$ with ratio a of the radii of the spheres S^k and S^{n-k-1} involved.

The poster results in some characterizations for Willmore hypercyclides, especially the ratio $a = a(n,k) > 1$ is given where hypertori of revolution are Willmore hypersurfaces. For $n = 3$, $k = 1$, we get Willmore tori with $a = \sqrt{2}$ which were already presented by Willmore in 1965.

Inversive planes with a 2-fold transitive automorphism group

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An *inversive (or Möbius) plane* is a geometrical incidence structure $\mathcal{M} = (\mathcal{P}, \mathcal{K}, \in)$, consisting a set of *points* $\mathcal{P}$ and a set of *circles* $\mathcal{K}$ satisfying the following axioms:

(i) For each three distinct points $P, Q, R \in \mathcal{P}$ there exists exactly one circle $k \in \mathcal{K}$ with $P, Q, R \in k$.

(ii) Let $P, Q \in \mathcal{P}$ and $k \in \mathcal{K}$ with $P \in k$ and $Q \notin k$. Then there exists exactly one $l \in \mathcal{K}$ with $P, Q \in l$ and $k \cap l = \{P\}$.

(iii) Each $k \in \mathcal{K}$ is incident with at least one $P \in \mathcal{P}$, and there are four points which are not concircular.

In particular, I am interested in inversive planes admitting a 2-fold transitive group of automorphism. One way to obtain such planes is to mix up several Miquelian planes in a suitable manner (for details, see [1, 2]). My main focus is to study the automorphism group of such planes and the automorphisms of order 2, to classify different isomorphism types and to find more examples.

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Quasi-periodic Riemann's boundary-value problem and its applications

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Nowadays, one of the directions in development of the theory of boundary-value problems is the research of problems in case the boundary conditions are set on an infinite number of boundary lines, on an enumerable set of lines in particular.

The solution of the modified Dirichlet problem, mixed problem for a plane, for a half-plane, and basic problems of the theory of elasticity as well was obtained. The research method of these problems is that by introducing auxiliary functions the solution of these problems is reduced to that of Riemann's boundary-value problem $\Phi^+(t) = G(t)\Phi^-(t) + g(t)$ with a constant coefficient $G(t)$ and an arbitary free term $g(t)$, that is, to a special case of the so-called quasi-periodic problem of Riemann. The method offered for the solution of these problems gave us an opportunity to achieve results in solving quasi-periodic Riemann's problem in general case, that is, in case of periodic coefficient $G(t)$ and arbitary free term $g(t)$.

Perturbation of solutions in the filtrational problems with free boundaries

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The inexactitude of initial information in the filtration problems leads to the necessity in study of solution modification at initial data variation.

In this paper, a parameter perturbation technique is used in order to receive quantitative estimations of free boundary problem decision variations.

Reaction of filtration flows with free boundaries to the variation of the filtration resistance field is studied. The groundwater flows under steady-state and transient conditions are examined. Boundary-value problems for head perturbation are formulated.

The problem of finding permeability perturbations, causing the greatest variations of a flow rate and a filtration region at different additional constraints, is put. Numerical solutions of these problems for model examples are obtained.

The results were obtained in cooperation with Professor A. V. Kosterin of Kazan State University.

The stress-strain state of rocks and mass transfer in deep layers

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A new approach for calculating pressure fields, permeability, and well productivity has been developed, taking into account the deformation of thin layer and of adjacent rocks. It is assumed that the pressure distribution in a layer is nonaxisymmetric, and that Young's modulus varies according to layer coordinates and time. This approach allows for the evaluation of well productivity with regard to two-phase seepage flow.

The initial problem can be devided into internal and external ones. An external problem concerns the stress-strain state of adjacent elastic rocks. A layer at a depth far exceeding its thickness is represented by an infinitely thin cut of a uniform elastic half-space; the rheological condition is realised on its boundaries.

The solution of the problem is obtained in integral form by using two-dimentional Fourier transform. The layer deformations are described by an integral operator of the liquid pressure field in a layer. Then the integral problem is reduced to the solution of a nonlinear two-dimentional partial parabolic equation for pressure.

This method is illustrated by the problem of the nonconcentric well in a circular layer.

The pressure in an unperturbed layer is assumed to be equal to zero. In initial time on the well contour it changes with a jump while on the external layer boundary it is equal to zero.

The problem is solved by the finite element method in combination with the iteration method. The relationships between the rate of productivity of extraction and injection wells under different layer pressure gradients are obtained. The pressure and permeability distributions in the seepage area are thus constructed.

The results were obtained in cooperation with Professor A. V. Kosterin and Professor E. V. Skvortsov of Kazan State University.

This project was carried out with the financial support of the Russian fund for fundamental research, project number 99-01-00466.

Stable planes

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Stable planes are locally compact *topological planes* with an open domain of intersection of lines. Classical examples are the projective planes $\mathcal{P}_2\mathbb{F}$, as well as the affine and hyperbolic planes, for alternative fields $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. A theorem by Löwen states that every stable plane has dimension 2, 4, 8, or 16. Every open subplane of a stable plane is a stable plane. Conversely, it is often a crucial question whether or not a given stable plane can be embedded as an open subplane in one of the classical planes.

The poster presents a series of 4-dimensional connected stable planes recently investigated by Maier and explains why they cannot be embedded in the projective plane $\mathcal{P}_2\mathbb{C}$. To that end, a translation mechanism, developed by Stroppel, between *sketched geometries* and *sketches*, that is, between geometrical and group theoretical methods, is introduced and profited from.

Cluster formation of spatial branching models

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The poster presents a part of the author's Ph.D. thesis.

We consider an interacting particle model, in which the particles living in $\mathbb{Z}^d$ underlie the following dynamics. The particles live an exponential time. During their life time the *migrate* according to a random walk; while at the end they either die or *branch* into two new particles situated at their parent particle's final site, each case with probability $1/2$. Both mechanisms occur independently for all particles, independently of each other and independently of the initial configuration.

If we start our model with a finite number of particles, it is easy to see that they all die out. For an infinite number of particles the question of the long term behavior is a more interesting one. It differs sharply in low and high dimensions: If $d \leq 2$, one gets local extinction, while for $d \geq 3$, the particles tend to a nontrivial equilibrium. It has been known for a while that local extinction goes along with clumping around a "typical surviving particle." This phenomenon is called *clustering*.

This poster is dedicated to the critical dimension, $d_c = 2$. A detailed description of phenomena concerning clustering including the growth rate of components with surviving particles, the age of the cluster in comparison to the system's age and its family structure is given. The concepts comprise *normalization* by the hight of a surviving component, *averaging* over large blocks of components and *rescaling* in space and time.

INTRODUCTION TO THE SESSION ON MATHEMATICAL MODELLING

Following the now-established tradition, we had again an inter-disciplinary session. This time it is on “mathematical modelling” and we have chosen three main areas: theoretical physics (Cecilia Jarlskog), biology (Helen Byrne), and visual numerical environment (Rosa Maria Spitaleri). In addition to these main talks, we also had two spontaneously contributed short talks: Laura Tedeschini Lalli on mathematics and music, and Lisbeth Fajstrup on geometry in computer science. All five speakers were able to convey to this general audience of mathematicians the excitement of their very different subjects, and to show us how many rather familiar concepts in mathematics can be put to use in so very different contexts.

I personally think that such an inter-disciplinary session is of tremendous use, not only for communication among established workers, but especially for the younger mathematicians. This same idea had been expanded and carried out in the two EWM workshops on renormalization (Paris) and moduli spaces (Oxford). One thing I have learnt since joining EWM is that women can cooperate well, and since there are still so few women mathematicians, promoting inter-disciplinary environments makes good sense. I for one was very happy to have learnt something about wound healing, about the numerical presentation of visual objects, the working of the Javanese gamelan, and the question of the dihomotopy of diloops when using my printer!

Cecilia Jarlskog’s talk, “Particle physicists’ beloved mathematics,” had very pretty transparencies which were generally greatly appreciated. Unfortunately, her heavy commitments, including chairing the Nobel Committee for Physics, prevented her from writing up her very enjoyable talk. For the others, their articles which follow speak eloquently of their subjects.

Tsou Sheung Tsun
Oxford

USING MATHEMATICS TO STUDY SOLID TUMOUR GROWTH

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1. Introduction. Hardly a day goes without the appearance of a press release claiming that a new cure for cancer has been discovered. Such media of interest is unsurprising given that cancer is now poised to overtake heart disease as the major cause of premature death in the Western World. Whilst many of these deaths are indirectly the result of improvements in healthcare (as life expectation rises the chances of succumbing to cancer increase), it is also true that treatment for many forms of cancer are still alarmingly ineffective. In the face of such news, biologists, clinicians, and pharmaceutical companies are now investing considerable effort in trying to improve the prognosis of patients diagnosed with cancer.

In order to develop effective treatments, it is important to identify the mechanisms controlling cancer growth, how they interact, and how they can most easily be manipulated to eradicate (or manage) the disease. In order to gain such insight, it is *usually* necessary to perform large numbers of time-consuming and intricate experiments—but not *always*. Through the development and solution of mathematical models that describe different aspects of solid tumour growth, applied mathematics has the potential to prevent excessive experimentation and also to provide biologists with complementary and valuable insight into the mechanisms that may control the development of solid tumours.

Whilst the application of mathematics to problems in industry has been an active area of research for many years, its application to medicine and biology is still a relatively new development. For example, most of the models of solid tumour growth were written in the last twenty years. In this paper, I review some of the major developments in modelling of solid tumour growth that have taken place over the past twenty years and indicate what I believe are the main directions for future mathematical research in this field. By reading the article I hope that you will, at least, learn some biology and, at best, be stimulated to learn more about the ways in which mathematics can help in the battle against cancer.

The outline of the remainder of the paper is as follows. Section 2 contains a brief description of the key stages of cancer growth and introduces the biological terminology. Sections 3, 4, and 5 contain reviews of different, but interrelated, models of avascular tumour growth. The paper concludes in Section 6

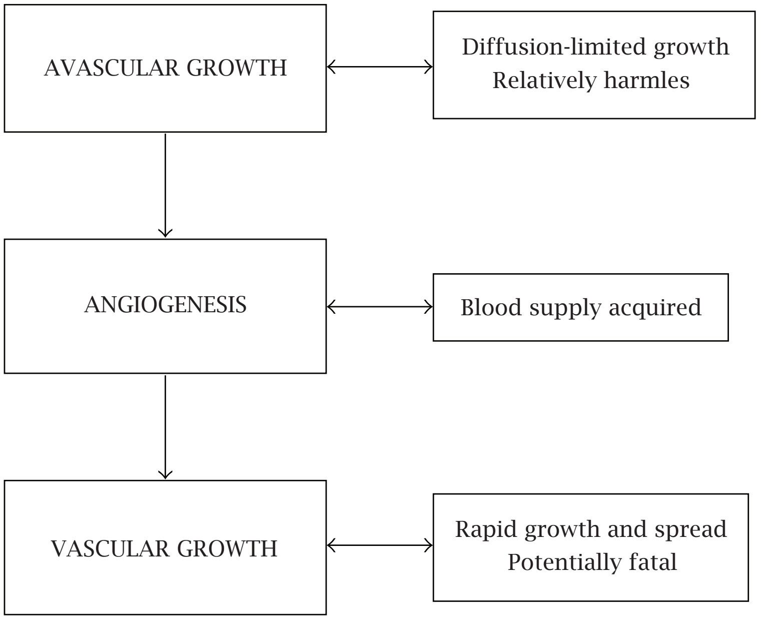

FIGURE 2.1. A schematic diagram showing how the stages of solid tumour growth are interrelated.

with a brief summary and a discussion of key directions for future mathematical research in solid tumour growth.

2. Background biology. Whilst the development of solid tumours is undoubtedly a complex process, involving many interacting mechanisms, it is still possible to identify two distinct phases of growth: the relatively benign phase of avascular growth which precedes the more aggressive, and potentially life-threatening, phase of vascular growth (see Figure 2.1). The main difference between these phases is that avascular tumours are devoid of blood vessels whereas vascular tumours are not. In the absence of a blood supply, avascular tumours receive vital nutrients, such as oxygen and glucose, and eliminate waste products via diffusive transport. Consequently, the size to which they grow is limited. Once connected to the host's blood supply, a vascular tumour is able to grow rapidly due to the presence of an almost limitless supply of vital nutrients. Such growth may impair the function of neighbouring organs. Additionally, tumour fragments that enter the blood supply are transported to other parts of the body where, if the environmental conditions are favourable, they will establish secondary tumours or metastases that further weaken the host. In order to make the transition from avascular to vascular growth the tumour undergoes a process known as angiogenesis.

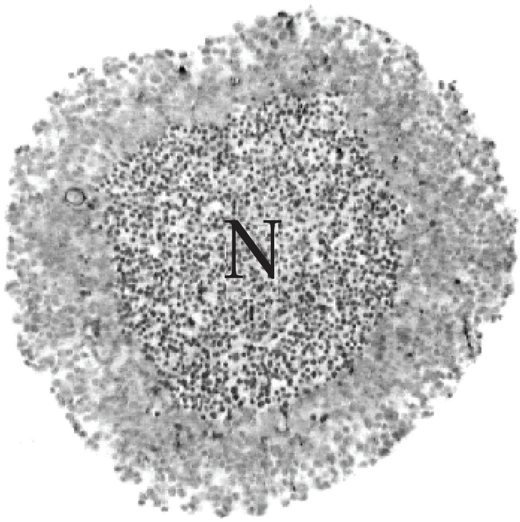

FIGURE 2.2. This figure shows a human tumour spheroid grown in vitro from a single breast cancer cell. It consists of a number of layers of viable proliferating tumour cells and an inner, necrotic (N) area where the low oxygen tension has caused substantial cell death.

In response to an externally-supplied nutrient, avascular tumours adopt a well-defined, radially-symmetric spatial structure: An outer rim of nutrient-rich, proliferating cells surrounds an intermediate hypoxic (i.e., low oxygen) annulus containing nutrient-poor, nonproliferating cells and a central necrotic core of nutrient-starved, dead cells (see Figure 2.2). In response to the stress of low oxygen, the hypoxic cells secrete a range of chemicals (or angiogenic factors) which diffuse out of the tumour and through the host tissue. When they reach neighbouring blood vessels, they activate the endothelial cells that line the blood vessels to proliferate and migrate preferentially towards the tumour, eventually furnishing it with a circulating blood supply so that vascular growth may commence [23].

Vascular tumours typically contain many cell types, including tumour cells, macrophages, and endothelial cells, which are embedded in a tissue matrix, whose composition and pattern of growth vary over time. For example, tumour cells close to blood vessels proliferate rapidly in the presence of abundant oxygen and nutrients. Where the demand for oxygen exceeds the delivery rate transient areas of hypoxia form. Being formed so quickly, most tumour blood vessels lack muscular tone and are prone to collapse when subjected to small pressure increases associated with tumour cell proliferation. With vessel collapse, the supply of oxygen and nutrients to a given region ceases and hypoxia occurs. As mentioned above, in response to hypoxia the tumours cells try to stimulate a vascular response from the host through the production of angiogenic factors. The resulting neovascularisation increases the local nutrient levels, enabling the tumour cells to recommence proliferation. In this way, the tumour's spatial structure changes dynamically, with periods of cell proliferation alternating with hypoxia in different parts of the tumour.

From the above description, it is clear that therapies which prevent or restrict the growth of blood vessels to solid tumours would be highly desirable. Whilst several anti-angiogenic factors have now been identified (e.g., angiostatin and endostatin), the results remain inconclusive. This is largely due to practical and ethical difficulties associated with organising clinical trials (e.g., it is almost impossible to get a large enough sample of patients with exactly the same type of tumour). Thus the conventional therapy for cancer patients remains a combination of surgery, radio- and chemo-therapy, the particular combination depending on the type of tumour being treated. However, by working in collaboration with clinicians and experimental biologists and developing realistic mathematical models of solid tumour growth, it may eventually be possible to optimise treatment for each individual.

3. Radially-symmetric avascular tumour growth. During the 1970s, experimentalists interested in tumour biology were focussing on the effect that changes in the concentration of externally-supplied nutrients such as oxygen and glucose had on the growth of avascular tumours [13, 31, 32]. They measured the radii of the approximately radially-symmetric tumours over time, sometimes supplementing their results with oxygen concentration profiles within the tumour and measurements of the necrotic and hypoxic radii. Presented with such data, it is not surprising that the key variables in mathematical models from that time were assumed to be the outer tumour radius $R(t)$ of the radially-symmetric tumour, the nutrient concentration $c(r,t)$ within the tumour volume and the radius of the interface between the proliferating and hypoxic regions $R_H(t)$ and the interface between the hypoxic and necrotic regions $R_N(t)$.

3.1. The mathematical model. When written in dimensionless form and in spherical polar coordinates, a typical model in which the width of the hypoxic region is neglected (i.e., $R_H(t) = R_N(t)$), has the following form:

$$\underbrace{\frac{\partial c}{\partial t} \sim 0}_{\substack{\text{quasi-steady}\\ \text{approximation}}} = \underbrace{\frac{D}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right)}_{\text{nutrient diffusion}} - \underbrace{\Gamma H(r-R_N)}_{\substack{\text{rate of nutrient}\\ \text{consumption}}}, \tag{3.1}$$

$$\underbrace{\frac{1}{3}\frac{d}{dt}(R^3) = R^2\frac{dR}{dt}}_{\substack{\text{rate of change}\\ \text{of tumour volume}}} = \underbrace{\int_0^R S(c)H(r-R_N)r^2 dr}_{\text{net rate of cell proliferation}} - \underbrace{\int_0^R N(c)H(R_N-r)r^2 dr}_{\text{rate of necrotic cell death}} \tag{3.2}$$

subject to the following boundary and initial conditions:

$$\frac{\partial c}{\partial r} = 0 \quad \text{on } r = 0, \tag{3.3}$$

$$c = c_\infty \quad \text{on } r = R(t), \tag{3.4}$$

$$c, \frac{\partial c}{\partial r} \quad \text{continuous across } r = R_{\mathrm{N}}(t), \tag{3.5}$$

$$c(R_{\mathrm{N}}(t), t) = c_{\mathrm{N}}, \tag{3.6}$$

$$R(t=0) = R_0. \tag{3.7}$$

In (3.1), we have adopted the usual approximation $\partial c/\partial t = 0$ which arises because a typical nutrient diffusion timescale is much shorter than the timescale of interest, the tumour doubling timescale [1, 16]. Since the necrotic core contains only dead cells, it is assumed that there is no nutrient consumption there. Denoting the Heaviside step-function by $H(\cdot)$ ($H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x \le 0$), we note that the term $\Gamma H(r - R_{\mathrm{N}})$ assumes that nutrient consumption by proliferating cells occurs at a constant rate Γ.

In (3.2), $S(c)$ and $N(c)$ denote cell proliferation and cell loss due to necrosis, respectively. Cell proliferation is assumed to be localised in the annulus $R_{\mathrm{N}} < r < R$ and represents the balance between mitosis (cell proliferation) and apoptosis (programmed or natural cell death). Necrotic cell loss is localised to nutrient-starved regions where $c < c_{\mathrm{N}}$. As simple examples, we fix

$$S(c) = s(c - \tilde{c}) \qquad \text{and} \qquad N(c) = 3s\lambda. \tag{3.8}$$

In (3.8), s and $\tilde{c}$ are positive constants and we interpret sc as the cell proliferation rate and $s\tilde{c}$ as the apoptotic rate. Additionally, $3s\lambda$ denotes the assumed constant rate of cell death within the necrotic core.

Substituting with $S(c)$ and $N(c)$ from (3.8), equation (3.2) becomes

$$R^2 \frac{dR}{dt} = s\int_0^R (c - \tilde{c}) r^2 dr - s\lambda R_{\mathrm{N}}^3. \tag{3.9}$$

Equations (3.3)–(3.7) close the model equations (3.1) and (3.9). Equation (3.3) is a symmetry condition, (3.4) fixes the nutrient concentration on the tumour boundary at the constant value c_∞, equation (3.5) ensures continuity of c and $\partial c/\partial r$ across $r = R_{\mathrm{N}}$, (3.6) defines R_{N} implicitly, and (3.8) defines the initial tumour radius.

3.2. Model simplifications. Due to the simple expressions used above to describe the cell proliferation rate and other kinetic terms, it is possible to write down explicit expressions for c, R, and R_{N}. For example, setting $D = 1$ in equation (3.1), when the tumour comprises proliferating cells only ($R_{\mathrm{N}} = 0$), we have

$$c(r,t) = c_\infty - \frac{\Gamma}{6}(R^2 - r^2), \qquad \frac{dR}{dt} = \frac{sR}{3}\left(c_\infty - \tilde{c} - \frac{\Gamma R^2}{15}\right). \tag{3.10}$$

Similarly, when the tumour comprises necrotic and proliferating cells ($R_{\mathrm{N}} > 0$), we have

$$c(r,t) = \begin{cases} c_{\mathrm{N}} & \text{for } r \in (0, R_{\mathrm{N}}(t)), \\ c_\infty - \frac{\Gamma}{6}(R^2 - r^2) + \frac{\Gamma R_{\mathrm{N}}^3}{3}\left(\frac{1}{r} - \frac{1}{R}\right) & \text{for } r \in (R_{\mathrm{N}}(t), R(t)), \end{cases}$$

where $R_{\mathrm{N}}(t)$ is defined in terms of $R(t)$ via the following algebraic expression:

$$c_\infty - c_{\mathrm{N}} = \frac{\Gamma}{6}(R^2 - R_{\mathrm{N}}^2) - \frac{\Gamma R_{\mathrm{N}}^2}{3R}(R - R_{\mathrm{N}}), \tag{3.11}$$

and the evolution of the outer tumour boundary is now specified in terms of $R(t)$ and $R_{\mathrm{N}}(t)$:

$$\frac{3R^2}{s}\frac{dR}{dt} = \left(c_\infty - \tilde{c} - \frac{\Gamma R^2}{6} - \frac{\Gamma R_{\mathrm{N}}^3}{3R}\right)(R^3 - R_{\mathrm{N}}^3) + \frac{\Gamma}{10}(R^5 - R_{\mathrm{N}}^5) + \frac{\Gamma R_{\mathrm{N}}^3}{2}(R^2 - R_{\mathrm{N}}^2) - N R_{\mathrm{N}}^3. \tag{3.12}$$

3.3. Model analysis. In real applications, the experimentally-determined functional forms used to describe the cell proliferation and death rates are more complicated than those used above. In such cases the model will not, in general, admit analytical solutions and one must resort to numerical methods. Whilst numerical solutions are of value, they often obscure the way in which different mechanisms interact. Insight into the behaviour of the system can be gained by studying special cases for which the model equations simplify. In this section, we illustrate how simple analytical techniques can be used to achieve this goal by studying three cases of physical interest:

(1) when the tumour is very small ($0 = R_{\mathrm{N}} < R \ll 1$),
(2) directly after the onset of necrosis ($0 < R_{\mathrm{N}} \ll R \sim O(1)$),
(3) when the width of the proliferating rim is small ($0 < R - R_{\mathrm{N}} \ll 1$).

Small tumour analysis. When $R_{\mathrm{N}} = 0$, equation (3.10) describes the tumour's growth rate and when $0 < R \ll 1$, it reduces to give

$$\frac{dR}{dt} = \frac{sR}{3}(c_\infty - \tilde{c}) + O(R^3). \tag{3.13}$$

Equation (3.13) shows how, when the tumour is small, its growth rate depends upon the balance between the rate of cell loss due to apoptosis and the rate of cell proliferation, the latter effect being controlled by the external nutrient concentration. In particular, if $\tilde{c} > c_\infty$ the tumour shrinks and we say that the tumour-free solution ($R = 0$) is stable. Conversely, if $\tilde{c} < c_\infty$ the trivial solution is unstable and growth is predicted.

Onset of necrosis. In order to characterise the tumour's growth after the onset of necrosis we introduce the small parameter $0 < \epsilon \ll 1$ and seek solutions to equations (3.11) and (3.12) of the form

$$R \sim R_0 + \epsilon R_1 + \epsilon^2 R_2 \qquad \text{and} \qquad R_{\mathrm{N}} \sim \epsilon R_{N1}. \tag{3.14}$$

Substituting with (3.14) in (3.11) and equating to zero terms of $O(\epsilon^n)$ yields the following expressions:

$$R_0^2 = \frac{6(c_\infty - c_N)}{\Gamma}, \qquad R_1 = 0, \qquad R_2 = \frac{3R_{N1}^2}{2R_0},$$

in which R_0^2 is the tumour radius at the onset of necrosis. These results show that when the necrotic core is small, variations in $R(t)$ are much smaller than variations in $R_N(t)$ ($O(\epsilon^2)$ versus $O(\epsilon)$). Substituting for R_2 and R_N in (3.12) yields a differential equation for R_2 which is singular in the limit as $\epsilon \to 0$. To regularise this equation, we introduce the short timescale $\tau = t/\epsilon^2$ and obtain

$$\frac{dR_2}{d\tau} = \frac{sR_0}{15}(3c_\infty - 5\tilde{c} + 2c_N) \equiv \Lambda.$$

This expression shows how, at the onset of necrosis, the tumour's growth rate depends upon the balance between cell proliferation, cell loss due to apoptosis and the nutrient concentration at which necrosis is initiated, but not on the rate of necrotic cell death λ. We conclude that when the necrotic core is small the tumour's growth rate is, to $O(\epsilon^2)$, independent of the rate of necrotic cell death and that persistence of the necrotic core will occur if $\Lambda > 0$.

Thin proliferating rim. Here we introduce $0 < \delta \ll 1$ and assume that

$$R_N = R(1 - \delta \bar{R}_{N1}) + O(\delta^2).$$

Substituting for R_N in (3.11) yields the following expression for the width of the proliferating rim $R - R_N \sim \delta R \bar{R}_{N1}$:

$$c_\infty - c_N = \frac{\Gamma}{2}\left(\delta R \bar{R}_{N1}\right)^2 = \frac{\Gamma}{2}(R - R_N)^2.$$

Using this result we deduce that a necessary condition for obtaining a tumour configuration with a thin proliferating rim is that $c_\infty = c_N + O(\delta^2)$, that is, the externally-supplied nutrient concentration must be very close in value to the nutrient concentration at which necrosis is initiated.

4. Asymmetric tumour growth and invasion. Improvements in medical imaging techniques, such as medical resonance imaging, have enabled clinicians to obtain high-quality pictures of tumours growing in vivo. These images indicate that, unlike avascular tumours grown in vitro, vascular tumours frequently possess irregular (fractal) boundaries [20]. Such information is now being used by clinicians to classify solid tumours, with irregular boundaries being characteristic of invasive and aggresive tumours. Given that avascular tumours are usually radially-symmetric, a natural question concerns the manner in which the symmetry of the tumours is broken and the fractal structures develop. Following [6, 16], in this section we show how the models presented in Section 3 can be extended to allow asymmetric growth.

New aspects of the model presented in this section are variables that describe cell velocity and pressure within the tumour and the inclusion of surface tension effects. These changes are motivated as follows. If the tumour is assumed incompressible, cell proliferation and death generate pressure differentials within the tumour which, in turn, cause cell motion, with cells moving from regions of high cell proliferation to regions of cell death. If we view the restraining force that maintains the tumour's compactness as a surface tension that acts on the tumour boundary, then the rate of growth of the tumour depends on the balance between the expansive force caused by the net cell proliferation rate within the tumour and the restraining force caused by surface tension.

4.1. The model. The mathematical model that we study is presented below in dimensionless form (for details, see [6, 16]). For simplicity, we assume that the tumour contains only live cells and, hence, that there is no necrotic core. The key variables are the nutrient concentration c, the tumour cell velocity $\boldsymbol{v}$, the pressure p and the position of the tumour boundary $\boldsymbol{r} = \boldsymbol{R}$, and their evolution is governed by the following system of partial differential equations:

$$0 = \nabla^2 c - \Gamma, \tag{4.1}$$

$$\boldsymbol{v} = -\mu \nabla p, \tag{4.2}$$

$$\nabla \cdot \boldsymbol{v} = -\mu \nabla^2 p = s(c - \tilde{c}), \tag{4.3}$$

$$\boldsymbol{n} \cdot \frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \cdot \boldsymbol{n} = -\mu \nabla p \cdot \boldsymbol{n} \quad \text{on } \boldsymbol{r} = \boldsymbol{R}. \tag{4.4}$$

Equation (4.1) generalises equation (3.1) for the case $R_{\mathrm{N}} = 0$. In (4.2), we treat the tumour's internal microstructure as a porous medium, and use Darcy's law to relate the velocity of the cells moving through the tumour to the internal pressure, the constant of proportionality μ denoting the motility of the tumour cells. Equation (4.3) expresses mass conservation within the tumour, when it is modelled as an incompressible fluid. For simplicity, we use the same proliferation rate $S(c)$ as adopted in Section 3 (see equation (3.8)). Also, since we can use Darcy's law ($\boldsymbol{v} = -\mu \nabla p$) to determine $\boldsymbol{v}$ from p, henceforth no explicit mention of $\boldsymbol{v}$ will be made. Finally, equation (4.4) defines the motion of points on the tumour boundary, where $\boldsymbol{r} = \boldsymbol{R}$. In (4.4), $\boldsymbol{n}$ denotes the unit outward normal vector.

The following boundary and initial conditions close equations (4.1)–(4.4):

$$\frac{\partial c}{\partial r} = 0 = \frac{\partial p}{\partial r} \quad \text{at } \boldsymbol{r} = \boldsymbol{0}, \tag{4.5}$$

$$c = c_\infty, \qquad p = 2\gamma\kappa \quad \text{on } \boldsymbol{r} = \boldsymbol{R}, \tag{4.6}$$

$$\boldsymbol{r} = \boldsymbol{R} \quad \text{prescribed at } t = 0. \tag{4.7}$$

Equations (4.5) ensure that c and p are bounded at $\boldsymbol{r} = \boldsymbol{0}$, whereas (4.6) fixes their values on the tumour boundary. The second term of equations (4.6) equates the pressure on the tumour boundary to the surface tension force $\gamma\kappa$ there. In (4.6), κ is the mean curvature and γ is a constant of proportionality. Equation (4.7) defines the initial size of the tumour.

4.2. Link with previous models. Under radial symmetry $r = R(t)$ denotes the tumour boundary and equations (4.3) and (4.4) become

$$-\frac{\mu}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) = s(c-\tilde{c}), \tag{4.8}$$

$$\frac{dR}{dt} = -\mu\frac{\partial p}{\partial r}\bigg|_{r=R}. \tag{4.9}$$

Integrating (4.8) once with respect to r and imposing (4.5) yields the following result:

$$-\mu\frac{\partial p}{\partial r} = \frac{1}{r^2}\int_0^r s(c-\tilde{c})r^2 dr.$$

Evaluating this expression at $r = R$ and substituting in (4.9) then supplies

$$R^2\frac{dR}{dt} = \int_0^R s(c-\tilde{c})r^2 dr.$$

This is identical to equation (3.2) of Section 3 for the case $R_{\mathrm{N}} = 0$, and generalises naturally to the case $R_{\mathrm{N}} > 0$. This result shows that our new model reduces to that of Section 3 under radial symmetry.

4.3. Linear stability analysis. Under radial symmetry, equations (4.1) and (4.3) admit solutions of the form

$$c = c_\infty - \frac{\Gamma}{6}(R^2 - r^2), \tag{4.10}$$

$$p = \frac{\gamma}{R} - \frac{s\Gamma}{120\mu}(R^2 - r^2)^2. \tag{4.11}$$

With $\partial/\partial t = 0$ in (4.4) it is possible to derive the following expression for the (nontrivial) steady-state tumour radius R:

$$\frac{\Gamma R^2}{15} = c_\infty - \tilde{c}. \tag{4.12}$$

As expected, this result can also be obtained by setting $d/dt = 0$ in equation (3.10).

