International Conference on Differential, Difference Equations and Their Applications

1–5 July 2002, Patras, Greece

Edited By: Panayiotis D. Siafarikas

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PREFACE

The International Conference on Differential, Difference Equations and Their Applications (ICDDEA), dedicated to Professor Evangelos K. Ifantis, took place from 1 to 5 July, 2002, at the Conference and Cultural Centre of the University of Patras, Patras, Greece. The aim of the conference was to provide a common meeting ground for specialists in differential equations, difference equations, and related topics, as well as in the rich variety of scientific applications of these subjects. This conference intended to cover, and was a forum for presentation and discussion of, all aspects of differential equations, difference equations, and their applications in the scientific areas of mathematics, physics, and other sciences.

In this conference, 101 scientists participated from 26 countries (Australia, Bulgaria, Canada, Chile, Czech Republic, Finland, France, Georgia, Great Britain, Greece, Hungary, Iran, Israel, Italy, Japan, Latvia, Mozambique, Netherlands, Norway, Poland, Romania, Singapore, Slovakia, Spain, Ukraine, and United States of America). The scientific program consisted of 4 plenary lectures, 37 invited lectures, and 42 research seminars. The contributions covered a wide range of subjects of differential equations, difference equations, and their applications.

The social program of the conference consisted of a welcome dinner and a guided visit at the Residence of Achaia Clauss winemakers, a Greek evening with traditional Greek dances and dinner at the private restaurant of the hotel Rio Beach, a visit at the castle of the picturesque city Nafpaktos with lunch at a traditional Greek taverna at the village of Eratini, a guided visit to ancient Delphi with dinner at a private restaurant at Nafpaktos, and a farewell dinner at the private restaurant "Parc de la Paix," near the conference cite.

This volume contains the papers that were accepted for publication after an ordinary refereeing process and according to the standards of the journal "Abstract and Applied Analysis"; we thank the referees, who helped us to guarantee the quality of the papers, for their work. Also I would like to thank the International Scientific Committee (Ondřej Došlý, Takasi Kusano, Andrea Laforgia, and Martin Muldoon) for their support and cooperation concerning the refereeing process, as well as the other members of the Local Organizing Committee (Chrysoula Kokologiannaki and Eugenia Petropoulou) for their help. Finally, I would like to thank the editors and especially the Editor-in-Chief, Professor Athanassios Kartsatos, of the journal "Abstract and Applied Analysis," which hosted the Proceedings of the ICDDEA.

We would also like to thank the Research Committee of the University of Patras, the Hellenic Ministry of Education and Religious Affairs, the Academy of Athens, the Municipality of Patras, the Ο.Π.Α.Π., the Prefecture of Western Greece, the Agricultural Bank of Greece, the Patras Papermills S.A., and the Supermarket Kronos S. A. S. I. for their financial support. The organizers would also like to thank the Directorate General of Antiquities - Museum Division for the free entrance to the archeological site and the museums of Delphi, and the Achaia Clauss Winemakers for the guided visit to its residence and for offering its residence and the wine for the welcome dinner. Finally the organizers would like to thank the Greek National Organization for offering their brochures and the following companies for offering their products during the whole week during which the conference took place: Athenian Brewery S.A., Loux Marlafekas S. A. soft drinks industry, 3E Hellenic Bottling Company, Tirnavos Wine Cooperatives, Tzafettas Greek Traditional Cheese, and the Plaisio Stores.

Last but not least, I would especially like to express my deep thanks to Lecturer Dr. Eugenia N. Petropoulou for her help concerning the procedure of the publication of this Proceedings.

Panayiotis D. Siafarikas
Guest Editor

EVANGELOS K. IFANTIS AND HIS WORK

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The aim of this work is to present briefly the scientific contribution of Professor Evangelos K. Ifantis in the research field of mathematics, to whom the "International Conference on Differential, Difference Equations and Their Applications" was dedicated. Professor Ifantis has worked in many areas of mathematics, including operator theory, difference equations, differential equations, functional equations, orthogonal polynomials and special functions, zeros of analytic functions, and analytic theory of continued fractions.

1. Who is E. K. Ifantis

Evangelos K. Ifantis was born in the village Palamas in Thessaly in central Greece, where he took elementary education. He finished high school in the city Karditsa of the same area and received his diploma in mathematics from the University of Athens in 1959. He attended a school at the Center of High Physical Studies and Philosophy of Science at the former Nuclear Research Center "Democritus." He prepared his Ph.D. thesis there, working alone, and received his Ph.D. from the University of Athens in 1969. In 1974 he was elected Professor of the Chair "Mathematics for Physicists" of the School of Natural Sciences of the University of Patras. From 1981 until now he belongs to the Department of Mathematics of the University of Patras. He retired on September 2002, after 28 years of academic service. He and his wife Eleni have two daughters (Klairi and Konstantina).

2. The work of E. K. Ifantis

2.1. Spectral theory of difference equations and the quantum mechanical phase problem. Evangelos K. Ifantis began his research with the following boundary value problem.

Find the values of E such that the functional equation

$$f(x+\alpha)+f(x-\alpha)+2\frac{b}{x}f(x)=2\left(E-\frac{b}{x}\right),\quad x,\alpha\in\mathbb{R}, \tag{2.1}$$

has solutions which satisfy the conditions $f(0)=0$, $f(\infty)=0$. This equation appears in energy band theory of the solid state physics and represents the motion of an electron in

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a lattice under the influence of a Coulomb potential. The solution of this problem has been found with classical methods and has been published in [2], which is the only paper of E. K. Ifantis in German. The eigenvalues have been found to be

$$E_k = \sqrt{1 + \frac{b^2}{\alpha^2(k+1)^2}}, \quad k = 0,1,2,\ldots, \; b \in \mathbb{R}, \; b \neq 0, \tag{2.2}$$

and the corresponding eigenvectors have the form

$$f_k(x) = \lambda_k^{-x/\alpha} \frac{x}{\alpha} P_k\left(\frac{x}{\alpha}\right), \tag{2.3}$$

where P_k is a polynomial of degree n. This shows in particular that the eigenvalues of the difference equation ($x = n$, $\alpha = 1$)

$$f(n+1) + f(n-1) + 2\frac{b}{n} f(n) = 2Ef(n), \quad n = 1,2,\ldots, \tag{2.4}$$

in the space $\ell_2(1,\infty)$ are the following:

$$E_k = \sqrt{1 + \frac{b^2}{(k+1)^2}}, \quad k = 0,1,2,\ldots, \tag{2.5}$$

for $b > 0$ and

$$E_k = -\sqrt{1 + \frac{b^2}{(k+1)^2}}, \quad k = 0,1,2,\ldots, \tag{2.6}$$

for $b < 0$. The question, if the corresponding to E_k eigenfunctions form a complete orthonormal set in ℓ_2, has led him to use the shift operator V defined in an abstract separable Hilbert space H and to reduce the above problem equivalently in an eigenvalue problem in H. The answer was negative and the result was published among others in [1, 3, 4, 8, 25].

At that time, the early 1970's, a problem studied in quantum physics was the problem of the quantization of the phase of the harmonic oscillator. In quantum mechanics, the phase problem begins with the definition of the phase operators C (cosine) and S (sine) which satisfy commutation rules analogous to the classical Poisson bracket relations

$$\{\cos\phi, H\} = \omega \sin\phi, \qquad \{\sin\phi, H\} = -\omega\cos\phi, \tag{2.7}$$

where $\phi = \arg(m\omega q + ip)$, and $H = (1/2m)[p^2 + (m\omega q)^2]$ is the classical harmonic oscillator. A well-known empirical rule applied here leads not to one operator but to a class of operators C and S which satisfy the commutation relations

$$[C,N] = iS, \qquad [S,N] = -iC, \tag{2.8}$$

where N, defined as $Ne_n = ne_n$, is the well-known number operator.

The problem was to choose suitable phase operators C and S leading to reasonable physical results. Another problem was to study the general common properties of the phase operators. To this, Professor Ifantis gave an abstract formulation in which the problem appeared as a peculiar case of the perturbation problem of continuous spectra. More precisely, the phase operators C and S have been written as

$$C = \frac{1}{2}(V^*A + AV), \qquad S = \frac{1}{2i}(V^*A - AV), \tag{2.9}$$

where V is the unilateral shift operator in an abstract separable Hilbert space with an orthonormal base e_n, $n = 1,2,\dots$, defined by $Ve_n = e_{n+1}$, $n = 1,2,\dots$, V^* its adjoint, and A a diagonal operator defined by $Ae_n = \alpha_n e_n$, $n = 1,2,\dots$, where, because of physical reasons, it was assumed that $\alpha_n > 0$ and $\lim_{n\to\infty} \alpha_n = 1$. The operator C was written in the form

$$C = \frac{1}{2}(V + V^*) + K, \tag{2.10}$$

where the spectrum of $(1/2)(V + V^*)$ is purely continuous and covers the close interval $[-1,+1]$, and the operator

$$K = \frac{1}{2}[(A - I)V + V^*(A - I)] \tag{2.11}$$

is selfadjoint and compact.

It was the first time that the well-known Weyl's theorem has been applied to the spectrum of difference equations. Recall that Weyl's theorem asserts that the essential spectrum of C is the same as the essential spectrum of $(1/2)(V + V^*)$, that is, the interval $[-1,1]$. The idea of applying Weyl's theorem came from the representation (2.10) and can be easily applied to the more general difference equation

$$\alpha_n f_{n+1} + \alpha_{n-1} f_n + b_n f_n = \lambda f_n. \tag{2.12}$$

Note that in matrix formulation, the result (2.10) can not be easily observed. Later, Weyl's theorem was used in the theory of orthogonal polynomials. Results concerning the quantum mechanical oscillator phase problem, the minimal uncertainty states for bounded observables, the nature of the spectrum of generalized oscillator phase operators and states, and minimizing the uncertainty product of the oscillator phase operators were published in a series of papers in [5, 6, 10] (three papers) and in [7]. Even up to now, people who work in this subject refer to these papers.

2.2. Zeros of analytic functions. Another problem at the beginning of the 1970s, in operator theory, was the study of the properties of the operator $T = V + C$, where V is the shift operator (in general a nonnormal isometry) and C a compact operator. Professor Ifantis defined the compact operator $Cf = (f,h)e_1$, $h = \sum_{n=1}^{\infty} c_n e_n$, an element of the Hilbert space H with the orthonormal base e_n, $n = 1,2,\dots$, and observed that the eigenvalues of T^* are related to the zeros of the function $P(z) = -1 + \sum_{n=1}^{\infty} \overline{c}_n z^n$, which belongs to the Hardy-Lebesgue space $H_2(\Delta)$. Thus Professor Ifantis in collaboration with his colleague in the Nuclear Research Center "Democritus" was led to a Hilbert space approach to the localization problem of the zeros of analytic functions in $H_2(\Delta)$. It was proved, among other things, that a lower bound for the zeros ϱ of $P(z)$ is given by $|\varrho| \geq 1/r(T) \geq 1/\|T\|$, where $r(T)$ is the spectral radius of T. Many well-known inequalities for the zeros of a polynomial (Cauchy, Walsh, Parodi, and others) have been derived from one source, the spectral radius of an operator in a Hilbert space. The results in this direction, together with a note concerning perturbations of a nonnormal isometry, were published in [9, 26, 27].

2.3. Differential and functional differential equations in the complex plane. In 1978, Professor Ifantis developed [11] an effective method for determining the existence of analytic solutions of linear functional differential equations of the form

$$(L+A)f(z) = 0, \tag{2.13}$$

where L is the operator

$$Lf(z) = \sum_{i=1}^{k} \phi_i(z)\frac{d^{k-i}}{dz^{k-i}}f(z), \tag{2.14}$$

defined in an appropriate dense domain of the Hardy-Lebesgue space $H_2(\Delta)$, and A is the operator

$$Af(z) = \sum_{i=1}^{\infty} \alpha_i(z)f(q^i z), \quad |q| \leq 1, \tag{2.15}$$

where $\phi_i(z)$, α_i, $i = 1,2,\ldots$, are analytic functions in an open set containing the closed unit disc.

He also studied systems of differential equations in the Hilbert space

$$H_2^k(\Delta) = H_2(\Delta) \times \cdots \times H_2(\Delta) \tag{2.16}$$

of the form

$$z^D \frac{df}{dz} = A_{ij}(z)f(z), \tag{2.17}$$

where $f(z) = (f_1(z), f_2(z),\ldots,f_k(z))$ is a vector function and $A_{ij}(z)$ is any matrix of bounded operators in $H_2(\Delta)$.

The method he developed reduces the existence problem of analytic solutions of (2.13) to a problem of finding the null space of a nonselfadjoint operator in an abstract separable Hilbert space. In particular, this method is also suitable for the study of entire solutions. A typical example is the following functional differential equation:

$$y'(x) = by(x) + \alpha y(\lambda x), \quad 0 \leq x < \infty, \tag{2.18}$$

$\alpha \in \mathbb{C}$, $b \in \mathbb{R}$, known as the pantograph equation. For $0 < |\lambda| \leq 1$, it is proved that (2.18) has a unique entire solution. For $|\lambda| > 1$, equation (2.18) has analytic solutions only for $\alpha = 0$ and $\alpha = -b/\lambda^{k-1}$, $k = 1,2,\ldots$. For $\alpha \neq 0$, the solutions of the pantograph equation (2.13) are polynomials of degree $k-1$, $k = 1,2,\ldots$. This extends a result of [55].

This method has been extended, later in 1987 [12, 13], in order to cover nonlinear ordinary differential equations of the form

$$(L+A)f(z) = G(z,f(z)), \tag{2.19}$$

with initial conditions $f(0)=\lambda$, $f'(0)=0$, where $G(z,f(z))$ is an analytic function of $f(z)$. For the nonlinear case, the method reduces the existence problem of families of analytic solutions of H to the study of (2.19) in the Banach space

$$H_1(\Delta) = \left\{ f : \Delta \longrightarrow \mathbb{C},\ f(z) = \sum_{n=1}^{\infty} \alpha_n z^{n-1},\ \text{analytic in } \Delta, \text{ with } \sum_{n=1}^{\infty} |\alpha_n| < +\infty \right\}, \tag{2.20}$$

which is embedded in $H_2(\Delta)$ and predicts solutions which converge absolutely on the close unit disc. The proofs are essentially based on a factorization of the operator $L+A$, which brings (2.19) to the form

$$(\tilde{L}+\tilde{A})f(z) = N(f), \tag{2.21}$$

where $\tilde{L}$ and $\tilde{A}$ denote the abstract forms of L and A, respectively, in an abstract separable Hilbert space and $N(f)$ is the abstract form of G, which has a k-invariant property playing an important role in the theory presented. Fixed point theory and bifurcation techniques can be applied, because the nonlinear operator G is Frechét differentiable in an open sphere of $H_1(\Delta)$. The most important is that the existence theorems have a constructive character and can provide an answer to the question "how small is the initial condition $f(0)=\lambda$?" As examples, some equations of particular interest, including the Emden equation, the one-dimensional Schrödinger equation, and others, are studied.

The results of these papers unify, extend, and improve some theorems previously obtained by many authors. Results concerning entire solutions of differential equations were published in [48].

At the same time, he published a paper [14] concerning solutions in ℓ_1 of nonlinear difference equations of the form

$$f(n+k) + \sum_{i=1}^{k} (\alpha_i + \alpha_i(n)) f(n+k-i) = G[f(n)], \tag{2.22}$$

under suitable conditions on $\alpha(n)$ and $G[f(n)]$.

2.4. Zeros of Bessel and mixed Bessel functions. In 1980, in my Ph.D. thesis (supervisor Professor Ifantis), the following was proved among other things: the number $\varrho \neq 0$ (in general complex) is a zero of the ordinary Bessel functions $J_\nu(z)$ of the first kind and order ν (in general complex), if and only if the equation

$$z^2 \frac{dy}{dz} + \left(-\frac{\varrho}{2} + (\nu+1)z \right) y(z) = -\frac{\varrho}{2} e^{-\varrho^2/2}, \quad y(0) = 1, \tag{2.23}$$

has a solution in $H_2(\Delta)$.

On the other hand, it was proved that the solvability of (2.23) in $H_2(\Delta)$ was equivalent to the eigenvalue problem of the compact operator

$$A_\nu = L_\nu(V + V^*) \tag{2.24}$$

in an abstract separable Hilbert space H with an orthonormal base e_n, $n = 1, 2, \dots$, where L_ν is the diagonal operator $L_\nu e_n = (1/(\nu+n))e_n$ and V, V^* the shift operator and its adjoint. This result led to an operator approach in an abstract separable Hilbert space for the study of the zeros of ordinary Bessel functions $J_\nu(z)$ and allowed us to give some alternative proofs of the properties of the Bessel functions, such as the Lommel-Hurwitz theorem, the Rayleigh formula, and so forth. Also, we obtained bounds for the real and complex zeros $j_{\nu,k}$ of the corresponding Bessel functions $J_\nu(z)$ and $\varrho_{\nu,k}$ of the mixed Bessel functions $\alpha J_\nu(z) + \beta z J'_\nu(z)$ and we discovered the following nonlinear differential equations for these zeros:

$$\frac{dj_{\nu,k}}{d\nu} = j_{\nu,k}(L_\nu x_\nu, x_\nu), \qquad \frac{d\varrho_{\nu,k}}{d\nu} = \varrho_{\nu,k}\frac{(L_\nu u_\nu, u_\nu) + \beta^2}{2(u_\nu, u_\nu) + \alpha\beta + \beta^{2\nu}}. \tag{2.25}$$

From the above differential equations, we obtained some differential inequalities and monotonicity properties, which are difficult to be found by classical methods, and improved many results obtained by other authors. The results in this direction were published in [20, 22, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 43, 49, 50, 51].

2.5. Variation of eigenvectors and eigenvalues when these depend on a real parameter and applications to the zeros of orthogonal polynomials. The method we used for the derivation of the differential equations for the zeros of Bessel and mixed Bessel functions was applied successfully by Professor Ifantis in 1988 in order to obtain *for the first time* a rigorous proof of the Hellmann-Feyman theorem concerning the differentiability of eigenvectors and eigenvalues, when these depend on a real parameter. The Hellmann-Feyman theorem is an old result which was used by the physicists formally. Also there are results concerning the concavity and the convexity of eigenvalues, Perron-Frobenius type tridiagonal operators with several applications to the zeros of orthogonal polynomials. The above results were published in [15, 16, 17, 23, 28, 37, 42, 44, 45, 46, 52].

2.6. Criteria for the nonselfadjointness of tridiagonal operators. A tridiagonal operator of the form

$$Te_n = \alpha_n e_{n+1} + \alpha_{n-1}e_{n-1} + b_n e_n, \tag{2.26}$$

where α_n is not bounded, is symmetric, defined on the dense domain consisting of finite linear combination of the base e_n. Always T admits selfadjoint extensions. The problem of finding conditions on α_n and b_n such that T is essentially selfadjoint (has a unique selfadjoint extension or not) remains open for large classes of sequences α_n and b_n. Ifantis found a new criterion on α_n and b_n such that T is not essentially selfadjoint [18].

2.7. Analytic theory of continued fractions. To every continued fraction of the form

$$K(z) = \cfrac{1}{z - b_1 - \cfrac{\alpha_1^2}{z - b_1 - \cfrac{\alpha_1^2}{z - \cdots}}}, \tag{2.27}$$

with $\alpha_i > 0$ and $b_i \in \mathbb{R}$, corresponds a tridiagonal operator of the form (2.26). It is known that if T is selfadjoint, then $K(z)$ converges to a finite value for every z in the set $\mathbb{C} - \Lambda(T)$, where $\Lambda(T)$ is the set of limit points of all zeros of the orthogonal polynomials which correspond to T. Professor Ifantis has studied conditions under which $\Lambda(T)$ is equal to the spectrum $\sigma(T)$ of T [29, 30].

2.8. Recent results of Professor Ifantis. Recent work of Professor Ifantis concerns the study of the spectrum of tridiagonal operators [19, 24, 47] and applications of orthogonal polynomials to semi-infinite Toda lattice [21, 53, 54].

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THE MATHEMATICIANS' SHARE IN THE GENERAL HUMAN CONDITION

NICOLAS K. ARTEMIADIS

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Mr. Vice Rector. Mr. Chairman, honorable guests and participants. Ladies and gentlemen.

First, I would like to thank the organizers of this conference for inviting me to give this inaugural address. It is, and always was, a pleasure to give a talk in this department, where I used to teach for many years and from where I retired as an emeritus professor.

Today, in my capacity as a regular member of the Academy of Athens, it is also a great pleasure to convey to all of you, ladies and gentlemen, on behalf of the Academy, warm greetings and to welcome you in Greece.

To Professor *Ευάγγελος Υφαντής*, to whom this conference is dedicated, I wish happiness and a prosperous life as emeritus. *Εύχομαι κ. Συνάδελφε*: *Σιδηροκέφαλος*, *Υγεία και συνεχίση του αξιόλογου ερευνητικού σου έργου*.

I come now to the subject of my lecture. A new century has already started. A new millennium has begun. So, it seemed natural to me that it would be appropriate to express some thoughts concerning the place of mathematics in a world of values and facts, something that we mathematicians seldom do, or more precisely to express some thoughts about the share of mathematicians in the general human condition.

This is a concern of the mathematician, but it is not a topic of mathematics. It is a topic of philosophy, if one agrees that philosophy is not a specialized science but a discipline that deals with the interaction of all human endeavors. I nevertheless hope to be able to make some valid observations.

Mathematics begins with an understanding of the abstract concept of a natural number (i.e., of the numbers 1, 2, 3, and so on) and the ability to count indefinitely. In this sense we may say that every human being is a mathematician. Modern historians hardly mention the mathematical component in the emergence of civilization. It was different in antiquity. In one of his plays, Aeschylus mentions: *αριθμόν έξοχον σοφισμάτων* (number, outstanding (concept) among the ingenious inventions). " With the exception of the concept of number, which is man's invention, everything else was created by God," Aristotle's

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saying, Aristotle (384–322 B.C.), could be the starting point for a survey of the role of mathematicians as an object of philosophical investigations.

I cannot and I do not want to discuss these things. However, I should like to touch, at least briefly, the work of some eminent philosophers who assigned to mathematics an extraordinary role in their systems.

Plato (428–348 B.C.) considers knowledge of mathematics to be a prerequisite of citizenship. He states that anyone who calls himself a civilized person should know that there exist incommensurable quantities in geometry. For example, it is impossible to find a unit of length such that both the side and the diagonal of a square are whole multiples of this unit. This requires a sophisticated proof and it is beyond the range of intuitive perception. But why should everybody know it? Plato wanted everybody to know that some facts are absolute certainties. To understand this need for certainty, one should read the plays of Aristophanes, which exhibit the emergence of nihilism in Plato's time. Nietzsche's doctrine, "Nothing is true. Everything is permitted," is illustrated in the play *Νεφέλαι* (The Clouds). In another play, *Ὄρνιθες* (The Birds), we see the human race entering into an alliance with birds in order to destroy the power of the gods. These plays were performed in honor of the god Dionysus. This very god receives, in another play, *Βάτραχοι* (The Frogs), a good trashing. So Plato tried to fight nihilism by exhibiting mathematics as a source of absolute truth and certainty.

Leibniz (1646–1716) was both an eminent mathematician and a philosopher. According to him, mathematics is the science that tells us what is possible. As far as the physical world is concerned, that is, that aspect of the world that Descartes called res extensa, this statement contains at least some truth. But according to Leibniz, God created our world by choosing among all possible worlds, in the sense that our world is "the best of all possible worlds."

The success of the exact sciences (which are based on the use of mathematics) has increased the range of our knowledge of the universe to a degree enormously beyond that available to Leibniz. Paradoxically, this has made many of us (including myself) more modest because our extensive knowledge has made us more aware of the range of our ignorance.

Like Leibniz, Descartes (1596–1650) was both a philosopher and an eminent mathematician. His philosophy is important to the history of the exact sciences through his dichotomy of the world into res extensa and res cogitans, but mathematics does not play an explicit role in his philosophy.

Spinoza (1632–1677) made an attempt to overcome this dualism by using not mathematics proper but at least the methods of mathematics. He proposed to derive definite philosophical truths from self-evident statements *more geometrico*, that is, in the manner of Euclid.

Spinoza provided deep and important insights, but we cannot safely say that this is due to his method, which does not qualify as a mathematical argument. In his main work, the *Ethics*, we find the statement "by God I understand being absolutely infinite." But what is "absolutely infinite"?

Spinoza did not know of the discovery of Georg Cantor (published in 1895), according to which there are smaller and larger infinitudes. There are more points in a finite interval of a straight line than there are natural numbers $1, 2, 3, \ldots$ (ad infinitum). There is even an infinite sequence of infinitudes, each larger than the previous one. Now, if we assume that there exists an infinitude containing all the previous ones, this assumption would lead to a contradiction. Now, we may be able to live with a contradiction, but we cannot tolerate it in a mathematical argument.

This last remark suggests that before relying on mathematics, it is necessary to understand both its potential and its limitations. So the question "what is mathematics?" comes up naturally.

I am not prepared to make an attempt at giving an epistemological definition of mathematics. I will only try to provide an intuitive understanding of mathematics.

Mathematics deals with concepts subject to the rules of logic, in particular to the postulate of the excluded middle. There exists at least one set of concepts of this type, namely, that of natural numbers. At this point, some comments and examples are appropriate.

It is not true that all statements involve concepts that are subject to logic. We cannot say that a person is either tall or not tall. Even if we give an artificial definition of tallness (say 1.84 m or more) we may run into trouble, because no measurement is absolutely precise. You see, there is a good reason why we have hundreds of thousands of laws. The law uses strangely defined concepts and has to be more and more casuistic to make them fit reality.

Nietzsche points out that only man-made concepts are subject to logic, while Kronecker, a 19th century mathematician, contrary to Aristotle and to Nietzche, says that "God made the whole numbers, all the rest is the work of man."

I think that these remarks will suffice. I will not go further by making statements about the "reality" of the natural numbers in philosophical (ontological) terms.

Mathematical research has two important and, I believe, unique characteristics: it involves an element of the infinite—being the only secular human activity to do so—and it produces an increasing wealth of problems with increasing abstraction. The element of the infinite in mathematics can be used to prove—in this case *more geometrico*, that is, in the way Euclid does—that the human mind is "superior" to any conceivable electronic computer. The human faculty of being able to "understand" is something that must be achieved by some noncomputational activity of the brain or mind. The description of the arguments needed here are very technical and they are linked with the name of one of the greatest mathematicians of our time, Kurt Gödel (1906–1978).

In an age where scientists as well as philosophers try to tell us that we are really nothing particular (a survival mechanism for our genes), they speak about strong artificial intelligence. Our mathematical abilities provide perhaps the simplest and strongest nonmetaphysical argument for our special position in nature.

To illustrate these remarks, and particularly the one concerning the "element of the infinite," we use the following example, which is a theorem of number theory.

> Every natural number I is the sum of the squares of at most four natural numbers. Unless $I + 1$ is divisible by 8, at most three squares suffice.

It is clear that no amount of direct calculations can prove this theorem because it involves infinitude of numbers. The proof is neither easy nor obvious and was given (for the first part of the theorem) in the late 18th century by Lagrange. This example clearly shows what is meant by "an element of the infinite."

The functions of mathematics may be described as an extension of some of the functions of language, and the reason is that our everyday language uses concepts that are not subject to the formulation of mathematical arguments and results.

The ability to name things—even nonmaterial ones, like feelings and sensations—is an act of abstraction and image making, and it is used by some philosophers to determine man's "specific difference" in the animal kingdom. Now, mathematics provides us with abstract images of things that are not accessible to the direct perception of our senses. A sophisticated and very important example of the image-making power of mathematics is the mathematical image of an atom with a nucleus and electrons. It consists entirely of formulas. But these formulas permit us to make predictions about the behavior of an atom. This is an enormous achievement and is but one example of the role of mathematics in physics, chemistry, and the branches of technology based on these sciences.

Mathematics can tell us that there are things we cannot do with the means at our disposal. For example, suppose we wish to seat the representatives, one for each, of the 180 members or so of the United Nations at a conference table. We cannot list the possible seating arrangements since their number would be greater than the number of electrons and protons in the known universe. Of course we are not particularly interested in such seating arrangements. But we might be interested in arrangements of genetic material in chromosomes where the numbers are large, too.

We now make some comments concerning "the phenomenon of mathematics," where the term "mathematics" is used in the strict sense: "the systematic derivation of theorems with the help of explicitly formulated arguments." Some mathematical insights are intuitively clear, for example, that a diameter divides a circle into two equal parts. Thales (ca. 624–548 B.C) has provided this. The fact that the side and the diagonal of a square are incommensurable is not at all intuitively clear. The Pythagorean School discovered it. A well-formulated proof of this and of related theorems appeared at the time of Plato and was due to his friend Theaetelus.

Although Babylonian, Indian, and Chinese scholars developed a body of mathematical knowledge, it is absolutely certain that mathematics is a creation of the Greeks. This does not mean the Athenians. With the exception of Theaetelus, none of the great Greek mathematicians lived in Athens. Euclid lived in Alexandria (Egypt). So did Apollonius. Archimedes lived in Syracuse (Sicily).

Nothing like the systematic work of Euclid and Apollonius is known from other civilizations of similar or earlier times. About Archimedes, the greatest applied mathematician of all time, Voltaire used to say, "There was more imagination in the head of Archimedes than in that of Homer."

What motivated these mathematicians? Not technology, not even astronomy; not a "practical" matter at all. It is true that Archimedes developed technological applications of mathematics, but the Romans, who certainly needed and used high technology, never contributed anything to mathematics. In fact, the systematic use of mathematics for the development of technology (excluding astronomy) started only in the 18th century. *The case for the development of mathematics was not usefulness.*

Earlier we compared some functions of mathematics to some functions of language. The analogy goes further. Language, too, is not merely an instrument of power or of usefulness. Nor is poetry. As far as mathematics is concerned, a good summary of its role appeared in an editorial (by Chandler and Edwards) in The Mathematical Intelligencer.

"It is a perennial problem for mathematicians to explain to the public at large what makes mathematics worthwhile if not its practicality. It is like explaining to someone who has never heard music what a lovely melody is .... Do let us try to teach the general public more of the sort of mathematics that they can use in everyday life but let us not allow them to think and certainly let us not slip into thinking that this is an essential quality of mathematics."

"There is a great cultural tradition to be preserved and enhanced. Each generation must learn the tradition anew. Let us take care not to educate a generation that will be deaf to the melodies that are the substance of our great mathematical culture."

In the past, some poets understood the beauty of mathematics. I already mentioned Aeschylus. Schiller calls it "divine." But examples of this type became rare, if not extinct, in modern times. The reason for this is of course the increasing inaccessibility of mathematics. Our latest products are available only to a very few people. However, *little would be left to human civilization, if we restrict it only to things that enjoy universal appreciation*.

There is one more aspect of mathematics that is usually mentioned as a mere curiosity. I believe it is more than that, since it relates to the idea of evolution. In several cases, scientists found the mathematical tools they needed, ready-made and available, sometimes, centuries earlier. The conics, for example (ellipse, parabola, hyperbola), have been thoroughly investigated by Apollonius in the third century B.C. and were available to Kepler in the 17th century A.D.

Another example is the theory of probability. First of all, it is strange that even a situation of complete disorder, that of random events, should be subject to mathematical laws. Second, what provoked the study of probability was a despised human activity, namely, gambling. One of the main contributors to the theory of probability was Pascal, who gave up mathematics because he thought that the only truly important thing in life was to work for the salvation of one's soul. And, finally, it turned out that the laws of probability are essential ingredients of the laws of nature. This insight started in the 19th century with Boltzmann and culminated in our century with the development of quantum theory.

I will close this lecture by trying to answer the question: what makes a mathematician? There exists a widespread resentment against mathematics. It is supposed to deal only with quantity or with computing. None of this is true, but I cannot explain that in a few words. The claim of the mathematician to be concerned with truth is frequently answered by saying that mathematical statements are not true, but merely correct. Nevertheless, it is true that human beings find the results of mathematics. Can anything be said about them? The answer is, "Not enough to enable us to recognize a mathematician if we meet one in a plane or at a party." But there exist properties without which a mathematician cannot exist. Some of them are a specific talent, an interest in the matter, and persistence to spend large amounts of time and energy. The mathematician needs an exceptionally great ability to stand frustration. His field is the only field with an "all-or-nothing" alternative. A piece of furniture may be more or less perfect. A theorem and a proof is either true or false.

It follows that the mathematician needs the support of a civilization that acknowledges as valuable the product of theory of pure thought.

One advantage a mathematician has is that his thoughts are eminently communicable, not perhaps from person to person, but certainly from nation to nation. Nothing is more international than the community of mathematicians. But I think it is rather time to stop here. I wish to all of you a very successful meeting and I declare this conference open.

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POSITIVE SOLUTIONS FOR SINGULAR DISCRETE BOUNDARY VALUE PROBLEMS

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We study the existence of zero-convergent solutions for the second-order nonlinear difference equation $\Delta(a_n\Phi_p(\Delta x_n)) = g(n,x_{n+1})$, where $\Phi_p(u) = |u|^{p-2}u$, $p > 1$, $\{a_n\}$ is a positive real sequence for $n \geq 1$, and g is a positive continuous function on $\mathbb{N}\times(0,u_0)$, $0 < u_0 \leq \infty$. The effects of singular nonlinearities and of the forcing term are treated as well.

1. Introduction

In this paper, we study decaying nonoscillatory solutions of the second-order difference equation

$$\Delta(a_n\Phi_p(\Delta x_n)) = g(n,x_{n+1}), \tag{1.1}$$

where Δ is the forward difference operator $\Delta x_n = x_{n+1} - x_n$, $\{a_n\}$ is a positive real sequence for $n \geq 1$, g is a positive continuous function on $\mathbb{N}\times(0,u_0)$, $0 < u_0 \leq \infty$, and $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$. The left-hand side in (1.1) is the one-dimensional discrete analogue of the p-Laplacian $u \to \operatorname{div}|\nabla u|^{p-2}\nabla u$ that appears in searching for radial solutions of nonlinear partial equations modelling various reaction-diffusion problems (see, e.g., [8]).

Observe that our assumptions on g allow us to consider the "singular case," that is, the case in which the nonlinearity g is unbounded with respect to the second variable in a right neighborhood of zero. From this point of view, a typical example is the nonlinear equation

$$\Delta(a_n\Phi_p(\Delta x_n)) = b_n\left[\Phi_q(x_{n+1})\right]^{-1} + r_n, \tag{1.2}$$

where $\{b_n\}$ and $\{r_n\}$ are real sequences with $b_n \geq 0$, $r_n \geq 0$, and $b_n + r_n > 0$ for $n \geq 1$ and $q > 1$.

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Equation (1.1) includes also the "regular case" with the forcing term

$$\Delta(a_n\Phi_p(\Delta x_n)) = b_n\Phi_q(x_{n+1}) + r_n. \tag{1.3}$$

Positive decreasing solutions of (1.3) when $b_n > 0$ and $r_n \equiv 0$ for $n \geq 1$ have been investigated in [5, 6].

Our aim is to study the existence of *decaying solutions of* (1.1), that is, positive solutions $\{x_n\}$ of (1.1) approaching zero as $n \to \infty$, in view of their crucial role in a variety of physical applications (see, e.g., [8]). By using a topological approach, we study mainly the effects of singular nonlinearities and those of the forcing term. Our results are also motivated also by some recent effects stated in the continuous case, see, for example, [1, 4, 9, 12] and the references therein. Our results complement the ones in [10, 11], where the existence of unbounded solutions of (1.1) is considered under the assumption $b_n < 0$. Finally, we recall that boundary value problems for equations in a discrete interval $[1,N_0]$ with singular nonlinear term in this interval have been considered recently in [2, 3].

2. Notation and preliminaries

A solution $\{x_n\}$ of (1.1) is said to be a *decaying solution* if $x_n > 0$, $\Delta x_n < 0$ eventually, and $\lim_n x_n = 0$. According to the asymptotic behavior of the quasidifference

$$x_n^{[1]} = a_n\Phi_p(\Delta x_n), \tag{2.1}$$

a decaying solution $\{x_n\}$ of (1.1) is called a *regularly decaying solution* or a *strongly decaying solution* according to $\lim_n x_n^{[1]} < 0$ or $\lim_n x_n^{[1]} = 0$, respectively. It is easy to show that every decaying solution $\{x_n\}$ of (1.1) satisfies, for every $n \geq 1$,

$$x_n > 0, \qquad \Delta x_n < 0. \tag{2.2}$$

Indeed, assume that (2.2) is verified for $n \geq N > 1$ and suppose there exists $n_0 < N$ such that $\Delta x_{n_0} \geq 0$, $\Delta x_i < 0$, $x_i > 0$, for $i = n_0 + 1,\dots,N$. From (1.1) we obtain

$$x_N^{[1]} = x_{n_0}^{[1]} + \sum_{i=n_0}^{N-1} g(i,x_{i+1}) > 0 \tag{2.3}$$

that implies $\Delta x_N > 0$, that is, a contradiction.

The set of decaying solutions will be denoted by $\mathbb{D}$ and those of *regularly decaying solutions* and *strongly decaying solutions* by $\mathbb{D}_R$ and $\mathbb{D}_S$, respectively. Clearly, $\mathbb{D} = \mathbb{D}_R \cup \mathbb{D}_S$ and

$$\begin{aligned} \mathbb{D}_R &= \Big\{\{x_n\} \text{ solution of } (1.1) : x_n > 0,\ \Delta x_n < 0,\ \lim_n x_n = 0,\ \lim_n x_n^{[1]} < 0\Big\}, \\ \mathbb{D}_S &= \Big\{\{x_n\} \text{ solution of } (1.1) : x_n > 0,\ \Delta x_n < 0,\ \lim_n x_n = 0,\ \lim_n x_n^{[1]} = 0\Big\}. \end{aligned} \tag{2.4}$$

Some notations are in order. Denote

$$Y_a = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{\Phi_{p^*}(a_n)}, \tag{2.5}$$

where p^* denotes the conjugate number of p, that is, $p^* = p/(p-1)$ or $1/p + 1/p^* = 1$.

When $Y_a < \infty$, denote by $\{A_n\}$ the sequence given by

$$A_n = \sum_{k=n}^{\infty} \frac{1}{\Phi_{p^*}(a_k)}. \tag{2.6}$$

We close this section by recalling the following lemma which is the discrete analogue of the Lebesgue dominated convergence theorem and plays an important role in proving topological properties of certain operators associated to the problem of existence of decaying solutions of (1.1).

Lemma 2.1. *Let $\{\alpha_{i,k}\}$ be a double real sequence, $\alpha_{i,k} \geq 0$, for $i,k \in \mathbb{N}$. Assume that the series $\sum_{k=1}^{\infty} \alpha_{i,k}$ totally converges, that is, there exists a sequence $\{\beta_k\}$ such that $\alpha_{i,k} \leq \beta_k$, $\sum_{k=1}^{\infty} \beta_k < \infty$, and let $\lim_{i\to\infty} \alpha_{i,k} = \varrho_k$ for every $k \in \mathbb{N}$. Then the series $\sum_{k=1}^{\infty} \varrho_k$ converges and*

$$\lim_{i\to\infty} \sum_{k=1}^{\infty} \alpha_{i,k} = \sum_{k=1}^{\infty} \varrho_k. \tag{2.7}$$

3. Regularly decaying solutions

In this section, we study the existence of solutions in the class $\mathbb{D}_R$. We start with a necessary condition. The following proposition holds.

Proposition 3.1. *If $\mathbb{D}_R \neq \emptyset$, then $Y_a < \infty$.*

Proof. Let $x = \{x_n\}$ be a solution of (1.1) in the class $\mathbb{D}_R$. Because $\{x_n^{[1]}\}$ is negative increasing and $\lim_n x_n^{[1]} = x_\infty^{[1]} < 0$, it holds that

$$a_n \Phi_p(\Delta x_n) < x_\infty^{[1]}. \tag{3.1}$$

This implies, for $n < N$,

$$\Phi_{p^*}(|x_\infty^{[1]}|) \sum_{j=n}^{N-1} \Phi_{p^*}\left(\frac{1}{a_j}\right) \leq x_n - x_N \tag{3.2}$$

that gives the assertion as $N \to \infty$. □

Remark 3.2. For any solution $\{x_n\} \in \mathbb{D}_R$, it holds that $a_n \Phi_p(\Delta x_n) \geq x_1^{[1]}$. Hence, from (3.2), we obtain the following upper and lower bounds:

$$-\Phi_{p^*}(x_\infty^{[1]}) A_n \leq x_n \leq -\Phi_{p^*}(x_1^{[1]}) A_n. \tag{3.3}$$

In addition, regularly decaying solutions $\{x_n\}$ are asymptotic to the sequence (2.6), that is,

$$\lim_n \frac{x_n}{A_n} = c_x, \quad 0 < c_x < \infty, \tag{3.4}$$

where $\Phi_p(c_x) = |x_\infty^{[1]}|$, as the Stolze theorem yields.

Assumption $Y_a < \infty$ is not sufficient for the existence of solutions in the class $\mathbb{D}_R$ as the following example shows.

Example 3.3. Consider the equation

$$\Delta\left(n^2\Phi_p(\Delta x_n)\right) = \frac{1}{x_{n+1}}. \tag{3.5}$$

Let $\{x_n\}$ be a solution of (3.5) in the class $\mathbb{D}_R$ and let $n_0 \geq 1$ such that $x_{n+1} < 1$ for $n > n_0$. Hence, for $n > n_0$,

$$\Delta\left(n^2\Phi_p(\Delta x_n)\right) > 1 \tag{3.6}$$

or

$$x_{n+1}^{[1]} > x_{n_0}^{[1]} + n - n_0 \tag{3.7}$$

that gives a contradiction as $n \to \infty$.

The following theorem holds.

Theorem 3.4. *Assume the following conditions:*

(i) $Y_a < \infty$;

(ii) *there exists a continuous function $F : \mathbb{N} \times (0,\delta] \to (0,\infty)$, $\delta < u_0$, monotone with respect to the second variable such that for $(n,v) \in \mathbb{N} \times (0,\delta]$,*

$$g(n,v) \leq F(n,v), \tag{3.8}$$

$$\sum_{n=1}^{\infty} F(n,A_{n+1}) < \infty. \tag{3.9}$$

Then (1.1) has solutions in the class $\mathbb{D}_R$. More precisely, for every $c \geq 1$, there exists a positive solution $\{x_n\}$ such that

$$\lim_n \frac{x_n}{A_n} = c, \tag{3.10}$$

where $\Phi_p(c) = \lim_n |x_n^{[1]}|$.

Proof. First, we prove the statement for F nonincreasing. Choose $n_0 \geq 1$ such that

$$\Phi_{p^*}(2)A_{n_0} < \delta, \tag{3.11}$$

$$\sum_{n=n_0}^{\infty} F(n,A_{n+1}) < 1. \tag{3.12}$$

Denote by $\ell^\infty_{n_0}$ the Banach space of all bounded sequences defined for $n \ge n_0$ and endowed with the topology of supremum norm. Let Ω be the nonempty subset of $\ell^\infty_{n_0}$ given by

$$\Omega = \{\{u_n\} \in \ell^\infty_{n_0} : A_n \le u_n \le \Phi_{p^*}(2)A_n\}. \tag{3.13}$$

Clearly, Ω is a bounded, closed, and convex subset of $\ell^\infty_{n_0}$. We define the mapping $T : \Omega \to \ell^\infty_{n_0}$ by

$$w_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\right)\Phi_{p^*}\left(1 + \sum_{i=j}^{\infty} g(i, u_{i+1})\right). \tag{3.14}$$

We prove that T satisfies the hypotheses of Schauder fixed-point theorem.

(a) The mapping T maps Ω into itself. Obviously, $A_n \le w_n$. Conditions (ii) and (3.12) imply

$$\sum_{j=n_0}^{\infty} g(j, u_{j+1}) \le \sum_{j=n_0}^{\infty} F(j, u_{j+1}) \le \sum_{j=n_0}^{\infty} F(j, A_{j+1}) \le 1, \tag{3.15}$$

and taking into account (3.14) and monotonicity of Φ_{p^*}, we have

$$w_n \le \sum_{j=n}^{\infty} \Phi_{p^*}\left(\frac{2}{a_j}\right) = \Phi_{p^*}(2)A_n. \tag{3.16}$$

(b) The mapping T is continuous in Ω. Let $\{U^{(i)}\}$ be a sequence in Ω converging to U in $\ell^\infty_{n_0}$. Because Ω is closed, $U \in \Omega$. Let $U^{(i)} = \{u_n^{(i)}\}$, $U = \{u_n\}$ and $W^{(i)} = T(U^{(i)}) = \{w_n^{(i)}\}$, $W = T(U) = \{w_n\}$. It holds for every integer $n \ge n_0$ that

$$\begin{aligned}
&\|T(U^{(i)}) - T(U)\| \\
&\quad = \sup_{n \ge n_0} |w_n^{(i)} - w_n| \\
&\quad \le \sup_{n \ge n_0} \sum_{k=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_k}\right) \left| \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} g\left(j, u_{j+1}^{(i)}\right)\right) - \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} g(j, u_{j+1})\right) \right| \\
&\quad \le \sum_{k=n_0}^{\infty} \alpha_{i,k},
\end{aligned} \tag{3.17}$$

where

$$\alpha_{i,k} = \Phi_{p^*}\left(\frac{1}{a_k}\right) \left| \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} g\left(j, u_{j+1}^{(i)}\right)\right) - \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} g(j, u_{j+1})\right) \right|. \tag{3.18}$$

From the continuity of g, we obtain

$$\lim_i g\left(j, u_{j+1}^{(i)}\right) = g(j, u_{j+1}) \quad \text{for } j \ge n_0, \tag{3.19}$$

and, in view of (ii) and the fact that $U^{(i)} \in \Omega$,

$$\left| g\left(j, u_{j+1}^{(i)}\right) \right| \le F(j, A_{j+1}). \tag{3.20}$$

Then the series $\sum_{j=k}^{\infty} g(j, u_{j+1}^{(i)})$ is totally convergent and, by Lemma 2.1,

$$\lim_i \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} g\left(j, u_{j+1}^{(i)}\right)\right) = \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} g(j, u_{j+1})\right), \tag{3.21}$$

that is,

$$\lim_i \alpha_{i,k} = 0 \quad \text{for every } k \ge n_0. \tag{3.22}$$

In addition, using (3.12), we find

$$\begin{aligned} \alpha_{i,k} &\le \left(\Phi_{p^*}\left(\frac{1}{a_k}\right)\right)\left(\Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} F\left(j, u_{j+1}^{(i)}\right)\right) + \Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} F(j, u_{j+1})\right)\right) \\ &\le 2\left(\Phi_{p^*}\left(\frac{1}{a_k}\right)\right)\Phi_{p^*}\left(1 + \sum_{j=k}^{\infty} F(j, A_{j+1})\right) \le 2\Phi_{p^*}(2)\left(\Phi_{p^*}\left(\frac{1}{a_k}\right)\right). \end{aligned} \tag{3.23}$$

Since $Y_a < \infty$, the series $\sum_{k=n_0}^{\infty} \alpha_{i,k}$ is totally convergent. Applying again Lemma 2.1, it follows from (3.17) and (3.22) that

$$\lim_i \|T(U^{(i)}) - T(U)\| \le \lim_i \sum_{k=n_0}^{\infty} \alpha_{i,k} = \sum_{k=n_0}^{\infty} \left[\lim_i \alpha_{i,k}\right] = 0. \tag{3.24}$$

Hence, T is continuous in Ω.

(c) The set $T(\Omega)$ is relatively compact. By a result in [7, Theorem 3.3], it is sufficient to prove that $T(\Omega)$ is uniformly Cauchy in the topology of $\ell_{n_0}^{\infty}$, that is, for every $\varepsilon > 0$, there exists an integer $n_\varepsilon \ge n_0$ such that $|w_{m_1} - w_{m_2}| < \varepsilon$ whenever $m_1, m_2 > \varepsilon$ for every $W = \{w_n\} \in T(\Omega)$. Let $W = T(U)$, $U = \{u_n\}$, and, without loss of generality, assume $m_1 < m_2$. From (3.14), we obtain

$$\begin{aligned} |w_{m_1} - w_{m_2}| &= \left| \sum_{j=m_1}^{m_2-1} \Phi_{p^*}\left(\frac{1}{a_j}\right)\Phi_{p^*}\left(1 + \sum_{i=j}^{\infty} g(i, u_{i+1})\right) \right| \\ &\le \left| \sum_{j=m_1}^{m_2-1} \Phi_{p^*}\left(\frac{1}{a_j}\right)\Phi_{p^*}\left(1 + \sum_{i=j}^{\infty} F(i, A_{i+1})\right) \right| \\ &\le \Phi_{p^*}(2) \sum_{j=m_1}^{m_2-1} \Phi_{p^*}\left(\frac{1}{a_j}\right), \end{aligned} \tag{3.25}$$

and the Cauchy criterion gives the relative compactness of $T(\Omega)$.

Hence, by Schauder fixed-point theorem, there exists $\{x_n\} \in \Omega$ such that $x_n = T(x_n)$ or, from (3.14),

$$x_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left[\frac{1}{a_j}\left(1+\sum_{i=j}^{\infty} g(i,x_{i+1})\right)\right]. \tag{3.26}$$

One can easily check that $\{x_n\}$ is a solution of (1.1) with $\Delta x_n < 0$, $\lim_n x_n = 0$, and $\lim_n x_n^{[1]} = -1$, and so $\{x_n\} \in \mathbb{D}_R$. Clearly, in view of Remark 3.2, $\{x_n\}$ satisfies (3.10) with $c = 1$.

To obtain the existence of a positive solution $\{x_n\}$ such that $\lim_n [x_n/A_n] = c > 1$, it is sufficient to observe that (3.9) and monotonicity of F imply that the series

$$\sum_{n=1}^{\infty} F(n,\lambda A_{n+1}) \tag{3.27}$$

is convergent for any $\lambda \geq 1$. Now, the assertion follows by considering in the subset

$$\Omega_\lambda = \{\{u_n\} \in \ell_{n_0}^{\infty} : \Phi_{p^*}(\lambda)A_n \leq u_n \leq \Phi_{p^*}(2\lambda)A_n\} \tag{3.28}$$

the operator $T : \{u_n\} \to \{w_n\}$ given by

$$w_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\right)\Phi_{p^*}\left(\lambda+\sum_{i=j}^{\infty} g(i,u_{i+1})\right) \tag{3.29}$$

and using an analogous argument as above.

In case F is nondecreasing on $(0,\delta]$, the proof is quite similar with some minor changes. It is sufficient to consider the subset Ω and the operator T as follows:

$$\begin{gathered} \Omega = \left\{\{u_n\} \in \ell_{n_0}^{\infty} : \frac{1}{2}A_n \leq u_n \leq A_n\right\}, \\ w_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\left(\frac{1}{2}+\sum_{i=j}^{\infty} g(i,u_{i+1})\right)\right), \end{gathered} \tag{3.30}$$

where n_0 is chosen such that

$$\sum_{n=n_0}^{\infty} F(n,A_{n+1}) < \frac{1}{2}. \tag{3.31}$$

The details are left to the reader. □

Remark 3.5. The existence of regularly decaying solutions $\{x_n\}$ satisfying (3.10) for $c \in (0,1)$ is guaranteed by the condition

$$\sum_{n=1}^{\infty} F(n,\Phi_{p^*}(c)A_{n+1}) < \infty \tag{3.32}$$

instead of (3.9) and can be proved using an analogous argument as given in the proof of Theorem 3.4.

For the special case of (1.2), assumption (ii) of Theorem 3.4 becomes

$$\sum_{n=1}^{\infty} b_n\left[\Phi_q(A_{n+1})\right]^{-1} < \infty, \qquad \sum_{n=1}^{\infty} r_n < \infty. \tag{3.33}$$

In this case, by applying Theorem 3.4 to (1.2), for every $c > 0$, we obtain the existence of solutions satisfying (3.10). In addition, for (1.2), conditions $Y_a < \infty$ and (3.33) become also necessary for the existence in $\mathbb{D}_R$ as the following result shows.

Corollary 3.6. *Equation (1.2) has solutions in the class $\mathbb{D}_R$ if and only if $Y_a < \infty$ and (3.33) hold.*

Proof. In view of Proposition 3.1 and Theorem 3.4, it is sufficient to prove that if $\mathbb{D}_R \neq \emptyset$, then (3.33) is verified. Let $\{x_n\}$ be a solution of (1.2) in the class $\mathbb{D}_R$. By the summation of (1.2) from n to $N-1$ and taking into account (3.3), we have

$$\begin{aligned} -x_n^{[1]} &= -x_N^{[1]} + \sum_{j=n}^{N-1} b_j\left[\Phi_q(x_{j+1})\right]^{-1} + \sum_{j=n}^{N-1} r_j \\ &> \lambda \sum_{j=n}^{N} b_j\left[\Phi_q(A_{j+1})\right]^{-1} + \sum_{j=n}^{N} r_j, \end{aligned} \tag{3.34}$$

where $\lambda = [\Phi_q[\Phi_{p^*}(-x_1^{[1]})]]^{-1}$. As $N \to \infty$, we obtain the assertion. □

Theorem 3.4 is applicable even if the nonlinearity g is bounded with respect to the dependent variable in a right neighborhood of zero, that is, the boundary value problem is "regular." In such a case, assumption (ii) of Theorem 3.4 can be simplified.

Corollary 3.7. *If $Y_a < \infty$ and*

$$\sum_{n=1}^{\infty} b_n \Phi_q(A_{n+1}) < \infty, \qquad \sum_{n=1}^{\infty} r_n < \infty, \tag{3.35}$$

then (1.3) has solutions in the class $\mathbb{D}_R$. More precisely, for every $c > 0$, there exists a positive solution $\{x_n\}$ such that (3.10) is verified with $\Phi_p(c) = \lim_n |x_n^{[1]}|$.

Proof. The assertion follows from Theorem 3.4 and Remark 3.5 by choosing $F(n,v) = b_n\Phi_q(v) + r_n$. □

4. Strongly decaying solutions

Here we study the existence of solutions in the class $\mathbb{D}_S$ for equations with possible singular nonlinearity. More precisely, in this section, we will assume that g satisfies the condition

$$\inf_{v \in (0,\delta]} g(i,v) = m_i > 0 \tag{4.1}$$

for infinitely many i, where δ is a positive constant, $\delta < u_0$. The following necessary conditions hold.

Proposition 4.1. *If $\mathbb{D}_S \neq \emptyset$, then*

$$\sum_{n=1}^{\infty} m_n < \infty, \tag{4.2}$$

$$\sum_{j=1}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\sum_{i=j}^{\infty} m_i\right) < \infty, \tag{4.3}$$

where m_j is given in (4.1).

Proof. Let $\{x_n\}$ be a solution of (1.1) in the class $\mathbb{D}_S$. Without loss of generality, we can assume $x_n < \delta$ for $n \geq 1$. Hence,

$$g(i, x_{i+1}) \geq \inf_{v \in (0,\delta]} g(i, v) = m_i. \tag{4.4}$$

By summing (1.1) from n to ∞, we obtain

$$-x_n^{[1]} = \sum_{i=n}^{\infty} g(i, x_{i+1}) \geq \sum_{i=n}^{\infty} m_i \tag{4.5}$$

that implies (4.2). By summing again from n to ∞, we have

$$x_n \geq \sum_{i=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\sum_{i=j}^{\infty} m_i\right), \tag{4.6}$$

and so (4.3) is proved. □

Remark 4.2. Because

$$\sum_{j=1}^{N} \Phi_{p^*}\left(\frac{1}{a_j}\sum_{i=j}^{N} m_j\right) \geq \Phi_{p^*}\left(\frac{1}{a_1}\sum_{i=1}^{N} m_j\right) = \Phi_{p^*}\left(\frac{1}{a_1}\right)\Phi_{p^*}\left(\sum_{i=1}^{N} m_j\right), \tag{4.7}$$

condition (4.3) implies (4.2).

A sufficient criterion for existence in $\mathbb{D}_S$ is given by the following theorem.

Theorem 4.3. *Assume (4.1) and (4.3). If there exists a continuous function $F : \mathbb{N} \times (0, \delta] \to (0, \infty)$, $0 < \delta < u_0$, nonincreasing with respect to the second variable such that, for $(n, v) \in \mathbb{N} \times (0, \delta]$,*

$$\begin{gathered} g(n, v) \leq F(n, v), \\ \sum_{n=1}^{\infty} \Phi_{p^*}\left[\frac{1}{a_n}\sum_{j=n}^{\infty} F(j, B_{j+1})\right] < \infty, \end{gathered} \tag{4.8}$$

where

$$B_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\sum_{i=j}^{\infty} m_i\right), \tag{4.9}$$

then (1.1) has solutions in the class $\mathbb{D}_S$.

Proof. Choose $n_0 \geq 1$ such that

$$B_{n_0} < \delta, \qquad \sum_{n=n_0}^{\infty} \Phi_{p^*}\left[\frac{1}{a_n}\sum_{j=n}^{\infty} F(j, B_{j+1})\right] < \delta. \tag{4.10}$$

Let Ω be the subset of $\ell_{n_0}^{\infty}$ given by

$$\Omega = \{\{u_n\} \in \ell_{n_0}^{\infty} : B_n \leq u_n \leq \delta\}. \tag{4.11}$$

In view of (4.1), it holds that $B_n > 0$. In addition, because $\{B_n\}$ is nonincreasing, from (4.10) the set Ω is nonempty. Clearly, Ω is bounded, closed, and convex in $\ell_{n_0}^{\infty}$. We define the mapping $T : \Omega \to \ell_{n_0}^{\infty}$ by

$$w_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left[\frac{1}{a_n}\sum_{j=n}^{\infty} g(j, u_{j+1})\right]. \tag{4.12}$$

Because

$$g(j, u_{j+1}) \geq \inf_{v \in (0,\delta]} g(j, v) = m_j, \tag{4.13}$$

we have

$$w_n \geq \sum_{j=n}^{\infty} \Phi_{p^*}\left[\frac{1}{a_n}\sum_{j=n}^{\infty} m_j\right] = B_n. \tag{4.14}$$

In addition, it holds for $j \geq n_0$ that

$$\sum_{j=n}^{\infty} g(j, u_{j+1}) \leq \sum_{j=n}^{\infty} F(j, u_{j+1}) \leq \sum_{j=n}^{\infty} F(j, B_{j+1}) \tag{4.15}$$

or, in view of (4.10),

$$w_n \leq \sum_{j=n}^{\infty} \Phi_{p^*}\left[\frac{1}{a_n}\sum_{j=n}^{\infty} F(j, B_{j+1})\right] < \delta. \tag{4.16}$$

Thus, $T(\Omega) \subseteq \Omega$. The continuity of T in Ω and the compactness of $\overline{T(\Omega)}$ follow by using a similar argument as in the proof of Theorem 3.4. Hence, by applying the Schauder fixed-point theorem, we obtain the existence of a fixed point $\{x_n\}$ of T. Clearly,

$$x_n = \sum_{j=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_j}\sum_{i=j}^{\infty} g(i, x_{i+1})\right), \tag{4.17}$$

and so $\{x_n\} \in \mathbb{D}_S$. $\square$

For the special case of singular equation (1.2) with $r_n = 0$ for $n \in \mathbb{N}$, Theorem 4.3 yields the following result.

Corollary 4.4. *Consider the equation*

$$\Delta(a_n \Phi_p(\Delta x_n)) = b_n [\Phi_q(x_{n+1})]^{-1} \tag{4.18}$$

with $b_n > 0$ *for infinitely many n. Assume*

$$\lim_{m\to\infty} \sum_{n=1}^{m} \Phi_{p^*}\left(\frac{1}{a_n}\sum_{k=n}^{m} b_k\right) < \infty \tag{4.19}$$

and denote

$$\beta_n = \sum_{i=n}^{\infty} \Phi_{p^*}\left(\frac{1}{a_i}\sum_{k=i}^{\infty} b_k\right). \tag{4.20}$$

If

$$\lim_{m\to\infty} \sum_{n=1}^{m} \Phi_{p^*}\left[\frac{1}{a_n}\sum_{j=n}^{m} b_j [\Phi_q(\beta_{j+1})]^{-1}\right] < \infty, \tag{4.21}$$

then (4.18) has solutions in the class $\mathbb{D}_S$.

The assumption in Theorem 4.3 (and Corollary 4.4) is not necessary for $\mathbb{D}_S \neq \varnothing$ as the following example shows.

Example 4.5. Consider the equation

$$\Delta^2 x_n = \frac{2}{n(n+1)^2(n+2)} (x_{n+1})^{-1}. \tag{4.22}$$

Clearly, (4.19) is satisfied. We have

$$\beta_n = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \frac{2}{j(j+1)^2(j+2)} < \sum_{i=n}^{\infty}\sum_{j=i}^{\infty} \frac{2}{j^4}. \tag{4.23}$$

Taking into account that for $n \in \mathbb{N}$, $n > 1$, and γ real positive constant, $\gamma > 1$, the following inequality holds

$$\sum_{i=n}^{\infty} \frac{1}{i^\gamma} < \int_{n-1}^{\infty} \frac{1}{x^\gamma} dx = \frac{1}{(\gamma-1)(n-1)^{\gamma-1}}, \tag{4.24}$$

from (4.23) we obtain

$$\beta_{n+1} < \sum_{i=n+1}^{\infty} \frac{2}{3(i-1)^3} = \frac{2}{3}\sum_{i=n}^{\infty} \frac{1}{i^3} < \frac{1}{3(n-1)^2}. \tag{4.25}$$

Hence

$$\sum_{n=1}^{\infty}\sum_{j=n}^{\infty} b_j (\beta_{j+1})^{-1} > \sum_{n=1}^{\infty}\sum_{j=n}^{\infty} \frac{6(j-1)^2}{j(j+1)^2(j+2)} = \infty, \tag{4.26}$$

and so condition (4.21) is not satisfied. But it is easy to verify that the sequence $\{x_n\}$, $x_n = 1/n$, is a solution of (4.22) and $\{x_n\} \in \mathbb{D}_S$.

The following result gives an application of Theorem 4.3 to the regular equation (1.3) with the forcing term.

Corollary 4.6. *If $r_n > 0$ for infinitely many n and*

$$\lim_{m\to\infty} \sum_{n=1}^{m} \Phi_{p^*}\left(\frac{1}{a_n}\sum_{k=n}^{m}(b_k + r_k)\right) < \infty, \tag{4.27}$$

then (1.3) has solutions in the class $\mathbb{D}_S$.

Proof. The assertion follows from Theorem 4.3 by choosing $F(n,v) = b_n + r_n$ and noting that (4.1) is satisfied because $m_i = r_i > 0$. □

5. Concluding remarks

(1) *The continuous case.* Decaying solutions of second-order nonlinear singular differential equations without the forcing term have been investigated in [9, 12]. Corollaries 3.6 and 4.4 can be regarded as the discrete counterparts of [9, Theorem 4.2] and [12, Theorem 5.2], respectively.

(2) *An effect of singular nonlinearities.* If $Y_a = \infty$ and

$$\lim_{m\to\infty} \sum_{n=1}^{m} \Phi_{p^*}\left(\frac{1}{a_n}\sum_{k=n}^{m} b_k\right) = \infty, \tag{5.1}$$

then, from Propositions 3.1 and 4.1, it follows that (4.18) does not possess any decaying solution. This fact cannot occur for equations with regular nonlinearity; for instance, the linear equation

$$\Delta^2 x_n = (1+n)^{-1} x_{n+1} \tag{5.2}$$

has strongly decaying solutions (see, e.g., [5, Corollary 3.3(a)]) and, in this case, $Y_a = \infty$ and (5.1) is verified.

(3) *An effect of the forcing term r_n.* As we have already noted, (1.3) without the forcing term r_n has been investigated in [5]. Comparing the results presented here and in [5], one can see that the existence of regularly decaying solutions of (1.3) remains valid for the equation with the forcing term r_n such that $\sum r_n < \infty$, while the existence of strongly decaying solutions of (1.3) is caused by the forcing term. More precisely, if (4.19) is satisfied, then (1.3) with $r_n \equiv 0$ and $p \le q$ does not have strongly decaying solutions, see [5, Theorem 2.3]. On the contrary, by Corollary 4.6, (1.3), with the forcing term r_n, $r_n > 0$, for infinitely many n, and satisfying (4.27), has strongly decaying solutions.

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SYMPLECTIC DIFFERENCE SYSTEMS: OSCILLATION THEORY AND HYPERBOLIC PRÜFER TRANSFORMATION

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We present basic methods of oscillation theory of symplectic difference systems (SDSs). A particular attention is devoted to the variational principle and to the transformation method. Hyperbolic Prüfer transformation for SDSs is established.

1. Introduction

In this paper, we deal with oscillatory properties and transformations of symplectic difference systems (SDSs)

$$z_{k+1} = \mathcal{S}_k z_k, \quad z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \ \mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \tag{1.1}$$

where $x, u \in \mathbb{R}^n$, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathbb{R}^{n\times n}$, and the matrix $\mathcal{S}$ is supposed to be symplectic, that is,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.2}$$

The last identity translates in terms of the block entries $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ as

$$\mathcal{A}^T\mathcal{C} = \mathcal{C}^T\mathcal{A}, \qquad \mathcal{B}^T\mathcal{D} = \mathcal{D}^T\mathcal{B}, \qquad \mathcal{A}^T\mathcal{D} - \mathcal{C}^T\mathcal{B} = I. \tag{1.3}$$

If $Z = \binom{X}{U}$, $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ are $2n \times n$ matrix solutions of (1.1) and $\mathcal{Z} = (Z\tilde{Z}) = \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U}\end{pmatrix}$, then using (1.2), we have

$$\Delta(\mathcal{Z}_k^T \mathcal{J} \mathcal{Z}_k) = \mathcal{Z}_{k+1}^T \mathcal{J} \mathcal{Z}_{k+1} - \mathcal{Z}_k^T \mathcal{J} \mathcal{Z}_k = \mathcal{Z}_k^T [\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k - \mathcal{J}] \mathcal{Z}_k = 0 \tag{1.4}$$

which means that $\mathcal{Z}_k$ are symplectic whenever this property is satisfied at one index, say $k = 0$. Consequently, (1.1) defines the discrete symplectic flow and this fact, together with (1.2), is the justification for the terminology *symplectic difference system*.

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SDSs cover, as particular cases, a large variety of difference equations and systems, among them the Sturm-Liouville second-order difference equation

$$\Delta(r_k \Delta x_k) + p_k x_{k+1} = 0, \quad r_k \neq 0,\ \Delta x_k = x_{k+1} - x_k, \tag{1.5}$$

the higher-order selfadjoint difference equation

$$\sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) = 0, \quad r_k^{[n]} \neq 0,\ \Delta^\nu = \Delta(\Delta^{\nu-1}), \tag{1.6}$$

and the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{1.7}$$

with $A, B, C \in \mathbb{R}^{n \times n}$, B and C symmetric (i.e., $B = B^T$, $C = C^T$), and $I - A$ invertible.

Our paper is organized as follows. In the remaining part of this section we recall, for the sake of later comparison, basic oscillatory properties of the Sturm-Liouville equation (1.5). Section 2 contains the so-called Roundabout theorem for (1.1) which forms the basis for the investigation of oscillatory properties of these systems. We also mention some results concerning transformations of (1.1). Section 3 is devoted to the illustration of the methods of oscillation theory of (1.1) and Section 4 contains a new result, the so-called *discrete hyperbolic Prüfer transformation*. We also formulate some open problems associated with this type of transformation.

Now, we recall basic facts of the oscillation theory of (1.5) as can be found, for example, in [1, 2, 11, 14]. We substitute $u = r\Delta x$ in (1.5). Then this equation can be written as a 2×2 Hamiltonian system (1.7)

$$\Delta \begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{r_k} \\ -p_k & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix} \tag{1.8}$$

and expanding the forward differences as a 2×2 symplectic system

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{1}{r_k} \\ -\dfrac{p_k}{r_k} & 1 - \dfrac{p_k}{r_k} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{1.9}$$

We say that an interval $(m, m+1]$ contains a *focal point* (an alternative terminology is *generalized zero*, see [13]) of a solution x of (1.5) if $x_m \neq 0$ and $r_m x_m x_{m+1} \leq 0$. Equation (1.5) is said to be *disconjugate* in the discrete interval $[0, N]$ if the solution $\tilde{x}$ given by the initial condition $\tilde{x}_0 = 0$, $\tilde{x}_1 = 1/r_0$ has no focal point in $(0, N+1]$. This equation is said to be *nonoscillatory* if there exists $n \in \mathbb{N}$ such that (1.5) is disconjugate on $[n, m]$ for every $m > n$, and it is said to be *oscillatory* in the opposite case.