It is straightforward to show that the nontrivial steady solution identified in (4.12) is stable with respect to time-dependent perturbations whereas the trivial solution is unstable. Hence, when investigating how symmetry-breaking perturbations affect the tumour's structure, we restrict attention to the nontrivial solution. We determine its response to asymmetric perturbations by introducing the small parameter $0 < \epsilon \ll 1$ and seeking solutions to (4.1)–(4.7) of the form

$$c_\epsilon \sim c(r) + \epsilon c_1(r,\theta,\phi,t),$$
$$p_\epsilon \sim p(r) + \epsilon p_1(r,\theta,\phi,t),$$
$$R_\epsilon \sim R + \epsilon R_1(\theta,\phi,t),$$

the time-independent $O(1)$-solutions (c, p, R) automatically satisfying equations (4.1)–(4.7) at leading order. Continuing to $O(\epsilon)$, we obtain equations governing the evolution of (c_1, p_1, R_1):

$$\nabla^2 c_1 = 0 = \mu \nabla^2 p_1 + s c_1, \tag{4.13}$$

$$\frac{\partial R_1}{\partial t} = -\mu \left(\frac{\partial p_1}{\partial r} + R_1 \frac{d^2 p}{dr^2} \right)_{r=R}, \tag{4.14}$$

with

$$\frac{\partial c_1}{\partial r} = 0 = \frac{\partial p_1}{\partial r} \quad \text{at } r = 0, \tag{4.15}$$

$$c_1(R, \theta, \phi, t) = -R_1 \frac{dc}{dr}\bigg|_{r=R}, \tag{4.16}$$

$$p_1(R, \theta, \phi, t) = -\frac{\gamma}{R^2}(2R_1 + \mathscr{L}(R_1))\bigg|_{r=R} - R_1 \frac{dp}{dr}\bigg|_{r=R}, \tag{4.17}$$

$$R_1(\theta, \phi, 0) = R_1^*(\theta, \phi), \quad \text{prescribed.} \tag{4.18}$$

In (4.17), $\mathscr{L}(\cdot)$ denotes the angular component of the Laplacian operator

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{\mathscr{L}(f)}{r^2}, \qquad \mathscr{L}(f) = \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}.$$

To determine the $O(\epsilon)$ contributions to the normal derivative of the pressure and the curvature on the tumour boundary, we have used results presented in the appendices of [8]. Following [6] we seek separable solutions to (4.13)–(4.18) of the form

$$\begin{aligned} c_1 &= \sum \chi_{lm}(t) r^l Y_{lm}(\theta, \phi), \\ p_1 &= \sum \left\{ \pi_{lm}(t) - \frac{s \chi_{lm} r^2}{2\mu(2l+3)} \right\} r^l Y_{lm}(\theta, \phi), \end{aligned} \tag{4.19}$$

$$R_1 = \sum \rho_{lm}(t) Y_{lm}(\theta, \phi), \tag{4.20}$$

where the spherical harmonics Y_{lm} satisfy $\mathscr{L}(Y_{lm}) = -l(l+1)Y_{lm}$ so that $\nabla^2 (r^l Y_{lm}) = 0$. Expressions relating the coefficients χ_{lm} and π_{lm} to ρ_{lm} are obtained by imposing (4.16) and (4.17) and exploiting the orthogonality of the spherical harmonics:

$$\chi_{lm} R^l = -\frac{\Gamma R}{3} \rho_{lm}, \qquad \pi_{lm} R^l = \frac{\gamma \rho_{lm}}{R^2}(l-1)(l+2) + \frac{s \chi_{lm} R^{l+2}}{2\mu(2l+3)}. \tag{4.21}$$

Substituting with (4.21) in (4.14), we deduce that

$$\frac{1}{s\rho_{lm}} \frac{d\rho_{lm}}{dt} = (l-1) \left(\frac{2s\Gamma R^2}{15(2l+3)} - \frac{\gamma\mu}{R^3} l(l+2) \right). \tag{4.22}$$

From (4.22), we note that all modes evolve independently: There is no coupling between the modes. Also, the system is insensitive to perturbations involving

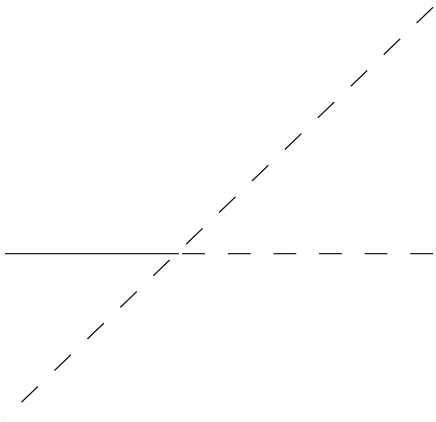

FIGURE 4.1. A schematic diagram of a transcritical bifurcation, showing how the stability of the solutions varies with the bifurcation parameter a: Stable solutions (solid line); unstable solutions (dashed line).

the first spherical harmonics. This is natural since modes having $l = 1$ correspond to a translation of the coordinate axes. It is also clear that the evolution of the modes is independent of m.

4.4. Model extensions. A weakness of the linear stability analysis presented above is its inability to distinguish between the growth rates of the $(2l+1)$ different spherical harmonics $Y_{lm}(\theta,\phi)$ associated with a particular value of l from a given family. In addition, the analysis fails to show how different modes interact. Recently, Byrne [4] has used weakly nonlinear analysis to resolve these problems in a neighbourhood of the bifurcation point at which the radially symmetric solution loses stability to spherical harmonics of order two $(Y_{2m}(\theta,\phi))$. It emerges that the system dynamics in a neighbourhood of this bifurcation point are governed by a transcritical bifurcation with D_3 symmetry [14]. Specifically, using a to denote the bifurcation parameter, the system dynamics can be written in the following form:

$$\frac{dz}{dt} = az - \bar{z}^2 \quad (z \in C), \tag{4.23}$$

where z is related to the amplitudes of the spherical harmonics $Y_{2m}(\theta,\phi)$. Now, this codimension zero bifurcation is its own universal unfolding and the pattern represented by its fixed points is shown in Figure 4.1. Interpreting these results, we deduce that in a neighbourhood of the bifurcation point none of the branches of asymmetric solutions are stable. Further, the identification of any asymmetric branches that are stable would necessitate the construction of numerical solutions of the original, fully-nonlinear problem.

Work in progress [8] involves extending the model of solid tumour growth originally developed by Greenspan [16] to include a quadratic overcrowding term in the tumour cell proliferation rate:

$$S(c) \longrightarrow s(c - \tilde{c} - \xi c^2).$$

We believe that, by using weakly nonlinear techniques, it should be possible to show that the presence of this new term results in an unfolding of the transcritical bifurcation of the form

$$\frac{dz}{dt} = az - \bar{z}^2 + b|z|^2 z, \tag{4.24}$$

where the parameter $b = b(\xi)$ vanishes when the overcrowding term is neglected ($\xi = 0$). Analysis of (4.24) leads naturally to the identification of conditions under which a branch of (locally) stable asymmetric solutions may exist. To illustrate this point and for comparison with Figure 4.1, in Figure 4.2, we sketch (4.24) when $b > 0$.

A key modelling assumption of the model of tumour growth discussed in this section was that Darcy's law provides a reasonable description of cell motion within a tumour. Recent work by Landman and Please [21] shows how Darcy's law, which is more commonly associated with flow through porous media, emerges naturally as a description of tumour cell migration.

Finally, there are many other ways in which our model of asymmetric tumour growth could be extended. For example, there are typically many different growth factors present within the tumour, (e.g., both oxygen and glucose are freely-diffusible nutrients that are important for sustaining viable tumour cells). In addition the tumour may contain other growth factors which are internal to cell and hence transported with them rather than diffusing freely. For example, cyclin is an intracellular chemical which must be present in sufficiently high levels in order for cell proliferation to occur [24].

5. Multiple cell populations. In the early to mid-eighties, Dorie et al. [9, 10] considered whether parameters relevant to tumour growth kinetics could be estimated by tracking the movement of labelled cells within a growing spheroid. In their experiments, inert polystyrene microspheres and radioactively labelled tumour cells were allowed to adhere to the outer surface of multicell spheroids. Changes in the distributions of the labelled particles within the tumour were studied and typical results are presented in [33, Figure 1]. These results show that the labelled microspheres migrate towards the centre of the tumour in a wavelike manner, with no microspheres discernible at the tumour boundary after 4 days. When the tumour cells are labelled instead, once again wavelike invasion is observed during the initial stages of migration. However, unlike the microspheres, by day 4 the labelled cells appear to be fairly evenly distributed throughout the tumour volume.

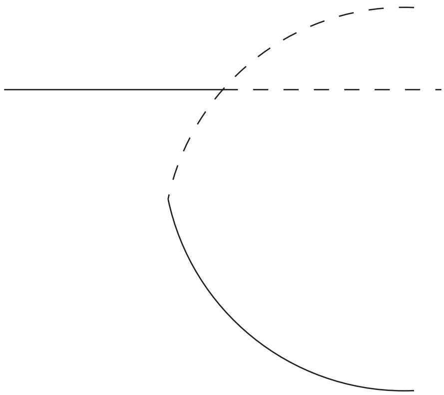

FIGURE 4.2. A schematic diagram of an unfolded transcritical bifurcation, showing how the stability of the solutions varies with the bifurcation parameter a for a fixed value of $b \neq 0$. Stable solutions on the nontrivial branch correspond to assymetric tumour configurations which are stable with respect to small time-dependent perturbations: Stable solutions (solid line); unstable solutions (dashed line).

A key feature of the experiments of Dorie et al., from a modelling viewpoint, is the presence of two distinct cell populations within the tumour, unlabelled and labelled tumour cells. In order to describe Dorie's experiments, we now show how the model presented in Section 4 can be extended to encompass the growth of tumours containing multiple cell types (for further details, see [33, 34]).

5.1. The model. The model that we study is presented below in dimensionless form (for details, see [33, 34]). The key physical variables are the densities of the unlabelled and labelled cells, which are denoted by n and m, respectively, the cell velocity $\boldsymbol{v}$, the nutrient concentration c and the tumour boundary $\boldsymbol{r} = \boldsymbol{R}$. For simplicity, we assume that the tumour consists of proliferation cells only and, hence, that there is no necrosis. Equations governing the evolution of $n, m, \boldsymbol{v}, c$, and $\boldsymbol{R}$ can be written as follows:

$$\frac{\partial n}{\partial t} + \underbrace{\nabla \cdot (\boldsymbol{v} n)}_{\substack{\text{convective} \\ \text{transport}}} = \underbrace{\mu_n \nabla^2 n}_{\substack{\text{random} \\ \text{motion}}} + \underbrace{s(c - \tilde{c})n}_{\substack{\text{net proliferation} \\ \text{rate}}}, \tag{5.1}$$

$$\frac{\partial m}{\partial t} + \nabla \cdot (\boldsymbol{v} m) = \mu_m \nabla^2 m + [s(c - \tilde{c})m]_+, \tag{5.2}$$

$$\nabla \cdot \boldsymbol{v} = (\mu_m - \mu_n)\nabla^2 m + s(c - \tilde{c})(1 - m) + [s(c - \tilde{c})m]_+, \tag{5.3}$$

$$0 = \nabla^2 c - \Gamma n - [\Gamma m]_+, \tag{5.4}$$

$$\boldsymbol{n} \cdot \frac{d\boldsymbol{R}}{dt} = \boldsymbol{n} \cdot \boldsymbol{v} \quad \text{on } \boldsymbol{r} = \boldsymbol{R}. \tag{5.5}$$

In equations (5.1) and (5.2) μ_n and μ_m are the random motility coefficients for the two cell types. Equation (5.3) is obtained by adding (5.1) and (5.2) under the assumption that there are no voids and hence that $n+m=$ constant ($\equiv 1$, by suitable rescaling) throughout the tumour volume. Equations (5.4) and (5.5) for c and $\boldsymbol{R}$ are analogues of equations (4.1) and (4.4).

In their experiments, Dorie et al. used two types of labelled cells: Inert microspheres and tumour cells. The functional forms for the rates at which the labelled cells proliferate and consume nutrient will depend crucially on which label is being used. We assume that tumour cells proliferate and consume nutrient in the same way as unlabelled tumour cells and that, being inert, labelled microspheres neither consume nutrient nor proliferate. We distinguish between these cases by defining

$$[f(x)]_+ = \begin{cases} f(x) & \text{tumour cells labelled,} \\ 0 & \text{microspheres labelled.} \end{cases}$$

We remark that, due to the no voids assumption, one of the dependent variables (either n or m) can be eliminated from our model. In the present context where we are focussing on the migration of labelled cells, it is appropriate to eliminate $n = 1-m$. We postpone a discussion of the boundary and initial conditions that are needed to close our model until it has been further simplified (see below).

Model simplification. Before continuing, we reformulate our model under the additional assumption of one-dimensional cartesian geometry, with x denoting distance from the tumour centre. It is also convenient to reorder the governing equations. In this way, we obtain the following simplified model:

$$0 = \frac{\partial^2 c}{\partial x^2} - \Gamma(1-m) - [\Gamma m]_+, \tag{5.6}$$

$$\frac{\partial v}{\partial x} = (\mu_m - \mu_n)\frac{\partial^2 m}{\partial x^2} + s(c-\tilde{c})(1-m) + [s(c-\tilde{c})m]_+, \tag{5.7}$$

$$\frac{dR}{dt} = v(R,t), \tag{5.8}$$

$$\begin{aligned} \frac{\partial m}{\partial t} + v\frac{\partial m}{\partial x} = {} & [\mu_m - (\mu_m - \mu_n)m]\frac{\partial^2 m}{\partial x^2} \\ & + (1-m)[s(c-\tilde{c})m]_+ - s(c-\tilde{c})m(1-m). \end{aligned} \tag{5.9}$$

Equations (5.6)–(5.9) are closed by imposing the following boundary and initial conditions:

$$c = c_\infty \quad \text{at } x = R(t), \tag{5.10}$$

$$\frac{\partial c}{\partial x} = 0 \quad \text{at } x = 0, \tag{5.11}$$

$$v = 0 \quad \text{at } x = 0, \tag{5.12}$$

$$\mu_m \frac{\partial m}{\partial x} - vm = 0 \quad \text{at } x = 0, R(t), \tag{5.13}$$

$$m(x,0) = m_{\text{in}}(x) \qquad \text{and} \qquad R(0) = R_0 \quad \text{prescribed.} \tag{5.14}$$

In equation (5.10), c_∞ is the external nutrient concentration, and so we assume that the nutrient concentration is continuous across $x = R(t)$. Equations (5.11), (5.12), and the first equation of (5.13) reflect the assumed symmetry of the tumour about $x = 0$. The second equation of (5.13) states that the flux of labelled cells across the outer tumour boundary is zero. Equations (5.14) define the initial distribution of labelled cells within the tumour and also its initial radius.

5.2. Link with previous models. If we assume that the tumour comprises a single cell type, and hence that $m = 0$ without loss of generality, then equation (5.9) is trivially satisfied and equations (5.6)-(5.8) reduce to the one-dimensional cartesian analogue of equations (4.1)–(4.4). This shows how the models of Sections 4 and 5 are related: The model of Section 4 corresponds to the special case of our current model for which only one cell type is present.

More generally, we remark that, in contrast to the analysis of Section 4, here we focus on one-dimensional tumour growth. Consequently, the velocity $\boldsymbol{v}$ possesses only one nonzero component and the no voids assumption ($n + m = 1$), together with equations (5.2)–(5.5), is sufficient to determine its evolution. In order to consider more complex (asymmetric) growth patterns, we would need to include additional, constitutive equations (such as Darcy's law) to determine the other nonzero components of the cell velocity.

5.3. Numerical results. Figures 5.1, 5.2, and 5.3 show how the nutrient concentration, cell velocity, and labelled cell density evolve when inert polystyrene microspheres are labelled. From Figures 5.1 and 5.2, we note that $c(x,t)$ and $v(x,t)$ rapidly attain steady-state profiles, with the nutrient concentration increasing monotonically with x and the velocity field having an internal minimum near $x = 0.6R(t)$. We note also that $v(x,t) \le 0$ and, hence, that convective transport drives cells towards $x = 0$. Figure 5.3 shows how the microspheres are internalised within the tumour. After spreading out ($t = 1$), they are transported, in a wavelike manner, towards the centre of the tumour ($t = 2$) where they eventually aggregate ($t = 3, 4, 5$).

We remark that the numerical results presented in Figures 5.1–5.3 show how the dependent variables c, v, and m vary over time and the scaled distance $0 \le \rho = x/R(t) \le 1$. Replotting these solutions in terms of $0 \le x \le R(t)$ does not alter the qualitative form of the solutions: It simply alters the width of the region containing the microspheres. Further, the tumour radius R rapidly attains a time-independent equilibrium value (results not shown). For all subsequent times the spatial transformation is the same. For these reasons, the numerical results were presented in terms of the scaled variables.

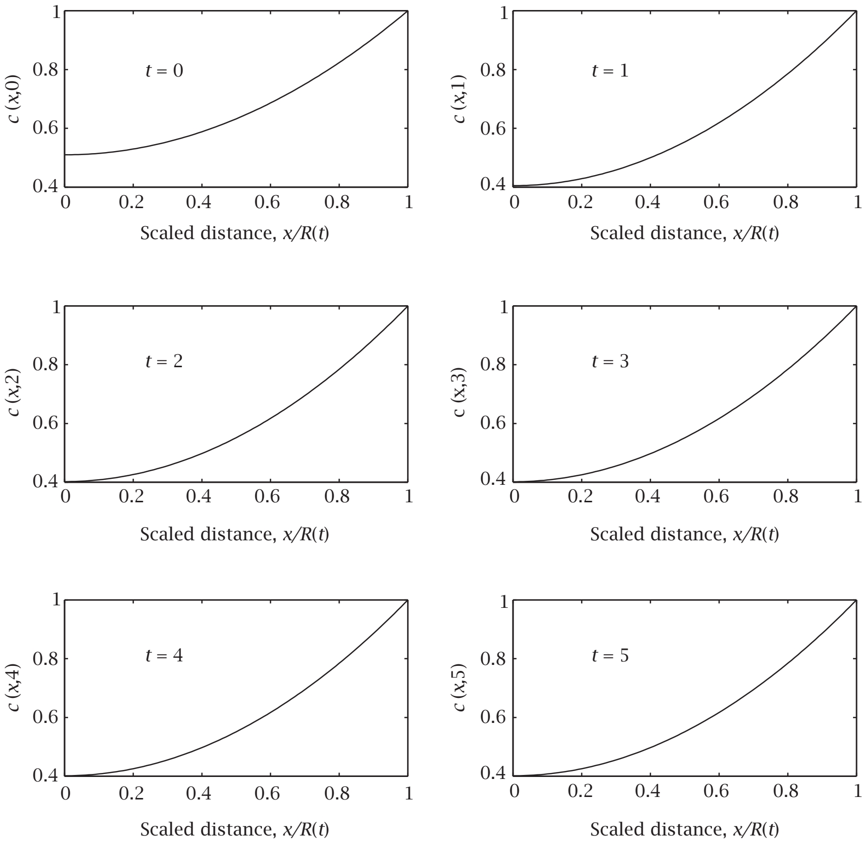

FIGURE 5.1. Numerical results describing the internalisation of labelled polystyrene microspheres within a growing tumour. Evolution of nutrient concentration at times $t = 0, 1, 2, 3, 4, 5$ (in dimensionless units) after internalisation. Parameter values: $s = 10$, $c_\infty = 1.0$, $d = 6.0$, $c_{\text{nec}} = 0.1$, $\Gamma = 0.5$, $\mu_n = 0.01$, $\mu_m = 0.008$, $R(0) = 1.4$.

Figure 5.4 shows how labelled tumour cells migrate within the tumour. The evolution of c and v are omitted since they are similar to the profiles depicted in Figures 5.1 and 5.2. Comparing Figures 5.3 and 5.4, we note that initially $(0 < t < 2)$ the speed of migration of the invading cells is the same for the microspheres and tumour cells. However, once they have penetrated to the centre of the tumour, the labelled tumour cells redistribute themselves evenly throughout the tumour.

5.4. Analytical results. Here we focus on the initial wavelike migration of the labelled cells and their limiting, or long-time, behaviour. Guided by the above numerical results, and in order to simplify the analysis, we assume that c, v, and R have evolved to steady-state profiles. To render the analysis

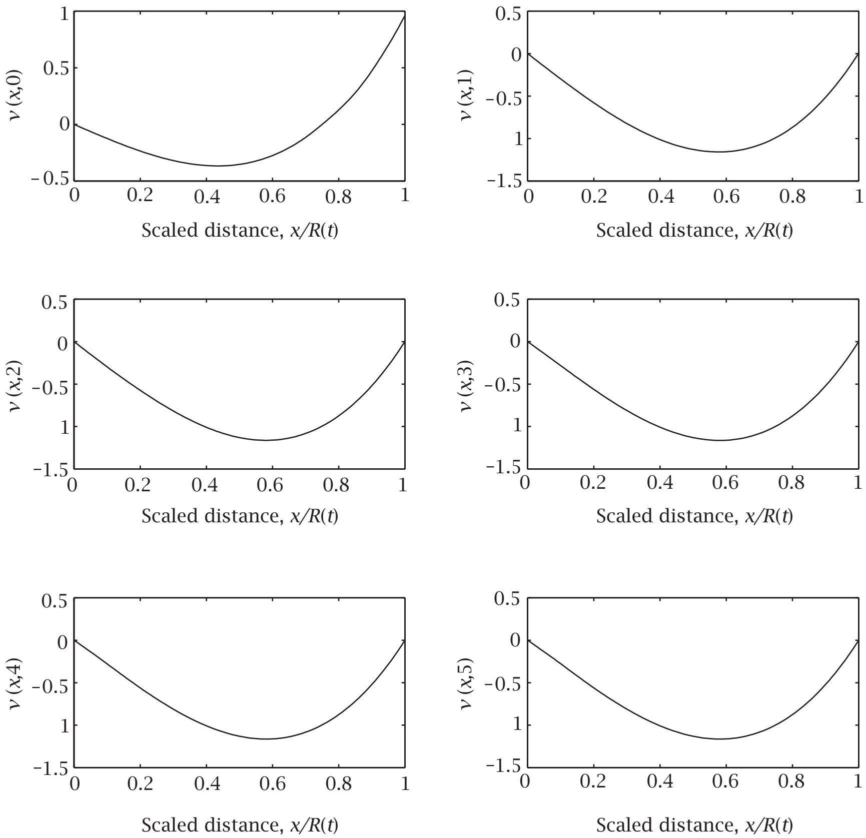

FIGURE 5.2. Numerical results describing the internalisation of labelled polystyrene microspheres within a growing tumour. Evolution of cell velocity at times $t = 0, 1, 2, 3, 4, 5$ (in dimensionless units) after internalisation. Parameter values: $s = 10$, $c_\infty = 1.0$, $d = 6.0$, $c_{\text{nec}} = 0.1$, $\Gamma = 0.5$, $\mu_n = 0.01$, $\mu_m = 0.008$, $R(0) = 1.4$.

tractable, we focus on a special case for which the number of labelled particles is small and may be characterised by the small parameter $0 < \epsilon \ll 1$. Specifically, we seek solutions to equations (5.6)–(5.14) of the form

$$\begin{aligned} c &= c_0(x) + O(\epsilon), & v &= v_0(x) + O(\epsilon), \\ R &= R_0 + O(\epsilon), & m &= \epsilon m_0(x,t) + O(\epsilon^2). \end{aligned} \tag{5.15}$$

By equating to zero coefficients of $O(1)$ in equations (5.6)–(5.8) equations for c_0, v_0, and R_0 are obtained. Since $m \sim O(\epsilon)$, these equations are independent of m_0 and, hence, their form is be unaffected by the labelling system being employed. The simplicity of the kinetic terms means that equations (5.6) and (5.7) are integrable at leading order and that c_0 and v_0 are defined in terms of

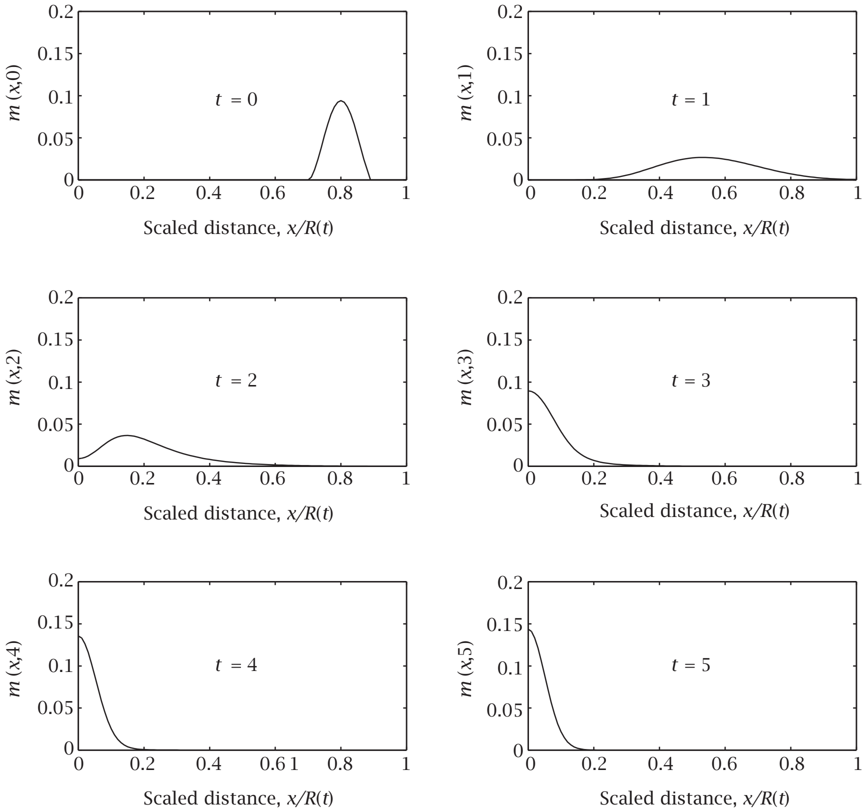

FIGURE 5.3. Numerical results describing the internalisation of labelled polystyrene microspheres within a growing tumour. Evolution of labelled microspheres at times $t = 0, 1, 2, 3, 4, 5$ (in dimensionless units) after internalisation. Parameter values: $s = 10$, $c_\infty = 1.0$, $d = 6.0$, $c_{\text{nec}} = 0.1$, $\Gamma = 0.5$, $\mu_n = 0.01$, $\mu_m = 0.008$, $R(0) = 1.4$.

R_0 as follows:

$$c_0(x) = c_\infty - \frac{\Gamma}{2}(R_0^2 - x^2), \qquad v_0(x) = sx\left(c_\infty - \tilde{c} - \frac{\Gamma}{2}\left(R_0^2 - \frac{x^2}{3}\right)\right). \tag{5.16}$$

Using (5.8), with $dR_0/dt = 0$, we deduce that

$$0 = R_0\left(c_\infty - \tilde{c} - \frac{\Gamma R_0^2}{3}\right), \tag{5.17}$$

and hence that for a nontrivial solution

$$R_0 = \sqrt{\frac{3}{\Gamma}(c_\infty - \tilde{c})}.$$

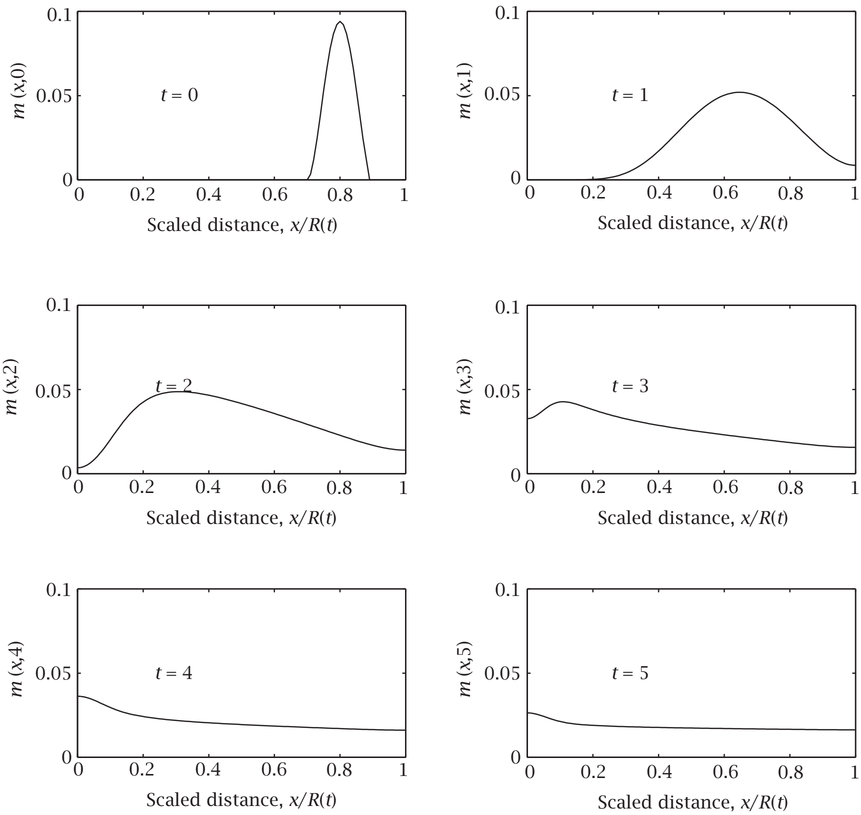

FIGURE 5.4. Numerical results describing the internalisation of labelled tumour cells within a growing tumour. The evolution of the nutrient concentration, the cell velocity, and the tumour radius (not shown) are qualitatively similar to those presented in Figure 2.2. The labelled tumour cells gradually distribute themselves uniformly throughout the tomour volume. Parameter values: $s = 10$, $c_\infty = 1.0$, $d = 6.0$, $c_{\text{nec}} = 0.1$, $\Gamma = 0.5$, $\mu_n = 0.01$, $\mu_m = 0.008$, $R(0) = 1.4$.

Comparing the above expression for R_0 with equations (3.10) and (4.12) shows clearly the effect that posing our tumour growth in different geometries has on the steady-state radius.

With R_0 determined by (5.17), the expression for v_0 can be rewritten as

$$v_0(x) = -\frac{s\Gamma x}{6}(R_0^2 - x^2).$$

Using this result, it is possible to show that $v_0(x)$ has a minimum turning point at $x = R_0/\sqrt{3} \sim 0.58R_0$. This is in good agreement with the numerical results presented in Figure 5.2.

Equation (5.9) is trivially satisfied at $O(1)$ when $m = \epsilon m_0(x)$, and therefore we must equate to zero coefficients of $O(\epsilon)$ to determine $m_0(x)$. In this way, we find that

$$\frac{\partial m_0}{\partial t} + v_0 \frac{\partial m_0}{\partial x} = \mu_m \frac{\partial^2 m_0}{\partial x^2} + F(c_0, m_0), \tag{5.18}$$

where

$$F(c_0, m_0) = \begin{cases} -s(sc_0 - \tilde{c})m_0 = \dfrac{dv_0}{dx} m_0 & \text{labelled microspheres,} \\ 0 & \text{labelled tumour cells.} \end{cases} \tag{5.19}$$

Wavelike migration. Both microspheres and tumour cells are much larger than nutrient molecules and, hence, their random motility coefficients will be much smaller than that of the nutrient so that in (5.18), $0 < \mu_m \ll 1$. When considering the initial, wavelike internalisation of the labelled particles we exploit this fact by neglecting random motion in equation (5.18). Then, to leading order, their initial migration can be described approximately by the following nonlinear wave equation:

$$\frac{\partial m_0}{\partial t} + v_0 \frac{\partial m_0}{\partial x} = F(c_0, m_0). \tag{5.20}$$

From (5.20), it is clear that the propagation of the labelled cells is driven by the velocity field. This, in turn, is created by differential cell proliferation and death rates. Now, for the asymptotic limit under consideration, v_0 is independent of the choice of cell labelling. Hence we deduce that the speed of migration of the labelled cells is also independent of whether microspheres or tumour cells are used. This prediction is in good agreement with the numerical results (see Figures 5.3 and 5.4). Given that (5.20) is independent of both μ_m and μ_n, the random motility coefficients for the labelled and unlabelled cells, respectively, we deduce that the initial migration of the labelled cells will be unaffected by the size of the labelled particles. This is consistent with the experimental results of Dorie et al. [9, 10].

It is possible to construct explicit solutions to equation (5.20) using the method of characteristics [33, 35]. When the microspheres are labelled, equation (5.20) admits the following solution:

$$m_0(x,t) = m_{\text{in}}\left(\frac{x}{\sqrt{Q}}\right) \exp\left\{-\frac{s\Gamma R_0^2 t}{3}\right\} Q^{-3/2}, \tag{5.21}$$

where

$$Q(x,t) = \left(1 - \frac{x^2}{R_0^2}\right) \exp\left\{-\frac{s\Gamma R_0^2 t}{3}\right\} + \frac{x^2}{R_0^2}.$$

By contrast, when the tumour cells are labelled

$$m_0(x,t) = m_{\text{in}}\left(\frac{x}{\sqrt{Q}}\right). \tag{5.22}$$

In Figure 5.5, we sketch these approximate solutions. From Figure 5.5(a), we observe how the pulse of labelled cells decreases in width as it travels towards the centre of the tumour. In order to preserve the total number of labelled cells the height of the pulse increases. Eventually, the height of this peak becomes so large that our asymptotic expansion ceases to be valid. Given that the initial data has compact support ($m_0(x,0) > 0 \forall x \in (x_L, x_T)$), an alternative way to see this is by following the characteristics through the endpoints of the domain of compact support. It is possible to show that $x_L(t)$, $x_T(t) \sim O(\exp\{-s\Gamma R_0^2 t/6\})$ as $t \to \infty$, that is, the width of the domain of compact support shrinks exponentially [33]. At the same time the height of the peak must increase in order to preserve the total number of microspheres within the tumour: $m_0 \sim O(\exp\{-s\Gamma R_0^2 t/6\})$. Eventually, the spatial gradients become so large that our approximate solution breaks down and second order effects, such as random motility, can no longer be neglected.

Figure 5.5(b) depicts the corresponding situation when the tumour cells are labelled. As in Figure 5.5(a), the width of the pulse of labelled cells decreases as it migrates towards the centre of the tumour. However, since the total number of cells is no longer preserved, our approximation predicts that the total number of cells actually falls. As in Figure 5.5(a), the approximation breaks down when the width of the peak becomes so small that terms involving second spatial derivatives, such as random motility, can no longer be neglected.