The next statement, usually referred to as the Roundabout theorem, shows that the discrete quadratic functional and the discrete Riccati equation play the same role in the oscillation theory of (1.5) as their continuous counterparts in the oscillation theory of the Sturm-Liouville differential equation

$$(r(t)x')' + p(t)x = 0. \tag{1.10}$$

PROPOSITION 1.1. *The following statements are equivalent:*

(i) *equation (1.5) is disconjugate on the interval* $[0,N]$,
(ii) *there exists a solution x of (1.5) having no focal point in* $[0,N+1]$,
(iii) *there exists a solution w of the Riccati equation (related to (1.5) by the substitution $w = r\Delta x/x$)*

$$\Delta w_k + p_k + \frac{w_k^2}{w_k + r_k} = 0 \tag{1.11}$$

which is defined for $k \in [0,N+1]$ and satisfies $r_k + w_k > 0$ for $k \in [0,N]$,
(iv) *the quadratic functional*

$$\sum_{k=0}^{N} \left\{ r_k (\Delta y_k)^2 - p_k y_{k+1}^2 \right\} > 0 \tag{1.12}$$

for every nontrivial $y = \{y_k\}_{k=0}^{N+1}$ with $y_0 = 0 = y_{N+1}$.

Note that the previous proposition actually shows that the Sturmian separation and comparison theory extend verbatim to (1.5), using the same argument as in the case of the differential equation (1.10).

2. Oscillation theory of SDSs

First, we turn our attention to Hamiltonian difference systems (1.7). Oscillation theory of these systems attracted considerable attention in late eighties and early nineties of the last century (see [8, 12] and the references given therein). Note that in both of these papers, it is assumed that the matrix B is positive definite and hence nonsingular. However, such Hamiltonian systems do not cover several important equations, for example, (1.6), in which case the matrix $B = \operatorname{diag}\{0,\dots,0,1/r_n^{[n]}\}$ in the Hamiltonian system corresponding to this equation. This difficulty was removed in the pioneering paper of Bohner [3], where the concept of the focal point of a matrix solution of (1.7) with B possibly singular was introduced. Later, this concept was extended to system (1.1) in [5] and reads as follows. We say that a conjoined basis $Z = \binom{X}{U}$ of (1.1) (i.e., a $2n \times n$ matrix solution such that $X^T U$ is symmetric and rank $\binom{X}{U} \equiv n$) has a *focal point* in an interval $(m, m+1]$, $m \in \mathbb{Z}$, if $\operatorname{Ker} X_{m+1} \not\subseteq \operatorname{Ker} X_m$ or " $\subseteq$ " holds, but $P_m := X_m X_{m+1}^{\dagger} \mathcal{B}_m \not\geq 0$, here Ker, †, and $\geq$ mean the kernel, the generalized inverse, and nonnegative definiteness of a symmetric matrix, respectively. Note that if the inclusion " $\subseteq$ " holds, then the matrix P_m is really symmetric (see [5]). System (1.1) is said to be *disconjugate* on $[0,N]$ if the solution $Z = \binom{X}{U}$ given by the initial condition $X_0 = 0$, $U_0 = I$ has no focal point in $(0,N+1]$. *Oscillation* and *nonoscillation* of (1.1) are defined via disconjugacy in the same way as for (1.5).

The following statement shows that, similar to the scalar case, certain discrete quadratic functional and Riccati-type difference equation play a crucial role in the oscillation theory of (1.1). This statement is proved in [5] and we present it here in a slightly modified form.

PROPOSITION 2.1. *The following statements are equivalent:*

(i) *system (1.1) is disconjugate in the interval* $[0,N]$,

(ii) *there exists a conjoined basis* $Z = \binom{X}{U}$ *without any focal point in* $[0,N+1]$ *and with* X_k *nonsingular in this interval,*

(iii) *there exists a symmetric solution* Q *of the Riccati matrix difference equation*

$$Q_{k+1} = (\mathcal{C}_k + \mathcal{D}_k Q_k)(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1} \tag{2.1}$$

which is defined for $k \in [0,N+1]$ *and the matrix* $P_k := \mathcal{B}_k^T(\mathcal{D}_k - Q_{k+1}\mathcal{B}_k)$ *is non negative definite for* $k \in [0,N]$,

(iv) *let* $\mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$. *The quadratic functional corresponding to (1.1)*

$$\begin{aligned} \mathcal{F}(z) &= \sum_{k=0}^{N} z_k^T \left\{ \mathcal{S}_k^T \mathcal{K} - \mathcal{K} \right\} z_k \\ &= \sum_{k=0}^{N} \left\{ x_k^T \mathcal{C}_k^T \mathcal{A}_k x_k + 2x_k^T \mathcal{C}_k^T \mathcal{B}_k u_k + u_k^T \mathcal{D}_k^T \mathcal{B}_k u_k \right\} \end{aligned} \tag{2.2}$$

is positive for every $z = \{z_k\}_{k=0}^{N+1}$ *satisfying* $\mathcal{K} z_{k+1} = \mathcal{K}\mathcal{S}_k z_k$, $\mathcal{K} z_0 = 0 = \mathcal{K} z_{N+1}$, *and* $\mathcal{K} z \not\equiv 0$, *that is, if we write* $z = \binom{x}{u}$, *then* $\mathcal{F}(x,u) > 0$ *for every* x, u *satisfying* $x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k$, $x_0 = 0 = x_{N+1}$, $x \not\equiv 0$.

It is not difficult to verify that if (1.1) is the rewritten equation (1.5), that is, $\mathcal{A} = 1$, $\mathcal{B} = 1/r$, $\mathcal{C} = -p$, and $\mathcal{D} = 1 - p/r$, then the objects appearing in the previous proposition reduce to their scalar counterparts mentioned in Proposition 1.1.

We finish this section with a short description of the transformation theory of (1.1). Let $\mathcal{R}_k$ be symplectic matrices and consider the transformation of (1.1)

$$z_k = \mathcal{R}_k \tilde{z}_k. \tag{2.3}$$

This transformation transforms (1.1) into the system

$$\tilde{z}_{k+1} = \tilde{\mathcal{S}}_k \tilde{z}_k, \qquad \tilde{\mathcal{S}}_k = \mathcal{R}_{k+1}^{-1} \mathcal{S}_k \mathcal{R}_k, \tag{2.4}$$

which is again a symplectic system as can be verified by a direct computation. The case when $\mathcal{R}$ is of the form

$$\mathcal{R}_k = \begin{pmatrix} H_k & 0 \\ G_k & H_k^{T-1} \end{pmatrix} \tag{2.5}$$

is of particular importance in oscillation theory of (1.1). In this case, transformation (2.3) preserves the oscillatory nature of transformed systems (see [5]) and if we write

$\tilde{\mathcal{S}} = \begin{pmatrix} \tilde{\mathcal{A}} & \tilde{\mathcal{B}} \\ \tilde{\mathcal{C}} & \tilde{\mathcal{D}} \end{pmatrix}$, then we have

$$\begin{aligned} \tilde{\mathcal{A}}_k &= H_{k+1}^{-1}(\mathcal{A}_k H_k + \mathcal{B}_k G_k), \quad \tilde{\mathcal{B}}_k = H_{k+1}^{-1}\mathcal{B}_k H_k^{T-1}, \\ \tilde{\mathcal{C}}_k &= H_{k+1}^T(\mathcal{C}_k H_k + \mathcal{D}_k G_k) - G_{k+1}^T(\mathcal{A}_k H_k + \mathcal{B}_k G_k), \\ \tilde{\mathcal{D}}_k &= H_{k+1}^T \mathcal{D}_k H_k^{T-1} - G_{k+1}^T \mathcal{B}_k H_k^{T-1}. \end{aligned} \tag{2.6}$$

Consequently, transformation (2.3), with $\mathcal{R}$ of the form (2.5), is a useful tool in the oscillation theory of (1.1); this system is transformed into an "easier" system and the results obtained for this "easier" system are then transformed back to the original system. For some oscillation results obtained in this way, we refer to [6, 9].

3. Oscillation theory of SDSs

In addition to the transformation method mentioned in Section 2, the Roundabout theorem (Proposition 2.1) suggests two other methods of the oscillation theory of these systems. The first one, the so-called *Riccati technique*, consists in the equivalence (i)$\Longleftrightarrow$(iii). The oscillation results for (1.7) with B positive definite, mentioned at the beginning of Section 2, were proved just using this method. However, as we have already mentioned, this method does not extend directly to a Hamiltonian system with B singular or to general SDSs. It is an open problem (which is the subject of the present investigation) how to modify this method in order to be applicable also in the more general situation.

The second principal method of the oscillation theory of (1.1), the so-called *variational principle*, is based on the equivalence of disconjugacy and positivity of quadratic functional (2.2), which is the equivalence (i)$\Leftrightarrow$(iv) in Proposition 2.1. Unlike the Riccati technique, this method extends to general SDSs almost without problems and the illustration of this extension is the main part of this section.

The discrete version of the classical Leighton-Wintner criterion for the Sturm-Liouville differential equation (1.10) states that the Sturm-Liouville difference equation (1.5) with $r_k > 0$ is oscillatory provided

$$\sum^{\infty} \frac{1}{r_k} = \infty, \qquad \sum^{\infty} p_k = \infty. \tag{3.1}$$

In this criterion, equation (1.5) is essentially viewed as a perturbation of the one-term (nonoscillatory) equation

$$\Delta(r_k \Delta x_k) = 0. \tag{3.2}$$

According to the equivalence of oscillation of (1.5) and the existence of a sequence (with zero boundary values) for which the associated quadratic functional (1.12) is nonpositive, for the oscillation of the "perturbed" equation (1.5), the sequence p_k must be, in a certain sense, sufficiently positive. The second condition in (3.1) is just a quantitative characterization of the "sufficient positivity" of p_k.

Now, we show how this criterion extends to (1.1). Let $\tilde{\mathscr{C}}_k$ be a sequence of symmetric nonpositive definite $n \times n$ matrices and consider the system

$$z_{k+1} = (\mathcal{S}_k + \tilde{\mathcal{S}}_k) z_k, \quad \tilde{\mathcal{S}}_k = \begin{pmatrix} 0 & 0 \\ \tilde{\mathscr{C}}_k \mathscr{A}_k & \tilde{\mathscr{C}}_k \mathscr{B}_k \end{pmatrix}, \tag{3.3}$$

as a perturbation of (1.1). The quadratic functional corresponding to (3.3) has the same class of admissible pairs x, u, as the functional corresponding to (1.1), and takes the form

$$\begin{aligned} \tilde{\mathcal{F}}_k(x,u) &= \sum_{k=0}^{N} \Big\{ x_k^T \big(\mathscr{C}_k + \tilde{\mathscr{C}}_k \mathscr{A}_k\big)^T \mathscr{A}_k x_k + 2x_k^T \big(\mathscr{C}_k + \tilde{\mathscr{C}}_k \mathscr{A}_k\big)^T \mathscr{B}_k u_k \\ &\qquad + u_k^T \big(\mathscr{D}_k + \tilde{\mathscr{C}}_k \mathscr{B}_k\big)_k^T \mathscr{B}_k u_k \Big\} \\ &= \sum_{k=0}^{N} \Big\{ x_k^T \mathscr{C}_k^T \mathscr{A}_k x_k + 2x_k^T \mathscr{C}_k^T \mathscr{B}_k u_k + u_k^T \mathscr{D}_k^T \mathscr{B}_k u_k \Big\} + \sum_{k=0}^{N} x_{k+1}^T \tilde{\mathscr{C}}_k x_{k+1}. \end{aligned} \tag{3.4}$$

In our extension of the Leighton-Wintner-type criterion to (1.1), we will need two additional concepts of the oscillation theory of these systems. System (1.1) is said to be *eventually controllable* if there exists $N \in \mathbb{N}$ such that the trivial solution $z = \binom{x}{u} \equiv \binom{0}{0}$ is the only solution for which $x_k = 0$ for $k \geq N$. A conjoined basis $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ is said to be the *recessive solution* of (1.1) at ∞ if there exists $N \in \mathbb{N}$ such that $\tilde{X}_k$ is nonsingular for $k \geq N$ and

$$\lim_{k \to \infty} \left(\sum_{j=N}^{k} \tilde{X}_{j+1}^{-1} \mathscr{B}_j \tilde{X}_j^{T-1} \right)^{-1} = 0. \tag{3.5}$$

Note that the principal solution at ∞ exists and it is unique (up to the right multiplication by a nonsingular constant matrix) whenever (1.1) is nonoscillatory and eventually controllable (see [5]).

Theorem 3.1. *Suppose that (1.1) is nonoscillatory, eventually controllable and let $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ be the principal solution at ∞ of this system. If there exists a vector $v \in \mathbb{R}^n$ such that*

$$\sum^{\infty} v^T \tilde{X}_{k+1}^T \tilde{\mathscr{C}}_k \tilde{X}_{k+1} v = -\infty, \tag{3.6}$$

then (3.3) is oscillatory.

We skip the proof of this statement which is based on a rather complicated construction of an admissible pair x, u for which $\tilde{\mathcal{F}}(x,u) < 0$ (for details, see [4]). We concentrate our attention on showing that the previous theorem is really an extension of the Leighton-Wintner criterion (3.1). Equation (3.2) can be written as the 2×2 symplectic system

$$\begin{pmatrix} x \\ u \end{pmatrix}_{k+1} = \begin{pmatrix} 1 & \dfrac{1}{r_k} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}_k \tag{3.7}$$

which plays the role of system (1.1). The perturbation quantity is $\tilde{\mathscr{C}} = -p$, that is, $\tilde{\mathscr{S}} = \left(\begin{smallmatrix} 0 & 0 \\ -p & -p/r \end{smallmatrix}\right)$, and hence the symplectic system (1.9) (which is the rewritten equation (1.5)) plays the role of (3.3). Now, the first condition in (3.1) is

$$\sum^{\infty} \frac{1}{r_k} = \infty \Longleftrightarrow \lim_{k\to\infty} \left(\sum^{k} \frac{1}{r_j} \right)^{-1} = 0 \tag{3.8}$$

which means that $\tilde{x}_k \equiv 1$ is the principal solution at ∞ of (3.2). Since $\tilde{\mathscr{C}} = -p$, clearly (3.6) with $v = 1$, $n = 1$, and $\tilde{X} = \tilde{x} = 1$ is equivalent to the second condition in (3.1). Hence, the Leighton-Wintner oscillation criterion (3.1) is really a consequence of Theorem 3.1.

4. Hyperbolic Prüfer transformation

The classical Prüfer transformation (established by Prüfer in [15]) is a useful tool in the qualitative theory of the second-order Sturm-Liouville differential equation

$$\left(r(t)x'\right)' + p(t)x = 0, \tag{4.1}$$

where r and p are continuous functions with $r(t) > 0$. By this transformation, any non trivial solution x of (4.1) and its quasiderivative rx' can be expressed in the form

$$x(t) = \rho(t)\sin\varphi(t), \qquad r(t)x'(t) = \rho(t)\cos\varphi(t), \tag{4.2}$$

where ρ and φ satisfy the first-order system

$$\varphi' = p(t)\sin^2\varphi + \frac{1}{r(t)}\cos^2\varphi, \qquad \rho' = \frac{1}{2}\sin 2\varphi(t)\left(\frac{1}{r(t)} - p(t)\right)\rho. \tag{4.3}$$

Since 1926, when the original paper of Prüfer appeared, the Prüfer transformation has been extended in various directions (see [7] and the references given therein). Here, we present another extension: the so-called *hyperbolic discrete Prüfer transformation* which is based on the following idea. If the Sturm-Liouville equation (4.1) possesses a solution x such that $(r(t)x'(t))^2 - x^2(t) > 0$ in some interval $I \subset \mathbb{R}$, then the solution x and its quasiderivative rx' can be expressed via the hyperbolic sine and cosine functions in the form

$$x(t) = \rho(t)\sinh\varphi(t), \quad r(t)x'(t) = \rho(t)\cosh\varphi(t) \tag{4.4}$$

in this interval, where the functions ρ and φ satisfy a first-order system similar to (4.3). The crucial role in our extension of this transformation is played by the so-called *hyperbolic symplectic system*, which is the SDS of the form

$$x_{k+1} = \mathscr{P}_k x_k + \mathscr{Q}_k u_k, \qquad u_{k+1} = \mathscr{Q}_k x_k + \mathscr{P}_k u_k, \tag{4.5}$$

that is, the $n \times n$ matrices $\mathscr{P}$ and $\mathscr{Q}$ satisfy

$$\mathscr{P}^T\mathscr{P} - \mathscr{Q}^T\mathscr{Q} = I, \qquad \mathscr{P}^T\mathscr{Q} - \mathscr{Q}^T\mathscr{P} = 0. \tag{4.6}$$

Note that the terminology *hyperbolic symplectic system* is motivated by the fact that in the scalar case $n = 1$, solutions of (4.5) are, in case $\mathcal{P}_k > 0$, of the form

$$x_k = \sinh\left(\sum^{k-1}\varphi_j\right), \qquad u_k = \cosh\left(\sum^{k-1}\varphi_j\right), \tag{4.7}$$

where φ_k is a sequence given by $\cosh\varphi_k = \mathcal{P}_k$, $\sinh\varphi_k = \mathcal{Q}_k$. For basic properties of solutions of hyperbolic symplectic systems, we refer to [10].

Theorem 4.1. *Suppose that (1.1) possesses a conjoined basis $\binom{X}{U}$ such that $U_k^T U_k - X_k^T X_k$ is positive definite for k in some discrete interval $[m,n]$, $m,n \in \mathbb{N}$. Then there exist nonsingular $n\times n$ matrices H_k and $n\times n$ matrices $\mathcal{P}_k$, $\mathcal{Q}_k$ satisfying (4.6), $k \in [m,n]$, such that*

$$X_k = S_k^T H_k, \qquad U_k = C_k^T H_k, \tag{4.8}$$

where $\binom{S}{C}$ is a conjoined basis of (4.5) satisfying $C^TC - S^TS = I$ (or, equivalently, $CC^T - SS^T = I$, $SC^T = CS^T$). The matrices $\mathcal{P}$ and $\mathcal{Q}$ are given by the formulas

$$\begin{aligned} \mathcal{P}_k &= H_{k+1}^{T-1}\left\{\left(\mathcal{C}_kX_k + \mathcal{D}_kU_k\right)^T U_k - \left(\mathcal{A}_kX_k + \mathcal{B}_kU_k\right)^T X_k\right\}H_k^{-1}, \\ \mathcal{Q}_k &= H_{k+1}^{T-1}\left\{\left(\mathcal{A}_kX_k + \mathcal{B}_kU_k\right)^T U_k - \left(\mathcal{C}_kX_k + \mathcal{D}_kU_k\right)^T X_k\right\}H_k^{-1}. \end{aligned} \tag{4.9}$$

Proof. Let H be any matrix satisfying $H^TH = U^TU - X^TX$, that is, $H = GD$, where D is the (unique) symmetric positive definite matrix satisfying $D^2 = U^TU - X^TX$ and G is any orthogonal matrix. Denote $\mathfrak{X} = (U+X)H^{-1}$, $\tilde{\mathfrak{X}} = (U-X)H^{-1}$. Then the fact that $\binom{X}{U}$ is a conjoined basis implies that

$$\begin{aligned} \mathfrak{X}_{k+1} &= \left(U_{k+1} + X_{k+1}\right)H_{k+1}^{-1} \\ &= \mathfrak{X}_kH_k\left(U_k + X_k\right)^{-1}\left(U_k^T - X_k^T\right)^{-1}\left(U_k^T - X_k^T\right)\left(U_{k+1} + X_{k+1}\right)H_{k+1}^{-1} \\ &= \mathfrak{X}_kH_k^{T-1}\left(U_k^T - X_k^T\right)\left(U_{k+1} + X_{k+1}\right)H_{k+1}^{-1} \\ &= \mathfrak{X}_k\left(\mathcal{P}_k^T + \mathcal{Q}_k^T\right). \end{aligned} \tag{4.10}$$

By the same computation, we get

$$\tilde{\mathfrak{X}}_{k+1} = \tilde{\mathfrak{X}}_k\left(\mathcal{P}_k^T - \mathcal{Q}_k^T\right). \tag{4.11}$$

Set

$$S_k = \frac{1}{2}\left(\mathfrak{X}_k^T - \tilde{\mathfrak{X}}_k^T\right), \qquad C_k = \frac{1}{2}\left(\mathfrak{X}_k^T + \tilde{\mathfrak{X}}_k^T\right). \tag{4.12}$$

Then we have

$$\begin{aligned} S_{k+1} &= \frac{1}{2}\left(\mathfrak{X}_{k+1}^T - \tilde{\mathfrak{X}}_{k+1}^T\right) = \frac{1}{2}\left[\left(\mathcal{P}_k + \mathcal{Q}_k\right)\mathfrak{X}_k^T - \left(\mathcal{P}_k - \mathcal{Q}_k\right)\tilde{\mathfrak{X}}_k^T\right] = \mathcal{P}_kS_k + \mathcal{Q}_kC_k, \\ C_{k+1} &= \frac{1}{2}\left(\mathfrak{X}_{k+1}^T + \tilde{\mathfrak{X}}_{k+1}^T\right) = \frac{1}{2}\left[\left(\mathcal{P}_k + \mathcal{Q}_k\right)\mathfrak{X}_k^T + \left(\mathcal{P}_k - \mathcal{Q}_k\right)\tilde{\mathfrak{X}}_k^T\right] = \mathcal{Q}_kS_k + \mathcal{P}_kC_k. \end{aligned} \tag{4.13}$$

Further,

$$\begin{aligned} C_kC_k^T - S_kS_k^T &= \frac{1}{4}(\mathcal{X}_k^T + \tilde{\mathcal{X}}_k^T)(\mathcal{X}_k + \tilde{\mathcal{X}}_k) - \frac{1}{4}(\mathcal{X}_k^T - \tilde{\mathcal{X}}_k^T)(\mathcal{X}_k - \tilde{\mathcal{X}}_k) \\ &= \frac{1}{2}(\mathcal{X}_k^T\tilde{\mathcal{X}}_k + \tilde{\mathcal{X}}_k^T\mathcal{X}_k) \\ &= \frac{1}{2}H_k^{T-1}\big[(U_k^T - X_k^T)(U_k + X_k) + (U_k^T + X_k^T)(U_k - X_k)\big]H_k^{-1} \\ &= \frac{1}{2}H_k^{T-1}(2U_k^TU_k - 2X_k^TX_k)H_k^{-1} = I, \end{aligned} \tag{4.14}$$

and similarly $C_kS_k^T - S_kC_k^T = 0$. The last two identities imply that the matrix $\left(\begin{smallmatrix} C & S \\ S & C \end{smallmatrix}\right)$ is symplectic. Hence, its transpose has the same property, which means that $\left(\begin{smallmatrix} S \\ C \end{smallmatrix}\right)$ is a conjoined basis and $C^TC - S^TS = I$ holds. Finally, from the hyperbolic system (4.5) and the identities for its solution $\left(\begin{smallmatrix} S \\ C \end{smallmatrix}\right)$, we have

$$\mathcal{P}_k = C_{k+1}C_k^T - S_{k+1}S_k^T, \qquad \mathcal{Q}_k = S_{k+1}C_k^T - C_{k+1}S_k^T, \tag{4.15}$$

and by a direct computation, we get $\mathcal{P}\mathcal{P}^T - \mathcal{Q}\mathcal{Q}^T = I$ and $\mathcal{P}\mathcal{Q}^T = \mathcal{Q}\mathcal{P}^T$, which, by the same argument as above, implies that also $\mathcal{P}^T\mathcal{P} - \mathcal{Q}^T\mathcal{Q} = I$ and $\mathcal{P}^T\mathcal{Q} - \mathcal{Q}^T\mathcal{P} = 0$. This completes the proof. □

Remark 4.2. Hyperbolic Prüfer transformation suggests an open problem in the transformation theory of (1.1), which can be explained as follows. In the hyperbolic Prüfer transformation, a conjoined basis $\left(\begin{smallmatrix} X \\ U \end{smallmatrix}\right)$ is expressed in the form (4.8), where $\left(\begin{smallmatrix} S \\ C \end{smallmatrix}\right)$ is a conjoined basis of the hyperbolic system (4.5). By the classical Prüfer transformation for (1.1) (established in [7]), a conjoined basis of (1.1) is expressed by (4.8), but $\left(\begin{smallmatrix} S \\ C \end{smallmatrix}\right)$ is a conjoined basis of the *trigonometric* SDS

$$S_{k+1} = \mathcal{P}_kS_k + \mathcal{Q}_kC_k, \qquad C_{k+1} = -\mathcal{Q}_kS_k + \mathcal{P}_kC_k \tag{4.16}$$

(similarly, as in the "hyperbolic" case, the terminology *trigonometric system* is justified by the fact that in the scalar case $n = 1$, solutions of (4.16) can be expressed via classical trigonometric sine and cosine functions). Observe that hyperbolic and trigonometric systems are SDSs whose matrices satisfy (in addition to (1.2))

$$\mathcal{S}^T\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\mathcal{S} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{4.17}$$

respectively,

$$\mathcal{S}^T\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\mathcal{S} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{4.18}$$

Now, let $\mathcal{N}$ be any $2n \times 2n$ matrix and denote by $\mathcal{G}_{\mathcal{N}}$ the subgroup of $2n \times 2n$ symplectic matrices satisfying

$$\mathcal{S}^T\mathcal{N}\mathcal{S} = \mathcal{N}. \tag{4.19}$$

The open problem is under what conditions on (1.1) any conjoined basis of this system can be expressed in the form (4.8) where $\binom{S}{C}$ is a conjoined basis of the SDS (1.1) whose matrix $\mathcal{S} \in \mathcal{G}_{\mathcal{N}}$.

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INTEGRAL REPRESENTATION OF THE SOLUTIONS TO HEUN'S BICONFLUENT EQUATION

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First, we trace the genesis of the canonical form of Heun's biconfluent equation. Second, we present a method which allows us to find an integral expression as a solution to our equation, and finally, using the properties of Meijer G-functions, we give an integral representation of a fundamental system of solutions to the biconfluent equation.

1. Preliminaries

Heun's differential equation and its confluent forms are used to build up new classes of solvable potentials. The Schrödinger equation formed with those potentials can be reduced to Heun's biconfluent differential equation. We list some examples:

(i) radial Schrödinger equation for the harmonic oscillator [15];
(ii) radial Schrödinger equation for the doubly anharmonic oscillator [4, 5, 10];
(iii) radial Schrödinger equation of a three-dimensional anharmonic oscillator [7, 8, 10];
(iv) radial Schrödinger equation of a class of confinement potentials [10, 16].

For other kinds of potentials, see [11, 12].

Recently, a very interesting and valuable monography was dedicated to Heun's equations [17]. Arscott [1] conjectures that solutions of Heun's equations are not expressible in terms of definite or contour integrals involving simpler functions. One should mention the work of Sleeman who gave a solution in the form of factorial series, which leads to Barnes-type contour integrals [18]. In the sequel, we will see that it is possible to give integral representations in terms of Mellin's kernel of solutions to biconfluent Heun's equation.

We start with the canonical form of a second-order differential equation with p ($p \geq 2$) elementary singular points ($p-1$ finite singularities and the ∞):

$$y'' + \sum_{r=1}^{p-1} \frac{2/p}{x-a_r} y' + \frac{\sum_{k=1}^{p-3} A_k x^{p-3-k}}{\prod_{r=1}^{p-1}(x-a_r)} y = 0. \tag{1.1}$$

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Many differential equations which occur in a large variety of problems arising from pure or applied mathematics or mathematical physics, often after appropriate algebraic or transcendental changes of variable, can be derived by the confluences of the singularities from (1.1). The classes of these equations are characterized by the Klein-Bőcher-Ince symbol $(\ell, q, r_1, r_2, \ldots, r_s)$ with

$$p = \ell + 2q + \sum_{k=1}^{s} (k+2) r_k, \tag{1.2}$$

where ℓ is the number of elementary singular points, q is the number of nonelementary regular singular points, and r_k is the number of irregular singular points of kind k. For the terminology, see [9].

If we set $p = 8$ by means of confluence process and after parametric reduction, we mention hereby some remarkable equations.

(1) Heun's equation $(0,4,0)$. The confluence of $a_7, a_6 \to 0$, $a_5, a_4 \to a$, $a_3, a_2 \to 1$, and $a_1 \to \infty$ leads to

$$y''(x) + \left(\frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-a}\right) y'(x) + \frac{\delta\eta(x-\lambda)}{x(x-1)(x-a)} y(x) = 0, \tag{1.3}$$

where α, β, γ, δ, λ, and a are six independent parameters and $\eta = \alpha + \beta + \gamma - \delta - 1$.

(2) Confluent Heun's equation $(0,2,1_2)$. The confluence of $a_7, a_6, a_5 \to \infty$, $a_4, a_3 \to 1$, and $a_2, a_1 \to 0$ leads to

$$\begin{aligned} &y''(x) + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right) y'(x) \\ &\quad + \frac{[\delta + (1/2)(\alpha+\gamma+2)]x + \eta + \beta/2 + (1/2)(\gamma-\alpha)(\beta+1)}{x(x-1)} y(x) = 0, \end{aligned} \tag{1.4}$$

with five independent parameters: α, β, γ, δ, and η.

(3) Biconfluent of Heun's equation $(0,1,1_4)$. The confluence of $a_7, a_6, a_5, a_4, a_3 \to \infty$ and $a_2, a_1 \to 0$ leads to

$$\begin{aligned} &xy''(x) + (1 + \alpha - \beta x - 2x^2) y'(x) \\ &\quad + \left[(\gamma - \alpha - 2)x - \frac{1}{2}(\delta + \beta + \alpha\beta)\right] y(x) = 0, \end{aligned} \tag{1.5}$$

with four independent parameters: α, β, γ, and δ.

(4) Double confluent of Heun's equation $(0,0,2_2)$. The confluence of $a_7, a_6, a_5 \to \infty$ and $a_4, a_3, a_2, a_1 \to 0$ leads to

$$\begin{aligned} &x^2 y''(x) + \left[1 + \alpha\left(x + \frac{1}{x}\right)\right] x y'(x) \\ &\quad + \left[\left(\beta - \frac{\alpha}{2}\right)\frac{1}{x} + \frac{\delta\alpha^2}{2} - \frac{1}{4} + \left(\gamma + \frac{\alpha}{2}\right)x\right] y(x) = 0, \end{aligned} \tag{1.6}$$

with four independent parameters: α, β, γ, and δ.

(5) Triconfluent of Heun's equation $(0,0,1_6)$. The confluence of $a_7, a_6, a_5, a_4, a_3, a_2, a_1 \to \infty$ leads to

$$y''(x) + (\gamma + 3x^2)y'(x) + [\alpha + (\beta - 3)x]y(x) = 0, \tag{1.7}$$

with three independent parameters: α, β, and γ.

2. The statement of the problem

Let $\mathcal{L}_x$ be a three-term differential operator [2, 3, 9]

$$\mathcal{L}_x = \frac{P_1(\theta)}{x} + P_0(\theta) + xR_1(\theta), \tag{2.1}$$

where $P_1(\theta)$, $P_0(\theta)$, and $R_1(\theta)$ are polynomials and $\theta = x(d/dx)$.

We are looking for a solution to

$$\mathcal{L}_x[y] = 0 \tag{2.2}$$

as

$$y(x) = \int_C K(x,t)Z(t)dt, \tag{2.3}$$

where

$$K(x,t) = \frac{K_1(xt)}{x} + K_0(xt) + xL_1(xt). \tag{2.4}$$

The path of integration C and the function $Z(t)$ will be defined in the sequel. We respectively introduce an auxiliary kernel and a companion differential operator:

$$\begin{aligned} \tilde{K}(x,t) &= t\tilde{K}_1(xt) + \tilde{K}_0(xt) + \frac{\tilde{L}_1(xt)}{t}, \\ \mathcal{M}_t &= t\tilde{P}_1(\theta) + \tilde{P}_0(\theta) + \frac{\tilde{R}_1(\theta)}{t}. \end{aligned} \tag{2.5}$$

In this last equation θ symbolizes the operator $t(d/dt)$. We have the following assumption:

$$\mathcal{L}_x[K(x,t)] = \mathcal{M}_t[\tilde{K}(x,t)]. \tag{2.6}$$

We denote by $\bar{\mathcal{M}}_t$ and $A(\tilde{K},Z)$, respectively, the formal adjoint of $\mathcal{M}_t$ and the concomitant (a bilinear functional of $\tilde{K}$, Z and their derivatives).

If

$$\bar{\mathcal{M}}_t[Z(t)] = 0, \tag{2.7}$$

and $A(\tilde{K},Z)]_C = 0$, then (2.3) is a solution to (2.2).

Setting $\zeta = xt$, (2.6) may be translated into the following system:

$$\begin{aligned} P_1(\theta-1)[K_1(\zeta)] &= \zeta^2\tilde{P}_1(\theta+1)[\tilde{K}_1(\zeta)],\\ P_1(\theta)[K_0(\zeta)]+P_0(\theta-1)[K_1(\zeta)] &= \zeta\{\tilde{P}_1(\theta)[\tilde{K}_0(\zeta)]+\tilde{P}_0(\theta+1)[\tilde{K}_1(\zeta)]\},\\ P_1(\theta+1)[L_1(\zeta)]+P_0(\theta)[K_0(\zeta)]&+R_1(\theta-1)[K_1(\zeta)]\\ &= \tilde{P}_1(\theta-1)[\tilde{L}_1(\zeta)]+\tilde{P}_0(\theta)[\tilde{K}_0(\zeta)]+\tilde{R}_1(\theta+1)[\tilde{K}_1(\zeta)],\\ \zeta\{P_0(\theta+1)[L_1(\zeta)]+R_1(\theta)[K_0(\zeta)]\} &= \tilde{P}_0(\theta-1)[\tilde{L}_1(\zeta)]+\tilde{R}_1(\theta)[\tilde{K}_0(\zeta)],\\ \zeta^2R_1(\theta+1)[L_1(\zeta)] &= \tilde{R}_1(\theta-1)[\tilde{L}_1(\zeta)]. \end{aligned} \tag{2.8}$$

According to the study in [2, 3], the previous system may be reduced to

$$P_1(\theta)[K_0(\zeta)] = \zeta\tilde{P}_0(\theta+1)[\tilde{K}_1(\zeta)], \tag{2.9}$$

$$P_0(\theta)[K_0(\zeta)] = \tilde{P}_0(\theta)[\tilde{K}_0(\zeta)], \tag{2.10}$$

$$\zeta R_1(\theta)[K_0(\zeta)] = \tilde{P}_1(\theta-1)[\tilde{L}_1(\zeta)]. \tag{2.11}$$

In the last system, we have three equations for four unknowns. To solve this system, we have to choose two basic equations and an interdependency relation between the components of the auxiliary kernel. Our choice will be guided by the kind of solution we are looking for.

3. Heun's biconfluent equation

The canonical form of an equation of class $(0,1,1_4)$ reads (see [6, 14])

$$xy''(x)+(1+\alpha-\beta x-2x^2)y'(x)+\left\{(\gamma-\alpha-2)x-\frac{1}{2}[\delta+(1+\alpha)\beta]\right\}y(x)=0. \tag{3.1}$$

Using the operator $\theta = x(d/dx)$, we get that

$$\left\{\frac{1}{x}\theta(\theta+\alpha)-\left(\beta\theta+-\frac{1}{2}[\delta+(1+\alpha)\beta]\right)+x(\gamma-\alpha-2-2\theta)\right\}[y]=0. \tag{3.2}$$

We set

$$P_1(\theta) = \theta(\theta+\alpha), \tag{3.3}$$

$$P_0(\theta) = -\beta(\theta+a), \tag{3.4}$$

where $a = (1/2)(\delta/\beta+\alpha+1)$, with $\beta \neq 0$, and

$$R_1(\theta) = -2(\theta+b), \tag{3.5}$$

where $b = (\alpha-\gamma+2)/2$.

According to the scheme described in Section 2, the companion operator reads

$$\mathcal{M}_t = \tilde{P}_0(\theta) = \theta + 1 + d, \tag{3.6}$$

where $d \in \mathbb{C}$.

To solve the system defined by (2.9), (2.10), and (2.11), we will take the first two equations as basic equations; the interdependency relation is

$$\tilde{K}_1 = \lambda \tilde{K}_0, \tag{3.7}$$

with $\lambda \in \mathbb{C}^*$. By elimination, we obtain

$$\{\tilde{P}_0(\theta - 1)P_1(\theta) - \lambda\zeta P_0(\theta + 1)\tilde{P}_0(\theta)\}[K_0] = 0. \tag{3.8}$$

Taking into account (3.3), (3.4), and (3.5), we have

$$\{\theta(\theta + \alpha)(\theta + d) + \beta\lambda\zeta(\theta + 2 + d)(\theta + a)\}[K_0] = 0. \tag{3.9}$$

If we take

$$\lambda = -\frac{1}{\beta}, \tag{3.10}$$

then K_0 satisfies

$$\{\theta(\theta + \alpha)(\theta + d) - \zeta(\theta + 2 + d)(\theta + a)\}[K_0] = 0, \tag{3.11}$$

which is nothing but a generalized hypergeometric differential equation whose solutions may be expressed as

$$K_0(\zeta) = {}_2F_2\left(\left.\begin{matrix} 2 + d, a \\ 1 + d, 1 + \alpha \end{matrix}\right| \zeta\right), \tag{3.12}$$

$$K_0(\zeta) = \zeta^{-\alpha}{}_2F_2\left(\left.\begin{matrix} 2 - \alpha + d, a - \alpha \\ 1 - \alpha + d, 1 - \alpha \end{matrix}\right| \zeta\right), \tag{3.13}$$

$$K_0(\zeta) = \zeta^{-d}{}_2F_2\left(\left.\begin{matrix} 2, a - d \\ 1 - d, 1 + \alpha - d \end{matrix}\right| \zeta\right). \tag{3.14}$$

3.1. First integral representation. In this subsection, we will use the kernel given by (3.12). First, we will compute the components of the auxiliary kernel. From (2.10), we have

$$\tilde{K}_0(\zeta) = [\tilde{P}_0(\theta)]^{-1}[P_0(\theta)[K_0(\zeta)]]. \tag{3.15}$$

If we take into account (3.4), (3.6), and (3.12), then (3.15) becomes

$$\tilde{K}_0(\zeta) = -\beta(\theta + 1 + d)^{-1}[(\theta + a)[{}_2F_2\left(\left.\begin{matrix} 2 + d, a \\ 1 + d, 1 + \alpha \end{matrix}\right| \zeta\right). \tag{3.16}$$

Using the properties of the operator θ, we get that

$$\tilde{K}_0(\zeta) = -\frac{a\beta\zeta}{1+d}\left\{\frac{1}{1+\alpha}\,{}_1F_1\left(\left.\begin{matrix}1+a\\2+\alpha\end{matrix}\right|\zeta\right) + \zeta^{-1}\,{}_1F_1\left(\left.\begin{matrix}a\\1+\alpha\end{matrix}\right|\zeta\right)\right\}, \tag{3.17}$$

with $\Re(1+d) > 0$.

According to (3.5), (3.6), and (3.12), $\tilde{L}_1$ satisfies the following differential equation:

$$\tilde{L}_1(\zeta) = -2\zeta(\theta+1+d)^{-1}\left[(\theta+b)\left[{}_2F_2\left(\left.\begin{matrix}2+d,a\\1+d,1+\alpha\end{matrix}\right|\zeta\right)\right]\right]. \tag{3.18}$$

The solution to the previous equation is

$$\tilde{L}_1(\zeta) = -\frac{2\zeta^2}{1+d}\left\{\frac{a}{1+\alpha}\,{}_1F_1\left(\left.\begin{matrix}1+a\\1+\alpha\end{matrix}\right|\zeta\right) + b\zeta^{-1}\,{}_1F_1\left(\left.\begin{matrix}a\\1+\alpha\end{matrix}\right|\zeta\right)\right\}, \tag{3.19}$$

with $\Re(1+d) > 0$.

Thus, the auxiliary kernel reads

$$\begin{aligned}\tilde{K}(x,t) = \frac{1}{1+d}\Bigg\{&\frac{axt^2}{1+\alpha}\left[1-\frac{2x+\beta}{t}\right]{}_1F_1\left(\left.\begin{matrix}1+a\\2+\alpha\end{matrix}\right|xt\right)\\&+t\left[a-\frac{2bx+a\beta}{t}\right]{}_1F_1\left(\left.\begin{matrix}a\\1+\alpha\end{matrix}\right|xt\right)\Bigg\},\end{aligned} \tag{3.20}$$

with $\Re(1+d) > 0$.

Using (3.6), the solution to (2.7) takes the form

$$Z(t) = t^d. \tag{3.21}$$

The concomitant associated with (2.6) is given by

$$A(\tilde{K},Z) = tZ(t)\tilde{K}(x,t). \tag{3.22}$$

The conjunction of (3.20), (3.21), and (3.22) leads to

$$\begin{aligned}A(\tilde{K},Z) = \frac{1}{1+d}\Bigg\{&\frac{axt^{3+d}}{1+\alpha}\left[1-\frac{2x+\beta}{t}\right]{}_1F_1\left(\left.\begin{matrix}1+a\\2+\alpha\end{matrix}\right|xt\right)\\&+t^{2+d}\left[a-\frac{2bx+a\beta}{t}\right]{}_1F_1\left(\left.\begin{matrix}a\\1+\alpha\end{matrix}\right|xt\right)\Bigg\},\end{aligned} \tag{3.23}$$

with $\Re(1+d) > 0$.

Now it is time to seek for a path of integration along which the concomitant will vanish. With this end in view, we need asymptotic expansion of generalized hypergeometric function which is obtainable via G-functions.

Proposition 3.1 [13]. *Recall that*

$$ {}_1F_1\left(\begin{matrix}A\\B\end{matrix}\,\middle|\,u\right)=\frac{\Gamma(B)}{\Gamma(A)}G^{1,1}_{1,2}\left(-u\,\middle|\,\begin{matrix}1-A\\0,1-B\end{matrix}\right), \tag{3.24}$$

where $G^{1,1}_{1,2}$ is a Meijer G-function.

If $a_1\notin\mathbb{N}$, $a_1\in\mathbb{C}$, and $\delta_1-\pi/2\le\operatorname{Arg}(u)\le\pi/2+\delta_2$, $\delta_1,\delta_2>0$, then

$$G^{1,1}_{1,2}\left(-u\,\middle|\,\begin{matrix}a_1\\b_1,b_2\end{matrix}\right)=u^{a_1-1}\left\{\sum_{k=0}^{N-1}M_ku^{-k}+O(u^{-N})\right\},\quad |u|\longrightarrow\infty, \tag{3.25}$$

with

$$\begin{gathered}M_k=\frac{\Gamma(1+b_1-a_1)}{\Gamma(a_1-b_2)}\frac{(1+b_1-a_1,k)(1+b_2-a_1,k)}{k!},\\ (\gamma,k)=\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}.\end{gathered} \tag{3.26}$$

Corollary 3.2. *If $A\notin\{1-n,\ n\in\mathbb{N}\}$, and $\delta_1-\pi/2\le\operatorname{Arg}(u)\le\pi/2+\delta_2$, $\delta_1,\delta_2>0$, then*

$$ {}_1F_1\left(\begin{matrix}A\\B\end{matrix}\,\middle|\,u\right)=u^{-A}\left\{\sum_{k=0}^{N-1}\tilde{M}_ku^{-k}+O(u^{-N})\right\},\quad |u|\longrightarrow\infty, \tag{3.27}$$

where $\tilde{M}_k=(\Gamma(B)/\Gamma(A))M_k$.

Finally, we have

$$\begin{aligned}A(\tilde{K},Z)=\frac{t^{2-a+d}}{1+d}\Bigg\{&ax^{-a}\left(1-\frac{2x+\beta}{t}\right)\left[\sum_{k=0}^{N-1}\tilde{M}_{1,k}(xt)^{-k}+O((xt)^{-N})\right]\\ &+x^{-a}\left(a-\frac{2bx+a\beta}{t}\right)\left[\sum_{k=0}^{N-1}\tilde{M}_{2,k}(xt)^{-k}+O((xt)^{-N})\right]\Bigg\},\end{aligned} \tag{3.28}$$

with

(i)

$$\begin{aligned}\tilde{M}_{1,k}&=\frac{\Gamma(2+\alpha)}{\Gamma(1+\alpha-a)}\frac{(1+a,k)(a-\alpha,k)}{k!},\\ \tilde{M}_{2,k}&=\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-a)}\frac{(a,k)(a-\alpha,k)}{k!},\end{aligned} \tag{3.29}$$

(ii) $a\notin\{1-n,\ n\in\mathbb{N}\}$ and $\Re(1+d)>0$,

(iii) $\delta_1+\pi/2\le\operatorname{Arg}(xt)\le3\pi/2+\delta_2$, $\delta_1,\delta_2>0$.

In accordance with what we have already seen, we have the following theorem.

Theorem 3.3. *If* $a \notin \{1-n,\ n\in\mathbb{N}\}$, $\Re(a-\alpha) > \Re(1+d) > 0$, $\delta_1 + \pi/2 \le \operatorname{Arg}(xt) \le 3\pi/2 + \delta_2$, $\delta_1, \delta_2 > 0$, *and* C *is the path of integration running from* ∞ *along the direction* $\operatorname{Arg}(t)$, *surrounding the origin and going back to* ∞ *following the same direction, then*

$$y(x) = \int_C {}_2F_2\left(\begin{matrix}2+d, a\\ 1+d, 1+\alpha\end{matrix}\middle| xt\right) t^d\,dt \tag{3.30}$$

is a solution of Heun's biconfluent equation.

3.2. The second integral representation. Now we will work with the kernel given by (3.13), *that is,*

$$K_0(\zeta) = \zeta^{-\alpha}{}_2F_2\left(\begin{matrix}2-\alpha+d, a-\alpha\\ 1-\alpha+d, 1-\alpha\end{matrix}\middle| \zeta\right). \tag{3.31}$$

Using the same technique as above, we get that

$$\begin{aligned}\tilde{K}(x,t) = \frac{1}{1-\alpha+d}\bigg\{ &\frac{(a-\alpha)x^{1-\alpha}t^{2-\alpha}}{1-\alpha}\left(\frac{1-2x}{t} - \frac{1}{\beta}\right){}_1F_1\left(\begin{matrix}1+a-\alpha\\ 2-\alpha\end{matrix}\middle| xt\right)\\ &+ x^{-\alpha}t^{1-\alpha}\left(\frac{a-\alpha-2(b-\alpha)x}{t} - \frac{a-\alpha}{\beta}\right){}_1F_1\left(\begin{matrix}a-\alpha\\ 1-\alpha\end{matrix}\middle| xt\right)\bigg\},\end{aligned} \tag{3.32}$$

with $\Re(a-\alpha+d) > 0$, and the concomitant takes the following expression:

$$\begin{aligned}A(\tilde{K},Z) = \frac{x^{1-\alpha}t^{3-\alpha+d}}{1-\alpha+d}\bigg\{ &\frac{a-\alpha}{1-\alpha}\left(\frac{1-2x}{t} - \frac{1}{\beta}\right){}_1F_1\left(\begin{matrix}1+a-\alpha\\ 2-\alpha\end{matrix}\middle| xt\right)\\ &+ \frac{1}{xt}\left(\frac{a-\alpha-2x(b-\alpha)}{t} - \frac{a-\alpha}{\beta}\right){}_1F_1\left(\begin{matrix}a-\alpha\\ 1-\alpha\end{matrix}\middle| xt\right)\bigg\}.\end{aligned} \tag{3.33}$$

Using the machinery of G-functions, we have the following proposition.

Proposition 3.4. *If* $a-\alpha \notin \{1-n,\ n\in\mathbb{N}\}$, $\Re(\alpha) < \Re(1+d) < \Re(a-1)$, $\delta_1 + \pi/2 \le \operatorname{Arg}(xt) \le 3\pi/2 + \delta_2$, $\delta_1, \delta_2 > 0$, *and* C *is the path of integration running from* ∞ *along the direction* $\operatorname{Arg}(t)$, *surrounding the origin and going back to* ∞ *following the same direction, then*

$$A(\tilde{K},Z)\rfloor_C = 0. \tag{3.34}$$

Under the hypothesis of the previous proposition, we get the following theorem.

Theorem 3.5. *Provided that the hypotheses of Proposition 3.4 are satisfied, then*

$$y(x) = \int_C (xt)^{-\alpha}{}_2F_2\left(\begin{matrix}2-\alpha+d, a-\alpha\\ 1-\alpha+d, 1-\alpha\end{matrix}\middle| xt\right) t^d\,dt \tag{3.35}$$

is a solution of Heun's biconfluent equation.

The conjunction of Theorems 3.3 and 3.5 gives a fundamental system of solutions to Heun's biconfluent equation.

Remarks. A similar study shows that

(i) the kernel $K_0(\zeta) = \zeta^{-d}{}_2F_2\left(\begin{smallmatrix} 2,a-d \\ 1-d,1+\alpha-d \end{smallmatrix} \middle| \zeta\right)$ does not lead to a solution;

(ii) the interdependency relations

$$\tilde{K}_1 = \gamma\tilde{L}_1, \qquad \tilde{K}_0 = \mu\tilde{L}_1 \tag{3.36}$$

do not allow to produce a solution to the biconfluent equation.

4. Case $\beta = 0$

The Heun's biconfluent equation reads

$$xy''(x) + (1+\alpha-2x^2)y'(x) + \left\{(\gamma-\alpha-2)x - \frac{\delta}{2}\right\}y(x) = 0 \tag{4.1}$$

or

$$\left\{\frac{\theta_x(\theta_x+\alpha)}{x} - \frac{\delta}{2} - 2x\left(\theta_x + \frac{2+\alpha-\gamma}{2}\right)\right\}[y(x)] = 0. \tag{4.2}$$

This situation gives rise to two subcases

4.1. Case $\delta = 0$. Heun's biconfluent equation becomes a simple hypergeometric equation, that is,

$$xy''(x) + (1+\alpha-2x^2)y'(x) + (\gamma-\alpha-2)xy(x) = 0, \tag{4.3}$$

which has

$$\begin{aligned} y_1(x) &= {}_1F_1\left(\begin{matrix} \dfrac{2+\alpha-\gamma}{4} \\ 1+\dfrac{\alpha}{2} \end{matrix} \middle| x^2\right), \\ y_2(x) &= x^{-\alpha}{}_1F_1\left(\begin{matrix} \dfrac{2-\alpha-\gamma}{4} \\ 1-\dfrac{\alpha}{2} \end{matrix} \middle| x^2\right) \end{aligned} \tag{4.4}$$

as a fundamental system of solutions that admits an integral representation of Mellin's type (see [2]).

4.2. Case $\delta \neq 0$. In this case, (3.8) becomes

$$\left\{\theta_\zeta(\theta_\zeta+\alpha)(\theta_\zeta+d) + \frac{\lambda\delta}{2}\zeta(\theta_\zeta+2+d)\right\}[K_0(\zeta)] = 0. \tag{4.5}$$

Choosing $\lambda = -2/\delta$, the previous equation takes the following form:

$$\{\theta_\zeta(\theta_\zeta+\alpha)(\theta_\zeta+d) - \zeta(\theta_\zeta+2+d)\}[K_0(\zeta)] = 0. \tag{4.6}$$

Hence

$$K_0(\zeta) = {}_1F_2\left(\begin{matrix}2+d\\1+d,1+\alpha\end{matrix}\middle|\zeta\right), \tag{4.7}$$

$$K_0(\zeta) = \zeta^{-\alpha}{}_1F_2\left(\begin{matrix}2-\alpha+d\\1-\alpha+d,1-\alpha\end{matrix}\middle|\zeta\right), \tag{4.8}$$

$$K_0(\zeta) = \zeta^{-d}{}_1F_2\left(\begin{matrix}2\\1-d,1+\alpha-d\end{matrix}\middle|\zeta\right). \tag{4.9}$$

Proceeding as in Section 3, we get the following proposition.

Proposition 4.1. *The pairs of auxiliary kernel and the concomitant associated with (2.6) are given by*

$$\begin{aligned}K(x,t) &= \frac{t}{1+d}\left\{\left[\frac{2-\delta}{2}-(2+\alpha-\gamma)\frac{x}{t}\right]{}_0F_1\left(\begin{matrix}-\\1+\alpha\end{matrix}\middle|xt\right)\right.\\&\qquad\left.-\frac{2x}{1+\alpha}{}_0F_1\left(\begin{matrix}-\\2+\alpha\end{matrix}\middle|xt\right)\right\},\\A(\tilde{K},Z) &= \frac{t^{2+d}}{1+d}\left\{\left[\frac{2-\delta}{2}-(2+\alpha-\gamma)\frac{x}{t}\right]{}_0F_1\left(\begin{matrix}-\\1+\alpha\end{matrix}\middle|xt\right)\right.\\&\qquad\left.-\frac{2x}{1+\alpha}{}_0F_1\left(\begin{matrix}-\\2+\alpha\end{matrix}\middle|xt\right)\right\},\end{aligned} \tag{4.10}$$

$$\begin{aligned}K(x,t) &= \frac{x^{-\alpha}t^{1-\alpha}}{1-\alpha+d}\left\{\left[1-\frac{\delta+2x(2-\alpha-\gamma)}{t}\right]{}_0F_1\left(\begin{matrix}-\\1-\alpha\end{matrix}\middle|xt\right)\right.\\&\qquad\left.-\frac{2x^2}{1-\alpha}{}_0F_1\left(\begin{matrix}-\\2-\alpha\end{matrix}\middle|xt\right)\right\},\\A(\tilde{K},Z) &= \frac{x^{-\alpha}t^{2+d-\alpha}}{1+d-\alpha}\left\{\left[1-\frac{\delta+2x(2-\alpha-\gamma)}{t}\right]{}_0F_1\left(\begin{matrix}-\\1-\alpha\end{matrix}\middle|xt\right)\right.\\&\qquad\left.-\frac{2x^2}{1-\alpha}{}_0F_1\left(\begin{matrix}-\\2-\alpha\end{matrix}\middle|xt\right)\right\}.\end{aligned} \tag{4.11}$$

Remark 4.2. The kernel given by (4.9) does not lead to an integral representation of a solution.

Theorem 4.3. *If* $0 < \Re(\alpha) < \Re 1 + d < (1/4)\Re(2\alpha-3)$, $\mu \le \operatorname{Arg}(xt) \le 2\pi-\mu$, $\mu > 0$, *and* C *is the path of integration running from* ∞ *along the direction* $\operatorname{Arg}(t)$, *surrounding the* *origin and going back to ∞ following the same direction, then*

$$\begin{aligned} y_1(x) &= \int_C {}_1F_2\left(\begin{matrix} 2+d \\ 1+d, 1+\alpha \end{matrix}\middle| xt\right) t^d\,dt, \\ y_1(x) &= \int_C (xt)^{-\alpha}\, {}_1F_2\left(\begin{matrix} 2-\alpha+d \\ 1-\alpha+d, 1-\alpha \end{matrix}\middle| xt\right) t^d\,dt, \end{aligned} \tag{4.12}$$

is a fundamental system of solutions to the Heun's biconfluent equation (4.1).

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ON THE EXTERIOR MAGNETIC FIELD AND SILENT SOURCES IN MAGNETOENCEPHALOGRAPHY

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Two main results are included in this paper. The first one deals with the leading asymptotic term of the magnetic field outside any conductive medium. In accord with physical reality, it is proved mathematically that the leading approximation is a quadrupole term which means that the conductive brain tissue weakens the intensity of the magnetic field outside the head. The second one concerns the orientation of the silent sources when the geometry of the brain model is not a sphere but an ellipsoid which provides the best possible mathematical approximation of the human brain. It is shown that what characterizes a dipole source as "silent" is not the collinearity of the dipole moment with its position vector, but the fact that the dipole moment lives in the Gaussian image space at the point where the position vector meets the surface of the ellipsoid. The appropriate representation for the spheroidal case is also included.

1. The magnetic field

The mathematical theory of magnetoencephalography (MEG) is governed by the equations of quasistatic theory of electromagnetism [11, 14, 15, 19, 20]. If we denote by V^- the region occupied by the conductive brain tissue, with conductivity $\sigma > 0$ and magnetic permeability $\mu_0 > 0$, then, as Geselowitz has shown [3, 9, 10], the magnetic field in the exterior of V^- region, V^+, due to the internal electric dipole current

$$\mathbf{J}^p(\mathbf{r}) = \mathbf{Q}\delta(\mathbf{r} - \mathbf{r}_0), \quad \mathbf{r}_0 \in V^-, \tag{1.1}$$

assumes the representation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} - \frac{\mu_0\sigma}{4\pi}\int_{\partial V^-} u^-(\mathbf{r}')\hat{\mathbf{n}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}'), \tag{1.2}$$

where $\mathbf{r} \in V^+$ and $\mathbf{Q}$ stands for the electric dipole moment.

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The scalar field u^- in the integrand of (1.2), over the boundary ∂V^- of V^-, describes the interior electric potential and solves the interior Neumann problem

$$\sigma \Delta u^-(\mathbf{r}) = \nabla \cdot \mathbf{J}^p(\mathbf{r}), \quad \mathbf{r} \in V^-, \tag{1.3a}$$

$$\frac{\partial}{\partial n} u^-(\mathbf{r}) = 0, \quad \mathbf{r} \in \partial V^-, \tag{1.3b}$$

where $\mathbf{J}^p$ is given by (1.1) and the boundary ∂V^- is assumed to be smooth.

Note that the solution of the boundary value problem (1.3) is unique up to an additive constant. Hence, the general solution of (1.3) has the form

$$u_c^-(\mathbf{r}) = c + u^-(\mathbf{r}), \quad \mathbf{r} \in V^-, \tag{1.4}$$

where u^- satisfies (1.3).

What we are going to show in the sequel is that, no matter what the shape of the smooth bounded boundary ∂V^- is, the leading term of the multipole expansion of (1.2) is not a dipole but a quadrupole term. Observe that an expansion of the source term, in (1.2) in terms of inverse powers of r, offers the leading dipole term

$$\frac{\mu_0}{4\pi} \mathbf{Q} \times \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} = \frac{\mu_0}{4\pi} \frac{\mathbf{Q} \times \hat{\mathbf{r}}}{r^2} + O\left(\frac{1}{r^3}\right), \quad r \longrightarrow \infty, \tag{1.5}$$

where $\mathbf{r} = r\hat{\mathbf{r}}$.

Similarly, the surface integral in (1.2) provides the expansion

$$\begin{aligned} &-\frac{\mu_0 \sigma}{4\pi} \int_{\partial V^-} u^-(\mathbf{r}')\hat{\mathbf{n}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}') \\ &\quad = -\frac{\mu_0 \sigma}{4\pi} \int_{\partial V^-} u^-(\mathbf{r}')\hat{\mathbf{n}}' ds(\mathbf{r}') \times \frac{\hat{\mathbf{r}}}{r^2} + O\left(\frac{1}{r^3}\right), \quad r \longrightarrow \infty. \end{aligned} \tag{1.6}$$

We will show that

$$\mathbf{Q} = \sigma \int_{\partial V^-} u^-(\mathbf{r})\hat{\mathbf{n}}\, ds(\mathbf{r}). \tag{1.7}$$

To this end we consider the Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{V^-} \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} dv(\mathbf{r}'), \quad \mathbf{r} \in V^+, \tag{1.8}$$

where the total current $\mathbf{J}$ is written as

$$\mathbf{J}(\mathbf{r}') = \mathbf{J}^p(\mathbf{r}') + \sigma \mathbf{E}^-(\mathbf{r}') = \mathbf{Q}\delta(\mathbf{r}' - \mathbf{r}_0) - \sigma \nabla_{\mathbf{r}'} u^-(\mathbf{r}') \tag{1.9}$$

and

$$\mathbf{E}^- = -\nabla u^- \tag{1.10}$$

is the interior electric field. The quasistatic form of the Ampere-Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{1.11}$$

implies that the total current is a solenoidal field, that is,

$$\nabla \cdot \mathbf{J} = 0. \tag{1.12}$$

Then condition (1.12) is used to prove the dyadic identity

$$\nabla \cdot (\mathbf{J} \otimes \mathbf{r}) = (\nabla \cdot \mathbf{J})\mathbf{r} + \mathbf{J} \cdot \nabla \otimes \mathbf{r} = \mathbf{J}, \tag{1.13}$$

in view of which

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int_{V^-} \mathbf{J}(\mathbf{r}') \times \left(-\nabla_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) dv(\mathbf{r}') \\
&= \frac{\mu_0}{4\pi} \nabla_{\mathbf{r}} \times \int_{V^-} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv(\mathbf{r}') \\
&= \frac{\mu_0}{4\pi} \nabla_{\mathbf{r}} \times \left[\frac{1}{r} \int_{V^-} \mathbf{J}(\mathbf{r}') dv(\mathbf{r}') + O\left(\frac{1}{r^2}\right) \right] \\
&= \frac{\mu_0}{4\pi} \nabla_{\mathbf{r}} \times \left[\frac{1}{r} \int_{V^-} \nabla_{\mathbf{r}'} \cdot (\mathbf{J}(\mathbf{r}') \otimes \mathbf{r}') dv(\mathbf{r}') + O\left(\frac{1}{r^2}\right) \right] \\
&= \frac{\mu_0}{4\pi} \nabla_{\mathbf{r}} \times \left[\frac{1}{r} \int_{\partial V^-} \hat{\mathbf{n}}' \cdot \mathbf{J}(\mathbf{r}') \otimes \mathbf{r}' ds(\mathbf{r}') + O\left(\frac{1}{r^2}\right) \right] \\
&= -\frac{\mu_0}{4\pi} \frac{\hat{\mathbf{r}}}{r^2} \times \int_{\partial V^-} \hat{\mathbf{n}}' \cdot \mathbf{J}(\mathbf{r}') \otimes \mathbf{r}' ds(\mathbf{r}') + O\left(\frac{1}{r^3}\right).
\end{aligned} \tag{1.14}$$

The fact that $\mathbf{r}_0 \in V^-$, the expression (1.9) for the current $\mathbf{J}$, and the boundary condition (1.3b) on ∂V^- imply that

$$\hat{\mathbf{n}}' \cdot \mathbf{J}(\mathbf{r}') = 0, \quad \mathbf{r}' \in \partial V^-. \tag{1.15}$$

Consequently, (1.14) concludes that

$$\mathbf{B}(\mathbf{r}) = O\left(\frac{1}{r^3}\right), \quad r \longrightarrow \infty. \tag{1.16}$$

In other words, the leading term of $\mathbf{B}$ in the exterior of V^- is a quadrupole for any smooth boundary ∂V^-. This result is compatible with physical reality.

Note that in the absence of conductive material, surrounding the source dipole current at $\mathbf{r}_0$, the expansion of $\mathbf{B}$ starts with a dipole term, that is , a term of order r^{-2}. But, in the presence of conductive material, the corresponding expansion starts with a quadrupole term, that is, a term of order r^{-3}. Hence, the conductive material partially "hides" the dipole.

As far as MEG measurements are concerned, this means that the conductive brain tissue weakens the intensity of the magnetic field exterior to the head.

This result is in accord with what is known for the special cases, where ∂V^- is a sphere [12, 17], a spheroid [1, 4, 5, 6, 7, 13], or an ellipsoid [2].

2. Silent sources

For the case of a sphere [17], where a complete expression for the magnetic field outside the sphere is known in the form

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}(\mathbf{Q}\times\mathbf{r}_0)\cdot\frac{[\tilde{\mathbf{I}} - \mathbf{r}\otimes\nabla]F(\mathbf{r})}{F^2(\mathbf{r})} \tag{2.1}$$

with

$$F(\mathbf{r}) = r\,|\mathbf{r}-\mathbf{r}_0|^2 + \mathbf{r}\cdot(\mathbf{r}-\mathbf{r}_0)\,|\mathbf{r}-\mathbf{r}_0|, \tag{2.2}$$

it is obvious that if $\mathbf{Q}$ is collinear to $\mathbf{r}_0$, then $\mathbf{B}$ vanishes. This is then characterized as a *silent source* since it represents a nontrivial activity of the brain that is not detectable in the exterior to the head space.

Unfortunately, the complete expression for $\mathbf{B}$, when ∂V^- is an ellipsoid, is not known and it seems far from being possible with the present knowledge of ellipsoidal harmonics. On the other hand, since the human brain is actually shaped in the form of an ellipsoid, with average semiaxes 6, 6.5, and 9 cm [18], even the leading analytic approximation [2] is of value.

In fact, the quadrupole term of $\mathbf{B}$ for a sphere, a prolate spheroid, and an ellipsoid can be written as

$$\mathbf{B}^q(\mathbf{r}) = \lim_{r\to\infty} r^3\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{8\pi}\mathbf{d}\cdot\tilde{\mathbf{G}}(\mathbf{r}), \tag{2.3}$$

where $\mathbf{d}$ is a vector which involves the location, the intensity, and the orientation of the source and $\tilde{\mathbf{G}}$ is a dyadic which is solely dependent on the geometry of the conductive medium. Hence, $\mathbf{d}$ represents the source and $\tilde{\mathbf{G}}$ represents the geometry.

In particular, if ∂V^- is a sphere of radius α, then

$$\mathbf{d} = \mathbf{d}_{\mathrm{sr}} = \mathbf{Q}\times\mathbf{r}_0, \tag{2.4}$$

$$\tilde{\mathbf{G}}_{\mathrm{sr}}(\mathbf{r}) = \frac{1}{r^3}(\tilde{\mathbf{I}} - 3\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}). \tag{2.5}$$

If ∂V^- is the prolate spheroid

$$\frac{x_1^2}{\alpha_1^2} + \frac{x_2^2 + x_3^2}{\alpha_2^2} = 1, \quad \alpha_2 < \alpha_1, \tag{2.6}$$

then

$$\mathbf{d} = \mathbf{d}_{\mathrm{sd}} = (\mathbf{Q}\times\mathbf{r}_0)\cdot\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1 + 2\mathbf{Q}\cdot\tilde{\mathbf{S}}\times\mathbf{r}_0\cdot(\tilde{\mathbf{I}} - \hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1) \tag{2.7}$$

with

$$\tilde{\mathbf{S}} = \frac{\alpha_1^2}{\alpha_1^2+\alpha_2^2}\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1 + \frac{\alpha_2^2}{\alpha_1^2+\alpha_2^2}(\tilde{\mathbf{I}} - \hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1), \tag{2.8}$$

and $\hat{\mathbf{G}}_{\mathrm{sd}}$ is some complicated dyadic function given in [13].

Finally, if ∂V^- is the triaxial ellipsoid

$$\frac{x_1^2}{\alpha_1^2}+\frac{x_2^2}{\alpha_2^2}+\frac{x_3^2}{\alpha_3^2}=1, \quad \alpha_3<\alpha_2<\alpha_1, \tag{2.9}$$

then

$$\mathbf{d}=\mathbf{d}_{\text{el}}=2\big(\mathbf{Q}\cdot\tilde{\mathbf{M}}\times\mathbf{r}_0\big)\cdot\tilde{\mathbf{N}} \tag{2.10}$$

with

$$\begin{aligned}\tilde{\mathbf{M}}&=\alpha_1^2\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1+\alpha_2^2\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_2+\alpha_3^2\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_3,\\ \tilde{\mathbf{N}}&=\frac{\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1}{\alpha_2^2+\alpha_3^2}+\frac{\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_2}{\alpha_1^2+\alpha_3^2}+\frac{\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_3}{\alpha_1^2+\alpha_2^2},\end{aligned} \tag{2.11}$$

where again $\tilde{\mathbf{G}}_{\text{el}}$ is given in terms of elliptic integrals and complicated expressions which can be found in [2].

Note that the dyadic $\tilde{\mathbf{M}}$ specifies the ellipsoid in the sense that the equation

$$\mathbf{r}\cdot\tilde{\mathbf{M}}^{-1}\cdot\mathbf{r}=1 \tag{2.12}$$

coincides with the ellipsoid (2.9), while the dyadic $\tilde{\mathbf{N}}$ characterizes the principal moments of inertia of the ellipsoid since

$$\tilde{\mathbf{N}}=\frac{m}{5}\tilde{\mathbf{L}}^{-1}, \tag{2.13}$$

where $\tilde{\mathbf{L}}$ is the inertia dyadic of the ellipsoid (2.9) and m is its total mass.