Limiting distributions of labelled cells. From the numerical simulations of Section 3 (see Figures 5.3 and 5.4), it is clear that for both labelling systems the particles eventually adopt a well-defined structure. In this section, we focus on the behaviour of equation (5.18) in the limit as $t \to \infty$. As our analysis will show, retention of the random motility term is important in this asymptotic limit. In order to investigate the long-time behaviour of the labelled particles, we now consider equation (5.18), with $\partial/\partial t = 0$,

$$0 = \mu_m \frac{d^2 m_0}{dx^2} - v_0 \frac{dm_0}{dx} + F. \tag{5.23}$$

When the labelled cells are inert microspheres equation (5.23) can be rewritten

$$0 = \frac{d}{dx}\left(\mu_m \frac{dm_0}{dx} - v_0 m_0\right).$$

Integrating with respect to x and imposing (5.13) yields

$$\frac{dm_0}{dx} = \frac{v_0 m_0}{\mu_m} = -\frac{s\Gamma x}{6\mu_m}(R_0^2 - x^2)m_0. \tag{5.24}$$

Since, for physically realistic solutions, $m_0 \geq 0$ we deduce that at equilibrium $m_0(x)$ is monotonically decreasing with x. Integrating again with respect to x yields

$$m_0(x) = M^* \exp\left\{-\frac{s\Gamma}{24\mu_m}x^2(2R_0^2 - x^2)\right\}, \tag{5.25}$$

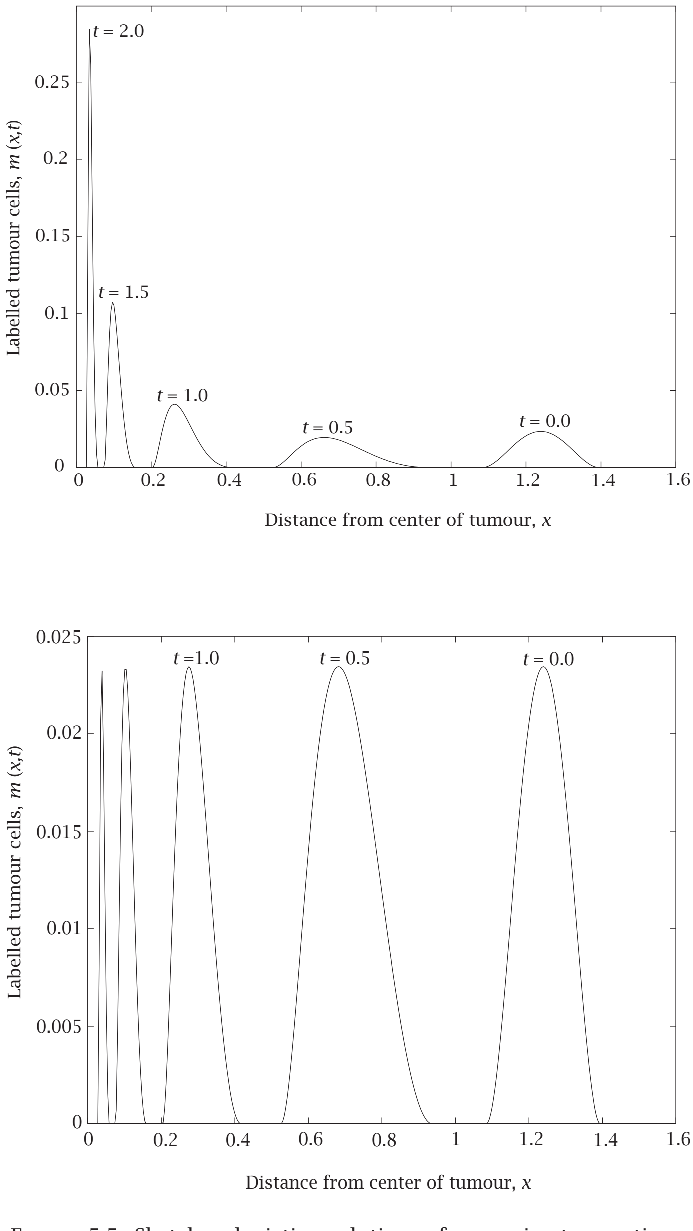

FIGURE 5.5. Sketches depicting solutions of approximate equations that describe initial migration of labelled particles within the tumour. (a) When microspheres are labelled, the pulse of labelled cells decreases in width and increases in height as it migrates towards the centre of the tumour. (b) When the tumour cells are labelled, the pulse decreases in width whilst remaining of constant height as it migrates towards the centre of the tumour. Parameter values: $s = 10.0$, $c_\infty = 1.0$, $\tilde{c} = 6.0$, $c_{\text{nec}} = 0.1$, and $\Gamma = 0.5$.

where the constant M^* is chosen so that the total number of labelled cells within the tumour remains fixed, that is,

$$\epsilon \int_0^1 m_0(x)\, dx = \int_0^1 m(x,0)\, dx = \int_0^1 m_{\text{in}}(x)\, dx.$$

From (5.25), we deduce that at equilibrium the microspheres are concentrated at the centre of the tumour ($x = 0$). This agrees with the experimental results of Dorie et al. [9, 10] and the numerical results presented above (see Figures 5.3 and 5.4). In addition, we note that as $\mu_m \to 0$ this equilibrium profile becomes steeper and more localised, approximating a delta-function in the limit as $\mu_m \to 0$. We remark further that if random motion is neglected ($\mu_m = 0$), then our model does not admit a classical steady solution. As discussed above, in this case the cells accumulate at the origin in a region (or boundary layer) whose width decreases exponentially with time. Eventually, $m_0(0,t) \sim O(\epsilon^{-1})$ and the asymptotic expansion breaks down. In practice, and on the basis of the experimental results of Dorie et al., it seems reasonable to assume that the cells undergo a minimal amount of random motion, and that this physical effect regularises any sharp increases in $\partial m_0/\partial x$ within such a boundary layer.

In (5.25), the multicell spheroid's growth parameters are combined in a single parameter grouping, $s\Gamma/\mu_m$. Consequently, fitting (5.15) to the experimentally observed limiting distribution of labelled microspheres will yield an estimate of $s\Gamma/\mu_m$. Now, the numerical simulations and the analytical solutions presented above indicate that the average migration speed of the microspheres during the early stages of the experiment yield estimates of the parameter grouping $s\Gamma$ which is independent of μ_m. Combining these two results should enable us to derive estimates of $s\Gamma$ and μ_m.

When the tumour cells are labelled, equation (5.23) can be written as

$$0 = \mu_m \frac{d^2 m_0}{dx^2} - v_0 \frac{dm_0}{dx} \Longrightarrow \frac{dm_0}{dx} = C^* \exp\left\{ -\int^x v_0(\bar{x})d\bar{x} \right\},$$

where the constant of integration C^* is determined by imposing (5.12). This yields

$$C^* = 0 \Longrightarrow m_0 = \text{constant}.$$

Hence we deduce that at equilibrium the labelled tumour cells are uniformly distributed throughout the tumour volume. This agrees with the experimental results of Dorie et al. [9, 10] and the numerical simulations of Section 3 (see Figures 5.4). Given that the limiting distribution of the labelled tumour cells is spatially uniform it yields no additional information about the growth kinetics of the underlying multicell spheroid. This contrasts with the situation when the microspheres are labelled.

5.5. Model extensions. There are many ways in which the mathematical model presented above could be extended. For example, by reinterpreting the definitions of the labelled and unlabelled cells, it should be possible to apply the model to avascular tumours that comprise multiple cell populations. The different subpopulations may distinguish cells that possess normal and mutant copies of a particular gene, such as the tumour suppressor gene *p53* [15, 29] which normally plays a key role in regulating apoptosis. Cells with abnormalities in *p53* are believed to widespread in cancer, occurring in almost 50% of tumours.

Other extensions which would bring the model more in line with the experiments of Dorie et al. include the following: Reformulating the model to describe the growth of a three-dimensional, radially-symmetric multicell spheroid; considering more developed tumour structures, involving necrotic and hypoxic regions; studying how the internalisation patterns are affected when the unlabelled cells are irradiated or pretreated with a drug. It would also be interesting to see how our model predictions are affected by incorporating effects to describe heterotypic interactions between the different types of cells. This could be accomplished by following the approach of [21, 28] and allowing the various cell populations to move with different velocities.

6. Discussion. In this paper, I have discussed three different, but interrelated, mathematical models of avascular tumour growth. By focussing on the increasing complexity of these models, I have tried to show how the field has developed over the past twenty years, in response to new biological results and deficiencies in existing models. I have also indicated several ways in which the models could be further adapted or extended to describe other, related situations.

Referring to the biological background presented in Section 2, it is clear that the processes of angiogenesis and vascular tumour growth have a more detrimental effect on patient well-being than avascular tumour growth. And yet, the models that I have presented have focussed entirely on avascular tumours. This restriction may be justified on a "learn to walk before you can run" principle, that is, until the mechanisms underpinning avascular tumour growth are elucidated, it will be difficult to develop realistic models of angiogenesis or vascular tumour growth. Whilst the development of vascular tumour growth models remains an open problem, good progress is now being made with modelling angiogenesis [2].

The original models of angiogenesis (for details, see [5, 25] and references therein) were formulated as systems of partial differential equations and were successful in reproducing many of its characteristic features, for example, acceleration of the endothelial cells that constitute blood vessels from the parent vessel towards the tumour accompanied by an increase in the density of capillary tips [12, 23]. However, the models were unable to resolve the detailed structure of the developing vasculature, a shortcoming that has been addressed in [2]. Anderson and Chaplain use a discretised version of the original system of partial differential equations to generate an equivalent cellular automata model in which the motion of individual endothelial cells can be monitored. Their numerical simulations show excellent qualitative agreement with experimental results.

The introduction of probabilistic effects into Chaplain and Anderson's models of angiogenesis [2] and other models of tumour invasion [3, 11] raise an important question regarding solid tumour growth: should we be developing deterministic or stochastic models? This is just one of a number of issues that remain to be resolved.

Other aspects of solid tumour growth that I have not been able to discuss here include invasion and cancer therapy. Whilst there are a few models describing invasion of solid tumour cells [3, 27, 30], the literature is replete with compartmental models (i.e., systems of coupled ordinary differential equations) that describe a tumour's response to chemotherapy (for details, see [22] and references therein). These have been successfully fitted to experimental and patient data in order to estimate kinetic parameters such as drug delivery rates. A weakness of such models is their failure to accurately account for the spatio-temporal heterogeneity that characterises vascular tumours. In order to predict the efficacy of gene-based therapies that are currently being developed, it is important to have realistic models of vascular tumours. For example, clinicians are currently trying to exploit the propensity for macrophages to migrate specifically to the notoriously drug-resistant hypoxic regions within solid tumours: They aim to genetically engineer macrophages that are activated under hypoxia to release anti-angiogenic and cytotoxic chemicals.

In conclusion, I believe that mathematical modelling will play an increasingly important role in helping biomedical researchers to gain useful insight into different aspects of solid tumour growth. At the same time, by studying such biological systems it should be possible to generate new models that either extend existing theories or are of independent mathematical interest. Whilst I remain sceptical about newspaper headlines proclaiming the discovery of the cure for cancer, I am optimistic that in the near future clinicians will have the necessary knowledge to develop effective treatments for individual cancer patients.

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DIFFERENTIAL MODELLING IN VISUAL NUMERICAL ENVIRONMENT

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Differential modelling is a widely used technique to understand physical phenomena, which has been getting a more and more powerful tool for handling complex application problems in several scientific fields.

And more and more often numerical methods are integrated in both special or general purpose tools equipped with interaction facilities and with advanced visualization techniques to evaluate huge quantity of numerical results. In this paper, we present basic numerical and visual components of the numerical visual environment we are developing to handle scientifically meaningful applications. In particular, we describe the multigrid finite difference approach, the multigrid methods and algorithms. These have been defined and experimented for the numerical solution of differential systems arising in image segmentation and grid generation which are fundamental steps of the computational field simulations. Visual tools are introduced and a few numerical results described and illustrated by figures.

1. Introduction. Numerical modelling of a physical phenomenon requires an effective design of the overall computational process, the definition of accurate and robust algorithms, an active control over the different computing steps and a fast investigation of computed intermediate and final results. Differential modelling is more and more often a tool to understand complexities of phenomena by saving human effort and/or money, since either it can work without concretely reproducing phenomena, their environment and characteristics in expensive physical models, or, when real models are inevitable, it can drive their design and experimental investigation in order to save resources.

In order to manage the growing complexity of application problems the scientists want to face, differential modelling is getting itself very complex and has to integrate a large set of methodologies. Numerical and visual algorithms are more and more often combined to improve the whole computational process.

Indeed, both continuous and discrete differential systems have been defined and combined to numerically model a large series of high-interesting problems by experts coming from industrial environment or human-health or other fields. Experts in scientific computing work in interdisciplinary projects to numerically compute solutions of PDEs in the more effective way.

As an instance of the developed numerical methods and algorithms, we illustrate the approach which couples classical finite difference approximation with multigrid computation. This is a complex but promising numerical approach to obtain effective algorithms. This computing method is based on the use of a hierarchy of grids to improve the capability of relaxation algorithms to smooth error components. We have been achieving interesting results in multigrid finite difference approximation of PDEs. We have defined and evaluated multigrid algorithms for the solution of systems arising in grid generation, fluid dynamics, and image segmentation problems. First we have obtained effective onegrid algorithms, then we coupled them with multigrid computation for numerical advances [8, 11, 13].

A visual computational environment integrates numerical and visual functions so that visual information can provide precious data to start, drive, evaluate application processes, and achieve important results.

Images can act as starting point to obtain data for differential modelling, as in hemodynamics, for instance, where angiograms are processed to reconstruct vessel geometries to model blood flows. On the other hand, synthetic images of computed data are a powerful tool for evaluating results and in case suggest how to modify the computational components of the process. Visual checks in appropriate computing points allow, for instance, [2, 7, 8]:

- fast evaluation over computing substeps during execution to be carried,
- value modification of numerical and visual parameters to be driven,
- understanding of huge quantities of final computed results to be accelerated,
- computed results to be compared with real images or other already acquired information,
- acquired knowledge on the studied phenomenon to be immediately and clearly transferred and judged by the experienced scientific communities.

The design of a modern visual numerical environment requires the definition and evolution of a collection of algorithms and software modules with large applicability and high integrability.

In the following, we illustrate the multigrid finite difference approach and the most recent developments in image processing and grid generation. We introduce visual tools, with specific and advanced tasks, which can be easily combined with numerical methods depending on the application for the realization of a simulation environment agile and efficient. We show also an instance of the large series of available numerical results [2, 7, 10].

2. Multigrid computation. Let Ω be a given domain and $\delta\Omega$ its boundary. We write the equations to be solved along with the associated boundary conditions in the form

$$Lw = F \quad \text{in } \Omega, \qquad \Lambda w = \Phi \quad \text{on } \delta\Omega, \tag{2.1}$$

where L is a nonlinear finite difference operator,
F is the right-hand side of the system,
Λ the operator defining the boundary conditions for each given problem,
Φ the assigned boundary values.

By defining finite difference approximation methods, we obtain the associated discrete problem

$$L^h w^h = F^h \quad \text{on } G^h, \qquad \Lambda^h w^h = \Phi^h \quad \text{on } \Gamma^h, \tag{2.2}$$

where G^h is a fixed grid covering Ω, with appropriate meshsize h, and Γ^h is the set of boundary points of G^h. In order to obtain accelerated solutions and convergence histories, we define the multigrid computation by adding an appropriate number of coarser grids to the computational grid with the chosen fineness h.

Set $h = h_M$ and assume the sequence of grids $G^0, G^1, \dots, G^M$, with decreasing meshsize: $h_0, h_1, \dots, h_M : h_0 > h_1 > \cdots > h_M$. Rewrite the problem (2.2), the discrete equation system along with the associated boundary conditions, in the form

$$\mathrm{L}^M w^M = F^M \quad \text{on } G^M, \qquad \Lambda^M w^M = \Phi^M \quad \text{on } \Gamma^M. \tag{2.3}$$

For each grid level l, $l = 0, 1, \dots, M-1$, the general form of the correction equation of the FAS algorithm and the related boundary conditions is [1, 3, 4]:

$$\mathrm{L}^l w^l = \tilde{F}^l \quad \text{on } G^l, \qquad \Lambda^l w^l = \tilde{\Phi}^l \quad \text{on } \Gamma^l, \tag{2.4}$$

where

$$\begin{aligned} \tilde{F}^l &= L^l\left(I_{l+1}^l w_a^{l+1}\right) + I_{l+1}^l\left(\tilde{F}^{l+1} - L^{l+1} w_a^{l+1}\right), \\ \tilde{\Phi}^l &= \Lambda^l\left(I_{l+1}^l w_a^{l+1}\right) + I_{l+1}^l\left(\tilde{\Phi}^{l+1} - \Lambda^{l+1} w_a^{l+1}\right), \end{aligned} \tag{2.5}$$

with w_a^{l+1} the current approximation on the finer grid, I_{l+1}^l the fine-to-coarse transfer operator (restriction) and Γ^l the set of boundary points of G^l. In the fine-grid correction step the current approximation is updated by the expression

$$\left(w_a^{l+1}\right)_{\text{new}} = \left(w_a^{l+1}\right)_{\text{old}} + I_l^{l+1}\left(w_a^l - I_{l+1}^l w_a^{l+1}\right), \tag{2.6}$$

where w_a^l is the current approximation on the level l and I_l^{l+1} is the coarse-to-fine transfer operator (prolongation).

3. Variational image segmentation. The definition and evaluation of numerical methods for image segmentation deal with the development of a visual numerical environment for advanced applications. We present the developed multigrid finite difference approximation method for the solution of Euler equations arising in variational image segmentation.

Image processing is a central technology for a wide range of applications and it requires effective computational techniques. Recently, processing approaches leading to variational methods and related partial differential equations have had a new development. We followed the Mumford and Shah's variational approach which is a well-based theory and motivates computational advances. We present the multigrid finite difference method which can solve the Euler equations associated to the sequence of functionals Γ-convergent to the functional of Mumford and Shah [5, 13, 14].

3.1. Euler equations. Let Ω be a bounded open set and f the image intensity with discontinuities, for instance, contours of objects or shadows. The variational approach of Mumford and Shah looks for a piecewise smooth approximation u of the function f, by minimization of an appropriate functional [5]. For the numerical solution of this problem, we assume the Euler equations associated to the kth functional $\mathscr{E}_k(u,z)$ of the sequence Γ-convergent to the functional of Mumford and Shah, in the following form [5, 13, 14]:

$$z^2\Delta u + \frac{\partial(z^2)}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial(z^2)}{\partial y}\frac{\partial u}{\partial y} = \frac{1}{\lambda}(u-f), \tag{3.1}$$

$$\Delta z + \frac{k^2}{4}(1-z) - \frac{k\lambda}{\alpha} z|\nabla u|^2 = 0, \tag{3.2}$$

where the function z is close to zero inside a tubular neighbourhood of the discontinuity set and to 1 outside. We assume the finite difference approximation of the equations (3.1) and (3.2) by central differencing in the form

$$\left(\frac{1}{\lambda} + \frac{4}{h^2}z^2\right)u - \frac{4}{h^2}z^2\tilde{u} - \frac{1}{2h^2}z(\bar{z}_x\bar{u}_x + \bar{z}_y\bar{u}_y) = \frac{1}{\lambda}f, \tag{3.3}$$

$$z\left(\frac{k\lambda}{4\alpha h^2}(\bar{u}_x^2 + \bar{u}_y^2) + \frac{4}{h^2} + \frac{k^2}{4}\right) - \frac{4}{h^2}\tilde{z} - \frac{k^2}{4} = 0, \tag{3.4}$$

where, for $w = (u,z)$,

$$\begin{aligned}
\tilde{w}(x,y) &= \frac{1}{4}(w(x+h,y) + w(x-h,y) + w(x,y+h) + w(x,y-h)),\\
\bar{w}_x(x,y) &= w(x+h,y) - w(x-h,y),\\
\bar{w}_y(x,y) &= w(x,y+h) - w(x,y-h).
\end{aligned}$$

We have defined and experimented finite difference onegrid and multigrid algorithms to process synthetic images and these algorithms can provide satisfactory solutions and show interesting convergence capabilities [13, 14].

3.2. Multigrid image segmentation. Let G_k^h be the grid covering the domain Ω, where h is the meshsize in both directions x and y, k the index appearing in the sequence of functionals $\mathscr{E}_k(u,z)$ and in the associated equations (3.1), (3.2), and Γ_k^h the set of the boundary points of G_k^h. We write the discrete image segmentation problem related to the system (3.3) and (3.4) in the compact form

$$L_k^h w_k = F_k^h \quad \text{on } G_k^h, \qquad \Lambda_k^h w_k = \Phi_k^h \quad \text{on } \Gamma_k^h, \tag{3.5}$$

where L_k^h is the finite difference operator associated with the system of equations (3.3) and (3.4),
F_k^h is the right-hand side in the system,
Λ_k^h the operator defining the boundary conditions for each given problem,
Φ_k^h the assigned boundary values.
Let Ω be the square domain $[0,1]\times[0,1]$. Let w_k^0 be a given initial approximation of the solution w_k of the problem (3.5) on the grid G_k^h. In order to compute the required approximate solution w_k^*, we have defined onegrid algorithms which apply ν_k sweeps of a relaxation algorithm [13]:

$$w_k^* = \mathcal{R}^{\nu_k}(w_k^0; L_k^h, F_k^h, \Lambda_k^h, \Phi_k^h), \tag{3.6}$$

where $\mathcal{R}$ is a relaxation procedure appropriate for nonlinear problems.

The goodness of the solution of the discrete image segmentation problem depends on the specific relationship between the meshsize h and the sequence index k in this problem. Since we are mainly interested in a good approximation of z in the tubular neighbourhood of the image contours, where z varies from zero to 1 (see [13, 14]), the meshsize h should allow an appropriate number of grid points to lie in this neighbourhood. We have assumed and used the parameter $p = hk$ to control the approximation of the rapid variation of z.

We have defined a multigrid algorithm by adding to the chosen computational grid G_k^h an appropriate number of coarser grids. By setting $h = h_M$ we assume the sequence of grids $G^0, G^1, \dots, G^M$, with decreasing meshsize: $h_0, h_1, \dots, h_M$ such that $h_0 > h_1 > \cdots > h_M$. By fixing and neglecting to write k the discrete problem (3.3) and (3.4) becomes of the form

$$L^M w^M = F^M \quad \text{on } G^M, \qquad \Lambda^M w^M = \Phi^M \quad \text{on } \Gamma^M,$$

where Γ^M is the set of boundary points of G^M. For each grid level l, $l = 0, 1, \dots,$ $M-1$ we can write the correction equation of the FAS algorithm and the related boundary conditions just in the same form (2.4) along with (2.5):

$$L^l w^l = \tilde{F}^l \quad \text{on } G^l, \qquad \Lambda^l w^l = \tilde{\Phi}^l \quad \text{on } \Gamma^l,$$

where

$$\tilde{F}^l = L^l(I_{l+1}^l w_a^{l+1}) + I_{l+1}^l(\tilde{F}^{l+1} - L^{l+1} w_a^{l+1}),$$
$$\tilde{\Phi}^l = \Lambda^l(I_{l+1}^l w_a^{l+1}) + I_{l+1}^l(\tilde{\Phi}^{l+1} - \Lambda^{l+1} w_a^{l+1}).$$

In the fine-grid correction step the current approximation is updated by the expression (2.6).

4. Numerical grid generation. Numerical grid generation is the crucial point of differential modelling. In order to achieve accurate and efficient simulations of a physical phenomenon in a complex domain, the scientist needs an appropriate discrete model of the domain, or in other words he has to appropriately place a discrete set of points (observers) in the domain to investigate the phenomenon in the best way. This task is often too time consuming and ineffective. A scientific community expert in grid generation has been and is working to improve this computational phase [6, 16, 17].

4.1. Elliptic grid generation equations. We have first developed multigrid algorithms for the solution of PDEs arising in numerical grid generation. We have been defining multigrid algorithms and evaluating their performance for effective handling of domain discretization problems by elliptic grid generation [16, 17]. In dealing with the general system

$$\Delta\xi^i = P^i\left(\xi^m, \frac{\partial \xi^m}{\partial x_j}, x_k\right), \quad i = m = k = j = 1,2,3 \quad \text{in } \Omega,$$

where the Laplacians of the curvilinear coordinates is assumed to be equal to appropriate functions of the curvilinear coordinates, their first derivatives and the cartesian coordinates, in a given physical domain Ω, we define the following two-dimensional continuous problem:

$$\begin{aligned} \Delta\xi &= P(\xi,\eta), \qquad \Delta\eta = Q(\xi,\eta) \quad \text{in } \Omega, \\ \xi &= f(x,y), \qquad \eta = g(x,y) \quad \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\partial\Omega$ is the boundary of Ω, and we assume that

$$\begin{aligned} P(\xi,\eta) = &-\sum_{i=1}^{n} a_i \operatorname{sign}(\xi-\xi_i)\exp\left(-c_i|\xi-\xi_i|\right) \\ &-\sum_{j=1}^{m} b_j \operatorname{sign}(\xi-\xi_j)\exp\left(-d_j\sqrt{(\xi-\xi_j)^2+(\eta-\eta_j)^2}\right), \end{aligned}$$

and an analogous control function $Q(\xi,\eta)$. By interchanging the independent and the dependent variables, we obtain the transformed system in the transformed computational domain Ω^*,

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} + J^2\left(P x_\xi + Q x_\eta\right) = 0, \tag{4.2}$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} + J^2\left(P y_\xi + Q y_\eta\right) = 0, \tag{4.3}$$

where

$$\alpha = x_\eta^2 + y_\eta^2, \qquad \beta = x_\xi x_\eta + y_\xi y_\eta, \qquad \gamma = x_\xi^2 + y_\xi^2, \qquad J = x_\xi y_\eta - x_\eta y_\xi.$$

We assume analogous Poisson's equations for the curvilinear coordinates ξ, η, and ζ of the three-dimensional computational space and the transformed system of the form

$$a_{11}r_{\xi\xi}+a_{22}r_{\eta\eta}+a_{33}r_{\zeta\zeta}+2\left(a_{12}r_{\xi\eta}+a_{13}r_{\xi\zeta}+a_{23}r_{\eta\zeta}+J^2\left(Pr_\xi+Qr_\eta+Rr_\zeta\right)\right)=0, \tag{4.4}$$

where $a_{ij} = \sum_{k=1}^{3} A_{k,i}A_{k,j}$, $A_{k,l}$, $k = l = 1,2,3$ are the cofactors and J is the determinant of the matrix $E = (r_\xi r_\eta r_\zeta)$, $r = (x,y,z)$.

Moreover,

$$P(\xi,\eta,\zeta) = -\sum_{j=1}^{m} b_{\xi_j}\,\mathrm{sign}\,(\xi-\xi_j)\exp\left(-d_{\xi_j}\left(c_{\xi_j}(\xi-\xi_j)^2\right)+c_{\eta_j}(\eta-\eta_j)^2+c_{\zeta_j}(\zeta-\zeta_j)^2\right)^{1/2}$$

and the functions $Q(\xi,\eta,\zeta)$ and $R(\xi,\eta,\zeta)$ have an analogous form with the coordinates η and ζ substituted in the function sign and the coefficients not necessary the same.

The transformed problem is completely defined by assuming appropriately transformed boundary conditions on $\partial\Omega^*$, the boundary of the transformed computational domain Ω^* either in two- or three-dimensional spaces.

The control function $P(\xi,\eta)$ allows ξ_i-lines, $i=1,\dots,n$ and (ξ_j,η_j)-points, $j=1,\dots,m$ to be activated as attraction sources with the following specific controls:

- the amplitude a_{ξ_i} controls the attraction of ξ-lines towards the ξ_i-lines,
- the amplitude b_{ξ_j} controls the attraction of ξ-lines towards the grid point (ξ_j,η_j),
- the decay factor c_{ξ_i} controls the extension of the ξ_i-line attraction,
- the decay factor d_{ξ_j} controls the extension of the (ξ_j,η_j)-point attraction,
- the function sign allows attraction to be exercised on both sides of each source.

4.2. Multigrid grid generation. Assume a sequence of grids $G^0,G^1,\dots,G^M$, with decreasing meshsize: $h_0,h_1,\dots,h_M$ such that $h_0 > h_1 > \cdots > h_M$, all approximating the current block B^*, along with its boundary Γ^*. Write the generating two- or three-dimensional quasi-linear systems (4.2), (4.3), or (4.4), along with the associated boundary conditions in the form

$$LX = 0 \quad \text{in } B^*, \qquad \Lambda X = \Phi \quad \text{on } \Gamma^*,$$

where $X = (x,y)$ or $X = r = (x,y,z)$, respectively. By standard central differencing we obtain the associated discrete problem

$$L^M X^M = 0 \quad \text{on } G^M, \qquad \Lambda^M X^M = \Phi^M \quad \text{on } \Gamma^M,$$

where Γ^M is the set of boundary points of G^M. The general form of the correction equation of the FAS algorithm and the related boundary conditions are

[7, 8, 11]:

$$L^l X^l = F^l \quad \text{on } G^l, \qquad \Lambda^l X^l = \tilde{\Phi}^l \quad \text{on } \Gamma^l,$$

where

$$F^l = L^l (I_{l+1}^l X_a^{l+1}) + I_{l+1}^l (F^{l+1} - L^{l+1} X_a^{l+1}),$$
$$\tilde{\Phi}^l = \Lambda^l (I_{l+1}^k X_a^{l+1}) + I_{l+1}^l (\tilde{\Phi}^{l+1} - \Lambda^{l+1} X_a^{l+1}),$$

X_a^{l+1} the current approximationon on the finer grid, I_{l+1}^l the fine-to-coarse transfer operator (restriction), and Γ^l the set of boundary points of G^l. In the fine-grid correction step the current approximation is updated by the expression

$$(X_a^{l+1})_{\text{new}} = (X_a^{l+1})_{\text{old}} + I_l^{l+1} (X_a^l - I_{l+1}^l X_a^{l+1}), \tag{4.5}$$

where X_a^l is the current approximation on the level l and I_l^{l+1} is the coarse-to-fine transfer operator (prolongation).

4.3. Multiblock and full-FAS algorithms. Multiblock generation results to be computationally more complex since information has to be provided block-by-block. We have to specify parameter vectors to store block connections, face ordering and orientation and other characteristics of the current decomposition of the domain, and the initial distributions of boundary grid points and an appropriate starting grid to carry out any iterative numerical generation algorithm.

Moreover, initial grid point distribution has to be provided on artificial boundaries, which do not have any physical boundary conditions.

We use interpolation between points on two boundaries, that is, grid lines joining points on two fixed boundaries through a number of blocks. We call fixed boundaries or free boundaries those physical boundaries used or not in this interpolation procedure, respectively. Therefore we differentiate three types of boundaries:

(a) fixed physical boundary,
(b) free physical boundary,
(c) artificial boundary.

In order to save the characteristics of the overall grid, we have overlapped two grid surfaces between any two adjacent blocks [15].

Full approximation storage (FAS) form and full multigrid computation have been added to enrich computational capabilities and improve algorithm performances [9].

The computational components for the multiblock and full-FAS multigrid elliptic grid generation in the two-dimensional form are as follows.

Grid generation methods

- *Multiblock grid generation:*

(1) subdivision of the physical domain Ω into an appropriate number of pieces, with appropriate connections, by defining artificial boundaries in Ω,
(2) mapping of the physical domain Ω on the transformed domain which is an appropriate assembly of $N \geq 1$ computational blocks B_l^*, $\Omega^* = \bigcup_l B_l^*$, $l = 1, 2, \ldots, N$,
(3) definition of criteria of solution acceptance,
(4) *elliptic grid generation*,
    (a) definition of the Poisson system (1) on the current block B_l,
    (b) activation of sources of attraction by defining the control functions in (1),
    (c) solving the transformed system (2) in the rectangle B_l^*,
        (i) application of a nonlinear multigrid solution algorithm, full approximation storage (FAS) algorithm,
        (ii) application of nonlinear relaxation algorithms with appropriate orderings: Lexicographical, red-black, four-colors, zebra-line,
        (iii) definition of criteria for anisotropy control,
        (iv) choice of a multigrid cycle (V or W),
    (d) updating of values on artificial boundaries of the next neighbouring block,
(5) iteration over all the blocks to achieve the desired solution according to the connection conditions.

- *Full multigrid algorithms*
  (1) restriction of the transformed system (2) on the coarsest grid G^0,
  (2) solution or relaxation of the restricted problem $L^0X^0 = F^0$ and $\Lambda^0X^0 = \Phi^0$,
  (3) interpolation of the coarser approximation to the finer grid,
  (4) execution of a multigrid cycle (V or W) at the reached finer level,
  (5) execution of the point 3 and 4 for all the finer grids, included G^M.

5. Visual tools. We are designing a visual numerical environment by the definition, development and evolution of a collection of methods and the related software modules with high integrability for differential modelling. In order to develop visual and numerical modules, with specific and advanced tasks, which can be easily combined depending on the application, we have carried out (a) the development of numerical methods, (b) the development of visualization techniques, (c) the design of human-computer interaction for computational processes, (d) the realization of interactive graphics systems, and (e) analysis of possible extensions to a multimodal approach [2, 7, 10].

The realization of a visual grid generation environment including variational image segmentation and numerical grid generation seems to be especially appropriate to carry out differential modelling. In practice regarding numerical and visual methods, we recall that the following main tools, for black-and-white 2D image segmentation and 2D and 3D grid generation, are available:

- *Solima* the modular code for onegrid variational image segmentation,
- *Mulima* the modular code for multigrid variational image segmentation,
- *Mulblock* the modular code which stores and save domain decompositions,
- *Mulgri* the modular code for the generation of boundary-fitted grids (BFGs) coupling elliptic generation with multigrid computation,
- *FullMulgri* the modular code coupling elliptic generation with full multigrid computation,
- *Ipargen* the modular code for isoparametric generation,
- *Visima* the module which allows visualization of input or computed images,
- *Visgen* the modular software to visualize generation results, which consists of the modules:
  - *visgri* the module which allows visualization of structured grids,
  - *vissou* the module which provides visualization of attraction sources by color (attractive ξ-lines, η-lines and points),
- *Vishis* the module which visualizes convergence histories, specialized for a given image or grid generation problem,
- GENSY the interactive graphic system for grid generation.

The *vishis* module allows to visualize convergence histories and immediately evaluate algorithmic performance. For instance, multigrid acceleration can be visually evaluated, that is, appropriate control over multigrid components and parameters by visualization of the error behaviour and decay can be easily exerted. Integrated in GENSY 1.1 it becomes very useful since the user can visualize residuals or errors during computation and change parameter values or the numerical strategy. It means computational steering [7].

The system GENSY. The interactive numerical and visual system GENSY consists of software modules which have been integrated to generate, visualize, and evaluate two- and three-dimensional grids [10, 12]. Both the expert and novice users can interactively apply the numerical methods and easily assign and update method parameters. They can also easily require visualization techniques, to exert both visual control over numerical computation and computed grids, and therefore modify and optimize the generation process on the base of the obtained visual information.

The design of an integrated system requires the definition of the basic numerical and visual objects of the system and the identification of the system functions for the management of such objects. We have equipped the numerical methods also with (i) the development of a visual language by definition of special elements for human-computer interaction, for instance, menus, icons, or other tools, (ii) the design of easy-to-use and user-friendly graphical user interfaces (GUI), mixed type.

In order to improve the elliptic grid generation, four icons have been designed to drive multigrid computation and control line-spacing of curvilinear coordinate systems [10]. From the practical point of view, the user should know just few things to use the system GENSY, since almost all the functions are automatically carried out or interactively chosen. The graphical windows, one of which is fixed on the screen and the others flexible in number and location for user needs, are reserved to visualization. User interactions designed for the system deal with the following two types of interactive control of the user:

Control over numerics: The user can control the numerical computation by visual information (display of block configuration, convergence histories, ...); control over method performance and method parameters can eventually require to update parameter values or abort execution at different points of the generating process, which can be either starting or running or over;

Control over graphics: The user can choose when and how display results provided by the numerical.

6. Numerical results. We have defined, implemented, and experimented multigrid algorithms having care of specific computational difficulties arising in each application field. We have investigated problems dealing, for instance, with anisotropy in grid generation or due to the presence and nature of the control function z in image segmentation. Large series of results are available for both grid generation and variational image segmentation and, on the light of the whole numerical experimentation, we might observe that algorithms can provide satisfactory solutions and show accelerated convergence.

We have applied the multigrid algorithm with appropriate numbers ν_1 and ν_2 of pre- and post-smoothing sweeps

$$\tilde{w}^{l+1} = \mathcal{R}^{\nu_1}(w_{\text{old}}^{l+1}; L^{l+1}, \tilde{F}^{l+1}, \Lambda^{l+1}, \tilde{\Phi}^{l+1}),$$

$$w_{\text{new}}^{l+1} = \mathcal{R}^{\nu_2}\left(\hat{w}^{l+1}; L^{l+1}, \tilde{F}^{l+1}, \Lambda^{l+1}, \tilde{\Phi}^{l+1}\right),$$

where w_{old}^{l+1} and w_{new}^{l+1} are the approximations on the level $l+1$ before and after each multigrid subcycle. This widely used procedure leads to obtain a better initial guess $\tilde{w}^{l+1}$ and improve the approximation $\hat{w}^{l+1} = (w_a^{l+1})_{\text{new}}$ or $\hat{w}^{l+1} = (X_a^{l+1})_{\text{new}}$ given by (2.6) or (4.5), respectively. We have assumed the Gauss-Seidel relaxation (GSr) which is known as a capable smoother of the error components, very effective to design multigrid algorithms. As successive displacement algorithm, GSr depends on the ordering of the system unknowns [3]. Basic ordering is the rotated lexicographical ordering where a grid point (x_1, y_1) precedes $(x_2, y_2) \Leftrightarrow x_1 < x_2$ or $(x_1 = x_2, y_1 < y_2)$. Moreover, a large series of results on how ordering affects multigrid grid generation algorithms has been achieved.

Just to give an idea of the kind of results which are available for interested readers, we show results of image segmentation application. We have applied

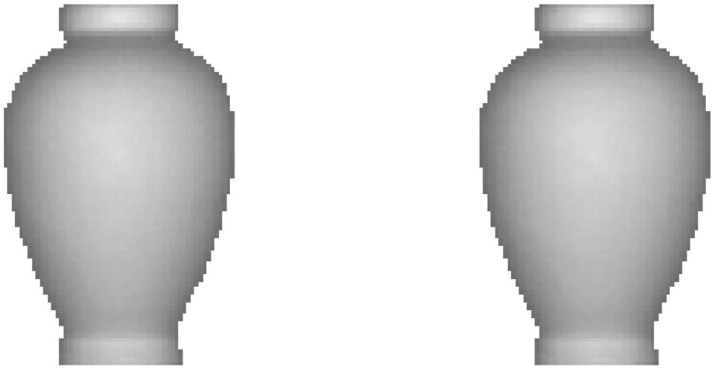
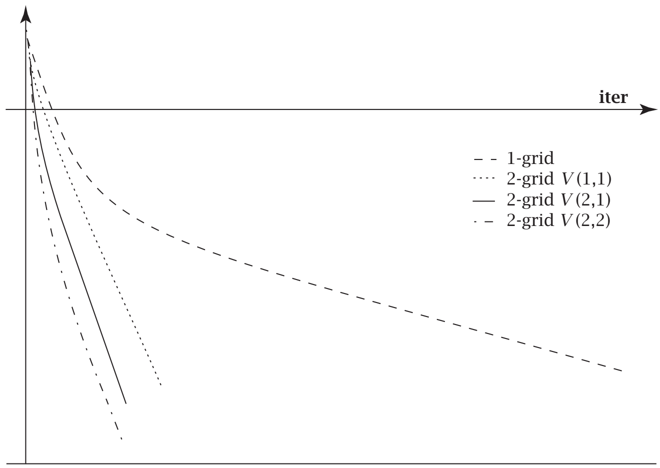


FIGURE 6.1. Amphora functions: Given image intensity f (left) and computed solution u_k^*(right).

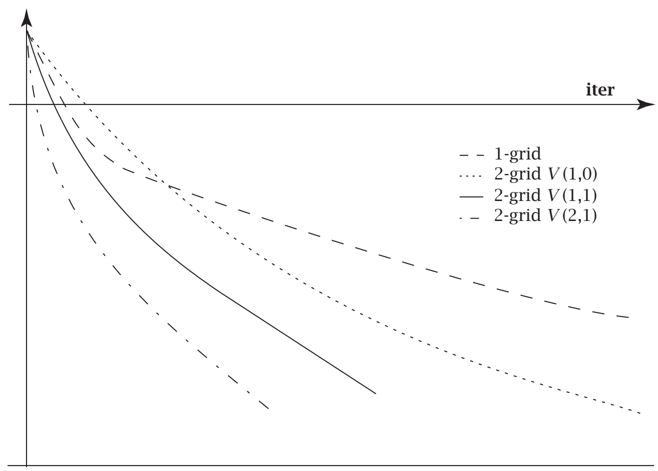
the multigrid algorithms to the segmentation of synthetic images with piecewise constant or varying brightness. We have solved segmentation problems related to (1) squares and circles in Ω, and (2) an amphora, all centered in Ω. We have generated images with different intensity values inside and outside geometric figures and we have experimented distinct resolutions, 64×64, 128×128, and 256×256, obtaining satisfactory results in all less or more complicated tests. Among images with varying intensity, we have segmented the 100×100 image of an amphora shown in Figure 6.1 (left). We can say that satisfactory solutions and convergence histories have been obtained. The grey value functions are normalized such that $f(x,y)\in[0,1]$. Figure 6.1 (left) shows the 100×100 (101×101 grid points) input image of the amphora, and on the right a computed solution u_k^*.

We have assumed as initial guess the approximation $w_k^0 = (u_k^0, z_k^0)$, with u_k^0 equal to the input image f and z_k^0 equal to 1 everywhere. Regarding convergence, we have compared convergence curves of method executions which represent values of each component of $(\log(\mathrm{res}_u), \log(\mathrm{res}_z))$ depending on the iteration number **iter**. As shown in Figure 6.2, the values of **iter** appear on the horizontal axis and the logarithmic values of the L_2 norm of each component of the residual $r_k^h = F_k^h - L_k^h w_k^h$, $r_k^h = (r_{u_k}^h, r_{z_k}^h)$, on the vertical axis. Figure 6.2 illustrates convergence histories depending on grid number and pre- and post-smoothing parameters and shows the accelerated performance of the twogrid with respect to the onegrid algorithm.

7. Conclusion. We have presented an approach to the differential modelling in visual numerical environment. We have introduced basic methodologies and their numerical and visual components. We have illustrated in particular numerical and visual methods, along with the related software tools, available for



(a)



(b)

FIGURE 6.2. Convergence histories for computed z_k^*: Square (a) and Amphora (b).

solving variational segmentation and grid generation problems. On the light of the experimental results, we observe that both numerical algorithms and visual techniques can provide satisfactory solutions, show accelerated convergence capabilities, and allow user driven solution processes. Specific computational difficulties of the application problems, due to the nature and characteristics of each specific phenomenon under study, can be appropriately handled by appropriate finite difference operator, the multigrid computation and visual investigation of results by effective human-computer interactions. Future extension and innovation of the presented visual computational environment can be promising for further improvements.

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MATHEMATICAL MODELLING IN MUSIC

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1. Introduction: From "sound" to "music" … mathematically? Mathematical modelling in music could broadly be framed in two realms, one dealing with the physics, and one dealing with the abstract, or symbolic, level of "music":

- mathematics and sound, dealing with the mathematical physics necessary or suitable to model the source of the sound;
- mathematics and composition, when an "alphabet" is agreed upon or taken as given, and the compositional rules are then studied, as a formal grammar or probabilistic process, or with combinatorial approaches.

The first approach is the one most found in textbooks, papers, and meetings. However, for example, at the meeting organized by Femmes et Mathematiques in Bordeaux, December 1998, the second approach was clearly stated and discussed with the appropriate mathematical tools.

The separation between the two realms is of course just for the sake of study, for the choice of the mathematical tools, and can be seen as the difference in time scale; the first approach looks at events in a smaller time scale than the second. On the other hand, the possible link between the two realms, and thus between the different time scales, poses the problem of the relation of the alphabet with the sound emitted by the source. Does an alphabet follow in some necessary or at least preferable way from a source?

Is there a finite repertoire of sounds that can be associated to one instrument rather than to another, owing to its physical characteristics? This is, in general, too abstract and too wide a question. But it is less vague than it seems at first sight. The entire theory of (our, western) music, rests on the hidden assumption that there is an alphabet, and that it consists of pitches, which are related to frequencies of vibration.

The existence of the alphabet itself rests on the possibility of discretizing frequencies in a natural way, and we call this discretization "musical scale." Still, we would like to do something to bridge the two approaches in mathematics, much as musicians do in music. There are phenomena of auditive organization in time that are retained as musical by some, and nonmusical (ritual, functional, …) by others; and there are even cultures in which there does not exist an activity denoted by a word corresponding to our "music." It is clear today that as soon as we use words such as "musical sounds," we are making a culturally determined choice, especially if "musical sounds" are understood as opposed to other sounds, similarly perceivable, and carrying meaning, but considered "nonmusical." There are as many "musics" as there are cultures. A mathematical model puts under mathematical processing a few quantitative characteristics of a phenomenon, to discover links which were previously ignored or not clear. The model consists of the individuations of some quantitative variables, and in the choice of the equations that bind them together, trusting, this way, to describe in mathematical terms an aspect of the phenomenon. It is obviously a good rule to remember that possible conclusions and implications, following such a formal treatment of the chosen variables, only regard the aspects subjected to the formal treatment, and that any other links, associations, or conclusions have to be discussed appropriately, checking to see that the mathematical model itself does not carry the implications as hypothesis. On the other hand, it is not always clear or simple to discuss which hypotheses are necessary and which can be seen with a more relative stance. Also, sometimes an existing model inspires another one, in a less linearly deductive fashion, in fact in a way that one should reconstruct with historical tools, and questions, we do not address this issue here. We would like to understand how mathematics has been used to reinforce cultural choices in fields removed from mathematics, and not subjected to it, such as the ones in music; how we got to the point of often defining as "musical," in common sense and in treatises of acoustics, only those sounds that satisfy very stringent periodicity conditions. We think that one of the problems has been the mathematical model, implicitly chosen, which favors the description of simple periods. Unveiling the model is a condition for the possibility to change it, and to choose from more of them. To this end, we propose a toy-model, for a mental experiment. We have "invented" a theoretical musical instrument (completely mathematical), that would not respect the conditions typically assumed in the classic (mathematical) descriptive models. Imagine then a people that played such instruments, century after century; the question now becomes which of the objective characteristics of the sound of the instruments would become structurally important; in what sense the continual use of certain instruments promotes (certainly does not dictate) one or another coherent and possible musical system, by way of transmission by the human ear.

2. Organization of the paper. In this paper, we discuss a simple model; we study the linear part of an operator of vibration. We will first look at the classical model of the "vibrating string." We discuss some of its characteristics that have been used to describe other vibrating bodies, and others that are specific and peculiar to the string. We then discuss two elements of the model, its one-dimensionality and its elasticity, to see how they each affect the spectrum. We put forward two more models, changing the vibrating string in either direction.

This work is purely theoretical. We then use this toy-model to speculate how the repetition of these vibrations over the centuries would favor extraction of time features; which ones, that is, would get retained and recognized. In particular, we look at how the physical principle of resonance would be more relative, as an organizer of repetition and expectation. In the entire work, we took the classical stance of looking at the vibrating instrument, and do not discuss the other poles of this activity, as the transducer or the complicated transforms that describe perception. Afterwards, we inquired whether there is an actual people who had been educated, by ear, to a music based on rigid more than on elastic instruments? Yes, there exists the entire "Gong-chime musical culture" of South-East Asia, which has fascinated western composers for over a century with its complexity and the quality of its sound. At the end of the paper, we report on the first results of fieldwork done with the gongsmiths that forge bronze instruments in Bali, confirming results we had obtained theoretically, and that seem to have eluded description in literature about musical instruments, so far. Moreover, the mathematical treatment of semirigid vibrations is in itself interesting, and one very soon finds open problems, as the dimensionality and the contours of the vibrating body change.

3. Change "music," change mathematical model

Definition 3.1. We will call "music," strictly for the ends of this paper, a collective meditation on Sound, codified in the activity of composition of sounds in time.

All mathematical treatment of sound starts from a mathematical model. The mathematical model on which the classical treatments are based, is the "vibrating string," that is the differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2},$$

where x is the position on the string, t is time, and the function for which we want to solve the equation is the displacement $u(t,x)$, describing the vibration of the string; c is a constant of the problem. This equation can also be derived as the limit of a difference equation describing a chain of harmonic oscillators, that is, a chain of mathematical "springs." Thinking of it this way, it is clear that when we think of a body vibrating as a mathematical vibrating string, and therefore when choosing this one equation, we think that the force linking to nearby elements is elastic, mathematically speaking, that is, of the type "spring," or "harmonic oscillator." In this case we say that the body resists tension, and we call it an "elastic body." Moreover, when $x \in \mathcal{R}$, this equation is one-dimensional in space. The equation of the vibrating string, besides being at the basis of the classical mathematical treatment of sound, has been a mathematical model of great impact, underlying many theories, some of which have no musical applications. For a review in this regard, see Odifreddi

[7]. In this paper, Odifreddi (a mathematician) follows the paths that the vibrating string has allowed or inspired, emphasizing that the existence of this model influenced western musical practice (via the theorists), and still today influences the development of recent physical theories. How, that is, around this model, more complex models are organized, also for phenomena very removed from it, physically and conceptually. Notice also that the seminal work of Marc Kac "Can you hear the shape of a drum?" uses the verb "hearing" in the mathematical sense of "How are the spectra different, if the operator is the Laplacian?" that is, the wave equation associated with a given surface. So, even in this mathematical sense, which has nothing to do with music, "to hear" is used to imply that the vibrations are elastic.

From sound to music. "Archetypal instruments," their spectrum. To the end of vibrations, air-columns behave as vibrating strings. We could therefore think of strings and air columns as "archetypal instruments" of our music, that is, of our musical culture. This is more or less accurate in Europe, and finds ample systematic treatment in eighteenth and nineteenth century orchestras, in written art-music. To change music, starting from this model, we can go in two directions: Let $x \in \mathcal{R}^2$, that is, think of the instrument as two-dimensional (it would then be an "elastic membrane"), or make the hypothesis that the vibrating body is not completely elastic, and offers resistance to folding and torsion, in this case it is called a "semirigid" body. Let us think what "music" could be inferred or deduced, that is, we look for an entire consistent system of slightly different vibrating bodies with these conditions. Helmholtz's concept is that the human ear behaves as a spectrum analyser, that is, that it is capable of detecting as different two sounds depending on how they can be approximated by superimposing simple vibrations, such as the sinusoidal ones. This hypothesis is based on the principle of resonance. While we do not know of experiments which have seriously tested this hypothesis, we make it explicit and keep it as a starting point for methodological reasons; the game here is to introduce few, but qualitatively important, changes to the basic model so as to proceed to systematically discuss their possible consequences, and the impact of the modelling assumptions on the results found.

Linear one-dimensional elastic music. The first characterization of the solutions of the vibrating string is their harmonic spectrum, they can be well approximated by superimposing simple vibrations (sinusoids) whose frequencies are in integer ratio. Physically, a harmonic spectrum allows measurements by resonance, because the frequencies are separated by a fixed distance, and its integer multiples. A people that would listen for centuries to mathematical strings and air columns, would therefore develop a subtle sensitivity to harmonic spectra, and to frequencies in rational ratio, developing the ability to align such frequencies by ear, and name "similar" two sounds when many of their components resonate. When two frequencies are close to each other, but not identical, their superimposition gives rise to the phenomenon of "beats"; our people of listeners and players of strings and air columns, would therefore develop a social adversion to beats, as a signal of possible evaluation errors in nearby frequencies, as possible errors in the individuation, understanding, and reproduction of the basic alphabet. Written art-music of our own tradition is labelled "learned," in that it is the one whose description has been most systematic and rationalized; this music is based in large part on strings and air columns, which have also been extensively described mathematically. The two descriptions could have well influenced each other. The nice speculation-story just narrated, is reasonable, but has been narrated after it has already happened; nearby sounds in a spectral sense are labelled "consonant" in our culture, and beats are regarded as unpleasant noise. The story, as I narrated it, has something positive in suggesting that the censorship against beats is a cultural phenomenon (regarding objective features of the signal), rather than a physiological-aesthetical one. Let us see what would happen with musics based on other "instruments."

An elastic two-dimensional music. Some assumptions of symmetry in the vibrating body allow for clearer, or easier, structure of the solutions and of their spectrum. So, let us first assume that the vibrating body is square, and suppose that the force restoring equilibrium is still elastic. The differential equation is the same, but the variable varies in the two-dimensional square, $x \in [0,1] \times [0,1]$, and the spectrum of its vibrations is given by two series of harmonic overtones and their combinations. A people educated on square instruments would first select as understandable and easily reproducible the first frequencies of the series:

$$f_{n,m} = \sqrt{n^2 + m^2}, \quad n, m = 0, 1, 2, \dots,$$

for example, $f_{0,1} = 1, f_{1,0} = 1, f_{1,1} = \sqrt{2}, \dots$.

Caution. A "square" music would select as recognizable by ear, and reproducible on instruments, an irrational frequency ratio, such as $\sqrt{2}$; such ratios have been carefully avoided in western musical practice of the last few centuries, precisely as difficult to intonate with the voice, and therefore unstable: It was mainly used in sequence or together with other ratios. Trace of this sanction is in the name diabolus in musica.

This trend of thought has already been considered by composers who, with the aid of computers, assign a spectrum and consider the ratios it prescribes. A warning, consequences can be drawn not only on the musical intervals, that is, the ratios between frequencies, but also on the principle of resonance as a main basis for musicality of a sound, and of a system of sounds. In particular, if we accept that irrational frequency ratios be a basis of a music unfolded by repetition and variation, at the same time we are drastically reducing the sanction against beats, that become a necessary, recognizable and controllable feature.

4. A semirigid, one-dimensional "musical instrument"

The equation. If we have reason to think that the vibrating body be semi-rigid, the force pulling to equilibrium is not of elastic type; its vibrations satisfy a fourth-order differential equation and we look for solutions of the type $u(x,t) = h(x)g(t)$,

$$\frac{\partial^4 u}{\partial x^4} + \mu^2 \cdot \frac{\partial^2 u}{\partial t^2} = 0$$

here, again $x \in \mathcal{R}$, that is, the spatial variable is one-dimensional, and the equation linking the variables, the mathematical model, goes under the name of "bar."

First consequence. There are no waves that travel with constant velocity and unmodified shape. A people educated with such instruments would develop a subtle discerning of spatial relations, even before the subtle discerning of ratios of frequencies. Also, this could well be one of the reasons numerical spectra of these instruments are difficult to perform and evaluate, because they necessarily involve an averaging. In fact, the analysis and numerical simulation of "bells" is a well-known hard problem.

Second novelty. There are many possibilities of "boundary conditions." For the vibrating string, the boundary conditions (i.e., conditions for $h(x)$ and its derivatives at edges or other particular points along the longitude of the vibrating body) are that the edges be fixed throughout the motion. In the mathematical bar one usually assumes a fixed boundary, that is, one thinks of a bar with a clamped border. When, as is reasonable, one considers a bar resting at some of its points on a support, or else suspended when the edges are free to vibrate, then the boundary conditions are different. If this bar is resting on two of its points, its vibrations meet obstacles in the "resting point." From a modelling point of view, this can be described in several ways, depending on how we think the deformation in its vicinity behaves. That is, we can give several "boundary conditions" for the differential equation of the bar, and all of them could be reasonably labelled "resting point." To fix ideas, we fix the origin of coordinates in the centre of the bar, and denote

l half-length of the bar ($x = l$ and $x = -l$ coordinates of its edges),
a coordinate of the suspension point,
$h(x)$ deplacement from equilibrium (at a given time).

We propose the following three different conditions as possible modellistic definitions of "suspension point":

(1) $h(a) = h(-a) = 0$ and $h''(l) = h''(-l) = 0$ (fixed point of suspension, edges free to rotate);

(2) $h'(a) = h'(-a) = 0$ and $h'''(l) = h'''(-l) = 0$ (humps in the points of suspension, cutting force nil at the edges);
(3) $h'(a) = h'(-a) = 0$ and $h''(l) = h''(-l) = 0$ (humps in the points of suspension, edges free to rotate).

Looking for particular solutions of the differential equation, of the type $u(x,t) = h(x)g(t)$, the partial differential equation written above splits into two ordinary differential equations, becoming:

$$g''(t) + \mu^2 k g(t) = 0, \qquad h^{(iv)}(x) - kh(x) = 0,$$

the first one with initial condition in time, and the second with boundary conditions to be chosen among the three just proposed, and k constant. We want to define as "musical instrument" a system of several bars, with different lengths, but all sharing the same structure of vibration, at least within the solutions considered. A system of bars having the same "sound color," reinforcing the musical culture of our imaginary people, repeating the same time-relations of sound at small time scale. We wonder, then, how difficult it would be to compute a point of suspension for the whole system of bars, that would guarantee their same vibration structure, regardless of their length. In mathematical terms, we are probing the dependence of the spectrum of the differential operator on the boundary conditions, and on the length of the bar.

Conditions of musicality, conditions of consistency. Towards the end of this study we have stated, as a condition of musicality, some consistency of the spectrum from one bar to another, when the length changes; this would ensure the possibility to build (theoretically) an instrument made of several "keys," all characterized by the same sound color, at least as regards the part due to the operator we are studying, excluding phenomena of nonlinearity and damping. This is an important exclusion, not just from a mathematical stand point, but because the entire perception of sound color is actually deeply linked with these strong temporal features. The research and measurements of Fletcher are today in this direction. We privilege the information coming to the ear from the spectrum of the differential operator, for methodological reasons; the theoretical instrument has to be compared with the model of a vibrating string, which we are reevaluating, and which is studied under analogous conditions. We will shortly wonder whether instruments actually existing in the world, are built this way. We should therefore make the hypothesis that the spectrum be repeated from key to key of the instrument, and that the people of players and listeners would therefore learn to draw out of their context some of its features, to repeat them in other temporal contexts; that is, this people would learn to "abstract" them. For the vibrating string, the spectrum does not change with the length of the string. For the first instrument, made of elastic membranes with fixed edges, this consistency condition becomes that the ratio between the lengths of the sides of the rectangle be preserved. The length of these sides determines the double series of overtones in the spectrum, and if their ratio is preserved, the entire spectrum rescales perfectly starting from other fundamentals, much as in the vibrating string. Possible irrational ratios would then be exactly retained. For the bar, if we consider it crucial that a musical instrument have a harmonic spectrum, among the rare conditions guaranteeing it is $l = a$, that is, those in which the point of suspension coincides with the edges. (We remark that treatises of musical acoustics often assume this to be true for the glockenspiel, small tuned metallophones. However, in an actual metallophone the bars are neither clamped nor resting at their edges.) Let us continue with this game, and impose the consistency of the spectrum from bar to bar, accepting nonharmonic spectra, if they come. The condition we find, for the bar to vibrate with a spectrum determined by the point of suspension, independently of its length, is that the ratio between length of the bar and point of suspension be preserved:

$$l = ba, \quad \text{with } b \text{ constant on all keys.}$$

In this case, the spectrum would not necessarily scale perfectly from one bar to the other, which would imply more complicated rules of abstraction by the ear than what we are used to. But the structure of the frequencies would be determined by the ratio a/l. This is the point we have reached with our theoretical study. We recall that in our simplified model we are assuming $x \in \mathcal{R}$. The two-dimensional semirigid problem presents substantial mathematical difficulties, even assuming a clasped edge.

5. Fieldwork research: The Gong-chimes musical culture of Southeast Asia, the artisan gongsmiths. On first measurement of the vibraphones in our symphonic orchestras, it does not appear that the proportionality conditions are respected, neither as rectangles, nor as longitudinal bars. Is this because the model is irrelevant, because the spectrum cannot be discerned by the human ear? Well, it is natural to ask whether there exists a people musically educated by instruments that are not based, as ours, mostly on vibrating strings and air columns, or their perturbations. Is there a music stemming from instruments in which the element of rigidity is predominant, and that have been played for centuries? Ethnomusicologists speak about a "musical culture of gongs and chimes," referring to Southeast Asia. Travellers report in their diaries of the "tingling music" as part of the landscape. Images of metal instruments and their players are carved in the sculptures of the Borobodur (Java, end of the VIII century), and in the temples of Angkor (Cambodia, XII century). The gamelan orchestras of Java and Bali, for example, consist of tens of metallophones, both rectangular and circular (i.e., chimes and gongs), of various widths. We know that Debussy, fascinated, spent days studying the music and the instruments of the Javanese gamelan playing at the International Exposition of Paris in 1889. The description of musical scales in Java and Bali is a deep problem, as yet not completely understood by ethnomusicologists, giving raise to difficult studies of its objectively measurable features. Music in this region is transmitted and learned aurally, by the ear, and it holds a very sophisticated tradition. The used interval patterns vary from village to village and from ensemble to ensemble, but within some limits. Even the intervals of fifth and octave are under discussion. We first went to measure the instruments in the STSI, the Academy for Performing Arts in Denpasar (Bali), and realized, to our surprise, that all the proportions were accurately respected. We then went to the gongsmiths who specialized in forging musical instruments in Bali, and documented the fieldwork in a video. Gamelan orchestras and ensembles are forged one at a time, the entire workshop works on the ensemble until all the instruments are finished. We observed, in particular, how the rectangular keys of the many genders were processed. Each bronze key has two holes, and two leather cords string the keys together through the holes, to form a gender's hanging keyboard. From the moment these points of suspension are measured and the holes are drilled in the workshop, the bars are then held by two fingers, holding the hole just drilled, and from then on, everything is checked by ear. The hole is therefore confirmed as crucial to the assessment of quality of sound. The entire process is then checked by ear by all the workers, as well as by a master gongsmith coordinating the workshop. This master gongsmith is also the one who accurately measures the point in which the hole is to be drilled. The three gongsmith workshops we visited, drilled in three different ways, at three different moments of the forging procedure and with three different techniques for measuring. However, in all cases the point of suspension was accurately calibrated at 1/4 of the total length. According to our numerical simulations, a ratio of $1:4$ (in our choice of coordinates, $l = 2a$), besides being easy to calculate in construction, would guarantee a nonharmonic spectrum. So, what about the beats? If the spectrum is nonharmonic, and if this is essential to the subtle discerning by ear, then we said beats should be a characteristic feature, not avoided, controlled instead. In Bali instruments go in pairs: they are forged (bar by bar) in pairs, tuned (bar by bar) in pairs, and played (in ensembles) always in pairs. Each pair of instruments consists of one tuned slightly lower and one tuned slightly higher. The two instruments, played together, form beats that are carefully sought and controlled by a master tuner, who establishes their velocity. This velocity of the beats is carefully sought by lowering or sharpening the tuning of the bars by means of filing. The same master then finishes the entire tuning of the orchestra, checking the whole halo of beats obtained by several instruments playing at the same time; this operation is called the "coming together." Orchestras are thus characterized by a sparkling halo of vibrato in beats, whose subtle control gives life to the sound. This halo of beats, so essential to Balinese music, is called ombah. With no ombah, or with an untuned ombah, a gamelan simply is not ready to be played, yet. Western listeners trying unconsciously to ignore this halo while listening to gamelan music, due to cultural conditioning of their ears, simply get frustrated, as if going to the Vatican and trying to ignore the marble and the gold. They are cancelling out of their aural experience exactly the time features that could guide them to the larger time scale, as if blinded by so much light. In the workshop on Sound organized by the Scuola Interculturale di Musica della Fondazione Cini, held in Venice in January 1997, one of the pioneering and leading experts of gamelan music, Mantle Hood referred to this as "they are wearing sunglasses over their ears." So, the question remains open as to what extent, and with which tools, the act of human listening and its deep cultural conditioning can be treated objectively.

Acknowledgements. This research has been the result of many different collaborations. It has been a joy to work with so many people animated by curiosity and imagination. The first discussions in this speculative direction took place with Francesco Guerra. The study of the vibrating bar with nonstandard boundary conditions was done with Silvia Notarangelo. The fieldwork in Bali, has been developed with Giovanni Giuriati, ethnomusicologist, and my husband. We would also like to thank Dr. Wayang Rai, Komang Sudirga, Wayang Beratha, Made Gableran, Wayang Mustika, musicians, tuners and gongsmiths, for their hospitality, and for opening their secrets. I am very grateful to Ibu Chandri, singer and first woman dalang (master puppeteer) in Bali, and to her daughter Komin, as the gentle and kind neighbors who whisked me to their bedrooms, unseen by the men, sharing more than catches the eye to teach me how to wear a sarong and kebaya elegantly enough to be accepted in the temples, and comfortably enough to enjoy the long festival nights in Bali.

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GEOMETRY AND TOPOLOGY IN CONCURRENT COMPUTING

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1. Introduction. This is a short introduction to a geometric model for concurrent computing. Except for Proposition 3.1, there is nothing new here.

It is probably well known to most mathematicians, that graph theory is used in computer science to describe for instance the different options for a process (see Figure 1.1). It can enter a loop and stay there for a number of rounds, it can skip the loop, there may be a choice between different branches in the graph, etc. This is a nice model, and it works well when one process is living in a world of its own, meaning that it will not share information or interact in other ways with other processes.

When there is more than one process, and these interact, it is not quite clear, which model to use. Some problems, that one has to face are:

(1) Not all processes can be allowed access to a shared resource at the same time:

- If two people, T_1 and T_2, are allowed to access a bank account at the same time, and they both want to check if there is any money, then withdraw it and update the status of the account, there is clearly a problem (for the bank at least) if T_1 checks the status, T_2 checks the status, then T_1 withdraws all the money she saw and so does T_2, before T_1 updates.
- If a printer server has access to three printers, only three processes can print at a time.

(2) The execution of two (or more) concurrent processes may have different results depending on which process gets the shared resource first, for example, who gets the money from the bank account.

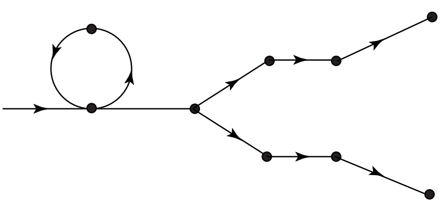

FIGURE 1.1. One process with a loop and a branching.

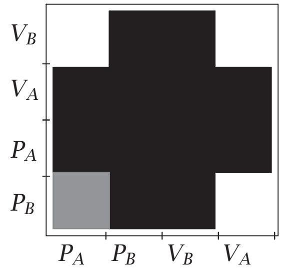

FIGURE 2.1. Example 2.1.

One way of getting around these problems is of course to run the processes in serial, that is, one after the other. But this is really not an option, it slows down the system considerably. So one has to find a balance: Running as much in parallel as possible without getting into the problems mentioned above. (Or at least keeping track of where they may occur.)

2. Geometric models. A graph is of course a geometric model, but there is much more geometry in the model we are studying.

EXAMPLE 2.1. Suppose two processes, T_1 and T_2, both want access to two shared resources, A and B. Suppose, moreover, that each resource will only allow the access of one process at a time. Then Dijkstra [1] suggested the following model: When T_1 accesses A, it puts a lock on it, and when it finishes it releases that lock. Locking is named P_A and releasing is V_A (from Dutch: Potlock and Vrei). The system

$$T_1 = P_A \cdot P_B \cdot V_B \cdot V_A, \qquad T_2 = P_B \cdot P_A \cdot V_A \cdot V_B,$$

has a geometric presentation or model as shown in Figure 2.1. All points are states of the system. The black area represents the *forbidden region*, that is, the states which are not allowed because both processes have access to A or B. An execution of the system is then a path from the lower left corner, the *initial state*, $(0,0)$, to the upper right corner, the *final state*, $(4,4)$. The path should be increasing in both coordinates, since time cannot run backwards. The grey area is the *unsafe region*, namely the states from which no increasing path can reach the final state. The maximal unsafe point $(1,1)$ is called a *deadlock*.

In Example 2.1, both processes are deterministic, that is, we always know what they want to do next, but in general, they may have loops or branchings, as in Figure 1.1, and this gives a more complicated geometry, for example, loops in more than one process give rise to tori. To get the right notion of execution paths—what the direction should be, when two paths are equivalent, that is, give the same outcome of the computation, a new kind of topology emerges.

Definition 2.2. Let X be a topological space.