Obviously, the ellipsoid is considered to be homogeneous, in which case its inertia dyadic reflects its geometrical characteristics.

It is worth noticing that the dyadic $\tilde{\mathbf{S}}$ divides the space into the 1D axis of revolution represented by $\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1$ and its 2D orthogonal complement represented by

$$\tilde{\mathbf{I}}-\hat{\mathbf{x}}_1\otimes\hat{\mathbf{x}}_1=\hat{\mathbf{x}}_2\otimes\hat{\mathbf{x}}_2+\hat{\mathbf{x}}_3\otimes\hat{\mathbf{x}}_3, \tag{2.14}$$

where all directions are equivalent (2D isotropy).

In the limit, as $\alpha_1\to\alpha$ and $\alpha_2\to\alpha$,

$$\begin{aligned}\tilde{\mathbf{S}}&\longrightarrow\frac{1}{2}\tilde{\mathbf{I}},\\ \mathbf{d}_{\text{sd}}&\longrightarrow\mathbf{Q}\times\mathbf{r}_0=\mathbf{d}_{\text{sr}}.\end{aligned} \tag{2.15}$$

Similarly, the complete geometrical anisotropy, carried by the ellipsoid, is expressed via the dyadics $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$, which dictate the characteristics of each principal direction in space.

In the limit, as $\alpha_1 \to \alpha$, $\alpha_2 \to \alpha$, and $\alpha_3 \to \alpha$, the following limits are obtained

$$\tilde{\mathbf{M}} \longrightarrow \alpha^2 \tilde{\mathbf{I}}, \qquad \tilde{\mathbf{N}} \longrightarrow \frac{1}{2\alpha^2}\tilde{\mathbf{I}}, \qquad \mathbf{d}_{\text{el}} \longrightarrow \mathbf{Q} \times \mathbf{r}_0 = \mathbf{d}_{\text{sr}}, \tag{2.16}$$

so that the spherical behavior is recovered.

Obviously, the vector $\mathbf{d}_{\text{sd}}$ for the spheroid and the vector $\mathbf{d}_{\text{el}}$ for the ellipsoid incorporate the modifications of the cross product (2.4) that are imposed by the particular geometry.

If the quadrupole contribution $\mathbf{B}^q$ is known, then

$$\mathbf{d} = \frac{8\pi}{\mu_0}\mathbf{B}^q(\mathbf{r}) \cdot \tilde{\mathbf{G}}^{-1}(\mathbf{r}), \tag{2.17}$$

where $\tilde{\mathbf{G}}$ is also known if the geometry is given.

This means that, if the spherical model is considered, then $\mathbf{Q}$ and $\mathbf{r}_0$ belong to the plane, through the origin, which is perpendicular to $\mathbf{d}_{\text{sd}}$.

For the case of the ellipsoid,

$$\mathbf{d}_{\text{el}} \cdot \tilde{\mathbf{N}}^{-1} = 2\mathbf{Q} \cdot \tilde{\mathbf{M}} \times \mathbf{r}_0, \tag{2.18}$$

which means that the modified $\mathbf{d}_{\text{el}}$ vector, that is, the vector $\mathbf{d}_{\text{el}} \cdot \tilde{\mathbf{N}}^{-1}$, defines a perpendicular plane on which both the modified moment $\mathbf{Q} \cdot \tilde{\mathbf{M}}$ and the position vector $\mathbf{r}_0$ lie.

The intermediate case of the spheroid shows that if $\mathbf{d}_{\text{sd}}$ is known, then we can extract information about the x_1-component of $\mathbf{Q} \times \mathbf{r}_0$ and the projection of $2\mathbf{Q} \cdot \tilde{\mathbf{S}} \times \mathbf{r}_0$ on the orthogonal complement of $\hat{\mathbf{x}}_1$.

This geometric analysis of the $\mathbf{d}$'s identifies the orientation of the silent sources.

For the simplest case of the sphere, a silent source is a dipole with a radial moment [17]. For the general case of the ellipsoid a silent source is a dipole with a modified moment $\mathbf{Q} \cdot \tilde{\mathbf{M}}$ parallel to $\mathbf{r}_0$. Then, since $\tilde{\mathbf{M}}^{-1}$ represents the Gaussian map [16], which takes a position vector on the surface of the ellipsoid to a vector in the normal to the surface direction at that point, it follows that $\mathbf{Q}$ will be silent if it is parallel to the normal of the ellipsoid in the direction of $\mathbf{r}_0$.

This silent direction for $\mathbf{Q}$ becomes parallel to $\mathbf{r}_0$ for the case of a sphere, but it is now clear that it is the normal to the surface direction, and not the collinearity with $\mathbf{r}_0$, that characterizes a dipole as silent.

Finally, we consider the spheroidal case. From (2.7), it follows that the vanishing of $\mathbf{d}_{\text{sd}}$ comes from the simultaneous solvability of the system

$$(\mathbf{Q} \times \mathbf{r}_0) \cdot \hat{\mathbf{x}}_1 = 0, \tag{2.19}$$

$$(\mathbf{Q} \cdot \tilde{\mathbf{S}} \times \mathbf{r}_0) \cdot \hat{\mathbf{x}}_2 = 0, \tag{2.20}$$

$$(\mathbf{Q} \cdot \tilde{\mathbf{S}} \times \mathbf{r}_0) \cdot \hat{\mathbf{x}}_3 = 0. \tag{2.21}$$

Condition (2.19) holds whenever the projections of $\mathbf{Q}$ and $\mathbf{r}_0$ on the x_2x_3-plane are parallel, while (2.20) and (2.21) hold whenever the projections of $\mathbf{Q} \cdot \tilde{\mathbf{S}}$ and $\mathbf{r}_0$ on the x_1x_3 and on the x_1x_2 planes are also parallel.

From (2.20) and (2.21), we obtain

$$\frac{\alpha_1^2}{\alpha_2^2}\frac{Q_1}{Q_2} = \frac{x_{01}}{x_{02}}, \qquad \frac{\alpha_1^2}{\alpha_2^2}\frac{Q_1}{Q_3} = \frac{x_{01}}{x_{03}}, \tag{2.22}$$

where $\mathbf{Q} = (Q_1, Q_2, Q_3)$ and $\mathbf{r}_0 = (x_{01}, x_{02}, x_{03})$.

Taking the ratio of (2.22), we obtain

$$\frac{Q_2}{Q_3} = \frac{x_{02}}{x_{03}}, \tag{2.23}$$

which is exactly what comes out of (2.19). Interpreting everything in geometrical language, we see that the vectors $\mathbf{Q}$ and $\mathbf{r}_0$ should be coplanar and they should lie on the meridian plane specified by $\mathbf{r}_0$. Then $\mathbf{Q}$ should point in the direction of the normal to the ellipse on this meridian plane in the direction of $\mathbf{r}_0$. We see, once more, that $\mathbf{Q}$ should be normal to the surface of the spheroid in the direction of $\mathbf{r}_0$. The only difference with the ellipsoid is that, as a consequence of the rotational symmetry, both $\mathbf{Q}$ and $\mathbf{r}_0$ always lie on a meridian plane.

As a final conclusion we remark that modeling the human brain, which is a genuine triaxial ellipsoid, by a sphere, the MEG measurements are misinterpreted, since detectable sources are considered as silent while at the same time information is lost from detectable sources that we think they are silent.

For a complete characterization of silent electromagnetic activity within the brain, which concerns not only a single dipole but any current distribution inside a spherical conductor, we refer to the work of Fokas, et al. [8].

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NEW SINGULAR SOLUTIONS OF PROTTER'S PROBLEM FOR THE 3D WAVE EQUATION

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In 1952, for the wave equation, Protter formulated some boundary value problems (BVPs), which are multidimensional analogues of Darboux problems on the plane. He studied these problems in a 3D domain Ω_0, bounded by two characteristic cones Σ_1 and $\Sigma_{2,0}$ and a plane region Σ_0. What is the situation around these BVPs now after 50 years? It is well known that, for the infinite number of smooth functions in the right-hand side of the equation, these problems do not have classical solutions. Popivanov and Schneider (1995) discovered the reason of this fact for the cases of Dirichlet's or Neumann's conditions on Σ_0. In the present paper, we consider the case of third BVP on Σ_0 and obtain the existence of many singular solutions for the wave equation. Especially, for Protter's problems in $\mathbb{R}^3$, it is shown here that for any $n \in \mathbb{N}$ there exists a $C^n(\bar{\Omega}_0)$ - right-hand side function, for which the corresponding unique generalized solution belongs to $C^n(\bar{\Omega}_0 \backslash O)$, but has a strong power-type singularity of order n at the point O. This singularity is isolated only at the vertex O of the characteristic cone $\Sigma_{2,0}$ and does not propagate along the cone.

1. Introduction

In 1952, at a conference of the American Mathematical Society in New York, Protter introduced some boundary value problems (BVPs) for the 3D wave equation

$$\Box u \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \tag{1.1}$$

in a domain $\Omega_0 \subset \mathbb{R}^3$. These problems are three-dimensional analogous of the Darboux problems (or Cauchy-Goursat problems) on the plane. The simply connected domain

$$\Omega_0 := \left\{ (x_1, x_2, t) : 0 < t < \frac{1}{2},\ t < \sqrt{x_1^2 + x_2^2} < 1 - t \right\} \tag{1.2}$$

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is bounded by the disk

$$\Sigma_0 := \{(x_1,x_2,t) : t=0,\ x_1^2+x_2^2<1\}, \tag{1.3}$$

centered at the origin $O(0,0,0)$ and by the two characteristic cones of (1.1)

$$\begin{aligned}\Sigma_1 &:= \left\{(x_1,x_2,t) : 0<t<\frac{1}{2},\ \sqrt{x_1^2+x_2^2}=1-t\right\},\\ \Sigma_{2,0} &:= \left\{(x_1,x_2,t) : 0<t<\frac{1}{2},\ \sqrt{x_1^2+x_2^2}=t\right\}.\end{aligned} \tag{1.4}$$

Similar to the plane problems, Protter formulated and studied [24] some 3D problems with data on the noncharacteristic disk Σ_0 and on one of the cones Σ_1 and $\Sigma_{2,0}$. These problems are known now as Portter's problems, defined as follows.

Protter's problems. Find a solution of the wave equation (1.1) in Ω_0 with the boundary conditions

(P1) $u|_{\Sigma_0\cup\Sigma_1}=0$,
(P1*) $u|_{\Sigma_0\cup\Sigma_{2,0}}=0$,
(P2) $u|_{\Sigma_1}=0$, $u_t|_{\Sigma_0}=0$,
(P2*) $u|_{\Sigma_{2,0}}=0$, $u_t|_{\Sigma_0}=0$.

Substituting the boundary condition on Σ_0 by the third-type condition $[u_t+\alpha u]|_{\Sigma_0}=0$, we arrive at the following problems.

Problems (P_α) *and* (P_α^*). Find a solution of the wave equation (1.1) in Ω_0 which satisfies the boundary conditions

(P_α) $u|_{\Sigma_1}=0$, $[u_t+\alpha u]|_{\Sigma_0\setminus O}=0$,
(P_α^*) $u|_{\Sigma_{2,0}}=0$, $[u_t+\alpha u]|_{\Sigma_0\setminus O}=0$,

where $\alpha\in C^1(\bar{\Sigma}_0\setminus O)$.

The boundary conditions of problem (P1*) (resp., of (P2*)) are the adjoined boundary conditions to such ones of (P1) (resp., of (P2)) for the wave equation (1.1) in Ω_0. Note that Garabedian in [10] proved the uniqueness of a classical solution of problem (P1). For recent results concerning Protter's problems (P1) and (P1*), we refer to [23] and the references therein. For further publications in this area, see [1, 2, 8, 14, 17, 18, 19, 21]. For problems (P_α), we refer to [11] and the references therein. In the case of the hyperbolic equation with the wave operator in the main part, which involves either lower-order terms or other type perturbations, problem (P_α) in Ω_0 has been studied by Aldashev in [1, 2, 3] and by Grammatikopoulos et al. [12]. On the other hand, Ar. B. Bazarbekov and Ak. B. Bazarbekov [5] give another analogue of the classical Darboux problem in the same domain Ω_0. Some other statements of Darboux-type problems can be found in [4, 6, 16] in bounded or unbounded domains different from Ω_0.

It is well known that, in contrast to the Darboux problem on the plane, the 3D problems (P1) and (P2) are not well posed. It is due to the fact that their adjoint homogeneous problems (P1*) and (P2*) have smooth solutions, whose span is infinite-dimensional (see, e.g., Tong [26], Popivanov and Schneider [22], and Khe [18]).

Now we formulate the following useful lemma, the proof of which is given in Section 2.

LEMMA 1.1. *Let* (ρ,φ,t) *be the polar coordinates in* $\mathbb{R}^3 : x_1 = \rho\cos\varphi$, $x_2 = \rho\sin\varphi$, *and* $x_3 = t$. *Let* $n \in \mathbb{N}$, $n \geq 4$,

$$H_k^n(\rho,t) = \sum_{i=0}^{k} A_i^k \frac{t(\rho^2 - t^2)^{n-3/2-k-i}}{\rho^{n-2i}},$$
$$E_k^n(\rho,t) = \sum_{i=0}^{k} B_i^k \frac{(\rho^2 - t^2)^{n-1/2-k-i}}{\rho^{n-2i}}, \tag{1.5}$$

where

$$A_i^k := (-1)^i \frac{(k-i+1)_i(n-1/2-k-i)_i}{i!(n-i)_i},$$
$$B_i^k := (-1)^i \frac{(k-i+1)_i(n+1/2-k-i)_i}{i!(n-i)_i}, \tag{1.6}$$

and $a_i := a(a+1)\cdots(a+i-1)$. *Then the functions*

$$V_k^{n,1}(\rho,t,\varphi) = H_k^n(\rho,t)\sin n\varphi, \qquad V_k^{n,2}(\rho,t,\varphi) = H_k^n(\rho,t)\cos n\varphi, \tag{1.7}$$

for $k = 0,1,\ldots,[n/2]-2$, *are classical solutions of the homogeneous problem (P1*) (i.e., for* $f \equiv 0$*), and the functions*

$$W_k^{n,1}(\rho,t,\varphi) = E_k^n(\rho,t)\sin n\varphi, \qquad W_k^{n,2}(\rho,t,\varphi) = E_k^n(\rho,t)\cos n\varphi, \tag{1.8}$$

for $k = 0,1,\ldots,[(n-1)/2]-1$, *are classical solutions of the homogeneous problem (P2*).*

A necessary condition for the existence of a classical solution for problem (P2) is the orthogonality of the right-hand side function f to all solutions $W_k^{n,i}$ of the homogeneous adjoined problem. In order to avoid an infinite number of necessary conditions in the frame of classical solvability, Popivanov and Schneider in [22, 23] gave definitions of a *generalized solution* of problem (P2) with an eventual singularity on the characteristic cone $\Sigma_{2,0}$, or only at its vertex O. On the other hand, Popivanov and Schneider [23] and Grammatikopoulos et al. [11] proved that for the right-hand side $f = W_0^{n,i}$ the corresponding unique generalized solution of problem (P_α) behaves like $(x_1^2 + x_2^2 + t^2)^{-n/2}$ around the origin O (for more comments about this subject, we refer to Remarks 1.4 and 1.6). Now we know some solutions, $W_k^{n,i}$, of the homogeneous adjoined problem (P2*), and if we take one of these solutions in the right-hand side of (1.1), then we have to expect that the generalized solution of problem (P_α) will also be singular, possibly with a different power type of singularity. An analogous result, in the case of problem (P1) and functions $V_k^{n,i}$, has been proved by Popivanov and Popov in [21]. Having this in mind, here we are looking for some new singular solutions of problem (P_α), which are different from those found in [11].

In the case of problem (P_α) with $\alpha(x) \neq 0$, there are only few publications, while for problem (P_α), concerning the wave equation (1.1), see the results of [11]. Moreover, some results of this type can also be found in Section 3.

For the homogeneous problem (P_α^*) even for the wave equation (except the case $\alpha \equiv 0$, i.e., except problem (P2*)), we do not know nontrivial solutions analogous to (1.7) and (1.8). In Section 2, we give an approach for finding nontrivial solutions. Relatively, we refer to Khe [18], who found nontrivial solutions for the homogeneous problems (P1*) and (P2*), but in the case of the Euler-Poisson-Darboux equation. These results are closely connected to such ones of Lemma 1.1.

In order to obtain our results, we formulate the following definition of a generalized solution of problem (P_α) with a possible singularity at O.

Definition 1.2. A function $u = u(x_1,x_2,t)$ is called a generalized solution of the problem

(P_α) $\Box u = f$, $u|_{\Sigma_1} = 0$, $[u_t + \alpha(x)u]|_{\Sigma_0} = 0$,

in Ω_0, if

(1) $u \in C^1(\bar{\Omega}_0 \backslash O)$, $[u_t + \alpha(x)u]|_{\Sigma_0 \backslash O} = 0$, and $u|_{\Sigma_1} = 0$,

(2) the identity

$$\int_{\Omega_0} (u_t v_t - u_{x_1} v_{x_1} - u_{x_2} v_{x_2} - f v)\,dx_1 dx_2 dt = \int_{\Sigma_0} \alpha(x)(uv)(x,0)\,dx_1 dx_2 \tag{1.9}$$

holds for all v in

$$V_0 := \{v \in C^1(\bar{\Omega}_0) : [v_t + \alpha v]\,|_{\Sigma_0} = 0,\ v = 0 \text{ in a neighborhood of } \Sigma_{2,0}\}. \tag{1.10}$$

Existence and uniqueness results for a generalized solution of problems (P1) and (P2) can be found in [23], while for problem (P_α), see [11].

In order to deal successfully with the encountered difficulties, as are singularities of generalized solutions on the cone $\Sigma_{2,0}$, we introduce the region

$$\Omega_\varepsilon = \Omega_0 \cap \{\varrho - t > \varepsilon\}, \quad \varepsilon \in [0,1), \tag{1.11}$$

which in polar coordinates becomes

$$\Omega_\varepsilon = \{(\varrho,\varphi,t) : t > 0,\ 0 \le \varphi < 2\pi,\ \varepsilon + t < \varrho < 1 - t\}. \tag{1.12}$$

Note that a generalized solution u, which belongs to $C^1(\bar{\Omega}_\varepsilon) \cap C^2(\Omega_\varepsilon)$ and satisfies the wave equation (1.1) in Ω_ε, is called a *classical solution* of problem (P_α) in Ω_ε, $\varepsilon \in (0,1)$. It should be pointed out that the case $\varepsilon = 0$ is totally different from the case $\varepsilon \neq 0$.

This paper is an extension of some results obtained in [11, 12] and, besides the introduction, involves two more sections. In Section 2, we formulate the 2D BVPs corresponding to the 3D Protter's problems. Using Riemann functions, we show the way for finding nontrivial solutions. For the same goal, we consider functions orthogonal to the Legendre one and formulate some open questions for finding more functions of this type in the frame of nontrivial solutions of problems (P1*), (P2*), and (P_α^*). Also, using the results of Sections 1 and 2, in Section 3, we study the existence of a singular generalized solution of 3D problem (P_α). To investigate the behavior of such singular solutions, we need some information about them. In Theorem 3.1, we state a maximum principle for the singular generalized solution of 2D problem ($P_{\alpha,2}$), corresponding to problem (P_α)

in Ω_0. This solution is a classical one in each domain Ω_ε, $\varepsilon \in (0,1)$. Note that this maximum principle can be applied even in the cases where the right-hand side changes its sign in the domain. (Theorem 1.3 deals exactly with this special situation.) Other maximum principles can be found in [6, 25]. Using information of this kind, we present singular generalized solutions which are smooth enough away from the point O, while at the point O, they have power-type singularity. More precisely, in Section 3, we prove the following theorem.

THEOREM 1.3. *Let $\alpha = \alpha(\rho) \in C^\infty(0,1] \cap C[0,1]$ and let $\alpha(\rho) \geq 0$ be an arbitrary function. Then, for each $n \in \mathbb{N}$, $n \geq 4$, there exists a function $f_n \in C^{n-3}(\bar{\Omega}_0) \cap C^\infty(\Omega_0)$, for which the corresponding unique generalized solution u_n of problem (P_α) belongs to $C^{n-1}(\bar{\Omega}_0 \backslash O)$ and satisfies the estimates*

$$\begin{aligned} &\left| u_n(x_1,x_2,|x|) \right| \geq \frac{1}{2} \left| u_n(2x_1,2x_2,0) \right| + |x|^{-(n-2)} \left| \cos n\left(\arctan \frac{x_2}{x_1} \right) \right|, \\ &\left| u_n\left(x_1,x_2,\frac{1-\tau n_1}{1+\tau n_1}|x| \right) \right| \geq |x|^{-(n-2)} \left| \cos n\left(\arctan \frac{x_2}{x_1} \right) \right|, \quad 0 \leq \tau \leq 1, \end{aligned} \tag{1.13}$$

where the constant $n_1 \in (0,1)$ depends only on n.

Remark 1.4. For the right-hand side of the wave equation equals $W_0^{n,2}$, the exact behavior of the *corresponding* singular solution $u_n(x_1,x_2,t)$ around the origin O is $(x_1^2 + x_2^2 + t^2)^{-n/2} \cos n(\arctan x_2/x_1)$ (see [11, 12]), while for the right-hand side equals $W_1^{n,2} = \partial^2/\partial t^2\{W_0^{n,2}\}$, the singularities are at least of type $(x_1^2 + x_2^2 + t^2)^{-(n-2)/2} \cos n(\arctan x_2/x_1)$ (see Theorem 1.3). The following open question arises: is this the exact type of singularity or not? If the last case is true, it would be possible, using an appropriate linear combination of both right-hand sides, to find a solution of the last lower-type singularity. Then the result of this kind could give an answer to Open Question (1).

Remark 1.5. It is interesting that for any parameter $\alpha(x) \geq 0$, involved in the boundary condition (P_α) on Σ_0, there are infinitely many singular solutions of the wave equation. Note that all these solutions have strong singularities at the vertex O of the cone $\Sigma_{2,0}$. These singularities of generalized solutions do not propagate in the direction of the bicharacteristics on the characteristic cone. It is traditionally assumed that the wave equation with right-hand side sufficiently smooth in $\bar{\Omega}_0$ cannot have a solution with an isolated singular point. For results concerning the propagation of singularities for second-order operators, see Hörmander [13, Chapter 24.5]. For some related results in the case of the plane Darboux problem, see [20].

Remark 1.6. Considering problems (P1) and (P2), Popivanov and Schneider [22] announced the existence of singular solutions for both wave and degenerate hyperbolic equations. First a priori estimates for singular solutions of Protter's problems (P1) and (P2), concerning the wave equation in $\mathbb{R}^3$, were obtained in [23]. In [1], Aldashev mentioned the results of [22] and, for the case of the wave equation in $\mathbb{R}^{m+1}$, showed that there exist solutions of problem (P1) (resp., (P2)) in the domain Ω_ε, which grow up on the cone $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (resp., $\varepsilon^{-(n+m-1)}$), and the cone $\Sigma_{2,\varepsilon} := \{\varrho = t + \varepsilon\}$ approximates $\Sigma_{2,0}$ when $\varepsilon \to 0$. It is obvious that, for $m = 2$, these results can be compared to the estimates of [11]. Finally, we point out that in the case of an equation which involves the wave operator and nonzero lower-order terms, Karatoprakliev [15] obtained a priori estimates, but only for the enough smooth solutions of problem (P1) in Ω_0.

We fix the right-hand side as a trigonometric polynomial of the order l:

$$f(x_1,x_2,t) = \sum_{n=2}^{l} \{ f_n^1(t,\rho)\cos n\varphi + f_n^2(t,\rho)\sin n\varphi \}. \tag{1.14}$$

We already know that the corresponding solution $u(x_1,x_2,t)$ may have behavior of type $(x_1^2+x_2^2+t^2)^{-l/2}$ at the point O. We conclude this section with the following questions.

Open Questions. (1) Find the exact behavior of all singular solutions at the point O, which differ from those of Theorem 1.3. In other words,

(i) are there generalized solutions for the right-hand side (1.14) with a higher order of singularity, for example, of the form $(x_1^2+x_2^2+t^2)^{-k-l/2}$, $k>0$?
(ii) are there generalized solutions for the right-hand side (1.14) with a lower order of singularity, for example, of the form $(x_1^2+x_2^2+t^2)^{k-l/2}$, $k>0$?

(2) Find appropriate conditions for the function f under which problem (P_α) has only classical solutions. We do not know any kind of such results even for problem (P2).

(3) From the a priori estimates, obtained in [11], for all solutions of problem (P_α), including singular ones, it follows that, as $\rho \to 0$, none of these solutions can grow up faster than the exponential one. The arising question is: are there singular solutions of problem (P_α) with exponential growth as $\rho \to 0$ or any such solution is of polynomial growth less than or equal to $(x_1^2+x_2^2+t^2)^{-l/2}$?

(4) Why there appear singularities for smooth right-hand side, even for the wave equation? Can we explain this phenomenon numerically?

In the case of problem (P1), the answers to Open Questions (1), (2), and (3) can be found in [21].

2. Nontrivial solutions for the homogeneous problems (P1*), (P2*), and (P_α^*)

Suppose that the right-hand side f of the wave equation is of the form

$$f(\rho,t,\varphi) = f_n^1(\rho,t)\cos n\varphi + f_n^2(\rho,t)\sin n\varphi, \quad n \in \mathbb{N}. \tag{2.1}$$

Then we are seeking solutions of the wave equation of the same form

$$u(\rho,t,\varphi) = u_n^1(\rho,t)\cos n\varphi + u_n^2(\rho,t)\sin n\varphi, \tag{2.2}$$

and due to this fact, the wave equation reduces to

$$(u_n)_{\rho\rho} + \frac{1}{\rho}(u_n)_\rho - (u_n)_{tt} - \frac{n^2}{\rho^2}u_n = f_n \tag{2.3}$$

in $G_0 = \{0<t<1/2;\ t<\rho<1-t\} \subset \mathbb{R}^2$.

Now introduce the new coordinates $x = (\rho + t)/2$, $y = (\rho - t)/2$ and set

$$v(x,y) = \rho^{1/2} u_n(\rho,t), \qquad g(x,y) = \rho^{1/2} f_n(\rho,t). \tag{2.4}$$

Then, denoting $\nu = n - (1/2)$, problems (P1*), (P2*), and (P^*_α) transform into the following problems.

Problems (P31), (P32), and (P3$_\alpha$). Find a solution $v(x,y)$ of the equation

$$v_{xy} - \frac{\nu(\nu+1)}{(x+y)^2} v = g \tag{2.5}$$

in the domain $D = \{0 < x < 1/2;\ 0 < y < x\}$ with the following corresponding boundary conditions:

(P31) $v(x,x) = 0$, $x \in (0,1/2)$ and $v(1/2,y) = 0$, $y \in (0,1/2)$,
(P32) $(v_y - v_x)(x,x) = 0$, $x \in (0,1/2)$ and $v(1/2,y) = 0$, $y \in (0,1/2)$,
(P3$_\alpha$) $(v_y - v_x)(x,x) - \alpha(x)v(x,x) = 0$, $x \in (0,1/2)$ and $v(1/2,y) = 0$, $y \in (0,1/2)$.

A basic tool for our treatment of problems (P3) is the Legendre functions P_ν (for more information, see [9]). Note that the function

$$R(x_1,y_1;x,y) = P_\nu\left(\frac{(x-y)(x_1-y_1) + 2x_1y_1 + 2xy}{(x_1+y_1)(x+y)}\right) \tag{2.6}$$

is a Riemann one for (2.5) (see Copson [7]), that is, with respect to the variables (x_1,y_1), it is a solution of (2.5) with $g = 0$, and

$$R(x,y_1;x,y) = 1, \qquad R(x_1,y;x,y) = 1. \tag{2.7}$$

Therefore, we can construct the function $u(x,y)$ in the following way. Integrating (2.5) over the characteristic triangle $\triangle$ with vertices $M(x,y) \in D$, $P(y,y)$, and $Q(x,x)$, and using the properties (2.7) of the Riemann function, we see that

$$\begin{aligned}
&\iint_\triangle R(x_1,y_1;x,y)g(x_1,y_1)\,dx_1\,dy_1\\
&\quad= \int_y^x \left[R(x_1,x_1;x,y)v_{x_1}(x_1,x_1) - R(x_1,y;x,y)v_{x_1}(x_1,y)\right]dx_1\\
&\qquad - \int_y^x \left[R_{y_1}(x,y_1;x,y)v(x,y_1) - R_{y_1}(y_1,y_1;x,y)v(y_1,y_1)\right]dy_1\\
&\quad= \int_y^x \left[R(x_1,x_1;x,y)v_{x_1}(x_1,x_1) + R_{y_1}(x_1,x_1;x,y)v(x_1,x_1)\right]dx_1\\
&\qquad - v(x,y) + v(y,y).
\end{aligned} \tag{2.8}$$

Hence

$$\begin{aligned}
v(x,y) = v(y,y) &+ \int_y^x \left[R(x_1,x_1;x,y)v_{x_1}(x_1,x_1) + R_{y_1}(x_1,x_1;x,y)v(x_1,x_1)\right]dx_1\\
&- \iint_\triangle R(x_1,y_1;x,y)g(x_1,y_1)\,dx_1\,dy_1.
\end{aligned} \tag{2.9}$$

In the case of $g = 0$, we obtain

$$\begin{aligned} v(x,y) = v(y,y) + \int_y^x \Big[P_\nu\Big(\frac{x_1^2+xy}{x_1(x+y)}\Big) v_{x_1}(x_1,x_1) \\ + P'_\nu\Big(\frac{x_1^2+xy}{x_1(x+y)}\Big) \frac{(x_1-x)(x_1+y)}{2x_1^2(x+y)} v(x_1,x_1)\Big] dx_1. \end{aligned} \tag{2.10}$$

Using the condition $v(x,0) = 0$, finally we find that

$$\begin{aligned} 0 &= \int_0^x P_\nu\Big(\frac{x_1}{x}\Big) v_{x_1}(x_1,x_1) + P'_\nu\Big(\frac{x_1}{x}\Big) \frac{(x_1-x)}{2x_1 x} v(x_1,x_1)\, dx_1 \\ &= \int_0^x P_\nu\Big(\frac{x_1}{x}\Big) \Big\{ v_{x_1}(x_1,x_1) - \frac{\partial}{\partial x_1}\Big[v(x_1,x_1) \frac{(x_1-x)}{2x_1} \Big]\Big\} dx_1 \end{aligned} \tag{2.11}$$

if we suppose, in addition, that $\lim t^{-1} v(t,t) = 0$, $t \to +0$. Thus,

$$\int_0^1 P_\nu(t) \Big\{ \frac{t+1}{t} v_x(tx,tx) + \frac{1-t}{t} v_y(tx,tx) - \frac{1}{xt^2} v(tx,tx) \Big\} dt = 0. \tag{2.12}$$

Suppose that there exist two functions ψ and ψ_1 such that

$$\psi(t)\psi_1(x) = \frac{t+1}{t} v_x(tx,tx) + \frac{1-t}{t} v_y(tx,tx) - \frac{1}{xt^2} v(tx,tx). \tag{2.13}$$

Then we are looking for a solution $\psi(t)$ of the equation

$$\int_0^1 P_\nu(t)\psi(t)\,dt = 0. \tag{2.14}$$

Now we are ready to formulate the following useful lemma.

Lemma 2.1. *The following identity holds:*

$$\int_0^1 t^p P_\nu(t)\,dt = 0, \quad p = \nu-2, \nu-4, \dots;\ p > -1. \tag{2.15}$$

Proof. As known, the Legendre functions $P_\nu(t)$ are solutions of the Legendre differential equation

$$(1-t^2)z'' - 2tz' + \nu(\nu+1)z = 0. \tag{2.16}$$

Using this fact, we see that

$$\begin{aligned}\nu(\nu+1)\int_0^1 t^p P_\nu(t)dt &= \int_0^1 t^p\left[(t^2-1)P_\nu'(t)\right]'dt\\&= -p\int_0^1 (t^{p+1}-t^{p-1})P_\nu'(t)dt\\&= p\int_0^1 (t^{p+1}-t^{p-1})P_\nu'(t)dt\\&= p\int_0^1 \left[(p+1)t^p-(p-1)t^{p-2}\right]P_\nu(t)dt\end{aligned}\tag{2.17}$$

if $p>1$. This means that

$$\left[\nu(\nu+1)-p(p+1)\right]\int_0^1 t^pP_\nu(t)dt = -p(p-1)\int_0^1 t^{p-2}P_\nu(t)dt,\quad p>1.\tag{2.18}$$

Since, for $p=\nu$, the left-hand side here is zero, clearly

$$\int_0^1 t^{\nu-2}P_\nu(t)dt = 0.\tag{2.19}$$

Using this fact and (2.18) with $p=\nu-2$, we conclude that

$$\int_0^1 t^{\nu-4}P_\nu(t)dt = 0,\quad \text{if } \nu-2>1,\tag{2.20}$$

and so the proof of the lemma follows by induction. □

Since, in our case, $\nu=n-1/2$, returning to problems (P1*), (P2*), and (P_α^*), we remark that, for each of these problems, we have the following conclusions.

Problem (P1).* On the line $\{y=x\}$, we have the condition $v(x,x)=0$. Thus, $(v_x+v_y)(x,x)=0$ and (2.13) becomes $\psi(t)\psi_1(x)=2v_x(tx,tx)$. It follows that in this case, by Lemma 2.1, possible solutions are the functions

$$v(x,x)=0,\qquad v_x(x,x)=x^p,\tag{2.21}$$

where $p=n-5/2, n-9/2,\dots,1/2$, if n is an odd number, or $p=n-5/2, n-9/2,\dots,-1/2$, if n is an even number. Thus, the solution $v(x,y)$ of the homogeneous problem (P1*) is explicitly found by (2.10) with values of v and v_x on $\{y=x\}$ given by (2.21).

Problem (P2).* In this case, for $y=x$, we have $(v_x-v_y)(x,x)=0$. Denote $h(x):=v(x,x)$, then $h'(x)=v_x(x,x)+v_y(x,x)$. Hence, we see that $v_x=v_y=h'/2$ and (2.13) becomes

$$\psi\left(\frac{z}{x}\right)\psi_1(x)=\frac{x}{z}h'(z)-\frac{x}{z^2}h(z)=x\left(\frac{h(z)}{z}\right)'.\tag{2.22}$$

By Lemma 2.1, possible solutions of the above equation are the functions

$$v(x,x)=x^p,\qquad v_x(x,x)=\frac{px^{p-1}}{2},\tag{2.23}$$

where $p = n - 1/2, n - 5/2, \dots, 5/2$, if n is an odd number, or $p = n - 1/2, n - 5/2, \dots, 3/2$, if n is an even number. The corresponding solution $v(x,y)$ of the homogeneous problem (P2*) is found again by (2.10) with values of $v(x,x)$ and $v_x(x,x)$ given by (2.23).

Problem (P_α^*). Denote $h(x) := v(x,x)$. Then together with the condition on the line $\{y = x\}$, we see that

$$h'(x) = v_x(x,x) + v_y(x,x), \qquad v_y(x,x) - v_x(x,x) - \alpha(x)v(x,x) = 0, \tag{2.24}$$

from where we have $v_y = (h' + \alpha h)/2$ and $v_x = (h' - \alpha h)/2$. In this case, (2.13) becomes

$$\psi\left(\frac{z}{x}\right)\psi_1(x) = x\left(\frac{h(z)}{z}\right)' - \alpha(z)h(z). \tag{2.25}$$

If $\alpha(z)$ is not identically zero, it is not obvious whether there are some nontrivial solutions of problem (P_α^*) or not.

Open problems. (1) Find a solution $\psi(t)$ of (2.14), different from those of (2.15), which gives a new nontrivial solution of problem (P1*) or (P2*).

(2) Using the way described above, find nontrivial solutions of problem (P_α^*), when $\alpha(x)$ is a nonzero function.

The representation (2.10), together with (2.21) and (2.23), gives us exact formulae for the solution of the homogeneous problems (P1*) and (P2*). Using Lemma 1.1, we obtain a different representation of the same solutions. The solutions $V_0^{n,i}$ and $W_0^{n,i}$ were found by Popivanov and Schneider, while the functions H_k^n and E_k^n can be found in [18] with a different presentation, where they are defined by using the Gauss hypergeometric function.

The following result implies Lemma 1.1.

Lemma 2.2. *The representations*

$$\frac{\partial}{\partial t}H_k^n(\rho,t) = 2(n-k-1)E_{k+1}^n(\rho,t), \tag{2.26}$$

$$\frac{\partial}{\partial t}E_k^n(\rho,t) = -2\left(n-k-\frac{1}{2}\right)H_k^n(\rho,t) \tag{2.27}$$

hold, where H_k^n and E_k^n represent derivatives of $E_0^n(\rho,t)$ with respect to t, that is,

$$\begin{aligned} H_k^n(\rho,t) &= \frac{(-1)^{k+1}}{(2n-2k-1)_{2k+1}}\left(\frac{\partial}{\partial t}\right)^{2k+1}\left(\frac{(\rho^2-t^2)^{n-1/2}}{\rho^n}\right), \\ E_k^n(\rho,t) &= \frac{(-1)^k}{(2n-2k)_{2k}}\left(\frac{\partial}{\partial t}\right)^{2k}\left(\frac{(\rho^2-t^2)^{n-1/2}}{\rho^n}\right). \end{aligned} \tag{2.28}$$

Proof. It is enough to check directly formulae (2.26) and (2.27). □

Proof of Lemma 1.1. We already know (see [23]) that $V_0^{n,i}$ and $W_0^{n,i}$ $(i = 1,2)$ are solutions of the wave equation (1.1). Using formulae (2.26) and (2.27), we conclude that $V_k^{n,i}$ and $W_k^{n,i}$ are also solutions of the wave equation. Thus, the functions $\rho^{1/2}H_k^n(t,\rho)$ and $\rho^{1/2}E_k^n(t,\rho)$ are solutions of the 2D equation (2.5). It is easy to see directly that

$$\frac{\partial(\rho^{1/2}E_k^n)}{\partial t}(\rho,0) = 0, \qquad (\rho^{1/2}E_k^n)(\rho,0) = \rho^{n-2k-1/2}\sum_{i=0}^{k} A_i^k. \tag{2.29}$$

These Cauchy conditions on $\{x = y\}$ (i.e., on $\{t = 0\}$) coincide with the conditions of (2.23) for $p = n - 2k - 1/2$ with the accuracy of a multiplicative constant. Moreover, because of the uniqueness of the solution of Cauchy problem for (2.5), the function $v(x,y)$ defined by (2.10), together with the conditions of (2.23) for $p = n - 2k - 1/2$, coincides with the function $(\sum_{i=0}^{k} A_i^k)^{-1}\rho^{1/2}E_k^n(\rho,t)$. □

3. New singular solutions of problem (P_α)

We are seeking a generalized solution of BVP (P_α) for the wave equation

$$\Box u = \frac{1}{\varrho}\left(\varrho u_\varrho\right)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = f(\varrho,\varphi,t), \tag{3.1}$$

which has some power type of singularity at the origin O. While in [11, 23] the function $W_0^{n,i}(\rho,t,\varphi)$ has been used systematically as the right-hand side function, we will try to use here, for the same reason, the function $W_1^{n,i}(\rho,t,\varphi)$. Due to the fact that the function $E_1^n(\rho,t)$ changes its sign inside the domain, the appearing situation causes some complications. Note first that, by Lemma 1.1, the functions

$$W_1^{n,2}(\varrho,\varphi,t) = \left\{\frac{\left(\varrho^2 - t^2\right)^{n-3/2}}{\varrho^n} - \frac{(n-3/2)}{(n-1)}\frac{\left(\varrho^2 - t^2\right)^{n-5/2}}{\varrho^{n-2}}\right\}\cos n\varphi, \quad n \geq 4, \tag{3.2}$$

with $W_1^{n,2} \in C^{n-3}(\bar{\Omega}_0)$, are classical solutions of problem (P_α^*) when $\alpha \equiv 0$.

To prove Theorem 1.3, consider now the special case of problem (P_α):

$$\Box u = \frac{1}{\varrho}\left(\varrho u_\varrho\right)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = W_1^{n,2}(\varrho,\varphi,t) \quad \text{in } \Omega_0, \tag{3.3}$$

$$u|_{\Sigma_1} = 0, \qquad [u_t + \alpha(\varrho)u]|_{\Sigma_0\setminus O} = 0. \tag{3.4}$$

Theorem 5.1 of [11] declares that problem (3.3), (3.4) has at most one generalized solution. On the other hand, by [11, Theorem 5.2], we know that for this right-hand side there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi \in C^{n-1}(\bar{\Omega}_0\setminus O), \tag{3.5}$$

which is a classical solution in Ω_ε, $\varepsilon \in (0,1)$. By introducing a new function

$$u^{(2)}(\varrho,t) = \varrho^{1/2}u^{(1)}(\varrho,t), \tag{3.6}$$

we transform (3.3) into the equation

$$u^{(2)}_{\varrho\varrho} - u^{(2)}_{tt} - \frac{4n^2-1}{4\varrho^2}u^{(2)} = \varrho^{1/2}E_1^n(\varrho,t), \tag{3.7}$$

with the string operator in the main part. The domain, corresponding to Ω_ε in this case, is

$$G_\varepsilon = \{(\varrho,t) : t > 0,\ \varepsilon + t < \varrho < 1 - t\}. \tag{3.8}$$

In order to use directly the results of [11], we introduce the new coordinates

$$\xi = 1 - \varrho - t, \qquad \eta = 1 - \varrho + t \tag{3.9}$$

and transform the singular point O into the point $(1,1)$.

From (3.7), we derive that

$$U_{\xi\eta} - \frac{4n^2-1}{4(2-\xi-\eta)^2}U = \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{1/2}F(\xi,\eta) \tag{3.10}$$

in $D_\varepsilon = \{(\xi,\eta) : 0 < \xi < \eta < 1-\varepsilon\}$, where

$$U(\xi,\eta) = u^{(2)}\big(\rho(\xi,\eta),t(\xi,\eta)\big), \qquad F(\xi,\eta) = E_1^n\big(\rho(\xi,\eta),t(\xi,\eta)\big). \tag{3.11}$$

In order to investigate the smoothness or the singularity of a solution for the original 3D problem (P_α) on $\Sigma_{2,0}$, we are seeking a classical solution of the corresponding 2D problem ($\mathrm{P}_{\alpha,2}$), not only in the domain D_ε but also in the domain

$$D_\varepsilon^{(1)} := \{(\xi,\eta) : 0 < \xi < \eta < 1,\ 0 < \xi < 1-\varepsilon\}, \quad \varepsilon > 0. \tag{3.12}$$

Clearly, $D_\varepsilon \subset D_\varepsilon^{(1)}$. Thus, we arrive at the Goursat-Darboux problem.

Problem ($P_{\alpha,2}$). Find a solution of the following BVP:

$$\begin{gathered} U_{\xi\eta} - c(\xi,\eta)U = g(\xi,\eta) \quad \text{in } D_\varepsilon^{(1)}, \\ U(0,\eta) = 0, \qquad \big[U_\eta - U_\xi + \alpha(1-\xi)U\big]\big|_{\eta=\xi} = 0. \end{gathered} \tag{3.13}$$

Here, the coefficients $c(\xi,\eta)$ and $g(\xi,\eta)$ are defined by

$$c(\xi,\eta) = \frac{4n^2-1}{4(2-\eta-\xi)^2} \in C^\infty\big(\bar{D}_\varepsilon^{(1)}\big), \quad n \ge 4,\ \varepsilon > 0, \tag{3.14}$$

$$g(\xi,\eta) = 2^{n-(5/2)}\left\{\frac{\big[(1-\xi)(1-\eta)\big]^{n-3/2}}{(2-\eta-\xi)^{n-1/2}} - \frac{(n-3/2)}{4(n-1)}\frac{\big[(1-\xi)(1-\eta)\big]^{n-5/2}}{(2-\eta-\xi)^{n-5/2}}\right\}, \tag{3.15}$$

where $g \in C^{n-3}(\bar{D}_\varepsilon^{(1)})$. In this case, it is obvious that $c(\xi,\eta) \geq 0$ in $\bar{D}_0 \backslash (1,1)$, but the function $g(\xi,\eta)$ is not nonnegative in D_0.

Note that, according to [11], solving problem $(\mathrm{P}_{\alpha,2})$ is equivalent to solving the following integral equation:

$$\begin{aligned} U(\xi_0,\eta_0) = {} & \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} [g(\xi,\eta)+c(\xi,\eta)U(\xi,\eta)]\,d\eta\,d\xi \\ & + 2\int_0^{\xi_0}\int_0^{\eta} [g(\xi,\eta)+c(\xi,\eta)U(\xi,\eta)]\,d\xi\,d\eta \\ & + \int_0^{\xi_0} \alpha(1-\xi)U(\xi,\xi)\,d\xi \quad \text{for } (\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}. \end{aligned} \tag{3.16}$$

For this reason, we define (see [11]) the following sequence of successive approximations $U^{(m)}$:

$$\begin{aligned} U^{(m+1)}(\xi_0,\eta_0) = {} & \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} [g(\xi,\eta)+c(\xi,\eta)U^{(m)}(\xi,\eta)]\,d\eta\,d\xi \\ & + 2\int_0^{\xi_0}\int_0^{\eta} [g(\xi,\eta)+c(\xi,\eta)U^{(m)}(\xi,\eta)]\,d\xi\,d\eta \\ & + \int_0^{\xi_0} \alpha(1-\xi)U^{(m)}(\xi,\xi)\,d\xi, \quad (\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}, \\ U^{(0)}(\xi_0,\eta_0) = {} & 0 \quad \text{in } D_\varepsilon^1. \end{aligned} \tag{3.17}$$

In [11], the uniform convergence of $U^{(m)}$ in each domain $D_\varepsilon^{(1)}$, $\varepsilon > 0$, has been proved. To use this fact here, we now formulate the following maximum principle, which is very important for the investigation of the singularity of a generalized solution of problem (P_α).

Theorem 3.1 (maximum principle). *Let* $c(\xi,\eta), g(\xi,\eta) \in C(\bar{D}_\varepsilon^{(1)})$, *let* $c(\xi,\eta) \geq 0$ *in* $\bar{D}_\varepsilon^{(1)}$, *let* $\alpha(\xi) \geq 0$ *for* $0 \leq \xi \leq 1$, *and*

(a) *let*

$$\int_0^{\xi_0}\int_{\xi_0}^{\eta_0} g(\xi,\eta)\,d\eta\,d\xi + 2\int_0^{\xi_0}\int_0^{\eta} g(\xi,\eta)\,d\xi\,d\eta \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}. \tag{3.18}$$

Then, for the solution $U(\xi,\eta)$ *of problem (3.13), it holds that*

$$U(\xi,\eta) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}. \tag{3.19}$$

(b) *If*

$$\int_0^{\xi_0} g(\xi,\eta_0)\,d\xi \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}, \tag{3.20}$$

then

$$U(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0 \quad \text{for } (\xi,\eta) \in \bar{D}_\varepsilon^{(1)}. \tag{3.21}$$

(c) *If* $g(\xi,\eta) \geq 0$ *in* $\bar{D}_\varepsilon^{(1)}$, *then*

$$U(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}. \tag{3.22}$$

Remark 3.2. Other variants of this maximum principle can be found in [11, 12]. In the cases which we consider below, the conditions of [11, 12] are not satisfied. For example, there are subdomains of $D_\varepsilon^{(1)}$ where $E_1^n < 0$.

Proof of Theorem 3.1. (a) Condition (3.18) says that for the first approximation $U^{(1)}$ of the sequence (3.17), we directly have $U^{(1)}(\xi_0,\eta_0) \geq 0$. Suppose that $(U^{(m)} - U^{(m-1)})(\xi_0,\eta_0) \geq 0$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned}
(U^{(m+1)} - U^{(m)})(\xi_0,\eta_0) &= \int_0^{\xi_0}\int_{\xi_0}^{\eta_0} c(\xi,\eta)(U^{(m)} - U^{(m-1)})(\xi,\eta)\,d\eta\,d\xi \\
&\quad + 2\int_0^{\xi_0}\int_0^{\eta} c(\xi,\eta)(U^{(m)} - U^{(m-1)})(\xi,\eta)\,d\xi\,d\eta \\
&\quad + \int_0^{\xi_0} \alpha(1-\xi)(U^{(m)} - U^{(m-1)})(\xi,\xi)\,d\xi \\
&\geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)},
\end{aligned} \tag{3.23}$$

and thus, by induction,

$$U(\xi_0,\eta_0) = \sum_{m=0}^{\infty} (U^{(m+1)} - U^{(m)})(\xi_0,\eta_0) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}. \tag{3.24}$$

(b) If condition (3.20) is satisfied, then it is easy to check that $U^{(1)}(\xi_0,\eta_0) \geq 0$ for any $(\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}$, and so, in view of (a), we see that $U(\xi_0,\eta_0) \geq 0$ for $(\xi_0,\eta_0) \in \bar{D}_\varepsilon^{(1)}$. Using the results of [11], we derive the following representation:

$$U_{\eta_0}(\xi_0,\eta_0) = \int_0^{\xi_0} g(\xi,\eta_0)\,d\xi + \int_0^{\xi_0} c(\xi,\eta_0)U(\xi,\eta_0)\,d\xi, \tag{3.25}$$

and hence we conclude that $U_{\eta_0} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$.

(c) If $g(\xi,\eta) \geq 0$ in $\bar{D}_\varepsilon^{(1)}$, then conditions (3.18) and (3.20) are obviously satisfied, and thus $U \geq 0$ and $U_{\eta_0} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$. The conclusion $U_{\xi_0} \geq 0$ in $\bar{D}_\varepsilon^{(1)}$ follows from the fact that (see [11])

$$\begin{aligned}
U_{\xi_0}(\xi_0,\eta_0) &= \alpha(1-\xi_0)U(\xi_0,\xi_0) + \int_0^{\xi_0} [g(\xi,\xi_0) + c(\xi,\xi_0)U(\xi,\xi_0)]\,d\xi \\
&\quad + \int_{\xi_0}^{\eta_0} [g(\xi_0,\eta) + c(\xi_0,\eta)U(\xi_0,\eta)]\,d\eta.
\end{aligned} \tag{3.26}$$

□

In order to prove our results, we make use of the following proposition.

Proposition 3.3. *Let $U(\xi,\eta)$ be the unique generalized solution for problem (3.13), where $c(\xi,\eta)$ and $g(\xi,\eta)$ are given by (3.14) and (3.15). Then $U(\xi,\eta)\in C^{n-1}(\bar{D}_0\backslash(1,1))$ and $U(\xi,\eta)\geq 0$ in $\bar{D}_0\backslash(1,1)$; in addition $U_\xi(\xi,\eta)\geq 0$, $U_\eta(\xi,\eta)\geq 0$ in some neighborhood of the point* $(1,1)$.

Proof. First note that in this case neither condition $g(\xi,\eta)\geq 0$ nor condition (3.20) is fulfilled. We will prove that condition (3.18) is satisfied. Introduce the polar coordinates (ρ,t) and consider the function $g(\rho,t)=\rho^{1/2}E_1^n(\rho,t)$ in the domain $G_0=\{(\varrho,t):t>0,\ t<\varrho<1-t\}$, then the representation formula (see (2.26))

$$\frac{\partial}{\partial t}\rho^{1/2}H_0^n(\rho,t)=2(n-1)\rho^{1/2}E_1^n(\rho,t)=2(n-1)g(\rho,t) \tag{3.27}$$

holds. Let $0\leq\rho_1\leq\rho_2\leq 1$. Using (3.27), it is easy to see that, for the first approximation $U^{(1)}$ of the solution, one has (see (3.17))

$$\begin{aligned}
2(n-1)&U^{(1)}\left(\frac{\rho_1+\rho_2}{2},\frac{\rho_2-\rho_1}{2}\right)\\
&=\int_{(1+\rho_1)/2}^{1}\rho^{1/2}H_0^n(\rho,1-\rho)d\rho+\int_{\rho_1}^{(1+\rho_1)/2}\rho^{1/2}H_0^n(\rho,\rho-\rho_1)\,d\rho\\
&\quad-\int_{(\rho_2+\rho_1)/2}^{\rho_2}\rho^{1/2}H_0^n(\rho,\rho_2-\rho)\,d\rho-\int_{\rho_1}^{(\rho_2+\rho_1)/2}\rho^{1/2}H_0^n(\rho,\rho-\rho_1)\,d\rho\\
&\quad+\int_{(1+\rho_2)/2}^{1}\rho^{1/2}H_0^n(\rho,1-\rho)d\rho+\int_{\rho_2}^{(1+\rho_2)/2}\rho^{1/2}H_0^n(\rho,\rho-\rho_2)\,d\rho.
\end{aligned} \tag{3.28}$$

Since $H_0^n\geq 0$, to prove that $U^{(1)}\geq 0$, it is enough to show that

$$I=\int_{(1+\rho_1)/2}^{1}\rho^{1/2}H_0^n(\rho,1-\rho)d\rho-\int_{(\rho_2+\rho_1)/2}^{\rho_2}\rho^{1/2}H_0^n(\rho,\rho_2-\rho)\,d\rho\geq 0. \tag{3.29}$$

For this purpose, we see that

$$\begin{aligned}
I&=\int_{(1+\rho_1)/2}^{1}(1-\rho)\rho^{-n+1/2}(2\rho-1)^{n-3/2}d\rho\\
&\quad-\int_{(\rho_2+\rho_1)/2}^{\rho_2}(\rho_2-\rho)\rho^{-n+1/2}\rho_2^{n-3/2}(2\rho-\rho_2)^{n-3/2}d\rho\\
&=\int_{(1+\rho_1)/2}^{1}(1-\rho)\rho^{-1}\left(2-\frac{1}{\rho}\right)^{n-3/2}d\rho\\
&\quad-\int_{(2+\rho_1-\rho_2)/2}^{1}(1-\rho)(\rho+\rho_2-1)^{-1}\rho_2^{n-3/2}\left(2-\frac{\rho_2}{\rho+\rho_2-1}\right)^{n-3/2}d\rho.
\end{aligned} \tag{3.30}$$

As a final step, notice that

$$2-\frac{1}{t}\geq 2-\frac{\rho_2}{t+\rho_2-1}\geq 0\quad\text{for }\frac{2+\rho_1-\rho_2}{2}\leq t\leq 1, \tag{3.31}$$

and therefore $I \geq 0$. So, we conclude that condition (3.18) is satisfied. It follows now, by Theorem 3.1, that $U(\rho,t) \geq 0$ in $G_0 \backslash (0,0)$. More precisely, for $\rho_2 < \delta$, the last term in (3.30) is small enough for small positive δ, and so

$$I \geq \int_{3/4}^{1} (1-\rho)\left(2-\frac{1}{\rho}\right)^{n-1/2} d\rho := c_0 > 0. \tag{3.32}$$

Thus, we find that $U^{(1)}(\rho,t) \geq c_0 > 0$ in a small neighborhood of the origin $(0,0)$. Therefore, for the solution $U(\xi,\eta)$ of problem $(\mathrm{P}_{\alpha,2})$ in coordinates (ξ,η), it follows that $U(\xi,\eta) \geq U^{(1)}(\xi,\eta) \geq c_0 > 0$ in the corresponding neighborhood of the point $(1,1)$. Using the representation

$$U_{\eta_0}(\xi_0,\eta_0) = \int_0^{\xi_0} \left\{ g(\xi,\eta_0) + \frac{4n^2-1}{4(2-\eta_0-\xi)^2} U(\xi,\eta_0) \right\} d\xi, \tag{3.33}$$

it is easy to see now that $U_{\eta_0}(\xi_0,\eta_0) \geq 0$ for $1-\delta \leq \xi_0 \leq \eta_0 \leq 1$ if $\delta > 0$ is small enough. Furthermore, using the representation (3.26) of $U_{\xi_0}(\xi_0,\eta_0)$, we can prove an analogous result for $U_{\xi_0}(\xi_0,\eta_0)$. □

Remark 3.4. In our opinion, the analogous result follows for all functions $E_k^n(\rho,t)$, $k = 2,3,\ldots,[n/2]-1$. As before, for $k > 0$, the function $E_k^n(\rho,t)$ changes its sign in the domain, but due to the monotonicity of the solution $U(\xi,\eta)$, the desired result would follow. Also, by using the more general formula

$$\frac{\partial}{\partial t} H_k^n(\rho,t) = 2(n-k-1)E_{k+1}^n(\rho,t), \tag{3.34}$$

this result could be obtained for $k > 1$ too.

Now we are ready to prove Theorem 1.3 formulated in the introduction.

Proof of Theorem 1.3. We will find the desired lower estimates for the singular solution $u(\rho,\varphi,t)$ of problem (3.3), (3.4). For the corresponding right-hand side $g(\xi,\eta)$, defined by (3.15), set

$$K = \int_{D_{1/2}^{(1)}} g^2(\xi,\eta)\,d\eta\,d\xi > 0. \tag{3.35}$$

Let $\varepsilon \in (0,1/2)$ be fixed. Then, for the *generalized solution* $U(\xi,\eta)$ of problem (3.13), it follows that

$$\begin{aligned} 0 < K &\leq \int_{D_\varepsilon^{(1)}} g^2(\xi,\eta)\,d\xi\,d\eta \\ &= \int_{D_\varepsilon^{(1)}} U_{\xi\eta}(\xi,\eta)g(\xi,\eta)\,d\xi\,d\eta - \int_{D_\varepsilon^{(1)}} c(\xi,\eta)U(\xi,\eta)g(\xi,\eta)\,d\xi\,d\eta \\ &=: I_1 + I_2, \end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
I_1 &= \int_0^{1-\varepsilon}\int_\xi^1 U_{\xi\eta}(\xi,\eta)g(\xi,\eta)d\eta\,d\xi \\
&= \int_0^{1-\varepsilon}\left[U_\xi(\xi,1)g(\xi,1)-U_\xi(\xi,\xi)g(\xi,\xi)\right]d\xi \\
&\quad -\int_{D_\varepsilon^{(1)}}(U_\xi g_\eta)(\xi,\eta)d\eta\,d\xi.
\end{aligned}\tag{3.37}$$

In view of (3.15), it is obvious that $g(\xi,1)=0$. Thus,

$$I_1 = -\int_0^{1-\varepsilon} U_\xi(\xi,\xi)g(\xi,\xi)d\xi - \int_{D_\varepsilon^{(1)}}(U_\xi g_\eta)(\xi,\eta)d\eta\,d\xi. \tag{3.38}$$

Since

$$\begin{aligned}
\int_{D_\varepsilon^{(1)}}(U_\xi g_\eta)(\xi,\eta)d\xi\,d\eta &= \int_0^{1-\varepsilon}\int_0^\eta (U_\xi g_\eta)(\xi,\eta)d\xi\,d\eta \\
&\quad +\int_{1-\varepsilon}^1\int_0^{1-\varepsilon}(U_\xi g_\eta)(\xi,\eta)d\xi\,d\eta \\
&= \int_0^{1-\varepsilon}\left[(Ug_\eta)(\eta,\eta)-(Ug_\eta)(0,\eta)\right]d\eta \\
&\quad +\int_{1-\varepsilon}^1\left[(Ug_\eta)(1-\varepsilon,\eta)-(Ug_\eta)(0,\eta)\right]d\eta \\
&\quad -\int_{D_\varepsilon^{(1)}}(Ug_{\xi\eta})(\xi,\eta)d\xi\,d\eta \\
&= \int_0^{1-\varepsilon}(Ug_\eta)(\eta,\eta)d\eta+\int_{1-\varepsilon}^1(Ug_\eta)(1-\varepsilon,\eta)d\eta \\
&\quad -\int_{D_\varepsilon^{(1)}}(Ug_{\xi\eta})(\xi,\eta)d\xi\,d\eta,
\end{aligned}\tag{3.39}$$

(3.38) becomes

$$\begin{aligned}
I_1 = &-\int_0^{1-\varepsilon}\left[U_\xi(\xi,\xi)g(\xi,\xi)+U(\xi,\xi)g_\eta(\xi,\xi)\right]d\xi \\
&-\int_{1-\varepsilon}^1 U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)d\eta+\int_{D_\varepsilon^{(1)}}(Ug_{\xi\eta})(\xi,\eta)d\xi\,d\eta.
\end{aligned}\tag{3.40}$$

An elementary calculation shows that

$$\begin{gathered}
g_{\xi\eta}(\xi,\eta)-c(\xi,\eta)g(\xi,\eta)=0, \\
g_\xi(\xi,\xi)=g_\eta(\xi,\xi)=\frac{1}{32(n-1)}(5-2n)(1-\xi)^{n-7/2}<0.
\end{gathered}\tag{3.41}$$

By (3.40) and (3.36), it follows that

$$
\begin{aligned}
0 < K \le I_1 + I_2 = &-\int_0^{1-\varepsilon} \left[U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\xi(\xi,\xi)\right]d\xi \\
&-\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)d\eta \\
&+\int_{D_\varepsilon^{(1)}} U(\xi,\eta)\left[g_{\xi\eta} - cg\right](\xi,\eta)d\xi\, d\eta.
\end{aligned}
\tag{3.42}
$$

Thus, we see that

$$
\begin{aligned}
0 < K \le I_1 + I_2 = &-\int_0^{1-\varepsilon} \left[U_\xi(\xi,\xi)g(\xi,\xi) + U(\xi,\xi)g_\xi(\xi,\xi)\right]d\xi \\
&-\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)d\eta,
\end{aligned}
\tag{3.43}
$$

where, as it is easy to check,

$$
g_\xi(\xi,\xi) = \frac{1}{2}\left[g(\xi,\xi)\right]_\xi. \tag{3.44}
$$

The function $U(\xi,\eta)$ is a classical solution of (3.13) in $\bar{D}_\varepsilon$, $\varepsilon \in (0,1)$, with

$$
U_\xi(\xi,\xi) = \frac{1}{2}\left[U(\xi,\xi)\right]_\xi + \frac{1}{2}\alpha(1-\xi)U(\xi,\xi). \tag{3.45}
$$

If we substitute (3.44) and (3.45) into (3.43), we get

$$
\begin{aligned}
K \le I_1 + I_2 &= -\frac{1}{2}\int_0^{1-\varepsilon} \left[U(\xi,\xi)g(\xi,\xi)\right]_\xi d\xi - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)g(\xi,\xi)d\xi \\
&\quad -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)d\eta \\
&= -\frac{1}{2}(Ug)(1-\varepsilon,1-\varepsilon) - \frac{1}{2}\int_0^{1-\varepsilon} \alpha(1-\xi)U(\xi,\xi)g(\xi,\xi)d\xi \\
&\quad -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)g_\eta(1-\varepsilon,\eta)d\eta.
\end{aligned}
\tag{3.46}
$$

Note that $\alpha(\xi) \ge 0$, $g(\xi,\xi) \ge 0$, and according to Proposition 3.3, we have

$$
U(\xi,\eta) \ge 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}, \qquad U_\eta(1-\varepsilon,\eta) \ge 0 \quad \text{for small enough } \varepsilon > 0. \tag{3.47}
$$

Calculating $g_\eta(1-\varepsilon,\eta)$ and denoting

$$
1 - \eta_\varepsilon := \varepsilon\frac{(2n-3)(2n+1) - 2\sqrt{2(2n-3)(2n+1)(n-1)}}{4n^2-1} := \varepsilon n_1, \tag{3.48}
$$

where the number $n_1 \in (0,1)$, we find

$$
\begin{aligned}
g_\eta(1-\varepsilon,\eta) < 0 \quad &\text{for } 1-\varepsilon < \eta < \eta_\varepsilon, \\
g_\eta(1-\varepsilon,\eta) > 0 \quad &\text{for } \eta_\varepsilon < \eta < 1.
\end{aligned}
\tag{3.49}
$$

This, together with (3.46), implies that

$$\begin{aligned}
K \le I_1 + I_2 &\le \int_{1-\varepsilon}^{\eta_\varepsilon} U(1-\varepsilon,\eta)\,|g_\eta(1-\varepsilon,\eta)|\,d\eta - \frac{1}{2}(Ug)(1-\varepsilon,1-\varepsilon) \\
&\quad - \int_{\eta_\varepsilon}^{1} U(1-\varepsilon,\eta)\,|g_\eta(1-\varepsilon,\eta)|\,d\eta \\
&\le U(1-\varepsilon,\eta_\varepsilon)\big[g(1-\varepsilon,1-\varepsilon) - g(1-\varepsilon,\eta_\varepsilon)\big] \\
&\quad - U(1-\varepsilon,\eta_\varepsilon)\big[g(1-\varepsilon,1) - g(1-\varepsilon,\eta_\varepsilon)\big] - \frac{1}{2}(Ug)(1-\varepsilon,1-\varepsilon) \\
&= \Big[U(1-\varepsilon,\eta_\varepsilon) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\Big]g(1-\varepsilon,1-\varepsilon)
\end{aligned} \tag{3.50}$$

because $g(1-\varepsilon,1)=0$. Moreover, since $g(1-\varepsilon,1-\varepsilon)=\varepsilon^{n-5/2}/8(n-1)$, we see that

$$0 < K \le \Big[U(1-\varepsilon,1-\varepsilon n_1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\Big]c_n\varepsilon^{n-(5/2)}. \tag{3.51}$$

Using the fact that $U \ge 0$ and $U_\eta \ge 0$, we obtain

$$0 < K \le U(1-\varepsilon,1-\tau\varepsilon n_1)c_n\varepsilon^{n-(5/2)}, \quad 0 \le \tau \le 1, \tag{3.52}$$

$$0 < K \le \Big[U(1-\varepsilon,1) - \frac{1}{2}U(1-\varepsilon,1-\varepsilon)\Big]c_n\varepsilon^{n-(5/2)}. \tag{3.53}$$

For $\xi = 1-\varepsilon$, $\eta = 1$, we have $\varrho = t = \varepsilon/2$ and (3.53) becomes

$$0 < K_1\varepsilon^{(5/2)-n} \le u_n^{(2)}\Big(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\Big) - \frac{1}{2}u_n^{(2)}(\varepsilon,0). \tag{3.54}$$

Finally, the inverse transformation gives

$$u_n^{(1)}(\rho,\rho) \ge \frac{1}{2}u_n^{(1)}(2\rho,0) + K_2\rho^{-(n-2)} \ge K_2\rho^{-(n-2)}, \tag{3.55}$$

where the positive constant K_2 depends only on n. Analogously, (3.52) gives

$$u_n^{(1)}\Big(\rho,\frac{1-\tau n_1}{1+\tau n_1}\rho\Big) \ge K_2\rho^{-(n-2)}, \quad 0 \le \tau \le 1. \tag{3.56}$$

Multiplying the function u_n by K_2^{-1}, we see that

$$\begin{gathered}
|u_n(\rho,\varphi,\rho)| \ge \frac{1}{2}|u_n(2\rho,\varphi,0)| + \rho^{-(n-2)}|\cos n\varphi| \ge \rho^{-n+2}|\cos n\varphi|, \\
\Big|u_n\Big(\rho,\varphi,\frac{1-\tau n_1}{1+\tau n_1}\rho\Big)\Big| \ge \rho^{-(n-2)}|\cos n\varphi|, \quad 0 \le \tau \le 1,
\end{gathered} \tag{3.57}$$

hold, and then (1.13) follows. The proof of the theorem is complete. □

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LINEAR DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAYS AND A FORCING TERM

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The paper discusses the asymptotic behaviour of all solutions of the differential equation $\dot{y}(t) = -a(t)y(t) + \sum_{i=1}^{n} b_i(t)y(\tau_i(t)) + f(t)$, $t \in I = [t_0, \infty)$, with a positive continuous function a, continuous functions b_i, f, and n continuously differentiable unbounded lags. We establish conditions under which any solution y of this equation can be estimated by means of a solution of an auxiliary functional equation with one unbounded lag. Moreover, some related questions concerning functional equations are discussed as well.

1. Introduction

In this paper, we study the problem of the asymptotic bounds of all solutions for the delay differential equation

$$\dot{y}(t) = -a(t)y(t) + \sum_{i=1}^{n} b_i(t)y\big(\tau_i(t)\big) + f(t), \quad t \in I = \big[t_0, \infty\big), \tag{1.1}$$

where a is a positive continuous function on I; b_i, f are continuous functions on I, τ_i are continuously differentiable functions on I fulfilling $\tau_i(t) < t$, $0 < \dot{\tau}_i(t) \le \lambda_i < 1$ for all $t \in I$ and $\tau_i(t) \to \infty$ as $t \to \infty$, $i = 1, \dots, n$.

The prototype of such equations may serve the equation with proportional delays

$$\dot{y}(t) = -ay(t) + \sum_{i=1}^{n} b_i y\big(\lambda_i t\big) + f(t), \quad t \ge 0, \tag{1.2}$$

where $a > 0$, $b_i \ne 0$, $0 < \lambda_i < 1$, $i = 1, \dots, n$, are real scalars. There are numerous interesting applications for (1.2) and its modifications, such as collection of current by the pantograph head of an electric locomotive, probability theory on algebraic structures or partition problems in number theory. Various special cases of (1.2) have been studied because of these applications, as well as for theoretical reasons (see, e.g., Bereketoglu and Pituk [1], Lim [11], Liu [12], or Ockendon and Taylor [15]).

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The study of these differential equations with proportional delays turned out to be the useful paradigm for the investigation of qualitative properties of differential equations with general unbounded lags. Some results of the above-cited papers have been generalized in this direction by Heard [7], Makay and Terjéki [13], and in [2, 3, 4]. For further related results on the asymptotic behaviour of solutions, see, for example, Diblík [5, 6], Iserles [8], or Krisztin [9].

In this paper, we combine standard methods from the theory of functional differential equations and some results of the theory of functional equations and difference equations to analyze the asymptotic properties of all solutions of (1.1). The main results are formulated in Sections 3 and 4. In Section 3, we derive the asymptotic estimate of all solutions of (1.1). Section 4 discusses some particular cases of (1.1) and improves the above derived estimate for these special cases. Both sections also present the illustrating examples involving, among others, (1.2).

2. Preliminaries

Let $t_{-1} := \min\{\tau_i(t_0),\ i = 1,2,\ldots,n\}$ and $I_{-1} := [t_{-1},\infty)$. By a solution of (1.1), we understand a real-valued function $y \in C(I_{-1}) \cap C^1(I)$ such that y satisfies (1.1) on I.

In the sequel, we introduce the notion of embeddability of given functions into an iteration group. This property will be imposed on the set of delays $\{\tau_1,\ldots,\tau_n\}$ throughout next sections.

Definition 2.1. Let $\psi \in C^1(I_{-1})$, $\dot{\psi} > 0$ on I_{-1}. Say that $\{\tau_1,\ldots,\tau_n\}$ can be embedded into an iteration group $[\psi]$ if for any τ_i there exists a constant d_i such that

$$\tau_i(t) = \psi^{-1}\big(\psi(t) - d_i\big), \quad t \in I. \tag{2.1}$$

Remark 2.2. The problem of embeddability of given functions $\{\tau_1,\ldots,\tau_n\}$ into an iteration group $[\psi]$ is closely related to the existence of a common solution ψ to the system of the simultaneous Abel equations

$$\psi\big(\tau_i(t)\big) = \psi(t) - d_i, \quad t \in I,\ i = 1,\ldots,n. \tag{2.2}$$

The complete solution of these problems have been described by Neuman [14] and Zdun [16]. These papers contain conditions under which (2.1) holds for any τ_i, $i = 1,\ldots,n$ (see also [10, Theorem 9.4.1]). We only note that the most important necessary condition is commutativity of any pair τ_i, τ_j, i, $j = 1,\ldots,n$. Notice also, that if τ_i are delays, then d_i must be positive.

3. The asymptotic bound of all solutions of (1.1)

The aim of this section is to formulate and prove the asymptotic estimate of all solutions of (1.1). We assume that all the assumptions imposed on a, b_i, τ_i, and f in Section 1 are valid.

Theorem 3.1. *Let* $\{\tau_1,\dots,\tau_n\}$ *be embedded into an iteration group* $[\psi]$. *Let* y *be a solution of (1.1), where* $a(t) \ge K/\exp\{\alpha\psi(t)\}$, $0 < \sum_{i=1}^{n} |b_i(t)| \le Ma(t)$ *for all* $t \in I$ *and suitable real constants* $K > 0$, $M > 0$, $\alpha < 1$. *If* $f(t) = O(\exp\{\beta\psi(t)\})$ *as* $t \to \infty$ *for a suitable real* β, *then*

$$y(t) = O\big(\exp\{\gamma\psi(t)\}\big) \quad \text{as } t \longrightarrow \infty,\ \gamma > \max\left(\alpha+\beta, \frac{\log M}{d_1},\dots,\frac{\log M}{d_n}\right), \tag{3.1}$$

where d_i, $i = 1,\dots,n$, *are given by (2.1).*

Proof. The substitution

$$s = \psi(t), \quad z(s) = \exp\{-\gamma\psi(t)\}y(t) \tag{3.2}$$

transforms (1.1) into the form

$$\begin{aligned} z'(s) = -\big[a(h(s))h'(s)+\gamma\big]z(s) &+ \sum_{i=1}^{n} b_i(h(s))\exp\{-\gamma d_i\}h'(s)z(\mu_i(s)) \\ &+ f(h(s))\exp\{-\gamma s\}h'(s), \end{aligned} \tag{3.3}$$

where "$'$" stands for d/ds, $h(s) = \psi^{-1}(s)$, and $\mu_i(s) = \psi(\tau_i(h(s))) = s - d_i$ on $\psi(I)$, $i = 1,\dots,n$. This form can be rewritten as

$$\begin{aligned} &\frac{d}{ds}\left[\exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}z(s)\right] \\ &\quad = \sum_{i=1}^{n} b_i(h(s))\exp\{-\gamma d_i\}h'(s)\exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}z(s-d_i) \\ &\qquad + \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}f(h(s))\exp\{-\gamma s\}h'(s), \end{aligned} \tag{3.4}$$

where $s_0 \in \psi(I)$ is such that $\gamma + a(h(s))h'(s) > 0$ for all $s \ge s_0$.

Put $\delta := \min(d_1,\dots,d_n) > 0$, $s_k := s_0 + k\delta$, $J_k := [s_{k-1}, s_k]$, $k = 1,2,\dots$. Let $s^* \in J_{k+1}$. The integration of (3.4) over $[s_k, s^*]$ yields

$$\begin{aligned} &\exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}z(s)\Big|_{s_k}^{s^*} \\ &\quad = \sum_{i=1}^{n}\int_{s_k}^{s^*} b_i(h(s))\exp\{-\gamma d_i\}h'(s)\exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}z(s-d_i)ds \\ &\qquad + \int_{s_k}^{s^*}\exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}f(h(s))\exp\{-\gamma s\}h'(s)ds, \end{aligned} \tag{3.5}$$

that is,

$$\begin{aligned}
z(s^*) = {} & \exp\left\{\gamma(s_k - s^*) - \int_{h(s_k)}^{h(s^*)} a(u)\,du\right\} z(s_k) \\
& + \exp\left\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\right\} \\
& \times \sum_{i=1}^{n} \int_{s_k}^{s^*} b_i(h(s)) \exp\{-\gamma d_i\} h'(s) \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\right\} z(s - d_i)\,ds \\
& + \exp\left\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\right\} \\
& \times \int_{s_k}^{s^*} \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\right\} f(h(s)) \exp\{-\gamma s\} h'(s)\,ds.
\end{aligned} \tag{3.6}$$

Put $M_k := \sup\{|z(s)|, s \in \cup_{p=1}^{k} J_p\}$, $k = 1, 2, \ldots$. Then one can estimate $z(s^*)$ as

$$\begin{aligned}
|z(s^*)| \le {} & M_k \exp\left\{\gamma(s_k - s^*) - \int_{h(s_k)}^{h(s^*)} a(u)\,du\right\} \\
& + M_k \exp\left\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\right\} \\
& \times \int_{s_k}^{s^*} \sum_{i=1}^{n} |b_i(h(s))| \exp\{-\gamma d_i\} h'(s) \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\right\} ds \\
& + \exp\left\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\right\} \\
& \times \int_{s_k}^{s^*} \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\right\} |f(h(s))| \exp\{-\gamma s\} h'(s)\,ds.
\end{aligned} \tag{3.7}$$

Noting that

$$\begin{gathered}
\sum_{i=1}^{n} |b_i(h(s))| \exp\{-\gamma d_i\} \le M \exp\{-\gamma d_i\} a(h(s)) < a(h(s)), \\
|f(h(s))| \exp\{-\gamma s\} \le K_1 \exp\{(\beta - \gamma)s\}, \quad K_1 > 0,
\end{gathered} \tag{3.8}$$

we can rewrite (3.7) as

$$\begin{aligned}
|z(s^*)| \le {} & M_k \exp\left\{\gamma(s_k - s^*) - \int_{h(s_k)}^{h(s^*)} a(u)\,du\right\} \\
& + M_k \exp\left\{-\int_{s_0}^{h(s^*)} a(u)\,du - \gamma s^*\right\} \\
& \times \int_{s_k}^{s^*} a(h(s)) h'(s) \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)\,du\right\} ds
\end{aligned}$$

$$
\begin{aligned}
&+K_1 \exp\left\{-\int_{s_0}^{h(s^*)} a(u)du - \gamma s^*\right\} \\
&\times \int_{s_k}^{s^*} \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\} h'(s)\exp\{(\beta-\gamma)s\}ds.
\end{aligned}
\tag{3.9}
$$

From here, we get

$$
\begin{aligned}
|z(s^*)| \le{}& M_k \exp\left\{\gamma(s_k - s^*) - \int_{h(s_k)}^{h(s^*)} a(u)du\right\} \\
&+ (M_k + K_2 \exp\{(\alpha+\beta-\gamma)s_k\}) \exp\left\{-\int_{s_0}^{h(s^*)} a(u)du - \gamma s^*\right\} \\
&\times \int_{s_k}^{s^*} a(h(s))h'(s)\exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}ds,
\end{aligned}
\tag{3.10}
$$

where $K_2 = K_1/K$. Using the assumptions imposed on a and $\dot{\tau}_i$, we can estimate the integral $I := \int_{s_k}^{s^*} a(h(s))h'(s)\exp\{\gamma s + \int_{s_0}^{h(s)} a(u)du\}ds$ as

$$
I \le \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}\Big|_{s_k}^{s^*}(1 + K_3 e^{-\omega s_k}), \quad K_3 > 0,\ \omega = 1-\alpha > 0 \tag{3.11}
$$

(for a similar situation see also [4]). Hence,

$$
\begin{aligned}
|z(s^*)| \le{}& M_k \exp\left\{\gamma(s_k - s^*) - \int_{h(s_k)}^{h(s^*)} a(u)du\right\} \\
&+ (M_k + K_2 \exp\{(\alpha+\beta-\gamma)s_k\}) \exp\left\{-\int_{s_0}^{h(s^*)} a(u)du - \gamma s^*\right\} \\
&\times \exp\left\{\gamma s + \int_{s_0}^{h(s)} a(u)du\right\}\Big|_{s_k}^{s^*}(1 + K_3 \exp\{-\omega s_k\}) \\
\le{}& M_k(1 + K_3\exp\{-\omega s_k\}) + K_2\exp\{(\alpha+\beta-\gamma)s_k\}(1 + K_3\exp\{-\omega s_k\}) \\
\le{}& M_k^*(1 + N\exp\{-\kappa s_k\}),
\end{aligned}
\tag{3.12}
$$

where $M_k^* := \max(M_k, K_2)$, $\kappa := \min(\omega, \gamma - \alpha - \beta) > 0$, and $N > 0$ is a constant large enough. Since $s^* \in J_{k+1}$ was arbitrary,

$$
M_{k+1}^* \le M_k^*(1 + N\exp\{-\kappa s_k\}) \le M_1^* \prod_{j=1}^{k}(1 + N\exp\{-\kappa s_j\}). \tag{3.13}
$$

Now, the boundedness of (M_k^*) as $k \to \infty$ implies via substitution (3.2) the asymptotic estimate (3.1). □

Remark 3.2. This remark concerns the possible extension of our results to differential equations with delays intersecting the identity at the initial point t_0. These equations form a wide and natural class of delay differential equations (see the following examples) and have many applications (some of them have been mentioned in Section 1). Since we are interested in the behaviour at infinity, it is obvious that the main notions and results of this paper can be easily reformulated to this case.

Example 3.3. Consider the equation

$$\dot{y}(t) = -\left(a + c\exp\{-t\}\right)y(t) + \sum_{i=1}^{n} b_i y(\lambda_i t) + f(t), \quad t \geq 0, \tag{3.14}$$

where $a > 0$, $c \geq 0$, $b_i \neq 0$, $0 < \lambda_i < 1$, $i = 1,\dots,n$ are constants and $f \in C([0,\infty))$ fulfils $f(t) = O(t^{\beta})$ as $t \to \infty$. Let $\psi(t) = \log t$, then functions $\{\lambda_1 t,\dots,\lambda_n t\}$ can be embedded into an iteration group $[\psi]$. Indeed,

$$\psi(\lambda_i t) = \psi(t) - \log \lambda_i^{-1}, \quad t > 0,\ i = 1,\dots,n. \tag{3.15}$$

Then, by Theorem 3.1, the estimate

$$y(t) = O(t^{\gamma}) \quad \text{as } t \longrightarrow \infty,\ \gamma > \max\left(\beta, \frac{\log \sum_{i=1}^{n} |b_i|/a}{\log \lambda_1^{-1}},\dots,\frac{\log \sum_{i=1}^{n} |b_i|/a}{\log \lambda_n^{-1}}\right) \tag{3.16}$$

holds for any solution y of (3.14).

4. Some particular cases of (1.1)

In this section, we first consider (1.1) in the homogeneous form

$$\dot{y}(t) = -a(t)y(t) + \sum_{i=1}^{n} b_i(t) y(\tau_i(t)), \quad t \in I. \tag{4.1}$$

Using a simple modification of the proof of Theorem 3.1, we improve the conclusion of this theorem for the case of (4.1). We assume that all the assumptions of Theorem 3.1 are valid (the assumptions on f are missing, of course). Using the same notation as in Theorem 3.1, we have the following theorem.

Theorem 4.1. *Let y be a solution of (4.1). Then*

$$y(t) = O\left(\exp\{\gamma\psi(t)\}\right) \quad \text{as } t \longrightarrow \infty,\ \gamma = \max\left(\frac{\log M}{d_1},\dots,\frac{\log M}{d_n}\right). \tag{4.2}$$

Proof. Following the proof of Theorem 3.1, we can see that the condition on γ in (3.1) becomes

$$\gamma > \max\left(\frac{\log M}{d_1},\dots,\frac{\log M}{d_n}\right) \tag{4.3}$$

in view of $f \equiv 0$ on I. Moreover, the inequality (3.8) can be replaced by

$$\sum_{i=1}^{n} |b_i(h(s))| \exp\{-\gamma d_i\} \le a(h(s)), \tag{4.4}$$

and this implies the validity of (4.2). □

Now, we consider (1.1) in another special form

$$\dot{y}(t) = -ay(t) + by(\tau(t)) + f(t), \quad t \in I, \tag{4.5}$$

where $a > 0$, $b \neq 0$ are constants, $\tau \in C^1(I)$, $\tau(t) < t$, $0 < \dot{\tau}(t) \le \lambda < 1$ for all $t \in I$, $\tau(t) \to \infty$ as $t \to \infty$, and $f \in C(I)$ fulfils $f(t) = O(\exp\{\beta\psi(t)\})$ as $t \to \infty$. Under these assumptions on τ, there exists a function $\psi \in C^1(I)$, $\dot{\psi} > 0$ on I such that

$$\psi(\tau(t)) = \psi(t) - \log\lambda^{-1}, \quad t \in I \tag{4.6}$$

(for this and related results concerning (4.6) see, e.g., [10]). Then applying Theorem 3.1 to (4.5), we can easily deduce that the property

$$y(t) = O(\exp\{\gamma\psi(t)\}) \quad \text{as } t \longrightarrow \infty,\ \gamma > \max\left(\alpha + \beta, \frac{\log(|b|/a)}{\log\lambda^{-1}}\right) \tag{4.7}$$

holds for any solution y of (4.5).

The asymptotic behaviour of (4.5) has been studied in [3]. If we put

$$\sigma := \frac{\log(|b|/a)}{\log\lambda^{-1}}, \tag{4.8}$$

then using the previous notation we can recall the following result.

Theorem 4.2 [3, Theorem 2.3]. *Consider (4.5), where $a > 0$, $b \neq 0$ are constants, $\tau, f \in C^1(I)$, $\tau(t) < t$, $0 < \dot{\tau}(t) \le \lambda < 1$ for all $t \in I$, $\tau(t) \to \infty$, $f(t) = O(\exp\{\beta\psi(t)\})$, and $\dot{f}(t) = O(\exp\{(\beta - 1)\psi(t)\})$ as $t \to \infty$. If y is a solution of (4.5), then*

$$y(t) = \begin{cases} O(\exp\{\sigma\psi(t)\}) & \text{as } t \longrightarrow \infty \text{ if } \beta < \sigma, \\ O(\exp\{\sigma\psi(t)\}\psi(t)) & \text{as } t \longrightarrow \infty \text{ if } \beta = \sigma, \\ O(\exp\{\beta\psi(t)\}) & \text{as } t \longrightarrow \infty \text{ if } \beta > \sigma. \end{cases} \tag{4.9}$$

It is easy to see that relations (4.9) yield sharper estimates of solutions than (4.7). On the other hand, we emphasize that the proof technique used in [3] is effective just for (4.5) and cannot be applied to more general equation (1.1). In the final part of this paper, we propose a simple way on how to extend the conclusions of Theorem 4.2 to some equation (4.5) with nonconstant coefficients. To explain the main idea, we consider (4.5), where the delayed argument is a power function.

Example 4.3. We consider the delay equation

$$\dot{y}(t) = -ay(t) + by(t^{\lambda}) + f(t), \quad t \ge 1, \tag{4.10}$$

where $a > 0$, $b \neq 0$, $0 < \lambda < 1$ are constants and $f \in C^1([1,\infty))$ fulfils the properties

$$f(t) = O((\log t)^{\beta}), \quad \dot{f}(t) = O((\log t)^{\beta-1}) \quad \text{as } t \longrightarrow \infty. \tag{4.11}$$

The corresponding Abel equation (4.6) has the form

$$\psi(t^{\lambda}) = \psi(t) - \log\lambda^{-1}, \quad t \geq 1, \tag{4.12}$$

and admits the function $\psi(t) = \log\log t$ as a solution with the required properties. Substituting this ψ into assumptions and conclusions of Theorem 4.2, we obtain the following result, where σ is given by (4.8), if y is a solution of (4.10), then

$$y(t) = \begin{cases} O((\log t)^{\sigma}) & \text{as } t \longrightarrow \infty \text{ if } \beta < \sigma, \\ O((\log t)^{\sigma} \log\log t) & \text{as } t \longrightarrow \infty \text{ if } \beta = \sigma, \\ O((\log t)^{\beta}) & \text{as } t \longrightarrow \infty \text{ if } \beta > \sigma. \end{cases} \tag{4.13}$$

Now, we consider the equation

$$\dot{y}(t) = -\frac{a}{t}y(t) + \frac{b}{t}y(t^{\lambda}) + f(t), \quad t \geq 1, \tag{4.14}$$

where a, b, and λ are the same as above and $f \in C^1([1,\infty))$.

Setting

$$s = \log t, \quad z(s) = y(t), \tag{4.15}$$

we can convert (4.14) into the form

$$z'(s) = -az(s) + bz(\lambda s) + f(\exp\{s\})\exp\{s\}, \quad s \geq 0. \tag{4.16}$$

Now if the forcing term in (4.16) fulfils the required asymptotic properties, then applying Theorem 4.2 to (4.16) and substituting this back into (4.15), we get that relations (4.13) are valid for any solution y of (4.14).

Remark 4.4. Following Example 4.3, we can extend asymptotic estimates (4.13) also to some other equations of the form

$$\dot{y}(t) = -\dot{\varphi}(t)[ay(t) - by(t^{\lambda})] + f(t), \quad t \geq 1, \tag{4.17}$$

where $\varphi \in C^1([1,\infty))$ and $\dot{\varphi} > 0$ on $[1,\infty)$. If we introduce the change of variables

$$s = \varphi(t), \quad z(s) = y(t), \tag{4.18}$$

then (4.17) can be transformed into

$$z'(s) = -az(s) + bz(\mu(s)) + f(\varphi^{-1}(s))(\varphi^{-1})'(s), \tag{4.19}$$

where $\mu(s) = \varphi((\varphi^{-1}(s))^{\lambda})$, $s \in \varphi(I)$. Now if the delayed argument and the forcing term in (4.19) fulfil the assumptions of Theorem 4.2, then we can apply this theorem to (4.19) and via substituting (4.18) obtain the validity of (4.13) for any solution y of (4.17).

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COMPARISON OF DIFFERENTIAL REPRESENTATIONS FOR RADIALLY SYMMETRIC STOKES FLOW

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Papkovich and Neuber (PN), and Palaniappan, Nigam, Amaranath, and Usha (PNAU) proposed two different representations of the velocity and the pressure fields in Stokes flow, in terms of harmonic and biharmonic functions, which form a practical tool for many important physical applications. One is the particle-in-cell model for Stokes flow through a swarm of particles. Most of the analytical models in this realm consider spherical particles since for many interior and exterior flow problems involving small particles, spherical geometry provides a very good approximation. In the interest of producing ready-to-use basic functions for Stokes flow, we calculate the PNAU and the PN eigensolutions generated by the appropriate eigenfunctions, and the full series expansion is provided. We obtain connection formulae by which we can transform any solution of the Stokes system from the PN to the PNAU eigenform. This procedure shows that any PNAU eigenform corresponds to a combination of PN eigenfunctions, a fact that reflects the flexibility of the second representation. Hence, the advantage of the PN representation as it compares to the PNAU solution is obvious. An application is included, which solves the problem of the flow in a fluid cell filling the space between two concentric spherical surfaces with Kuwabara-type boundary conditions.

1. Introduction

Slow motion of a mass of particles relative to a viscous fluid has been studied extensively because of its importance in practical applications. In order to construct tractable mathematical models of the flow systems involving particles, it is necessary to conform to a number of simplifications. A dimensionless criterion, which determines the relative importance of inertial and viscous effects, is the Reynolds number [3]. Stokes equations for the steady flow of a viscous, incompressible fluid at small Reynolds number (creeping flow) have been known for over one and a half centuries (1851). They connect the vector velocity with the scalar total pressure field [3]. The total pressure and vorticity fields are harmonic, while the velocity is biharmonic and divergence-free.

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Complications often arise because of the complex geometry encountered in assemblages composed of particles of arbitrary shape. There are many efficient methods in use to solve this kind of problems with Stokes flow, such as numerical computation, streamfunction techniques, and analytic-function methods [10]. One of the largest physical areas of importance concerns the construction of particle-in-cell models which are useful in the development of simple but reliable analytical expressions for heat and mass transfer in swarms of particles. The technique of cell models is based on the idea according to which a large enough porous concentration of particles within a fluid can be represented by many separate unit cells where every cell contains one particle. Thus, the consideration of a full-dimensional porous media is being referred to as that of a single particle and its fluid cover. That way, the mathematical formulation of any physical problem is significantly simplified. For many interior and exterior flow problems involving small particles, spherical geometry [6] provides very good approximation and stands for the simplest geometry that can be employed. Although relative physical problems enjoy rotational symmetry, we retain the nonaxisymmetric character of three-dimensional (3D) flows.

The introduction of differential representations of the solutions of Stokes equations [1, 8, 9, 10] serves to unify our own approach on all 3D incompressible fluid motions. Based on the previous formulation of cell models, the problem is now focused on the use of the appropriate representation that coincides with the physical problem. The major advantage of the differential representations is that they provide us with the flow fields for Stokes flow in terms of harmonic potentials. The most famous general spatial solutions are the PN solution [8, 10], the Boussinesq-Galerkin solution [1, 10], and the PNAU solution [9]. Recently, a method of connecting 3D differential representations has been developed [2], where the PN and the Boussinesq-Galerkin differential representations were interrelated and connection formulae between the corresponding spherical harmonic and biharmonic potentials were developed.

Here we are interested in the connection of the PN solution with the PNAU representation in spherical coordinates. This is made possible by connecting the appropriate eigenfunctions that generate the flow fields through these representations. Our aim is to calculate the nonaxisymmetric flow fields, generated by the vector spherical harmonic eigenfunctions [4, 7], through the PN representation and then to face the inverse problem of determining those vector spherical harmonic and biharmonic eigenfunctions [4, 6, 10], which lead to the same velocity and total pressure fields via the PNAU representation. Furthermore, both the internal and the external flow problems are being examined. The above procedure cannot be inverted as a consequence of the flexibility that the PN representation enjoys as it compares to the PNAU solution. This indicates that the use of the PN differential representation forms a more complete way to solve particle-in-cell flow problems.

As a demonstration of the usefulness and the possibilities offered by the PN representation, we derive the solution of the problem of creeping flow through a swarm of stationary spherical particles, embedded within an otherwise quiescent Newtonian fluid that moves with constant uniform velocity in the axial direction using the Kuwabara-type boundary conditions [5].

2. Fundamentals of stokes flow

Stokes flow which is characterized by steady, nonaxisymmetric 3D, creeping ($\mathrm{Re} \ll 1$), incompressible (density $\varrho = \text{const}$), and viscous (dynamic viscosity $\mu = \text{const}$) motion around particles embedded within smooth, bounded domains $\Omega(\mathbb{R}^3)$ is governed by the following set of partial differential equations [3]:

$$\mu\Delta\mathbf{v}(\mathbf{r}) - \nabla \mathrm{P}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{2.1}$$

$$\nabla \cdot \mathbf{v}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{2.2}$$

where $\mathbf{v}(\mathbf{r})$ is the biharmonic velocity field, $\mathrm{P}(\mathbf{r})$ is the harmonic total pressure field, and $\mathbf{r}$ stands for the position vector. An immediate consequence of (2.1) is that, for creeping flow, the generated pressure is compensated by the viscous forces while equation (2.2) secures the incompressibility of the fluid. Once the velocity field is obtained, the harmonic vorticity field is defined as

$$\boldsymbol{\omega}(\mathbf{r}) = \nabla \times \mathbf{v}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{2.3}$$

Papkovich and Neuber [8] proposed the following 3D differential representation of the solutions for Stokes flow, in terms of the harmonic potentials $\boldsymbol{\Phi}(\mathbf{r})$ and $\Phi_0(\mathbf{r})$:

$$\begin{aligned} \mathbf{v}^{\mathrm{PN}}(\mathbf{r}) &= \boldsymbol{\Phi}(\mathbf{r}) - \frac{1}{2}\nabla\left(\mathbf{r} \cdot \boldsymbol{\Phi}(\mathbf{r}) + \Phi_0(\mathbf{r})\right), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \\ \mathrm{P}^{\mathrm{PN}}(\mathbf{r}) &= \mathrm{P}_0^{\mathrm{PN}} - \mu\nabla \cdot \boldsymbol{\Phi}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \end{aligned} \tag{2.4}$$

whereas $\mathrm{P}_0^{\mathrm{PN}}$ is a constant pressure and

$$\Delta\boldsymbol{\Phi}(\mathbf{r}) = \mathbf{0}, \quad \Delta\Phi_0(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{2.5}$$

On the other hand, Palaniappan et al. [9] assumed another 3D differential representation for the solutions of Stokes equations as a function of the harmonic and biharmonic potentials $\mathrm{A}(\mathbf{r})$ and $\mathrm{B}(\mathbf{r})$, respectively:

$$\begin{aligned} \mathbf{v}^{\mathrm{PNAU}}(\mathbf{r}) &= \nabla \times \nabla \times \left(\mathbf{r}\mathrm{A}(\mathbf{r})\right) + \nabla \times \left(\mathbf{r}\mathrm{B}(\mathbf{r})\right), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \\ \mathrm{P}^{\mathrm{PNAU}}(\mathbf{r}) &= \mathrm{P}_0^{\mathrm{PNAU}} + \mu(1 + \mathbf{r} \cdot \nabla)\Delta\mathrm{A}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \end{aligned} \tag{2.6}$$

where $\mathrm{P}_0^{\mathrm{PNAU}}$ is a constant pressure, while

$$\Delta^2\mathrm{A}(\mathbf{r}) = 0, \quad \Delta\mathrm{B}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{2.7}$$

and Δ and ∇ stand for the Laplacian and the gradient operators, respectively.

In what follows, we find the interrelation of these differential representations in order to obtain connection formulae between the spherical harmonic $(\boldsymbol{\Phi}, \Phi_0, \mathrm{B})$ and biharmonic (A) eigenfunctions. Putting it in a different way, given an eigenmode of one of the representations, we look for the particular combination of eigenmodes of the other representation that generates the same velocity and total pressure fields. Initially, the physically important internal and external fields of the velocity and the total pressure $(\mathbf{v}, \mathrm{P})$ are constructed using the representations (2.4) and (2.6). Since spherical geometry is employed, we are using vector spherical harmonics [7] in order to simplify our calculations.

Furthermore, in order to demonstrate the usefulness of the PN differential representation, we use it to solve the Stokes flow problem within a fluid cell limited between two concentric spherical surfaces. In this way, we are led to recover the solution of the Kuwabara-type problem [5] for the small Reynolds number flow around spheres embedded in a viscous fluid.

3. Vector spherical harmonic and biharmonic eigenfunctions

Introducing the spherical coordinate system [6] ($\zeta = \cos\theta$, $-1 \le \zeta \le 1$),

$$x_1 = r\sqrt{1-\zeta^2}\cos\varphi, \qquad x_2 = r\sqrt{1-\zeta^2}\sin\varphi, \qquad x_3 = r\zeta, \tag{3.1}$$

where $0 \le r < +\infty$, $0 \le \theta \le \pi$, and $0 \le \varphi < 2\pi$, we define the sphere B_r for $r > 0$ as the set

$$B_r = \{\mathbf{r} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \le r^2\}. \tag{3.2}$$

The outward unit normal vector on the surface of the sphere $r = r_0$ is furnished by the formula

$$\hat{\mathbf{n}}(r_0,\zeta,\varphi) = \left(\sqrt{1-\zeta^2}\cos\varphi, \sqrt{1-\zeta^2}\sin\varphi, \zeta\right) = \frac{\mathbf{r}(r_0,\zeta,\varphi)}{r_0}, \tag{3.3}$$

where for any nondegenerate sphere B_{r_0}, we have $r_0 > 0$. Furthermore, $|\zeta| \le 1$. The differential operators ∇ and Δ, in spherical coordinates, assume the forms

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} - \frac{\sqrt{1-\zeta^2}}{r}\hat{\boldsymbol{\zeta}}\frac{\partial}{\partial \zeta} + \frac{1}{r\sqrt{1-\zeta^2}}\hat{\boldsymbol{\varphi}}\frac{\partial}{\partial \varphi}, \tag{3.4}$$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \zeta}\left[(1-\zeta^2)\frac{\partial}{\partial \zeta}\right] + \frac{1}{r^2(1-\zeta^2)}\frac{\partial^2}{\partial \varphi^2}, \tag{3.5}$$

while $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\zeta}}$, and $\hat{\boldsymbol{\varphi}}$ stand for the coordinate unit vectors of our system for $r > 0$ and $|\zeta| \le 1$.

For every value of $n = 0, 1, 2, \ldots$, there exist $(2n+1)$ linearly independent spherical surface harmonics [4] given by

$$Y_n^{ms}(\hat{\mathbf{r}}) = P_n^m(\zeta)\begin{cases}\cos m\varphi, & s = e,\\ \sin m\varphi, & s = o,\end{cases} \tag{3.6}$$

for $m = 0, 1, 2, \ldots, n$, $|\zeta| \le 1$, $\varphi \in [0, 2\pi)$, where

$$\oint_{S^2} Y_n^{ms}(\hat{\mathbf{r}})Y_{n'}^{m's'}(\hat{\mathbf{r}})dS(\hat{\mathbf{r}}) = \frac{4\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{nn'}\delta_{mm'}\delta_{ss'}\frac{1}{\varepsilon_m}, \tag{3.7}$$

with δ_{ij}, $i = n, m, s$, $j = n', m', s'$, the Kronecker delta, ε_m the Neumann factor ($\varepsilon_m = 1$, $m = 0$, and $\varepsilon_m = 2$, $m \ge 1$), and s denoting the even (e) or the odd (o) character of the spherical surface harmonics; $P_n^m = P_n^m(\zeta)$ are the associated Legendre functions of the first kind [4] given by the relation

$$P_n^m(\zeta) = \frac{(1-\zeta^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\zeta^{n+m}} (\zeta^2 - 1)^n, \quad |\zeta| < 1, \tag{3.8}$$

where n denotes the degree and m the order.

In spherical coordinates, the linear space of harmonic functions can be expressed via the complete set of internal and external solid spherical harmonics, that is,

$$\Delta g(\mathbf{r}) = 0 \Longleftrightarrow g(\mathbf{r}) = \begin{cases} r^n Y_n^{ms}(\hat{\mathbf{r}}), \\ r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}}), \end{cases} \tag{3.9}$$

for $n \geq 0$, $m = 0, 1, \ldots, n$, and $s = e, o$. Similarly, according to the representation theorem of Almansi (1897) [10], every biharmonic function permits an appropriate decomposition into two harmonic functions $h_1(\mathbf{r})$ and $h_2(\mathbf{r})$, that is,

$$h(\mathbf{r}) = h_1(\mathbf{r}) + r^2 h_2(\mathbf{r}) \quad \text{with } \Delta h_1(\mathbf{r}) = \Delta h_2(\mathbf{r}) = 0. \tag{3.10}$$

For every $-1 \leq \zeta \leq 1$ and $\varphi \in [0, 2\pi)$, the vector spherical surface harmonics [7] which are defined by the relations

$$\mathbf{P}_n^{ms}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_n^{ms}(\hat{\mathbf{r}}), \tag{3.11}$$

$$\mathbf{B}_n^{ms}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{n(n+1)}} \left[-\sqrt{1-\zeta^2}\, \hat{\boldsymbol{\zeta}} \frac{\partial}{\partial \zeta} + \frac{1}{\sqrt{1-\zeta^2}} \hat{\boldsymbol{\varphi}} \frac{\partial}{\partial \varphi} \right] Y_n^{ms}(\hat{\mathbf{r}}), \tag{3.12}$$

$$\mathbf{C}_n^{ms}(\hat{\mathbf{r}}) = -\frac{1}{\sqrt{n(n+1)}} \hat{\mathbf{r}} \times \left[-\sqrt{1-\zeta^2}\, \hat{\boldsymbol{\zeta}} \frac{\partial}{\partial \zeta} + \frac{1}{\sqrt{1-\zeta^2}} \hat{\boldsymbol{\varphi}} \frac{\partial}{\partial \varphi} \right] Y_n^{ms}(\hat{\mathbf{r}}), \tag{3.13}$$

for any $n \geq 0$, $m = 0, 1, \ldots, n$, and $s = e, o$, are pointwise perpendicular; that is,

$$\mathbf{P}_n^{ms} \cdot \mathbf{C}_n^{ms} = \mathbf{C}_n^{ms} \cdot \mathbf{B}_n^{ms} = \mathbf{B}_n^{ms} \cdot \mathbf{P}_n^{ms} = 0. \tag{3.14}$$

Moreover they satisfy the orthogonality relations

$$\begin{aligned} \oint_{S^2} \mathbf{P}_n^{ms}(\hat{\mathbf{r}}) \cdot \mathbf{P}_{n'}^{m's'}(\hat{\mathbf{r}}) dS(\hat{\mathbf{r}}) &= \oint_{S^2} \mathbf{B}_n^{ms}(\hat{\mathbf{r}}) \cdot \mathbf{B}_{n'}^{m's'}(\hat{\mathbf{r}}) dS(\hat{\mathbf{r}}) \\ &= \oint_{S^2} \mathbf{C}_n^{ms}(\hat{\mathbf{r}}) \cdot \mathbf{C}_{n'}^{m's'}(\hat{\mathbf{r}}) dS(\hat{\mathbf{r}}) \\ &= \frac{4\pi}{2n+1} \frac{(n+m)}{(n-m)} \delta_{nn'} \delta_{mm'} \delta_{ss'} \frac{1}{\varepsilon_m}, \end{aligned} \tag{3.15}$$

where

$$\varepsilon_m = \begin{cases} 1, & m = 0, \\ 2, & m \geq 1. \end{cases} \tag{3.16}$$

Thus, for any $\mathbf{r} \in \Omega(\mathbb{R}^3)$, the internal vector spherical harmonics [7] are provided as

$$\mathbf{N}_n^{(i)ms}(\mathbf{r}) = \nabla\left(r^{n+1} Y_{n+1}^{ms}(\hat{\mathbf{r}})\right) = \sqrt{(n+1)(n+2)}r^n \mathbf{B}_{n+1}^{ms}(\hat{\mathbf{r}}) + (n+1)r^n \mathbf{P}_{n+1}^{ms}(\hat{\mathbf{r}}) \tag{3.17}$$

for $n = 0,1,2,\ldots$, $m = 0,1,\ldots,n+1$, and $s = e,o$;

$$\mathbf{M}_n^{(i)ms}(\mathbf{r}) = \nabla \times \left(\mathbf{r} r^n Y_n^{ms}(\hat{\mathbf{r}})\right) = \sqrt{n(n+1)} r^n \mathbf{C}_n^{ms}(\hat{\mathbf{r}}) \tag{3.18}$$

for $n = 1,2,\ldots$, $m = 0,1,\ldots,n$, and $s = e,o$;

$$\mathbf{G}_n^{(i)ms}(\mathbf{r}) = r^{2n+1} \mathbf{N}_n^{(e)ms}(\mathbf{r}) = \sqrt{n(n-1)} r^n \mathbf{B}_{n-1}^{ms}(\hat{\mathbf{r}}) - n r^n \mathbf{P}_{n-1}^{ms}(\hat{\mathbf{r}}) \tag{3.19}$$

for $n = 0,1,2,\ldots$, $m = 0,1,\ldots,n-1$, and $s = e,o$. On the other hand, the external vector spherical harmonics [7] assume the forms

$$\mathbf{N}_n^{(e)ms}(\mathbf{r}) = \nabla\left(r^{-n} Y_{n-1}^{ms}(\hat{\mathbf{r}})\right) = \sqrt{n(n-1)} r^{-(n+1)} \mathbf{B}_{n-1}^{ms}(\hat{\mathbf{r}}) - n r^{-(n+1)} \mathbf{P}_{n-1}^{ms}(\hat{\mathbf{r}}) \tag{3.20}$$

for $n = 1,2,\ldots$, $m = 0,1,\ldots,n-1$, and $s = e,o$;

$$\mathbf{M}_n^{(e)ms}(\mathbf{r}) = \nabla \times \left(\mathbf{r} r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right) = \sqrt{n(n+1)} r^{-(n+1)} \mathbf{C}_n^{ms}(\hat{\mathbf{r}}) \tag{3.21}$$

for $n = 1,2,\ldots$, $m = 0,1,\ldots,n$, and $s = e,o$;

$$\begin{aligned}\mathbf{G}_n^{(e)ms}(\mathbf{r}) &= r^{-(2n+1)} \mathbf{N}_n^{(i)ms}(\mathbf{r}) \\ &= \sqrt{(n+1)(n+2)} r^{-(n+1)} \mathbf{B}_{n+1}^{ms}(\hat{\mathbf{r}}) + (n+1) r^{-(n+1)} \mathbf{P}_{n+1}^{ms}(\hat{\mathbf{r}})\end{aligned} \tag{3.22}$$

for $n = 0,1,2,\ldots$, $m = 0,1,\ldots,n+1$, and $s = e,o$. Then, the following complete expansion of any vector function $\mathbf{u}(\mathbf{r})$ which belongs to the kernel space of the operator Δ is obtained:

$$\begin{aligned}\mathbf{u}(\mathbf{r}) = & \sum_{s=e,o} a_0^{(i)0s} \mathbf{N}_0^{(i)0s}(\mathbf{r}) + \sum_{s=e,o} a_0^{(i)1s} \mathbf{N}_0^{(i)1s}(\mathbf{r}) \\ & + \sum_{s=e,o} c_0^{(e)0s} \mathbf{G}_0^{(e)0s}(\mathbf{r}) + \sum_{s=e,o} c_0^{(e)1s} \mathbf{G}_0^{(e)1s}(\mathbf{r}) \\ & + \sum_{n=1}^{\infty} \sum_{m=0}^{n+1} \sum_{s=e,o} a_n^{(i)ms} \mathbf{N}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \sum_{s=e,o} a_n^{(e)ms} \mathbf{N}_n^{(e)ms}(\mathbf{r}) \\ & + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{s=e,o} b_n^{(i)ms} \mathbf{M}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{s=e,o} b_n^{(e)ms} \mathbf{M}_n^{(e)ms}(\mathbf{r}) \\ & + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \sum_{s=e,o} c_n^{(i)ms} \mathbf{G}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty} \sum_{m=0}^{n+1} \sum_{s=e,o} c_n^{(e)ms} \mathbf{G}_n^{(e)ms}(\mathbf{r})\end{aligned} \tag{3.23}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$. In the interest of making this work more complete and independent, we provide in an appendix some relations between the vector spherical harmonics. The relevant information and recurrence relations for the associated Legendre functions of the first kind can be found in [4].

4. PN eigenflows

In view of equations (2.4), (2.5) and (3.9), (3.23), the harmonic eigenfunctions $\mathbf{\Phi}(\mathbf{r})$ and $\Phi_0(\mathbf{r})$, $\mathbf{r} \in \Omega(\mathbb{R}^3)$, with constant coefficients $a_n^{(i)ms}$, $b_n^{(i)ms}$, $c_n^{(i)ms}$, $a_n^{(e)ms}$, $b_n^{(e)ms}$, $c_n^{(e)ms}$, and $d_n^{(i)ms}$, $d_n^{(e)ms}$, respectively,

$$\begin{aligned}\mathbf{\Phi}(\mathbf{r}) = &\sum_{n=0}^{\infty}\sum_{m=0}^{n+1}\sum_{s=e,o} a_n^{(i)ms}\mathbf{N}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} b_n^{(i)ms}\mathbf{M}_n^{(i)ms}(\mathbf{r}) \\ &+ \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{s=e,o} c_n^{(i)ms}\mathbf{G}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{s=e,o} a_n^{(e)ms}\mathbf{N}_n^{(e)ms}(\mathbf{r}) \\ &+ \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} b_n^{(e)ms}\mathbf{M}_n^{(e)ms}(\mathbf{r}) + \sum_{n=0}^{\infty}\sum_{m=0}^{n+1}\sum_{s=e,o} c_n^{(e)ms}\mathbf{G}_n^{(e)ms}(\mathbf{r}),\end{aligned} \tag{4.1}$$

$$\Phi_0(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} d_n^{(i)ms}\left(r^n Y_n^{ms}(\hat{\mathbf{r}})\right) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} d_n^{(e)ms}\left(r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right) \tag{4.2}$$

generate the velocity and total pressure fields $\mathbf{v}^{\mathrm{PN}}$, P^{PN}. In terms of (3.11)–(3.22) and (A.1)–(A.11), the PN flow fields are written as

$$\begin{aligned}\mathbf{v}^{\mathrm{PN}}(\mathbf{r}) = &\sum_{n=0}^{\infty}\sum_{m=0}^{n+1}\sum_{s=e,o}\left[-\frac{(n-1)}{2}a_n^{(i)ms} - \frac{1}{2}d_{n+1}^{(i)ms} + \frac{(n+2)(2n+5)}{2(2n+3)}c_{n+2}^{(i)ms}r^2\right]\mathbf{N}_n^{(i)ms}(\mathbf{r}) \\ &+ \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{s=e,o}\left[\frac{(n+2)}{2}a_n^{(e)ms} - \frac{1}{2}d_{n-1}^{(e)ms} - \frac{(n-1)(2n-3)}{2(2n-1)}c_{n-2}^{(e)ms}r^2\right]\mathbf{N}_n^{(e)ms}(\mathbf{r}) \\ &+ \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\left[b_n^{(i)ms}\right]\mathbf{M}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\left[b_n^{(e)ms}\right]\mathbf{M}_n^{(e)ms}(\mathbf{r}) \\ &+ \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{s=e,o}\left[\frac{(n-1)}{(2n-1)}c_n^{(i)ms}\right]\mathbf{G}_n^{(i)ms}(\mathbf{r}) \\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{n+1}\sum_{s=e,o}\left[\frac{(n+2)}{(2n+3)}c_n^{(e)ms}\right]\mathbf{G}_n^{(e)ms}(\mathbf{r}),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\mathrm{P}^{\mathrm{PN}}(\mathbf{r}) = \mathrm{P}_0^{\mathrm{PN}} + \mu\Bigg\{&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}(n+1)(2n+3)c_{n+1}^{(i)ms}\left(r^n Y_n^{ms}(\hat{\mathbf{r}})\right) \\ &+ \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} n(2n-1)c_{n-1}^{(e)ms}\left(r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right)\Bigg\}\end{aligned} \tag{4.4}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$.

5. PNAU eigenflows

According to (2.6), (2.7) and (3.9), (3.10), (3.23), the biharmonic and harmonic eigenfunctions $\mathrm{A}(\mathbf{r})$ and $\mathrm{B}(\mathbf{r})$, $\mathbf{r} \in \Omega(\mathbb{R}^3)$, with constant coefficients $f_n^{(i)ms}$, $g_n^{(i)ms}$, $f_n^{(e)ms}$, $g_n^{(e)ms}$, and $e_n^{(i)ms}$, $e_n^{(e)ms}$, respectively,

$$\begin{aligned}\mathrm{A}(\mathbf{r}) = {} & \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} f_n^{(i)ms}\left(r^n Y_n^{ms}(\hat{\mathbf{r}})\right) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} f_n^{(e)ms}\left(r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right) \\ & + r^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} g_n^{(i)ms}\left(r^n Y_n^{ms}(\hat{\mathbf{r}})\right) + r^2 \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} g_n^{(e)ms}\left(r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right),\end{aligned} \tag{5.1}$$

$$\mathrm{B}(\mathbf{r}) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} e_n^{(i)ms}\left(r^n Y_n^{ms}(\hat{\mathbf{r}})\right) + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} e_n^{(e)ms}\left(r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right) \tag{5.2}$$

generate the PNAU velocity and total pressure fields $\mathbf{v}^{\mathrm{PNAU}}$ and $\mathrm{P}^{\mathrm{PNAU}}$ by virtue of (3.11)–(3.22) as well as (A.1)–(A.11). That is,

$$\begin{aligned}\mathbf{v}^{\mathrm{PNAU}}(\mathbf{r}) = {} & \sum_{n=0}^{\infty}\sum_{m=0}^{n+1}\sum_{s=e,o}\left[(n+2) f_{n+1}^{(i)ms} + \frac{(n+2)(2n+5)}{(2n+3)} g_{n+1}^{(i)ms} r^2\right]\mathbf{N}_n^{(i)ms}(\mathbf{r}) \\ & + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{s=e,o}\left[-(n-1) f_{n-1}^{(e)ms} - \frac{(n-1)(2n-3)}{(2n-1)} g_{n-1}^{(e)ms} r^2\right]\mathbf{N}_n^{(e)ms}(\mathbf{r}) \\ & + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\left[e_n^{(i)ms}\right]\mathbf{M}_n^{(i)ms}(\mathbf{r}) + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o}\left[e_n^{(e)ms}\right]\mathbf{M}_n^{(e)ms}(\mathbf{r}) \\ & + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{s=e,o}\left[\frac{2(n-1)}{(2n-1)} g_{n-1}^{(i)ms}\right]\mathbf{G}_n^{(i)ms}(\mathbf{r}) \\ & + \sum_{n=0}^{\infty}\sum_{m=0}^{n+1}\sum_{s=e,o}\left[\frac{2(n+2)}{(2n+3)} g_{n+1}^{(e)ms}\right]\mathbf{G}_n^{(e)ms}(\mathbf{r}),\end{aligned} \tag{5.3}$$

$$\begin{aligned}\mathrm{P}^{\mathrm{PNAU}}(\mathbf{r}) = \mathrm{P}_0^{\mathrm{PNAU}} + \mu\Bigg\{ & \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} 2(n+1)(2n+3) g_n^{(i)ms}\left(r^n Y_n^{ms}(\hat{\mathbf{r}})\right) \\ & + \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{s=e,o} 2n(2n-1) g_n^{(e)ms}\left(r^{-(n+1)} Y_n^{ms}(\hat{\mathbf{r}})\right)\Bigg\},\end{aligned} \tag{5.4}$$

for every $\mathbf{r} \in \Omega(\mathbb{R}^3)$.

6. Comparison of the PN and PNAU representations

In this section, our aim is to find the exact harmonic and biharmonic potentials given by equations (4.1), (4.2) and (5.1), (5.2), which lead to the same velocity and total pressure fields. From this point of view, we look for connection formulae for the differential representations that secure the identities

$$\mathbf{v}^{\mathrm{PN}}(\mathbf{r}) = \mathbf{v}^{\mathrm{PNAU}}(\mathbf{r}), \quad \mathrm{P}^{\mathrm{PN}}(\mathbf{r}) = \mathrm{P}^{\mathrm{PNAU}}(\mathbf{r}), \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{6.1}$$

By virtue of (6.1), we proceed by interrelating the PN flow fields (4.3), (4.4) with the corresponding PNAU flow fields (5.3), (5.4). This correlation leads to connection formulae that interrelate the corresponding constant coefficients of the potentials (4.1), (4.2), (5.1), and (5.2). What is actually happening is that the connection of the velocity and total pressure fields has been transferred to the corresponding connection of the constant coefficients of the potentials. Indeed, after some calculations, we obtain the relations

$$c_{n+1}^{(i)ms} = 2g_n^{(i)ms} \quad \text{for } n = 0,1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.2}$$

$$c_{n-1}^{(e)ms} = 2g_n^{(e)ms} \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.3}$$

$$b_n^{(i)ms} = e_n^{(i)ms} \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.4}$$

$$b_n^{(e)ms} = e_n^{(e)ms} \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.5}$$

$$(n-2)a_{n-1}^{(i)ms} + d_n^{(i)ms} = -2(n+1)f_n^{(i)ms} \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.6}$$

$$(n+3)a_{n+1}^{(e)ms} - d_n^{(e)ms} = -2nf_n^{(e)ms} \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.7}$$

which establish the connection between the PN and PNAU representations at the coefficient level. The cases that do not follow the general relations (6.2)–(6.7) for $n = 0$ are treated separately. These concern the coefficients

$$g_0^{(e)0e},\ e_0^{(i)0e},\ e_0^{(e)0e},\ d_0^{(i)0e},\ f_0^{(i)0e},\ f_0^{(e)0e} \in \mathbb{R}. \tag{6.8}$$

Furthermore, the interrelation of the total pressures implies the equation of the constant pressures defined earlier, that is,

$$\mathrm{P}_0^{\mathrm{PN}} = \mathrm{P}_0^{\mathrm{PNAU}}. \tag{6.9}$$

Flows of zero vorticity are irrotational flows. Consequently, irrotational fields force the corresponding terms of the potentials, or of the flow fields, to vanish. Then, according to (4.3) and (5.3) of the velocity fields, in view of (2.3) and the relations (A.7), (A.9), and (A.11), the following constant coefficients are set to zero on the basis of orthogonality arguments:

$$c_{n+1}^{(i)ms} = g_n^{(i)ms} = 0 \quad \text{for } n = 0,1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.10}$$

$$c_{n-1}^{(e)ms} = g_n^{(e)ms} = 0 \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.11}$$

$$b_n^{(i)ms} = e_n^{(i)ms} = 0 \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o, \tag{6.12}$$

$$b_n^{(e)ms} = e_n^{(e)ms} = 0 \quad \text{for } n = 1,2,\ldots,\ m = 0,1,\ldots,n,\ s = e,o. \tag{6.13}$$

Even though the biharmonic part of the biharmonic potential A and the harmonic potential B are connected directly to the **G**-component and the **M**-component of the harmonic potential $\boldsymbol{\Phi}$, respectively, as shown from the general equations (6.2)–(6.5), the procedure of interrelation is not invertible. The reason for this lack of invertibility is due to the general connection relations (6.6) and (6.7), where the harmonic part of the biharmonic potential A is given through the **N**-component of the harmonic potential $\boldsymbol{\Phi}$ and through the harmonic potential Φ_0. The transformation from one representation to the other is not obtainable analytically in the sense that one can start with the PN differential representation and regain the results from the PNAU differential solution through the relations above, but one cannot come the opposite way since two sets of internal and external constant coefficients of the PN solution cannot be determined. Consequently, we deal with a higher number of degrees of freedom for the PN differential representation, a fact that implies the flexibility of the PN representation. In other words, for the same eigenflow, the PN representation lives in a higher-dimensional space than the PNAU one.

7. Application: the Kuwabara sphere-in-cell model

In order to demonstrate the usefulness of the PN differential representation ((2.4), (2.5) or (4.3), (4.4)), we use it to solve the axisymmetric Stokes flow problem through a swarm of stationary spherical particles, embedded within an otherwise quiescent Newtonian fluid that moves with constant uniform velocity in the polar direction. In other words, according to the idea of particle-in-cell models described in the introduction, we are interested in solving the creeping flow within a fluid cell limited between two concentric spherical surfaces.

Two concentric spheres are considered. The inner one, indicated by S_α, at $r = \alpha$, is solid and stationary. It lives within a spherical layer, which is confined by the outer sphere indicated by S_b, at $r = b$. A uniformly approaching velocity of magnitude U, in the negative direction of the x_3-axis, generates the axisymmetric flow in the fluid layer between the two spheres. The boundary conditions assume the forms

$$v_r = 0 \quad \text{on } r = \alpha, \tag{7.1}$$

$$v_\zeta = 0 \quad \text{on } r = \alpha, \tag{7.2}$$

$$v_r = -U\zeta \quad \text{on } r = b, \tag{7.3}$$

$$\omega_\varphi \equiv \hat{\boldsymbol{\varphi}} \cdot \boldsymbol{\omega} = 0 \quad \text{on } r = b, \tag{7.4}$$

where v_r and v_ζ are the r and ζ components of the axisymmetric PN velocity field and ω_φ refers to the φ component of the vorticity field given by (2.3). Equations (7.1) and (7.2) express the nonslip flow condition. Equation (7.3) implies that there is a flow across the boundary of the fluid envelope S_b. Furthermore, according to the Kuwabara argument, the vorticity is assumed to vanish on the external sphere, as shown by equation (7.4). This completes the statement of a well-posed boundary value problem.

Since the PN representation covers 3D flow fields, for 2D flows, as in our case, we are obliged to make a considerable reduction considering rotational symmetry. This is attainable and requires the same velocity field on every meridian plane. That is, the velocity is independent of the azimuthal angle φ:

$$\partial \mathbf{v}^{\mathrm{PN}}(\mathbf{r})\partial\varphi = \mathbf{0}, \quad \mathbf{r} \in \Omega(\mathbb{R}^3), \tag{7.5}$$

and its vector lives on a meridian plane:

$$\hat{\boldsymbol{\varphi}} \cdot \mathbf{v}^{\mathrm{PN}}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega(\mathbb{R}^3). \tag{7.6}$$

Now, imposing the axisymetric conditions (7.5) and (7.6) to our representation, the velocity field (4.3) is written in a suitable form:

$$\mathbf{v}^{\mathrm{PN}}(r,\zeta) = v_r^{\mathrm{PN}}(r,\zeta)\hat{\mathbf{r}} + v_\zeta^{\mathrm{PN}}(r,\zeta)\hat{\boldsymbol{\zeta}}, \quad r > 0,\ |\zeta| \le 1, \tag{7.7}$$

where the components of the velocity are expressed in terms of the radial component and Legendre functions of the first kind via

$$\begin{aligned} v_r^{\mathrm{PN}}(r,\zeta) = \sum_{n=0}^{\infty} \frac{1}{2}\Big\{ &(n+1)\left(\frac{(n+3)}{(2n+3)}\tilde{c}_{n+1}^{(e)} + \tilde{d}_n^{(e)}\right) r^{-(n+2)} + \frac{n(n+1)}{(2n-1)}\tilde{c}_{n-1}^{(e)} r^{-n} \\ &- n\left(\frac{(n-2)}{(2n-1)}\tilde{c}_{n-1}^{(i)} + \tilde{d}_n^{(i)}\right) r^{n-1} - \frac{n(n+1)}{(2n+3)}\tilde{c}_{n+1}^{(i)} r^{n+1}\Big\} P_n(\zeta), \end{aligned} \tag{7.8}$$

$$\begin{aligned} v_\zeta^{\mathrm{PN}}(r,\zeta) = \sum_{n=1}^{\infty} \frac{1}{2}\Big\{ &\left(\frac{(n+3)}{(2n+3)}\tilde{c}_{n+1}^{(e)} + \tilde{d}_n^{(e)}\right) r^{-(n+2)} + \frac{(n-2)}{(2n-1)}\tilde{c}_{n-1}^{(e)} r^{-n} \\ &+ \left(\frac{(n-2)}{(2n-1)}\tilde{c}_{n-1}^{(i)} + \tilde{d}_n^{(i)}\right) r^{n-1} + \frac{(n+3)}{(2n+3)}\tilde{c}_{n+1}^{(i)} r^{n+1}\Big\} P_n^1(\zeta), \end{aligned} \tag{7.9}$$

while for the total pressure we obtain, from equation (4.4),

$$\mathrm{P}^{\mathrm{PN}}(r,\zeta) = \mathrm{P}_0^{\mathrm{PN}} - \mu \sum_{n=0}^{\infty} \left\{(n+1)\tilde{c}_{n+1}^{(i)} r^n - n\tilde{c}_{n-1}^{(e)} r^{-(n+1)}\right\} P_n(\zeta), \quad r > 0,\ |\zeta| \le 1. \tag{7.10}$$

The vorticity field given in (2.3), in view of (7.7), (7.8), and (7.9), is easily confirmed to be expressible in

$$\boldsymbol{\omega}^{\mathrm{PN}}(r,\zeta) = \hat{\boldsymbol{\varphi}}\omega_\varphi^{\mathrm{PN}}(r,\zeta), \quad r > 0,\ |\zeta| \le 1, \tag{7.11}$$

whereas

$$\omega_\varphi^{\mathrm{PN}}(r,\zeta) = \sum_{n=1}^{\infty} \left\{\tilde{c}_{n+1}^{(i)} r^n + \tilde{c}_{n-1}^{(e)} r^{-(n+1)}\right\} P_n^1(\zeta). \tag{7.12}$$

The constant coefficients $\tilde{c}_n^{(i)}$, $\tilde{c}_n^{(e)}$, $\tilde{d}_n^{(i)}$, and $\tilde{d}_n^{(e)}$, $n \ge 0$, must be determined from the appropriate boundary conditions.

In order to apply the boundary conditions (7.1)–(7.4), we use the expressions (7.8), (7.9), and (7.12) as well as certain recurrence and orthogonality relations for the Legendre functions [4]. After some extensive algebra, one obtains a complicated system of linear algebraic equations involving the unknown constant coefficients, where only the first term provides us with the solution and then we obtain the corrected solution of the Kuwabara-type boundary value problem [5], that is,

$$\mathbf{v}^{\mathrm{PN}}(r,\zeta) = v_r^{\mathrm{PN}}(r,\zeta)\hat{\mathbf{r}} + v_\zeta^{\mathrm{PN}}(r,\zeta)\hat{\boldsymbol{\zeta}}, \tag{7.13}$$

$$v_r^{\mathrm{PN}}(r,\zeta) = \frac{U\zeta}{2K}\left[\frac{3}{5\ell^3}\left(\frac{r}{\alpha}\right)^2 - \left(2+\frac{1}{\ell^3}\right) + 3\left(\frac{\alpha}{r}\right) - \left(1-\frac{2}{5\ell^3}\right)\left(\frac{\alpha}{r}\right)^3\right], \tag{7.14}$$

$$v_\zeta^{\mathrm{PN}}(r,\zeta) = -\frac{U\sqrt{1-\zeta^2}}{2K}\left[\frac{6}{5\ell^3}\left(\frac{r}{\alpha}\right)^2 - \left(2+\frac{1}{\ell^3}\right) + \frac{3}{2}\left(\frac{\alpha}{r}\right) + \frac{1}{2}\left(1-\frac{2}{5\ell^3}\right)\left(\frac{\alpha}{r}\right)^3\right], \tag{7.15}$$

$$\boldsymbol{\omega}^{\mathrm{PN}}(r,\zeta) = \hat{\boldsymbol{\varphi}}\frac{3U\sqrt{1-\zeta^2}}{2\alpha K}\left[-\frac{1}{\ell^3}\left(\frac{r}{\alpha}\right) + \left(\frac{\alpha}{r}\right)^2\right], \tag{7.16}$$

$$\mathrm{P}^{\mathrm{PN}}(r,\zeta) = \mathrm{P}_0^{\mathrm{PN}} + \frac{3\mu U\zeta}{2\alpha K}\left[\frac{2}{\ell^3}\left(\frac{r}{\alpha}\right) + \left(\frac{\alpha}{r}\right)^2\right], \tag{7.17}$$

where $\ell = b/\alpha > 1$, $K = (\ell-1)^3(1+3\ell+6\ell^2+5\ell^3)/5\ell^6$, and α, b are the radii of the concentric spheres. We remark here on the simple way one can obtain the solution preserving at the same time the mathematical rigor.

8. Conclusions

A method for connecting two differential representations for nonaxisymmetric Stokes flow was developed. Based on this method, we examined the Papkovich-Neuber (PN) [8, 10] and the Palaniappan et al. (PNAU) [9] differential representations, which offer solutions for such flow problems in spherical geometry. The important physical flow fields (velocity, total pressure) are presented in terms of vector spherical harmonics. Furthermore, interrelation of the flow fields leads to connection formulae for the constant coefficients of the potentials, using the corresponding potentials as a function of spherical eigenfunctions. An immediate consequence of the interrelation of our representations for Stokes flow is that this procedure cannot be inverted. Consequently, one can always calculate the flow fields via the PNAU representation once the PN eigenmodes are known, but one cannot obtain relations that provide the PN potentials through the harmonic and biharmonic PNAU potentials.

An application of the present theory to an axisymmetric Stokes flow problem in a spherical cell (as a mean of modeling flow through a swarm of spherical particles) with the help of the PN differential representation was provided. An extension of the problem presented here to the case of ellipsoidal geometry for the creeping flow of small ellipsoidal particles is under current investigation.

Appendix

For completeness, we present the following relations between the vector surface and solid spherical harmonics:

$$\mathbf{B}_n^{ms} \times \mathbf{P}_n^{ms} = Y_n^{ms}\mathbf{C}_n^{ms}, \quad \mathbf{P}_n^{ms} \times \mathbf{C}_n^{ms} = Y_n^{ms}\mathbf{B}_n^{ms}, \tag{A.1}$$

$$\mathbf{B}_n^{ms} = \hat{\mathbf{r}} \times \mathbf{C}_n^{ms}, \quad \mathbf{C}_n^{ms} = \mathbf{B}_n^{ms} \times \hat{\mathbf{r}}, \tag{A.2}$$

$$\mathbf{P}_n^{ms}(\hat{r}) = \frac{r^{-n+1}}{(2n+1)}\mathbf{N}_{n-1}^{(i)ms}(\mathbf{r}) - \frac{r^{n+2}}{(2n+1)}\mathbf{N}_{n+1}^{(e)ms}(\mathbf{r}), \tag{A.3}$$

$$\mathbf{B}_n^{ms}(\hat{\mathbf{r}}) = \sqrt{\frac{n+1}{n}}\frac{r^{-n+1}}{(2n+1)}\mathbf{N}_{n-1}^{(i)ms}(\mathbf{r}) + \sqrt{\frac{n}{n+1}}\frac{r^{n+2}}{(2n+1)}\mathbf{N}_{n+1}^{(e)ms}(\mathbf{r}), \tag{A.4}$$

$$\mathbf{C}_n^{ms}(\hat{\mathbf{r}}) = \frac{r^{-n}}{2\sqrt{n(n+1)}}\mathbf{M}_n^{(i)ms}(\mathbf{r}) + \frac{r^{n+1}}{2\sqrt{n(n+1)}}\mathbf{M}_n^{(e)ms}(\mathbf{r}), \tag{A.5}$$

for $n = 0, 1, 2, \ldots$, $m = 0, 1, \ldots, n+1$, $s = e, o$, and $\mathbf{r} \in \Omega(\mathbb{R}^3)$. Finally, for the vector spherical harmonics, one can easily derive the following relations:

$$\nabla \cdot \mathbf{N}_n^{(i)ms}(\mathbf{r}) = 0, \qquad \nabla \cdot \mathbf{N}_n^{(e)ms}(\mathbf{r}) = 0, \tag{A.6}$$

$$\nabla \times \mathbf{N}_n^{(i)ms}(\mathbf{r}) = \mathbf{0}, \qquad \nabla \times \mathbf{N}_n^{(e)ms}(\mathbf{r}) = \mathbf{0}, \tag{A.7}$$

$$\nabla \cdot \mathbf{M}_n^{(i)ms}(\mathbf{r}) = 0, \qquad \nabla \cdot \mathbf{M}_n^{(e)ms}(\mathbf{r}) = 0, \tag{A.8}$$

$$\nabla \times \mathbf{M}_n^{(i)ms}(\mathbf{r}) = (n+1)\mathbf{N}_{n-1}^{(i)ms}(\mathbf{r}), \qquad \nabla \times \mathbf{M}_n^{(e)ms}(\mathbf{r}) = -n\mathbf{N}_{n+1}^{(e)ms}(\mathbf{r}), \tag{A.9}$$

$$\begin{aligned} \nabla \cdot \mathbf{G}_n^{(i)ms}(\mathbf{r}) &= -n(2n+1)r^{n-1}Y_{n-1}^{ms}(\hat{\mathbf{r}}), \\ \nabla \cdot \mathbf{G}_n^{(e)ms}(\mathbf{r}) &= -(n+1)(2n+1)r^{-(n+2)}Y_{n+1}^{ms}(\hat{\mathbf{r}}), \end{aligned} \tag{A.10}$$

$$\nabla \times \mathbf{G}_n^{(i)ms}(\mathbf{r}) = -(2n+1)\mathbf{M}_{n-1}^{(i)ms}(\mathbf{r}), \qquad \nabla \times \mathbf{G}_n^{(e)ms}(\mathbf{r}) = (2n+1)\mathbf{M}_{n+1}^{(e)ms}(\mathbf{r}). \tag{A.11}$$

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ZERO-DISPERSION LIMIT FOR INTEGRABLE EQUATIONS ON THE HALF-LINE WITH LINEARISABLE DATA

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We study the zero-dispersion limit for certain initial boundary value problems for the defocusing nonlinear Schrödinger (NLS) equation and for the Korteweg-de Vries (KdV) equation with dominant surface tension. These problems are formulated on the half-line and they involve linearisable boundary conditions.

1. An initial boundary value problem for soliton equations

In recent years, there has been a series of results of Fokas and collaborators on boundary value problems for soliton equations (see [3] for a comprehensive review). The method of Fokas in [3] goes beyond existence and uniqueness. In fact, it reduces these problems to Riemann-Hilbert factorisation problems in the complex plane, thus generalising the existing theory which reduces initial value problems to Riemann-Hilbert problems via the method of inverse scattering. One of the main advantages of the Riemann-Hilbert formulation is that one can use recent powerful results on the asymptotic behaviour of solutions to these problems (as some parameter goes to infinity) to derive asymptotics for the solution of the associated soliton equation. For the study of the long-time asymptotics, such methods were pioneered by Its and then made rigorous and systematic by Deift and Zhou; the method is known as "nonlinear steepest descent" in analogy with the linear steepest descent method which is applicable to asymptotic problems for Fourier-type integrals (see, e.g., [2]). A generalisation of the steepest descent method developed in [1] is able to give rigorous results for the so-called "semiclassical" or "zero-dispersion" limit of the solution of the Cauchy problem for $(1+1)$-dimensional integrable evolution equations, in the case where the Lax operator is selfadjoint. The method has been further extended in [9] for the "nonselfadjoint" case.