(1) A collection $\mathcal{U}(X)$ of pairs $(U, \leq_U)$ with partially ordered open subsets U covering X is a *local partial order* on X if it satisfies the following: For every $x \in X$, there is a nonempty partially ordered open neighbourhood $W(x) \subset X$ such that the restrictions of $\leq_U$ to $W(x)$ coincide for all $U \in \mathcal{U}(X)$, that is, for all $U \in \mathcal{U}(X)$ and for all $y, z \in W(x) \cap U$,

$$y \leq_U z \Longleftrightarrow y \leq_W z.$$

(2) Two local partial orders $\mathcal{U}(X)$ and $\mathcal{V}(X)$ on X are equivalent if their *union* $\mathcal{U}(X) \cup \mathcal{V}(X)$ is a local partial order.

(3) A topological space X together with an equivalence class of local partial orders is called a *locally partially ordered space*. If there is a covering $\mathcal{U}$ in the equivalence class such that all $(U, \leq_U) \in \mathcal{U}$ are PO-spaces (the relation $\leq_U$ is closed), then X is a local PO-space.

Remark 2.3. Two local partial orders $\mathcal{U}(X)$ and $\mathcal{V}(X)$ are equivalent if and only if for every $x \in X$ there is a nonempty open neighbourhood $W(x) \subset X$ such that the restrictions of $\leq_U$ and $\leq_V$ to $W(x)$ coincide for all $U \in \mathcal{U}(X)$ and $V \in \mathcal{V}(X)$. We say that $\leq_X$ is well defined on $W(x)$.

We will assume that X is a local PO-space in the following; even if some statements make sense for the more general locally partially ordered spaces. We need local partial orders to deal with loops.

Example 2.4. The circle $S^1 = \{e^{i\theta} \in \mathbb{C}\}$ has a local partial order: The open subsets $U_1 = \{e^{i\theta} \in S^1 \mid \pi/4 < \theta < 7\pi/4\}$ and $U_2 = \{e^{i\theta} \in S^1 \mid 5\pi/4 < \theta < 11\pi/4\}$ are (partially) ordered by the order on the θ's. Notice that the relation on S^1 generated by these local partial orders by taking the transitive closure is $x \leq y$ for any pair x, y. Hence we do not take the transitive closure!

Definition 2.5. Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be local PO-spaces. A continuous map $f : X \to Y$ is called a *dimap* (directed map) if for all $x \in X$ there is a subset $V(f(x)) \subset Y$ on which $\leq_Y$ is well defined and a subset $U(x) \subset f^{-1}(V(f(x)))$ on which $\leq_X$ is well defined and such that for all $y, z \in U(x) : y \leq_X z \Rightarrow f(y) \leq_Y f(z)$ A *dipath* in X is a dimap $f : I \to X$ from the unit interval $I \in \mathbb{R}$ with the natural (global) order $\leq$.

Definition 2.6. For X a local PO-space, and $x \in X$, we define

(1) The future $\uparrow x = \{y \in X \mid \text{there is a dipath from } x \text{ to } y\}$.
(2) The past $\downarrow x = \{y \in X \mid \text{there is a dipath from } y \text{ to } x\}$.

Definition 2.7. Let X be a local PO-space. Let $\mathcal{F} \subseteq X$. Then $x \in X$ is a *deadlock* with respect to $\mathcal{F}$ if $\uparrow x = \{x\}$ and $x \notin \mathcal{F}$. If $\mathcal{F} \cap \uparrow x = \emptyset$, then x is *unsafe* with respect to $\mathcal{F}$ and $\mathcal{F}$ is *unreachable* from x.

The model for a concurrent system is then a certain local PO-space X. There is an initial and a final point, and an execution is a dipath in X from the initial point to the final point. To be more specific, X is constructed as a product of graphs as the one in Figure 1.1, from which one then removes a region corresponding to the forbidden region. This is a special case of what Pratt [6] calls an HDA, a Higher Dimensional Automaton.

3. Dihomotopy. A priori, there are uncountably many dipaths in Figure 2.1, and we want to distinguish only those for which the corresponding executions may give different results. This equivalence is *dihomotopy*, that is, deformations of dipaths. In the following, X is a local PO-space.

Definition 3.1. Let x_0 and $x_1 \in X$. The set of dipaths from x_0 to x_1 is denoted $\vec{P}_1(X,x_0,x_1)$.

Definition 3.2. Let I denote the unit-interval. Let $\alpha_1 : I \to X$ and $\alpha_2 : I \to X$ be paths in X from x_0 to x_1.

(1) A continuous map $H : I \times I \to X$ is called a *dihomotopy* between α_1 and α_2 if every partial map $H_s(t) = H(t,s) : I \to X, t \in I$, is a dipath from x_0 to x_1 and if $H(t,0) = \alpha_1(t)$ and $H(t,1) = \alpha_2(t)$. We write: $\alpha_1 \vec{\sim} \alpha_2$.

(2) The set of dihomotopy classes of dipaths in X from x_0 to x_1 is denoted $\vec{\pi}_1(X,x_0,x_1)$.

Remark 3.3. The local partial order need *not* be preserved with respect to the variables s. Dihomotopy is an equivalence relation.

So is this really different from the usual homotopy theory? Yes, it is:

(1) In ordinary homotopy, we study loops, and this would then correspond to elements of $\vec{\pi}_1(X,x_0,x_0)$, called *diloops*. If X has a global partial order, there are no diloops.

(2) We can concatenate two dipaths, if one begins where the other ends, and this gives a map $\vec{\pi}_1(X,x_0,x_1) \times \vec{\pi}_1(X,x_1,x_2) \to \vec{\pi}_1(X,x_0,x_2)$, but there is in general no inverse to an equivalence class of dipaths. Hence there is no group structure on the equivalence classes of dipaths from a given initial point to a given final point.

(3) Dihomotopy counts more than the number of "holes," as one can see in Figure 3.1.

(4) Two dipaths may be homotopic in the usual sense, but not dihomotopic [4].

Diloops are much more rigid than loops in ordinary homotopy.

Proposition 3.1. *A diloop $\gamma \in \vec{P}_1(X,p,p)$ is dihomotopic to the constant dimap p if and only if γ is constant.*

Proof. Let $\gamma : I \to X$ be a diloop in a local PO-space X. Suppose that $H : I \times I \to X$ is a contraction of γ, that is, $H(t,0) = \gamma(t)$ and $H(t,1) = p$. Suppose, moreover, without loss of generality, that $\gamma_s(t) = H(s,t)$ is nontrivial for $s < 1$. Let $U \subset X$ be an open neighborhood of p and suppose U is a partially ordered

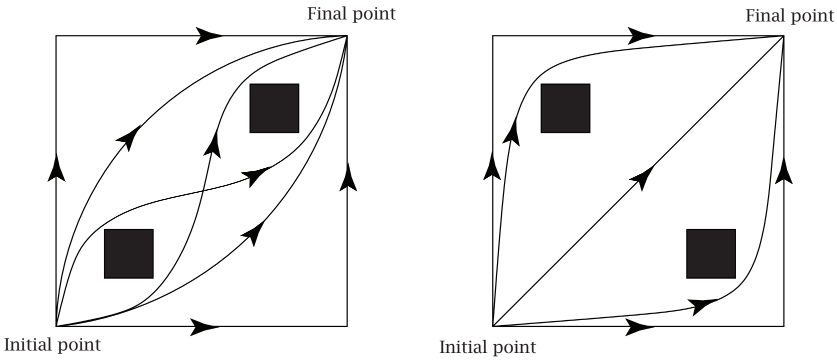

FIGURE 3.1. Two holes in $\mathbb{R}^2$. The first has four dihomotopy classes of dipaths. The second has three dihomotopy classes of dipaths.

space. Then $H^{-1}(U)$ is open in $I \times I$ and contains $I \times 1$. By compactness of I, there is an ϵ such that for $1-\epsilon < s \leq 1$ any dipath $\gamma_s(t) = H(t,s)$ is in U.

Since U is partially ordered, there is transitivity: $\gamma_s(t_1) \leq \gamma_s(1) = \gamma_s(0) \leq \gamma_s(t_2)$ for all $t_1, t_2 \in I$ and hence, since $\gamma_s(t)$ is nontrivial, there are $t_1, t_2 \in I$ such that $\gamma_s(t_1) \neq \gamma_s(t_2)$ but $\gamma_s(t_1) \leq \gamma_s(t_2)$ and $\gamma_s(t_2) \neq \gamma_s(t_1)$. This is a contradiction. □

4. Results. We used the geometric model to construct an algorithm which detects deadlocks and unsafe regions in *PV* models, without loops, where the geometric model is the n-cube I^n in $\mathbb{R}^n$ with some n-rectangles removed [3]. There is a way of translating a system with loops to several systems without loops (by taking "diconnected coverings" in the proper sense) [2], so we can now handle systems with loops too.

In [5], Gunawardena used homotopy theory to prove that for *PV* systems without loops, if all processes lock all objects before they start releasing any (i.e., there is a sequence of P_{A_i} and then a sequence of V_{A_j}), then any dipath in the corresponding subset of I^n is dihomotopic to a dipath along the edges of I^n, that is, any execution is equivalent to a serial execution. This is called safety in database theory and was already known, but the proof was very tedious. We gave a more general statement in this direction in [4].

We are working on classifying dipaths up to dihomotopy, and for two processes without loops, this is in [7]. There are different candidates for higher dimensional dihomotopy invariants, and there is in general much work to do in both the computer science direction and the mathematics direction.

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INTRODUCTION TO THE SESSION ON DISCRETE MATHEMATICS

The session illustrated the wide spectrum of areas covered by Discrete Mathematics and the many applications of discrete methods; in particular, the invited talks showed how different areas are interrelated by the use of these methods.

The talk by Maylis Delest, *Combinatorics, information visualisation and algebraic languages* gave a vivid view of the application of combinatorial methods to information visualization, thus connecting combinatorics and computer science.

Combinatorial notions in topology came into play in Ulrike Tillmann's talk on *Combinatorics of the surface category and TQFTs*, where the combinatorics underlying the surface category was analyzed.

Andrea Blunck's talk on *Finite circle planes* belonged to the area of Finite Geometry; she considered finite circle planes as combinatorial structures, leading in particular to enumeration problems.

Typical discrete structures studied in algebra are finite groups; in her talk, *Conjugacy class sizes—some implications for finite groups*, Rachel Camina reported on the study of finite groups satisfying certain arithmetical conditions on the conjugacy class sizes.

Christine Bessenrodt

COMBINATORICS, INFORMATION VIZUALISATION, AND ALGEBRAIC LANGUAGES

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1. Introduction. Let Ω be a class of combinatorial objects. We suppose that they are enumerated by the integer a_n according to the value n of some parameter p. Let $f(t) = \sum_{n=0} a_n t^n$ be the corresponding generating function. One of the main problems addressed by combinatorics is finding $f(t)$ and its properties knowing a recurrence on a_n or even a sequence of the first values $a_0, a_1, a_2, \ldots$. Many books are devoted to this field [24, 32]. With computer algebra systems, new techniques have been set up for getting results [1, 2]. Internet network users may ask online information on a sequence of values [27]. This last service is an encyclopedia of integer sequences but also it gives useful references to objects that are counted by sequences. Enumerative combinatorics is focussed on getting more inside the formula using bijection with object classes. We give an old trite example due to Euler in order to enlighten what we call *getting inside formula*. Let s_n be defined by

$$s_n = \sum_{i=0}^{n} (2i+1).$$

Of course, we have $s_n = (n+1)^2$. This result can be obtained by algebra but also explained by a geometrical construction. For each i, the value $(2i+1)$ is represented by a hook of $(2i+1)$ cells in the plane $\mathbb{N} \times \mathbb{N}$. Then, the s_n value is constructed by putting the hooks upon each other. See Figure 1.1.

In this paper, we focus on methods intensively studied by the *Combinatorics Bordeaux School* and some of their applications. After some definitions and notations, we describe in Section 3 the DSV methodology and in Section 4 two extensions that are object grammars and Q-grammars. At the end, we show applications of these techniques to information visualization.

2. Definitions and notations. This section summarizes briefly the notions needed for understanding this paper. A more complete background can be acquired from [3, 4, 23]. Let X be a nonempty set called alphabet. The elements of X are called letters. A word is a finite sequence of letters from X. The empty word is usually denoted by ϵ. Let u and v be two words on X,

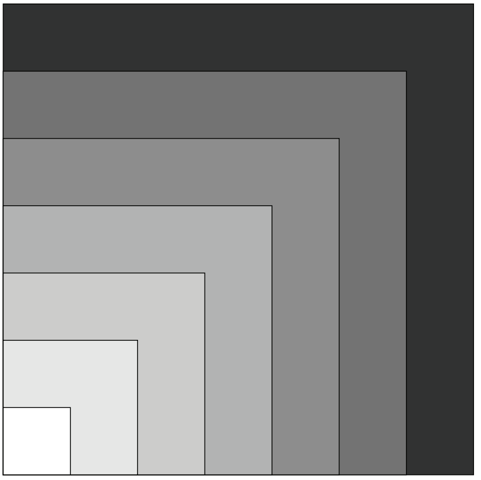


FIGURE 1.1. Euler proof for $n = 6$, $s_n = 49$.

$u = u_1 \cdots u_p$ and $v = v_1 \cdots v_q$. We define the concatenation of two words to be $uv = u_1 \cdots u_p v_1 \cdots v_q$. We denote by X^* the free monoid generated by X, that is, the set of all words on X endowed with the operation of concatenation. The number of occurrences of the letter x in the word u is denoted by $|u|_x$. The number of letters of a word w is called length of w and is denoted by $|w|$. A language is a subset of X^*. To every language $\mathcal{L}$, one can associate a noncommutative formal power series

$$L = \sum_{w \in \mathcal{L}} w,$$

that is, an element of the algebra $\mathbb{Z}\langle\langle X \rangle\rangle$ of noncommutative formal power series with variables in X and coefficients in $\mathbb{Z}$.

DEFINITION 2.1. An algebraic grammar is a 4-tuple $G = \langle N, X, P, s \rangle$ such that N and X are two disjoint alphabets called, respectively, the nonterminal and the terminal alphabet, s is an element of N called axiom, and P is a set of pairs (α, β) with $\alpha \in N$ and $\beta \in (N \cup X)^*$ called production rules and is denoted by $\alpha \to \beta$.

Let α be in N and u in $(N \cup X)^*$, $u = u_1 \alpha u_2$. A derivation in G is a rewriting of u as $v = u_1 \beta u_2$ with $\alpha \to \beta$. This is denoted by $u \to v$. We say that a word w is deriving from a nonterminal symbol α in G if there exists a sequence of derivations which rewrites α as w. This will be denoted by $\alpha \xrightarrow{*} w$. The set $L(G)$ of words generated by s is called the algebraic language generated by G. In general, there may exist several grammars for a given algebraic language.

EXAMPLE 2.2. The main example in combinatorics is the Dyck language, not because of its complexity, but because of the frequency with which it occurs in

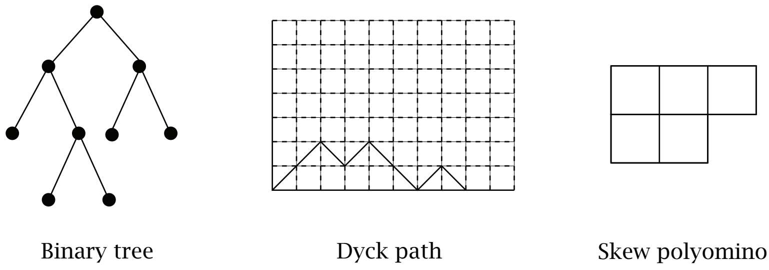


FIGURE 2.1. Combinatorial objects encoded by the word xxyxyyxy.

different settings. It encodes numerous and diverse structures such as trees, paths, polyominoes. See Figure 2.1. Its words are generated by the grammar G_1 given by

$$N = \{D\}, \qquad X = \{x, y\}, \qquad s = D,$$

and the production rules

$$D \longrightarrow xDyD, \quad D \longrightarrow \epsilon.$$

This example gives rise to unambiguous algebraic grammars, that is, algebraic grammars in which every word is obtained only once from the axiom using the production rules in a left-right derivation that is deriving first the leftmost terminal. In such cases, the formal power series associated to the language verifies equations which follow directly from the production rules. In our example,

$$D = xDyD + \epsilon.$$

In the following, all the grammars are considered to be unambigous.

3. DSV methodology. This methodology stems from an idea of M. P. Schützenberger from 1959 [25, 26]. This method is now known as the DSV-methodology, following M. P. Schützenberger's wish expressed to Viennot [30].

Let X be an alphabet, $X = \{x_1, \dots, x_k\}$. The commutative image of a series produces, from a noncommutative formal power series, a commutative one, called an enumerative series of the language $\mathscr{L}$. This is defined by

$$\chi_0(\mathscr{L}) = \sum_{i_1, i_2, \dots, i_k \in \mathbb{N}^k} n_{i_1, i_2, \dots, i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$$

such that $n_{i_1, i_2, \dots, i_k}$ is the number of words w in $\mathscr{L}$ such that $|w|_{x_j} = i_j$ for each j in $[1 \cdots k]$. In this way, we obtain an application from the Boolean semi-ring $\mathbb{B}\langle\langle X\rangle\rangle$ to the semi-ring $\mathbb{N}\langle\langle X\rangle\rangle$ of commutative formal power series with variables in X. We will often denote by L the series $\chi_0(\mathscr{L})$. The application χ_0 is not a morphism but the following theorem holds.

Theorem 3.1. *The image of an algebraic language under χ_0 is an algebraic series which is one of the components of the solution of the system of equations obtained via χ_0 from an unambiguous grammar of the language.*

Example 3.2. The computation on the Dyck language classically leads to the equation:

$$D(x,y) = xyD(x,y) + 1.$$

An elementary computation shows that

$$D(x,y) = \frac{1-\sqrt{1-4xy}}{2x^2},$$

from which we deduce easily that the number of Dyck words of length 2n is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Numerous results, in several areas, were obtained by this method: Polyominos [10, 12] tRNA structures [28]. Some overviews can be found in [7, 31].

Remark 3.3. Some algebraic languages cannot be associated to unambiguous grammars. Flajolet [16] has shown that their generating series are related to transcendental series.

So, the first step in the Schützenberger methodology, namely the encoding, requires particular insight: One must find a bijection between the objects and an algebraic language. We remark that, frequently, the language which is obtained in the bijection process turns out to be closely related to the Dyck language. We will see in Section 4 an explanation of this fact. Thus, in the following, we describe bijections linked directly to Dyck words. Let us consider the set B of binary trees. If b is in B, then it admits the following recursive description: Either $b = (\mathrm{root}(b), L(b), R(b))$ where $\mathrm{root}(b)$ is an internal node called root and $L(b)$ ($R(b)$, respectively) is the left (right, respectively) tree, or b is a single point called leaf. To encode a tree by a Dyck word, traverse the tree in left first depth-first order (or prefix order, that is, visiting first the root, then the left subtree, then the right subtree). During the traversal, write x at each internal node and y at each leaf, except the last one. This is the classical bijection between binary trees and Dyck words [29]. One deduces the following well-known result.

Theorem 3.4. *The number of binary trees having $n+1$ leaves is the Catalan number C_n.*

We now consider the set of paths in $\mathbb{N} \times \mathbb{N}$ which are sequences of points $(s_0, s_1, \ldots, s_n)$. The pairs (s_i, s_{i+1}) are called elementary steps. They are North-East (South-East, respectively) if $s_i = (k, k')$ and $s_{i+1} = (k+1, k'+1)$ (respectively, $s_{i+1} = (k+1, k'-1)$). The height of the step s_i is k'. These paths are

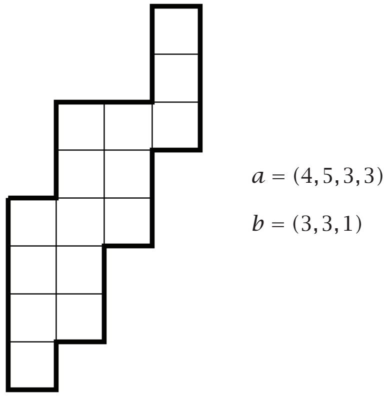


FIGURE 3.1. Euler proof for $n = 6$, $s_n = 49$.

clearly in bijective correspondence with Dyck words: Simply follow the path from s_0 to s_n, encoding each NE step by x, and each SE step by y. The word obtained in this manner is a Dyck word if and only if $s_0 = (0,0)$ and $s_n = (\lfloor n/2 \rfloor, 0)$. The corresponding path is frequently referred to a Dyck path. Another subject where the Schützenberger methodology gives good results is that of polyomino enumeration. For surveys, we refer the interested reader to [7, 18, 31]. A polyomino can be described as a finite connected union of cells (unit squares) in the plane $\mathbb{N} \times \mathbb{N}$, without cut points. A column (row, respectively) of a polyomino is the intersection of the polyomino with an infinite vertical (horizontal, respectively) strip of cells. A polyomino is column-convex (row-convex, respectively) if every column (row, respectively) is connected. A skew polyomino is both row- and column-convex and for each one of its columns there is

- no column on its right with a cell lower than its lowest cell,
- no column on its left with a cell higher than its highest cell.

An analysis of these constraints leads to an alternate definition of a skew polyomino, as a pair of integer sequences $(a_1,\dots,a_n)$ and $(b_1,\dots,b_{n-1})$, where a_i is the number of cells belonging to the ith column and $b_i + 1$ is the number of adjacent cells from columns i and $i+1$. In Figure 3.1, is displayed a skew polyomino and the two sequences a and b. These two sequences can be viewed as the heights of the peaks (step North-East followed by a step South-East) and the heights of the troughs (step South-East followed by a step North-East) in a Dyck path. So encoding a skew polyomino by a Dyck word is straightforward and it is easy to deduce the next result.

THEOREM 3.5. *The number of skew polyominos whose perimeter equals* $2n+2$ *is the Catalan number* C_n.

This result was already known a long time ago. The bijection from the Schützenberger methodology merely explains combinatorially the link between

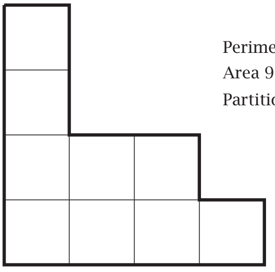


FIGURE 4.1. A Ferrer diagram.

polyominos and Catalan numbers. The first new result [12], in enumerative combinatorics obtained by this methodology, pertains to convex polyominos.

THEOREM 3.6. *The number p_{2n} of convex polyominos having a perimeter $2n+8$ is*

$$p_{2n} = (2n+11)4^n - 4(2n+1)\binom{2n}{n}.$$

This result was first proved with bijection with languages constructed from Dyck one and heavy computations. A totally bijective proof was given by Mireille Bousquet-Mélou [5].

4. Extension of DSV-methodology

4.1. Q-grammars. The study of compilers in computer science shows that the semantic attribute method described by Irons [20, 21] and then by Knuth [22] allows the translation of words from an algebraic language. Most of the resulting translations, however, are not algebraic languages. In the context of enumerative combinatorics, the same set of objects may lead to an algebraic generating function if counted according to a certain parameter, and to a nonalgebraic one if counted according to another. For example, the generating function for Ferrers diagrams (that is representation of partition of an integer, see Figure 4.1) is algebraic according to the perimeter of the diagram

$$f(x) = \frac{x^2}{1-2x}$$

and not algebraic according to the number of cells

$$f(q) = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

The interest presented by the attribute method lies in the fact that translation is defined locally, on each production rule of the grammar. Formally, in the combinatorics background, we have the

Definition 4.1. Let $G = (X, N, P, S)$ be a grammar. For each $U \in N$, an attribute family defined on G is given by a finite set T_U of attributes.

- Each attribute $\tau \in T_U$ has a domain D_τ; the cartesian product $\prod_{\tau \in T_U} D_\tau$ is denoted by $\mathcal{D}_U$;
- for each attribute $\tau \in T_U$ and for each derivation in P of U, $R : U \to w_0 U_1 \cdots U_k w_k$, a computation rule is defined $f_{\tau,R}$, that is a function from $\mathcal{D}_{U_1} \times \cdots \times \mathcal{D}_{U_k}$ into D_τ.

$(G, (T_U)_{U \in N})$ is called an attribute grammar.

For each word $w \in L(G)$, and each attribute τ, the function describes the recursive computation of $\tau(w)$.

It can be shown (see [9]) that if the attribute system is well defined (in a sense that we will not explain here), then a system of q-equations can be obtained directly from the q-grammarattribute grammar. Adding attributes to a grammar introduces nonalgebraic substitutions in the commutative equations. Here, we just give a trite example.

Example 4.2. The language coding Ferrers diagrams is the language encoding their profile by means of words of the form $w = aub$ written on the alphabet $\{a, b\}$. A grammar for this language is

$$G = \langle \{S, L\}, \{a, b\}, \{S \longrightarrow aLb, L \longrightarrow aL, L \longrightarrow bL, L \longrightarrow \epsilon\}, S \rangle.$$

The attribute grammar (G, τ) defined below computes the number of cells based on this encoding:

$$\begin{aligned}
S &\longrightarrow aLb, & \tau(S) &= q^{|\tau(L)|_a + |\tau(L)|_b + 1} ab\tau(L), \\
L &\longrightarrow aL, & \tau(L) &= q^{|\tau(L)|_b} a\tau(L), \\
L &\longrightarrow bL, & \tau(L) &= b\tau(L), \\
L &\longrightarrow \epsilon, & \tau(L) &= 1.
\end{aligned}$$

It is easy to show that the system of q-equations is

$$S(a, b; q) = qaL(aq, bq; q)b,$$
$$L(a, b; q) = aL(a, bq; q) + bL(a, b; q) + 1,$$

from which one can deduce the well-known generating function

$$s(a, b; q) = \sum_{n=0}^{\infty} \frac{a^n q^{n+1}}{(1 - qb)(1 - q^2 b) \cdots (1 - q^{n+1} b)}.$$

Systems of q-equations can be obtained by this method, but solving them remains challenging even in cases when they give very nice results (see [11]). We must point out a general solution for q-equation given by M. Bousquet-Mélou in [6].

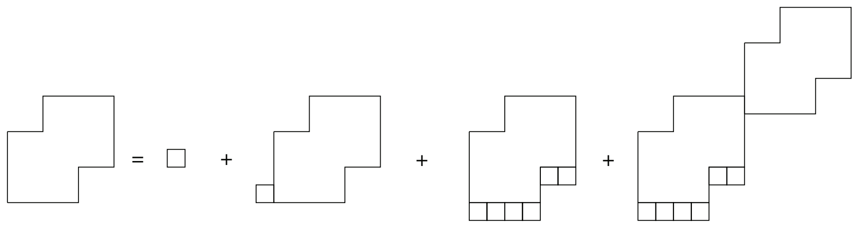


FIGURE 4.2. An object grammar for skew polyominoes.

4.2. Object grammars. Another extension of the DSV-methodology is object grammars, due to Dutour and Fédou [15]. They describe a Schützenberger method without word that is based only on the unambigous recursive decomposition of the combinatorial objects. Other methods have a similar approach see [17]. We give a definition, then we describe some applications. Let $\mathscr{E}$ a set of combinatorial objects.

DEFINITION 4.3. Let $\{E_i\}_{i=1,k}$ and E be subsets of $\mathscr{E}$. An object operation is an application from $E_1 \times E_2 \times \cdots \times E_k$ in E.

DEFINITION 4.4. An object grammar is a 4-tuple $G = \langle \mathscr{F}, \mathscr{T}, P, \int \rangle$ such that

- $\mathscr{F}$ is a set of subsets of $\mathscr{E}$,
- $\mathscr{T}$ is a finite set of terminal objects in $\mathscr{E}$,
- P is a set of object operations defined on k-tuples of $\mathscr{F}$ with value in $\mathscr{F}$,
- $\int$ is in $\mathscr{F}$ and is called axiom.

Clearly if $\mathscr{T}$ is an alphabet and $\mathscr{E} = \mathscr{T}^*$, then an algebraic language can be defined by an object grammar. As for algebraic language, one can define the derivation of an object from $\int$ in the object grammar G.

EXAMPLE 4.5. We come back to skew polyominoes. Let $\mathscr{E}$ be the set of polyominoes, and let $\mathscr{S}$ be the set of skew polyominoes. We define

- $\mathscr{F} = \{\mathscr{S}\}$,
- $\mathscr{T} = \{\square\}$, that is, the polyomino with only one cell,
- $\int = \mathscr{S}$.

Let O_1 and O_2 be skew polyominoes, then the objects operations are the following:

- ϕ_1 consists of adding to O_1 a column with one cell on its left in order to get a new skew polyomino,
- ϕ_2 consists of adding one cell to each column of O_1,
- ϕ_3 consists of gluing two polyominoes O_1 and O_2 by the rightmost upper cell of O_1 and the leftmost downer cell of O_2.

Clearly, we have the recursive equation

$$\mathscr{S} = \square + \phi_1(\mathscr{S}) + \phi_2(\mathscr{S}) + \phi_3(\phi_2(\mathscr{S}), \mathscr{S})$$

that is pictured in Figure 4.2.

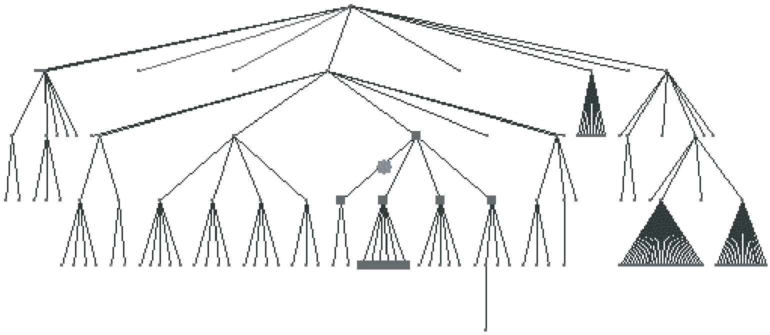


FIGURE 4.3. A view of the Latour software.

In enumerative combinatorics, Catalan numbers and Dyck words are involved in a lot of constructions. One central result in objects grammars enlightens this fact.

Let $G = \langle \mathcal{F}, \mathcal{T}, P, \int \rangle$ be an object grammar such that $\mathcal{F} = \mathcal{D}$. Dutour and Fédou associate a characteristic polynomial $g(x)$ to G. We define it for a one-dimensional system. It is obtained by substituting in the right part of the rule associated to $\mathcal{D}$ each occurence of $\mathcal{D}$ by x and each terminal object by 1, forgetting the object operations.

EXAMPLE 4.6. From the previous equation, we get $g(x) = 1 + 2x + x^2$.

Using the notion of substitution and constructing the solution, Dutour and Fédou proved the following theorem.

THEOREM 4.7. *Two one-dimensional grammars of degree almost two are isomorphic by the substitution process.*

As a consequence, for a large class of combinatorial objects, bijections can be constructed from the Dyck language. Nice examples are given in [15]. Of course, the notion of a polynomial can be extended to a system having higher degree than one. Moreover, they give an efficient tool for the random generation of objects. The Maple package is available at

http://dept-info.labri.u-bordeaux.fr/˜dutour/QALGO

5. Application to information visualization. In this section, we describe the software Latour [19] developped by CWI and LaBRI. This software deals with tree visualization. The problem of displaying and interacting with large set of information can be abstracted to the same problem for graphs. Latour is devoted to the special case of tree. A lot of software try to have a very good drawing of the structure, [13]. The scale of information visualization raises up structures having frequently several thousands of nodes. Insteed of trying

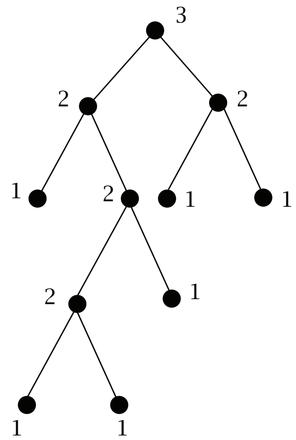


FIGURE 5.1. Strahler number computation for binary trees.

to solve the complete problem, the Latour's approach is to use enumerative combinatorics in order to construct tools for exploring or folding parts of the structure.

In Figure 4.3, a tree is shown with Latour menus. The main goal in Latour is to construct measures on the tree that help the user in watching information. Here, we want to enlighten only two measures: Guiding the user in a zoom function and folding automatically subtrees that are too big or too small.

5.1. Strahler numbers as a user guide. Strahler numbers are very classical numbers in lots of fields as biology, computer science [31]. They were defined by the geographers Norton and Strahler in order to give a mathematical definition for fluvial bassin. The definition for Strahler numbers on binary trees can be done as follow.

DEFINITION 5.1. Let $b = (\mathrm{root}(b), L(b), R(b))$ be a binary tree. To each node $v \in b$, the Strahler number $S(v)$ of v is

- if v is a leaf then $S(v) = 1$,
- if $S(\mathrm{root}(L(b))) = S(\mathrm{root}(R(b)))$ then $S(v) = S(\mathrm{root}(L(b))) + 1$,
- else $S(v) = \max(S(\mathrm{root}(L(b))), S(\mathrm{root}(R(b))))$.

The Strahler number of the tree is $S(\mathrm{root}(b))$.

An example is displayed on Figure 5.1. Many extensions of Strahler numbers can be set for plane trees. One of them, due to Fédou, is meaningful according to the computer science definition that is the minimum number of registers for computing an arithmetical expression [8]. In this case, the Strahler numbers are rather given by an algorithm than by a formula. Roughly, the value of the

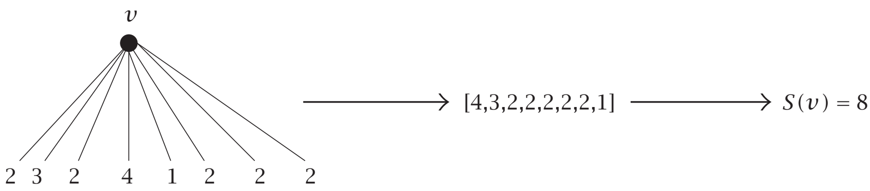
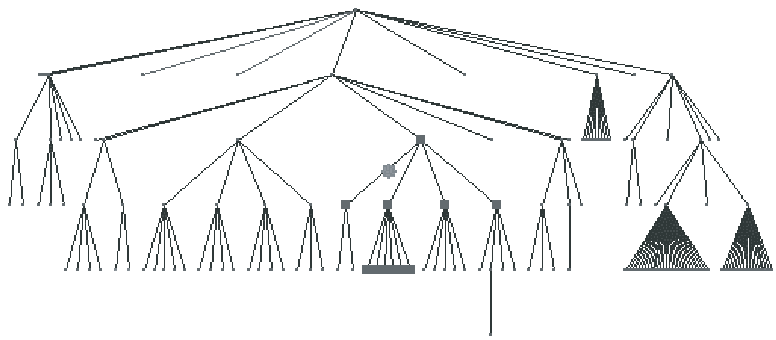
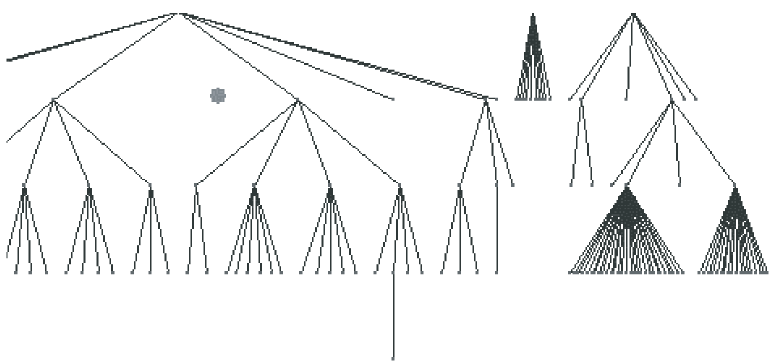


FIGURE 5.2. Strahler number computation for plane trees.



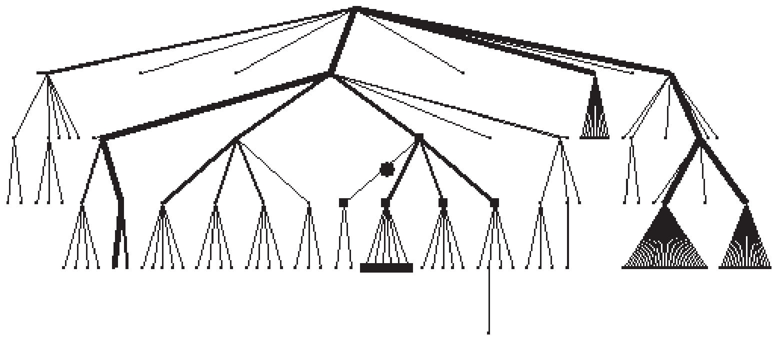
(a) Simple view.



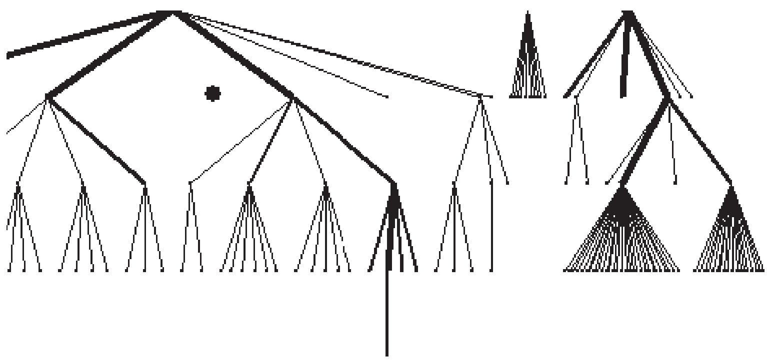
(b) Simple zoom view.

FIGURE 5.3. Views from latour.

leaves are 1. Suppose that a vertex v of the tree has sons $v_1, v_2, \ldots, v_k$ then rank the $v_{i\,i=1\cdots k}$ in decreasing order, it gives a list of values $u(i)_{i=1\cdots k}$. Then



(a) Strahler view.



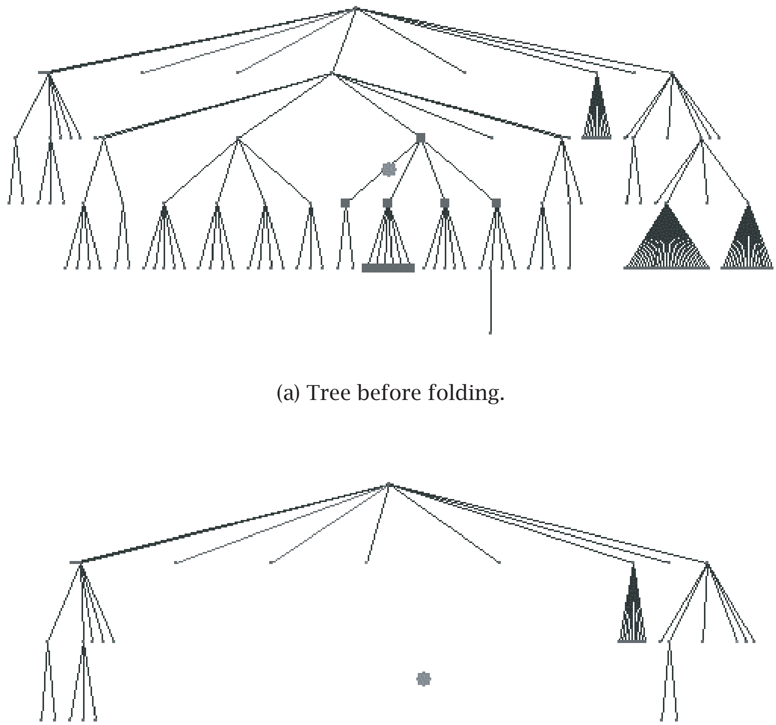
(b) Strahler zoom.

FIGURE 5.4. Views from latour.

do

```
s:=u(1);
free:=u(1)-1;
for i from 2 to k do
  if free<u(i) then s:=s+1
               else free:=free-1
  fi
od;
```

Then the variable s contains the value of $S(v)$ (see Figure 5.2).



(a) Tree before folding.

(b) Tree after folding confidence level 5%.

FIGURE 5.5. Latour: a fold.

A part of the mathematical study of this parameter on trees, coloring edges according to this parameter in a tree visualization software guide the user during a zoom. It shows him which relative importance has the zoom window in the whole tree. We have experimented this technic in several fields (data structures for compilers, file hierarchical systems, …). In Figure 5.3 (respectively, Figure 5.4), we give two views of the same tree with zoom without (respectively, with) Strahler measure.

5.2. Folding trees using leaves numbers. The number of leaves of a plane tree is a very classical parameter. We have the well-known result.

THEOREM 5.2. *The number of tree B having n nodes and k leaves is*

$$C_{n,k} = \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}.$$

Let f_n be the random variable number of leaves in a tree having n nodes. Then, as soon as n is greater than 10, f_n as a normal distribution with mean $n/2$ and standard deviation $\sqrt{(n/8)}$. This approximation can be deduced based on a much more general (and highly nontrivial) theorem described in a paper of M. Dmrota [14].

In Latour, we use this distribution for folding automatically subtrees that are "unusually" large or small with respect to the number of leaves. We must point out, that, at this stage, we do not take into account the number of leaves of the full tree. This can change drastically the probability law. In the future, the fold tool will offer to the user to take into account the number of leaves in the general tree and also two others features:

- the maximum degree of the nodes,
- the maximal length of paths such that each node has only one son.

Anyway, users agree on this tool. In Figure 5.5, we show the folding effect. The Latour software is available at http://www.cwi.nl/InfoVisu.

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COMBINATORICS OF THE SURFACE CATEGORY AND TQFTS

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The $(1+1)$-dimensional cobordism category of closed 1-dimensional manifolds and oriented surfaces is a most basic mathematical structure. It has played a fundamental role in string theory and conformal field theory. We give a brief account of its structure and indicate what structure it gives vector spaces and other algebraic objects on which it (or an embellished version of it) acts. We explain how such functors lead to invariants of 2- and 3-dimensional manifolds and give an application to topological conformal field theory.

1. $(1+1)$**-dimensional TQFTs.** Let $\mathscr{C}$ be the category of closed 1-dimensional manifolds up to diffeomorphisms and oriented cobordisms up to diffeomorphisms. The objects of $\mathscr{C}$ are thus in one-to-one correspondence with the natural numbers, $n \in \mathbb{N}$ representing n copies of the unit circle, and the morphisms are oriented surfaces $F : n \to m$ with boundary $\partial F = n \sqcup m$ which may have any number of components $c(F)$ with sum of their genera $g(F)$. Denote by

$$
\begin{aligned}
D &: 1 \longrightarrow 0, && \text{the disk;}\\
P &: 2 \longrightarrow 1, && \text{the pair of pants surface;}\\
C &: 1 \longrightarrow 1, && \text{the cylinder: The identity morphism;}\\
T &: 1 \longrightarrow 1, && \text{the torus with two boundary components.}
\end{aligned}
$$

Composition in $\mathscr{C}$ is given by gluing along common boundary components. There is also a monoidal functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$ given by disjoint union. An important observation is that this monoidal product is symmetric.

Let $\mathscr{C}_b$ be the subcategory in which no component of F is a cobordism to the empty manifold zero (thus D is not in $\mathscr{C}_b$), and let $\mathscr{C}_1$ be the subcategory with the single object 1 and all cobordisms connected (neither D nor P are in $\mathscr{C}_1$). Note $\mathscr{C}_1$ is isomorphic to the category with one object and morphism set the natural numbers representing the genus of the surface.

Definition 1.1. A $(1+1)$-dimensional topological quantum field theory (TQFT) is a nontrivial monoidal functor $\mathscr{A}$ from $\mathscr{C}$ to the category of vector spaces with monoidal structure given by tensor product.

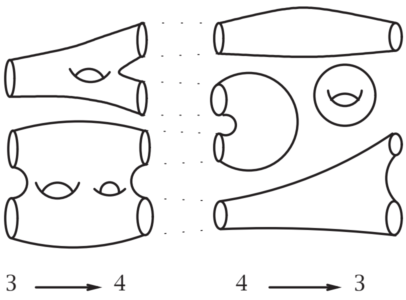

FIGURE 1.1. Composition in $\mathscr{C}$.

The following theorem is folklore and can be proved as an exercise. The converse holds as well (cf. [1]).

THEOREM 1.2. *$\mathscr{A}(1)$ is a finite-dimensional commutative Frobenius algebra: The product is given by $\mathscr{A}(P)$ and the trace by $\mathscr{A}(D)$.*

A Frobenius algebra is an algebra with trace such that the composite of product and trace is a non-degenerate inner product.

Note that $\mathscr{A}$ determines a topological invariant of closed surfaces: As $\mathscr{A}$ is monoidal $\mathscr{A}(0) = \mathbf{C}$ and hence the linear function associated to a closed surface is an element of $\mathbf{C}$, our invariant. It is well known that the universal topological invariant of a closed surface is its Euler characteristic χ. Define a functor Φ from $\mathscr{C}$ to the category with one object and morphisms the integers by assigning to a cobordism $F : n \to m$ the number

$$\Phi(F) := \frac{1}{2}\big(m - n - \chi(F)\big) = g(F) + m - c(F).$$

This functor detects most of the combinatorics of $\mathscr{C}$: When F is in $\mathscr{C}_b$, $\Phi(F) \geq 0$, and the restriction of Φ to $\mathscr{C}_b$ is left adjoint to the inclusion functor $\mathscr{C}_1 \to \mathscr{C}_b$. Φ factors through the localization $\mathscr{C}[\mathscr{C}^{-1}]$ of $\mathscr{C}$ in which all morphisms are formally inverted. It induces an isomorphism of endomorphism sets $\mathscr{C}[\mathscr{C}^{-1}](n,n) \simeq \mathbb{Z}$ for each $n \geq 0$. See [8] for details and applications.

REMARK 1.3. Similarly one can define TQFTs in higher dimensions leading to invariants of higher dimensional manifolds. Particularly interesting examples are found in dimensions $2+1$ and $3+1$ (see [2, 7] for surveys of the motivating examples).

2. $(1+1)$**-dimensional 2-TQFTs.** We embellish $\mathscr{C}$ by considering also morphisms of cobordisms. The result is a 2-category which we denote by $\mathscr{S}$. Given a cobordism F, its set of automorphisms in $\mathscr{S}$ is defined to be the mapping

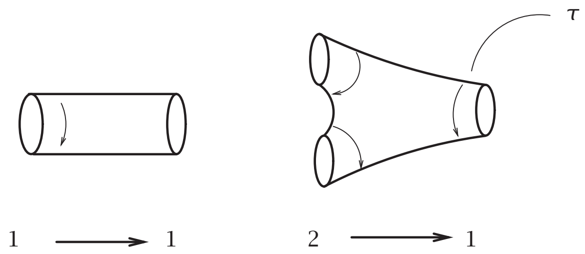

FIGURE 2.1. Generators in $\Gamma(C)$ and $\Gamma(P)$.

class group

$$\Gamma(F) = \pi_0 \mathrm{Diff}^+(F, \partial F),$$

the components of the group of orientation preserving diffeomorphisms of F that leave the boundary pointwise fixed. For example, $\Gamma(C)$ is $\mathbb{Z}$ and $\Gamma(P)$ is the pure ribbon braid group on two strands, $\mathbb{Z}^3$; these two groups are generated by diffeomorphisms that result when twisting one of the boundary components by a full turn. We denote the element corresponding to the twist around the target circle of P by τ. The composition and the symmetric monoidal structures of $\mathscr{C}$ can be extended to $\mathscr{S}$ in an obvious way.

The target of a functor from $\mathscr{S}$ is naturally another 2-category. We are also now interested in finding invariants of closed three manifolds. Thus vector spaces need to occur in the categorical hierachy not as objects but as morphisms. This motivates the following definitions.

A $\mathbf{C}$-category is a category in which the morphisms sets are $\mathbf{C}$-vector spaces and composition is bilinear; for example, the category of finite dimensional $\mathbf{C}$-vector spaces $\mathrm{Vect}_{\mathbf{C}}$, and the category of finitely generated projective modules $\mathrm{Mod}(A)$ over a $\mathbf{C}$-algebra A. The tensor product of two $\mathbf{C}$-categories is formed by taking the Cartesian product of the objects and the usual tensor product of the morphism spaces. The unit element for this tensor product is thus the category with one object and morphism set $\mathbf{C}$.

We need to impose two technical conditions on the $\mathbf{C}$-categories: (1) that coproducts exist, and (2) that the kernel of idempotents exist. Such categories are called complete. Every $\mathbf{C}$-category has a unique completion; for example, the completion of the unit element is just $\mathrm{Vect}_{\mathbf{C}}$. Let $\mathbf{C}$-CAT denote the 2-category with complete $\mathbf{C}$-categories as objects, linear functors as morphisms, and natural transformations as morphisms between morphisms.

DEFINITION 2.1. A $(1+1)$-dimensional 2-TQFT is a (nontrivial) monoidal functor $\mathscr{A}$ of 2-categories from $\mathscr{S}$ to $\mathbf{C}$-CAT.

THEOREM 2.2 (see [9]). *$\mathscr{A}(1)$ is a Frobenius category. Furthermore, $\mathscr{A}(1)$ is equivalent to* $\mathrm{Mod}(A)$ *of a semi-simple algebra A, where A is the endomorphism set of a particular element in $\mathscr{A}(1)$.*

A Frobenius category is the categorical version of a Frobenius algebra. In particular, $\mathscr{A}(1)$ is braided monoidal (the product given by $\mathscr{A}(P)$, the braiding by the natural tranformation $\mathscr{A}(\tau)$), and balanced (the generator of $\Gamma(C) = \mathbb{Z}$ induces a natural transformation of the identity functor).

It can be deduced that $\mathscr{A}$ determines a modular functor in the sense of [6]. Hence, it determines a topological invariant of closed 3-mainfolds by unpublished work of Kontsevich and Walker which shows that every modular functor determines a TQFT in dimension $2+1$.

3. CFTs and TCFTs. Replace $\mathscr{C}$ now by the category $\mathscr{M}$ in which the surface F is replaced by the moduli space of complex surfaces $\mathscr{M}(F)$ of topological type F. Thus a morphism from n to m is now a Riemann surface. Composition and symmetric monoidal structure are defined just as in $\mathscr{C}$.

The morphism sets of $\mathscr{M}$ have a natural topology. One way to study $\mathscr{M}$ algebraically is via the 2-category $\mathscr{S}$ of the previous section. Note that

$$\mathscr{M}(F) \simeq B\Gamma(F)$$

whenever F is a connected surface with at least one boundary component; here $BG = K(G,1)$ denotes the classifying space of the discrete group G. Another way to study $\mathscr{M}$ algebraically is to replace the spaces $\mathscr{M}(F)$ by their chain complexes $C_*\mathscr{M}(F)$. We will denote the resulting category by $C_*\mathscr{M}$.

Definition 3.1. (1) A conformal field theory (CFT) is a (nontrivial) monoidal functor $\mathscr{A}$ from $\mathscr{M}$ to the category of Hilbert spaces.

(2) A topological conformal field theory (TCFT) is a (nontrivial) monoidal functor $\mathscr{A}$ from $C_*\mathscr{M}$ to the category of $\mathbf{C}$-chain complexes.

CFTs are very hard to study (see [5]). We restrict ourselves to TCFTs. The chain complex $V := \mathscr{A}(1)$ is known as the BRST-complex and its homology H_*V is the space of physical states. Note that each element in $H_*\mathscr{M}(F)$ for $F : n \to m$ defines an operation $H_*V^{\otimes n} \to H_*V^{\otimes m}$.

Theorem 3.2 (see [3])**.** *H_*V is a Batalin-Vilkovisky algebra.*

Batalin-Vilkovisky algebras are graded Poisson algebras with a differential. The product (of degree 0) on H_*V is given by the image of the unit in $\mathbb{Z} = H_0\mathscr{M}(P)$; the Lie bracket $[\ ,\ ]$ (of degree 1) is the image of τ in $\Gamma(P) = \mathbb{Z}^3 = H_1\mathscr{M}(P)$; the differential $\triangle$ (of degree 1) is the image of the unit in $\Gamma(C) = \mathbb{Z} = H_1\mathscr{M}(C)$.

Getzler's theorem above takes only the operations coming from genus zero surfaces into account. The following theorem, however, says that these are not indicative for the structure of H_*V.

Theorem 3.3 (see [10])**.** *Let t be image under $\mathscr{A}$ of the unit in $\mathbb{Z} = H_0\mathscr{M}(T)$. Then in the localization $H_*V[t^{-1}]$, the differential $\triangle$ and the Lie bracket $[\ ,\ ]$*

are trivial. More precisely, in H_*V

$$t^3\triangle = 0, \qquad t^3[\ ,\ ] = 0.$$

The essential ingredient of the proof is Harer's stability theorem [4]. It says that the homology of the moduli spaces is independent of the genus and the number of boundary components in dimensions less than half the genus. Thus twisting a boundary component of a surface with sufficiently high genus has the same effect on homology as when that boundary component is closed by gluing in a disk. The twist, however, becomes homotopic to the identity in the resulting surface. As $\triangle$ is of degree 1, genus 3 is sufficient and hence $t^3\triangle = 0$. Similarly, τ vanishes in homology when the surface has at least genus 3, and hence $t^3[\ ,\ ] = 0$.

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CONJUGACY CLASS SIZES—SOME IMPLICATIONS FOR FINITE GROUPS

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1. Introduction. Let G be a finite group and x be an element of G. The *conjugacy class of x in G* is defined to be

$$x^G = \{g^{-1}xg : g \in G\}.$$

Using the orbit-stabiliser theorem, it is easy to see that the size of the conjugacy class $|x^G|$ is equal to $|G : C_G(x)|$ where $C_G(x) = \{g \in G : g^{-1}xg = x\}$ is the centraliser of x in G, and thus $|x^G|$ is sometimes called the *index of x in G*. A common theme in group theory has been to deduce structural results about G given arithmetical conditions on the indices of the elements of G. One of the earliest and most important results is the following.

Lemma 1.1 (Burnside's p^α lemma [2]). *Let G be a finite group and suppose there exists $x \in G$ with $|x^G| = p^\alpha$ for some prime p and some $\alpha \in \mathbb{N}$. Then G is not simple.*

In 1953 Baer characterised all finite groups G in which every element of prime power order has prime power index.

Theorem 1.2 (see [1]). *Every element of prime power order in the finite group G has prime power index in G, if and only if, G is the direct product of groups $G_1, \dots, G_n$ with the following properties:*

(a) *The orders of G_i and G_j are relatively prime for $i \neq j$.*

(b) *If G_i is not of prime power order, then the order of G_i is divisible by exactly two different primes and its Sylow subgroups are abelian.*

He went on to say: "The question may be raised as to the characterization of those groups whose p-elements, for just one prime p, have prime power index. The present discussion does not seem to throw much light on this question."

An answer to this question is given in the next section.

In the final section the question of whether a group can be recognised as being nilpotent, given the sizes of its conjugacy classes, is considered.

Additional motivation for studying the effect of conjugacy class sizes on the structure of a group is given by the fact that the number of conjugacy classes of G is equal to the number of irreducible characters of G over $\mathbb{C}$. Denote this set of irreducible characters by $\mathrm{Irr}(G)$. Many results have been proved concerning the influence of the arithmetical structure of the set of character degrees of G on the structure of G. It is natural to ask about possible analogous results with conjugacy class sizes replacing character degrees. This is the approach taken by Chillag and Herzog [5]. For example, Willems has proved the following (see [10]).

Character version. If $\chi(1)$ is not divisible by 4 for all $\chi \in \mathrm{Irr}(G)$, then either G is soluble or it has a normal soluble subgroup with factor group A_7.

Chillag and Herzog proved the following analogous result.

Conjugacy class version. If $|x^G|$ is not divisible by 4 for all $x \in G$, then G is soluble.

The original proof of this result uses the classification of finite simple groups. For a classification-free proof see [4].

Notation. Let p be a prime. A p-element is an element of order p^n for some $n \in \mathbb{N}$. Similarly, a p-subgroup is a subgroup of order p^n for some $n \in \mathbb{N}$. A p'-subgroup is a subgroup of order coprime to p.

2. q-Baer groups.

The following definition is inspired by Baer's question.

Definition 2.1. Let G be a finite group and q a prime dividing $|G|$. G is called a q-Baer group if all q-elements of G are of prime power index.

Note that direct q'-factors of G tend to be ignored as they have no effect on this property.

Example 2.2. (a) Let G be nilpotent and suppose q divides $|G|$, then G is a q-Baer group. This follows from the fact that G is a product of its Sylow subgroups. In particular each q-element is of q-power index.

(b) Let V be a 2-dimensional vector space over the finite field $\mathbb{F}_3$ of 3 elements. Then V is additively isomorphic to $\mathbb{F}_9$ the finite field of 9 elements. The nonzero elements of $\mathbb{F}_9$, denoted by $\mathbb{F}_9^*$, form a multiplicative group of order 8 and multiplication by an element of $\mathbb{F}_9^*$ defines an isomorphism of V. Thus, in particular, if you take the element of order 2 in $\mathbb{F}_9^*$ you have defined an action of C_2 the cyclic group of order 2, on V and you can construct the semi-direct product VC_2, this is a 2-Baer and a 3-Baer group.

More generally, let p and q be distinct primes. Let V be a vector space over the finite field $\mathbb{F}_q$ and let P be a p-group with an irreducible representation over V. Then the extension VP is a q-Baer group (note P need not be abelian).

(c) Similar to (b), let V be a 3-dimensional vector space over $\mathbb{F}_2$, so V is additively isomorphic to $\mathbb{F}_8$. Then, multipication by an element of $\mathbb{F}_8^*$ yields an action of C_7 on V and we have constructed a semi-direct product VC_7 which is both a 2-Baer and a 7-Baer group. However, if we now extend VC_7 by the field automorphism of $\mathbb{F}_8$ given by $x \mapsto x^2$ which has order 3 (and is also an automorphism of multiplicative $\mathbb{F}_8^*$), then we have a group of order 8.7.3. Moreover, although this extension is still a 2-Baer group, each 2-element has index 7, it is no longer a 7-Baer group, each 7-element has index 24 and it is not a 3-Baer group, each 3-element has index 28.

Before stating the theorem that characterises q-Baer groups, we introduce some standard definitions.

Definition 2.3. Let q be a prime. Then $\mathrm{O}_q(G)$ denotes the largest normal q-subgroup of G and $\mathrm{O}_{q'}(G)$ the largest normal q'-subgroup.

Definition 2.4. A finite group G is q-soluble if every composition factor of G whose order is divisible by q is abelian.

If G is q-soluble it has a normal series

$$1 < H_1 < H_2 < \cdots < H_n = G$$

such that the factors alternate between being q'-groups and q-groups. The minimum number of q-factors in such a normal series is the q-length of the group. Thus if G is q-soluble of q-length 1, it has a normal series of the form

$$1 \leq \mathrm{O}_{q'}(G) \leq K \leq G,$$

where $K/\mathrm{O}_{q'}(G)$ is a q-group and G/K is a q'-group.

We are now ready to state the theorem.

Theorem 2.5 (see [4]). *Let G be a q-Baer group for some prime q. Then*
(a) *G is q-soluble with q-length 1, and*
(b) *there is a unique prime p such that each q-element has p-power index. Further, let Q be a Sylow q-subgroup of G.*
(c) *If $p = q$ then Q is a direct factor of G.*
(d) *If $p \neq q$ then Q is abelian, $\mathrm{O}_p(G)Q$ is normal in G and $G/\mathrm{O}_{q'}(G)$ is soluble.*

Using the following lemma, which is readily proved, we can see how Baer's result follows from the above theorem. Suppose G is both a q-Baer and a p-Baer group for primes $p \neq q$ and that all q-elements have p-power index. Then, by the theorem and lemma, PQ is a direct factor of G where P is an abelian Sylow p-subgroup and Q is an abelian Sylow q-subgroup. Baer's theorem follows.

Lemma 2.6 (see [4]). *Let G be a q-Baer group and a p-Baer group for primes $p \neq q$. Suppose that all q-elements have p-power index. Then all p-elements have q-power index.*

The first step in the proof of the theorem is to generalise the following result of Wielandt [1].

Lemma 2.7 (Wielandt's lemma). *Let G be a finite group and $x \in G$. Suppose x is a p-element and $|x^G| = p^n$ for some prime p and some $n \in \mathbb{N}$, then $x \in \mathrm{O}_p(G)$.*

Lemma 2.8 (Generalisation of Wielandt's lemma)**.** *Let G be a finite group and $x \in G$. Suppose $|x^G| = p^n$ for some prime p and some $n \in \mathbb{N}$, then $[x^G, x^G] \subseteq \mathrm{O}_p(G)$.*

That Wielandt's result follows from the above can be seen as follows. Suppose Wielandt's hypotheses hold, by the generalisation $[x^G, x^G] \subseteq \mathrm{O}_p(G)$. This implies that $\mathrm{O}_p(G)\langle x\rangle$ is a subnormal p-subgroup of G and hence is in $\mathrm{O}_p(G)$, giving Wielandt's result.

To prove the generalisation the following result of Kazarin is used. Its proof uses the theory of modular characters and is independent of the classification of finite simple groups, and thus so is the proof of the characterisation of q-Baer groups.

Theorem 2.9 (see [9])**.** *Let $x \in G$, where G is a group of finite order. Suppose $|x^G| = p^n$ for some prime p and some $n \in \mathbb{N}$. Then $\langle x^G\rangle$ is a soluble subgroup of G.*

3. Nilpotent groups. First, we introduce some notation. In 1953, Itô [8] defined the *conjugate type vector* of a finite group G to be the r-tuple $\{n_1, n_2, \dots, n_r\}$ where $n_1 > n_2 > \cdots > n_r = 1$ are the numbers that occur as conjugacy class sizes of G. We define the product of two conjugate type vectors as

$$\{n_1, n_2, \dots, n_r\} \times \{m_1, m_2, \dots, m_s\} = \left\{n_i m_j \mid 1 \le i \le r,\ 1 \le j \le s\right\}.$$

Note that if G and H are finite groups with conjugate type vectors $\bar{n} = \{n_1, n_2, \dots, n_r\}$ and $\bar{m} = \{m_1, m_2, \dots, m_s\}$, respectively, then $G \times H$ has conjugate type vector $\bar{n} \times \bar{m}$.

A nilpotent group G is a product of its Sylow subgroups, so its conjugate type vector is of the form

$$\left\{p_1^{a_{1,1}}, \dots, p_1^{a_{1,r_1}}\right\} \times \cdots \times \left\{p_n^{a_{n,1}}, \dots, p_n^{a_{n,r_n}}\right\},$$

where the p_i are distinct primes and $a_{i,j} \in \mathbb{N}$. The question is, whether a group with such a conjugate type vector is nilpotent. So far two results in this direction have been proved.

Theorem 3.1 (see [3])**.** *Let p and q be distinct primes and $a, b \in \mathbb{N}$. A group with conjugate type vector*

$$\{p^a, 1\} \times \{q^b, 1\}$$

is nilpotent.

Theorem 3.2 (see [4])**.** *Let $p_1, \dots, p_r$ be distinct primes. A group with conjugate type vector*

$$\{p_1, 1\} \times \cdots \times \{p_r, 1\}$$

is nilpotent.

To answer this question in full it is useful to know which conjugate type vectors it is possible for a p-group to have. This question has been answered by Cossey and Hawkes.

Theorem 3.3 (see [6]). *Let p be a prime and S a finite set of p-powers containing 1. Then there exists a p-group of nilpotency class 2 for which S (ordered appropriately) is its conjugate type vector.*

It is worth noting that Isaacs has proved an analogous result for character degrees [7]. However, the question, whether you can recognise a group as being nilpotent from its conjugate type vector, is still open.

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FINITE CIRCLE PLANES

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1. Definition and examples. Circle planes are generalizations of the classical *real Möbius plane*, that is, the geometry of points and circles on the 2-sphere. Some of the essential incidence properties of this geometry appear as axioms of an abstract circle plane. Note that there are different axiomatic approaches, for example, the concepts of *Möbius plane* and of *Benz plane* (see below). Our definition of *circle planes* follows A. Herzer, which means that they are exactly the 2*-dimensional chain spaces* (see [4]).

Definition 1.1. A *circle plane* is an incidence structure $\Sigma = (\mathbb{P}, \mathscr{C})$, where $\mathbb{P} \neq \emptyset$ is the set of *points* of Σ and $\mathscr{C}$ is a set of certain subsets of $\mathbb{P}$ called *circles*, such that the following axioms are satisfied:

CP1: If $p, q, r \in \mathbb{P}$ are pairwise distant, then there is a unique $C \in \mathscr{C}$ with $p, q, r \in C$.

Here two points are called *distant*, if they are different and joined by at least one circle.

CP2: For each $p \in \mathbb{P}$ the *residue* $\Sigma_p := (\mathbb{P}_p, \mathscr{C}_p)$, with $\mathbb{P}_p := \{q \in \mathbb{P} \mid q$ distant to $p\}$, $\mathscr{C}_p := \{C \setminus \{p\} \mid p \in C \in \mathscr{C}\}$, is a *partial affine plane*, that is, an affine plane with some parallel classes of lines being removed.

We start with some small examples.

Example 1.2. Let $\Sigma = (\mathbb{P}, \mathscr{C})$ be the incidence structure with 9 points and 6 circles given in Figure 1.1, where the lines of the figure are the "circles" of Σ. Three points of Σ are pairwise distant, exactly if they constitute one of the 6 circles, so CP1 holds. The residue Σ_p at some point $p \in \mathbb{P}$ looks as in Figure 1.1.

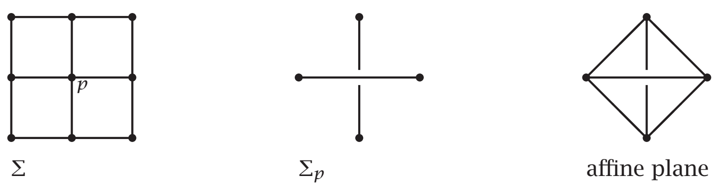

Figure 1.1.

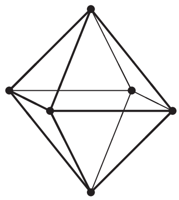

FIGURE 1.2.

So this residue is the affine plane of order 2, where two parallel classes of lines are missing. The same is true for every other point of Σ, and axiom CP2 is satisfied as well. Altogether, Σ is a circle plane.

In addition, Σ fulfils the following condition: The relation "not distant" is the union of two equivalence relations. Both equivalence relations have 3 equivalence classes of 3 elements each, which are exactly the 6 lines that are missing in the first figure in order to make Σ the affine plane of order 3.

A circle plane $\Sigma = (\mathbb{P}, \mathscr{C})$, where in each residue Σ_p exactly two parallel classes of lines are missing, and where "not distant" is the union of two equivalence relations, is called a *Minkowski plane*. So the circle plane of Example 1.2 is a Minkowski plane. Compare [7] for another, but equivalent, definition.

EXAMPLE 1.3. Consider the incidence structure $\Sigma = (\mathbb{P}, \mathscr{C})$, whose points are the 6 vertices and whose circles are the 8 faces of an octahedron (see Figure 1.2).

One can easily verify that this is also a circle plane. Here each residue is the affine plane of order 2 with one parallel class of lines missing. Moreover, the relation "not distant" is an equivalence relation.

A circle plane $\Sigma = (\mathbb{P}, \mathscr{C})$ as in Example 1.3, where in each residue Σ_p exactly one parallel class of lines is missing, and where "not distant" is an equivalence relation, is called a *Laguerre plane*. Compare again [7] for a different, but equivalent, approach.