In a recent paper [8], Kamvissis, by making use of the nonlinear steepest descent method, has studied the "zero-dispersion" limit of the initial boundary value problem for the $(1+1)$-dimensional, integrable, defocusing, nonlinear Schrödinger (NLS) equation on the half-line, for quite general initial and boundary data. In this paper, we consider

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the simplest case of "linearisable" data. More precisely, we consider the two "archetypal" soliton equations

$$\begin{gathered} ihu_t(x,t) + h^2 u_{xx}(x,t) - 2\,|\,u(x,t)\,|^2 u(x,t) = 0, \quad x \geq 0,\ t \geq 0, \\ u(x,0) = u_0(x) \in \mathbb{S}(\mathbb{R}^+), \quad 0 < x < \infty, \end{gathered} \tag{1.1}$$

with the linearisable boundary condition

$$u_x(0,t) - \chi u(0,t) = 0, \quad t > 0, \tag{1.2}$$

for some constant $\chi \geq 0$, where h is the semiclassical parameter which is assumed to be small and positive and $\mathbb{S}(\mathbb{R}^+)$ denotes the Schwartz class on $[0,\infty)$;

$$\begin{gathered} u_t(x,t) + u_x(x,t) + 6uu_x(x,t) - h^2 u_{xxx}(x,t) = 0, \quad x \geq 0,\ t \geq 0, \\ u(x,0) = u_0(x) \in \mathbb{S}(\mathbb{R}^+), \quad 0 < x < \infty, \end{gathered} \tag{1.3}$$

with the linearisable boundary condition

$$u(0,t) = \chi, \quad u_{xx}(0,t) = \chi + 3\chi^2,\ t \geq 0, \tag{1.4}$$

for some constant χ, where h is the dispersion parameter which is assumed to be small and positive.

It is well known that these equations admit a "Lax-pair" formulation. Namely, these equations are the compatibility condition for the equations $L\mu = 0$ and $B\mu = 0$, where L and B are differential operators on a Hilbert space. In the NLS equation case, for example, they are given by

$$\begin{gathered} L = \begin{pmatrix} \partial_x - ik & iu \\ -i\bar{u} & \partial_x + ik \end{pmatrix}, \\ B = \begin{pmatrix} ih\partial_t + 4ik^2 + i|u|^2 & -2ku - iu_t \\ -2k\bar{u} + i\bar{u}_t & ih\partial_t - i|u|^2 \end{pmatrix}. \end{gathered} \tag{1.5}$$

Here, the bar denotes complex conjugation, k is the spectral variable, and $u = u(x,t)$ is the solution of (1.1).

The traditional method of solving initial value problems for soliton equations that admit a Lax-pair formulation is to focus on the operator L and apply the theory of scattering and inverse scattering to this operator.

On the other hand, one of the main ideas of the method of Fokas is that for initial boundary value problems, the two operators L and B should be on an equal footing. The scattering transform should be applied to both operators *simultaneously*, while a so-called *global relation* has to be imposed on the data to ensure compatibility (see relation (2.6)). The global relation will ensure existence, uniqueness, and the validity of the Riemann-Hilbert formulation.

2. The Riemann-Hilbert problem

As shown in [4], initial boundary value problems for integrable evolution PDEs can be reduced to a Riemann-Hilbert factorisation problem, under the special assumption that the so-called global relation, a condition on the given data, holds (see relation (2.6)).

Consider first the NLS equation with general boundary conditions. Namely, either $u(0,t)$, $u_x(0,t)$, or a relation between $u(0,t)$ and $u_x(0,t)$ is given. The situation for Korteweg-de Vries (KdV) equation is similar where *two* boundary conditions are given.

Let Σ be the contour $\mathbb{R} \cup i\mathbb{R}$ with the following orientation:

(i) the real axis is oriented from left to right,
(ii) the positive imaginary axis is oriented from infinity towards zero,
(iii) the negative imaginary axis is oriented from infinity towards zero.

We use the following convention: the +-side of an oriented contour is always to its left, according to the given orientation.

Letting M_+ and M_- denote the limits of M on Σ from left and right, respectively, we define the Riemann-Hilbert factorisation problem

$$M_+(x,t,k) = M_-(x,t,k)J(x,t,k), \tag{2.1}$$

where

$$\begin{gathered} J(x,t,k) = \begin{cases} J_4^{-1}, & k \in \mathbb{R}^+, \\ J_1^{-1}, & k \in i\mathbb{R}^+, \\ J_3^{-1}, & k \in i\mathbb{R}^-, \end{cases} \\ J_2 = J_3 J_4^{-1} J_1, \quad k \in \mathbb{R}^-, \end{gathered} \tag{2.2}$$

with

$$\begin{gathered} J_1 = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta} & 1 \end{pmatrix}, \\ J_3 = \begin{pmatrix} 1 & -\bar{\Gamma}(\bar{k})e^{-2i\Theta} \\ 0 & 1 \end{pmatrix}, \\ J_4 = \begin{pmatrix} 1 & -\gamma(k)e^{-2i\Theta} \\ \bar{\gamma}(k)e^{2i\Theta} & 1 - |\gamma(k)|^2 \end{pmatrix}, \\ \Theta(x,t,k) = \frac{\theta}{h}, \quad \theta = kx + 2k^2 t. \end{gathered} \tag{2.3}$$

The functions γ and Γ are defined in terms of the spectral functions of the problem (see [5, (2.25), (2.28)]), with important analyticity properties (see [5, (2.21), (2.22)]). In particular,

$$\Gamma(k) = \frac{1}{a(k)\big(a(k)(\bar{A}(\bar{k})/\bar{B}(\bar{k})) - b(k)\big)}, \tag{2.4}$$

where a, b are the spectral functions for the x-problem and A, B are the spectral functions for the t-problem. The functions a, b are analytic and bounded in the upper half-plane, while A, B are analytic and bounded in the first and third quadrants of the k-plane.

The solution of the NLS equation can be recovered from the solution of (2.1) as follows:

$$u(x,t) = 2ih \lim_{k\to\infty} \left(kM^{12}(x,t,k)\right), \tag{2.5}$$

where the index 12 denotes the (12)-entry of a matrix.

The following "global relation" is imposed on the scattering data:

$$a(k)B(k) - b(k)A(k) = e^{4ik^2T}c(k), \tag{2.6}$$

where $c(k)$ is analytic and bounded for $\operatorname{Im} k > 0$, and $c(k) = O(1/k)$ as $k \to \infty$. Here, T is the time up to which we solve the initial boundary value problem for NLS. In general, A, B are functions of T.

There exists a complicated relation between $u(0,t)$ and $u_x(0,t)$; the global relation is the expression of this in the spectral space.

In our particular case (problem (1.1)), $T = \infty$ and the global relation becomes

$$a(k)B(k) - b(k)A(k) = 0 \tag{2.7}$$

for $\arg(k) \in [0,\pi/2]$.

The KdV is treated similarly. The contour Σ^{KdV} consists of the real line oriented from left to right, together with the curves

$$\begin{aligned} l_+ &= \left\{k = k_R + ik_I,\ k_I > 0,\ \frac{1}{4} + 3k_R^2 - k_I^2 = 0\right\}, \\ l_- &= \left\{k = k_R + ik_I,\ k_I < 0,\ \frac{1}{4} + 3k_R^2 - k_I^2 = 0\right\} \end{aligned} \tag{2.8}$$

oriented from right to left. Instead of (2.1), the Riemann-Hilbert problem becomes

$$M_+^{\mathrm{KdV}}(x,t,k) = M_-^{\mathrm{KdV}}(x,t,k)J^{\mathrm{KdV}}(x,t,k), \tag{2.9}$$

where

$$\begin{aligned} J^{\mathrm{KdV}}(x,t,k) &= \begin{cases} \left(J_1^{\mathrm{KdV}}\right)^{-1}, & k \in l_-, \\ \left(J_3^{\mathrm{KdV}}\right)^{-1}, & k \in l_+, \end{cases} \\ J_2^{\mathrm{KdV}} &= J_3^{\mathrm{KdV}}\left(J_4^{\mathrm{KdV}}\right)^{-1}J_1^{\mathrm{KdV}}, \quad k \in \mathbb{R}, \end{aligned} \tag{2.10}$$

with

$$\begin{aligned} J_1^{\mathrm{KdV}} &= \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta^{\mathrm{KdV}}} & 1 \end{pmatrix}, \\ J_3^{\mathrm{KdV}} &= \begin{pmatrix} 1 & -\bar{\Gamma}(\bar{k})e^{-2i\Theta^{\mathrm{KdV}}} \\ 0 & 1 \end{pmatrix}, \\ J_4^{\mathrm{KdV}} &= \begin{pmatrix} 1 & -\gamma(k)e^{-2i\Theta^{\mathrm{KdV}}} \\ \bar{\gamma}(k)e^{2i\Theta^{\mathrm{KdV}}} & 1-|\gamma(k)|^2 \end{pmatrix}, \\ \Theta^{\mathrm{KdV}}(x,t,k) &= \frac{\theta^{\mathrm{KdV}}}{h}, \quad \theta^{\mathrm{KdV}} = -kx+(k+4k^3)t. \end{aligned} \tag{2.11}$$

Here, Γ is still defined via (2.4) and the spectral functions a, b, A, and B still satisfy (2.7), where of course these functions are now expressed in terms of the KdV spectral problems.

3. Linearisable data

In general, the global relation together with the definition of $A(k)$ and $B(k)$ imply a nonlinear Volterra integral equation for the missing boundary values. For example, it is shown in [5] that in the case of the defocusing NLS equation with $q(0,t) = f_0(t)$ given, the unknown boundary value $q_x(0,t) = f_1(t)$ satisfies a nonlinear Volterra equation which has a global solution.

We note that the analogous step for linear evolution equations is solved by algebraic manipulations [3]. This is a consequence of the invariance of the unknown terms in the global relation under $k \to \nu(k)$, where $\omega(\nu(k)) = \omega(k)$ and ω is k^2 and $k+4k^3$ for the NLS and KdV, respectively. Unfortunately, the global relation now involves the solution of the t-problem $\Phi(t,k)$ which in general breaks the invariance. However, for a particular class of boundary conditions, this invariance survives. This is precisely the class of "linearisable problems," namely a class of problems for which $A(k)$ and $B(k)$ can be explicitly written in terms of $a(k)$ and $b(k)$.

It is shown in [4] that for the NLS equation with the boundary condition (1.2),

$$\frac{B(k)}{A(k)} = -\frac{2k+i\chi}{2k-i\chi}\frac{b(-k)}{a(-k)}, \tag{3.1}$$

while for the KdV with the boundary condition (1.4),

$$\frac{B(k)}{A(k)} = \frac{f(k)b(\nu(k)) - a(\nu(k))}{f(k)a(\nu(k)) - b(\nu(k))}, \tag{3.2}$$

where

$$\begin{aligned} \nu^2 + k\nu + k^2 + \frac{1}{4} &= 0, \\ f(k) &= \frac{\nu+k}{\nu-k}\left(1 - \frac{4\nu k}{\chi}\right). \end{aligned} \tag{3.3}$$

Now we restrict ourselves to NLS equation first. It is then easy to see that B/A is analytic and $O(1/k)$ in the first quadrant. Hence, the coefficient Γ is analytic (at least) in the first quadrant of the k-plane, and bounded there.

This has an important consequence. Noting the decay properties of the term $\exp(2i\Theta)$ as $k \to \infty$ in the first quadrant, it is immediate that the positive imaginary axis of the contour can be deformed clockwise to the positive real part. This deformation is exact, not approximate.

Similarly, the negative imaginary part of the contour can be deformed to the negative real axis. We end up with a Riemann-Hilbert problem with jumps only along the real axis. In fact, let

$$
\begin{aligned}
&N(x,t,k) = M(x,t,k), \quad \arg(k) \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),\\
&N(x,t,k) = M(x,t,k)J_1^{-1}, \quad \arg(k) \in \left(0, \frac{\pi}{2}\right),\\
&N(x,t,k) = M(x,t,k)J_3^{-1}, \quad \arg(k) \in \left(\frac{3\pi}{2}, 2\pi\right).
\end{aligned} \tag{3.4}
$$

The Riemann-Hilbert problem becomes

$$
\begin{aligned}
&N_+(x,t,k) = N_-(x,t,k)J(x,t,k), \quad k \in \mathbb{R}, \qquad \lim_{k\to\infty} N(x,t,k) = I,\\
&\text{where } J_N(x,t,k) = \begin{pmatrix} 1 & (\gamma - \bar{\Gamma})(k)e^{-2i\Theta} \\ -(\bar{\gamma} - \Gamma)(k)e^{2i\Theta} & 1 - |\gamma(k)|^2 \end{pmatrix},
\end{aligned} \tag{3.5}
$$

and formula (2.5) holds with N instead of M. Again, we stress that problems (2.1) and (3.5) are exactly equivalent, not just approximately equivalent.

Note that for $k \geq 0$, $\gamma(k) - \bar{\Gamma}(k) = 0$. This follows from the definition in (2.4) and the global relation (2.6). For negative k, $\gamma(k) - \bar{\Gamma}(k) = (bA - aB)/(\bar{a}A - \bar{b}B)$ has all the smoothness and decay properties that are required from a bona fide reflection coefficient corresponding to a realised potential. For example, if the initial data u_0 belongs to the Schwartz class of $\mathbb{R}^+$, then a and b are also Schwartz, while A and B are smooth and bounded. Hence, $R = \gamma - \bar{\Gamma}$ belongs to the Schwartz class of $\mathbb{R}$. There is a unique potential v_0 corresponding to R, which is a continuation of u_0. The Riemann-Hilbert problem (3.5) then gives the evolution of the solution to the NLS equation under initial data v_0.

Thus, we have shown that, in the linearisable case for NLS equation, the half-line problem can be recovered from the solution of the full real line problem by appropriately continuing the initial data.

The above observations are an immediate consequence of the question of studying the semiclassical limit of NLS equation. Since the initial boundary value problem can be considered as a restriction of an initial value problem and since the initial value problem for NLS equation is well understood (studied by Jin et al. in [6, 7]), the results for the initial boundary value problem are recovered immediately.

More precisely, with the introduction of a small dispersive constant h, the changes in the Riemann-Hilbert problem (2.1) will be as follows:

(i) x, t will be replaced by x/h, t/h;
(ii) the coefficients γ, Γ will now be dependent on h.

Since the deformation leading to (3.5) is exact, that is, there is no error (possibly dependent on h), the reduction to a Riemann-Hilbert problem on the line is possible, exactly as in the $h = 1$ case.

Phenomenologically, one sees that the half-plane $x, t \geq 0$ can be divided into two regions. In the first "smooth" region, strong semiclassical limits exist and satisfy the formally limiting system. In the NLS equation case, letting $\rho = |u|^2$, $\mu = h\,\mathrm{Im}(\bar{u}u_x)$, the strong limits $\tilde{\rho}$, $\tilde{\mu}$, as $h \to 0$, exist and satisfy

$$\tilde{\rho}_t + \tilde{\mu}_x = 0, \qquad \tilde{\mu}_t + \left(\frac{\tilde{\mu}^2}{\tilde{\rho}} - \frac{\tilde{\rho}^2}{2}\right)_x = 0. \tag{3.6}$$

In the second "turbulent" region, fast oscillations appear that can be described in terms of slowly modulating finite-gap solutions. Only weak limits exist for ρ, μ as $h \to 0$ and they can be expressed in terms of the solutions of the so-called Whitham system. Rigorous asymptotic formulae for ρ, μ are also easily available (see, e.g., [9]).

In the case of the KdV equation, even with the appropriate changes of contour and phase, the situation is more complicated. It is not true anymore, even for linearisable data, that the half-line problem can be recovered from the solution of the full real line problem by appropriately continuing the initial data, for any value of the small dispersion parameter h. However, we will show that this reduction is possible *asymptotically* as $h \to 0$.

Denote by D_1 and D_2 the domains bounded by $l_+ \cup \mathbb{R}$ and $l_- \cup \mathbb{R}$, respectively. In order to follow the argument used above for the NLS equation, we need to "deform" the curves l_- and l_+ to the real line. This is not possible because $\bar{\Gamma}(\bar{k})e^{-2i\Theta^{\mathrm{KdV}}}$ and $\Gamma(k)e^{2i\Theta^{\mathrm{KdV}}}$ are not bounded as $k \to \infty$ in the domains D_1 and D_2, respectively. What we can do is to "conjugate away" the jumps on l_- and l_+ by introducing an auxiliary Riemann-Hilbert problem which can be reduced to a scalar (hence explicitly solvable) Riemann-Hilbert problem. This is possible because of the triangularity of the jump matrices J_1^{KdV} and J_3^{KdV}.

In the end, we still obtain a Riemann-Hilbert problem on the real line, which, however, is not of the "standard" KdV form, at least for general h. As long as we are only interested in the asymptotics $h \to 0$, though, the reduction to a "standard" Riemann-Hilbert problem is possible. This means that the already existing analysis of the zero-dispersion limit of KdV on the full line (see [10, 11, 12, 13]) is after all applicable.

More precisely, following [8], we consider the following Riemann-Hilbert problem:

$$L_+(x,t,k) = L_-(x,t,k)J^{\mathrm{KdV}}(x,t,k), \tag{3.7}$$

where

$$J^{\mathrm{KdV}}(x,t,k) = \left(J_3^{\mathrm{KdV}}\right)^{-1}, \quad k \in l_+, \tag{3.8}$$

with

$$J_3^{\mathrm{KdV}} = \begin{pmatrix} 1 & -\bar{\Gamma}(\bar{k})e^{-2i\Theta^{\mathrm{KdV}}} \\ 0 & 1 \end{pmatrix}, \quad \Theta^{\mathrm{KdV}}(x,t,k) = \frac{\theta^{\mathrm{KdV}}}{h}, \tag{3.9}$$

where

$$\theta^{\mathrm{KdV}} = -kx + (k+4k^3)t, \tag{3.10}$$

such that $\lim_{k\to\infty} L(x,t,k) = I$.

This problem can be solved explicitly as follows:

$$L(x,t,k) = \begin{pmatrix} 1 & l(x,t,k) \\ 0 & 1 \end{pmatrix}, \tag{3.11}$$

where

$$l(x,t,k) = \frac{1}{2\pi i}\int_{l^+} \frac{-\bar{\Gamma}(\bar{s})e^{-2i\Theta^{\mathrm{KdV}}(x,t,s)}ds}{s-k}. \tag{3.12}$$

Similarly, consider

$$U_+(x,t,k) = U_-(x,t,k)\left(J_1^{\mathrm{KdV}}\right)^{-1}(x,t,k), \quad k \in l_-, \tag{3.13}$$

with

$$J_1^{\mathrm{KdV}} = \begin{pmatrix} 1 & 0 \\ \Gamma(k)e^{2i\Theta^{\mathrm{KdV}}} & 1 \end{pmatrix}, \tag{3.14}$$

such that $\lim_{k\to\infty} U(x,t,k) = I$. We now have

$$U(x,t,k) = \begin{pmatrix} 1 & 0 \\ u(x,t,k) & 1 \end{pmatrix}, \tag{3.15}$$

where

$$u(x,t,k) = \frac{1}{2\pi i}\int_{l^-} \frac{\Gamma(s)e^{2i\Theta^{\mathrm{KdV}}(x,t,s)}ds}{s-k}. \tag{3.16}$$

Now set

$$N(x,t,k) = \begin{cases} M(x,t,k)U^{-1}(x,t,k), & k \in D_1, \\ M(x,t,k)L^{-1}(x,t,k), & k \in D_2, \\ M(x,t,k), & \text{otherwise}. \end{cases} \tag{3.17}$$

Then $N(x,t,k)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$, with $\lim_{k\to\infty} N(x,t,k) = I$, and across $\mathbb{R}$ the jump is given by

$$N_+(x,t,k) = N_-(x,t,k)L(x,t,k)J(x,t,k)U^{-1}(x,t,k). \tag{3.18}$$

An easy deformation argument shows that $L = I + O(h)$, $U = I + O(h)$ as $h \to 0$ (one can simply deform the contour "upwards" or "downwards" and the real part of the phase will become negative). This means that the factor L can be ignored asymptotically. (This is not entirely obvious at this point. The full-line zero-dispersion analysis of KdV involves the introduction of a so-called *g-function* and a conjugation of the Riemann-Hilbert problem by a term $e^{\sigma_3 g/h}$. Only after obtaining the new "conjugated" Riemann-Hilbert problem one is allowed to use the fact that $L, U = I + O(h)$ as $h \to 0$. See [8] for details.)

Again, the half-plane $x, t \geq 0$ can be divided into two regions. In the first "smooth" region, a strong zero-dispersion limit $\tilde{u} = \lim_{h\to 0} u$ exists and satisfies the formally limiting system

$$\tilde{u}_t(x,t) + \tilde{u}_x(x,t) + 6\tilde{u}\tilde{u}_x(x,t) = 0. \tag{3.19}$$

In the second "turbulent" region, fast oscillations appear that can be described in terms of slowly modulating finite-gap solutions. Only a weak limit exists for u as $h \to 0$ and it can be expressed in terms of the solutions of the Whitham system for KdV. A rigorous asymptotic formula for u is also easily available (see, e.g., [13]).

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ON SAMPLING EXPANSIONS OF KRAMER TYPE

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We treat some recent results concerning sampling expansions of Kramer type. The link of the sampling theorem of Whittaker-Shannon-Kotelnikov with the Kramer sampling theorem is considered and the connection of these theorems with boundary value problems is specified. Essentially, this paper surveys certain results in the field of sampling theories and linear, ordinary, first-, and second-order boundary value problems that generate Kramer analytic kernels. The investigation of the first-order problems is tackled in a joint work with Everitt. For the second-order problems, we refer to the work of Everitt and Nasri-Roudsari in their survey paper in 1999. All these problems are represented by unbounded selfadjoint differential operators on Hilbert function spaces, with a discrete spectrum which allows the introduction of the associated Kramer analytic kernel. However, for the first-order problems, the analysis of this paper is restricted to the specification of conditions under which the associated operators have a discrete spectrum.

1. Introduction

This paper surveys certain results in the area of sampling theories and linear, ordinary, first- and second-order boundary value problems that produce Kramer analytic kernels.

1.1. Notations. The symbol $\mathbf{H}(U)$ represents the class of Cauchy analytic functions that are holomorphic (analytic and regular) on the open set $U \subseteq \mathbb{C}$, that is, $\mathbf{H}(\mathbb{C})$ represents the class of all entire or integral functions on $\mathbb{C}$. The symbol $I = (a,b)$ denotes an arbitrary open interval of $\mathbb{R}$; the use of "loc" restricts a property to compact subintervals of $\mathbb{R}$. All the functions $f : (a,b) \to \mathbb{C}$ are taken to be Lebesgue measurable on (a,b), all integrals are in the sense of Lebesgue, and AC denotes absolute continuity with respect to Lebesgue measure.

If w is a weight function on I, then the Hilbert function space $L^2(I;w)$ is the set of all complex-valued, Lebesgue measurable functions $f : I \to \mathbb{C}$ such that $\int_a^b w|f|^2 \equiv \int_a^b w(x)|f(x)|^2 dx < +\infty$ and then, with due regard to equivalence classes, the norm and

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inner product are given by

$$\|f\|_w^2 := \int_I w|f|^2, \qquad (f,g)_w := \int_a^b w(x)f(x)\bar{g}(x)dx. \tag{1.1}$$

1.2. The W.S.K. sampling theorem. This sampling theorem owes its first appearance to Whittaker, in 1915. The same result was obtained later and independently by Kotel'nikov, in 1933, and by Shannon, in 1949. So, it is presently known in the mathematical literature as the W.S.K. theorem (see [31, 35, 40]). However, there are more names who have legitimate claims to be included and for a historical review, we refer to [26, 27]. Turning to the seminal paper by Shannon, this theorem, the proof of which is found in [35], reads as follows.

Theorem 1.1 (see [35]). *If a signal (function) $f(t)$ contains no frequencies higher than $W/2$ cycles per second, that is, is band limited to $[-\pi W, \pi W]$, which means that $f(t)$ is of the form*

$$f(t) = \int_{-\pi W}^{\pi W} g(x)\exp(ixt)dx, \tag{1.2}$$

then $f(t)$ is completely determined by giving its ordinates at a sequence of points spaced $1/W$ apart and $f(t)$ is the sum of its "scaled" cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{W}\right)\frac{\sin\pi(Wt-n)}{\pi(Wt-n)}. \tag{1.3}$$

Remark 1.2. This is the first of the sampling theory results; the signal f cannot change to a substantially new value in a time less than half a cycle of its highest frequency, $W/2$ cycles per second. And moreover, the collection of "samples" $\{f(n/W) : n = 0, \pm 1, \pm 2, \ldots\}$ specifies g via its Fourier series, since the general Fourier coefficient of g (in (1.2)) is $f(n/W)$, and then g specifies f via (1.2). So, if f can be "measured" at the sampling points $\{n/W : n \in \mathbb{Z}\}$, which are equidistantly spaced over the whole real line $\mathbb{R}$, then f can be reconstructed uniquely at every point of the real line $\mathbb{R}$. The engineering principle established in this way leads to the assertion that certain functions whose frequency content is bounded are equivalent to an information source with discrete time. This has a major application in signal analysis, and in order to obtain, in general, a great appreciation of the broad scope of sampling theory, we refer, for example, to [4, 5, 6, 26, 28, 30, 33].

The contents of the paper are as follows: Section 2 gives an analytical background information about the original and the analytic form of the Kramer theorem followed by a discussion concerning quasidifferential problems and operators; Section 3 gives an account of results with respect to the generation of Kramer analytic kernels from first-order boundary value problems, but without mentioning the spectral properties that yield a discrete spectrum of the associated operators; and finally, Section 4 deals with results about the connection of second-order linear ordinary boundary value problems and the Kramer sampling theorem.

2. Introduction to the analytical background

2.1. The original and the analytic form of the Kramer theorem. In 1959, Kramer published the following remarkable result, the proof of which is given in [32].

Theorem 2.1 (Kramer theorem). *Suppose that* $f(t) := \int_I K(x,t)g(x)dx$, $t \in \mathbb{R}$, *for some* $g \in L^2(I)$, *where* I *is an open interval of* $\mathbb{R}$ *and the kernel* $K : I \times \mathbb{R} \to \mathbb{R}$ *satisfies the properties that, for each real* t, $K(\cdot,t) \in L^2(I)$, *and there exists a countable set of reals* $\{t_n : n \in \mathbb{Z}\}$ *such that* $\{K(\cdot,t_n) : n \in \mathbb{Z}\}$ *forms a complete orthogonal set on* $L^2(I)$. *Then*

$$f(t) = \sum_{n \in Z} f(t_n)S_n(t), \qquad S_n(t) := \frac{\int_I K(x,t)\bar{K}(x,t_n)dx}{\int_I |K(\cdot,t_n)|^2 dx}. \tag{2.1}$$

And moreover, the conditions on the kernel are met by certain solutions of selfadjoint eigenvalue problems, where the parameter t *is an eigenvalue parameter; the eigenvalues are chosen to be the sampling points and the complete orthogonal system of eigenfunctions, the set of functions* $\{K(x,\lambda_n) : n \in \mathbb{Z}\}$.

Remark 2.2. (i) Each eigenvalue problem that produces a complete set of eigenfunctions and also real simple and countably infinite many eigenvalues is suitable for the Kramer theorem. For a study of Kramer kernels constructed from boundary value problems, see, for example, [7, 32].

(ii) A certain class of boundary value problems transforms the W.S.K. sampling theorem (Theorem 1.1) into a particular case of the Kramer theorem. For example, take under consideration the selfadjoint, regular eigenvalue problem, for $\sigma > 0$, $\lambda \in \mathbb{R}$:

$$-iy'(x) = \lambda y(x), \quad x \in [-\sigma,\sigma],\ y(-\sigma) = y(\sigma). \tag{2.2}$$

The eigenvalues are given by $\lambda_n = n\pi/\sigma$, $n \in \mathbb{Z}$, and the corresponding eigenfunctions are $y_n(x) = \exp(in\pi x/\sigma)$, $n \in \mathbb{Z}$. The general solution $K(x,\lambda) = \exp(ix\lambda)$ of the differential equation is a suitable kernel for Theorem 1.1. So, if f is of the form

$$f(\lambda) = \int_{-\sigma}^{\sigma} \exp(ix\lambda)g(x)dx, \quad g \in L^2(-\sigma,\sigma),\ \lambda \in \mathbb{R}, \tag{2.3}$$

then there exists the sampling representation

$$f(\lambda) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{\sigma}\right)\frac{\sin(\sigma\lambda - n\pi)}{(\sigma\lambda - n\pi)}. \tag{2.4}$$

(iii) The Kramer kernel that arises from the above example has a significant property. This property also emerges in a number of other cases of symmetric boundary value problems and is not predicted in the statement of Kramer's theorem, that is, $K(x,\cdot) \in \mathbf{H}(\mathbb{C})$, $x \in I$ (see Section 1.1). For additional details of the previous boundary value problem, see the results in [15, Section 5.1].

The following theorem gives an analytic form of the Kramer theorem in the way that allows analytic dependence of the kernel on the sampling parameter.

Theorem 2.3. *Let $I = (a,b)$ be an arbitrary open interval of $\mathbb{R}$ and let w be a weight function on I. Let the mapping $K : I \times \mathbb{C} \to \mathbb{C}$ satisfy the following properties:*

(1) $K(\cdot,\lambda) \in L^2(I;w)$ $(\lambda \in \mathbb{C})$,
(2) $K(x,\cdot) \in \mathbf{H}(\mathbb{C})$ $(x \in (a,b))$,
(3) *there exists a sequence* $\{\lambda_n \in \mathbb{R} : n \in \mathbb{Z}\}$ *satisfying*
  (i) $\lambda_n < \lambda_{n+1}$ $(n \in \mathbb{Z})$,
  (ii) $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$,
  (iii) *the sequence of functions* $\{K(\cdot,\lambda_n) : n \in \mathbb{Z}\}$ *forms a locally linearly independent and a complete orthogonal set in the Hilbert space* $L^2(I;w)$,
(4) *the mapping* $\lambda \mapsto \int_a^b w(x)|K(x,\lambda)|^2 dx$ *is locally bounded on* $\mathbb{C}$.

Define the set of functions $\{K\}$ *as the collection of all functions* $F : L^2(I;w) \times \mathbb{C} \to \mathbb{C}$ *determined by, for* $f \in L^2(I;w)$,

$$F(f;\lambda) \equiv F(\lambda) := \int_a^b w(x)K(x,\lambda)f(x)dx \quad (\lambda \in \mathbb{C}). \tag{2.5}$$

Then for all $F \in \{K\}$,

(a) $F(f,\cdot) \in \mathbf{H}(\mathbb{C})$ $(f \in L^2(I;w))$;
(b) *if* $S_n : \mathbb{C} \to \mathbb{C}$ *is defined by, for all* $n \in \mathbb{Z}$,

$$S_n(\lambda) := ||K(\cdot,\lambda_n)||_w^{-2} \int_a^b w(x)K(x,\lambda)\bar{K}(x,\lambda_n)dx \quad (\lambda \in \mathbb{C}), \tag{2.6}$$

then $S_n \in \mathbf{H}(\mathbb{C})$;
(c) $F(f,\lambda) \equiv F(\lambda) = \sum_{n\in\mathbb{Z}} F(\lambda_n)S_n(\lambda)$, *for all* $F \in \{K\}$, *where the series is absolutely convergent, for each* $\lambda \in \mathbb{C}$, *and locally uniformly convergent on* $\mathbb{C}$.

Proof. For the proof of this theorem see [18, Theorem 2 and Corollary 1]; the ideas for these results come from [10] and [21, Theorem 1.1]. □

Remark 2.4. (i) Suitable problems for the above theorem are, for example, regular selfadjoint eigenvalue problems of *n*th-order and singular selfadjoint problems of second-order in the limit-circle endpoint case (for classifications of eigenvalue problems, see [34], and for information concerning Kramer analytic kernels, see, e.g., [15, 19, 41]).

(ii) As outlined in Remark 2.2(ii), the W.S.K. theorem can be seen as a particular case of Kramer's result for a certain class of problems. So, the question arises whether these two theorems are equivalent to each other or not. The link of the W.S.K. "sampling results" and Kramer's theorem has been the concern of many authors. The first person who dealt with this problem was Campbell in 1964 (see [7]). Later, there is a lot to be found in the literature; see, for example, [29, 42]. Also, an extensive historical perspective of the equivalence of Kramer and W.S.K. theorems for second-order boundary value problems is given in [24]; there also may be found some results for the Bessel and the general Jacobi cases.

2.2. Quasidifferential problems and operators. The environment of the general theory of quasiderivatives is the best for the study of symmetric (selfadjoint) boundary value problems which, as noticed in Remark 2.4(i), are a source for the generation of Kramer analytic kernels. Furthermore, all the classical differential expressions appear as special cases of quasidifferential expressions; for confirmation, we refer to [13, 14, 20, 25, 34]. Finally, the Shin-Zettl quasidifferential expressions are considered to be the most general ordinary linear differential expressions so far defined, for order $n \in \mathbb{N}$ and $n \geq 2$; for details see [9, 11, 22, 23, 36, 37, 38, 43]. Accordingly, the general formulation of quasidifferential boundary value problems will be performed as follows.

Let $I = (a,b)$ be an open interval of the real line $\mathbb{R}$. Let M_n be a linear ordinary differential expression. In the classical case, M_n is of finite order $n \geq 1$ on I with complex-valued coefficients, and of the form

$$M_n[f] = p_n f^{(n)} + p_{n-1} f^{(n-1)} + \cdots + p_1 f' + p_0 f, \tag{2.7}$$

where $p_j : I \to \mathbb{C}$ with $p_j \in L^1_{\text{loc}}(I)$, $j = 0,1,\ldots,n-1$, n, and further $p_n \in \text{AC}_{\text{loc}}(I)$ with $p_n(x) \neq 0$, for almost all $x \in I$. For the special case $n = 1$, see details in [12].

In the more general quasidifferential case, the expression M_n is defined as in [23] and [14, Section I]. For $n \geq 2$, the expression $M_n := M_A$ is determined by a complex Shin-Zettl matrix $A = [a_{rs}] \in Z_n(I)$ with the domain $D(M_n)$ of M_A defined by

$$\begin{gathered} D(M_A) := \left\{ f : I \longrightarrow \mathbb{C} : f_A^{[r-1]} \in \text{AC}_{\text{loc}}(I), \text{ for } r = 1,2,\ldots,n \right\}, \\ M_A[f] := i^n f_A^{[n]} \quad (f \in D(M_A)), \end{gathered} \tag{2.8}$$

where the quasiderivatives $f_A^{[j]}$, for $j = 1,2,\ldots,n$, are taken relative to the matrix $A \in Z_n(I)$. For these results and additional properties, see the notes [9]. In this investigation, M_A is Lagrange symmetric in the notation of [9, 20].

Every classical ordinary linear differential expression M_n, as in (2.7), can be written as a quasidifferential expression M_A, as in (2.8), with the same order $n \geq 2$. The first-order differential expressions are essentially classical in form. Therefore, we can assume that when $n \geq 2$, M_n is a quasidifferential expression specified by an appropriate Shin-Zettl matrix $A \in Z_n(I)$. When $n = 1$, we consider M_1 as a classical expression and the analysis given here works also in this case.

Now, the Green's formula for M_n has the form

$$\int_\alpha^\beta \{\bar{g} M_n[f] - f \overline{M_n[g]}\} = [f,g](\beta) - [f,g](\alpha) \quad (f,g \in D(M_n)), \tag{2.9}$$

for any compact subinterval $[\alpha,\beta]$ of (a,b). Here the skew-symmetric sesquilinear form $[\cdot,\cdot]$ is taken from (2.9); that is, it maps $D(M_n) \times D(M_n) \to \mathbb{C}$ and is defined, for $n \geq 2$, by

$$[f,g](x) := i^n \sum_{r=1}^n (-1)^{r-1} f^{[n-r]}(x) \overline{g^{(r-1)}}(x) \quad (x \in (a,b),\ f,g \in D(M_n)) \tag{2.10}$$

and, for $n = 1$, by

$$[f,g](x) := i\rho(x)f(x)\bar{g}(x) \quad (x \in (a,b),\ f,g \in D(M_1)). \tag{2.11}$$

From the Green's formula (2.9), it follows the limits

$$\begin{aligned} [f,g](a) &:= \lim_{x\to a^+}[f,g](x),\\ [f,g](b) &:= \lim_{x\to b^-}[f,g](x), \end{aligned} \tag{2.12}$$

both exist and are finite in $\mathbb{C}$.

The spectral differential equations associated with the pairs $\{M_n, w\}$, where w is a given nonnegative weight (see Section 1.1), are

$$M_n[y] = \lambda w y \quad \text{on } (a,b) \tag{2.13}$$

with the spectral parameter $\lambda \in \mathbb{C}$. The solutions of (2.13) are considered in the Hilbert function space $L^2((a,b);w)$ (see Section 1.1). In order to define symmetric boundary value problems in this space, linear boundary conditions of the form (see (2.9), (2.10), (2.11), and (2.12))

$$[y,\beta_r] \equiv [y,\beta_r](b) - [y,\beta_r](a) = 0, \quad r = 1,2,\dots,d, \tag{2.14}$$

have to be connected, where the family $\{\beta_r,\ r = 1,2,\dots,d\}$ is a linearly independent set of maximal domain functions chosen to satisfy the symmetry condition

$$[\beta_r,\beta_s](b) - [\beta_r,\beta_s](a) = 0 \quad (r,s = 1,2,\dots,d). \tag{2.15}$$

The integer $d \in \mathbb{N}_0$ is the common deficiency index of (2.13) determined in $L^2((a,b);w)$ and gives the number of boundary conditions needed for the boundary value problem ((2.13), (2.14)) to be symmetric, that is, to produce a selfadjoint operator in $L^2((a,b);w)$. This boundary value problem generates a uniquely determined unbounded selfadjoint operator T in the space $L^2((a,b);w)$; see [23].

If the problem is regular on an interval (a,b), in which case this interval has to be bounded, then $d = n$ and the generalized boundary conditions (2.14) require the pointwise values of the solution y and its quasiderivatives at the endpoints a and b. For this regular case when the order $n = 2m$ is even and the Lagrange symmetric matrix is real valued, see details in [34]. In the case $n = 1$, the index d can take the values 0 or 1, but the value 0 is rejected (see Remark 3.3). In the case $n = 2$, essentially the Sturm-Liouville case, the index d may take the values 0, 1, or 2; this value depends on the regular/limit-point/limit-circle classification, in $L^2(I;w)$, at the endpoints a and b of the differential expression M_n (cf. [39, Chapter II]).

For the connection between the classical and quasidifferential systems, we refer to [14].

3. First-order problems

In this section, we investigate in greater details the link between the Kramer sampling theorem and linear ordinary differential equations of first-order. The results we present here are given in [19]. We only point out that the development of our operator theory as a source for the construction of Kramer analytic kernels is not given here; see [19] for details of these Kramer kernels. The operator theory required is to be found in [1, 2, 8]; for the classical theory of selfadjoint extensions of symmetric operators as based on Hilbert space constructions, see [34].

3.1. Differential equations and operators. The selfadjoint boundary value problems considered here are generated by the general first-order Lagrange symmetric linear differential equation which defines the differential expression M_1 and is of the form

$$\begin{aligned} M_1[y](x) &:= i\rho(x)y'(x) + \frac{1}{2}i\rho'(x)y(x) + q(x)y(x) \\ &= \lambda w(x)y(x), \quad \forall x \in (a,b), \end{aligned} \tag{3.1}$$

where $-\infty \leq a < b \leq +\infty$ and $\lambda \in \mathbb{C}$ is the spectral parameter. Also,

$$\begin{gathered} \rho, q, w : (a,b) \longrightarrow \mathbb{R}, \\ \rho \in \mathrm{AC}_{\mathrm{loc}}(a,b), \quad \rho(x) > 0, \quad \forall x \in (a,b), \\ q, w \in L^1_{\mathrm{loc}}(a,b), \\ w(x) > 0, \quad \text{for almost all } x \in (a,b). \end{gathered} \tag{3.2}$$

Under conditions (3.2), the differential equation (3.1) has the following initial value properties; let $c \in (a,b)$ and $\gamma \in \mathbb{C}$, then there exists a unique mapping $y : (a,b) \times \mathbb{C} \to \mathbb{C}$ with

(i) $y(\cdot,\lambda) \in \mathrm{AC}_{\mathrm{loc}}(a,b)$, for all $\lambda \in \mathbb{C}$,
(ii) $y(x,\cdot) \in \mathbf{H}$, for all $x \in (a,b)$,
(iii) $y(c,\lambda) = \gamma$, for all $\lambda \in \mathbb{C}$,
(iv) $y(\cdot,\lambda)$ satisfies (3.1), for almost all $x \in (a,b)$ and all $\lambda \in \mathbb{C}$.

However, direct formal integration shows that the required solution y is given by

$$y(x,\lambda) = \gamma\sqrt{\frac{\rho(c)}{\rho(x)}} \exp\left(\int_c^x \frac{(\lambda w(t) - q(t))}{i\rho(t)} dt\right), \quad \forall x \in (a,b),\ \forall \lambda \in \mathbb{C}. \tag{3.3}$$

Remark 3.1 (see [19, Lemma 2.1]). A necessary and sufficient condition to ensure that the solution $y(\cdot,\lambda) \in L^2((a,b);w)$, for all $\lambda \in \mathbb{C}$, is

$$\int_a^b \frac{w(t)}{\rho(t)} dt < +\infty. \tag{3.4}$$

We notice that if there are any selfadjoint operators T in $L^2((a,b);w)$ generated by M_1 (see (3.1)), then all such operators have to satisfy the inclusion relation

$$T_0 \subseteq T = T^* \subseteq T_1 = T_0^*, \tag{3.5}$$

where T_0 and T_1 are the minimal and maximal operators, respectively, generated by M_1. From the general theory of unbounded operators in Hilbert space, such selfadjoint operators exist if and only if the deficiency indices (d^-, d^+) of T_0 are equal; see [34, Chapter IV]. Thus for selfadjoint extensions of T_0 to exist, there are only two possibilities:

(i) $d^- = d^+ = 0$,
(ii) $d^- = d^+ = 1$.

Remark 3.2 (see [19, Lemma 4.1]). (i) The indices $d^- = d^+ = 0$ if and only if, for some $c \in (a,b)$, $w/\rho \notin L^1(a,c]$ and $w/\rho \notin L^1[c,b)$.

(ii) The indices $d^- = d^+ = 1$ if and only if $w/\rho \in L^1(a,b)$.

Remark 3.3. (a) In the case of Remark 3.2(i), if we define the operator T by $T := T_0^* = T_0$, then T is the (unique) selfadjoint operator in $L^2((a,b);w)$ generated by the differential expression M_1 of (3.1). The selfadjoint boundary value problem, in this case, consists only of the differential equation (3.1). In fact, the spectrum of T is purely continuous and occupies the whole real line, that is, $\sigma(T) = C\sigma(T) = \mathbb{R}$. We note that this case can give no examples of interest for sampling theories. As an example in $L^2(-\infty,+\infty)$, consider $iy'(x) = \lambda y(x)$, for all $x \in (-\infty,+\infty)$.

(b) In the case of Remark 3.2(ii), which covers all regular cases of (3.1) and all singular cases when condition (3.4) is satisfied, the general Stone/von Neumann theory of selfadjoint extensions of closed symmetric operators in Hilbert space proves that there is a continuum of selfadjoint extensions $\{T\}$ of the minimal operator T_0 with $T_0 \subset T \subset T_1$. These extensions can be determined by the use of the generalized Glazman-Krein-Naimark (GKN) theory for differential operators as given in [12, Section 4, Theorem 1]. The domain of any selfadjoint extension T of T_0 can be obtained as a restriction of the domain of the maximal operator T_1. These restrictions are obtained by choosing an element $\beta \in D(T_1)$ such that β arises from a nonnull member of the quotient space $D(T_1)/D(T_0)$ with the symmetric property (recall (2.15)) $[\beta,\beta](b^-) - [\beta,\beta](a^+) = 0$. With this boundary condition function $\beta \in D(T_1)$, the domain $D(T)$ is now determined by

$$D(T) := \{f \in D(T_1) : [f,\beta](b^-) - [f,\beta](a^+) = 0\}, \tag{3.6}$$

and the selfadjoint operator is defined by $Tf := w^{-1}M[f]$, for all $f \in D(T)$.

For an example of such a boundary condition function β, see [19, Section 4, (4.20)].

Now, the selfadjoint boundary value problem consists of considering the possibility of finding nontrivial solutions $y(\cdot,\lambda)$ of the differential equation (3.1) with the property $y(\cdot,\lambda) \in L^2((a,b);w)$ that satisfies the boundary condition

$$[y(\cdot,\lambda),\beta](b^-) - [y(\cdot,\lambda),\beta](a^+) = 0. \tag{3.7}$$

The solution of this problem depends upon the nature of the spectrum $\sigma(T)$ of the selfadjoint operator T determined by the choice of the boundary condition element β.

In the case of Remark 3.2(ii), it is shown in [19, Theorem 5.1] that the spectrum of $\sigma(T)$ of any selfadjoint extension T of T_0 is discrete, simple, and has equally spaced eigenvalues on the real line of the complex spectral plane.

3.2. Kramer analytic kernels. The results in [19] read as follows.

Theorem 3.4. *Suppose that (3.1) satisfies (3.2) and also (3.4) to give equal deficiency indices $d^- = d^+ = 1$. Let the selfadjoint operator T be determined by imposing a coupled boundary condition (3.6) on the domain $D(T_1)$ of the maximal operator T_1 using a symmetric boundary condition function β as in Remark 3.3(b). Denote the spectrum $\sigma(T)$ of T by $\{\lambda_n : n \in \mathbb{Z}\}$. Define the mapping $K : (a,b)\times\mathbb{C} \to \mathbb{C}$ by, where $c \in (a,b)$ is fixed,*

$$K(x,\lambda) := \frac{1}{\sqrt{\rho(x)}} \exp\left(\int_c^x \frac{\lambda w(t) - q(t)}{i\rho(t)}\,dt\right), \quad \forall x \in (a,b),\ \lambda \in \mathbb{C}. \tag{3.8}$$

Then the kernel K, together with the point set $\{\lambda_n : n \in \mathbb{Z}\}$, satisfies all the conditions required for the application of Theorem 2.3 to yield K as a Kramer analytic kernel in the Hilbert space $L^2((a,b);w)$.

Proof. See [19]. □

For an example of this general result, we refer to [19, Theorem 7.1] (cf. Remark 2.2(ii)). This example is considered in [15] too.

4. Second-order problems

In this section, we deal with the generation of Kramer analytic kernels from second-order linear ordinary boundary value problems. The results given here can be found in [15].

4.1. Sturm-Liouville theory. Sturm-Liouville boundary value problems are effective in generating Kramer analytic kernels. These problems concern the classic Sturm-Liouville differential equation

$$-\left(p(x)y'(x)\right)' + q(x)y(x) = \lambda w(x)y(x) \quad (x \in I = (a,b)), \tag{4.1}$$

where $-\infty \le a \le b \le +\infty$ and $\lambda \in \mathbb{C}$ is the spectral parameter. Also,

$$\begin{gathered} p,q,w : (a,b) \longrightarrow \mathbb{R}, \\ p^{-1},q,w \in L^1_{\text{loc}}(a,b), \\ w(x) > 0, \quad \text{for almost all } x \in (a,b). \end{gathered} \tag{4.2}$$

For a discussion on the significance of these conditions, see [16, page 324]. For the general theory of Sturm-Liouville boundary value problems, see [39, Chapters I and II]. Accordingly, we impose a structural condition.

Condition 4.1. The endpoint a of the differential equation (4.1) is to be regular or limit-circle in $L^2(I;w)$; independently, the endpoint b is to be regular or limit-circle in $L^2(I;w)$ (cf. [21]).

Remark 4.2. The endpoint classification of Condition 4.1 leads to a minimal, closed, symmetric operator in $L^2(I;w)$ generated by (4.1) with deficiency indices $d^{\pm} = 2$; in turn, this requires that all selfadjoint extensions A of this minimal, symmetric operator are determined by applying two linearly independent, symmetric boundary conditions and either

(i) both conditions are separated with one applied at a and with one applied at b, or
(ii) both conditions are coupled.

4.1.1. Regular or limit-circle case with separated boundary conditions. This case of Condition 4.1 and Remark 4.2(i) concerns the results of [21]. The Sturm-Liouville differential equation is given by (4.1) and satisfies (4.2). The separated boundary conditions are

$$[y,\kappa_-](a) = [y,\kappa_+](b) = 0, \tag{4.3}$$

where, for a given pair of functions $\{\kappa_-,\chi_-\}$, the following conditions are fulfilled:

(C1) $\kappa_-,\chi_- : (a,b) \Rightarrow \mathbb{R}$ are maximal domain functions,
(C2) $[\kappa_-,\chi_-](a) = 1$.

The pair $\{\kappa_+,\chi_+\}$ satisfies analogous conditions at the endpoint b.

This symmetric boundary value problem gives a selfadjoint differential operator T with the following properties:

(a) T is unbounded in $L^2((a,b);w)$,
(b) the spectrum of T is real and discrete with limit points at $+\infty$ or $-\infty$ or both,
(c) the spectrum of T is simple,
(d) the eigenvalues and eigenvectors satisfy the boundary value problem.

The results in [21] are given by the following theorem.

Theorem 4.3. *Let the coefficients p, q, and w satisfy the conditions (4.2); let the Sturm-Liouville quasidifferential equation (4.1) satisfy the endpoint classification of Condition 4.1; let the separated boundary conditions be given by (4.3), where the boundary condition functions $\{\kappa_-,\chi_-\}$ and $\{\kappa_+,\chi_+\}$ satisfy conditions (C1) and (C2); let the selfadjoint differential operator T be determined by the separated, symmetric boundary value problem; let the simple, discrete spectrum of T be given by $\{\lambda_n : n \in \mathbb{Z}\}$ with $\lim_{n\to\pm\infty}\lambda_n = \pm\infty$; let $\{\psi_n : n \in \mathbb{Z}\}$ be the eigenvectors of T; and let the pair of basis solutions $\{\phi_1,\phi_2\}$ of (4.1) satisfy the initial conditions, for some point $c \in (a,b)$:*

$$\begin{aligned} \phi_1(c,\lambda) &= 1, \quad (p\phi_1')(c,\lambda) = 0, \\ \phi_2(c,\lambda) &= 0, \quad (p\phi_2')(c,\lambda) = 1. \end{aligned} \tag{4.4}$$

Define the analytic Kramer kernel $K_- : (a,b) \times \mathbb{C} \to \mathbb{C}$ by

$$K_-(x,\lambda) := [\phi_1(\cdot,\lambda),\kappa_-](a)\phi_2(x,\lambda) - [\phi_2(\cdot,\lambda),\kappa_-](a)\phi_1(x,\lambda). \tag{4.5}$$

Then

(i) *$K_-(\cdot,\lambda)$ is a solution of (4.1), for all $\lambda \in \mathbb{C}$, and $K_-(\cdot,\lambda) \in \mathbb{R}$ ($\lambda \in \mathbb{R}$);*
(ii) *$K_-(\cdot,\lambda)$ is an element of the maximal domain and in particular of $L^2((a,b);w)$;*
(iii) *$[K_-(\cdot,\lambda),\kappa_-](a) = 0$;*
(iv) *$[K_-(\cdot,\lambda),\kappa_+](b) = 0$ if and only if $\lambda \in \{\lambda_n : n \in \mathbb{Z}\}$;*
(v) *$K_-(x,\cdot) \in \mathbf{H}(\mathbb{C})$ ($x \in (a,b)$);*

(vi) $K_-(\cdot,\lambda_n) = k_n\psi_n$, *where* $k_n \in \mathbb{R}\backslash\{0\}$ $(n \in \mathbb{Z})$;

(vii) K_- *is unique up to multiplication by a factor* $e(\cdot) \in \mathbf{H}(\mathbb{C})$ *with* $e(\lambda) \neq 0$ $(\lambda \in \mathbb{C})$ *and* $e(\lambda) \in \mathbb{R}$ $(\lambda \in \mathbb{R})$.

Remark 4.4. The notation K_- is chosen for technical reasons; there is a kernel K_+ with similar properties, but with a and κ_- replaced by b and κ_+.

For an example, we refer to [15, Section 5.2, Example 5.8]. This example can also be found in [21].

4.1.2. Regular or limit-circle case with coupled boundary conditions. This case of Condition 4.1 and Remark 4.2(ii) covers the results of [16]. Here the situation is different. Let (4.1) satisfy (4.2) and let the boundary conditions be given by

$$\mathbf{y}(b) = e^{i\alpha}T\mathbf{y}(a), \quad \text{for some } \alpha \in [-\pi,\pi] \tag{4.6}$$

with the 2×2 matrix $T = [t_{rs}]$, where $t_{rs} \in \mathbb{R}$ $(r,s = 1,2)$, $\det(T) = 1$, and the 2×1 vector $\mathbf{y}$ is defined by

$$\mathbf{y}(t) := \begin{pmatrix} [y,\theta](t) \\ [y,\phi](t) \end{pmatrix} \quad (t \in (a,b)); \tag{4.7}$$

T is called boundary condition matrix. The functions θ and ϕ are chosen such that

(i) θ and ϕ are real-valued maximal domain functions;

(ii) $[\theta,\phi](a) = \lim_{t\to a^+}[\theta,\phi](t) = 1$;

(iii) $[\theta,\phi](b) = \lim_{t\to b^-}[\theta,\phi](t) = 1$.

For example, θ and ϕ can be real-valued solutions of (4.1) on (a,b).

The boundary conditions (4.6) are coupled and selfadjoint, for each endpoint either regular or limit-circle, and are in canonical form (see [3]).

Let the pair of basis solutions $\{u,v\}$ of the differential equation (4.1) be specified by the possibly singular initial conditions (cf. [3]), for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} &[u,\theta](a,\lambda) = 0, \qquad [u,\phi](a,\lambda) = 1, \\ &[v,\theta](a,\lambda) = 1, \qquad [v,\phi](a,\lambda) = 0. \end{aligned} \tag{4.8}$$

To define a differential operator A, choose any boundary condition matrix T and any $\alpha \in [-\pi,\pi]$; the boundary value problem gives a selfadjoint differential operator with the properties (a), (b), and (d) and, in place of (c), the property that the multiplicity of the spectrum is either 1 or 2.

Remark 4.5. For complex boundary conditions, that is, when $0 < \propto < \pi$ or $-\pi < \propto < 0$, the spectrum is always simple. In the case of real boundary conditions, that is, $\alpha = -\pi, 0, \pi$, the spectrum may or may not be simple (see [3]).

The complex case. According to the comments made in Remark 4.5, the results in [16] are divided into two parts. The first part is referred to as the complex case when $0 < \alpha < \pi$ or $-\pi < \alpha < 0$ and this gives the following theorem.

Theorem 4.6. *Let (4.1) satisfy Condition 4.1, where the coefficients p, q, and w satisfy (4.2) and let the symmetric, coupled, and complex boundary condition be given by, see (4.6),*

$$\mathbf{y}(b) = e^{i\alpha} T\mathbf{y}(a), \quad \text{for some } \alpha \in (-\pi,0) \cup (0,\pi); \tag{4.9}$$

let A be the unique selfadjoint, unbounded differential operator in $L^2((a,b);w)$, specified by (4.1) and (4.6); let the discrete spectrum σ of A be represented by $\{\lambda_n : n \in \mathbb{Z}\}$ with $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$, *and let $\{\psi_n : n \in \mathbb{Z}\}$ represent the corresponding eigenfunctions. Let the analytic function $D(T,\cdot) : \mathbb{C} \to \mathbb{C}$ be defined by, with solutions u, v determined by (4.8),*

$$\begin{aligned} D(T,\lambda) :=& t_{11}[u(\cdot,\lambda),\phi](b) + t_{22}[v(\cdot,\lambda),\theta](b) \\ &- t_{12}[v(\cdot,\lambda),\phi](b) - t_{21}[u(\cdot,\lambda),\theta](b). \end{aligned} \tag{4.10}$$

Then

(i) $D(T,\cdot) \in \mathbf{H}(\mathbb{C})$;
(ii) *λ is an eigenvalue of A if and only if λ is a zero of $D(T,\lambda) - 2\cos(\alpha)$;*
(iii) *the zeros of $D(T,\lambda) - 2\cos(\alpha)$ are real and simple;*
(iv) *the eigenvalues of A are simple.*

Let the above-stated definitions and conditions hold; then the boundary value problem (4.1) and (4.6) generate two independent analytic Kramer kernels K_1 and K_2:

$$\begin{aligned} K_1(x,\lambda) &:= ([u(\cdot,\lambda),\theta](b) - e^{i\alpha}t_{12})v(x,\lambda) - ([v(\cdot,\lambda),\theta](b) - e^{i\alpha}t_{11})u(x,\lambda), \\ K_2(x,\lambda) &:= ([u(\cdot,\lambda),\phi](b) - e^{i\alpha}t_{22})v(x,\lambda) - ([v(\cdot,\phi),\theta](b) - e^{i\alpha}t_{21})u(x,\lambda). \end{aligned} \tag{4.11}$$

Proof. See [16]. □

The real case. The second part of [16] is concerned with real boundary value problems, that is, $\alpha = -\pi, 0, \pi$, for which the following structural condition holds (see Remark 4.5).

Condition 4.7. In the real case $\alpha = -\pi, 0, \pi$, all the eigenvalues are assumed to be simple.

The results in this case are similar to the results stated in Theorem 4.6 except that a phenomenon of degeneracy may occur; see [16, Section 8, Definition 3].

Theorem 4.8. *Let all the conditions of Theorem 4.6 hold with the addition of conditions (4.6); let the kernels K_1 and K_2 be given by (4.11) and the phenomenon of degeneracy be defined as in* [16]. *For $r = 1,2$, let the subspaces $L_r^2((a,b);w)$ of $L^2((a,b);w)$ be defined by*

$$L_r^2((a,b);w) := \operatorname{span}\{\psi_n;\ n \in \mathbb{Z}_r\} \quad (r = 1,2). \tag{4.12}$$

Then

(i) *every eigenvalue in $\{\lambda_n : n \in \mathbb{Z}\}$ is nondegenerate for at least one K_r;*
(ii) *for $r = 1,2$, the kernel K_r is an analytic Kramer kernel for the subspace $L_r^2((a,b);w)$;*
(iii) *$K(x,\lambda) = \alpha_1 K_1(x,\lambda) + \alpha_2 K_2(x,\lambda)$ $(x \in (a,b);\ \lambda \in \mathbb{C})$ is an analytic Kramer kernel for the whole space $L^2((a,b);w)$, for $\alpha_1, \alpha_2 \in \mathbb{R}$.*

Proof. See [16]. □

Remark 4.9. The case when the multiplicity of the spectrum $\sigma(A)$ is 2 is fully examined in [17].

Examples for both the above complex and real cases can be found in [16] and also in [15, Section 5.2]. In all the examples, the regular differential equation

$$-y''(x) = \lambda y(x) \quad (x \in [-\pi,\pi]) \tag{4.13}$$

is considered and $\theta(x) = \cos x$ and $\phi(x) = \sin x$ are chosen so as to give the boundary conditions (see (4.6))

$$\mathbf{y}(\pi) \equiv \begin{pmatrix} y'(\pi) \\ -y(\pi) \end{pmatrix} = e^{i\alpha} T \begin{pmatrix} y'(-\pi) \\ -y(-\pi) \end{pmatrix} \equiv e^{i\alpha} T \mathbf{y}(-\pi). \tag{4.14}$$

The functions u and v that satisfy the initial conditions are

$$\begin{aligned} u(x,\lambda) &= -\cos\left(\sqrt{\lambda}(x+\pi)\right), \\ v(x,\lambda) &= \frac{1}{\sqrt{\lambda}} \sin\left(\sqrt{\lambda}(x+\pi)\right). \end{aligned} \tag{4.15}$$

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A DENSITY THEOREM FOR LOCALLY CONVEX LATTICES

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Let E be a real, locally convex, locally solid vector lattice of (AM)-type. First, we prove an approximation theorem of Bishop's type for a vector subspace of such a lattice. Second, using this theorem, we obtain a generalization of Nachbin's density theorem for weighted spaces.

1. Introduction

In this paper, we introduce the concept of antisymmetric ideal with respect to a pair (A,F), when A is a subset of the real part of the center of E, and F is a vector subspace of E. This notion is a generalization, for locally convex lattices, of the notion of antisymmetric set from the theory of function algebras.

Further, we study some properties of the family of antisymmetric ideals. For example, we show that every element of this family contains a unique minimal element belonging to this family.

The main result of this paper is Theorem 4.2 which states that for every $x \in E$ we have $x \in \overline{F}$ if and only if $\pi_I(x) \in \overline{\pi_I(F)}$ for any minimal (A,F)-antisymmetric ideal I, where π_I denotes the canonical mapping $E \to E/I$.

This theorem is a Bishop's type approximation theorem and generalizes a similar result for $C(X)$.

Finally, we show that if the pair (A,F) fulfils some supplementary conditions, then F is dense in E, and also show how Nachbin's density theorem for weighted spaces follows from this theorem.

2. Preliminaries

In the sequel, E denotes a real, locally convex, locally solid vector lattice of (AM)-type. For every closed ideal I of E, we will denote by π_I the canonical mapping $E \to E/I$ and by π_I' it's adjoint. The center $Z(E)$ of E is the algebra of all order-bounded endomorphisms on E, that is, those $U \in L(E,E)$ for which there exists $\lambda_U > 0$ such that $|U(x)| \leq \lambda_U |x|$, for all $x \in E$. The real part of the center is $\operatorname{Re} Z(E) = Z(E)_+ - Z(E)_+$.

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Definition 2.1. For every closed ideal I of E and every $U \in \operatorname{Re} Z(E)$, $\pi_I(U) : E/I \to E/I$ is defined by

$$\pi_I(U)(\pi_I(x)) = \pi_I(U(x)), \quad x \in E. \tag{2.1}$$

It is easily seen that the operator $\pi_I(U)$ is well defined. For every $A \subset Z(E)$, we denote

$$\pi_I(A) = \{\pi_I(U);\ U \in A\}. \tag{2.2}$$

Remark 2.2. If $A \subset \operatorname{Re} Z(E)$, then $\pi_I(A) \subset \operatorname{Re} Z(E/I)$.

Indeed, if $U \in A$, then, for every $x \in E$, we have

$$\begin{aligned} \left|\pi_I(U)(\pi_I(x))\right| &= \left|\pi_I(U(x))\right| = \pi_I(\left|U(x)\right|) \\ &\leq \pi_I(\lambda_U |x|) = \lambda_U \pi_I(|x|) = \lambda_U \left|\pi_I(x)\right|, \end{aligned} \tag{2.3}$$

hence $\pi_I(U) \in Z(E/I)$.

Definition 2.3. Let I and J be two closed ideals of E such that $I \subset J$. Then the following two mappings can be defined: $\pi_{IJ} : E/I \to E/J$ given by

$$\pi_{IJ}[\pi_I(x)] = \pi_J(x), \quad x \in E, \tag{2.4}$$

and $M_{IJ} : \operatorname{Re} Z(E/I) \to \operatorname{Re} Z(E/J)$ given by

$$M_{IJ}(U)(\pi_J(x)) = \pi_{IJ}(U(\pi_I(x))), \quad U \in \operatorname{Re} Z(E/I). \tag{2.5}$$

As a consequence of the inequality,

$$\begin{aligned} \left|M_{IJ}(U)(\pi_J(x))\right| &= \left|\pi_{IJ}(U(\pi_I(x)))\right| \\ &= \pi_{IJ}(\left|U(\pi_I)(x)\right|) \leq \pi_{IJ}(\lambda_U \left|\pi_I(x)\right|) \\ &= \lambda_U \pi_{IJ}(\left|\pi_I(x)\right|) = \lambda_U \pi_J(|x|) = \lambda_U \left|\pi_J(x)\right|, \end{aligned} \tag{2.6}$$

for every $x \in E$, the range of M_{IJ} is included in $\operatorname{Re} Z(E/J)$.

3. Antisymmetric ideals

Let A be a subset of $\operatorname{Re} Z(E)$ containing 0 and let F be a vector subspace of E.

Definition 3.1. A closed ideal I of E is said to be antisymmetric with respect to the pair (A,F) if, for every $U \in \pi_I(A)$ with the property $U[\pi_I(F)] \subset \pi_I(F)$, it follows that there exists a real number α such that $U = \alpha \mathbf{1}_{E/I}$, where $\mathbf{1}_{E/I}$ is the identity operator on E/I.

Of course, E itself is an antisymmetric ideal with respect to the pair (A,F) for every $A \subset \operatorname{Re} Z(E)$ and every vector subspace F of E.

Further, we denote by $\mathcal{A}_{A,F}(E)$ the family of all (A,F)-antisymmetric ideals of E.

Now we consider the particular case $E = C(X,\mathbb{R})$, where X is a compact Hausdorff space. It is well known that there is a one-to-one correspondence between the class of the closed ideals of $C(X,\mathbb{R})$ and the class of the closed subsets of X. Namely, for every closed subset S of X, the set $I_S = \{f \in C(X,\mathbb{R});\ f|S = 0\}$ is a closed ideal of $C(X,\mathbb{R})$ and every closed ideal of $C(X,\mathbb{R})$ has this form.

Definition 3.2. Let A be a subset of $C(X,\mathbb{R})$ with $0 \in A$ and let F be a closed subset of $C(X,\mathbb{R})$. A closed subset S of X is said to be antisymmetric with respect to the pair (A,F) if every $f \in A$ with the property $f \cdot g|S \in F|S$ for every $g \in F$ is constant on S.

Remark 3.3. A closed subset S of X is (A,F)-antisymmetric if and only if the corresponding ideal I_S is (A,F)-antisymmetric in the sense of Definition 3.1.

Indeed, it is sufficient to observe that $\pi_{I_S}(a) = a|S$ for every subset S of X.

Lemma 3.4. *Let (I_α) be a family of elements of $\mathcal{A}_{A,F}(E)$ such that $J = \sum_\alpha I_\alpha \neq E$. Then*

$$I = \cap_\alpha I_\alpha \in \mathcal{A}_{A,F}(E). \tag{3.1}$$

Proof. If $U \in \pi_I(A)$ has the property $U[\pi_I(F)] \subset \pi_I(F)$, then

$$M_{II_\alpha}(U)(\pi_{I_\alpha}(F)) = \pi_{II_\alpha}[U(\pi_I(F))] \subset \pi_{II_\alpha}[\pi_I(F)] = \pi_{I_\alpha}(F). \tag{3.2}$$

Let $V \in A$ be such that $U = \pi_I(V)$. For every $x \in E$, we have

$$\begin{aligned} M_{II_\alpha}(U)(\pi_{I_\alpha}(x)) &= \pi_{II_\alpha}[U(\pi_I(x))] = \pi_{II_\alpha}[\pi_I(V)(\pi_I(x))] \\ &= \pi_{II_\alpha}[\pi_I(V(x))] = \pi_{I_\alpha}[V(x)] = \pi_{I_\alpha}(V)(\pi_{I_\alpha}(x)). \end{aligned} \tag{3.3}$$

Thus, $M_{II_\alpha}(U) = \pi_{I_\alpha}(V) \in \pi_{I_\alpha}(A) \subset \operatorname{Re} Z(E/I_\alpha)$ and $M_{II_\alpha}(U)(\pi_{I_\alpha}(F)) \subset \pi_{I_\alpha}(F)$. Since $I_\alpha \in \mathcal{A}_{A,F}(E)$, it follows that an $a_\alpha \in \mathbb{R}$ exists such that $M_{II_\alpha}(U) = a_\alpha \cdot \mathbf{1}_{E/I_\alpha}$.

On the other hand, we have

$$M_{IJ}(U) = M_{I_\alpha J}[M_{II_\alpha}(U)] = a_\alpha \cdot \mathbf{1}_{E/I_\alpha}. \tag{3.4}$$

Since $J \neq E$, it follows that $a_\alpha = a$ (constant) for any α. Therefore,

$$M_{II_\alpha}(U) = a \cdot \mathbf{1}_{E/I_\alpha} = a \cdot M_{II_\alpha}(\mathbf{1}_{E/I}), \tag{3.5}$$

hence,

$$M_{II_\alpha}(U - a \cdot \mathbf{1}_{E/I}) = 0, \tag{3.6}$$

for any α, and this involves $U = a \cdot \mathbf{1}_{E/I}$. □

Corollary 3.5. *Every $I \in \mathcal{A}_{A,F}(E)$ contains a unique minimal ideal $\widetilde{I} \in \mathcal{A}_{A,F}(E)$.*

Proof. Let $I \in \mathcal{A}_{A,F}(E)$ be such that $I \neq E$ and let $\widetilde{I} = \cap\{J \in \mathcal{A}_{A,F}(E);\ J \subset I\}$. According to Lemma 3.4, $\widetilde{I} \in \mathcal{A}_{A,F}(E)$. It is now obvious that $\widetilde{I} \subset I$ and $\widetilde{I}$ is minimal. □

Further, we denote by $\widetilde{\mathcal{A}}_{A,F}(E)$ the family of all minimal closed ideals of E, antisymmetric with respect to the pair (A,F).

4. Bishop's type approximation theorem

LEMMA 4.1. *Let A be a subset of* $\operatorname{Re} Z(E)$ *with* $0 \in A$, *let F be a vector subspace of E, and let V be a convex and solid neighborhood of the origin of E, which is also a sublattice. If* $f \in \operatorname{Ext}\{V^0 \cap F^0\}$ *and* $I = \{x \in E;\ |f|(|x|) = 0\}$, *then* $I \in \mathcal{A}_{A,F}(E)$.

Proof. Let $U \in \pi_I(A)$ be such that $U[\pi_I(F)] \subset \pi_I(F)$. We can suppose that $0 \leq U \leq \mathbf{1}_{E/I}$. Since $f \in I^0$, there exists $g \in (E/I)'$ such that $f = \pi_1' g$. Obviously, $g \in \{[\pi_I(V)]^0 \cap [\pi_I(F)]^0\}$. We denote $g_1 = U'g$, $g_2 = (\mathbf{1}_{E/I} - U)'g$, and $a_i = \inf\{\lambda > 0 : g_i \in \lambda[\pi_I(V)]^0\} = \sup\{|g_i(y)| :\ y \in \pi_I(V)\}$, for $i = 1,2$.

Since $g = g_1 + g_2 \in (a_1 + a_2)[\pi_I(V)]^0$, it follows that $f \in (a_1 + a_2)V^0$, hence $a_1 + a_2 \geq 1$.

On the other hand, for any $y_1, y_2 \in \pi_I(V)$, we have

$$
\begin{aligned}
|g_1(y_1)| + |g_2(y_2)| &= |g(U(y_1))| + |g(\mathbf{1}_{E/I} - U)(y_2)| \\
&\leq |g|(U(|y_1| \vee |y_2|)) + (\mathbf{1}_{E/I} - U)(|y_1| \vee |y_2|) \\
&= |g|(|y_1| \vee |y_2|).
\end{aligned}
\tag{4.1}
$$

Since $\pi_I(V)$ is a sublattice and $g \in [\pi_I(V)]^0$, it follows that $|y_1| \vee |y_2| \in \pi_I(V)$, hence $|g|(|y_1| \vee |y_2|) \leq 1$.

Therefore, $|g_1(y_1)| + |g_2(y_2)| \leq 1$ for any $y_1, y_2 \in \pi_I(V)$ and this yields $a_1 + a_2 \leq 1$, hence $a_1 + a_2 = 1$.

Now, we observe that if $|g|(|y|) = 0$, then $y = 0$. Indeed, let $x \in E$ be such that $y = \pi_I(x)$.

We have $0 = |g|(|\pi_I(x)|) = |\pi_I' g|(|x|) = |f|(|x|)$.

If follows that $x \in I$, hence $y = \pi_I(x) = 0$.

This remark involves that if $g_1 = U'g = 0$, then $U = 0$ and, analogously, $g_2 = (\mathbf{1}_{E/I} - u)'g = 0$ implies $U = \mathbf{1}_{E/I}$.

Therefore, we can suppose that $g_i \neq 0$ for $i = 1,2$, and hence $a_i > 0$, $i = 1,2$. Further, we have

$$
g = a_1 \frac{g_1}{a_1} + a_2 \frac{g_2}{a_2}, \qquad \frac{g_i}{a_i} \in [\pi_I(V)]^0 \cap [\pi_I(F)]^0, \quad i = 1,2. \tag{4.2}
$$

Since $g \in \operatorname{Ext}\{[\pi_1(V)]^0 \cap [\pi_I(F)]^0\}$, either $g = g_1/a_1$ or $g = g_2/a_2$. In the first case, $(U - a_1\mathbf{1}_{E/I})'(g) = 0$.

The last equality yields $U = a_1\mathbf{1}_{E/I}$. □

The main result concerning antisymmetric ideals is the following Bishop's type approximation theorem.

THEOREM 4.2. *Let E be a real, locally convex, locally solid vector lattice of (AM)-type,* $A \subset \operatorname{Re} Z(E)$ *with* $0 \in A$, *and let F be a vector subspace of E. Then, for any $x \in E$,*

$$
x \in \overline{F} \Longleftrightarrow \pi_I(x) \in \overline{\pi_I(F)} \tag{4.3}
$$

for every $I \in \widetilde{\mathcal{A}}_{A,F}(E)$.

Proof. The necessity is clear. We suppose that $\pi_I(x) \in \overline{\pi_I(F)}$ for any $I \in \widetilde{\mathcal{A}}_{A,F}(E)$ and that $x \notin \overline{F}$. Then, there exists $f \in E'$ such that $f(x) \neq 0$ and $f(y) = 0$ for any $y \in F$.

Let V be a solid, convex neighborhood of the origin which is also a sublattice of E. By the Krein-Milman theorem, we may assume that $f \in \mathrm{Ext}\{V^0 \cap F^0\}$. If we denote $J = \{x \in E;\ |f|(|x|) = 0\}$, then, according to Lemma 4.1, we have $J \in \widetilde{\mathcal{A}}_{A,F}(E)$. On the other hand, by Corollary 3.5, it follows that there exists $J_0 \in \widetilde{\mathcal{A}}_{A,F}(E)$ such that $J_0 \subset J$. Since $\pi_{J_0}(x) \in \overline{\pi_{J_0}(F)}$ and $f \in J_0^0 \cap F^0$, we have $f(x) = 0$, and this contradicts the choice of f. □

Theorem 4.3. *Let E be a real, locally convex, locally solid vector lattice of (AM)-type, let A be a subset of* $\mathrm{Re}\,Z(E)$ *with $0 \in A$, and let F be a vector subspace of E with the properties*

(i) *$AF \subset F$,*
(ii) *F is not included in any maximal ideal of E,*
(iii) *every closed (A,F)-antisymmetric ideal I of E with the property $\pi_I(A) \subset \mathbb{R} \cdot \mathbf{1}_{E/I}$ is a maximal ideal.*

Then $\overline{F} = E$.

Proof. Let $x \in E$ and $I \in \widetilde{\mathcal{A}}_{A,F}(E)$. Hypothesis (i) involves that $\pi_I(A)[\pi_I(F)] \subset \pi_I(F)$, and since I is (A,F)-antisymmetric, we have $\pi_I(U) = \alpha_U \cdot \mathbf{1}_{E/I}$ for any $U \in A$. Now, from (iii), it results that I is a maximal ideal and thus that the dimension of $\pi_I(E)$ is one.

Since $F \subset E$, we have either $\pi_I(F) = \{0\}$ or $\pi_I(F) = \pi_I(E)$.

From (ii), it results that $\pi_I(F) \neq \{0\}$. Therefore, we have $\pi_I(F) = \pi_I(E)$ and thus $\pi_I(x) \in \pi_I(F)$ for any $I \in \widetilde{\mathcal{A}}_{A,F}(E)$. According to Theorem 4.2, it follows that $x \in \overline{F}$. □

5. The case of weighted spaces

Typical examples of locally convex lattices are the weighted spaces.

Let X be a locally compact Hausdorff space and let V be a Nachbin family on X, that is, a set of nonnegative upper semicontinuous functions on X directed in the sense that, given $v_1, v_2 \in V$ and $\lambda > 0$, a $v \in A$ exists such that $v_i \leq \lambda v$, $i = 1,2$. We denote by $CV_0(X)$ the corresponding weighted spaces, that is,

$$CV_0(X) = \{f \in C(X,\mathbb{R});\ fv \text{ vanishes at infinity for any } v \in V\}. \tag{5.1}$$

The weighted topology on $CV_0(X)$ is denoted by ω_V and it is determined by the seminorms $\{p_v\}_{v \in V}$, where

$$p_v(f) = \sup\{|f(x)|\,v(x):\ x \in X\}, \quad \text{for any } f \in CV_0(X). \tag{5.2}$$

The topology ω_V is locally convex and has a basis of open neighborhoods of the origin of the form

$$D_v = \{f \in CV_0(x):\ p_v(f) < 1\}. \tag{5.3}$$

Clearly, $CV_0(X)$ is a locally convex, locally solid vector lattice of (AM)-type with respect to the topology ω_V and to the ordering $f \leq g$ if and only if $f(x) \leq g(x)$, $x \in X$.

A result of Goullet de Rugy [1, Lemma 3.8] states that for every closed ideal I of $CV_0(X)$ there exists a closed subset Y of X such that

$$I = \{f \in CV_0(X) : f|Y = 0\}. \tag{5.4}$$

Therefore, there exists a one-to-one map from the family of closed ideals of $CV_0(X)$ onto the family of closed subsets of X.

If X is a compact Hausdorff space and $V = \{1\}$, then $CV_0(X) = C(X, \mathbf{R})$ and the weighted topology ω_V coincides with the uniform topology of $C(X, \mathbb{R})$.

Further, we denote by $C_b(X, \mathbb{R})$ the algebra of all real bounded continuous functions on X.

As in the case of $C(X)$, we have the following definition.

Definition 5.1. Let A be a subset of $C_b(X)$ with $0 \in A$ and let F be a vector subspace of $CV_0(X)$. A closed subset S of X is called antisymmetric with respect to the pair (A, F) if and only if the corresponding ideal

$$I_S = \{f \in CV_0(X) : f|S = 0\} \tag{5.5}$$

is an (A, F)-antisymmetric ideal, and this means that every $a \in A$ with the property $\alpha \cdot h|S \in F|S$, for any $h \in F$, is constant on S.

It is easily seen that every $x \in X$ belongs to a maximal (A, F)-antisymmetric set S_x. At the same time, if $x \neq y$, we have either $S_x = S_y$ or $S_x \cap S_y = \varnothing$.

Theorem 4.2 then involves the following theorem.

Theorem 5.2. *Let A and F be as in Definition 5.1. Then, a function $f \in CV_0(X)$ belongs to F if and only if $f|S_x \in \overline{F|S_x}$ for any $x \in X$.*

The following theorem is a generalization of Nachbin's density theorem for weighted spaces in the real case.

Theorem 5.3. *Let A be a subset of $C_b(X, \mathbb{R})$ with $0 \in A$ and let F be a vector subspace of $CV_0(X)$ with the properties*

(i) *$AF \subset F$,*
(ii) *A separates the points of X,*
(iii) *for every $x \in X$, there is an $f \in F$ such that $f(x) \neq 0$.*

Then $\overline{F} = CV_0(X)$.

Proof. Since the centre of the lattice $E = CV_0(X)$ is the algebra $C_b(X)$ of all continuous bounded functions on X (see, e.g., [2]), it follows that $A \subset \operatorname{Re} Z(E)$. On the other hand, from (iii), it follows that F is not included in any maximal ideal. Since $AF \subset F$ and A separates the points of X, it results that every (A, F)-antisymmetric subset S of X is a singleton, and thus the corresponding ideal I_S is a maximal ideal. Thus the hypotheses of Theorem 4.3 are satisfied and so Theorem 5.3 is proved. $\square$

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ON PERIODIC-TYPE SOLUTIONS OF SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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We establish nonimprovable, in a certain sense, sufficient conditions for the existence of a unique periodic-type solution for systems of linear ordinary differential equations.

1. Formulation of the problem and statement of the main results

Let n_1 and n_2 be natural numbers, $\omega > 0$, $\Lambda_i \in \mathbb{R}^{n_i \times n_i}$ $(i = 1,2)$ nonsingular matrices, and $\mathcal{P}_{ik} : \mathbb{R} \to \mathbb{R}^{n_i \times n_k}$ $(i,k = 1,2)$ and $q_i : \mathbb{R} \to \mathbb{R}^{n_i}$ $(i = 1,2)$ matrix and vector functions whose components are Lebesgue integrable on each compact interval. We consider the problem on the existence and uniqueness of a solution of the linear differential system

$$\frac{dx_i}{dt} = \mathcal{P}_{i1}(t)x_1 + \mathcal{P}_{i2}(t)x_2 + q_i(t) \quad (i = 1,2), \tag{1.1}$$

satisfying the conditions

$$x_i(t+\omega) = \Lambda_i x_i(t) \quad \text{for } t \in \mathbb{R} \ (i = 1,2). \tag{1.2}$$

When Λ_1 and Λ_2 are unit matrices, this problem becomes the well-known problem on a periodic solution which has been the subject of numerous studies (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and the references therein).

In this paper, sufficient conditions for the unique solvability of problem (1.1), (1.2) are established, which are nonimprovable in a certain sense and in particular provide new results on the existence of a unique ω-periodic solution of system (1.1).

The following notation is used in the paper:

(1) $\mathbb{R}$ is the set of real numbers;

(2) $\mathbb{R}^n$ is the n-dimensional real Euclidean space;

(3) $x = (\xi_i)_{i=1}^n \in \mathbb{R}^n$ is the column vector with components $\xi_1, \dots, \xi_n$,

$$|x| = \left(|\xi_i|\right)_{i=1}^n, \qquad \|x\| = \left(\sum_{i=1}^n \xi_i^2\right)^{1/2}; \tag{1.3}$$

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(4) $x \cdot y$ is the scalar product of vectors $x, y \in \mathbb{R}^n$;
(5) $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices $X = (\xi_{ik})_{i,k=1}^{m,n}$ with components ξ_{ik} ($i = 1,\dots, m; k = 1,\dots,n$),

$$|X| = (|\xi_{ik}|)_{i,k=1}^{m,n}, \qquad \|X\| = \left(\sum_{k=1}^{n}\sum_{i=1}^{m}\xi_{ik}^2\right)^{1/2}; \tag{1.4}$$

(6) X^* is the transposed matrix of the matrix X;
(7) E_n is the unit $n \times n$ matrix;
(8) $\det(X)$ is the determinant of the matrix X;
(9) $r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$;
(10) if $X \in \mathbb{R}^{n\times n}$, then $\lambda_0(X)$ is a minimal eigenvalue of the matrix $(1/2)(X + X^*)$.

Inequalities between the matrices and the vectors are understood componentwise. Throughout the paper, it will be assumed that

$$\mathcal{P}_{ik}(t+\omega) = \Lambda_i \mathcal{P}_{ik}(t)\Lambda_k^{-1}, \quad q_i(t+\omega) = \Lambda_i q_i(t) \quad \text{for } t \in \mathbb{R}\ (i,k = 1,2). \tag{1.5}$$

For each $i \in \{1,2\}$, consider the differential system

$$\frac{dx}{dt} = \mathcal{P}_{ii}(t)x \tag{1.6}$$

and denote by X_i its fundamental matrix satisfying the initial condition

$$X_i(0) = E_{n_i}. \tag{1.7}$$

If, however, the matrix $\Lambda_i - X_i(\omega)$ is nonsingular, then it is assumed that

$$G_i(t,\tau) = X_i(t)\big(X_i^{-1}(\omega)\Lambda_i - E_{n_i}\big)^{-1}X_i^{-1}(\tau). \tag{1.8}$$

For each $i \in \{1,2\}$, we define a matrix function $\Lambda_{i0} : [0,3\omega] \to \mathbb{R}^{n_i\times n_i}$ in the following manner:

$$\Lambda_{i0}(s) = E_{n_i} \quad \text{for } 0 \le s \le \omega, \tag{1.9}$$

$$\Lambda_{i0}(s) = |\Lambda_i^k| \quad \text{for } k\omega < s \le (k+1)\omega\ (k = 1,2). \tag{1.10}$$

Theorem 1.1. *Let*

$$\det\big(\Lambda_i - X_i(\omega)\big) \ne 0 \quad (i = 1,2), \tag{1.11}$$

and there exists a nonnegative matrix $A \in \mathbb{R}^{n_1\times n_1}$ such that $r(A) < 1$, and

$$\int_t^{t+\omega}\int_\tau^{\tau+\omega} \big|G_1(t,\tau)\mathcal{P}_{12}(\tau)G_2(\tau,s)\mathcal{P}_{12}(s)\big|\Lambda_{10}(s)\,ds\,d\tau \le A \quad \textit{for } 0 \le t \le \omega. \tag{1.12}$$

Then problem (1.1), (1.2) has a unique solution.

Example 1.2. Let $n_1 = n_2 = 1$, $\Lambda_1 = \Lambda_2 = 1$, $q_i(t) \equiv 0$, $\mathcal{P}_{i1}(t) \equiv p_i(t)$, and $\mathcal{P}_{i2}(t) = -p_i(t)$, where $p_i : \mathbb{R} \to]0,+\infty[$ $(i = 1,2)$ are the integrable on $[0,\omega]$ ω-periodic functions. Then conditions (1.5), (1.11), and (1.12), where $A = 1$, are fulfilled. On the other hand, in the considered case, system (1.1) has the form

$$\frac{dx_i}{dt} = p_i(t)(x_1 - x_2) \quad (i = 1,2) \tag{1.13}$$

and therefore problem (1.1), (1.2) has an infinite set of solutions

$$\{(x_1,x_2) : x_1(t) \equiv x_2(t) \equiv c,\ c \in \mathbb{R}\}. \tag{1.14}$$

This example shows that the condition $r(A) < 1$ in Theorem 1.1 is nonimprovable and it cannot be replaced by the condition $r(A) \le 1$.

Theorem 1.3. *Let*

$$X_i(\omega) = \Lambda_1, \qquad \det\left(\Lambda_2 - X_2(\omega)\right) \ne 0, \tag{1.15}$$

$$\det\left(Q_0\right) \ne 0, \tag{1.16}$$

where

$$\begin{aligned} Q_0 &= \int_0^\omega X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)Q(\tau)d\tau, \\ Q(t) &= \int_t^{t+\omega} G_2(t,s)\mathcal{P}_{21}(s)X_1(s)ds. \end{aligned} \tag{1.17}$$

Let, further, there exist a nonnegative matrix $A \in \mathbb{R}^{n_2\times n_2}$ such that $r(A) < 1$, and

$$\int_t^{t+\omega}\left[H(t,\tau) + \int_\tau^{\tau+\omega} \left|Q(t)Q_0^{-1}X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)\right| H(\tau,s)ds\right]d\tau \le A \quad \text{for } 0 \le t \le \omega, \tag{1.18}$$

where

$$H(t,\tau) = \int_0^\tau \left|G_2(t,\tau)\mathcal{P}_{21}(\tau)X_1(\tau)X_1^{-1}(s)\mathcal{P}_{12}(s)\right|\Lambda_{20}(s)ds. \tag{1.19}$$

Then problem (1.1), (1.2) has a unique solution.

Example 1.4. Consider the problem

$$\begin{aligned} &\frac{dx_1}{dt} = B_1x_2, \qquad \frac{dx_2}{dt} = \varepsilon B_2x_1 + Bx_2, \\ &x_i(t+\omega) = x_i(t) \quad \text{for } t \in \mathbb{R}\ (i = 1,2), \end{aligned} \tag{1.20}$$

where ε is a positive constant, $B_1 \in \mathbb{R}^{n_1\times n_2}$, $B_2 \in \mathbb{R}^{n_2\times n_1}$, $B \in \mathbb{R}^{n_2\times n_2}$, and $\det(B) \ne 0$. This problem is obtained from problem (1.1), (1.2) when $\Lambda_i = E_{n_i}$ $(i = 1,2)$, $\mathcal{P}_1$ is a zero matrix, $\mathcal{P}_{12}(t) \equiv B_1$, $\mathcal{P}_{21}(t) \equiv \varepsilon B_2$, $\mathcal{P}_{22}(t) \equiv B$, and $q(t) \equiv 0$. It is obvious that conditions

(1.5) and (1.15) are fulfilled for this problem. On the other hand, by virtue of (1.17) and (1.19), we have

$$Q(t) \equiv \varepsilon B^{-1}B_2, \qquad Q_0 = \varepsilon\omega B_1B^{-1}B_2,$$
$$H(t,\tau) = \tau\varepsilon\left|\left(\exp(-\omega B) - E_{n_2}\right)^{-1}\exp\left((t-\tau)B\right)B_2B_1\right|. \tag{1.21}$$

Therefore, condition (1.16) is fulfilled if and only if

$$\det\left(B_1B^{-1}B_2\right) \neq 0. \tag{1.22}$$

If the latter inequality is fulfilled, then, by Theorem 1.3, there exists $\varepsilon_0 > 0$ such that, for arbitrary $\varepsilon \in]0,\varepsilon_0[$, problem (1.20) has only a trivial solution. If $\det(B_1B^{-1}B_2) = 0$, then, for arbitrary ε, problem (1.20) has an infinite set of solutions

$$\left\{(x_1,x_2) : x_1(t) \equiv cx_{10},\ x_2(t) = cx_{20},\ c \in \mathbb{R}\right\}, \tag{1.23}$$

where $x_{10} \in \mathbb{R}^{n_1}$ is the eigenvector of the matrix $B_1B^{-1}B_2$ corresponding to the zero eigenvalue and $x_{20} = -\varepsilon B^{-1}B_2x_{10}$.

Example 1.4 shows that condition (1.16) is essential and cannot be omitted.

Theorem 1.5. *Let there exist a matrix $A \in \mathbb{R}^{n_1\times n_2}$, symmetric matrices $A_i \in \mathbb{R}^{n_i\times n_i}$ $(i = 1,2)$, and an integrable function $\delta : [0,\omega] \to [0,+\infty[$ such that*

$$\Lambda_2^*A\Lambda_1 = A, \qquad \Lambda_i^*A_i\Lambda_i = A_i \quad (i = 1,2) \tag{1.24}$$

and the following inequalities are fulfilled almost everywhere on $[0,\omega]$:

$$\lambda_0\left(A_1\mathcal{P}_{11}(t) + A^*\mathcal{P}_{21}(t)\right) \geq \delta(t), \qquad \lambda_0\left(A_2\mathcal{P}_{22}(t) + A\mathcal{P}_{12}(t)\right) \geq \delta(t), \tag{1.25}$$
$$\delta(t) \geq p(t), \tag{1.26}$$

where

$$p(t) = \frac{1}{2}\left(\left\|A_1\mathcal{P}_{12}(t) + A^*\mathcal{P}_{22}(t)\right\| + \left\|A_2\mathcal{P}_{21}(t) + A\mathcal{P}_{11}(t)\right\|\right). \tag{1.27}$$

If, moreover,

$$\int_0^\omega \left(\delta(t) - p(t)\right)dt > 0, \tag{1.28}$$

then problem (1.1), (1.2) has a unique solution.

Example 1.2 shows that conditions (1.5), (1.24), (1.25), and (1.26) do not guarantee the unique solvability of problem (1.1), (1.2). Therefore, condition (1.28) in Theorem 1.5 is essential and cannot be omitted.

2. Auxiliary propositions

In this section, we consider the problem

$$\frac{dx}{dt} = \mathcal{P}(t)x + q(t), \tag{2.1}$$

$$x(t+\omega) = \Lambda x(t) \quad \text{for } t \in \mathbb{R}, \tag{2.2}$$

assuming that $\Lambda \in \mathbb{R}^{n\times n}$ is a nonsingular matrix, and $\mathcal{P} : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $q : \mathbb{R} \to \mathbb{R}^n$ are matrix and vector functions with components Lebesgue integrable on $[0,\omega]$ and satisfying the conditions

$$\mathcal{P}(t+\omega) = \Lambda \mathcal{P}(t)\Lambda^{-1} \quad \text{for } t \in \mathbb{R}, \tag{2.3}$$

$$q(t+\omega) = \Lambda q(t) \quad \text{for } t \in \mathbb{R}. \tag{2.4}$$

We denote by X the fundamental matrix of the homogeneous differential system

$$\frac{dx}{dt} = \mathcal{P}(t)x, \tag{2.5}$$

satisfying the initial condition

$$X(0) = E_n. \tag{2.6}$$

Condition (2.3) immediately implies the following lemma.

Lemma 2.1. *The matrix function X satisfies the identity*

$$X(t+\omega) = \Lambda X(t)\Lambda^{-1}X(\omega) \quad \textit{for } t \in \mathbb{R}. \tag{2.7}$$

Lemma 2.2. *Problem (2.5), (2.2) has only a trivial solution if and only if*

$$\det\left(\Lambda - X(\omega)\right) \neq 0. \tag{2.8}$$

Proof. Let x be an arbitrary solution of system (2.5). Then

$$x(t) = X(t)c \quad \text{for } t \in \mathbb{R}, \tag{2.9}$$

where $c \in \mathbb{R}^n$. Hence, by Lemma 2.1, it follows that x is a solution of problem (2.5), (2.2) if and only if

$$\left(\Lambda X(t) - \Lambda X(t)\Lambda^{-1}X(\omega)\right)c = 0 \quad \text{for } t \in \mathbb{R}. \tag{2.10}$$

However, for the latter identity to be fulfilled, it is necessary and sufficient that c be a solution of the system of algebraic equations

$$\left(\Lambda - X(\omega)\right)c = 0. \tag{2.11}$$

Therefore, problem (2.5), (2.2) has only a trivial solution if and only if the latter system has only a trivial solution, that is, if (2.8) is fulfilled. □

Lemma 2.3. *Problem (2.1), (2.2) is uniquely solvable if and only if the corresponding homogeneous problem (2.5), (2.2) has only a trivial solution, that is, if inequality (2.8) is fulfilled. Moreover, if (2.8) is fulfilled, then the solution of problem (2.1), (2.2) admits the representation*

$$x(t) = \int_t^{t+\omega} G(t,s)q(s)ds \quad \text{for } t \in \mathbb{R}, \tag{2.12}$$

where

$$G(t,s) = X(t)\big(X^{-1}(\omega)\Lambda - E_n\big)^{-1}X^{-1}(s). \tag{2.13}$$

Proof. By Lemma 2.2, to prove Lemma 2.3, it is sufficient to establish that if inequality (2.8) is fulfilled, then the vector function x given by equality (2.12) is a solution of problem (2.1), (2.2).

According to (2.7) and (2.13), we have

$$\begin{aligned}
\frac{\partial G(t,s)}{\partial t} &= \mathcal{P}(t)G(t,s) \quad \text{for } s \in \mathbb{R} \text{ and almost all } t \in \mathbb{R},\\
G(t,t+\omega)\Lambda - G(t,t) &= X(t)\big(X^{-1}(\omega)\Lambda - E_n\big)^{-1}\big(X^{-1}(t+\omega)\Lambda - X^{-1}(t)\big)\\
&= X(t)\big(X^{-1}(\omega)\Lambda - E_n\big)^{-1}\big(X^{-1}(\omega)\Lambda - E_n\big)X^{-1}(t)\\
&= E_n \quad \text{for } t \in \mathbb{R},\\
G(t+\omega,s+\omega) &= \Lambda X(t)\Lambda^{-1}X(\omega)\big(X^{-1}(\omega)\Lambda - E_n\big)^{-1}\\
&\quad \times \big(\Lambda^{-1}X(\omega)\big)^{-1}X^{-1}(t)\Lambda^{-1}\\
&= \Lambda G(t,s)\Lambda^{-1} \quad \text{for } s \in \mathbb{R},\ t \in \mathbb{R}.
\end{aligned} \tag{2.14}$$

If, along with these identities, we also take into consideration condition (2.4), then, from (2.12), we obtain

$$\begin{aligned}
\frac{dx(t)}{dt} &= \mathcal{P}(t)x(t) + G(t,t+\omega)q(t+\omega) - G(t,t)q(t)\\
&= \mathcal{P}(t)x(t) + \big(G(t,t+\omega)\Lambda - G(t,t)\big)q(t)\\
&= \mathcal{P}(t)x(t) + q(t) \quad \text{for almost all } t \in \mathbb{R}^n,\\
x(t+\omega) &= \int_{t+\omega}^{t+2\omega} G(t+\omega,s)q(s)ds = \int_t^{t+\omega} G(t+\omega,s+\omega)q(s+\omega)ds\\
&= \Lambda \int_t^{t+\omega} G(t,s)q(s)ds = \Lambda x(t) \quad \text{for } t \in \mathbb{R}.
\end{aligned} \tag{2.15}$$

Thus x is a solution of problem (2.1), (2.2). □

3. Proofs of the main results

Proof of Theorem 1.1. By Lemma 2.3, it is sufficient to show that the homogeneous problem

$$\frac{dx_i}{dt} = \mathcal{P}_{i1}(t)x_1 + \mathcal{P}_{i2}(t)x_2, \tag{3.1}$$

$$x_i(t+\omega) = \Lambda_i x_i(t) \quad \text{for } t \in \mathbb{R} \ (i = 1,2) \tag{3.2}$$

has only a trivial solution.

Let (x_1, x_2) be an arbitrary solution of this problem. By virtue of Lemma 2.3, condition (1.11) and the equalities

$$\begin{aligned} \mathcal{P}_{12}(t+\omega)x_2(t+\omega) &= \Lambda_1 \mathcal{P}_{12}(t)x_2(t), \\ \mathcal{P}_{21}(t+\omega)x_1(t+\omega) &= \Lambda_2 \mathcal{P}_{21}(t)x_1(t) \quad \text{for almost all } t \in \mathbb{R} \end{aligned} \tag{3.3}$$

guarantee the validity of the representations

$$\begin{aligned} x_1(t) &= \int_t^{t+\omega} G_1(t,s)\mathcal{P}_{12}(s)x_2(s)ds, \\ x_2(t) &= \int_t^{t+\omega} G_2(t,s)\mathcal{P}_{21}(s)x_1(s)ds. \end{aligned} \tag{3.4}$$

Therefore,

$$x_1(t) = \int_t^{t+\omega} \int_\tau^{\tau+\omega} G_1(t,\tau)\mathcal{P}_{12}(\tau)G_2(\tau,s)\mathcal{P}_{21}(s)x_1(s)ds. \tag{3.5}$$

Let

$$\begin{aligned} x_1(t) &= \left(x_{1k}(t)\right)_{k=1}^{n_1}, \\ \rho_k &= \max\left\{ \left| x_{1k}(t) \right| : 0 \le t \le \omega \right\} \quad (k = 1,\dots,n_1),\ \rho = \left(\rho_k\right)_{k=1}^{n_1}. \end{aligned} \tag{3.6}$$

Then by (1.9), (1.10) for $i = 1$, we have

$$\left| x_1(s) \right| \le \Lambda_{10}(s)\rho \quad \text{for } 0 \le s \le 3\omega. \tag{3.7}$$

If, along with this, we also take into consideration inequality (1.12), then, from representation (3.5), we obtain

$$\left| x_1(t) \right| \le A\rho \quad \text{for } 0 \le t \le \omega. \tag{3.8}$$

Hence $\rho \le A\rho$ and, therefore,

$$\left(E_{n_1} - A\right)\rho \le 0. \tag{3.9}$$

According to the condition $r(A) < 1$ and the nonnegativeness of the matrix A, the matrix $E_{n_1} - A$ is nonsingular and $(E_{n_1} - A)^{-1}$ is nonnegative. Hence the multiplication of the latter vector inequality by $(E_{n_1} - A)^{-1}$ gives $\rho \le 0$. Therefore, $\rho = 0$, that is,

$$x_1(t) = 0 \quad \text{for } 0 \le t \le \omega. \tag{3.10}$$

By virtue of this equality, from (3.4), it follows that $x_i(t) = 0$ for $t \in \mathbb{R}$ $(i = 1,2)$. □

Proof of Theorem 1.3. Let (x_1,x_2) be an arbitrary solution of problem (3.1), (3.2). Then by the Cauchy formula, we have

$$x_1(t) = X_1(t)c + \int_0^t X_1(t)X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)x_2(\tau)d\tau, \tag{3.11}$$

where $c \in \mathbb{R}^{n_1}$. On the other hand, by Lemma 2.3, the nonsingularity of the matrix $\Lambda_2 - X_2(\omega)$ and the equality

$$\mathcal{P}_{21}(t+\omega)x_1(t+\omega) = \Lambda_2\mathcal{P}_{21}(t)x_1(t) \quad \text{for almost all } t \in \mathbb{R} \tag{3.12}$$

guarantee the validity of the representation

$$x_2(t) = \int_t^{t+\omega} G_2(t,\tau)\mathcal{P}_{21}(\tau)x_1(\tau)d\tau. \tag{3.13}$$

Hence, by virtue of equalities (1.17) and (3.11), it follows that

$$x_2(t) = Q(t)c + \int_t^{t+\omega} z(t,\tau)d\tau, \tag{3.14}$$

where

$$z(t,\tau) = \int_0^\tau G_2(t,\tau)\mathcal{P}_{21}(\tau)X_1(\tau)X_1^{-1}(s)\mathcal{P}_{12}(s)x_2(s)ds. \tag{3.15}$$

By Lemma 2.1 and the equality $X_1(\omega) = \Lambda_1$, we have

$$X_1(t+\omega) = \Lambda_1 X(t) \quad \text{for } t \in \mathbb{R}. \tag{3.16}$$

Therefore, from (3.11), we find

$$x_1(t+\omega) = \Lambda_1 X_1(t)c + \Lambda_1\int_0^{t+\omega} X_1(t)X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)x_2(\tau)d\tau. \tag{3.17}$$

Hence, by (3.2), it follows that

$$x_1(t) = X_1(t)c + \int_0^{t+\omega} X_1(t)X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)x_2(\tau)d\tau. \tag{3.18}$$

If now we again apply representation (3.11), then it becomes clear that the identity

$$\int_t^{t+\omega} X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)x_2(\tau)d\tau = 0 \quad \text{for } t \in \mathbb{R} \tag{3.19}$$

is valid.

Using (3.14), from the latter identity, we find

$$\widetilde{Q}(t)c = -\int_t^{t+\omega}\int_\tau^{\tau+\omega} X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)z(\tau,s)\,ds\,d\tau, \tag{3.20}$$

where

$$\widetilde{Q}(t) = \int_t^{t+\omega} X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)Q(\tau)\,d\tau. \tag{3.21}$$

By Lemma 2.1,

$$G_2(t+\omega,s+\omega) = \Lambda_2 G_2(t,s)\Lambda_2^{-1}. \tag{3.22}$$

If, along with this identity, we also take into account identities (1.5) and (3.16), then we obtain

$$Q(t+\omega) = \int_t^{t+\omega} G_2(t+\omega,s+\omega)\mathcal{P}_{21}(s+\omega)X_1(s+\omega)\,ds = \Lambda_2 Q(t). \tag{3.23}$$

Therefore, from (1.17) and (3.21), we have

$$\begin{aligned}\widetilde{Q}(t) &= \int_t^{\omega} X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)Q(\tau)\,d\tau \\ &\quad + \int_0^{t} X_1^{-1}(\tau+\omega)\mathcal{P}_{12}(\tau+\omega)Q(\tau+\omega)\,d\tau \\ &= \int_t^{\omega} X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)Q(\tau)\,d\tau \\ &\quad + \int_0^{t} X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)Q(\tau)\,d\tau = Q_0 \quad \text{for } t\in\mathbb{R}.\end{aligned} \tag{3.24}$$

By virtue of this fact and condition (1.16), from (3.11), (3.14), and (3.20), we get

$$\begin{aligned}x_1(t) &= \int_0^t X_1(t)X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)x_2(\tau)\,d\tau \\ &\quad - X_1(t)\int_0^{\omega}\int_\tau^{\tau+\omega} Q_0^{-1}X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)z(\tau,s)\,ds\,d\tau,\end{aligned} \tag{3.25}$$

$$x_2(t) = \int_t^{t+\omega}\left(z(t,\tau) - \int_\tau^{\tau+\omega} Q(t)Q_0^{-1}X_1^{-1}(\tau)\mathcal{P}_{12}(\tau)z(\tau,s)\,ds\right)d\tau. \tag{3.26}$$

Let $x_2(t) = (x_{2k}(t))_{k=1}^{n_2}$,

$$\rho_k = \max\{\,|x_{2k}(t)| : 0\le t\le\omega\} \quad (k=1,\dots,n_2),\ \rho = (\rho_k)_{k=1}^{n_2}. \tag{3.27}$$

Then, by (1.9), (1.10) for $i=2$, we have

$$|x_2(s)| \le \Lambda_{20}(s)\rho \quad \text{for } 0\le s\le 3\omega. \tag{3.28}$$

By this inequality and the notation (1.19) and (3.15), we have

$$|z(t,\tau)| \le H(t,\tau)\rho \quad \text{for } t\in\mathbb{R},\ 0\le\tau\le 3\omega. \tag{3.29}$$

Due to this estimate and inequality (1.18), from (3.26), we find

$$|x_2(t)| \le A\rho \quad \text{for } 0 \le t \le \omega. \tag{3.30}$$

Hence it is clear that $\rho \le A\rho$ and, therefore,

$$(E_{n_2} - A)\rho \le 0. \tag{3.31}$$

By virtue of the condition $r(A) < 1$ and the nonnegativeness of the matrix A, the latter inequality implies $\rho = 0$. Therefore,

$$x_2(t) = 0, \quad z(t,\tau) = 0 \quad \text{for } 0 \le t \le \omega,\ 0 \le \tau \le 3\omega, \tag{3.32}$$

due to which we find from (3.2) and (3.25) that $x_i(t) = 0$ for $t \in \mathbb{R}$ $(i = 1,2)$. Thus problem (3.1), (3.2) has only a trivial solution. By Lemma 2.3, this fact guarantee the unique solvability of problem (1.1), (1.2). □

Proof of Theorem 1.5. By virtue of Lemma 2.3, it is sufficient to establish that problem (3.1), (3.2) has only a trivial solution.

Let (x_1,x_2) be an arbitrary solution of problem (3.1), (3.2) and

$$u(t) = \frac{1}{2}\big(A_1x_1(t)\cdot x_1(t) + A_2x_2(t)\cdot x_2(t)\big) + Ax_1(t)\cdot x_2(t). \tag{3.33}$$

Then

$$\begin{aligned} u'(t) &= A_1x_1'(t)\cdot x_1(t) + A_2x_2'(t)\cdot x_2(t) + Ax_1'(t)\cdot x_2(t) + A^*x_2'(t)\cdot x_1(t) \\ &= \big(A_1\mathcal{P}_{11}(t) + A^*\mathcal{P}_{21}(t)\big)x_1(t)\cdot x_1(t) \\ &\quad + \big(A_2\mathcal{P}_{22}(t) + A\mathcal{P}_{12}(t)\big)x_2(t)\cdot x_2(t) + \big(A_1\mathcal{P}_{12}(t) + A^*\mathcal{P}_{22}(t)\big)x_2(t)\cdot x_1(t) \\ &\quad + \big(A_2\mathcal{P}_{21}(t) + A\mathcal{P}_{11}(t)\big)x_1(t)\cdot x_2(t) \quad \text{for almost all } t \in \mathbb{R}. \end{aligned} \tag{3.34}$$

However, by conditions (1.25) and the Schwartz inequality, for almost all $t \in [0,\omega]$, we have

$$\begin{gathered} \big(A_1\mathcal{P}_{11}(t) + A^*\mathcal{P}_{21}(t)\big)x_1(t)\cdot x_1(t) \ge \delta(t)\|x_1(t)\|^2, \\ \big(A_2\mathcal{P}_{22}(t) + A\mathcal{P}_{12}(t)\big)x_2(t)\cdot x_2(t) \ge \delta(t)\|x_2(t)\|^2, \\ \big(A_1\mathcal{P}_{12}(t) + A^*\mathcal{P}_{22}(t)\big)x_2(t)\cdot x_1(t) + \big(A_2\mathcal{P}_{21}(t) + A\mathcal{P}_{11}(t)\big)x_1(t)\cdot x_2(t) \\ \le 2p(t)\|x_1(t)\|\|x_2(t)\| \le p(t)\Big(\|x_1(t)\|^2 + \|x_2(t)\|^2\Big), \end{gathered} \tag{3.35}$$

where p is the function given by equality (1.27). Therefore,

$$u'(t) \ge \big(\delta(t) - p(t)\big)\Big(\|x_1(t)\|^2 + \|x_2(t)\|^2\Big) \quad \text{for almost all } t \in [0,\omega]. \tag{3.36}$$

On the other hand, by virtue of (1.24) and (3.2), we have

$$\begin{aligned} u(\omega) &= \frac{1}{2}\left(A_1\Lambda_1 x_1(0)\cdot\Lambda_1 x_1(0) + A_2\Lambda_2 x_2(0)\cdot\Lambda_2 x_2(0)\right) \\ &\quad + A\Lambda_1 x_1(0)\cdot\Lambda_2 x_2(0) \\ &= \frac{1}{2}\left(\Lambda_1^* A_1\Lambda_1 x_1(0)\cdot x_1(0) + \Lambda_2^* A_2\Lambda_2 x_2(0)\cdot x_2(0)\right) \\ &\quad + \Lambda_2^* A\Lambda_1 x_1(0)\cdot x_2(0) = u(0). \end{aligned} \tag{3.37}$$

Thus

$$0 = \int_0^\omega u'(t)dt \ge \int_0^\omega (\delta(t) - p(t))\left(\|x_1(t)\|^2 + \|x_2(t)\|^2\right)dt. \tag{3.38}$$

Hence, by virtue of conditions (1.26) and (1.28), it follows that there exists $t_0 \in [0,\omega]$ such that

$$x_i(t_0) = 0 \quad (i = 1,2). \tag{3.39}$$

Therefore, $x_i(t) = 0$ for $t \in \mathbb{R}$ $(i = 1,2)$ since system (3.1) with the zero initial conditions has only a trivial solution. □

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ON THE SOLUTIONS OF NONLINEAR INITIAL-BOUNDARY VALUE PROBLEMS

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We deal with the general initial-boundary value problem for a second-order nonlinear nonstationary evolution equation. The associated operator equation is studied by the Fredholm and Nemitskii operator theory. Under local Hölder conditions for the nonlinear member, we observe quantitative and qualitative properties of the set of solutions of the given problem. These results can be applied to different mechanical and natural science models.

1. Introduction

The generic properties of solutions of the second-order ordinary differential equations were studied by Brüll and Mawhin in [2], Mawhin in [7], and by Šeda in [8]. Such questions were solved for nonlinear diffusional-type problems with the Dirichlet-, Neumann-, and Newton-type conditions in [5, 6].

In this paper, we study the set structure of classic solutions, bifurcation points and the surjectivity of an associated operator to a general second-order nonlinear evolution problem by the Fredholm operator theory. The present results allow us to search the generic properties of nonparabolic models which describe mechanical, physical, reaction-diffusion, and ecology processes.

2. The formulation of the problem and basic notions

Throughout this paper, we assume that the set $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$. The real number T is positive and $Q := (0,T] \times \Omega$, $\Gamma := (0,T] \times \partial\Omega$.

We use the notation D_t for $\partial/\partial t$, D_i for $\partial/\partial x_i$, D_{ij} for $\partial^2/\partial x_i \partial x_j$, where $i, j = 1,\ldots,n$, and $D_0 u$ for u. The symbol $\operatorname{cl} M$ means the closure of a set M in $\mathbb{R}^n$.

We consider the nonlinear differential equation (possibly of a nonparabolic type)

$$D_t u - A(t,x,D_x)u + f(t,x,u,D_1 u,\ldots,D_n u) = g(t,x) \tag{2.1}$$

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for $(t,x) \in Q$, where the coefficients a_{ij}, a_i, a_0, for $i, j = 1,\dots,n$, of the second-order linear operator

$$A(t,x,D_x)u = \sum_{i,j=1}^{n} a_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} a_i(t,x)D_iu + a_0(t,x)u \tag{2.2}$$

are continuous functions from the space $C(\operatorname{cl} Q,\mathbb{R})$. The function f is from the space $C(\operatorname{cl} Q \times \mathbb{R}^{n+1},\mathbb{R})$ and $g \in C(\operatorname{cl} Q,\mathbb{R})$.

Together with (2.1), we consider the following general homogeneous boundary condition:

$$B_3(t,x,D_x)u|_\Gamma := \sum_{i=1}^{n} b_i(t,x)D_iu + b_0(t,x)u|_\Gamma = 0, \tag{2.3}$$

where the coefficients b_i, for $i = 1,\dots,n$, and b_0 are continuous functions from $C(\operatorname{cl}\Gamma,\mathbb{R})$.

Furthermore, we require for the solution of (2.1) to satisfy the homogeneous initial condition

$$u|_{t=0} = 0 \quad \text{on } \operatorname{cl}\Omega. \tag{2.4}$$

Remark 2.1. In the case where $b_i = 0$, for $i = 1,\dots,n$, and $b_0 = 1$ in (2.3), we get the Dirichlet problem studied in [5].

If we consider the vector function $\nu := (0,\nu_1,\dots,\nu_n) : \operatorname{cl}\Gamma \to \mathbb{R}^{n+1}$ and the value $\nu(t,x)$ which means the unit inner normal vector to $\operatorname{cl}\Gamma$ at the point $(t,x) \in \operatorname{cl}\Gamma$ and we let $b_i = \nu_i$ for $i = 1,\dots,n$ on $\operatorname{cl}\Gamma$, then problem (2.1), (2.3), (2.4) represents the Newton or Neuman problem investigated in [6].

Our considerations are concerned with a broad class of nonparabolic operators.

In the following definitions, we will use the notations

$$\begin{aligned}
\langle u\rangle^{s}_{t,\mu,Q} &:= \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\mu}},\\
\langle u\rangle^{y}_{x,\nu,Q} &:= \sup_{\substack{(t,x),(t,y)\in \operatorname{cl} Q\\ x\neq y}} \frac{|u(t,x)-u(t,y)|}{|x-y|^{\nu}},\\
\langle f\rangle^{s,y,v}_{t,x,u} &:= |f(t,x,u_0,u_1,\dots,u_n) - f(s,y,v_0,v_1,\dots,v_n)|,\\
\langle f\rangle^{s,y,v(s,y)}_{t,x,u(t,x)} &:= |f[t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)],\\
&\quad - f[s,y,v(s,y),D_1v(s,y),\dots,D_nv(s,y)]|,
\end{aligned} \tag{2.5}$$

where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ are from $\mathbb{R}^n$, $|x-y| = [\sum_{i=1}^{n}(x_i-y_i)^2]^{1/2}$, and $\mu,\nu \in \mathbb{R}$.

We will need the following Hölder spaces (see [4, page 147]).

Definition 2.2. Let $\alpha \in (0,1)$.

(1) By the symbol $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ we denote the vector space of continuous functions $u : \mathrm{cl}\,Q \to \mathbb{R}$ which have continuous derivatives $D_i u$ for $i = 1,\dots,n$ on $\mathrm{cl}\,Q$, and the norm

$$\begin{aligned} \|u\|_{(1+\alpha)/2,1+\alpha,Q} := &\sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i u(t,x) \right| + \langle u \rangle^{s}_{t,(1+\alpha)/2,Q} \\ &+ \sum_{i=1}^{n} \langle D_i u \rangle^{s}_{t,\alpha/2,Q} + \sum_{i=1}^{n} \langle D_i u \rangle^{y}_{x,\alpha/2,Q} \end{aligned} \tag{2.6}$$

is finite.

(2) The symbol $C_{(t,x)}^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ means the vector space of continuous functions $u : \mathrm{cl}\,Q \to \mathbb{R}$ for which there exist continuous derivatives $D_t u$, $D_i u$, $D_{ij} u$ on $\mathrm{cl}\,Q$, $i,j = 1,\dots,n$, and the norm

$$\begin{aligned} \|u\|_{(2+\alpha)/2,2+\alpha,Q} = &\sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i u(t,x) \right| + \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_t u(t,x) \right| \\ &+ \sum_{i,j=1}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_{ij} u(t,x) \right| + \sum_{i=1}^{n} \langle D_i u \rangle^{s}_{t,(1+\alpha)/2,Q} + \langle D_t u \rangle^{s}_{t,\alpha/2,Q} \\ &+ \sum_{i,j=1}^{n} \langle D_{ij} u \rangle^{s}_{t,\alpha/2,Q} + \langle D_t u \rangle^{y}_{x,\alpha,Q} + \sum_{i,j=1}^{n} \langle D_{ij} u \rangle^{y}_{x,\alpha,Q} \end{aligned} \tag{2.7}$$

is finite.

(3) The symbol $C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ means the vector space of continuous functions $u : \mathrm{cl}\,Q \to \mathbb{R}$ for which the derivatives D_t, $D_i u$, $D_t D_i u$, $D_{ij} u$, $D_{ijk} u$, $i,j,k = 1,\dots,n$, are continuous on $\mathrm{cl}\,Q$, and the norm

$$\begin{aligned} \|u\|_{(3+\alpha)/2,3+\alpha,Q} := &\sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i u(t,x) \right| + \sum_{i,j=1}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_{ij} u(t,x) \right| \\ &+ \sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_t D_i u(t,x) \right| + \sum_{i,j,k=1}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_{ijk} u(t,x) \right| \\ &+ \langle D_t u \rangle^{s}_{t,(1+\alpha)/2,Q} + \sum_{i,j=1}^{n} \langle D_{ij} u \rangle^{s}_{t,(1+\alpha)/2,Q} \\ &+ \sum_{i=1}^{n} \langle D_t D_i u \rangle^{s}_{t,\alpha/2,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u \rangle^{s}_{t,\alpha/2,Q} \\ &+ \sum_{i=1}^{n} \langle D_t D_i u \rangle^{y}_{x,\alpha,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u \rangle^{y}_{x,\alpha,Q} \end{aligned} \tag{2.8}$$

is finite.

The above-defined norm spaces are Banach ones.

Definition 2.3 (the smoothness condition $(\mathrm{S}_3^{1+\alpha})$). Let $\alpha \in (0,1)$. The differential operators $A(t,x,D_x)$ from (2.1) and $B_3(t,x,D_x)$ from (2.3) satisfy *the smoothness condition* $(\mathrm{S}_3^{1+\alpha})$ if, respectively,

(i) the coefficients a_{ij}, a_i, a_0 from (2.1), for $i,j = 1,\dots,n$, belong to the space $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ and $\partial\Omega \in C^{3+\alpha}$,

(ii) the coefficients b_i from (2.3), for $i=1,\dots,n$, belong to the space $C_{t,x}^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,\Gamma,\mathbb{R})$.

Definition 2.4 (the complementary condition (C)). If at least one of the coefficients b_i, for $i = 1,\dots,n$, of the differential operator $B_3(t,x,D_x)$ in (2.3) is not zero, then $B_3(t,x,D_x)$ satisfies the *complementary condition (C)*.

Now, we are prepared to formulate hypotheses for deriving fundamental lemmas.

Definition 2.5. (1) Fredholm conditions.

(A1) Consider the operator $A_3 : X_3 \to Y_3$, where

$$A_3 u = D_t u - A(t,x,D_x)u, \quad u \in X_3, \tag{2.9}$$

and the operators $A(t,x,D_x)$ and $B_3(t,x,D_x)$ satisfy the smoothness condition $(\mathrm{S}_3^{1+\alpha})$ for $\alpha \in (0,1)$ and the complementary condition (C). Here, we consider the vector spaces

$$\begin{aligned} D(A_3) &:= \{u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\,Q,\mathbb{R});\ B_3(t,x,D_x)u|_\Gamma = 0,\ u|_{t=0}(x) = 0 \text{ for } x \in \mathrm{cl}\,Q\}, \\ H(A_3) &:= \{v \in C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R});\ B_3(t,x,D_x)v(t,x)|_{t=0,x\in\partial\Omega} = 0\} \end{aligned} \tag{2.10}$$

and Banach subspaces (of the given Hölder spaces)

$$\begin{aligned} X_3 &= \left(D(A_3), \|\cdot\|_{(3+\alpha)/2,3+\alpha,Q}\right), \\ Y_3 &= \left(H(A_3), \|\cdot\|_{(1+\alpha)/2,1+\alpha,Q}\right). \end{aligned} \tag{2.11}$$

(A2) There is a second-order linear homeomorphism $C_3 : X_3 \to Y_3$ with

$$C_3 u = D_t u - C(t,x,D_x)u, \quad u \in X_3, \tag{2.12}$$

where

$$C(t,x,D_x)u = \sum_{i,j=1}^{n} c_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} c_i(t,x)D_i u + c_0(t,x)u, \tag{2.13}$$

satisfying the smoothness condition $(\mathrm{S}_3^{1+\alpha})$. The operator C_3 is not necessarily a parabolic one.

(2) Local Hölder and compatibility conditions.

Let $f := f(t,x,u_0,u_1,\dots,u_n) : \operatorname{cl} Q \times \mathbb{R}^{n+1} \to \mathbb{R}$, $\alpha \in (0,1)$, and let p, q, p_r, for $r = 0,1,\dots,n$ be nonnegative constants. Here, D represents any compact subset of $(\operatorname{cl} Q) \times \mathbb{R}^{n+1}$. For f, we need the following assumptions:

(B1) let $f \in C^1(\operatorname{cl} Q \times \mathbb{R}^{n+1},\mathbb{R})$ and let the first derivatives $\partial f/\partial x_i$, $\partial f/\partial u_j$ be locally Hölder continuous on $\operatorname{cl} Q \times \mathbb{R}^{n+1}$ such that

$$\begin{aligned} \left\langle \frac{\partial f}{\partial x_i} \right\rangle_{t,x,u}^{s,y,v} &\le p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r \left| u_r - v_r \right|, \\ \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{t,x,u}^{s,y,v} &\le p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r \left| u_r - v_r \right|, \end{aligned} \tag{2.14}$$

for $i = 1,\dots,n$, $j = 0,1,\dots,n$, and any D;

(B2) let $f \in C^3(\operatorname{cl} Q \times \mathbb{R}^{n+1},\mathbb{R})$ and let the local growth conditions for the third derivatives of f hold on any D:

$$\begin{aligned} \left\langle \frac{\partial^3 f}{\partial \tau \partial x_i \partial u_j} \right\rangle_{t,x,u}^{t,x,v} &\le \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial \tau \partial u_j \partial u_k} \right\rangle_{t,x,u}^{t,x,v} &\le \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial x_i \partial x_l \partial u_j} \right\rangle_{t,x,u}^{t,x,v} &\le \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial x_i \partial u_j \partial u_k} \right\rangle_{t,x,u}^{t,x,v} &\le \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial u_j \partial u_k \partial u_r} \right\rangle_{t,x,u}^{t,x,v}, &\le \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \end{aligned} \tag{2.15}$$

where $\beta_s > 0$ for $s = 0,1,\dots,n$ and $i,l = 1,\dots,n$; $j,k,r = 0,1,\dots,n$;

(B3) the equality of compatibility

$$\sum_{i=1}^{n} b_i(t,x) D_i f(t,x,0,\dots,0) + b_0(t,x) f(t,x,0,\dots,0)|_{t=0,x\in\partial\Omega} = 0 \tag{2.16}$$

holds.

(3) Almost coercive condition.

Let, for any bounded set $M_3 \subset Y_3$, there exist a number $K > 0$ such that for all solutions $u \in X_3$ of problem (2.1), (2.3), (2.4) with the right-hand sides $g \in M_3$, the following alternative holds:

(C1) either

(α_1) $\|u\|_{(1+\alpha)/2,1+\alpha,Q} \leq K$, $f := f(t,x,u_0) : \mathrm{cl}\,Q \times \mathbb{R} \to \mathbb{R}$, and the coefficients of the operators A_3 and C_3 (see (2.1) and (A2)) satisfy the equations

$$a_{ij} = c_{ij}, \quad a_i = c_i, \quad \text{for } i,j = 1,\dots,n, \qquad a_0 \neq c_0 \quad \text{on } \mathrm{cl}\,Q, \tag{2.17}$$

or

(α_2) $\|u\|_{(2+\alpha)/2,2+\alpha,Q} \leq K$, $f := f(t,x,u_0,u_1,\dots,u_n) : \mathrm{cl}\,Q \times \mathbb{R}^{n+1} \to \mathbb{R}$, and the coefficients of the operators A_3 and C_3 satisfy the relations

$$a_{ij} = c_{ij} \quad \text{for } i,j = 1,\dots,n, \qquad a_i \neq c_i \quad \text{for at least one } i = 1,\dots,n \tag{2.18}$$

on $\mathrm{cl}\,Q$.

Remark 2.6. (1) Especially, condition (A2) is satisfied for the diffusion operator

$$C_3 u = D_t u - \triangle u, \quad u \in X_3, \tag{2.19}$$

or for any uniformly parabolic operator C_3 with sufficiently smooth coefficients. However, the operator C_3 is not necessarily uniform parabolic.

(2) The local Hölder conditions in (B1) and (B2) admit sufficiently strong growths of f in the last variables $u_0,u_1,\dots,u_n$. For example, they include exponential and power-type growths.

Definition 2.7. (1) A couple $(u,g) \in X_3 \times Y_3$ will be called *the bifurcation point of the mixed problem* (2.1), (2.3), (2.4) if u is a solution of that mixed problem and there exists a sequence $\{g_k\} \subset Y_3$ such that $g_k \to g$ in Y_3 as $k \to \infty$, and problem (2.1), (2.3), (2.4) for $g = g_k$ has at least two different solutions u_k, v_k for each $k \in \mathbb{N}$ and $u_k \to u$, $v_k \to u$ in X_3 as $k \to \infty$.

(2) The set of all solutions $u \in X_3$ of (2.1), (2.3), (2.4) (or the set of all functions $g \in Y_3$) such that (u,g) is a bifurcation point of problem (2.1), (2.3), (2.4) will be called *the domain of bifurcation (the bifurcation range)* of that problem.

3. Fundamental lemmas

Lemma 3.1. *Let conditions (A1) and (A2) hold (see Definition 2.5). Then,*

(1) $\dim X_3 = +\infty$*;*

(2) *the operator* $A_3 : X_3 \to Y_3$ *is a linear bounded Fredholm operator of the zero index.*

Proof. (1) To prove the first part of this lemma, we use the decomposition theorem from [9, page 139].

Let X be a linear space and let $x^* : X \to \mathbb{R}$ be a linear functional on X such that $x^* \neq 0$. Furthermore, let $M = \{x \in X;\ x^*(x) = 0\}$ and let $x_0 \in X - M$. Then, every element $x \in X$ can be expressed by the formula

$$x = \left[\frac{x^*(x)}{x^*(x_0)}\right] x_0 + m, \quad m \in M, \tag{3.1}$$

that is, there is a one-dimensional subspace L_1 of X such that $X = L_1 \oplus M$.

If we now let

$$M_1 := \left\{u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\,Q,\mathbb{R}) =: H^{3+\alpha};\ B_3(t,x,D_x)u|_\Gamma = 0\right\}, \tag{3.2}$$

which is the linear subspace of $H^{3+\alpha}$, then there exists a linear subspace L_1 of $H^{3+\alpha}$ with $\dim L_1 = 1$ such that $H^{3+\alpha} = L_1 \oplus M_1$. Similarly, if we take $M_2 := \{u \in M_1;\ u|_{t=0} = 0$ on $\mathrm{cl}\,\Omega\}$, then there is a subspace L_2 of M_1 with $\dim L_2 = 1$ such that $M_1 = L_2 \oplus M_2$. Hence, we have $H^{3+\alpha} = L_1 \oplus L_2 \oplus D(A_3)$. Since $\dim H^{3+\alpha} = +\infty$, we get that $\dim X_3 = +\infty$.

(2) (a) In the first step, we prove the boundedness of the linear operator A_3. To this end, we observe the norm $\|A_3u\|_{(1+\alpha)/2,1+\alpha,Q}$ for $u \in D(A_3)$. From the assumption $(\mathrm{S}_3^{1+\alpha})$ we get for $k = 0,1,\dots,n$,

$$\sup_{(t,x)\in \mathrm{cl}\,Q} \left|D_kA_3u(t,x)\right| \le K_1\|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_1 > 0. \tag{3.3}$$

Applying again the smoothness assumption $(\mathrm{S}_3^{1+\alpha})$, the mean value theorem for the functions u and D_iu, and the boundedness of Q, we obtain for the second member of the above-mentioned norm the following estimation:

$$\begin{aligned}\langle A_3u\rangle^s_{t,(1+\alpha)/2,Q} &= \sup_{\substack{(t,x),(s,x)\in \mathrm{cl}\,Q\\ t\neq s}} \frac{\left|A_3u(t,x) - A_3u(s,x)\right|}{|t-s|^{(1+\alpha)/2}}\\ &\le K_2\|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_2 > 0.\end{aligned} \tag{3.4}$$

For the third member of the norm (2.6), we estimate for $k = 1,\dots,n$ as follows:

$$\begin{aligned}\langle D_kA_3u\rangle^s_{t,\alpha/2,Q} &= \sup_{\substack{(t,x),(s,x)\in \mathrm{cl}\,Q\\ t\neq s}} \frac{\left|D_kA_3u(t,x) - D_kA_3u(s,x)\right|}{|t-s|^{\alpha/2}}\\ &\le K_3\|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_3 > 0.\end{aligned} \tag{3.5}$$

An estimation of the last member in (2.6) for A_3u is given by the following inequality for $k = 1,\dots,n$:

$$\begin{aligned}\langle D_kA_3u\rangle^y_{x,\alpha/2,Q} &= \sup_{\substack{(t,x),(t,y)\in \mathrm{cl}\,Q\\ x\neq y}} \frac{\left|D_kA_3u(t,x) - D_kA_3u(t,y)\right|}{|x-y|^{\alpha/2}}\\ &\le K_4\|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_4 > 0.\end{aligned} \tag{3.6}$$

From the estimations (3.3), (3.4), (3.5), and (3.6), we can conclude that

$$||A_3u||_{Y_3} = ||A_3u||_{(1+\alpha)/2,1+\alpha,Q} \le K(n,T,\alpha,\Omega,a_{ij},a_i,a_0)\|u\|_{X_3}. \tag{3.7}$$

(b) To prove that A_3 is a Fredholm operator with the zero index, we express it in the form

$$A_3u = C_3u + \left[C(t,x,D_x) - A(t,x,D_x)\right]u =: C_3u + T_3u, \tag{3.8}$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism and C is the linear operator from (A2). By the decomposition Nikolskii theorem [10, page 233], it is sufficient to show that $T_3 : X_3 \to Y_3$ is a linear completely continuous operator.

The complete continuity of T_3 can be proved by the Ascoli-Arzelá theorem (see [11, page 141]).

From $(S_3^{1+\alpha})$, the uniform boundedness of the operator

$$T_3u = \sum_{i,j=1}^{n} [c_{ij}(t,x) - a_{ij}(t,x)]D_{ij}u + \sum_{i=1}^{n} [c_i(t,x) - a_i(t,x)]D_iu \\ + [c_0(t,x) - a_0(t,x)]u \tag{3.9}$$

follows by the same way as the boundedness of the operator A_3 in the previous part (1). Thus, for all $u \in M \subset X_3$, where M is a set bounded by the constant $K_1 > 0$, we obtain the estimate

$$||T_3u||_{Y_3} \le K(n, \alpha T, \Omega, a_{ij}, c_{ij}, a_i, c_i, a_0, c_0)\,\|u\|_{X_3} \le KK_1. \tag{3.10}$$

Using the smoothness condition of the operators A and C, we get the inequalities

$$\begin{aligned}
|T_3u(t,x) - T_3u(s,y)| \le & \sum_{i,j=1}^{n} \big|[c_{ij} - a_{ij}](t,x) - [c_{ij} - a_{ij}](s,y)\big|\,|D_{ij}u(t,x)| \\
& + \sum_{i,j=1}^{n} |c_{ij}(s,y) - a_{ij}(s,y)|\,|D_{ij}u(t,x) - D_{ij}u(s,y)| \\
& + \sum_{i=1}^{n} \big|[c_i - a_i](t,x) - [c_i - a_i](s,y)\big|\,|D_iu(t,x)| \\
& + \sum_{i=1}^{n} |c_i(s,y) - a_i(s,y)|\,|D_iu(t,x) - D_iu(s,y)| \\
& + \big|[c_0 - a_0](t,x) - [c_0 - a_0](s,y)\big|\,|u(t,x)| \\
& + |c_0(s,y) - a_0(s,y)|\,|u(t,x) - u(s,y)| \\
\le & \, 4K_1Kn^2\big[|t-s|^{\alpha/2} + |x-y|^{\alpha}\big] \\
& + 2K_1Kn\big[\big(|t-s|^{\alpha/2} + |x-y|^{\alpha}\big) + \big(|t-s|^{(1+\alpha)/2} + |x-y|\big)\big] \\
& + 2K_1K\big[\big(|t-s|^{\alpha/2} + |x-y|^{\alpha}\big) + \big(|t-s| + |x-y|\big)\big],
\end{aligned} \tag{3.11}$$

where K_1, K are positive constants. Hence, the equicontinuity of $T_3M \subset Y_3$ follows. This finishes the proof of Lemma 3.1. □

Lemma 3.1 implies the following alternative.

Corollary 3.2. *Let L mean the set of all second-order linear differential operators*

$$A_3 = D_t - A(t,x,D_x) : X_3 \longrightarrow C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R}) \tag{3.12}$$

satisfying conditions (C) and $(S_2^{1+\alpha})$. *Then, for each* $A_3 \in L$, *the mixed homogeneous problem* $A_3u = 0$ *on* Q, *(2.3), and (2.4) has a nontrivial solution or any* $A_3 \in L$ *is a linear bounded Fredholm operator of the zero-index mapping* X_3 *onto* Y_3.

The following lemma establishes the complete continuity of the Nemitskii operator from the nonlinear part of (2.1).

Lemma 3.3. *Let assumptions (B1) and (B3) be satisfied. Then the Nemitskii operator* $N_3 : X_3 \to Y_3$ *defined by*

$$(N_3u)(t,x) = f[t,x,u(t,x),D_1u(t,x),\ldots,D_nu(t,x)] \tag{3.13}$$

for $u \in X_3$ *and* $(t,x) \in \mathrm{cl}\,Q$ *is completely continuous.*

Proof. Let $M_3 \subset X_3$ be a bounded set. By the Ascoli-Arzelá theorem, it is sufficient to show that the set $N_3(M_3)$ is uniformly bounded and equicontinuous. We will use assumption (B3) to prove the inclusion $N_3(M_3) \subset Y_3$.

Take $u \in M_3$. According to assumption (B1), we obtain the local boundedness of the function f and of its derivatives $\partial f/\partial x_i$ on $(\mathrm{cl}\,Q) \times \mathbb{R}^{n+1}$ for $i = 1,\ldots,n$. From this and from the equation

$$D_i(N_3u)(t,x) = \left\{D_if[\cdot] + \sum_{l=0}^{n} \frac{\partial f}{\partial u_l}[\cdot]D_iD_lu\right\}[\cdot,\cdot,u,D_1u,\ldots,D_nu](t,x), \tag{3.14}$$

we have the estimation

$$\sup_{(t,x)\in \mathrm{cl}\,Q} |D_i(N_3u)(t,x)| \le K_1 \tag{3.15}$$

for $i = 0,1,\ldots,n$ with a positive sufficiently large constant K_1 not depending on $u \in M_3$.

Using the differentiability of f and the mean value theorem in the variable t for the difference of the derivatives of u, we can write

$$\langle N_3u\rangle^{s}_{t,(1+\alpha)/2,Q} \le K_1. \tag{3.16}$$

Similarly, by (2.14), we have

$$\langle D_iN_3u\rangle^{s}_{t,\alpha/2,Q} \le K_1, \qquad \langle D_iN_3u\rangle^{y}_{x,\alpha,Q} \le K_1, \tag{3.17}$$

for $i = 1,\ldots,n$ and $u \in M_3$. The previous estimations yield the inequality

$$||N_3u||_{Y_3} \le K_1 \tag{3.18}$$

for all $u \in M_3$.