From the two circle planes of the examples above, one can construct infinitely many new—but not very interesting—circle planes by using the following obvious fact.

REMARK 1.4. Let $\Sigma_1 = (\mathbb{P}_1, \mathscr{C}_1)$ and $\Sigma_2 = (\mathbb{P}_2, \mathscr{C}_2)$ be two circle planes with disjoint point sets. Then also the *union* $\Sigma_1 \cup \Sigma_2 := (\mathbb{P}_1 \cup \mathbb{P}_2, \mathscr{C}_1 \cup \mathscr{C}_2)$ is a circle plane.

We are going to restrict ourselves to *connected* circle planes, where any two points p, q belong to a finite sequence $p = p_0, p_1, \ldots, p_n = q$ of points with p_{i-1} distant to p_i $(i = 1, \ldots, n)$. Note that this is no substantial restriction because each circle plane is the union of its *connected components*, which in turn are circle planes.

The circle planes of the examples above are obviously connected. One can show that in fact every Minkowski plane and every Laguerre plane is connected.

2. Combinatorics of finite circle planes. A circle plane $\Sigma = (\mathbb{P}, \mathscr{C})$ is called *finite*, if $\mathbb{P}$ is a finite set. Then obviously also the circle set $\mathscr{C}$ is finite. Finite circle planes can be considered as combinatorial structures.

Let $C, D \in \mathscr{C}$ be two circles of a finite circle plane $\Sigma = (\mathbb{P}, \mathscr{C})$ that meet in a point $p \in \mathbb{P}$. Then $C \setminus \{p\}$ and $D \setminus \{p\}$ are lines of the residue Σ_p and hence lines of a finite affine plane. In particular, both C and D have $n+1$ points, where $n \geq 2$ is the order of this affine plane.

Now assume that Σ is connected. Then any two circles belong to a finite sequence of circles that meet, and hence have the same number of points, say $n+1$. As a consequence, each residue Σ_p is obtained from an affine plane of order n. Note that not all $n \in \mathbb{N}$ are admissible orders of finite affine planes (see [3]).

In a similar way, one can show that in a connected finite circle plane Σ the number $m \geq 1$ of circles joining two distant points p, q is independent of the choice of p and q. Since m is the number of lines through q in the residue Σ_p, we obtain that in Σ_p there are m parallel classes of lines of the underlying affine plane left, or, in other words, $n+1-m$ parallel classes are missing. In particular, we have $m \leq n+1$.

We summarize our observations as follows.

Proposition 2.1. *Let $\Sigma = (\mathbb{P}, \mathscr{C})$ be a connected finite circle plane. Then there are $n, m \in \mathbb{N}$ with $m \leq n+1$, such that*

- *each circle contains exactly $n+1$ points,*
- *each pair of distant points is joined by exactly m circles.*

The pair (n, m) is called the order of Σ.
If Σ is of order (n, m), then for each $p \in \mathbb{P}$ the residue Σ_p contains n^2 points and m parallel classes that consist of n lines each. So in Σ there are exactly n^2 points distant to p and mn circles through p. Moreover, if Σ has v points, then the number of circles of Σ equals $vmn/(n+1)$.

The statement on the number of circles is obtained by counting the incident point-circle pairs in Σ.

The circle planes of Examples 1.2 and 1.3 have orders $(2,1)$ and $(2,2)$, respectively. More generally, a finite Minkowski plane always has order $(n, n-1)$, and a finite Laguerre plane has order (n, n).

3. Finite Möbius planes. Apart from Laguerre and Minkowski planes, there is a third type of circle plane that has been treated in detail in the literature, namely, Möbius planes (compare, e.g., [3], where they are called inversive planes).

Let $\Sigma = (\mathbb{P}, \mathscr{C})$ be a connected circle plane. Then Σ is called a *Möbius plane*, if each residue Σ_p is an affine plane (i.e., no parallel classes of lines are missing).

Most authors only consider Möbius, Laguerre, and Minkowski planes (see [7]). They are subsumed under the name of *Benz planes*, because it was W. Benz who first studied them in a uniform way (see [1]). Pictures of small Benz planes can be found in [6].

Now let Σ be a Möbius plane. Any two different points of some residue Σ_p are distant, because their joining line in the affine plane Σ_p extends to a circle. The condition that Σ is connected yields that any two different points of Σ are distant. Hence in Möbius planes the relations "distant" and "different" coincide. In particular, the point set of a Möbius plane is the point set of an affine plane plus one extra point.

The order of a finite Möbius plane Σ always has the form $(n, n+1)$. One says for short that Σ has *order* n. If Σ is of order n, then it is a 3*-design* with parameters $(n^2+1, n+1, 1)$, meaning that Σ has n^2+1 points, each *block* (i.e., each circle) contains $n+1$ points, and any 3 points lie together in exactly one block.

In [5, Theorem 4.9], it is shown that also the converse holds.

Theorem 3.1. *The finite Möbius planes are exactly the* 3*-designs with parameters* $(n^2+1, n+1, 1)$*, where* $n \geq 2$*.*

The finite Laguerre and Minkowski planes do not admit such a nice combinatorial characterization, since the relation "distant" is more involved in these cases.

Using Theorem 3.1, one can easily check the following. Compare [3] or [4] for an analogue that also holds in the infinite case.

Theorem 3.2. *Let* $\mathrm{PG}(3,n)$ *be the* 3*-dimensional projective space of order* n *(over the finite field with* n *elements). A set* $\mathbb{O}$ *of* n^2+1 *points of* $\mathrm{PG}(3,n)$ *is called an ovoid, if each line of* $\mathrm{PG}(3,n)$ *meets* $\mathbb{O}$ *in either* 0*,* 1*, or* 2 *points.*

For an ovoid $\mathbb{O}$ *in* $\mathrm{PG}(3,n)$*, consider the set* $\mathscr{C}(\mathbb{O}) := \{\mathbb{O} \cap \mathscr{E} \mid \mathscr{E} \text{ plane in } \mathrm{PG}(3,n), |\mathbb{O} \cap \mathscr{E}| \geq 3\}$ *of plane sections of* $\mathbb{O}$*. Then the incidence structure*

$$\Sigma(\mathbb{O}) := (\mathbb{O}, \mathscr{C}(\mathbb{O}))$$

is a Möbius plane, called ovoidal Möbius plane.

The classical example of an ovoid is an elliptic quadric. The circles of the associated Möbius plane are the regular conics on the quadric. A class of non-classical ovoids are the so-called Suzuki-Tits ovoids (described, e.g., in [3, pages 52, 53]), which exist only if $n = 2^{2k-1}$.

4. Chain geometries. There is also an analytical approach to circle planes. The *chain geometries* $\Sigma(K,R)$ over associative algebras with 1 are chain spaces (compare [4]). If the K-algebra R is 2-dimensional over the field K, then $\Sigma(K,R)$ is a circle plane.

Definition 4.1. Let R be an associative algebra with 1 over the field K, such that $1_K = 1_R$. Let $\mathrm{GL}_2(R)$ be the group of invertible 2×2 matrices over R. The *projective line* over the ring R is the set

$$\mathbb{P}(R) := \left\{ R(a,b) \mid a,b \in R, \exists\, c,d \in R : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) \right\}.$$

We embed $\mathbb{P}(K)$ in $\mathbb{P}(R)$ via $K(k,l) \mapsto R(k,l)$. The group $\mathrm{GL}_2(R)$ acts on $\mathbb{P}(R)$ from the right in a natural way. A subset $\mathbb{P}(K)^\gamma$, $\gamma \in \mathrm{GL}_2(R)$, of $\mathbb{P}(R)$ is called a *chain*. Let $\mathscr{C}(K,R)$ be the set of all chains in $\mathbb{P}(R)$. Then the incidence structure

$$\Sigma(K,R) := \big(\mathbb{P}(R), \mathscr{C}(K,R)\big)$$

is called the *chain geometry* over (K,R).

There are exactly three types of 2-dimensional algebras (K,R). These give rise to Möbius, Laguerre, and Minkowski planes, respectively (see [1]).

Theorem 4.2. *Let R be a 2-dimensional algebra with 1 over the field K, and let $\Sigma = \Sigma(K,R)$ be the chain geometry over (K,R). Then one of the following cases occurs:*

(1) *The ring R is a field (a quadratic extension of K). Then Σ is a Möbius plane.*

(2) *The ring R is isomorphic to the ring $K(\varepsilon) := K + K\varepsilon$, with $\varepsilon^2 = 0$, of dual numbers over K. Then Σ is a Laguerre plane.*

(3) *The ring R is isomorphic to the direct product $K \times K$ (with componentwise addition and multiplication). Then Σ is a Minkowski plane.*

The Benz planes $\Sigma(K,R)$ are called *miquelian*, because they can be characterized by Miquel's configuration theorem (cf. [4]).

If the field K is finite, then the associated Benz planes are finite as well.

The three types of 2-dimensional K-algebras can be distinguished as follows. If R is a field, then R has no maximal ideals $\neq \{0\}$. The ring $R = K(\varepsilon)$ has exactly one maximal ideal, namely $K\varepsilon$; hence $K(\varepsilon)$ is a local ring. The ring $K \times K$ has exactly the two maximal ideals $K \times \{0\}$ and $\{0\} \times K$. Note that the number of nontrivial maximal ideals of R coincides with the number of parallel classes of lines missing in the residues Σ_p of $\Sigma = \Sigma(K,R)$.

We conclude with an example of a connected finite circle plane which is not a Benz plane.

Example 4.3 (see [2]). Let $K = \mathrm{GF}(4)$ be the field with 4 elements, considered as a subring of the ring $R = M(2\times 2, F)$ of 2×2 matrices over the field $F = \mathrm{GF}(2)$. Although R is not a K-algebra, one can define $\Sigma = \Sigma(K,R)$ as above.

Then Σ is a connected circle plane of order $(4,2)$, with 35 points and thus with $35 \cdot 2 \cdot 4/(4+1) = 56$ circles. In particular, Σ is not a Benz plane.

A similar construction works for arbitrary quadratic field extensions. However, apart from Example 4.3 one does not obtain circle planes: In all other cases three pairwise distant points are joined by more than one circle (compare [2]).

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INTRODUCTION TO THE SESSION ON HILBERT PROBLEMS

In 1900, when he was 38 years old, David Hilbert presented 23 mathematical problems to the International Congress of Mathematicians in Paris. These problems still today challenge mathematicians all over the world. Most of the problems are partially solved, some are answered in the negative, and some have been restated or generalized; new interpretations have been solved. Hilbert's problems have generated (and still generate) much new mathematical research, see, for example, *Mathematical developments arising from Hilbert problems*, in Proceedings of Symposia in Pure Mathematics, vol. 28, edited by F. E. Browder, Amer. Math. Soc. Providence, Rhode Island 1976. There is also a collection on Hilbert's Problems, edited by P. S. Alexandrov, 1969, in Russian, which has been translated into German. A new book on the Hilbert problems will appear in 2000: *We must know, we shall know; a History of the Hilbert Problems* by Jeremy J. Gray, Oxford Univ. Press, 2000 (to appear).

List of the Hilbert problems. (1) Cantor's problem of the cardinal number of the continuum (the continuum hypothesis).

(2) The compatibility of the arithmetical axioms.

(3) The equality of the volumes of two tetrahedra of equal bases and equal altitudes (see Ruth Kellerhals' lecture).

(4) Problem of the straight line as the shortest distance between two points.

(5) Lie's concept of continuous group transformations without the assumption of the differentiability of the functions defining the group.

(6) Mathematical treatment of the axioms of physics.

(7) Irrationality and transcendence of certain numbers.

(8) Problems of prime numbers (the distribution of primes and the Riemann hypothesis).

(9) Proof of the most general law of reciprocity in an arbitrary number field (see class field theory developed by Hilbert, Takagi, Artin, and others; norm rest symbols computed by Shafarevich in 1950, and further developments as in algebraic K-theory).

(10) Determination of the solvability of a Diophantine equation (see Marie-Francoise Roy's lecture).

(11) Generalization of the theory of quadratic forms over the rational numbers to an arbitrary number field (the Hasse principle 1923/24, arithmetic and algebraic theory of quadratic forms).

(12) Extension of Kroneker's theorem on abelian fields to any realm of rationality (see Norbert Schappacher, *On the history of Hilbert's twelfth problem:*

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(13) Impossibility of the solution of the general equation of 7th degree by means of functions of only two arguments.

(14) Proof of the finiteness of certain complete systems of functions (proved by Emmy Noether in 1926 for invariants of finite groups; disproved, in general, by M. Nagata in 1958).

(15) Rigorous foundation of Schubert's enumerative calculus.

(16) Problem of the topology of algebraic curves and surfaces (see Marie-Francoise Roy's lecture).

(17) Expression of definite forms by squares (see Marie-Francoise Roy's lecture).

(18) Building up of space from congruent polyhedra (n-dimensional crystallography groups, fundamental domains, sphere packing problem).

(19) Are the solutions of regular problems in the calculus of variations always necessarily analytic?

(20) The general problem of boundary values.

(21) Proof of the existence of linear differential equations having a prescribed monodromic group.

(22) Uniformization of analytic relations by means of automorphic functions.

(23) Further development of the methods of the calculus of variations.

References and links related to the Hilbert problems can be found at the URL:

http://www.mathematik.uni-bielefeld.de/˜kersten/hilbert/problems.html

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OLD AND NEW ABOUT HILBERT'S THIRD PROBLEM

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This is a summary of a survey talk about Hilbert's third problem on scissors congruence and analogous questions in hyperbolic geometry. The interested reader finds some selected publications for further information in the bibliography.

1. Introduction and some history. In the list of 23 problems proposed by David Hilbert during the Second International Congress of Mathematicians held in Paris in 1900, the third problem plays a special role, and does so in several respects.

In contrast to the other problems, the third one deals with elementary geometrical questions about the foundations of geometry. Actually, in 1899, Hilbert had just finished writing the book *Grundlagen der Geometrie* and was interested in how to teach geometry. In this context, Hilbert mentioned that—contrary to the planar case—volume computations in three-dimensional Euclidean geometry are always based on some limiting process and on methods of exhaustion. He asked for a rigorous proof that one cannot construct a theory of polyhedral volume without the continuity axiom. His precise formulation goes as follows.

The equality of the volumes of two tetrahedra of equal bases and equal altitudes. *In two letters to Gerling, Gauss expresses his regret that certain theorems of solid geometry depend upon the method of exhaustion, that is, in modern phraseology, upon the axiom of continuity (or upon the axioms of Archimedes). Gauss mentions in particular the theorem of Euclid, that triangular pyramids of equal altitudes are to each other as their bases. Now, the analogous problem in the plane has been solved. Gerling also succeeded in proving the equality of volume of symmetrical polyhedra by dividing them into congruent parts. Nevertheless, it seems to me probable that a general proof of this kind for the theorem of Euclid just mentioned is impossible, and it should be our task to give a rigorous proof of its impossibility. This would be obtained, as soon as we succeeded in "specifying two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra, and which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra."*

In the very same year, Max Dehn confirmed Hilbert's conjecture by constructing two polyhedra of equal volume which are *not* equidecomposable. I will come back to Dehn's solution below.

For a long while, the problem was forgotten until some Swiss mathematicians started to work on related questions. Among those were

– Jean-Pierre Sydler, a student of Heinz Hopf at the ETH Zürich, who extended the work of Dehn in a completely satisfactory way. For this, he obtained the Gold Medal of the Danish Academy of Sciences in 1966;

– Hugo Hadwiger and his group at the University of Bern who contributed by extending Hilbert's problem to Euclidean spaces of arbitrary dimensions.

In 1974, during the conference on *mathematical developments arising from Hilbert problems* at DeKalb, there was a talk held about the third problem. Unfortunately, the speaker did not submit his manuscript for publication in the proceedings (see [12]).

In recent years, the mathematicians B. Jessen, A. Thorup, J. Dupont of the Danish school, C.-H. Sah, P. Cartier, J. Milnor, J.-L. Cathelineau, A. Goncharov, and others showed active interest in this circle of questions.

2. Reformulation and results. Although Hilbert's third problem deals with solid geometry and polyhedral volume, it is basically an *algebraic* problem. Let $X^n = S^n, E^n$ or H^n be the standard space of constant curvature $1, 0$, or -1. The *scissors congruence* or *polytope group* $\mathcal{P}(X^n)$ of X^n is the abelian group generated by the symbols $[P]$ for each polytope $P \subset X^n$ subject to the relations

$$[P \sqcup Q] = [P] + [Q], \quad \text{where } P \sqcup Q \text{ is the disjoint union of } P, Q;$$
$$[g(P)] = [P], \quad \forall g \in \operatorname{Iso}(X^n).$$

The problem can now be stated as follows: *Find a complete system of invariants for the scissors congruence or groups.* There is the following criterion.

Proposition 2.1 (Zylev). $[P] = [Q] \Leftrightarrow P \sim Q$ *equidecomposable, that is,* $\exists P = P_1 \sqcup \cdots \sqcup P_n, Q = Q_1 \sqcup \cdots \sqcup Q_n$ *such that* $g_k(P_k) = Q_k$ *for some* $g_1, \ldots, g_n \in \operatorname{Iso}(X^n)$.

For $n = 2$ and in all geometries, a classical result due to Farkas Bolyai and P. Gerwien says that polygonal area separates points in $\mathcal{P}(X^2)$.

Lemma 2.2. *Let* $P, Q \subset X^2$ *be two polygons. Then,* $[P] = [Q]$ *if and only if* $vol_2(P) = vol_2(Q)$.

3. Dehn's solution for E^3 and the theorem of Dehn-Sydler. We now present a short outline of Dehn's proof and note that he profited from a hint of Bricard. Dehn discovered—beside polyhedral volume—another scissors congruence invariant, the so-called *Dehn invariant*.

Let $P \subset E^3$ be a Euclidean polyhedron with edges $e_1, \ldots, e_r$ of lengths $l_1, \ldots, l_r$ and dihedral angles $\alpha_1, \ldots, \alpha_r$ attached at $e_1, \ldots, e_r$. The Dehn invariant is then defined by

$$D(P) = \sum_{i=1}^{r} l_i \otimes_{\mathbb{Z}} \alpha_i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}.$$

It is obvious that $D(\text{prism}) = 0$. Now, a necessary condition for two polyhedra $P, Q \subset E^3$ to be equidecomposable is that

$$\mathrm{vol}_3(P) = \mathrm{vol}_3(Q), \qquad D(P) = D(Q). \tag{$*$}$$

Dehn's solution consists of the construction of the following counter-example. Let $P := S_{\text{reg}}(2\alpha)$ be a regular tetrahedron of edge length 1, that is, $\cos(2\alpha) = 1/3$ and α is irrational. On the other hand, choose a regular cube Q with $\mathrm{vol}_3(Q) = \mathrm{vol}_3(P)$. Then, P and Q cannot be scissors congruent since $D(Q) = 0$ while $D(P) = 6 \otimes 2\alpha \neq 0$ (see $(*)$).

In 1965, after 20 years of hard work, Sydler proved that the conditions $(*)$ are also sufficient.

Theorem 3.1 (Dehn-Sydler). *Let $P, Q \subset E^3$ be two polyhedra. Then, $[P] = [Q]$ if and only if $\mathrm{vol}_3(P) = \mathrm{vol}_3(Q)$ and $D(P) = D(Q)$.*

In 1968, Jessen [9] found a much simpler proof of Sydler's result. Moreover, only a few years later, he discovered that the analogous problem for E^4 can be reduced to the case of E^3 as follows.

Let $P \subset E^4$ be a Euclidean polytope with polygonal faces $p_1, \ldots, p_r$ of areas $f_1, \ldots, f_r$ and with dihedral angles $\alpha_1, \ldots, \alpha_r$ attached at $p_1, \ldots, p_r$. Consider the *Dehn invariant* defined, similarly as above, by

$$D(P) = \sum_{i=1}^{r} f_i \otimes_{\mathbb{Z}} \alpha_i \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}.$$

Theorem 3.2 (Jessen). *Let P, $Q \subset E^4$ be two polytopes. Then, $[P] = [Q]$ if and only if $\mathrm{vol}_4(P) = \mathrm{vol}_4(Q)$ and $D(P) = D(Q)$.*

The proof is based essentially on the reducibility to the 3-dimensional result by using the properties that

(a) in E^4: $P \sim [0,1] \times Q$ for some polyhedron $Q \subset E^3$;

(b) $D(P) = D(I \times Q) = D(Q)$.

However, for arbitrary spaces $X^n = S^n$, E^n or H^n, $n \geq 3$, and $X^n \neq E^3, E^4$, the *generalized third problem of Hilbert* asking for a complete system of invariants for $\mathcal{P}(X^n)$ is unresolved.

4. Some developments concerning $\mathcal{P}(H^n)$. In the last few years, Hilbert's third problem experienced some revival. This is mainly due to the interplay with the cohomology of Lie groups made discrete, number theory and polylogarithms, algebraic K—theory and Borel groups, and the motivic interpretation of the non-Euclidean Dehn complex. For example, it was shown that the theorem of Dehn-Sydler is equivalent to the fact that

$$H_2(SO(3), \mathbb{R}^3) = 0.$$

In this article, it is impossible to discuss these relations (see [3, 7]). However, in the following, I would like to indicate how some *geometrical* ideas of Sydler and Jessen can be adapted to describe $\mathcal{P}(H^3)$.

Consider hyperbolic space H^n with boundary ∂H^n of *points at infinity*. For this space, there are different polytope groups. While $\mathcal{P}(H^n)$ denotes the usual polytope group, $\mathcal{P}(\overline{H^n})$ is built upon polytopes with vertices possibly at infinity, and $\mathcal{P}(\overline{H^n})_\infty$ is generated by hyperbolic simplices all of whose vertices are at infinity (such simplices are termed *ideal*).

Moreover, in any n-space of constant curvature, one can decompose convex polytopes and simplices into *orthoschemes*; these are certain orthogonal simplices which generalize the notion of right-angled triangles in some way. They possess exactly two among the $n+1$ vertices which might be at infinity (in the extremal case, they are termed *doubly asymptotic*). Orthoschemes are very basic and fundamental in the following sense.

Proposition 4.1 (Debrunner-Sah). (1) $\mathcal{P}(H^n)$ *is generated by orthoschemes.*
(2) $\mathcal{P}(\overline{H^{2m+1}})$ *is generated by doubly asymptotic orthoschemes.*
(3) *For* $n \geq 3 : \mathcal{P}(H^n) \simeq \mathcal{P}(\overline{H^n}) \simeq \mathcal{P}(\overline{H^n})_\infty$.

Now, the notion of the Dehn invariant can be extended, to include the case of asymptotic, hyperbolic polyhedra. For example, consider an ideal tetrahedron

$$S_\infty(z) \subset \overline{H^3} = \left(P_1(\mathbb{C}) \times \mathbb{R}_+, ds^2 = \frac{|dz|^2}{(\operatorname{Im} z)^2}\right)$$

with vertices $0, 1, \infty, z$ in the upper half space model. $S_\infty(z)$ has three pairs of dihedral angles attached at opposite edges, namely

$$\alpha_1 := \arg z, \qquad \alpha_2 := \arg\left(1 - \frac{1}{z}\right), \qquad \alpha_3 := \arg \frac{1}{1-z} = \pi - (\alpha_1 + \alpha_2).$$

It can be seen that the formula

$$D(S_\infty(z)) = 2 \sum_{i=1}^{3} \log(2 \sin \alpha_i) \otimes_{\mathbb{Z}} \alpha_i$$

extends Dehn's invariant of a hyperbolic tetrahedron if all vertices tend to boundary points. Hence, the generalized third Hilbert problem for hyperbolic space goes as follows.

Conjecture 4.2. $P \sim Q$ *in* $\overline{H^3} \Leftrightarrow \mathrm{vol}_3(P) = \mathrm{vol}_3(Q)$, $D(P) = D(Q)$.

Sydler's original papers (cf. [14]) are difficult to read. The simplification of Jessen still reflects the principal geometrical idea as expressed by the role of orthogonal simplices and the fundamental lemma. That is, consider an orthoscheme $R(a,b;\lambda) \subset E^3$ defined by the parameters

$$a = \sin^2 \alpha_1, \quad b = \sin^2 \alpha_3 \quad \text{whence } \cos^2 \alpha_2 = a \cdot b,$$

as well as

$$\lambda := l_1 \cdot \tan \alpha_1 = l_2 \cdot \cot \alpha_2 = l_3 \cdot \tan \alpha_3.$$

Here, α_1, α_2, and α_3 denote the non-right dihedral angles of R attached at edges of lengths l_1, l_2, l_3 (see Figure 4.1).

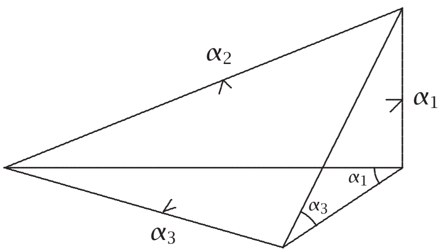

FIGURE 4.1.

It follows that

$$\mathrm{vol}_3(R(a,b;\lambda)) = \frac{\lambda^3}{6}(v(ab) - v(a) - v(b)), \quad \text{where } v(x) := \frac{1-x}{x}.$$

For $0 < a,b,c < 1$, put

$$X := R(a,b;\lambda) + R(ab,c;\lambda), \qquad Y := R(a,c;\lambda) + R(ac,b;\lambda).$$

An easy calculation shows that $\mathrm{vol}_3(X) = \mathrm{vol}_3(Y)$ and $D(X) = D(Y)$.

THEOREM 4.3 (The fundamental lemma). *$X \sim Y$, that is, $R(a,b;\lambda) + R(ab,c;\lambda) \sim R(a,c;\lambda) + R(ac,b;\lambda)$.*

Given this lemma the Dehn-Sydler theorem, according to Jessen, is roughly proven as follows:

(1) Let $\mathcal{X} < \mathcal{P}(E^3)$ be generated by the prisms in E^3, that is, $\mathcal{X} \subset \ker(D)$, and suppose $\mathcal{P}(E^3)/\mathcal{X}$ admits the structure of a real vector space. Moreover, suppose

$$\mathrm{vol}_3 : \mathcal{X} \longrightarrow \mathbb{R} \quad \text{is bijective.}$$

Then, one shows that $\mathcal{X} = \ker(D)$ which finally yields that

$$\mathrm{vol}_3 \times D : \mathcal{P}(E^3) \longrightarrow \mathbb{R} \times (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}) \quad \text{is injective.}$$

(2) Next, one observes that the class of orthoschemes are generators of $\mathcal{P}(E^3)/\mathcal{Z}$ and that the fundamental lemma provides several important algebraic relations. A crucial implication of these are the following properties, stated and proved in a very elegant algebraic setting by Jessen and Thorup.

Theorem 4.4 (Jessen-Thorup). *Let $F:(0,1)\times(0,1)\longrightarrow V$ be a mapping to a real vector space V satisfying*

$$F(a,b)=F(b,a),\qquad F(a,b)+F(ab,c)=F(b,c)+F(a,bc).$$

Then, there is a mapping $f:(0,1)\longrightarrow V$ such that $F(a,b)=f(ab)-f(a)-f(b)$.

Theorem 4.5 (Jessen-Thorup). *Let $G:(0,\infty)\times(0,\infty)\longrightarrow V$ be such that*

$$G(a,b)=G(b,a),\qquad G(a,b)+G(a+b,c)=G(b,c)+G(a,b+c),$$
$$G(ac,bc)=cG(a,b).$$

Then, there is a mapping $g:(0,\infty)\longrightarrow V$ such that $G(a,b)=g(a+b)-g(a)-g(b)$.

(3) To finish the proof, take a polyhedral basis $\{Q_r\}$ of $\mathcal{P}(E^3)/\mathcal{Z}$ so that, for each polyhedron P, we have

$$P\sim_{\mathcal{Z}}\sum_r m_rQ_r\quad\text{for some } m_r\in\mathbb{R}.$$

In particular, we obtain $R\sim_{\mathcal{Z}}\sum_r F_rQ_r$ with $F_r=F_r(a,b)$ satisfying the condition of Jessen-Thorup in Theorem 4.4. Therefore, there is a function $f_r(x)$ which is additive, annulates π and is such that

$$f_r(P):=\sum_{l\text{ edge of }P} l\otimes f_r(\alpha)$$

represents Dehn's invariant. Since R is a generator, one deduces that $P\sim_{\mathcal{Z}}\sum_r f_r(P)Q_r$. Finally, one gets $A\sim_{\mathcal{Z}}\sum_r f_r(A)Q_r=\sum_r f_r(B)Q_r\sim_{\mathcal{Z}}B$.

Now turn to the hyperbolic analogue. Let $R(a,b;\mu)\subset H^3$ denote a hyperbolic orthoscheme with parameters a,b as above and consider the additional parameter

$$\mu=\frac{\cos^2\alpha_2-\sin^2\alpha_1\sin^2\alpha_3}{\cos^2\alpha_1\cos^2\alpha_3}=:\tan^2\theta,\qquad \theta\in\left[0,\frac{\pi}{2}\right].$$

One checks that

$$\mu=\tanh l_1\cdot\tan\alpha_1=\tanh l_2\cdot\cot\alpha_2=\tanh l_3\cdot\tan\alpha_3,$$

and that

$$\cos^2\alpha_2=a\bigcirc_\mu b:=ab+\mu^2(1-a)(1-b).$$

For $0 < a,b,c < 1$, put

$$U := R(a,b;\mu) + R(a \bigcirc_\mu b,c;\mu), \qquad V := R(a,c;\mu) + R(a \bigcirc_\mu c,b;\mu).$$

Again, it follows that $\mathrm{vol}_3(U) = \mathrm{vol}_3(V)$ and $D(U) = D(V)$.

QUESTION 4.6 (Analogue of the fundamental lemma for H^3)**.** $U \sim V$, that is, $R(a,b;\mu) + R(a \bigcirc_\mu b,c;\mu) \sim R(a,c;\mu) + R(a \bigcirc_\mu c,b;\mu)$?

In order to simplify the question, the following observation is useful. Computing the volume of R (an expression in *dilogarithm functions*), one finds that

$$\mathrm{vol}_3(R(a,b;\mu)) = \mathrm{vol}_3(R_\infty(a)) + \mathrm{vol}_3(R_\infty(b)) - \mathrm{vol}_3\left(R_\infty(a \bigcirc_\mu b)\right),$$

where $R_\infty(a)$ denotes a simply asymptotic orthoscheme with dihedral angles α_1, θ, $(\pi/2) - \theta$. Therefore, a way to study Hilbert's third problem for hyperbolic 3-space, is to investigate the above question for $\mathcal{P}(\overline{H^3})$ and to profit from the isomorphisms in Proposition 4.1.

5. Algebraic K-theoretical aspects—a brief account. The group $\mathcal{P}(H^3)$ admits further equivalent and very elegant interpretations. The following is a very short tour around these ideas (due to Sah, Dupont, Thurston, and others).

Let $\mathcal{P}(\mathbb{C})$ be the abelian group generated by $\{z\}$, $z \in \mathbb{C}$, such that

(1) $\{0\} = \{1\} = 0$;

(2) $\forall a \neq b \in \mathbb{C} \setminus \{0,1\} : \{a\} - \{b\} + \{a/b\} - \{1-a/1-b\} + \{b(1-a)/a(1-b)\} = 0$.

This group can be isomorphically identified with the group $\mathcal{T}(\mathbb{C})$ generated by ideal tetrahedra, that is, $\mathcal{T}(\mathbb{C})$ is the group of 4-tuples (p_0,p_1,p_2,p_3), $p_i \in P_1(\mathbb{C})$, with

(1′) $(g(p_0),g(p_1),g(p_2),g(p_3)) = (p_0,p_1,p_2,p_3)$, $\forall g \in \mathrm{PSL}(2,\mathbb{C})$,

(2′) $\forall p_0,\dots,p_4 \in \mathbb{C}$ disjoint:

$$\sum_{i=1}^{4} (-1)^i (p_0,\dots,\widehat{p_i},\dots,p_4) = 0.$$

The identification is then performed by using the map

$$(p_0,p_1,p_2,p_3) \longmapsto \left\{z := r(p_0,p_1,p_2,p_3)\right\} \quad (r \text{ denotes cross-ratio}).$$

Furthermore, one has

$$\mathrm{vol}_3(p_0,p_1,p_2,p_3) = \mathrm{vol}_3\left(\infty,0,1,r(p_0,p_1,p_2,p_3)\right) = D(z),$$

where

$$D(z) := \log|z| \arg(1-z) - \mathrm{Im}\, Li_2(z)$$

denotes the Bloch-Wigner dilogarithm (a modification of the classical dilogarithm $Li_2(z)$) which satisfies the 5-term relation of Spence-Abel

$$\sum_{i=1}^{4}(-1)^i D\Big(r(p_0,\ldots,\widehat{p_i},\ldots,p_4)\Big)=0.$$

Now, the following isomorphisms can be established.

$$\mathcal{P}(H^3)\simeq\mathcal{P}(\overline{H^3})_\infty\simeq\mathcal{P}(\mathbb{C})/_{\langle\{z\}+\{\bar z\}\rangle}\simeq\mathcal{P}(\mathbb{C})^-,$$

where the exponent $^-$ describes the eigenspace to the eigenvalue -1 with respect to complex conjugation. Next, consider the following mappings:

$$\rho:\mathcal{P}(\mathbb{C})\longrightarrow\Lambda^2_{\mathbb{Z}}\mathbb{C}^\times=\mathbb{C}^\times\otimes_{\mathbb{Z}}\mathbb{C}^\times/\langle a\otimes b+b\otimes a\rangle,\qquad \rho(\{z\}):=z\wedge(1-z),$$
$$\widetilde{D}:\mathcal{P}(\mathbb{C})\longrightarrow\mathbb{R},\qquad \widetilde{D}(\{z\}):=D(z).$$

They are related to volume and Dehn's invariant according to the following picture:

(a) The composition $\mathcal{P}(H^3)\longrightarrow\mathcal{P}(\mathbb{C})^-\xrightarrow{\widetilde{D}}\mathbb{R}$ is the volume.

(b) The composition $\mathcal{P}(H^3)\longrightarrow\mathcal{P}(\mathbb{C})^-\xrightarrow{\rho^-}(\Lambda^2_{\mathbb{Z}}\mathbb{C}^\times)^-\simeq\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{R}/\pi\mathbb{Z}$ is Dehn's invariant.

Finally, consider the *Bloch group* defined by $B(\mathbb{C}):=\ker\big(\mathcal{P}(\mathbb{C})\xrightarrow{\rho}\Lambda^2_{\mathbb{Z}}\mathbb{C}^\times\big)$. Again, let $B(\mathbb{C})^-$ denote the negative part of $B(\mathbb{C})$ with respect to the involution induced by the conjugation (actually, the Bloch group can be identified with the group of polyhedra with vanishing Dehn invariant). In this setting, Hilbert's third problem for H^3 can be reformulated very efficiently as follows:

Is the mapping $\widetilde{D}:B(\mathbb{C})^-\longrightarrow\mathbb{R}$ *injective?*

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THE ROLE OF HILBERT PROBLEMS IN REAL ALGEBRAIC GEOMETRY

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Real algebraic geometry studies sets defined by systems of polynomial equations over the reals [5]. Three problems of Hilbert [18] are related to important aspects of real algebraic geometry.

We first examine these problems, what is known about their solutions, and what developments they led to. We end with a short discussion on the role they played in the development of real algebraic geometry during this century.

In many cases, the references given in this text point to text books or survey papers where simple proofs and full references can be found rather than to the original papers.

1. Hilbert's 17th problem. It is obvious that a polynomial which is a sum of squares is everywhere nonnegative. A very natural question is the following:

- Is every polynomial, everywhere nonnegative, a sum of squares?

A polynomial everywhere nonnegative is not always a sum of squares of polynomials. This result is due to Hilbert [16] but Motzkin [34] was the first in 1967 to construct explicit examples of this situation. For example

$$P = Z^6 + X^4Z^2 + X^2Y^4 - 3X^2Y^2Z^2$$

is everywhere nonnegative and is not a sum of squares of polynomials. This is not too hard to prove since the degree of the polynomials to look for is at most 3.

Minkowski sugggested to Hilbert the following reformulation, which is Hilbert's 17th problem:

- Is every polynomial, everywhere nonnegative, a sum of squares of rational functions?

Now, since denominators are allowed, the space of search for an expression as a sum of squares is much bigger, and there are no a priori limitations on the degrees to consider.

Emil Artin's positive answer [2], in 1925, is one of the most convincing successes of modern algebra, which was starting at that time.

In order to prove a result about the reals, the method of the proof uses much more abstract objects, namely real closures of the field of rational functions.

Artin's proof goes this way:

• Consider a polynomial P which is not a sum of squares of rational functions with real coefficients.

• Since P is not a sum of squares, the set of sums of squares is a proper cone of the field of rational functions which does not contain P (a cone of a ring contains the squares, is closed under addition and multiplication and a proper cone also does not contain -1).

• Using Zorn's lemma, and taking a maximal proper cone which does not contain P, we get a total order on the field of rational functions for which P is negative.

• Taking the real closure of the field of rational functions for this order, which is the biggest possible ordered field extending the given order and algebraic over the field of rational functions, we get a field in which P takes negative values.

• It remains to prove that if P takes negative values in a real closed field containing the reals, P takes negative values over the reals.

So, for example, Motzkin's polynomial above is a sum of squares of rational functions, even though it may not be easy to write it explicitly so.

Artin's proof which we just sketched is the starting point of the abstract theory of the reals. Real closed fields originally defined by Artin and Schreier [3] as ordered fields with no algebraic ordered extension, have later been characterized by Tarski [46] as fields which are ordered, where every positive element is a square and every odd degree polynomial has a root. The real closure of an ordered field is the smallest real closed field containing it.

Many problems remain after Artin's proof. Among them:

- quantitative aspects: how many squares?
- effectivity problems: is there an algorithm checking that a given polynomial is everywhere nonnegative and providing the sum of squares?
- complexity problems: what are the best possible bounds on the degrees of the sum of squares?

A bound on the number of squares was proved by Pfister in 1967 [36]: 2^n squares are enough if n is the number of variables of the polynomial P. It is remarkable that the degree of the polynomial plays no role in the bound on the number of squares needed. The proof uses Pfister's theory of multiplicative quadratic forms [36].

The exact bound is far from being known, since the only known lower bound on the number of squares needed is $n+2$.

Since Artin's proof is based on Zorn's lemma, no explicit bound can be easily extracted from its inspection (see [10, 27] though). The explicit construction of the sum of squares is a difficult problem and has attracted much attention and many contributions (see [10, 11] for a detailed account of the literature on this topic).

Hilbert's 17th problem can be seen as part of a much more general problem, which is to provide a dictionary between algebra and geometry in the real case.

In complex algebraic geometry, this dictionary is very classical and given by the correspondance between algebraic sets and radical ideals (a particularly easy to read presentation can be found in [8]). Artin's result relates nonnegativity, which is a geometric property, to sums of squares, which is an algebraic notion. Stengle's positivstellensatz [45], proved in 1974, provides a quite general version of a dictionary between algebra and geometry in the real case. The version we present here can be found in [5].

Weak Positivstellensatz. Let R be a real closed field, $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ are three finite subsets of $R[X_1,\ldots,X_n]$, $\mathcal{C}$ the cone generated by $\mathcal{F}$, $\mathcal{M}$ the monoid generated by $\mathcal{G}$, and $\mathcal{I}$ the ideal generated by $\mathcal{H}$.

The following are equivalent:

(i) The set

$$\{x \in R^n \mid \forall f \in \mathcal{F} f(x) \geq 0, \forall g \in \mathcal{G}\; g(x) \neq 0, \forall h \in \mathcal{H}\; h(x) = 0\}$$

is empty.

(ii) There exist $f \in \mathcal{C}$, $g \in \mathcal{M}$, and $h \in \mathcal{I}$ such that $f + g^2 + h = 0$.

This result can be interpreted as follows: if a family of inequalities and equalities is incompatible, there exists an algebraic certificate testifying it.

Using a classical trick due to Rabinovitch the following result [45] follows.

Positivstellensatz. Let $\mathcal{G}$ be a finite subset of $R[X_1,\ldots,X_n]$, and let

$$W = \{x \in R^n \mid \forall g \in \mathcal{G}\; g(x) \geq 0\}.$$

Let $\mathcal{C}$ be the cone of $R[X_1,\ldots,X_n]$ generated by $\mathcal{G}$, and let $f \in R[X_1,\ldots,X_n]$. Then $\forall x \in W f(x) > 0 \Leftrightarrow \exists g, h \in \mathcal{C}$ such that $fg = 1 + h$.

As an immediate consequence of the Positivstellensatz, when $\mathcal{G} = \emptyset$, one recovers that a nonnegative polynomial is a quotient of sums of squares, hence also a sum of squares.

The proof of these results rely on Zorn's lemma and uses the notion of a prime cone [5, 7].

It is natural to wonder whether there exists an algorithm producing the algebraic identities announced. The answer is yes [26]; however there remains a lot to do to provide an efficient algorithm [28].

2. Hilbert's 16th problem. The first part of this problem is to determine

- the different possible shapes of an algebraic curve, or more generally of a real hypersurface in projective space.

We do not discuss the second part of the problem here.

Consider a nonsingular algebraic hypersurface H of degree d in $\mathbb{R}P^n$. This polynomial has a set of zeroes $\mathbb{R}H$ in the real projective space $\mathbb{R}P^n$. The first part of Hilbert's 16th problem asks what are the possible topological types of the pairs $(\mathbb{R}P^n, \mathbb{R}H)$ (for a given degree d). For the curves of $\mathbb{R}P^2$ the complete answer is known only up to degree 7. For the surfaces of $\mathbb{R}P^3$ the complete answer is known only up to degree 4.

To make progress on this problem it is necessary to work in two main directions: first to find restrictions on the topological types of $(\mathbb{R}P^n, \mathbb{R}H)$, second to construct hypersurfaces with the types which are not forbidden.

In the study of the topology of real algebraic curves, it is traditional to pay special attention to M-curves. An M-curve is a curve of $\mathbb{R}P^2$ such that its set of real points has the maximal number of connected components; this number is equal to $(d-1)(d-2)/2+1$ for degree d by Harnack's theorem [14].

An *oval* of a real algebraic projective curve Γ is a connected component of Γ whose complement is not connected. The orientable connected component of this complement is the *interior* of the oval.

Harnack's theorem gives no restriction on the relative position of the ovals. The relative position of ovals in the projective place can be described in terms of *nest*, when an oval contains another oval in its interior. The *depth* of an oval Ω of Γ is the number of ovals containing Ω in their interiors. Bezout's theorem creates restrictions. An M-curve of degree 4 with 4 ovals cannot have a nest, because otherwise a line having at least 6 points of intersection with the curve could be easily constructed. So the only configuration for an M-curve of degree 4 is 4 ovals without any nest.

This simple argument is no more sufficient in degree 6. This case has only been completely solved in 1971. The M-curves of degree 6 have 11 ovals. Bezout's theorem proves that an oval of depth 2 is impossible and also that there exists at most one oval with other ovals in its interior. A construction due to Harnack gives an M-curve with the following configuration: an oval containing another oval in its interior and the nine other ovals outside, without any nest. This configuration is denoted by $1\langle 1\rangle \coprod 9$. A construction by Hilbert [17] gives a different configuration denoted $1\langle 9\rangle \coprod 1$: an oval containing nine ovals in its interior, the last oval being outside.

For a long time people wondered whether there were other configurations for an M-curve of degree 6, until Gudkov constructed such a curve with a configuration $1\langle 5\rangle \coprod 5$ [13].

These three configurations are the only possible for the M-curves of degree 6. This is a consequence of a famous congruence conjectured by Gudkov and proved by Rokhlin [41, 48].

Let Γ be a nonsingular real algebraic curve of even degree $2k$ in $\mathbb{P}_2(\mathbb{R})$. It is possible to choose a homogeneous equation F of Γ, such that $F(x,y,z) \leq 0$ for every $(x:y:z)$ outside all the ovals of Γ. Then, with

$$B_+ = \{(x:y:z) \in \mathbb{P}_2(\mathbb{R}) \mid F(x,y,z) \geq 0\}.$$

Rokhlin-Gudkov's congruence. The Euler-Poincaré characteristic of B_+ is congruent to k^2 modulo 8.

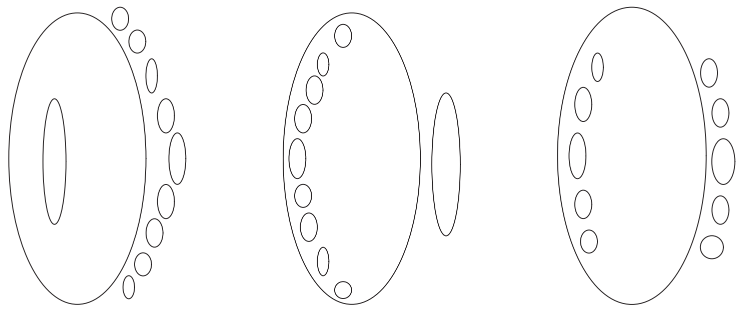

FIGURE 2.1.

$\chi(B_+)$, the Euler-Poincaré characteristic of B_+, can be computed in the following way. An oval of Γ is *even* (respectively, *odd*) if its depth is even (respectively, odd). The number of even (respectively, odd) ovals of Γ is denoted by p (respectively, n). Then $\chi(B_+) = p - n$.

For an M-curve of degree 6, $p + n = 11$ and, using the congruence $p - n \equiv 9$ (mod 8), the only possibilites are $(p = 10,\ n = 1)$, $(p = 6,\ n = 5)$, or $(p = 2,\ n = 9)$. Thus the three configurations described above are the only possible.

The numbers p and n have been considered first by Virginia Ragsdale (1) [37] who proposed the conjecture that for every curve of degree $2k$,

$$p \le \frac{3k(k-1)}{2} + 1, \qquad n \le \frac{3k(k-1)}{2}.$$

The spectacular development of the topology of real algebraic varieties in the 1970s implies new restrictions on the topology of a real algebraic variety. V. Arnold [1], V. Rokhlin [41, 42, 43], and V. Kharlamov [22, 23, 24] have obtained important general obstructions. The discovery of new invariants for the varieties of dimension 4 by Seberg and Witten and the proof of Thom's conjecture in 1994 [25] have implied new restrictions in the topology of real algebraic curves [31].

Although a lot of work had been done on obstructions, the methods of construction had not changed much from the 19th century. In 1980, Viro proposed a completely new method to construct real algebraic varieties [20, 21]. The combinatorial patchwork, which is a particular case of Viro's method, gives a recipe to construct hypersurfaces using a simple combinatorial procedure. For simplicity the recipe is explained here for curves but the general case is completely similar.

In order to construct a real algebraic curve in $\mathbb{R}P^2$, the following combinatorial data are given.

Let d be a positive integer and T the triangle

$$\{(x, y) \in \mathbb{R}^2 : x \ge 0, y \ge 0, x + y \le d\}.$$

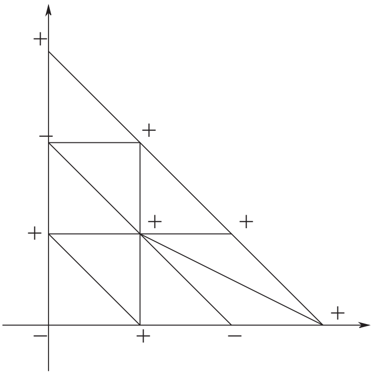

FIGURE 2.2.

The number d is the degree of the curve constructed and the triangle T is the Newton polygon of the curve.

Suppose that T is triangulated in such a way that all the vertices of the triangulation have integer coordinates. Suppose also that a sign distribution, $a_{i,j} = \pm$ is given at the vertices of the triangulation, (see Figure 2.2). A piecewise linear curve L in $\mathbb{R}P^2$ is constructed as follows.

Take copies $T_x = s_x(T)$, $T_y = s_y(T)$, $T_{xy} = s(T)$ of T, where $s = s_x \circ s_y$ and s_x, s_y are reflections against the axis of coordinates. Extend the triangulation of T to a symmetric triangulation of $T \cup T_x \cup T_y \cup T_{xy}$, and extend the sign distribution to a sign distribution at the vertices of the extended triangulation with the following rule: When a vertex is transformed into its mirror image with respect to a coordinate axis, its sign is preserved when the distance to the axis is even and changed when this distance is odd (see Figure 2.3).

If a triangle of the triangulation has vertices with different signs, a segment is drawn from the middle of the edges to isolate $+$ from $-$. The union of these segments is denoted by L' and is contained in $T \cup T_x \cup T_y \cup T_{xy}$ (see Figure 2.3). Sides of $T \cup T_x \cup T_y \cup T_{xy}$ are glued using s. The space T_* so obtained is homeomorphic to $\mathbb{R}P^2$. The curve L is the image of L' in T_*.

A pair (T_*, L) is a chart of a real algebraic curve C in $\mathbb{R}P^2$, if there exists a homeomorphism between the pairs (T_*, L) and $(\mathbb{R}P^2, \mathbb{R}C)$.

Suppose now that the triangulation of T is convex. This means that there exist a piecewise convex function $\nu : T \longrightarrow \mathbb{R}$ linear on each triangle of the triangulation and not linear on the union of two triangles.

VIRO'S THEOREM. If the triangulation of T is convex, there exists a nonsingular real algebraic curve C of degree d in $\mathbb{R}P^2$ with chart (T_*, L).

A curve with chart (T_*, L) is called a T-curve.

It is easy to verify that the triangulation of

$$\{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ x + y \leq 3\}$$

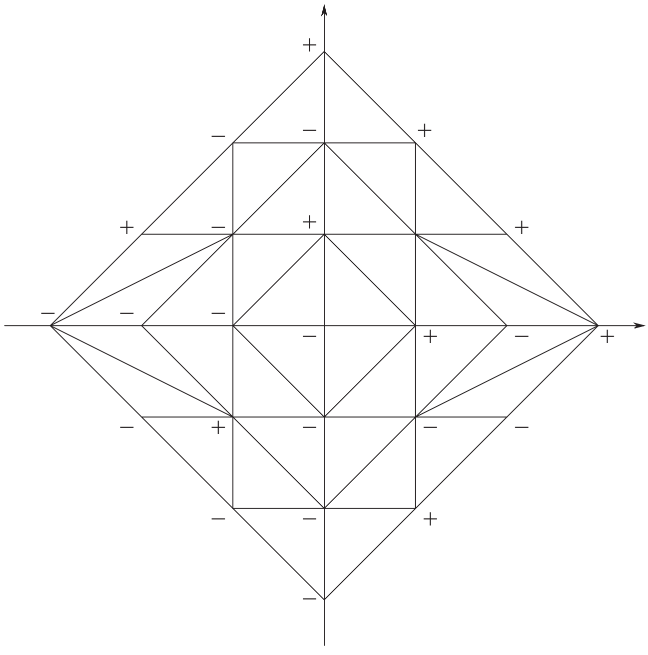

FIGURE 2.3.

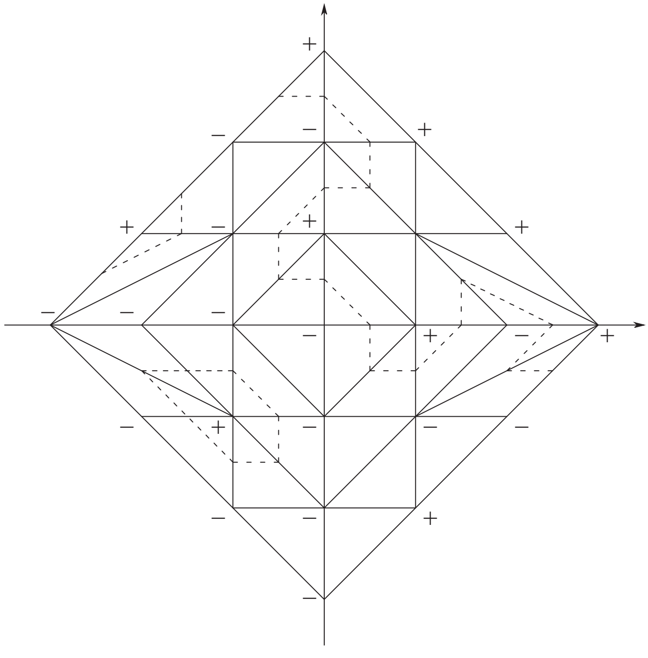

FIGURE 2.4.

in Figure 2.2 is convex. Thus, Figure 2.3 is a topological picture of a curve of degree 3 in $\mathbb{R}P^2$. The set of real points of this curve has two connected components in $\mathbb{R}P^2$.

The class of T-curves is quite rich. For example, all curves of $\mathbb{R}P^2$ up to degree 6 are T-curves. For every degree, there exists a T-curve which is an M-curve. T-curves have been constructed as counter-examples to Ragsdale conjecture [19, 20, 21]. There are also real algebraic curves of $\mathbb{R}P^2$ which are not T-curves [19].

3. Hilbert's 10th problem. The problem is to answer the following question:

• Is there an algorithm deciding the existence of integer solutions to a set of Diophantine equations?

This is for a set of polynomial equations with integer coefficients.

The answer is of course yes for univariate polynomials since there are easy bounds on the roots in terms of the coefficients.

The negative answer for the general problem was proved by a young russian mathematician Matiyasevich [29, 30] in 1972. His work used results of Davis, Putnam, and Robinson (2) [9, 40].

Interestingly the corresponding problems when real solutions are looked for has a positive answer. The decision problem over the reals:

• Is there an algorithm deciding the existence of real solutions to a set of polynomial equations with integer coefficients?

was solved with a yes answer by Tarski [46] and Seidenberg [44].

Again the existence of an algorithm raises complexity questions. This is an active field of research [4, 6, 12, 15, 39].

4. Brief discussion. Hilbert's problems have played an important role in the development of real algebraic geometry. The previous discussions illustrate this point. However, some very important ideas were developed without any connection to these problems. Morse, for example, related the change in topology to the existence and local behaviour of critical points of a function [32] (see Figure 4.1). Morse theory plays a key role in the quantitative [33, 35, 47] (3) and algorithmic aspects of real algebraic geometry [4, 12].

Bibliographic Information

(1) Virginia Ragsdale (1870–1945). She graduated from Guilford College in 1892. She won the first scholarship established by Bryn Mawr College for a Guilford woman graduating with the highest degree average. She took her Bachelor degree in 1896 and was awarded the Bryn Mawr European Fellowship. She chose to go to Gottingen for one year, to study under Hilbert and Klein. She completed her Ph.D. in Bryn Mawr in 1906. Her paper "*On the arrangement of the real branches of plane algebraic curves*" [37] was published immediately after her dissertation. She was instructor, assistant professor, professor,

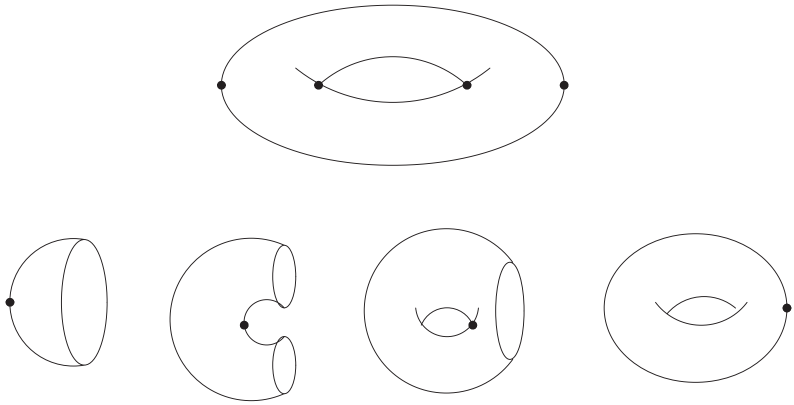

FIGURE 4.1.

and finally head of the department at Woman's College in North Carolina. She retired from teaching in 1928 and took care of her ill mother, taking the responsibility of the home.

(2) Julia Robinson (1919–1985). She took her Ph. D. under Tarski in 1948. She made a major contribution to the solution of Hilbert's 10th problem [9, 40]. In 1976 she became the first woman mathematician to be elected to the National Academy of Sciences. In 1982 she was nominated to the presidency of the American Mathematical Society. A book written by her sister Constance Reid is devoted to her [38].

(3) Olga Oleinik. A former professor at Moscow University, now retired, she lives in Russia. She worked in several fields of mathematics: topology of real algebraic varieties, PDEs. Over 70 years old, she is still publishing papers.

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Appendices

Appendix A

MINUTES OF THE GENERAL ASSEMBLY

Place: Kloster Loccum, Loccum, Germany.
Time: September 3, 1999, 14.05–18.40.
Present: Twenty members from ten countries, three nonmembers.

1. Opening of the general assembly. Moderators: Catherine (Cathy) Hobbs, UK and Marjatta Näätänen, Finland.

The ninth meeting of the EWM was announced in the EWM newsletter on April 1998 and March 1999, and on the EWM web pages. The requirement for majority was also met. Thus Cathy Hobbs declared the general assembly to be legal.

Cathy Hobbs welcomed everybody to the general assembly.

As a member of the standing committee, Cathy Hobbs also thanked people for their work during the past two years.

Hobbs and Näätänen proposed an agenda for the general assembly. The agenda was accepted.

2. Appointing people to various tasks during the general assembly

2.1. Appointing people to take notes at the assembly.
Nadja Kutz, Germany
Maren Riemewschneider, Germany
Rachel Camina, UK
Sandra Pott, UK

2.2. Appointing two people to check the minutes.
Emilia Mezzetti, Italy
Tsou Sheung Tsun, UK

2.3. Appointing people to count the votes.
Ina Kersten, Germany
Lisbeth Fajstrup, Denmark

3. Approving new members. The moderators suggested that those present who were not yet members of the EWM could join now if they so wished. Their fee for 1999 would be waived. The suggestion was approved.

The general assembly approved those who had joined after the previous general assembly in Trieste, Italy, and those who joined at the place of the present meeting.

4. Approving the minutes of the previous general assembly. The minutes of the previous general assembly held during the eighth meeting of the EWM were approved.

5. Electing two auditors and a deputy auditor. The former auditors Kirsi Peltonen, Finland, and Seija Kämäri, Finland, were re-elected. No deputy auditor was seen necessary.

6. Confirming the financial statement and discharging those responsible for liabilities. Marjatta Näätänen gave the financial reports for 1997 and 1998 (appendix), which were confirmed. The auditors have suggested that those responsible for liabilities should be discharged which the general assembly did.

7. Deciding fees. The general assembly agreed to keep fees the same as before: 1 Euro (low), 20 Euro (standard), and 50 Euro (high).

8. Electing standing committee and convenor and deputy convenor(s) for 2000–2001. In the statutes of the EWM, it is stipulated that the term of half of the members will expire and the other half will continue.

The continuing members of the standing committee, which all wanted to continue, were as follows:

Christine Bessenrodt, Germany
Cathy Hobbs
Irene Sciriha, Malta
Betül Tanbay, Turkey
Tsou Sheung Tsun
Inna Yemelyanova, Russia

The current standing committee proposed the following as new members:

Polina Agranovich, Ukraine
Laura Fainsilber, Sweden
Laura Tedeschini-Lalli, Italy
Lyudmila Bordag, Germany
Marjatta Näätänen
Irene Pieper-Seier, Germany

Of these six, Agranovich and Fainsilber have been members of the standing committee for 1998–1999.

The general assembly agreed with the standing committee's proposal and elected those mentioned above.

Laura Tedeschini-Lalli pointed out that it was regrettable that there would be no French or Spanish members in the standing committee, given the importance of the Femmes et Mathematiques, especially. The moderators agreed with this, but pointed out that there were no French or Spanish members present at the general assembly to be asked to.

9. Electing the convenor and deputy convenors. The newly elected standing committee proposed Irene Sciriha as the convenor. However, she had not given her consent by the time of the general assembly. Therefore, it was agreed that the standing committee could elect the convenor among themselves in the case that Irene Sciriha would refuse (Irene Sciriha has since agreed to be the convenor). Laura Fainsilber and Christine Bessenrodt were elected as deputy convenors. It was decided that they should contact the standing committee to elect the convenor.

10. Electing international coordinators. The following persons were proposed:

Marie Demlova, Czech Republic (Central and East)
Laura Fainsilber (North)
Rosa-Maria Spitaleri, Italy (South and West)
Tatiana Vasilieva (Russia)

Their election was approved.

11. Confirming the regional coordinators. By the time of the general assembly not all the regional coordinators could have been contacted. Therefore, it was decided that Riitta Ulmanen, Finland, would get in touch with those who had not given their answer so far. The general assembly confirmed the following regional coordinators:

Andrea Blunck, Austria	Nerute Kligiene, Lithuania
Elena Gavrilova, Bulgaria	Irene Sciriha, Malta
Marie Demlova, Czech Republic	Coby Geijsel, The Netherlands
Lisbeth Fajstrup, Denmark	Ragni Piene, Norway
Helle Hein, Estonia	Magdalena Jaroszewska, Poland
Marja Kankaanrinta, Finland	Emilia Petrisor, Romania
Christine Charretton, France	Galina Riznichenko, Russia
Sybille Handrock, Germany	Rosa Maria Miro Roig, Spain
Cathy Hobbs and Sandra Pott, UK	Gerd Brandell, Sweden
Maria Leftaki, Greece	Karin Baur, Switzerland
Barbara Fantechi, Italy	Betül Tanbay, Turkey
Daina Taimina, Latvia	Lyudmila Kirichenko, Ukraine

12. Choosing time, place, and organizing committee for the next meeting. (Malta has since been chosen to the place of the next meeting.)

12.1. Place. The general assembly discussed possible places for the next meeting in 2001. The following places were suggested:

- Cambridge
- Canary Islands
- Crete

- Estonia, Tartu
- Malta
- Switzerland
- Sweden or Denmark

For the year 2003, Croatia and Italy, Trieste, were mentioned as possible places.

12.2. Organizing committee. The following persons were chosen for the organizing committee:

Tsou Sheung Tsun, coordinator
Marie-Francoise Coste Roy, France
Christine Bessenrodt
Laura Fainsilber

It was decided that the organizing committee would complete itself if necessary. Also, the decision on the place and time was left to the organizing committee to decide following the guidelines given above in Subsection 12.1.

12.3. Suggested themes for the next meeting:

- probability/statistics: applied to economics,
- uses of geometry (interdisciplinary),
- financial mathematics,
- nonlinearity,
- cohomology theories (expository),
- engineering mathematics,
- history of mathematics,
- Lie theory,
- Cauchy-Riemann geometry,
- commutative algebra,
- noncommutative geometry.

Nonmathematical topics:

- math education,
- mathematics and the public.

In addition to plenary talks, also short, prepared communications related to the topic in question were suggested.

13. Setting up commissions.

- Link with the European Mathematical Society (EMS): Emilia Mezzetti
- Link with the Association for Women in Mathematics (AMS): Christine Bessenrodt
- Link with third world countries: Laura Tedeschini-Lalli
- Newsletter: Nadia Larsen, Denmark, Maren Riemewschneider, Germany
- Proceedings: Rachel Camina, UK, Lisbeth Faijstrup, Denmark
- Web page: Olga Capriotti, Italy
- E-mail network: Sarah Rees, UK

- Funding committee: Cathy Hobbs (Treasurer), Bettina Kürner, Germany, Emilia Mezzetti

14. Proceedings. The general assembly discussed the contents of the meeting proceedings.

The proceedings would comprise of all the contributions of invited speakers. Also, talks presented at Kloster Loccum which the speakers wished to be included, would be in the proceedings.

It was agreed that these latter contributions to the proceedings, and articles to the newsletter as well, could/would be sent to be refereed by a commission or a referee. It was also recommended that those willing to present their posters in the proceedings should write a half a page resume on the subject. Polina Agranovich will be in charge of editing the poster section.

15. Regional activities. Reports on regional activities were presented on September 2.

16. Thematic activities. Tsou Sheung Tsun gave a short report on the workshop on Moduli Spaces in Mathematics and Physics held in Oxford, July 2–3, 1998. A request for continuing these workshops was presented.

Reports on this and other workshops and meetings held after the eighth general meeting are in the Newsletter number 6, March 1999.

The following interdisciplinary meetings were suggested:

- a meeting on nonlinearity in Russia. Possibilities to organize this meeting will be examined and Laura Fainsilber will be the contact person;
- a satellite meeting to the ICM to be held in Barcelona. The satellite meeting could be a session for students approaching their Ph.D. and those close after that. Emilia Mezzetti will contact Rosa Maria Miro Roig, Spain, on this;
- Rosa Maria Spitaleri, Italy, will promote the EWM on meetings she is organizing on mathematical modelling and related subjects.

17. Open discussion

17.1. It was considered useful if the EWM had, to promote its goals and ideas, a poster to be used, for example, in conferences and a flyer (a handbill) to be handed out in other suitable situations, too. It was decided that the EWM should declare a competition to design these. Possibilities to get funding for this purpose from companies and other sources should be examined.

17.2. The EWM should send out a questionnaire on age and career. The common concept being that mathematicians do their best work at an early age. The EWM wants to contradict this preassumption. Therefore a questionnaire should be made and send out to mathematicians of all ages and both genders.

17.3. It was suggested that names of Ph.D. supervisors should be on the EWM web page with their contact information.

18. Closing the general assembly. Cathy Hobbs thanked everyone for attending and declared the general assembly closed.

NETIQUETTE RULES AND GENERAL USE OF THE EWM ELECTRONIC NETWORK

At the discussion of the EWM e-mail network we worked out the netiquette rules below—it was agreed to send them regularily to the list.

The EWM-lists are part of the mailbase system, for details, see the following URL:

http://www.mailbase.ac.uk

1. How to join ewm-discuss. Send an e-mail (no subject line) to

mailbase@mailbase.ac.uk

with the following content:

JOIN ewm-discuss *Emmy Noether*
stop

where, instead of *Emmy Noether* you should put *your own name* (not your login, nor e-mail address!).

2. How to leave ewm-discuss. Send an e-mail (no subject line) with the following content:

LEAVE ewm-discuss
stop

The same procedure applies to the list ewm-all or ewm-uk. NOTE that the lists ewm-discuss and ewm-uk are *subsets (= sublist)* of ewm-all, that is, if you (as an element) leave that lists, but wish to stay in ewm-all you have to sign up for ewm-all. So if you want to be part of, for example, ewm-discuss sign up for ewm-all too—you will not get e-mails twice and you can go on and off ewm-discuss without leaving ewm-all.

3. Netiquette rules. The mailbase system is governed by the Mailbase Acceptable Use Policy at

http://www.mailbase.ac.uk/docs/aup.html

In addition to the above policy, please observe the following rules:

- At most one e-mail per day per person

For ewm-all

- Send information only.

(This includes job offers, conference announcements, etc., short requests and quick updates on topics of ewm-discuss.)

For ewm-discuss

- Write the topic you are commenting on in the subject line.

(e.g., woman in Afghanistan, Age limits, etc.)

- Please do not repeat arguments.

Read the archives concerning the topic you are going to comment on at the following URL:

http://www.mailbase.ac.uk

- Please try to be short and as clear and productive as possible.

If you have an interesting topic for discussion send it to ewm-discuss!

Every member of the ewm-lists is responsible for the liveliness and efficiency of our network.

Be active!

Nadja Kutz

Appendix B

COMMITTEE MEMBERS

(1999–2001)

Standing committee

- Irene Sciriha (Convenor); e-mail: irene@maths.um.mt
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