With respect to (B1), for any $u \in M_3$ and $(t,x),(s,y) \in \operatorname{cl} Q$ such that $|t-s|^2 + |x-y|^2 < \delta^2$ with a sufficiently small $\delta > 0$, we have

$$\left| N_3 u(t,x) - N_3 u(s,y) \right| < \epsilon, \quad \epsilon > 0, \tag{3.19}$$

which is the equicontinuity of $N_3(M_3)$. This finishes the proof of Lemma 3.3. □

Lemma 3.4. *Let assumptions (A1), (A2), (B1), (B3), and (C1) hold. Then the operator $F_3 = A_3 + N_3 : X_3 \to Y_3$ is coercive.*

Proof. We need to prove that if the set $M_3 \subset Y_3$ is bounded in Y_3, then the set of arguments $F_3^{-1}(M_3) \subset X_3$ is bounded in X_3.

In both cases (α_1) and (α_2), we get for all $u \in F_3^{-1}(M_3)$,

$$||N_3 u||_{(1+\alpha)/2,1+\alpha,Q} \leq K_1, \tag{3.20}$$

where $K_1 > 0$ is a sufficiently large constant. Hence,

$$||A_3 u||_{Y_3} \leq K_1 \tag{3.21}$$

for any $u \in F_3^{-1}(M_3)$.

Hypothesis (A2) ensures the existence and uniqueness of the solution $u \in X_3$ of the linear equation

$$C_3 u = y, \tag{3.22}$$

and for any $y \in Y_3$,

$$\|u\|_{X_3} \leq K_1 \|y\|_{Y_3}. \tag{3.23}$$

If we write

$$\begin{aligned} C_3 u = A_3 u &+ \sum_{i,j=1}^{n} \left[a_{ij}(t,x) - c_{ij}(t,x)\right] D_{ij} u \\ &+ \sum_{i=1}^{n} \left[a_i(t,x) - c_i(t,x)\right] D_i u + \left[a_0(t,x) - c_0(t,x)\right] u, \end{aligned} \tag{3.24}$$

then in both cases and for each $u \in F_3^{-1}(M_3)$, we obtain

$$\|y\|_{Y_3} \leq ||C_3 u||_{Y_3} \leq K_1, \tag{3.25}$$

whence, by inequality (3.23), we can conclude that the operator F_3 is coercive. □

Lemma 3.5. *Let the Nemitskii operator $N_3 : X_3 \to Y_3$ from (3.13) satisfy conditions (B2) and (B3). Then the operator N_3 is continuously Fréchet-differentiable, that is, $N_3 \in C^1(X_3, Y_3)$ and it is completely continuous.*

Proof. From (B2), we obtain (B1) which implies by Lemma 3.3 the complete continuity of N_3. To obtain the first part of the assertion of this lemma, we need to prove that the Fréchet derivative $N_3' : X_3 \to L(X_3, Y_3)$ defined by the equation

$$N_3'(u)h(t,x) = \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}(t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)]D_jh(t,x) \tag{3.26}$$

for $u,h \in X_3$ is continuous on X_3. Thus, we must prove, for every $v \in X_3$, that

$$\forall \epsilon > 0 \ \exists \delta(\epsilon,v) > 0, \quad \forall u \in X_3, \ \|u - v\|_{X_3} < \delta : \sup_{h \in X_3, \|h\|_{X_3} \le 1} \left\| [N_3'(u) - N_3'(v)]h \right\|_{Y_3} < \epsilon. \tag{3.27}$$

Using the norms (2.6), (2.8) and the estimation $\|u - v\|_{X_3} < \delta$, we have for the first term of (3.27) by the mean value theorem,

$$\begin{aligned}
&\sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i[N_3'(u) - N_3'(v)]h(t,x) \right| \\
&\quad \le \sum_{i,j=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \Bigg[\left\langle \frac{\partial^2 f}{\partial x_i \partial u_j} \right\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} \left| D_jh(t,x) \right| \\
&\qquad\qquad + \sum_{k=0}^{n} \left\langle \frac{\partial^2 f}{\partial u_j \partial u_k} \right\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} \left| D_{ik}u \right| \cdot \left| D_jh \right|(t,x) \\
&\qquad\qquad + \sum_{k=0}^{n} \left| \frac{\partial^2 f}{\partial u_j \partial u_k}(t,x,v(t,x),\dots) \right| \left| D_{ik}u - D_{ik}v \right| \left| D_jh \right|(t,x) \\
&\qquad\qquad + \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} \left| D_{ij}h(t,x) \right| \Bigg] < K\delta, \quad K > 0.
\end{aligned} \tag{3.28}$$

For the second term of (3.27), we estimate as follows:

$$\begin{aligned}
&\langle [N_3'(u) - N_3'(v)]h \rangle_{t,(1+\alpha)/2,Q}^{s} \\
&\quad \le \sum_{j=0}^{n} \sup_{\mathrm{cl}\,Q,\, t\ne s} |t-s|^{-(1+\alpha)/2} \Bigg[\left| \int_s^t D_\tau \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau \right| \left| D_jh(t,x) \right| \\
&\qquad\qquad + \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \left| \int_s^t D_\tau D_jh(\tau,x)\,d\tau \right| \Bigg] \\
&\quad \le K\delta, \quad K > 0.
\end{aligned} \tag{3.29}$$

Here, we have used the mean value theorem for $\partial^2 f/\partial\tau\partial u_j$, $\partial^2 f/\partial u_j \partial u_k$, and $\partial f/\partial u_j$ for $j,k = 0,1,\dots,n$.

The third term of (3.27) gives by (2.15),

$$
\begin{aligned}
&\sum_{i=1}^{n}\langle D_i\{[N_3'(u)-N_3'(v)]h\}\rangle_{t,\alpha/2,Q}^{s}\\
&\quad\le \sum_{i=1}^{n}\sum_{j=0}^{n}\sup_{\mathrm{cl}\,Q,t\neq s}|t-s|^{-\alpha/2}\\
&\qquad\quad\times\Bigg\{\Bigg|\int_s^t D_\tau\left\langle\frac{\partial^2 f}{\partial x_i\partial u_j}\right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}d\tau\Bigg|\,|D_jh(t,x)|\\
&\qquad\qquad+\left\langle\frac{\partial^2 f}{\partial x_i\partial u_j}\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}\Bigg|\int_s^t D_\tau D_jh(\tau,x)d\tau\Bigg|\\
&\qquad\qquad+\sum_{k=0}^{n}\Bigg[\Bigg|\int_s^t D_\tau\left\langle\frac{\partial^2 f}{\partial u_j\partial u_k}\right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}d\tau\Bigg|\,|D_{ik}u|\,|D_jh|(t,x)\\
&\qquad\qquad\quad+\Bigg|\int_s^t D_\tau\left[\frac{\partial^2 f}{\partial u_j\partial u_k}(\tau,x,v,\dots)d\tau\right]\Bigg|\\
&\qquad\qquad\quad\times|D_{ik}u(t,x)-D_{ik}v(t,x)|\,|D_jh(t,x)|\\
&\qquad\qquad\quad+\left\langle\frac{\partial^2 f}{\partial u_j\partial u_k}\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}|D_{ik}u(t,x)-D_{ik}u(s,x)|\,|D_jh(t,x)|\\
&\qquad\qquad\quad+\Bigg|\frac{\partial^2 f}{\partial u_j\partial u_k}(s,x,v,\dots)\Bigg|\\
&\qquad\qquad\quad\times|D_{ik}u(t,x)-D_{ik}v(t,x)-[D_{ik}u(s,x)-D_{ik}v(s,x)]|\,|D_jh(t,x)|\\
&\qquad\qquad\quad+\left\langle\frac{\partial^2 f}{\partial u_j\partial u_k}\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}|D_{ik}u(s,x)|\,\Bigg|\int_s^t D_\tau D_jh(\tau,x)d\tau\Bigg|\\
&\qquad\qquad\quad+\Bigg|\frac{\partial^2 f}{\partial u_j\partial u_k}(s,x,v,\dots)\Bigg|\,|D_{ik}u(s,x)-D_{ik}v(s,x)|\\
&\qquad\qquad\quad\times\Bigg|\int_s^t D_\tau D_jh(\tau,x)d\tau\Bigg|\\
&\qquad\qquad\quad+\Bigg|\int_s^t D_\tau\left\langle\frac{\partial f}{\partial u_j}\right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}d\tau\Bigg|\,|D_{ij}h(t,x)|\\
&\qquad\qquad\quad+\left\langle\frac{\partial f}{\partial u_j}\right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}|D_{ij}h(t,x)-D_{ij}h(s,x)|\Bigg]\Bigg\}\\
&\quad\le K\left(\sum_{s=0}^{n}\delta^{\beta_s}+\delta\right),\quad K>0.
\end{aligned}\tag{3.30}
$$

Making the corresponding changes, the last term of (3.27), by condition (B2), gives the required estimation:

$$\sum_{i=1}^{n} \langle D_i\{[N_3'(u) - N_3'(v)]h\}\rangle_{x,\alpha,Q}^{y}. \tag{3.31}$$

This finishes the proof of Lemma 3.5. □

4. Generic properties for continuous operators

On a mutual equivalence between the solution of the given initial-boundary value problem and an operator equation, we have the following lemma.

Lemma 4.1. *Let $A_3 : X_3 \to Y_3$ be the linear operator from Lemma 3.1, let $N_3 : X_3 \to Y_3$ be the Nemitskii operator from Lemma 3.3, and let $F_3 = A_3 + N_3 : X_3 \to Y_3$. Then,*

(1) *the function $u \in X_3$ is a solution of the initial-boundary value problem (2.1), (2.3), (2.4) for $g \in Y_3$ if and only if $F_3 u = g$;*
(2) *the couple $(u,g) \in X_3 \times Y_3$ is the bifurcation point of the initial-boundary value problem (2.1), (2.3), (2.4) if and only if $F_3(u) = g$ and $u \in \Sigma$, where Σ means the set of all points of X_3 at which F_3 is not locally invertible.*

Proof. (1) The first equivalence directly follows from the definition of the operator F_3 and of the mixed problem (2.1), (2.3), (2.4).

(2) If (u,g) is a bifurcation point of the mixed problem (2.1), (2.3), (2.4) and u_k, v_k, and g_k for $k = 1,2,\ldots$ have the same meaning as in Definition 2.7, then with respect to (1) we have $F_3(u) = g$, $F_3(u_k) = g_k = F_3(v_k)$. Thus, F_3 is not locally injective at u. Hence, F_3 is not locally invertible at u, that is, $u \in \Sigma$. Conversely, if F_3 is not locally invertible at u and $F_3(u) = g$, then F_3 is not locally injective at u. Indirectly, from Definition 2.7, we see that the couple (u,g) is a bifurcation point of (2.1), (2.3), (2.4). □

Lemma 4.2. *Let*

(i) *the operator $A(t,x,D_x) \neq 0$ from (2.1) and the operator $B_3(t,x,D_x)$ from (2.3) satisfy the smoothness condition $(\mathrm{S}_3^{1+\alpha})$;*
(ii) *the nonlinear part f of (2.1) belong to $C(\operatorname{cl} Q \times \mathbb{R}^{n+1}, \mathbb{R})$;*
(iii) *the operator $A_3 + N_3 : X_3 \to Y_3$ be nonconstant.*

Then, for any compact set of the right-hand sides $g \in Y_3$ from (2.1), the set of all solutions of problem (2.1), (2.3), (2.4) is compact (possibly empty).

Proof. Following the proof of Lemma 3.1, we see that $\dim X_3 = +\infty$ and the linear operator $A_3 : X_3 \to Y_3$ is continuous and accordingly closed. From hypothesis (ii) the Nemitskii operator $N_3 : X_3 \to Y_3$ given in (4.9) is closed too. By [8, Proposition 2.1], the operator $F_3 = A_3 + N_3 : X_3 \to Y_3$ is proper, and with respect to Lemma 4.1 we get our assertion. □

Theorem 4.3. *Under assumptions (A1), (A2) and (B1), (B3), the following statements hold for problem (2.1), (2.3), (2.4):*

(a) *the operator* $F_3 = A_3 + N_3 : X_3 \to Y_3$ *is continuous;*
(b) *for any compact set of the right-hand sides* $g \in Y_3$ *from (2.1), the corresponding set of all solutions is a countable union of compact sets;*
(c) *for* $u_0 \in X_3$*, there exist neighborhoods* $U(u_0)$ *of* u_0 *and* $U(F_3(u_0))$ *of* $F_3(u_0) \in Y_3$ *such that for each* $g \in U(F_3(u_0))$*, there is a unique solution of (2.1), (2.3), (2.4) if and only if the operator* F_3 *is locally injective at* u_0*.*

Moreover, if (C1) is assumed, then

(d) *for each compact set of* Y_3*, the corresponding set of all solutions is compact (possibly empty).*

Proof. Assertion (a) is evident by Lemmas 3.1 and 3.3.

Using the Nikolškii theorem for A_3, we can write

$$F_3 = C_3 + \left(T_3 + N_3\right), \tag{4.1}$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism and is proper (see [8, Proposition 2.1]) and $T_3 + N_3 : X_3 \to Y_3$ is a completely continuous mapping.

Now take the compact sets $K \subset Y_3$ and $F_3^{-1}(K)$. Then there exists a sequence of the closed and bounded sets $M_n \subset F_3^{-1}(K) \subset X_3$ for $n = 1,2,\dots$ such that $\bigcup_{n=1}^{\infty} M_n = F_3^{-1}(K)$.

According to [8, Proposition 2.2], the restrictions $F_3|_{M_n}$ for $n = 1,2,\dots$ are proper mappings and $\left(F_3|_{M_n}\right)^{-1}(K) = M_n$ is a compact set. Hence, the operator F_3 is σ-proper, which gives the result (b).

Assertion (d) is a direct consequence of [8, Proposition 2.2].

Suppose now that F_3 is injective in a neighborhood $U(u_0)$ of $u_0 \in X_3$. From the decomposition (4.1) the mapping

$$C_3^{-1} F_3 = I + C_3^{-1}\left(T_3 + N_3\right), \tag{4.2}$$

where $I : X \to Y$ is the identity, is completely continuous and injective in $U(u_0)$. On the basis of the Schauder domain invariance theorem (see [3, page 66]), the set $C_3^{-1}F_3(U(u_0))$ is open in X_3 and the restriction $C_3^{-1}F_3|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $C_3^{-1}F_3(U(u_0))$. Therefore, F_3 is locally invertible. From Lemma 4.1 we obtain (c).

The most important properties of the mapping F_3, whereby A_3 is a linear bounded Fredholm operator of zero index, N_3 is completely continuous, and F_3 is coercive, give the following theorem. □

Theorem 4.4. *If hypotheses (A1), (A2), (B1), (B3), and (C1) are satisfied, then for the initial-boundary value problem (2.1), (2.3), (2.4), the following statements hold.*

(e) *For each* $g \in Y_3$*, the set* S_{3g} *of all solutions is compact (possibly empty).*
(f) *The set* $R(F_3) = \{g \in Y_3 :$ *there exists at least one solution of the given problem*$\}$ *is closed and connected in* Y_3*.*
(g) *The domain of bifurcation* D_{3b} *is closed in* X_3 *and the bifurcation range* R_{3b} *is closed in* Y_3*.* $F_3(X_3 - D_{3b})$ *is open in* Y_3*.*
(h) *If* $Y_3 - R_{3b} \neq \emptyset$*, then each component of* $Y_3 - R_{3b}$ *is a nonempty open set (i.e., a domain).*

The number n_{3g} of solutions is finite, constant (it may be zero) on each component of the set $Y_3 - R_{3b}$, that is, for every g belonging to the same component of $Y_3 - R_{3b}$.

(i) *If $R_{3b} = 0$, then the given problem has a unique solution $u \in X_3$ for each $g \in Y_3$ and this solution continuously depends on g as a mapping from Y_3 onto X_3.*

(j) *If $R_{3b} \neq \emptyset$, then the boundary of the F_3-image of the set of all points from X_3 in which the operator F_3 is locally invertible is a subset of the F_3-image of the set of all points from X_3 in which F_3 is not locally invertible, that is,*

$$\partial F_3(X_3 - D_{3b}) \subset F_3(D_{3b}) = R_{3b}. \tag{4.3}$$

Proof. Statement (e) follows immediately from Theorem 4.3(d).

(f) Let the sequence $\{g_n\}_{n\in\mathbb{N}} \subset R(F_3) \subset Y_3$ converge to $g \in Y_3$ as $n \to \infty$. By Theorem 4.3(d), there is a compact set of all solutions $\{u_\gamma\}_{\gamma\in I} \subset X_3$ (I is an index set) of the equations $F_3(u) = g_n$ for all $n = 1,2,\dots$. Then there exists a sequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_\gamma\}_{\gamma\in I}$ converging to $u \in X_3$ for which $F_3(u_{n_k}) = g_{n_k} \to g$. Since the operator F_3 is proper, whence it is closed, we have $F_3(u) = g$. Hence, $g \in R(F_3)$ and $R(F_3)$ is a closed set.

The connectedness of $R(F_3) = F_3(X_3)$ follows from the fact that $R(F_3)$ is a continuous image of the connected set X_3.

(g) According to Lemma 4.1(2), $D_{3b} = \Sigma_3$ and $R_{3b} = F_3(D_{3b})$. Since $X_3 - \Sigma_3$ is an open set, D_{3b} and its continuous image R_{3b} are closed sets in X_3 and Y_3, respectively.

Since $X_3 - D_{3b}$ is a set of all points in which the mapping F_3 is locally invertible, then it ensures that to each $u_0 \in X_3 - D_{3b}$ there is a neighborhood $U_1(F_3(u_0)) \subset F_3(X_3 - D_{3b})$, which means that the set $F_3(X_3 - D_{3b})$ is open.

(h) The set $Y_3 - R_{3b} = Y_3 - F_3(D_{3b}) \neq 0$ is open in Y_3, then each of its components is nonempty and open.

The second part of (h) follows from Ambrosetti theorem [1, page 216].

(i) Since $R_{3b} = \emptyset$, the mapping F_3 is locally invertible in X_3. From [8, Proposition 2.2], we get that F_3 is a proper mapping. Then the global inverse mapping theorem [12, page 174] proves this statement.

(j) By (f) and (g), we have ($\Sigma_3 = D_{3b}$)

$$F_3(X_3) = F_3(\Sigma_3) \cup F_3(X_3 - \Sigma_3) = F_3(\Sigma_3) \cup \overline{F_3(X_3 - \Sigma_3)} = \overline{F(X_3)}. \tag{4.4}$$

Furthermore, $\partial F_3(X_3 - \Sigma_3) = \overline{F(X_3 - \Sigma_3)} - F(X_3 - \Sigma_3)$, and thus the previous equality implies assertion (j). □

Theorem 4.5. *Under assumption (A1), (A2), (B1), (B3), and (C1), each of the following conditions is sufficient for the solvability of problem (2.1), (2.3), (2.4) for each $g \in Y_3$:*

(k) *for each $g \in R_{3b}$, there is a solution u of (2.1), (2.3), (2.4) such that $u \in X_3 - D_{3b}$;*

(l) *the set $Y_3 - R_{3b}$ is connected and there is a $g \in R(F_3) - R_{3b}$.*

Proof. First of all, we see that conditions (k) and (l) are mutually equivalent to the following conditions:

(k′) $F_3(D_{3b}) \subset F_3(X_3 - D_{3b})$,
(l′) $Y_3 - R_{3b}$ is a connected set and

$$F_3(X_3 - D_{3b}) - R_{3b} \neq \emptyset, \tag{4.5}$$

respectively ($D_{3b} = \Sigma_3$).

Then it is sufficient to show that conditions (k′) and (l′), respectively, are sufficient for the surjectivity of the operator $F_3 : X_3 \to Y_3$.

(k′) From the first equality of (4.4), we obtain $F_3(X_3) = F_3(X_3 - D_{3b})$. Hence, $R(F_3)$ is an open as well as a closed subset of the connected space Y_3. Thus, $R(F_3) = Y_3$.
(l′) By Theorem 4.4(h), $\operatorname{card} F_3^{-1}(\{q\}) = \text{const} =: k \geq 0$ for every $q \in Y_3 - R_{3b}$.

If $k = 0$, then $F_3(X_3) = R_{3b}$ and $F_3(X_3 - D_{3b}) \subset R_{3b}$. This is a contradiction to (4.5). Then $k > 0$ and $R(F_3) = Y_3$. □

The other surjectivity theorem is true.

Theorem 4.6. *Let hypotheses (A1), (A2), (B1), (B3), and (C1) hold and*

(i) *there exists a constant $K > 0$ such that all solutions $u \in X_3$ of the initial-boundary value problem for the equation*

$$C_3u + \mu[A_3u - C_3u + N_3u] = 0, \quad \mu \in (0,1), \tag{4.6}$$

with data (2.3), (2.4), fulfil one of conditions (α_1) and (α_2) of the almost coercive condition (C1), then

(m) *problem (2.1), (2.3), (2.4) has at least one solution for each $g \in Y_3$;*
(n) *the number n_{3g} of solutions of (2.1), (2.3), (2.4) is finite, constant, and different from zero on each component of the set $Y_3 - R_{3b}$ (for all g belonging to the same component of $Y_3 - R_{3b}$).*

Proof. (m) It is sufficient to prove the surjectivity of the mapping $F_3 : X_3 \to Y_3$. By Lemma 3.1, we can write

$$F_3 = A_3 + N_3 = C_3 + (T_3 + N_3), \tag{4.7}$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism from X_3 onto Y_3 and $T_3 + N_3 : X_3 \to Y_3$ is a completely continuous operator. Then the operator

$$C_3^{-1}F_3 = I + C_3^{-1}(T_3 + N_3) : X_3 \longrightarrow X_3 \tag{4.8}$$

is completely continuous and condensing (see [12, page 496]). The set $\Sigma_3 = D_{3b}$ is the set of all points $u \in X_3$ where $C_3^{-1}F_3$, as well as F_3, is not locally invertible.

Denote $S_1 \subset X_3$ a bounded set. Then $C_3(S_1) =: S$ is bounded in Y_3, and by Lemma 3.4, $F_3^{-1}(S) = F_3^{-1}(C_3(S_1)) = (C_3^{-1} \circ F_3)^{-1}(S_1)$ is a bounded set in X_3. Thus, the operator $C_3^{-1} \circ F_3$ is coercive.

Now we show that condition (i) implies the conditions from [8, Theorem 3.2, Corollary 3.3, and Remark 3.1] for $F(u) = C_3^{-1} \circ F_3(u)$ and $C(u) = G(u) = u$, $u \in X_3$.

In fact, as $C_3^{-1} \circ F_3(u) = ku$ if and only if $F_3(u) = kC_3(u)$, we get for $k < 0$,

$$C_3u + (1-k)^{-1}\left[A_3u - C_3u + N_3u\right] = 0, \tag{4.9}$$

where $(1-k)^{-1} \in (0,1)$.

In case (α_1), there is a constant $K > 0$ such that for all solutions $u \in X_3$ of (4.9),

$$\|u\|_{(1+\alpha)/2,1+\alpha,Q} \leq K, \tag{4.10}$$

and in case (α_2),

$$\|u\|_{(2+\alpha)/2,2+\alpha,Q} \leq K. \tag{4.11}$$

Furthermore, by the same method as in Lemma 3.4, we get the estimation

$$\|u\|_{X_3} < K_1, \quad K_1 > 0, \tag{4.12}$$

for all solutions $u \in X_3$ of $C_3^{-1} \circ F_3u = ku$. Hence, we get the surjectivity of F_3 and thus (m).

(n) From Theorem 4.4(h) and the surjectivity of F_3, it follows that there is $n_{3g} \neq 0$. This finishes the proof of Theorem 4.6. □

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ON A NONLOCAL CAUCHY PROBLEM FOR DIFFERENTIAL INCLUSIONS

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We establish sufficient conditions for the existence of solutions for semilinear differential inclusions, with nonlocal conditions. We rely on a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler and on the Schaefer's fixed-point theorem combined with lower semicontinuous multivalued operators with decomposable values.

1. Introduction

In this paper, we are concerned with proving the existence of solutions of differential inclusions, with nonlocal initial conditions. More precisely, in Section 2, we consider the following differential inclusion, with nonlocal initial conditions:

$$y' \in F(t, y), \quad t \in J = [0, b], \tag{1.1a}$$

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0, \tag{1.1b}$$

where $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $y_0 \in \mathbb{R}^n$, and $0 \leq t_1 < t_2 < \cdots < t_p \leq b$, $p \in \mathbb{N}$, $c_k \neq 0$, $k = 1, 2, \ldots, p$.

The single-valued case of problem (1.1) was studied by Byszewski [5], in which a new definition of mild solution was introduced. In the same paper, it was remarked that the constants c_k, $k = 1, \ldots, p$, from condition (1.1b) can satisfy the inequalities $|c_k| \geq 1$, $k = 1, \ldots, p$. As pointed out by Byszewski [4], the study of initial value problems with nonlocal conditions is of significance since they have applications in problems in physics and other areas of applied mathematics.

The initial value problem (1.1) was studied by Benchohra and Ntouyas [1] in the semilinear case where the right-hand side is assumed to be convex-valued. Here, we drop this restriction and consider problem (1.1) with a nonconvex-valued right-hand side. By using the fixed-point theorem for contraction multivalued maps due to Covitz and Nadler [7] and the Schaefer's theorem combined with a selection theorem of Bressan

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and Colombo for lower semicontinuous (l.s.c.) multivalued operators with decomposable values, existence results are proposed for problem (1.1).

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this paper.

We denote by $\mathcal{P}(E)$ the set of all subsets of E normed by $\|\cdot\|_{\mathcal{P}}$ and by $C(J,\mathbb{R}^n)$ the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}. \tag{1.2}$$

Also, $L^1(J,\mathbb{R}^n)$ denotes the Banach space of measurable functions $y : J \to \mathbb{R}^n$ which are Lebesgue integrable and normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt. \tag{1.3}$$

Let A be a subset of $J \times \mathbb{R}^n$. The set A is $\mathcal{L} \otimes \mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $N \times D$, where N is Lebesgue measurable in J and D is Borel measurable in $\mathbb{R}^n$. A subset B of $L^1(J,\mathbb{R}^n)$ is decomposable if, for all $u, v \in B$ and $N \subset J$ measurable, the function $u\chi_N + v\chi_{J-N} \in B$, where χ denotes the characteristic function.

Let E be a Banach space, X a nonempty closed subset of E, and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. The operator G is l.s.c. if the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is open for any open set C in E. The operator G has a fixed point if there is $x \in X$ such that $x \in G(x)$. For more details on multivalued maps, we refer to Deimling [8], Górniewicz [10], Hu and Papageorgiou [11], and Tolstonogov [13].

Definition 1.1. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ be a multivalued operator. The operator N has property (BC) if

(1) N is l.s.c.;
(2) N has nonempty closed and decomposable values.

Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C(J,\mathbb{R}^n) \longrightarrow \mathcal{P}(L^1(J,\mathbb{R}^n)) \tag{1.4}$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J,\mathbb{R}^n) : w(t) \in F(t,y(t)) \text{ for a.e. } t \in J\}. \tag{1.5}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 1.2. Let $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. The multivalued map F is of l.s.c. type if its associated Niemytzki operator $\mathcal{F}$ is l.s.c. and has nonempty closed and decomposable values.

Next, we state a selection theorem due to Bressan and Colombo [3].

THEOREM 1.3 (see [3]). *Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ be a multivalued operator which has property (BC). Then N has a continuous selection, that is, there exists a (single-valued) continuous function $g : Y \to L^1(J,\mathbb{R}^n)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

Let (X,d) be a metric space. We use the following notations:

$$\begin{aligned} P(X) &= \{Y \in \mathcal{P}(X) : Y \neq \varnothing\}, \\ P_{cl}(X) &= \{Y \in P(X) : Y \text{ closed}\}, \\ P_b(X) &= \{Y \in P(X) : Y \text{ bounded}\}, \\ P_{cp}(X) &= \{Y \in P(X) : Y \text{ compact}\}. \end{aligned} \tag{1.6}$$

Consider $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{1.7}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized metric space.

Definition 1.4. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \leq \gamma d(x,y) \quad \text{for each } x, y \in X; \tag{1.8}$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

For more details on multivalued maps and the proofs of known results cited in this section, we refer to Deimling [8], Górniewicz [10], Hu and Papageorgiou [11], and Tolstonogov [13].

In the sequel, we will use the following fixed-point theorem for contraction multivalued operators given by Covitz and Nadler [7] (see also Deimling [8, Theorem 11.1]).

LEMMA 1.5. *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{fix} N \neq \varnothing$.

2. Main results

Definition 2.1. Assume that $\sum_{k=1}^{p} c_k \neq -1$. A function $y \in C(J,\mathbb{R}^n)$ is called a mild solution of (1.1) if there exists a function $v \in L^1(J,\mathbb{R}^n)$ such that $v(t) \in F(t,y(t))$ a.e. on J, and

$$y(t) = A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} v(s)ds\right) + \int_0^t v(s)ds, \tag{2.1}$$

where $A = (1 + \sum_{k=1}^{p} c_k)^{-1}$.

We will need the following assumptions:

(H1) $F : J \times \mathbb{R}^n \to P_{cp}(\mathbb{R}^n)$ has the property that $F(\cdot, y) : J \to P_{cp}(\mathbb{R}^n)$ is measurable for each $y \in \mathbb{R}^n$;

(H2) there exists $l \in L^1(J, \mathbb{R}^+)$ such that

$$\begin{gathered} H_d(F(t,y), F(t,\overline{y})) \le l(t)|y - \overline{y}| \quad \text{for almost each } t \in J,\ y, \overline{y} \in \mathbb{R}^n, \\ d(0, F(t,0)) \le \ell(t) \quad \text{for almost each } t \in J; \end{gathered} \tag{2.2}$$

(H3) assume that

$$\sum_{k=1}^{p} c_k \ne -1; \tag{2.3}$$

(H4) $|A| \sum_{k=1}^{p} |c_k| L(t_k) + L(b) < 1$, where $L(t) = \int_0^t l(s)ds$.

Theorem 2.2. *Assume that hypotheses (H1), (H2), (H3), and (H4) are satisfied. Then problem (1.1) has at least one mild solution on J.*

Proof. Transform problem (1.1) into a fixed-point problem. Consider the multivalued operator $N : C(J, \mathbb{R}^n) \to \mathcal{P}(C(J, \mathbb{R}^n))$ defined by

$$N(y) := \left\{ h \in C(J, \mathbb{R}^n) : h(t) = A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} g(s)ds \right) + \int_0^t g(s)ds : g \in S_{F,y} \right\}, \tag{2.4}$$

where

$$S_{F,y} = \{ g \in L^1(J, \mathbb{R}^n) : g(t) \in F(t, y(t)) \text{ for a.e. } t \in J \}. \tag{2.5}$$

We will show that N satisfies the assumptions of Lemma 1.5. The proof will be given in two steps.

Step 1. We prove that $N(y) \in P_{cl}(C(J, \mathbb{R}^n))$ for each $y \in C(J, \mathbb{R}^n)$.

Indeed, let $(y_n)_{n \ge 0} \in N(y)$ such that $y_n \to \tilde{y}$ in $C(J, \mathbb{R}^n)$. Then $\tilde{y} \in C(J, \mathbb{R}^n)$ and there exist $g_n \in S_{F,y}$ such that

$$y_n(t) = A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} g_n(s)ds \right) + \int_0^t g_n(s)ds. \tag{2.6}$$

Using the fact that F has compact values, and from (H2), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J, E)$ and hence $g \in S_{F,y}$. Then for each $t \in [0, b]$,

$$y_n(t) \longrightarrow \tilde{y}(t) = A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} g(s)ds \right) + \int_0^t g(s)ds. \tag{2.7}$$

So, $\tilde{y} \in N(y)$.

Step 2. We prove that $H_d(N(y_1), N(y_2)) \le \gamma \|y_1 - y_2\|_\infty$ for each $y_1, y_2 \in C(J, \mathbb{R}^n)$ (where $\gamma < 1$).

Let $y_1, y_2 \in C(J,\mathbb{R}^n)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t, y_1(t))$ such that

$$h_1(t) = A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} g_1(s)ds\right) + \int_0^t g_1(s)ds, \quad t \in J. \tag{2.8}$$

From (H2), it follows that

$$H_d(F(t,y_1(t)),F(t,y_2(t))) \le l(t)\,|y_1(t) - y_2(t)|, \quad t \in J. \tag{2.9}$$

Hence, there is $w \in F(t, y_2(t))$ such that

$$|g_1(t) - w| \le l(t)\,|y_1(t) - y_2(t)|, \quad t \in J. \tag{2.10}$$

Consider $U : J \to \mathcal{P}(\mathbb{R}^n)$ given by

$$U(t) = \{w \in \mathbb{R}^n : |g_1(t) - w| \le l(t)\,|y_1(t) - y_2(t)|\}. \tag{2.11}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_2(t))$ is measurable (see [6, Proposition III.4]), there exists $g_2(t)$ a measurable selection for V. So, $g_2(t) \in F(t, y_2(t))$ and

$$|g_1(t) - g_2(t)| \le l(t)\,|y_1(t) - y_2(t)| \quad \text{for each } t \in J. \tag{2.12}$$

We define for each $t \in J$,

$$h_2(t) = A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} g_2(s)ds\right) + \int_0^t g_2(s)ds, \quad t \in J. \tag{2.13}$$

Then we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \left| A\sum_{k=1}^{p} c_k \int_0^{t_k} [g_1(s) - g_2(s)]ds + \int_0^t [g_1(s) - g_2(s)]ds \right| \\ &\le |A| \sum_{k=1}^{p} |c_k|\,||y_1 - y_2||_\infty \int_0^{t_k} \ell(s)ds \\ &\quad + ||y_1 - y_2||_\infty \int_0^t l(s)ds \\ &\le \left(|A| \sum_{k=1}^{p} |c_k|\,L(t_k) + L(b)\right)||y_1 - y_2||_\infty. \end{aligned} \tag{2.14}$$

Then

$$||h_1 - h_2||_\infty \le \left(|A| \sum_{k=1}^{p} |c_k|\,L(t_k) + L(b)\right)||y_1 - y_2||_\infty. \tag{2.15}$$

By the analogous relation obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1),N(y_2)) \le \left(|A| \sum_{k=1}^{p} |c_k|\,L(t_k) + L(b)\right)||y_1 - y_2||_\infty. \tag{2.16}$$

From (H4), we have that

$$\gamma := |A| \sum_{k=1}^{p} |c_k| L(t_k) + L(b) < 1. \tag{2.17}$$

Then N is a contraction, and thus, by Lemma 1.5, it has a fixed point y which is a mild solution to (1.1). □

Remark 2.3. Consider the Bielecki-type norm (see [2]) on $C(J,\mathbb{R}^n)$, defined by

$$\|y\|_{\mathcal{B}} = \max_{t\in J}\{|y(t)|e^{-\tau L(t)}\}, \tag{2.18}$$

where $L(t) = \int_0^t l(s)ds$. Since

$$e^{-\tau L(b)}\|y\|_\infty \le \|y\|_{\mathcal{B}} \le \|y\|_\infty, \tag{2.19}$$

the norms $\|y\|_{\mathcal{B}}$ and $\|y\|_\infty$ are equivalent.

Then we can prove Step 2 of Theorem 2.2, that is, $H_d(N(y_1),N(y_2)) \le \gamma\|y_1 - y_2\|_{\mathcal{B}}$ for each $y_1, y_2 \in C(J,\mathbb{R}^n)$, where

$$\gamma = \frac{1}{\tau}\left(|A|\sum_{k=1}^{p}|c_k|e^{\tau L(t_k)} + 1\right). \tag{2.20}$$

Indeed, we have

$$\begin{aligned}
||h_1 - h_2||_{\mathcal{B}} &= \max_{t\in J} e^{-\tau L(t)}\left| A\sum_{k=1}^{p} c_k \int_0^{t_k}[g_1(s)-g_2(s)]ds \right.\\
&\qquad\qquad \left. + \int_0^t [g_1(s)-g_2(s)]ds\right|\\
&\le |A|\sum_{k=1}^{p}|c_k|\,||y_1-y_2||_{\mathcal{B}}\int_0^{t_k}\ell(s)e^{\tau L(s)}ds\\
&\quad + ||y_1-y_2||_{\mathcal{B}}\int_0^t l(s)e^{\tau L(s)}ds\\
&\le \left(|A|\sum_{k=1}^{p}|c_k|\frac{e^{\tau L(t_k)}}{\tau} + \frac{1-e^{-\tau L(b)}}{\tau}\right)||y_1-y_2||_{\mathcal{B}}\\
&\le \left(|A|\sum_{k=1}^{p}|c_k|\frac{e^{\tau L(t_k)}}{\tau} + \frac{1}{\tau}\right)||y_1-y_2||_{\mathcal{B}}.
\end{aligned} \tag{2.21}$$

We can choose τ such that $\gamma < 1$. In this case, (H4) must be deleted.

By the help of the Schaefer's fixed-point theorem combined with the selection theorem of Bressan and Colombo for l.s.c. maps with decomposable values, we will present an existence result for problem (1.1). Before this, we introduce the following hypotheses which are assumed hereafter:

(H5) $F : J \times C(J,\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ is a nonempty compact-valued multivalued map such that
  (a) $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
  (b) $u \mapsto F(t,u)$ is l.s.c. for a.e. $t \in J$;

(H6) for each $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \le h_r(t) \quad \text{for a.e. } t \in J,\ u \in \mathbb{R}^n \text{ with } |u| \le r. \tag{2.22}$$

In the proof of Theorem 2.5, we will need the next auxiliary result.

Lemma 2.4 (see [9]). *Let $F : J \times C(J,\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty, compact values. Assume that (H5) and (H6) hold. Then F is of l.s.c. type.*

Theorem 2.5. *Suppose, in addition to hypotheses (H5) and (H6), that the following also holds:*

(H7) *Assume that $\|F(t,y)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,y)\} \le p(t)\psi(|y|)$ for almost all $t \in J$ and all $y \in \mathbb{R}^n$, where $p \in L^1(J,\mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with*

$$\int^{\infty} \frac{du}{\psi(u)} = \infty. \tag{2.23}$$

Then the initial value problem (1.1) has at least one solution on J.

Proof. By Lemma 2.4, (H5) and (H6) imply that F is of l.s.c. type. Then, from Theorem 1.3, there exists a continuous function $f : C(J,\mathbb{R}^n) \to L^1(J,\mathbb{R}^n)$ such that $f(y) \in \mathcal{F}(y)$ for all $y \in C(J,\mathbb{R}^n)$.

We consider the problem

$$\begin{gathered} y'(t) = f(y)(t), \quad t \in J, \\ y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0. \end{gathered} \tag{2.24}$$

We remark that if $y \in C(J,\mathbb{R}^n)$ is a solution of problem (2.24), then y is a solution to problem (1.1).

Transform problem (2.24) into a fixed-point problem by considering the operator $N_1 : C(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ defined by

$$N_1(y)(t) := A\left(y_0 - \sum_{k=1}^{p} c_k \int_0^{t_k} f(y)(s)ds\right) + \int_0^t f(y)(s)ds. \tag{2.25}$$

We will show that N_1 is a compact operator.

Step 1. The operator N_1 is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C(J,\mathbb{R}^n)$. Then

$$\begin{aligned}\left|N_1(y_n)(t)-N_1(y)(t)\right| &\leq |A|\sum_{k=1}^{p}|c_k|\int_0^{t_k}|f(y_n)(s)-f(y)(s)|\,ds\\&\quad+\int_0^t|f(y_n)(s)-f(y)(s)|\,ds\\&\leq |A|\sum_{k=1}^{p}|c_k|\int_0^{b}|f(y_n)(s)-f(y)(s)|\,ds\\&\quad+\int_0^b|f(y_n)(s)-f(y)(s)|\,ds.\end{aligned}\tag{2.26}$$

Since the function f is continuous, then

$$\|N_1(y_n)-N_1(y)\|_\infty \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.\tag{2.27}$$

Step 2. The operator N_1 maps bounded sets into bounded sets in $C(J,\mathbb{R}^n)$.

Indeed, it is enough to show that there exists a positive constant c such that for each $y\in B_q=\{y\in C(J,E):\|y\|_\infty\leq q\}$, one has $\|N_1(y)\|_\infty\leq c$. By (H6), we have for each $t\in J$,

$$\begin{aligned}|N_1(y)(t)| &\leq |A|\left(|y_0|+\sum_{k=1}^{p}|c_k|\int_0^{t_k}|f(y)(s)|\,ds\right)+\int_0^t|f(y)(s)|\,ds\\&\leq |A|\left(|y_0|+\sum_{k=1}^{p}|c_k|\,\|h_q\|_{L^1}\right)+\|h_q\|_{L^1(J,\mathbb{R}_+)}.\end{aligned}\tag{2.28}$$

Step 3. The operator N_1 maps bounded sets into equicontinuous sets of $C(J,\mathbb{R}^n)$.

Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, and $B_q=\{y\in C(J,\mathbb{R}^n):\|y\|_\infty\leq q\}$ a bounded set of $C(J,E)$. Thus,

$$|N_1(y)(\tau_2)-N_1(y)(\tau_1)|\leq\int_{\tau_1}^{\tau_2}h_q(s)ds.\tag{2.29}$$

As $\tau_2\to\tau_1$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1, 2, and 3, together with the Arzelá-Ascoli theorem, we can conclude that N_1 is completely continuous.

Step 4. Now, it remains to show that the set

$$\mathscr{E}(N_1):=\{y\in C(J,\mathbb{R}^n):y=\lambda N_1(y)\text{ for some }0<\lambda<1\}\tag{2.30}$$

is bounded.

Let $y\in\mathscr{E}(N_1)$. Then $y=\lambda N_1(y)$ for some $0<\lambda<1$ and

$$y(t)=\lambda A\left(y_0-\sum_{k=1}^{p}c_k\int_0^{t_k}f(y)(s)ds\right)+\lambda\int_0^t f(y)(s)ds,\quad t\in J.\tag{2.31}$$

This implies, by (H7), that for each $t \in J$, we have

$$|y(t)| \leq |A||y_0| + |A| \sum_{k=1}^{p} |c_k| \int_0^{t_k} p(t)\psi(|y(t)|)dt + \int_0^t p(s)\psi(|y(s)|)ds. \tag{2.32}$$

We take the right-hand side of the above inequality as $v(t)$, then we have

$$\begin{gathered} v(0) = |A||y_0| + |A| \sum_{k=1}^{p} |c_k| \int_0^{t_k} p(t)\psi(|y(t)|)dt, \quad |y(t)| \leq v(t),\ t \in J, \\ v'(t) = p(t)\psi(|y(t)|), \quad t \in J. \end{gathered} \tag{2.33}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq p(t)\psi(v(t)), \quad t \in J. \tag{2.34}$$

This implies that for each $t \in J$,

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq \int_0^b p(s)ds < +\infty. \tag{2.35}$$

This inequality, together with hypothesis (H7), implies that there exists a constant d such that $v(t) \leq d$, $t \in J$, and hence $\|y\|_\infty \leq d$, where d depends only on the functions p and ψ. This shows that $\mathscr{E}(N_1)$ is bounded. As a consequence of Schaefer's theorem [12], we deduce that N_1 has a fixed point y which is a solution to problem (2.24). Then y is a solution to problem (1.1). □

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EXACT SOLUTIONS OF THE SEMI-INFINITE TODA LATTICE WITH APPLICATIONS TO THE INVERSE SPECTRAL PROBLEM

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Several inverse spectral problems are solved by a method which is based on exact solutions of the semi-infinite Toda lattice. In fact, starting with a well-known and appropriate probability measure μ, the solution $\alpha_n(t)$, $b_n(t)$ of the Toda lattice is exactly determined and by taking $t=0$, the solution $\alpha_n(0)$, $b_n(0)$ of the inverse spectral problem is obtained. The solutions of the Toda lattice which are found in this way are finite for every $t>0$ and can also be obtained from the solutions of a simple differential equation. Many other exact solutions obtained from this differential equation show that there exist initial conditions $\alpha_n(0)>0$ and $b_n(0)\in\mathbb{R}$ such that the semi-infinite Toda lattice is not integrable in the sense that the functions $\alpha_n(t)$ and $b_n(t)$ are not finite for every $t>0$.

1. Introduction

We write the semi-infinite Toda lattice as follows:

$$\frac{d\alpha_n(t)}{dt}=\alpha_n(t)\big(b_{n+1}(t)-b_n(t)\big), \tag{1.1}$$

$$\frac{db_n(t)}{dt}=2\big(\alpha_n^2(t)-\alpha_{n-1}^2(t)\big),\quad t\geq 0,\ n=1,2,\dots \tag{1.2}$$

and we ask for solutions which satisfy the initial conditions

$$\alpha_n(0)=\alpha_n,\qquad b_n(0)=b_n, \tag{1.3}$$

where α_n, b_n are real sequences with $\alpha_n>0$. In an attempt to compute the functions $b_n(t)$ and $\alpha_n(t)$ in some problems where the existence and uniqueness of a solution is proved, we observed that many solutions of the initial value problem (1.1), (1.2), (1.3) satisfy the relation

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)}=\delta b_1(t)+c, \tag{1.4}$$

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where δ, c are real numbers with $\delta \geq 0$ and the dot means differentiability with respect to t. Then it follows easily that

$$\begin{aligned} \alpha_n^2(t) &= \left(\frac{n(n-1)}{2}\delta + n\right)\alpha_1^2(t), \\ b_n(t) &= \left[(n-1)\delta + 1\right]b_1(t) + (n-1)c. \end{aligned} \tag{1.5}$$

An example, where relation (1.4) holds and the functions $\alpha_1(t)$ and $b_1(t)$ can be exactly determined, has been published among others in [7]. Here, we present many examples starting from the well-known probability measures which determine uniquely the sequences in (1.3). All these examples provide alternative solutions of important well-known inverse spectral problems. (For details about the inverse spectral problem, see Section 3.) In other words, the solution of many well-known and important inverse spectral problems is obtained from one source, a class of exactly solvable Toda lattices.

The solutions of the Toda lattice which are found in this way are finite for every $t > 0$. These global solutions can also be obtained from the solutions of the differential equation

$$\ddot{b}_1(t) = 2(\delta b_1(t) + c)\dot{b}_1(t). \tag{1.6}$$

Any solution of this equation satisfies (1.4). Given the initial conditions $b_1(0)$ and $\alpha_1(0) > 0$ of the Toda lattice, we find the solution $b_1(t)$ of (1.6) which satisfies these conditions. We have the possibility to choose δ and c according to the form of the solution we want to construct. Moreover, from (1.6), we can find solutions of the Toda lattice with poles. Thus, many solutions of (1.6) show that there exist initial conditions $\alpha_n(0)$ and $b_n(0)$ such that the Toda lattice is not global integrable in the sense that the functions $\alpha_n(t)$ and $b_n(t)$ are not finite for every $t > 0$ (see Example 2.1 and Remark 4.10). In Section 2, we give the proofs of (1.5) and (1.6) and we obtain from (1.5) and (1.6) several forms of exact solutions of the Toda lattice. In Section 3, we define the inverse spectral problem and present preliminary results which we need. For many measures whose support is not bounded from above, the standard method of determining the function $b_1(t)$, and consequently $\alpha_1(t)$, $b_2(t)$, and so on, fails. With respect to the inverse spectral problem, we avoid this difficulty by using Theorem 3.2 (see also Remark 3.3). In Section 4, we give five examples of inverse spectral problems which can be solved by the method which is based on the determination of exact solutions of the Toda lattice. Note that all the exact solutions obtained in Section 2, by solving (1.6), and in Section 4, by solving several inverse spectral problems, are solutions with unbounded initial conditions.

2. Solutions of $\ddot{b}_1(t) = 2(\delta b_1(t) + c)\dot{b}_1(t)$

First, we give the proofs of relations (1.5) and (1.6).

From (1.1) (for $n = 1$) and (1.4), we have

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = \delta b_1(t) + c = b_2(t) - b_1(t) \tag{2.1}$$

or

$$b_2(t) = (\delta + 1)b_1(t) + c. \tag{2.2}$$

From (2.2) and (1.2), we find

$$\dot{b}_2(t) = (\delta+1)\dot{b}_1(t) = 2\left(\alpha_2^2(t) - \alpha_1^2(t)\right). \tag{2.3}$$

Since, from (1.2), we have

$$\dot{b}_1(t) = 2\alpha_1^2(t), \tag{2.4}$$

we obtain

$$\alpha_2^2(t) = (\delta+2)\alpha_1^2(t). \tag{2.5}$$

Continuing in this way, we find $b_3(t) = (2\delta+1)b_1(t) + 2c$ and $\alpha_3^2(t) = (3\delta+3)\alpha_1^2(t)$. Formulas (1.5) are obtained by induction. From (2.4), we obtain

$$\ddot{b}_1(t) = 4\alpha_1(t)\dot{\alpha}_1(t). \tag{2.6}$$

Thus from (2.4), (2.6), and (1.4), we obtain (1.6).

Equation (1.6) can be easily integrated. We present below several forms of the obtained solutions. The forms of the solutions depend on the initial conditions $\alpha_1(0)$ and $b_1(0)$ and the values of δ and c.

Case 1 ($\delta = 0$, $c = 0$). In this case, the solution of (1.6) is $b_1(t) = 2\alpha_1^2(0)t + b_1(0)$. From this and (2.4), we find $\alpha_1(t) = \alpha_1(0)$ and from (1.5), we obtain

$$\alpha_n(t) = \sqrt{n}\alpha_1(0), \quad b_n(t) = b_1(t) = 2\alpha_1^2(0)t + b_1(0), \quad n = 1,2,\ldots. \tag{2.7}$$

The solution in this case is finite for every $t > 0$ (global solution).

Case 2 ($\delta = 0$, $c \neq 0$). In this case, the solution of (1.6) is

$$b_1(t) = \frac{\alpha_1^2(0)}{c}\left(e^{2ct} - 1\right) + b_1(0) \tag{2.8}$$

and the solution $\alpha_n(t)$, $b_n(t)$ of the system (1.1), (1.2) is

$$\begin{aligned} \alpha_n(t) &= \sqrt{n}\alpha_1(0)e^{ct}, \\ b_n(t) &= \frac{\alpha_1^2(0)}{c}\left(e^{2ct} - 1\right) + b_1(0) + (n-1)c. \end{aligned} \tag{2.9}$$

Case 3 ($\delta > 0$, $c = 0$). The solution of (1.6) depends on the value

$$A_0 = 2\alpha_1^2(0) - \delta b_1^2(0). \tag{2.10}$$

If $A_0 = 0$, the global solution of (1.6) has the form

$$b_1(t) = \frac{b_1(0)}{1 - \delta b_1(0)t}, \quad b_1(0) < 0 \tag{2.11}$$

and the solution $\alpha_n(t)$, $b_n(t)$ of the system (1.1), (1.2) is given by

$$\alpha_n(t) = \sqrt{n(n\delta - \delta + 2)}\frac{-\sqrt{\delta}b_1(0)}{2(1-\delta b_1(0)t)},$$
$$b_n(t) = n\frac{\delta b_1(0)}{1-\delta b_1(0)t} + \frac{(1-\delta)b_1(0)}{1-\delta b_1(0)t}. \tag{2.12}$$

If $A_0 > 0$, the solution of (1.6) is

$$b_1(t) = \sqrt{\frac{A_0}{\delta}}\tan\left(\sqrt{\delta A_0}t + \Gamma_1\right), \tag{2.13}$$

$$\alpha_n(t) = \sqrt{n(n\delta - \delta + 2)A_0}\frac{1}{2\cos\left(\sqrt{\delta A_0}t + \Gamma_1\right)},$$
$$b_n(t) = \left(n + \frac{1}{\delta} - 1\right)\sqrt{A_0\delta}\tan\left(\sqrt{A_0\delta}t + \Gamma_1\right), \tag{2.14}$$

where $\Gamma_1 = \arctan(\sqrt{\delta/A_0}b_1(0))$. In order to have a global solution of the form (2.14), the condition $\cos(\sqrt{A_0\delta}t + \Gamma_1) \neq 0$, for all $t > 0$, should hold.

If $A_0 < 0$, the solution of (1.6) is given by

$$b_1(t) = \frac{\Gamma_2(-\sqrt{-A_0\delta})e^{-2t\sqrt{-A_0\delta}} - \sqrt{-A_0\delta}}{\delta\left(1 - \Gamma_2 e^{-2t\sqrt{-A_0\delta}}\right)} \tag{2.15}$$

and the functions $\alpha_n(t)$ and $b_n(t)$ have the form

$$\alpha_n(t) = \sqrt{n(n\delta - \delta + 2)\Gamma_2(-A_0)}\frac{e^{-t\sqrt{-A_0\delta}}}{1 - \Gamma_2 e^{-2t\sqrt{-A_0\delta}}},$$
$$b_n(t) = [(n-1)\delta + 1]\frac{\Gamma_2(-\sqrt{-A_0\delta})e^{-2t\sqrt{-A_0\delta}} - \sqrt{-A_0\delta}}{\delta\left(1 - \Gamma_2 e^{-2t\sqrt{-A_0\delta}}\right)}, \tag{2.16}$$

where $\Gamma_2 = (\delta b_1(0) + \sqrt{-A_0\delta})/(\delta b_1(0) - \sqrt{-A_0\delta})$. We have a global solution if and only if

$$1 - \frac{\delta b_1(0) + \sqrt{-A_0\delta}}{\delta b_1(0) - \sqrt{-A_0\delta}}e^{-2t\sqrt{-A_0\delta}} \neq 0, \quad t \geq 0. \tag{2.17}$$

Case 4 ($\delta > 0$, $c \neq 0$). In this case, the solution of (1.6) and consequently the form of the solution of the system (1.1), (1.2) depends on the value

$$A = (2\alpha_1^2(0) - \delta b_1^2(0) - 2cb_1(0))\delta - c^2. \tag{2.18}$$

If $A = 0$, the solution of (1.6) and the functions $\alpha_n(t)$ and $b_n(t)$ have the form

$$b_1(t) = \frac{2cB_1e^{2ct}}{\delta(1 - B_1e^{2ct})}, \tag{2.19}$$

$$\begin{aligned} \alpha_n(t) &= \sqrt{\frac{n(n\delta - \delta + 2)B_1}{\delta}} \frac{ce^{ct}}{1 - B_1e^{2ct}}, \\ b_n(t) &= nc\frac{1 + B_1e^{2ct}}{1 - B_1e^{2ct}} + \frac{cB_1e^{2ct}(2 - \delta) - c\delta}{\delta(1 - B_1e^{2ct})}, \end{aligned} \tag{2.20}$$

where $B_1 = \delta b_1(0)/(\delta b_1(0) + 2c)$. In order to construct a global solution of the form (2.20), we must choose δ, c, $b_1(0)$, and $\alpha_1(0)$ such that $\delta b_1(0) + 2c \neq 0$, $A = (2\alpha_1^2(0) - \delta b_1^2(0) - 2cb_1(0))\delta - c^2 = 0$, and $1 - B_1e^{2ct} \neq 0$, for every $t > 0$.

If $A > 0$, we have

$$b_1(t) = \frac{\sqrt{A}}{\delta} \tan\left(\sqrt{A}t + B_2\right) - \frac{c}{\delta}, \tag{2.21}$$

$$\begin{aligned} \alpha_n(t) &= \sqrt{\frac{n(n\delta - \delta + 2)A}{\delta}} \frac{1}{2\cos\left(\sqrt{A}t + B_2\right)}, \\ b_n(t) &= \left(n + \frac{1}{\delta} - 1\right)\sqrt{A}\tan\left(\sqrt{A}t + B_2\right) - \frac{c}{\delta}, \end{aligned} \tag{2.22}$$

where $B_2 = \arctan((\delta b_1(0) + c)/\sqrt{A})$. In order to have a global solution of the form (2.22), the condition $\cos(\sqrt{A}t + B_2) \neq 0$, for all $t > 0$, should hold. Finally, for $A < 0$, the solutions are exponential with

$$b_1(t) = \frac{B_3(c - \sqrt{-A})e^{-2t\sqrt{-A}} - \sqrt{-A} - c}{\delta(1 - B_3e^{-2t\sqrt{-A}})}, \tag{2.23}$$

$$\begin{aligned} \alpha_n(t) &= \sqrt{\frac{n(n\delta - \delta + 2)B_3(-A)}{\delta}} \frac{e^{-t\sqrt{-A}}}{1 - B_3e^{-2t\sqrt{-A}}}, \\ b_n(t) &= [(n-1)\delta + 1]b_1(t) + (n-1)c, \end{aligned} \tag{2.24}$$

where $B_3 = (\delta b_1(0) + c + \sqrt{-A})/(\delta b_1(0) + c - \sqrt{-A})$ and $b_1(t)$ is given by (2.23). In order to have a solution without poles, the condition

$$1 - \frac{\delta b_1(0) + c + \sqrt{-A}}{\delta b_1(0) + c - \sqrt{-A}} e^{-2t\sqrt{-A}} \neq 0, \quad t \geq 0, \tag{2.25}$$

must be satisfied.

Example 2.1. Taking $\delta = 2$, $c = 1 - \lambda$, $b_1(0) = 1$, and $\alpha_1(0) = \sqrt{\lambda}$, $0 < \lambda < 1$, we find $A = -(\lambda - 9)(\lambda - 1) < 0$ for $0 < \lambda < 1$, $B_3 = (3 - \lambda + \sqrt{-A})/(3 - \lambda - \sqrt{-A}) > 1$. This means that condition (2.25) is not satisfied for every $t > 0$. We conclude that the Toda lattice, with initial conditions $b_1(0) = 1$, $\alpha_1(0) = \sqrt{\lambda}$, and $b_n(0)$, $\alpha_n(0)$ given by (1.5) for $t = 0$, $\delta = 2$, and $c = 1 - \lambda$, is not integrable in the sense that the functions $\alpha_n(t)$ and $b_n(t)$ are not finite for every $t > 0$.

Example 2.2. Taking $\delta = 2$, $c = 1-\lambda$, $b_1(0) = -1$, $\alpha_1(0) = \sqrt{\lambda}$, $0 < \lambda < 1$, $\alpha_n(0) = n\sqrt{\lambda}$, and $b_n(0) = -(1+\lambda)n+\lambda$, we have $A = -(1-\lambda)^2$, $B_3 = \lambda$. Then condition (2.25) is satisfied and $b_1(t)$ is given by

$$b_1(t) = -\frac{1-\lambda}{1-\lambda e^{-2(1-\lambda)t}}. \tag{2.26}$$

This example is a particular case of Example 4.5. In fact, (2.26) is the same with (4.11) for $\beta = 0$ and $\kappa = 1-\lambda > 0$.

3. The inverse spectral problem

It is well known that any probability measure μ on the real line with finite moments and infinite support determines uniquely a pair of real sequences α_n, b_n with $\alpha_n > 0$ and a class of orthonormal polynomials $P_n(x)$ ($\int_{-\infty}^{\infty} P_n(x)P_m(x)d\mu(x) = \delta_{nm}$), which satisfy the relation

$$\begin{gathered} \alpha_n P_{n+1}(x) + \alpha_{n-1}P_{n-1}(x) + b_n P_n(x) = xP_n(x), \\ P_0(x) = 0, \qquad P_1(x) = 1. \end{gathered} \tag{3.1}$$

Conversely, for any pair of real sequences α_n, b_n with $\alpha_n > 0$, there exists at least one probability measure μ such that the polynomials (3.1) are orthonormal. The measure μ is unique if and only if the tridiagonal operator $L(0)$, defined on finite linear combination of an orthonormal basis e_n, $n = 1,2,\ldots$, of a Hilbert space H:

$$\begin{aligned} L(0)e_n &= \alpha_n e_{n+1} + \alpha_{n-1}e_{n-1} + b_n e_n, \\ L(0)e_1 &= \alpha_1 e_2 + b_1 e_1, \end{aligned} \tag{3.2}$$

is (essentially) selfadjoint (see [1, 8, 11] for these subjects and their relationships). If $L(0)$ is selfadjoint, then there exists a one parameter family E_t, $-\infty < t < \infty$, of orthogonal projections on H such that for every x, $\|x\| = 1$, the function $F(t) = (E_t x, x)$, where $(\cdot,\cdot)$ means scalar product, is a distribution function, that is, a nondecreasing function which is continuous on the right and satisfies $F(-\infty) = 0$, $F(\infty) = 1$. In particular, for $x = e_1$, the distribution function

$$F(t) = (E_t e_1, e_1) \tag{3.3}$$

is the distribution function which corresponds to the unique probability measure μ, that is, μ and F are connected by (see Theorem 3.1)

$$\mu((-\infty, t]) = F(t). \tag{3.4}$$

The direct problem of orthogonal polynomials is the following.

Given the real sequences $\alpha_n > 0$ and b_n, find the measure of orthogonality of the polynomials which are defined by (3.1). This problem has a long history. Only the problem of finding conditions on α_n and b_n, such that the above problem has a unique solution (note that at least one solution always exists), is connected with many important problems in analysis, for instance, the moment problem, the problem of selfadjoint extensions of an unbounded symmetric operator, and others [1, 8, 11] (see also [5]). Note that the measure of orthogonality is a probability measure on the real line with finite moments and support consisting of infinitely many points.

The inverse problem of orthogonal polynomials is the following.

Given a probability measure μ on the real line with finite moments and infinite support, find the coefficients α_n and b_n which define the polynomials P_n in (3.1). Sometimes we say the inverse problem of the operator $L(0)$ or the inverse problem of μ instead of the inverse problem of the polynomials (3.1).

There exists a standard procedure, which determines uniquely the sequences α_n and b_n but this procedure involves many and arduous calculations, and exact solutions are very difficult to be found. In fact, multiplication by $P_n(x)$ in (3.1) and integration gives

$$b_n = \int_{-\infty}^{\infty} xP_n^2(x)d\mu(x). \tag{3.5}$$

First, we find from (3.5) that $b_1 = \int_{-\infty}^{\infty} x\,d\mu(x)$. Consequently, taking $n = 1$ in (3.1) and multiplying the relation $\alpha_1 P_2(x) + b_1 = x$ by $P_2(x)$, we obtain $\alpha_1^2 = \int_{-\infty}^{\infty} x^2 d\mu(x) - b_1^2$. After this, knowing the polynomial $P_2(x)$, we determine b_2 from (3.5). Then we find α_2, the polynomial $P_3(x)$, and so on. The inverse problem of a selfadjoint operator is the problem of finding the operator when some of its properties are given, for instance, its spectrum. It is well known that the spectrum is not always enough for the solution of this problem. In the present case, the spectrum of $L(0)$ is not enough to determine uniquely the sequences α_n, b_n. However, the knowledge of the distribution function $(E_t e_1, e_1)$ is enough, because of the following well-known theorem, which we prove for completeness.

Theorem 3.1. *Let $L(0)$, defined by (3.2), be selfadjoint and let E_t, $-\infty < t < \infty$ be its spectral family. Then the measure which corresponds to the distribution function $(E_t e_1, e_1)$ is the unique measure of orthogonality of the polynomials P_n defined by (3.1).*

Proof. Let $L(0) = T$. Then T can be written as

$$T = \int_{-\infty}^{\infty} t\,dE_t, \tag{3.6}$$

in the sense that

$$(Tx, y) = \int_{-\infty}^{\infty} t\,d(E_t x, y) \tag{3.7}$$

for every x, y in the definition domain of T. Then the operator $P_m(T)P_n(T)$ can be written as follows:

$$\begin{aligned} P_m(T)P_n(T) &= \int_{-\infty}^{\infty} P_m(t)P_n(t)dE_t, \\ (P_m(T)P_n(T)e_1, e_1) &= \int_{-\infty}^{\infty} P_m(t)P_n(t)d(E_t e_1, e_1). \end{aligned} \tag{3.8}$$

The operator $P_n(T)$ acting on the element e_1 produces the vector e_n, that is,

$$P_n(T)e_1 = e_n, \quad n = 1, 2, \ldots. \tag{3.9}$$

Relation (3.9) is obvious for $n = 1$ and for $n \geq 2$ it follows from (3.2) and the relation $\alpha_n P_{n+1}(T) + \alpha_{n-1} P_{n-1}(T) + b_n P_n(T) = T P_n(T)$, by induction. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} P_m(t) P_n(t) d(E_t e_1, e_1) &= (P_m(T) P_n(T) e_1, e_1) \\ &= (P_n(T) e_1, P_m(T) e_1) \\ &= (e_n, e_m) = \delta_{n,m}.\end{aligned} \tag{3.10}$$

□

The measure which corresponds to the distribution function (3.3) is called spectral measure of the tridiagonal operator $L(0)$.

Another theorem that we will need is the following one.

Theorem 3.2. *Assume that the operator*

$$L(0) : L(0) e_n = \alpha_n e_{n+1} + \alpha_{n-1} e_{n-1} + b_n e_n, \quad n = 1, 2, \ldots, \tag{3.11}$$

is essentially selfadjoint with spectral measure μ. Then the operator

$$L_1(0) : L_1(0) e_n = \alpha_n e_{n+1} + \alpha_{n-1} e_{n-1} - b_n e_n \tag{3.12}$$

is also essentially selfadjoint with spectral measure $\mu\tau^{-1}$, where $\tau(x) = -x$.

Proof. By Theorem 3.1 and by the equivalence of the properties "essential selfadjointness" of $L(0)$ and "uniqueness of the measure of orthogonality" of the corresponding polynomials, it is enough to prove that $\mu\tau^{-1}$ is a measure of orthogonality of the polynomials

$$\begin{gathered}\alpha_n R_{n+1}(x) + \alpha_{n-1} R_{n-1}(x) - b_n R_n(x) = x R_n(x), \\ R_0(x) = 0, \qquad R_1(x) = 1,\end{gathered} \tag{3.13}$$

provided that μ is a measure of orthogonality of the polynomials (3.1). It is easy to see that the polynomials $R_n(x)$ and $P_n(x)$ are related by

$$R_n(x) = (-1)^n P_n(-x). \tag{3.14}$$

Assume that

$$\int_{-\infty}^{\infty} P_n(x) P_m(x) d\mu = \delta_{n,m}. \tag{3.15}$$

Then by a well-known property in measure theory, we have

$$\begin{aligned}\int_{-\infty}^{\infty} R_n(x) R_m(x) d\mu\tau^{-1} &= \int_{-\infty}^{\infty} R_n(\tau(x)) R_m(\tau(x)) d\mu \\ &= \int_{-\infty}^{\infty} R_n(-x) R_m(-x) d\mu \\ &= (-1)^{n+m} \int_{-\infty}^{\infty} P_n(x) P_m(x) d\mu = \delta_{n,m}.\end{aligned} \tag{3.16}$$

□

Given the initial conditions α_n and b_n of the Toda lattice, we assume that the operator $L(0)$ is (essentially) selfadjoint which means that the measure μ of orthogonality of the polynomials (3.1) is unique.

The method of the inverse spectral problem works as follows: system (1.1), (1.2) is equivalent to the equation

$$\frac{dL(t)}{dt} = M(t)L(t) - L(t)M(t), \tag{3.17}$$

where $L(t)$ and $M(t)$ are the tridiagonal operators

$$\begin{aligned} L(t)e_n &= \alpha_n(t)e_{n+1} + \alpha_{n-1}(t)e_{n-1} + b_n(t)e_n, \\ M(t)e_n &= \alpha_n(t)e_{n+1} - \alpha_{n-1}(t)e_{n-1}. \end{aligned} \tag{3.18}$$

Under suitable assumptions on α_n, b_n, one finds the spectral measure $\mu^{(t)}$ of the operator $L(t)$ [7]. Note that the spectral measure μ can be found by solving the direct problem of the operator $L(0)$. For the Toda lattice, as it is written in (1.1), (1.2), $\mu^{(t)}$ has the form

$$d\mu^{(t)}(x) = \frac{e^{2xt}d\mu(x)}{\int_{-\infty}^{\infty} e^{2xt}d\mu(x)}, \quad t \geq 0. \tag{3.19}$$

The solution of the Toda lattice is obtained by solving the inverse problem of $L(t)$. In fact, starting from the spectrum measure $\mu^{(t)}$, we consider the linearly independent elements $1, x, x^2, \ldots$ of the space $L_2(\mu^{(t)})$, the orthogonalization of which by the use of the Gram-Schmidt method gives the orthogonal polynomials $P_n(t,x)$ which satisfy

$$\begin{gathered} \alpha_n(t)P_{n+1}(t,x) + \alpha_{n-1}(t)P_{n-1}(t,x) + b_n(t)P_n(t,x) = xP_n(t,x), \\ P_0(t,x) = 0, \qquad P_1(t,x) = 1, \end{gathered} \tag{3.20}$$

with $\alpha_n(t) > 0$ and $b_n(t)$ real. By a well-known procedure, the sequences $\alpha_n(t)$, $b_n(t)$ can be found from the above recurrence relation. Moreover, they satisfy system (1.1), (1.2). In our case, it is enough to determine exactly the function $b_1(t)$, which is given by

$$b_1(t) = \frac{\int_{-\infty}^{\infty} xe^{2xt}d\mu(x)}{\int_{-\infty}^{\infty} e^{2xt}d\mu(x)}, \quad t \geq 0. \tag{3.21}$$

What we need for the solution of the Toda lattice is the spectral measure μ of $L(0)$. If the spectrum of $L(0)$ is discrete with eigenvalues λ_n and normalized eigenvectors x_n, then

$$\mu(\{\lambda_n\}) = \left|(e_1, x_n)\right|^2 = \sigma_n^2 \tag{3.22}$$

and $b_1(t)$ is given by

$$b_1(t) = \frac{\sum_{n=1}^{\infty} \lambda_n e^{2\lambda_n t}\sigma_n^2}{\sum_{n=1}^{\infty} e^{2\lambda_n t}\sigma_n^2}. \tag{3.23}$$

In Section 4, we begin with a probability measure μ without knowing the initial conditions $\alpha_n(0)$, $b_n(0)$. Then we determine the element $b_1(t)$ from (3.21) or (3.23) and examine the validity of the relation (1.4). After this, the solution of the Toda lattice is given by (1.5). The solution of the inverse problem of μ is given by

$$\begin{aligned} \alpha_n^2(0) &= \left(\frac{n(n-1)}{2}\delta + n\right)\alpha_1^2(0), \\ b_n(0) &= \left[(n-1)\delta + 1\right]b_1(0) + (n-1)c. \end{aligned} \tag{3.24}$$

Remark 3.3. The usefulness of Theorem 3.2 is that if the support of the measure μ lies in the interval $[\alpha,\infty)$, $\alpha \in \mathbb{R}$, and if it is not a bounded set, then the integrals in (3.21) may not be finite. In this case, we find the solution $\alpha_n(0)$, $b_n(0)$ of the inverse spectral problem of the measure $\mu\tau^{-1}$, $\tau(x) = -x$, whose support is bounded from above. Then the solution of the inverse spectral problem of μ, due to Theorem 3.2, is $\alpha_n(0)$, $-b_n(0)$, $n = 1,2,\ldots$.

4. Examples

In all the following examples, we begin with a probability measure μ on the real line with finite moments $\mu_n = \int_{-\infty}^{\infty} x^n d\mu(x)$, $\mu_0 = 1$, and infinite support.

Example 4.1. Consider the probability measure μ whose distribution function is given by

$$F(x) = \frac{1}{\Gamma(\alpha+1)}\int_0^x \xi^\alpha e^{-\xi} d\xi, \quad \alpha > -1, \tag{4.1}$$

where Γ is the gamma function. The moments

$$\mu_k = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty x^k x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)}, \quad k = 0,1,2,\ldots, \tag{4.2}$$

are finite. Moreover, we can see that there exists a positive number r such that the series

$$\sum_{m=1}^{\infty} \frac{\mu_m r^m}{m!} \tag{4.3}$$

converges and by a well-known criterion (see, e.g., [2, Theorem 30.1]), the moment problem is determined or equivalently the operator $L(0)$ is selfadjoint (see [1]).

Since the integrals of (3.21) do not exist for this measure, we consider the measure $\mu\tau^{-1}$, $\tau(x) = -x$ instead of μ. We find from (3.21)

$$\begin{aligned} b_1(t) &= -\frac{\alpha+1}{2t+1}, \qquad \alpha_1(t) = \frac{\sqrt{\alpha+1}}{2t+1}, \\ \frac{\dot{\alpha}_1(t)}{\alpha_1(t)} &= \frac{2}{\alpha+1} b_1(t) \qquad \left(\delta = \frac{2}{\alpha+1},\ c = 0\right). \end{aligned} \tag{4.4}$$

Thus, from (1.5), we obtain

$$\begin{aligned} \alpha_n(t) &= \frac{\sqrt{n(n+\alpha)}}{2t+1}, \\ b_n(t) &= -\frac{2n+\alpha-1}{2t+1}, \quad t \geq 0,\ n = 1,2,\dots. \end{aligned} \tag{4.5}$$

Conclusion 4.2. The solution of the Toda lattice with initial conditions

$$\begin{aligned} \alpha_n(0) &= \sqrt{n(n+\alpha)}, \\ b_n(0) &= -(2n+\alpha-1) \end{aligned} \tag{4.6}$$

is given by (4.5). The solution of the inverse problem of $\mu\tau^{-1}$ is given by (4.6). Due to Theorem 3.2, the solution of the inverse problem of the measure μ is

$$\begin{aligned} \alpha_n(0) &= \sqrt{n(n+\alpha)}, \\ b_n(0) &= (2n+\alpha-1). \end{aligned} \tag{4.7}$$

Remark 4.3. The solution of the inverse problem that we studied in this example is well known in the theory of Laguerre polynomials defined by

$$\begin{gathered} \sqrt{n(n+\alpha)}P_{n+1}(x) + \sqrt{(n-1)(n-1+\alpha)}P_{n-1}(x) + (2n+\alpha-1)P_n(x) = xP_n(x), \\ P_0(x) = 0, \qquad P_1(x) = 1. \end{gathered} \tag{4.8}$$

In fact, it is well known that the measure of orthogonality of $P_n(x)$ is unique and its distribution function is given by (4.1) (see [3]).

Remark 4.4. In the following examples, we have a difficulty to establish the convergence of the series (4.3). We avoid this difficulty as follows: suppose that we start with a measure of the form (3.19) and we have found exactly a solution of the Toda lattice $\alpha_n(t)$, $b_n(t)$ for $t \geq 0$. This means that we have found exactly the coefficients of the polynomials $P_n(t,x)$ which are orthonormal with respect to the measure $\mu^{(t)}$. In all the examples that we will give, we can see from the coefficients $\alpha_n(t)$, $b_n(t)$ that $\mu^{(t)}$ is the unique measure of orthogonality for every $t \geq 0$. In fact, the well-known criterion of Carleman [1] can easily be applied. For $t = 0$, we find the coefficients of the polynomials whose measure of orthogonality is the measure μ. In this way, we solve both the inverse and the direct spectral problem of μ.

Example 4.5. Consider the discrete probability measure

$$\mu(\{-\kappa n+\beta\}) = (1-\lambda)\lambda^{n-1}, \quad 0 < \lambda < 1,\ n = 1,2,\dots,\ \kappa > 0. \tag{4.9}$$

Obviously, its support is infinite consisting of the points $-\kappa+\beta$, $-2\kappa+\beta$, $-3\kappa+\beta,\ldots$, and the moments $\mu_k = (1-\lambda)\sum_{n=1}^{\infty}(-\kappa n+\beta)^k\lambda^{n-1}$, $k = 0,1,2,\ldots$, are finite. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{2xt}\,d\mu(x) &= (1-\lambda)\sum_{n=1}^{\infty} e^{-2\kappa nt+2\beta t}\lambda^{n-1} \\
&= \frac{(1-\lambda)e^{2\beta t}}{\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda}{e^{2\kappa t}}\right)^n = \frac{(1-\lambda)e^{2\beta t}}{e^{2\kappa t}-\lambda}, \\
\int_{-\infty}^{\infty} xe^{2xt}\,d\mu(x) &= \sum_{n=1}^{\infty}(-\kappa n+\beta)e^{-2\kappa nt}e^{2\beta t}(1-\lambda)\lambda^{n-1} \\
&= \frac{\beta(1-\lambda)e^{2\beta t}}{e^{2\kappa t}-\lambda} - \frac{\kappa(1-\lambda)e^{2\beta t}}{e^{2\kappa t}(1-\mu)^2}, \quad \mu = \frac{\lambda}{e^{2\kappa t}}.
\end{aligned} \tag{4.10}$$

Thus, from (3.21), we obtain

$$b_1(t) = \beta - \frac{\kappa}{1-\lambda e^{-2\kappa t}}. \tag{4.11}$$

From (1.2), we find

$$\frac{\dot{b}_1(t)}{2} = \alpha_1^2(t) = \frac{\lambda\kappa^2 e^{-2\kappa t}}{\left(1-\lambda e^{-2\kappa t}\right)^2}, \qquad \alpha_1(t) = \frac{\kappa\sqrt{\lambda}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}} \tag{4.12}$$

and after some manipulation, we obtain

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = 2b_1(t)+\kappa-2\beta. \tag{4.13}$$

Now, from (1.5), we find the solution of the Toda lattice which is

$$\begin{aligned}
\alpha_n(t) &= \frac{n\kappa\sqrt{\lambda}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}}, \\
b_n(t) &= (2n-1)\left(\beta - \frac{\kappa}{1-\lambda e^{-2\kappa t}}\right) + \kappa(n-1) - 2\beta(n-1).
\end{aligned} \tag{4.14}$$

The inverse problem of $L(0)$ can be solved by setting $t = 0$ in (4.14), that is,

$$\begin{aligned}
\alpha_n(0) &= \frac{n\kappa\sqrt{\lambda}}{1-\lambda}, \\
b_n(0) &= -\frac{(1+\lambda)\kappa n}{1-\lambda} + \beta + \frac{\kappa}{1-\lambda} - \kappa.
\end{aligned} \tag{4.15}$$

Due to Theorem 3.2, the solution of the inverse problem of the measure

$$\mu(\{\kappa n-\beta\}) = (1-\lambda)\lambda^{n-1}, \quad 0<\lambda<1,\ n = 1,2,\ldots,\ \kappa>0 \tag{4.16}$$

is

$$\alpha_n(0) = \frac{n\kappa\sqrt{\lambda}}{1-\lambda}, \qquad b_n(0) = \frac{(1+\lambda)\kappa n}{1-\lambda} - \beta - \frac{\kappa}{1-\lambda} + \kappa. \tag{4.17}$$

Remark 4.6. For $\beta = 0$, $\kappa = 1$, the inverse problem of the measure in (4.16) has been solved in [4] by a different method. Also for $\beta = 0$, $\kappa = 1 - \lambda$, the inverse problem of the measure in (4.16) can be solved by using a result of Stieltjes in [9, 10]. In fact, Stieltjes considered the continued fraction

$$F(z,\lambda) = \cfrac{1}{z + \cfrac{1}{1 + \cfrac{\lambda}{z + \cfrac{2}{1 + \cfrac{2\lambda}{z + \cfrac{3}{1 + \cfrac{3\lambda}{z + \ddots}}}}}}} \tag{4.18}$$

This fraction, by the identity

$$z + c_1 - \frac{c_1 c_2}{c_2 + k_1} = z + \frac{c_1}{1 + c_2/k_1}, \tag{4.19}$$

where $\alpha_n^2 = c_{2n-1}c_{2n}$ and $b_n = c_{2n-2}c_{2n-1}$, $b_1 = c_1$, can be transformed into the fraction

$$F(z,\lambda) = \cfrac{1}{z + b_1 - \cfrac{\alpha_1^2}{z + b_2 - \cfrac{\alpha_2^2}{z + b_3 - \ddots}}} = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z + x} \tag{4.20}$$

Stieltjes gives in [9, 10] what we call nowadays the Stieltjes transform of the measure (4.16):

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{z + x} = \sum_{n=1}^{\infty} \frac{(1-\lambda)\lambda^{n-1}}{z + n(1-\lambda)}. \tag{4.21}$$

From the analytic theory of continued fractions [6], it follows that the coefficients α_n, b_n of the orthogonal polynomials corresponding to μ are given by $\alpha_n = \sqrt{\lambda} n$ and $b_n = n(1+\lambda) - \lambda$.

In fact, it is well known that if the tridiagonal operator $L(0)$ is selfadjoint with spectral measure μ, then the Stieltjes transform of μ is given by

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{z + x} = \cfrac{1}{z + b_1 - \cfrac{\alpha_1^2}{z + b_2 - \cfrac{\alpha_2^2}{z + b_3 - \ddots}}} \tag{4.22}$$

In case the measure is discrete with mass points $\lambda_1, \lambda_2, \dots$, the Stieltjes transform is given by

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{z+x} = \sum_{k=1}^{\infty} \frac{\mu(\{\lambda_k\})}{z+\lambda_k}. \tag{4.23}$$

Conversely, from the solution of the inverse problem, the Stieltjes transform of the measure (4.16) follows. This transform was presented by Stieltjes without proof in [9, 10] (see also [12, page 367]).

Example 4.5 is a particular case of the following example.

Example 4.7. Consider the probability measure

$$\mu(\{\beta - \kappa n\}) = \frac{(\alpha)_{n-1}(1-\lambda)^{\alpha}\lambda^{n-1}}{(n-1)!}, \quad \alpha > 0,\ 0 < \lambda < 1,\ n = 1, 2, \dots, \tag{4.24}$$

where $(\alpha)_{n-1} = \alpha(\alpha+1)\cdots(\alpha+n-2)$. The support of this measure is infinite and the moments

$$\mu_\kappa = \sum_{n=1}^{\infty} \frac{(-\kappa n+\beta)^{\kappa}\alpha(\alpha+1)\cdots(\alpha+n-2)(1-\lambda)^{\alpha}\lambda^{n-1}}{(n-1)!} \tag{4.25}$$

are finite.

From (4.24), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2xt}\,d\mu &= \sum_{n=1}^{\infty} \frac{e^{2t(-\kappa n+\beta)}(\alpha)_{n-1}\lambda^{n-1}(1-\lambda)^{\alpha}}{(n-1)!} = \frac{(1-\lambda)^{\alpha}e^{2\beta t-2\kappa t}}{\left(1-\lambda e^{-2\kappa t}\right)^{\alpha}}, \\ \int_{-\infty}^{\infty} xe^{2xt}\,d\mu &= (1-\lambda)^{\alpha}e^{2\beta t-2\kappa t}\frac{\left[(\kappa\lambda-\kappa\alpha\lambda-\beta\lambda)e^{-2\kappa t}+\beta-\kappa\right]}{\left(1-\lambda e^{-2\kappa t}\right)^{\alpha+1}}. \end{aligned} \tag{4.26}$$

Thus,

$$\begin{aligned} b_1(t) &= \beta + \frac{\lambda\kappa e^{-2\kappa t}(1-\alpha)-\kappa}{1-\lambda e^{-2\kappa t}}, \\ \dot{b}_1(t) &= \frac{2\lambda\kappa^2\alpha e^{-2\kappa t}}{\left(1-\lambda e^{-2\kappa t}\right)^2} = 2\alpha_1^2(t), \\ \alpha_1(t) &= \frac{\kappa\sqrt{\lambda\alpha}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}}, \\ \frac{\dot{\alpha}_1(t)}{\alpha_1(t)} &= \frac{2}{\alpha}b_1(t) + \frac{2\kappa}{\alpha} - \frac{2\beta}{\alpha} - \kappa. \end{aligned} \tag{4.27}$$

From this relation, we see that in this case we have $\delta = 2/\alpha$ and $c = 2\kappa/\alpha - 2\beta/\alpha - \kappa$. Then from (1.5), it follows that

$$\begin{aligned} \alpha_n(t) &= \frac{\kappa\sqrt{n(n+\alpha-1)\lambda}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}}, \\ b_n(t) &= \left(\frac{-\kappa(1+\lambda e^{-2\kappa t})}{1-\lambda e^{-2\kappa t}}\right)n + \left(1-\frac{2}{a}\right)b_1(t) - \frac{2\kappa}{\alpha} + \frac{2\beta}{\alpha} + \kappa. \end{aligned} \tag{4.28}$$

The solution of the inverse problem is given by

$$\begin{aligned} \alpha_n(0) &= \frac{\kappa\sqrt{n(n+\alpha-1)\lambda}}{1-\lambda}, \\ b_n(0) &= \frac{-\kappa(1+\lambda)n + \lambda\kappa(2-\alpha) + \beta(1-\lambda)}{1-\lambda}. \end{aligned} \tag{4.29}$$

We note that this inverse spectral problem is well known in the theory of the measure of orthogonality of the Meixner polynomials (see [3, page 175]), which we find here by an alternative method.

Using Theorem 3.2, we can derive the solution of the inverse problem of the measure (4.24). This solution is given by

$$\begin{aligned} \alpha_n(0) &= \frac{\kappa\sqrt{n(n+\alpha-1)\lambda}}{1-\lambda}, \\ b_n(0) &= \frac{\kappa(1+\lambda)n - \lambda\kappa(2-\alpha) - \beta(1-\lambda)}{1-\lambda}. \end{aligned} \tag{4.30}$$

Example 4.8. Consider the probability measure

$$\mu(\{\beta - \gamma n\}) = \frac{e^{-\alpha}\alpha^{n-1}}{(n-1)!}, \quad \alpha > 0,\ \gamma > 0,\ n = 1, 2, \ldots. \tag{4.31}$$

From (4.31), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2xt}\, d\mu &= e^{-\alpha+2\beta t-2\gamma t+\alpha e^{-2\gamma t}}, \\ \int_{-\infty}^{\infty} x e^{2xt}\, d\mu &= e^{-\alpha+2\beta t-2\gamma t+\alpha e^{-2\gamma t}}\left(\beta - \gamma - \alpha\gamma e^{-2\gamma t}\right). \end{aligned} \tag{4.32}$$

Thus,

$$b_1(t) = \beta - \gamma - \alpha\gamma e^{-2\gamma t}. \tag{4.33}$$

From (4.33),

$$\dot{b}_1(t) = 2\alpha\gamma^2 e^{-2\gamma t} = 2\alpha_1^2(t), \tag{4.34}$$

and from (1.2), we find

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = -\gamma. \tag{4.35}$$

From (4.35), we see that in this case we have $\delta = 0$ and $c = -\gamma$. Then from (1.5), it follows that

$$\alpha_n(t) = \gamma\sqrt{n\alpha}e^{-\gamma t}, \qquad b_n(t) = \beta - \gamma - \alpha\gamma e^{-2\gamma t} - (n-1)\gamma. \tag{4.36}$$

The solution of the inverse problem is

$$\alpha_n(0) = \gamma\sqrt{n\alpha}, \qquad b_n(0) = \beta - \alpha\gamma - n\gamma \tag{4.37}$$

for $\beta = 1/\gamma$, $\alpha = 1/\gamma^2$. Thus, we have obtained the example studied in [7]. Here we note that we obtain an alternative derivation of the measure of orthogonality of the Charlier polynomials (see appendix in [7]).

In this example, we solved the inverse problem of the measure (4.31). Due to Theorem 3.2, we can see that the solution of the inverse problem of the measure

$$\mu(\{\gamma n - \beta\}) = \frac{e^{-\alpha}\alpha^{n-1}}{(n-1)!}, \quad \alpha > 0,\ \gamma > 0,\ n = 1,2,\dots, \tag{4.38}$$

is

$$\alpha_n(0) = \gamma\sqrt{n\alpha}, \qquad b_n(0) = -\beta + \alpha\gamma + n\gamma. \tag{4.39}$$

Example 4.9. Consider the probability measure μ whose distribution function is given by

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-(\xi-m)^2/2\sigma^2}\,d\xi, \quad m,\sigma > 0. \tag{4.40}$$

The moments $\mu_k = (1/\sigma\sqrt{2\pi})\int_{-\infty}^{\infty} x^k e^{-(x-m)^2/2\sigma^2}\,dx$ are finite.

We calculate the integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{2xt}\,d\mu &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2xt-(x-m)^2/2\sigma^2}\,dx = e^{2t(m+\sigma^2 t)},\\
\int_{-\infty}^{\infty} xe^{2xt}\,d\mu &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{2xt-(x-m)^2/2\sigma^2}\,dx = (m+2\sigma^2 t)e^{2t(m+\sigma^2 t)}.
\end{aligned} \tag{4.41}$$

Thus relation (3.21) gives

$$b_1(t) = m + 2\sigma^2 t \tag{4.42}$$

and so

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = 0 \quad (\delta = c = 0). \tag{4.43}$$

Then from (1.5), we obtain

$$\alpha_n(t) = \sigma\sqrt{n}, \quad b_n(t) = b_1(t) = m + 2\sigma^2 t, \quad n = 1,2,\dots. \tag{4.44}$$

As a consequence, the solution of the inverse problem of μ is

$$\alpha_n(0) = \sigma\sqrt{n}, \quad b_n(0) = m, \quad n = 1,2,\dots. \tag{4.45}$$

Due to Theorem 3.2, the solution of the inverse problem of the measure $\mu\tau^{-1}$ is

$$\alpha_n(0) = \sigma\sqrt{n}, \quad b_n(0) = -m, \quad n = 1, 2, \ldots. \tag{4.46}$$

Remark 4.10. In [7], we proved that the Toda lattice has a unique solution $\alpha_n(t)$, $b_n(t)$ provided that the tridiagonal operator $L(0)$ is (essentially) selfadjoint and bounded from above. This means that the support of the measure μ is a set bounded from above. Examples showed that there exist spectral measures μ with support not bounded from above such that the integrals in (3.21) do not exist (see, e.g., the measure μ in Example 4.1 or the measure $\mu\tau^{-1}$ in Examples 4.5, 4.7, and 4.8). What can be said for the integrability of these systems? From (1.6), we see that in these cases the solution $\alpha_n(t)$, $b_n(t)$ has poles for some $t > 0$.

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NONANALYTIC SOLUTIONS OF THE KdV EQUATION

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To Professor E. K. Ifantis on the occasion of his 67th birthday

We construct nonanalytic solutions to the initial value problem for the KdV equation with analytic initial data in both the periodic and the nonperiodic cases.

1. Introduction

It is well known that the solution to the Cauchy problem of the KdV equation with an analytic initial profile is analytic in the space variable for a fixed time (see Trubowitz [11] and Kato [7]). However, analyticity in the time variable fails. Here, we will present several examples demonstrating this phenomenon of the KdV equation. More precisely, we will show that the initial value problem

$$\begin{gathered} \partial_t u + \partial_x^3 u + u\partial_x u = 0, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R} \text{ or } \mathbb{T},\ t \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where $\varphi(x)$ is an appropriate analytic function, cannot have an analytic solution in t for fixed x, say $x = 0$. By replacing x with $-x$, we see that it suffices to consider the equivalent problem

$$\partial_t u = \partial_x^3 u + u\partial_x u, \tag{1.2}$$

$$u(x,0) = \varphi(x). \tag{1.3}$$

If $u(x,t)$ was analytic in t at $t = 0$, then it could be written as a power series of the following form:

$$u(x,t) = \sum_{j=0}^{\infty} \frac{\partial_t^j u(x,0)}{j!} t^j \tag{1.4}$$

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with a nonzero radius of convergence. In particular, there must be some constant A such that

$$\partial_t^n u(x,0) \leq A^n n! \tag{1.5}$$

for every positive integer n.

In computing the values of $\partial_t^n u(x,0)$, we will rely on the fact (to be demonstrated shortly) that if $u(x,t)$ is a solution to (1.2), then $\partial_t^j u(x,t)$ can be written as a polynomial of $u(x,t), \partial_x u(x,t), \partial_x^2 u(x,t), \dots, \partial_x^{3j} u(x,t)$.

To motivate the discussion that will follow, we will first look at $\partial_t u(x,t)$ and $\partial_t^2 u(x,t)$. The initial value problem (1.2) and (1.3) gives

$$\partial_t u(x,0) = \varphi'''(x) + \varphi(x)\varphi'(x). \tag{1.6}$$

Then, differentiating (1.2), we obtain

$$\begin{aligned}
\partial_t^2 u &= \partial_t\big[\partial_x^3 u + u\partial_x u\big] = \partial_x^3 \partial_t u + u\partial_x \partial_t u + \partial_x u \partial_t u \\
&= \partial_x^3\big(\partial_x^3 u + u\partial_x u\big) + u\partial_x\big(\partial_x^3 u + u\partial_x u\big) + \partial_x u\big(\partial_x^3 u + u\partial_x u\big) \\
&= \partial_x^6 u + u\partial_x^4 u + 3\partial_x^3 u\partial_x u + 3\partial_x^2 u\partial_x^2 u + \partial_x^3 u\partial_x u \\
&\quad + u\partial_x^4 + u\partial_x u\partial_x u + uu\partial_x^2 u + \partial_x u\partial_x^3 u + u\partial_x u\partial_x u \\
&= \partial_x^6 u + 2u\partial_x^4 u + 5\partial_x u\partial_x^3 u + 3\partial_x^2 u\partial_x^2 u + uu\partial_x^2 u + 2u\partial_x u\partial_x u,
\end{aligned} \tag{1.7}$$

and hence

$$\partial_t^2 u(x,0) = \varphi^{(6)} + 2\varphi\varphi^{(4)} + 5\varphi^{(1)}\varphi^{(3)} + 3\varphi^{(2)}\varphi^{(2)} + \varphi\varphi\varphi^{(2)} + 2\varphi\varphi^{(1)}\varphi^{(1)}. \tag{1.8}$$

Now, suppose that the initial data is a function of the form

$$u(x,0) = \varphi(x) = (a-x)^{-d}, \tag{1.9}$$

where a is some (complex-valued) constant. Then we have

$$\partial_t u(x,0) = d(d+1)(d+2)(a-x)^{-(d+3)} + d(a-x)^{-(2d+1)}, \tag{1.10}$$

and we make our key observation: the exponent of $(a-x)$ will be the same for both terms if and only if

$$d = 2. \tag{1.11}$$

In a similar way, we compute for the terms of $\partial_t^2 u$ at $t=0$,

$$\begin{aligned}
\partial_x^6 u &= d(d+1)(d+2)(d+3)(d+4)(d+5)(a-x)^{-(6+d)},\\
u\partial_x^4 u &= d(d+1)(d+2)(d+3)(a-x)^{-(4+2d)},\\
\partial_x u\partial_x^3 u &= d\cdot d(d+1)(d+2)(a-x)^{-(4+2d)},\\
\partial_x^2 u\partial_x^2 u &= d(d+1)\cdot d(d+1)(a-x)^{-(4+2d)},\\
uu\partial_x^2 u &= d(d+1)(a-x)^{-(2+3d)},\\
u\partial_x u\partial_x u &= d\cdot d(a-x)^{-(2+3d)},
\end{aligned} \tag{1.12}$$

and we again see that for all the terms in the expression of $\partial_t^2 u(x,0)$ to have equal exponents, we must have $d=2$.

Next, we will show that with this choice of d, the "homogeneity" degree of all the terms in the expression of $\partial_t^j u(x,0)$ is the same number which is equal to $3j+2$. More precisely, if, for a term of the form

$$(\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u)\cdots(\partial_x^{\alpha_\ell} u), \quad \alpha_m \in \{0,1,2,\dots\}, \tag{1.13}$$

we assign the "homogeneity" degree

$$(\alpha_1+2)+(\alpha_2+2)+\cdots+(\alpha_\ell+2) = |\alpha|+2\ell, \tag{1.14}$$

then we have the following lemma.

Lemma 1.1. *If $u(x,t)$ is a solution to the initial value problem (1.2) and (1.3), then*

$$\partial_t^j u = \partial_x^{3j} u + \sum_{|\alpha|+2\ell=3j+2} C_\alpha (\partial_x^{\alpha_1} u)\cdots(\partial_x^{\alpha_\ell} u) \tag{1.15}$$

with $C_\alpha \geq 0$.

Here, for a multi-index $\alpha=(\alpha_1,\dots,\alpha_\ell)$, we use the notation $|\alpha| = \alpha_1+\cdots+\alpha_\ell$.

Proof. Our computations above show that Lemma 1.1 is true for $j=0,1,2$. Next, we assume that it is true for j and we will show that it is true for $j+1$. We have

$$\partial_t^{j+1} u = \partial_x^{3j}(\partial_t u) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha \partial_t\left[(\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u)\cdots(\partial_x^{\alpha_\ell} u)\right]. \tag{1.16}$$

The first term is equal to

$$\partial_x^{3j}(\partial_x^3 u + u\partial_x u) = \partial_x^{3(j+1)} u + \partial_x^{3j}(u\partial_x u), \tag{1.17}$$

where $\partial_x^{3(j+1)} u$ is the leading term in the expression of $\partial_t^{j+1} u$ and $\partial_x^{3j}(u\partial_x u)$, and by using Leibniz rule, it can be written as a sum of terms of the form $(\partial_x^{\alpha_1} u)(\partial_x^{\alpha_2} u)$ of homogeneity degree $3(j+1)+2$. Also, using the product rule for differentiation, each term of the sum in (1.15) gives

$$\begin{aligned}\partial_t\big[(\partial_x^{\alpha_1}u)(\partial_x^{\alpha_2}u)\cdots(\partial_x^{\alpha_\ell}u)\big]\\ =\partial_x^{\alpha_1}\partial_t u\cdot(\partial_x^{\alpha_2}u)\cdots(\partial_x^{\alpha_\ell}u)+\cdots+(\partial_x^{\alpha_1}u)(\partial_x^{\alpha_2}u)\cdots\partial_x^{\alpha_\ell}\partial_t u.\end{aligned} \tag{1.18}$$

Finally, replacing $\partial_t u$ by $\partial_x^3 u+u\partial_x u$ in each term of the last sum gives the desired result. For example, the first term is equal to

$$\partial_x^{\alpha_1}\big[\partial_x^3 u+u\partial_x u\big]\cdot(\partial_x^{\alpha_2}u)\cdots(\partial_x^{\alpha_\ell}u), \tag{1.19}$$

and each term that results by applying $\partial_x^{\alpha_1}$ on $\partial_x^3 u+u\partial_x u$ has nonnegative coefficients and homogeneity degree equal to

$$\alpha_1+5+(\alpha_2+2)+\cdots+(a_\ell+2)=|\alpha|+3=3j+2+3=3(j+1)+2. \tag{1.20}$$

□

2. Nonperiodic case

Using the initial condition

$$u(x,0)=(a-x)^{-2}, \tag{2.1}$$

we find that

$$\partial_x^{\alpha_m}u(x,0)=(\alpha_m+1)!(a-x)^{-(\alpha_m+2)}. \tag{2.2}$$

Therefore, at $t=0$, relations (1.15) and (2.2) give that

$$\partial_t^j u(x,0)=\Bigg((3j+1)!+\sum_{|\alpha|+2\ell=3j+2}C_\alpha\big[(\alpha_1+1)!\big]\cdots\big[(\alpha_\ell+1)!\big]\Bigg)(a-x)^{-(3j+2)} \tag{2.3}$$

or

$$\partial_t^j u(x,0)=\big[(3j+1)!+b_j\big](a-x)^{-(3j+2)}, \tag{2.4}$$

where $b_j\geq 0$.

Finally, using (2.4), we see that

$$\big|\partial_t^j u(x,0)\big|\geq|a-x|^{-(3j+2)}(3j)!. \tag{2.5}$$

Inequality (2.5) shows that $u(x,t)$ cannot be analytic near $t=0$ for any fixed $x\neq a$.

Observe that if $a\in\mathbb{R}$, then $u(x,0)=(a-x)^{-2}$ is real-valued and analytic in $\mathbb{R}-\{a\}$. So, one may ask if there are nonanalytic solutions to KdV when the initial data are analytic everywhere in $\mathbb{R}$.

Globally analytic data. If $a \in \mathbb{C} - \mathbb{R}$, then $u(x,0) = (a-x)^{-2}$ is analytic in $\mathbb{R}$. In particular, if we choose $a = i$ and $x = 0$, then we have

$$\partial_t^j u(0,0) = i^{-(3j+2)}\left[(3j+1)! + b_j\right]. \tag{2.6}$$

However, in this case, the KdV solution is complex-valued. Thus, one may ask the question if we can have real-valued initial data which are analytic on $\mathbb{R}$ and for which the KdV solution is not analytic in t.

Real-valued globally analytic data. Next we choose

$$u(x,0) = \Re(i-x)^{-2}. \tag{2.7}$$

Then

$$\partial_x^k u(x,0) = (k+1)!\Re(i-x)^{-2-k}, \tag{2.8}$$

$$\partial_x^k u(0,0) = -(k+1)!\Re i^{-k} = \begin{cases} -1, & k = 4j, \\ 1, & k = 4j+2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.9}$$

Using (1.15) and (2.8), we have

$$\begin{aligned} &\partial_t^j u(x,0) \\ &\quad = \Re(3j+1)!(i-x)^{-(3j+2)} \\ &\qquad + \sum_{|\alpha|+2\ell=3j+2} C_\alpha\left((\alpha_1+1)!\Re(i-x)^{-2-\alpha_1}\right)\cdots\left((\alpha_k+1)!\Re(i-x)^{-2-\alpha_k}\right), \end{aligned} \tag{2.10}$$

so

$$\begin{aligned} &\partial_t^j u(0,0) \\ &\quad = (3j+t)!\Re i^{-(3j+2)} \\ &\qquad + \sum_{|\alpha|+2\ell=3j+2} C_\alpha\left[(\alpha_1+1)!\right]\left[\Re i^{-(2+\alpha_1)}\right]\cdots\left[(\alpha_\ell+1)!\right]\left[\Re i^{-(2+\alpha_\ell)}\right]. \end{aligned} \tag{2.11}$$

We have

$$\begin{gathered} \Re i^{-(3j+2)} = (-1)^{1/2(3j+2)}, \\ \left[\Re i^{-(2+\alpha_1)}\right]\cdots\left[\Re i^{-(2+\alpha_\ell)}\right] = (-1)^\ell\left(\Re i^{-\alpha_1}\right)\cdots\left(\Re i^{-\alpha_\ell}\right). \end{gathered} \tag{2.12}$$

If α_m is an odd number for some m, then the last product equals zero, while if α_m is even for all m, then the last product equals

$$(-1)^\ell(-1)^{(1/2)(\alpha_1+\cdots+\alpha_\ell)} = (-1)^{(1/2)(3j+2)}. \tag{2.13}$$

Therefore, if j is even, then

$$\partial_t^j u(0,0) = -(-1)^{3j/2}\left[(3j+1)! + b_j\right], \tag{2.14}$$

where $b_j \geq 0$, which shows that the solution $u(x,t)$ which exists (see, e.g., [8, 9, 10]) cannot be analytic in t near $t = 0$ when $x = 0$.

3. Periodic case

Now, for the periodic case, define

$$g(x) = \frac{-e^{ix}}{2 - e^{ix}} = -\sum_{k=1}^{\infty} 2^{-k} e^{ikx}. \tag{3.1}$$

Then

$$g^{(n)}(x) = -\sum_{k=1}^{\infty} 2^{-k} (ik)^n e^{ikx}, \tag{3.2}$$

$$g^{(n)}(0) = i^{n+2} A_n, \tag{3.3}$$

where

$$A_n = \sum_{k=1}^{\infty} 2^{-k} k^n > 2^{-n} n^n. \tag{3.4}$$

Let $u(x,t)$ be a solution to the initial value problem (1.2) and (1.3) with initial data $\phi(x) = g(x)$. Then, by (1.15), we have

$$\begin{aligned} \partial_t^j u(0,0) &= g^{(3j)}(0) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha \left(g^{(\alpha_1)}(0)\right) \cdots \left(g^{(\alpha_\ell)}(0)\right) \\ &= \left(A_{3j} + \sum_{|\alpha|+2\ell=3j+2} C_\alpha A_{\alpha_1} \cdots A_{\alpha_\ell}\right) \left(i^{3j+2}\right), \end{aligned} \tag{3.5}$$

and by (3.4), we have that for any j,

$$\left| \partial_t^j u(0,0) \right| \geq A_{3j} > 2^{-3j} (3j)^{3j} > (j!)^3. \tag{3.6}$$

Therefore, $u(x,t)$ is not analytic in the t-variable at the point $(0,0)$.

Real-valued solutions. Let $u(x,t)$ be the solution to the Cauchy problem (1.2) and (1.3) with initial data $\phi(x) = \Re g(x)$. By (1.15), we have

$$\partial_t^j u(0,0) = \Re g^{(3j)}(0) + \sum_{|\alpha|+2\ell=3j+2} C_\alpha \Re\left(g^{(\alpha_1)}(0)\right) \cdots \Re\left(g^{(\alpha_\ell)}(0)\right). \tag{3.7}$$

Now, by (3.3), we note that

$$\Re g^{(n)}(0) = \begin{cases} g^{(n)}(0), & \text{for } n \text{ even}, \\ 0, & \text{for } n \text{ odd}, \end{cases} \tag{3.8}$$

and thus, any product of the form

$$\Re\left(g^{(\alpha_1)}(0)\right)\cdots\Re\left(g^{(\alpha_\ell)}(0)\right) \tag{3.9}$$

must be equal to either 0 or

$$\left(g^{(\alpha_1)}(0)\right)\cdots\left(g^{(\alpha_\ell)}(0)\right). \tag{3.10}$$

Therefore, if we assume that j is even, then for some sequence of real, nonnegative coefficients D_α (specifically where $D_\alpha \in \{0, C_\alpha\}$), we have

$$\begin{aligned}\partial_t^j u(0,0) &= g^{(3j)}(0) + \sum_{|\alpha|+2\ell=3j+2} D_\alpha \left(g^{(\alpha_1)}(0)\right)\cdots\left(g^{(\alpha_\ell)}(0)\right)\\ &= \left(A_{3j} + \sum_{|\alpha|+2\ell=3j+2} D_\alpha A_{\alpha_1}\cdots A_{\alpha_\ell}\right)\left(i^{3j+2}\right).\end{aligned} \tag{3.11}$$

It follows that for any even j,

$$\left|\partial_t^j u(0,0)\right| \geq A_{3j} > 2^{-3j}(3j)^{3j} > (j!)^3. \tag{3.12}$$

Therefore, the solution $u(x,t)$ which exists (see, e.g., [1]) is not analytic in the t-variable at the point $(0,0)$.

4. Concluding remarks

One of the motivations for this work has been the results in [5, 4]. There, it was proved that, unlike the KdV, the Cauchy problem for the evolution equation

$$\partial_t u - \partial_t \partial_x^2 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, \quad x \in \mathbb{T},\ t \in \mathbb{R}, \tag{4.1}$$

with analytic initial data is analytic in both the space and the time variables, globally in x and locally in t. This equation was introduced independently by Fuchssteiner and Fokas [3] and by Camassa and Holm [2] as an alternative to KdV modeling shallow water waves. In the past decade, it has been the subject of extensive studies from the analytic as well as the geometric and algebraic points of view.

Finally, we note that one may obtain nonanalytic solutions to the KdV by using other analytic initial data. For example, G. Łysik in a private communication mentioned that in the nonperiodic case, he can show that the Cauchy problem for the KdV with initial data $\varphi(x) = 1/(1+x^2)$ is not analytic (like in the heat equation). For more results about the analyticity and smoothing effects of the KdV, we refer the reader to the paper of Kato and Ogawa [6] and the references therein.

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SUBDOMINANT POSITIVE SOLUTIONS OF THE DISCRETE EQUATION $\Delta u(k+n) = -p(k)u(k)$

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A delayed discrete equation $\Delta u(k+n) = -p(k)u(k)$ with positive coefficient p is considered. Sufficient conditions with respect to p are formulated in order to guarantee the existence of positive solutions if $k \to \infty$. As a tool of the proof of corresponding result, the method described in the author's previous papers is used. Except for the fact of the existence of positive solutions, their upper estimation is given. The analysis shows that every positive solution of the indicated family of positive solutions tends to zero (if $k \to \infty$) with the speed not smaller than the speed characterized by the function $\sqrt{k} \cdot (n/(n+1))^k$. A comparison with the known results is given and some open questions are discussed.

1. Introduction and motivation

In this contribution, the delayed scalar linear discrete equation

$$\Delta u(k+n) = -p(k)u(k) \tag{1.1}$$

with fixed $n \in \mathbb{N} \setminus \{0\}$, $\mathbb{N} := \{0,1,\dots\}$, and variable $k \in N(a)$, $N(a) := \{a, a+1, \dots\}$, $a \in \mathbb{N}$, is considered. The function $p : N(a) \to \mathbb{R}$ is supposed to be positive. We are interested in the existence of positive solutions of (1.1). As a tool of the proof, the method described in [2, 5] is used.

Equation (1.1) can be considered as a discrete analogue of the delayed linear differential equation of the form

$$\dot{x}(t) = -c(t)x(t-\tau) \tag{1.2}$$

with positive coefficient c on $I = [t_0, \infty)$, $t_0 \in \mathbb{R}$, which was considered in many works. We mention at least the books by Győri and Ladas [14] and by Erbe et al. [12] and the papers by Domshlak and Stavroulakis [9], by Elbert and Stavroulakis [11], by Győri and Pituk [16], and by Jaroš and Stavroulakis [18]. Note that close problems were investigated, for example, by Castillo [3], Čermák [4], Kalas and Baráková [19], and Slyusarchuk [22].

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In [6], it was investigated that if (1.2) admits a positive solution $\tilde{x}$ on an interval I, then it admits on I *two positive solutions* x_1 and x_2, satisfying

$$\lim_{t\to\infty}\frac{x_2(t)}{x_1(t)}=0. \tag{1.3}$$

Moreover, every solution x of (1.2) on I is represented by the formula

$$x(t)=Kx_1(t)+O\big(x_2(t)\big), \tag{1.4}$$

where $K\in\mathbb{R}$ depends on x and O is the Landau order symbol. In this formula, the solutions x_1, x_2 can be changed to any couple of positive on I solutions $\tilde{x}_1, \tilde{x}_2$ of (1.2) satisfying the property

$$\lim_{t\to\infty}\frac{\tilde{x}_2(t)}{\tilde{x}_1(t)}=0 \tag{1.5}$$

(see [6, pages 638-639]). This invariance property led to the following terminology: if (x_1,x_2) is a fixed couple of positive solutions (having the above-indicated properties) of (1.2), then the solution x_1 is called a *dominant solution* and the solution x_2 is called a *subdominant solution*. Subdominant solutions can serve as an analogy to "small solutions" as they are used, for example, in the book by Hale and Verduyn Lunel [17], and dominant solutions express an analogy to the notion of "special solution" which is used in many investigations (see, e.g., Rjabov [20]).

In the present contribution, we will give sufficient conditions for the existence of positive solutions of (1.1). We will discuss known sufficient conditions too, and we will show that our conditions have a more general character than the previous ones. Otherwise the method of the proof of corresponding result permits to express an estimation of the considered positive solution. Taking into account the fact that this solution tends to zero (if $k\to\infty$) with speed not smaller than the speed characterized by the function $\sqrt{k}\cdot(n/(n+1))^k$, we can conclude that this solution is an analogy to the notion of subdominant solution introduced above, in the case of scalar delayed linear differential equations. Moreover, the supporting motivation for the terminology used is the fact that our result does not hold for nondelayed equations of type (1.1), that is, it does not hold if $n=0$. This is in full accordance with differential equations again, since obviously the subdominant solution does not appear if $\tau=0$, in (1.2), that is, it does not appear in the case of ordinary differential equations.

2. Preliminary

We consider the scalar discrete equation

$$\Delta u(k+\tilde{n})=f\big(k,u(k),u(k+1),\dots,u(k+\tilde{n})\big), \tag{2.1}$$

where $f(k,u_0,u_1,\dots,u_{\tilde{n}})$ is defined on $N(a)\times\mathbb{R}^{\tilde{n}+1}$, with values in $\mathbb{R}$, $a\in\mathbb{N}$, and $\tilde{n}\in\mathbb{N}$.

Together with the discrete equation (2.1), we consider an initial problem. It is posed as follows: for a given $s\in\mathbb{N}$, we are seeking the solution of (2.1) satisfying $\tilde{n}+1$ initial conditions

$$u(a+s+m) = u^{s+m} \in \mathbb{R}, \quad m = 0,1,\dots,\tilde{n}, \tag{2.2}$$

with prescribed constants u^{s+m}.

We recall that the solution of the initial problem (2.1), (2.2) is defined as an infinite sequence of numbers

$$\begin{aligned}&\{u(a+s) = u^s,\ u(a+s+1) = u^{s+1},\dots,\\ &\ u(a+s+\tilde{n}) = u^{s+\tilde{n}}, u(a+s+\tilde{n}+1), u(a+s+\tilde{n}+2),\dots\}\end{aligned} \tag{2.3}$$

such that, for any $k \in N(a+s)$, equality (2.1) holds.

The existence and uniqueness of the solution of the initial problem (2.1), (2.2) are obvious for every $k \in N(a+s)$. If the function f satisfies the Lipschitz condition with respect to u-arguments, then the initial problem (2.1), (2.2) depends continuously on the initial data [1].

We define, for every $k \in N(a)$, a set $\omega(k)$ as

$$\omega(k) := \{u \in \mathbb{R} : b(k) < u < c(k)\}, \tag{2.4}$$

where $b(k)$, $c(k)$, $b(k) < c(k)$ are real functions defined on $N(a)$.

The following theorem is taken from the investigation in [2].

Theorem 2.1. *Suppose that $f(k,u_0,u_1,\dots,u_{\tilde{n}})$ is defined on $N(a)\times\mathbb{R}^{\tilde{n}+1}$ with values in $\mathbb{R}$ and for all $(k,u_0,u_1,\dots,u_{\tilde{n}}),(k,v_0,v_1,\dots,v_{\tilde{n}}) \in N(a)\times\mathbb{R}^{\tilde{n}+1}$:*

$$|f(k,u_0,u_1,\dots,u_{\tilde{n}}) - f(k,v_0,v_1,\dots,v_{\tilde{n}})| \le \lambda(k)\sum_{i=0}^{\tilde{n}} |u_i - v_i|, \tag{2.5}$$

where $\lambda(k)$ is a nonnegative function defined on $N(a)$. If, moreover, the inequalities

$$f(k,u_0,u_1,\dots,u_{\tilde{n}-1}, b(k+\tilde{n})) - b(k+\tilde{n}+1) + b(k+\tilde{n}) < 0, \tag{2.6}$$

$$f(k,u_0,u_1,\dots,u_{\tilde{n}-1}, c(k+\tilde{n})) - c(k+\tilde{n}+1) + c(k+\tilde{n}) > 0 \tag{2.7}$$

hold for every $k \in N(a)$, every $u_0 \in \omega(k)$, and $u_1 \in \omega(k+1),\dots,u_{\tilde{n}-1} \in \omega(k+\tilde{n}-1)$, then there exists an initial problem

$$u^*(a+m) = u^*_m \in \mathbb{R}, \quad m = 0,1,\dots,\tilde{n}, \tag{2.8}$$

with

$$u^*_0 \in \omega(a), u^*_1 \in \omega(a+1),\dots,u^*_n \in \omega(a+\tilde{n}) \tag{2.9}$$

such that the corresponding solution $u = u^*(k)$ of (2.1) satisfies the inequalities

$$b(k) < u^*(k) < c(k), \tag{2.10}$$

for every $k \in N(a)$.

3. Existence of subdominant positive solutions

In this section, we prove the existence of a positive solution of (1.1). In the proof of the corresponding theorem (see Theorem 3.2 below), the following elementary lemma concerning asymptotic expansion of the indicated function is necessary. The proof is omitted since it can be done easily with the aid of binomial formula.

LEMMA 3.1. *For $k \to \infty$ and fixed $\sigma, d \in \mathbb{R}$, the following asymptotic representation holds:*

$$\left(1+\frac{d}{k}\right)^{\sigma} = 1+\frac{\sigma d}{k}+\frac{\sigma(\sigma-1)d^2}{2k^2}+\frac{\sigma(\sigma-1)(\sigma-2)d^3}{6k^3}+O\left(\frac{1}{k^4}\right). \tag{3.1}$$

THEOREM 3.2 (subdominant positive solution). *Let $a \in \mathbb{N}$ and $n \in \mathbb{N} \setminus \{0\}$ be fixed. Suppose that there exists a constant $\theta \in [0,1)$ such that the function $p : N(a) \to \mathbb{R}$ satisfies the inequalities*

$$0 < p(k) \leq \left(\frac{n}{n+1}\right)^{n} \cdot \left(\frac{1}{n+1}+\frac{\theta n}{8k^2}\right), \tag{3.2}$$

for every $k \in N(a)$. Then there exist a positive integer $a_1 \geq a$ and a solution $u = u(k)$, $k \in N(a_1)$, of (1.1) such that the inequalities

$$0 < u(k) < \sqrt{k} \cdot \left(\frac{n}{n+1}\right)^{k} \tag{3.3}$$

hold for every $k \in N(a_1)$.

Proof. In the proof, Theorem 2.1 with $\tilde{n} = n$ is used. We define

$$\begin{gathered} f\left(k, u(k), u(k+1), \ldots, u(k+n)\right) := -p(k)u(k), \\ b(k) := 0, \qquad c(k) := \sqrt{k} \cdot \left(\frac{n}{n+1}\right)^{k}, \end{gathered} \tag{3.4}$$

for every $k \in N(a)$. In this case (see (2.4)),

$$\omega(k) := \{u \in \mathbb{R} : b(k) < u < c(k)\} \equiv \left\{u \in \mathbb{R} : 0 < u < \sqrt{k} \cdot \left(\frac{n}{n+1}\right)^{k}\right\}. \tag{3.5}$$

Due to the linearity of equation (1.1), the Lipschitz-type condition (2.5) is obviously satisfied with $\lambda(k) \equiv p(k)$. We verify that the inequality of type (2.6) holds. It is easy to see that, for every $k \in N(a)$, $\tilde{n} = n$,

$$f\left(k, u_0, u_1, \ldots, u_{n-1}, b(k+n)\right) - b(k+n+1) + b(k+n) = -p(k)u_0 < 0 \tag{3.6}$$

since the function p is, by (3.2), positive and u_0 is a positive term too since $u_0 \in \omega(k)$.

We start the verification of inequality (2.7). We get, for sufficiently large $k \in N(a)$ and for $\tilde{n} = n$,

$$f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\ = -p(k)u_0 - \sqrt{k+n+1} \cdot \left(\frac{n}{n+1}\right)^{k+n+1} + \sqrt{k+n} \cdot \left(\frac{n}{n+1}\right)^{k+n}. \tag{3.7}$$

Since $u_0 \in \omega(k)$, that is,

$$-u_0 > -\sqrt{k} \cdot n^k/(n+1)^k, \quad k \in N(a), \tag{3.8}$$

we get

$$\begin{aligned} &f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\ &\quad > -p(k)\sqrt{k} \cdot \left(\frac{n}{n+1}\right)^k - \left(\frac{n}{n+1}\right)^k \cdot \left(\frac{n}{n+1}\right)^{n+1} \sqrt{k+n+1} \\ &\quad + \left(\frac{n}{n+1}\right)^k \cdot \left(\frac{n}{n+1}\right)^n \sqrt{k+n} = \mathcal{H}_1 \end{aligned} \tag{3.9}$$

with

$$\mathcal{H}_1 := \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[-p(k) - \left(\frac{n}{n+1}\right)^{n+1} \cdot \sqrt{1 + \frac{n+1}{k}} + \left(\frac{n}{n+1}\right)^n \cdot \sqrt{1 + \frac{n}{k}} \right]. \tag{3.10}$$

Now applying formula (3.1) twice, with $\sigma = 1/2$, $d = n+1$, to the expression

$$\sqrt{1 + \frac{n+1}{k}} \tag{3.11}$$

and, with $\sigma = 1/2$, $d = n$, to the expression

$$\sqrt{1 + \frac{n}{k}}, \tag{3.12}$$

we obtain

$$\begin{aligned} \mathcal{H}_1 = &\left(\frac{n}{n+1}\right)^k \sqrt{k} \\ &\times \left[-p(k) - \left(\frac{n}{n+1}\right)^{n+1} \cdot \left(1 + \frac{n+1}{2k} - \frac{(n+1)^2}{8k^2} + \frac{(n+1)^3}{16k^3} + O\left(\frac{1}{k^4}\right)\right) \right. \\ &\left. + \left(\frac{n}{n+1}\right)^n \cdot \left(1 + \frac{n}{2k} - \frac{n^2}{8k^2} + \frac{n^3}{16k^3} + O\left(\frac{1}{k^4}\right)\right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \Bigg[- p(k) - \left(\frac{n}{n+1}\right)^{n+1} + \left(\frac{n}{n+1}\right)^n \\
&\qquad + \frac{1}{k}\left(\frac{-n^{n+1}}{2(n+1)^n} + \frac{n^{n+1}}{2(n+1)^n}\right) + \frac{1}{k^2}\left(\frac{n^{n+1}}{8(n+1)^{n-1}} - \frac{n^{n+2}}{8(n+1)^n}\right) \\
&\qquad + \frac{1}{k^3}\left(\frac{-n^{n+1}}{16(n+1)^{n-2}} + \frac{n^{n+3}}{16(n+1)^n}\right) + O\left(\frac{1}{k^4}\right)\Bigg] \\
&= \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \Bigg[- p(k) + \left(\frac{n}{n+1}\right)^n \frac{-n+n+1}{n+1} + \frac{1}{k^2}\frac{n^{n+1}(n+1) - n^{n+2}}{8(n+1)^n} \\
&\qquad + \frac{1}{k^3}\frac{-n^{n+1}(n+1)^2 + n^{n+3}}{16(n+1)^n} + O\left(\frac{1}{k^4}\right)\Bigg] = \mathcal{H}_2
\end{aligned}
\tag{3.13}
$$

with

$$
\begin{aligned}
\mathcal{H}_2 := \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \Bigg[& - p(k) + \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} + \frac{1}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n \\
& + \frac{1}{16k^3}\frac{-2n^{n+2} - n^{n+1}}{(n+1)^n} + O\left(\frac{1}{k^4}\right)\Bigg].
\end{aligned}
\tag{3.14}
$$

Due to inequality (3.2), we obtain that

$$
\begin{aligned}
\mathcal{H}_2 \geq & \left(\frac{n}{n+1}\right)^k \sqrt{k} \\
& \cdot \Bigg[- \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{\theta n}{8k^2}\right) + \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \\
& \qquad + \frac{1}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n + \frac{1}{16k^3}\frac{-2n^{n+2} - n^{n+1}}{(n+1)^n} + O\left(\frac{1}{k^4}\right)\Bigg] \\
= & \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \mathcal{H}_3
\end{aligned}
\tag{3.15}
$$

with

$$
\mathcal{H}_3 := \frac{1-\theta}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n - \frac{1}{16k^3}\frac{n^{n+1}(1+2n)}{(n+1)^n} + O\left(\frac{1}{k^4}\right). \tag{3.16}
$$

Now, it is obvious that there exists an integer $a_1 \geq a$ such that the inequality $\mathcal{H}_3 > 0$ holds for every $k \in N(a_1)$. Consequently,

$$
f\left(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)\right) - c(k+n+1) + c(k+n) > 0, \tag{3.17}
$$

that is, inequality (2.7) holds for every $k \in N(a_1)$. So, all the suppositions of Theorem 2.1 are met with $a := a_1$, $\tilde{n} = n$. Then, following its affirmation, there exists an initial problem

$$u^*(a_1 + m) = u_m^* \in \mathbb{R}, \quad m = 0, 1, \dots, n, \tag{3.18}$$

with

$$u_0^* \in \omega(a_1), u_1^* \in \omega(a_1 + 1), \dots, u_n^* \in \omega(a_1 + n) \tag{3.19}$$

such that the corresponding solution $u = u^*(k)$ of (1.1) satisfies the inequalities

$$b(k) = 0 < u^*(k) < c(k) = \sqrt{k} \cdot \left(\frac{n}{n+1}\right)^k, \tag{3.20}$$

for every $k \in N(a_1)$, that is, (3.3) holds. The theorem is proved. □

4. Comparisons and concluding remarks

We remark that analogous (in a sense) problems are discussed, for example, in [10, 13, 14, 15, 21]. The following known result (see [14, page 192]) will be formulated with a notation adapted with respect to our notation.

Theorem 4.1. *Assume* $n \in \mathbb{N} \setminus \{0\}$, $p(k) > 0$ *for* $k \geq 0$, *and*

$$p(k) \leq \frac{n^n}{(n+1)^{n+1}}. \tag{4.1}$$

Then the difference equation (1.1), where $k = 0, 1, 2, \dots$, *has a positive solution*

$$\{u(0), u(1), u(2), \dots\}. \tag{4.2}$$

Comparing this result with the result given by Theorem 3.2, we conclude that inequality (3.2) is a substantial improvement over (4.1) since the choice $\theta = 0$ in (3.2) gives inequality (4.1). Moreover, inequality (3.2), unlike inequality (4.1), involves the variable k on the right-hand side. As noted in [14, page 179], for $p(k) \equiv p = \text{const}$, inequality (4.1) is sharp in a sense, since in this case the necessary and sufficient condition for the oscillation of all solutions of (1.1) is the inequality

$$p > \frac{n^n}{(n+1)^{n+1}}. \tag{4.3}$$

Inequality (3.2) can be considered as a discrete analogy of the inequality

$$c(t) \leq \frac{1}{e} + \frac{1}{8et^2} \tag{4.4}$$

(t is supposed to be sufficiently large) used in [11, Theorem 3], in order to give a guarantee of the existence of a positive solution of (1.2).

5. Open questions

We indicate problems, still unsolved , whose solution will lead to progress in the considered theory.

Open Question 5.1. Does the affirmation of Theorem 2.1 remain valid if $\theta = 1$? In other words, can inequality (3.2) be replaced by a weaker one

$$0 < p(k) \leq \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{n}{8k^2}\right)? \tag{5.1}$$

Open Question 5.2. As a motivation for the following problem, we state this known fact: equation (1.1) with "limiting" value of coefficient (corresponding to $\theta = 0$), that is, the equation

$$\Delta u(k+n) = -\frac{n^n}{(n+1)^{n+1}} \cdot u(k), \tag{5.2}$$

admits two positive and asymptotically noncomparable solutions: a dominant one (we use a similar terminology as involved in Section 1)

$$u_1(k) = k \cdot \left(\frac{n}{n+1}\right)^k \tag{5.3}$$

and a subdominant one

$$u_2(k) = \left(\frac{n}{n+1}\right)^k, \tag{5.4}$$

since

$$\lim_{k\to\infty} \frac{u_2(k)}{u_1(k)} = \lim_{k\to\infty} \frac{1}{k} = 0. \tag{5.5}$$

In this connection, the next problem arises: is it possible to prove (under the same conditions as indicated in Theorem 3.2) the existence of the second solution $u^\star(k)$ of the equation

$$\lim_{k\to\infty} \frac{u(k)}{u^\star(k)} = 0? \tag{5.6}$$

In other words, is the couple of solutions $u^\star(k)$ and $u(k)$ a couple of dominant and subdominant solutions?

Open Question 5.3. Together with the investigation of linear discrete problems, the development of methods for the investigation of nonlinear discrete problems is a very important problem too. Is it, for example, possible (based on the similarity of continuous and discrete methods) to obtain analogies of the results of the investigation of singular problems for ordinary differential equations performed in [7, 8] in the discrete case?

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ELECTROMAGNETIC FIELDS IN LINEAR AND NONLINEAR CHIRAL MEDIA: A TIME-DOMAIN ANALYSIS

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We present several recent and novel results on the formulation and the analysis of the equations governing the evolution of electromagnetic fields in chiral media in the time domain. In particular, we present results concerning the well-posedness and the solvability of the problem for linear, time-dependent, and nonlocal media, and results concerning the validity of the local approximation of the nonlocal medium (optical response approximation). The paper concludes with the study of a class of nonlinear chiral media exhibiting Kerr-like nonlinearities, for which the existence of bright and dark solitary waves is shown.

1. Introduction

Chiral media are isotropic birefringent substances that respond to either electric or magnetic excitation with both electric and magnetic polarizations. Such media have been known since the end of the nineteenth century (e.g., the study of chirality by Pasteur) and find a wide range of applications from medicine to thin film technology. The understanding of the properties of such media, the differences from ordinary dielectrics, and their possible applications requires detailed mathematical modelling. The mathematical modelling of chiral media is done through the modification of the constitutive relations for normal dielectrics. While for a normal dielectric material the electric displacement D depends solely on the electric field E, and the magnetic field B depends solely on the magnetic induction H, in a chiral medium, D and B depend on a combination of E and H, [9, 11]. In most cases of interest these constitutive laws are nonlocal relations containing E and H. This is a common model for time-dispersive chiral media. Also these constitutive laws may be either linear or nonlinear relations of the fields corresponding to the modelling of linear or nonlinear chiral media, respectively.

Most of the mathematical work on chiral media so far treats the time-harmonic case; see [3] and the references therein. It is the aim of this paper to collect and review some recent results as well as to present some novel ones on the mathematical study of linear and nonlinear chiral media in the time domain. The structure of the paper is as follows.

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We first present some general well-posedness results for models of linear nonlocal chiral media. Then, we introduce a well-known local approximation to nonlocal chiral media, the Drude-Born-Fedorov (DBF) approximation, and study its validity. In the case where the medium under consideration presents a periodic spatial structure, with rapidly varying physical parameters, we study the problem of homogenization, exhibiting that the solution of the problem converges to the solution of a related problem for an effective spatially homogeneous medium whose (constant) parameters are determined.

So far, attention has mainly focused on linear media, with or without time dispersion. However, there is a rapidly growing interest on nonlinear chiral media. The study of such systems is still in its initial stages and very little work has been done in this direction; see, for example, [7, 14]. In the last section we present some recent results on the evolution of electromagnetic fields in chiral media with cubic nonlinearity, in the weak-dispersion, low-chirality limit, where a set of four coupled partial differential equations of the nonlinear Schrödinger (NLS) type for the evolution of the slowly varying envelopes of the electromagnetic fields is derived and the existence in certain limits of vector solitons of the dark-bright type is established.

2. Formulation of the problem for linear media

In this section, we establish the equations governing the evolution of electromagnetic fields in chiral media.

We will start with the Maxwell postulates for a general medium; see, for example, [10]. For a chiral material we have the following constitutive relations that connect the various fields

$$D = \epsilon E + \epsilon_1 \star E + \zeta \star H, \qquad B = \mu H + \mu_1 \star H + \xi \star E, \tag{2.1}$$

where by $\star$ we denote the convolution operator, that is, $\alpha \star U = \int_0^t \alpha(x,\tau)U(x,t-\tau)d\tau$.

For a linear medium, the condition that the fields B and D are divergence-free is equivalent to the condition that the fields E and H are divergence-free. Thus the equations for the evolution of the fields in a chiral medium will take the form

$$\begin{aligned}
\operatorname{curl} E &= -\frac{\partial}{\partial t}(\mu H + \mu_1 \star H + \xi \star E) + F, \\
\operatorname{curl} H &= \frac{\partial}{\partial t}(\epsilon E + \epsilon_1 \star E + \zeta \star H) + G, \\
\operatorname{div} H &= \operatorname{div} E = 0,
\end{aligned} \tag{2.2}$$

supplemented with the initial conditions

$$E(x,0) = 0, \qquad H(x,0) = 0. \tag{2.3}$$

This initial value problem will be called hereafter Problem I. The above formulation is valid in unbounded space. The problem may be treated also in domains Ω with sufficiently smooth boundary $\partial\Omega$ using a boundary condition that corresponds to the physical situation at hand. We will treat here the boundary condition for a perfect conductor.

In this case the Maxwell equations will have to be complemented with the boundary conditions [10]

$$n \times E = 0, \quad n \cdot H = 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

where n is the unit outward normal vector to $\partial\Omega$. We now treat the solvability of this problem. First we write it in a more compact form. We define the matrices

$$A = \begin{bmatrix} \epsilon I_3 & 0 \\ 0 & \mu I_3 \end{bmatrix}, \qquad K = \begin{bmatrix} \epsilon_1 I_3 & \zeta I_3 \\ \xi I_3 & \mu_1 I_3 \end{bmatrix}, \tag{2.5}$$

where I_3 is the 3×3 unit matrix and 0 is the zero matrix. We further introduce the six-vector notation

$$\mathscr{E} = (E, H), \qquad \mathscr{D} = (D, B), \qquad \mathscr{F} = (F, G), \tag{2.6}$$

and the differential operator

$$L = \begin{bmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{bmatrix}, \tag{2.7}$$

where again 0 is the zero 3×3 matrix. The domain of this operator is taken to be

$$D(L) = \{\Phi \mid \Phi = (\phi, \psi) \in X,\ \operatorname{curl}\phi \in \left(L^2(\Omega)\right)^3,\ \operatorname{curl}\psi \in \left(L^2(\Omega)\right)^3,\ n \times \phi = 0 \text{ on } \partial\Omega\}, \tag{2.8}$$

where X is the linear space $X := L^2 := L^2(\Omega)^3 \oplus L^2(\Omega)^3$ which is a Hilbert space when equipped with the inner product

$$\langle \mathcal{U}, \mathcal{V} \rangle = \int_\Omega \epsilon u_1 \cdot \bar{v}_1 dx + \int_\Omega \mu u_2 \cdot \bar{v}_2 dx = \langle \epsilon u_1, v_1 \rangle_0 + \langle \mu u_2, v_2 \rangle_0, \tag{2.9}$$

where $\mathcal{U}, \mathcal{V} \in L^2$, with $\mathcal{U} = (u_1, u_2)$, $\mathcal{V} = (v_1, v_2)$, and the overbar denotes complex conjugation.

In this notation, Problem I assumes the form

$$\frac{d}{dt}(A\mathscr{E} + K \star \mathscr{E}) = L\mathscr{E} + \mathscr{F} \tag{2.10}$$

which is to be solved for given $\mathscr{F}$ and for homogeneous initial conditions $\mathscr{E}(x, 0) = 0$.

We will use the Laplace transform $\hat{u}(s) = \int_0^\infty u(t) e^{-st} dt$ defined for a real function $u : \mathbb{R} \to \mathbb{R}$ and for a complex variable $s = \sigma + i\eta$, provided the integral exists [5]. In the following we denote by $\mathscr{L}(\mathbb{R})$ the (linear) space of functions $u \in L^1_{\text{loc}}(\mathbb{R})$ such that $\operatorname{supp} u \subset [0, \infty)$, and for which the set

$$I_u = \left\{\sigma \in \mathbb{R} : \int_0^\infty |u(t)| e^{-\sigma t} dt < \infty\right\} \tag{2.11}$$

is not empty. We also define the space

$$\mathscr{L}_0 = \{\mathcal{U} \in L^1_{\text{loc}}(\mathbb{R}, L^2) : ||\mathcal{U}(\cdot)|| \in \mathscr{L}(\mathbb{R})\}. \tag{2.12}$$

In the sequel we impose the following three assumptions on the data of the problem:

(A1) ϵ and μ are positive and bounded functions of x;

(A2) $\mathcal{F}$ has a well-defined Laplace transform (i.e., $\mathcal{F} \in \mathcal{L}_0$);

(A3) the Laplace transform of K exists and converges to the zero matrix as $\sigma \to \infty$ in any matrix norm.

We take the Laplace transform of (2.10), and using the properties of the Laplace transform and multiplying by A^{-1} from the left, we obtain

$$(N - sI)\hat{\mathcal{E}} = s\hat{K}_0\hat{\mathcal{E}} - \hat{\mathcal{F}}_0, \tag{2.13}$$

where

$$N = A^{-1}L, \qquad \hat{K}_0 = A^{-1}\hat{K}, \qquad \hat{\mathcal{F}}_0 = A^{-1}\hat{\mathcal{F}}. \tag{2.14}$$

This is an equivalent form of the original problem (2.10).

To study the solvability of problem (2.13) we must study the properties of the differential operator N. The domain of this operator is $D[N] = D[L]$ and it can be shown that the operator N is unbounded, densely defined, and iN is selfadjoint. Furthermore, if $\mathrm{Re}(s) \neq 0$, the operator $N - sI$ is invertible and the norm of the inverse satisfies the estimate

$$\left|\left|(N - sI)^{-1}\right|\right| \leq \frac{1}{|\sigma|}. \tag{2.15}$$

The proofs of these claims are similar to the ones provided for the case of unbounded domains [6].

So, for $s = \sigma \in \mathbb{R}^+$, (2.13) is equivalent to the equation

$$\hat{\mathcal{E}} = s(N - sI)^{-1}\hat{K}_0\hat{\mathcal{E}} - (N - sI)^{-1}\hat{\mathcal{F}}_0. \tag{2.16}$$

But (2.16) is in the form of a fixed point problem, $T\mathcal{U} = \mathcal{U}$, for the affine operator $T : L^2 \to L^2$ where

$$T\mathcal{U} = s(N - sI)^{-1}\hat{K}_0\mathcal{U} - (N - sI)^{-1}\hat{\mathcal{F}}_0. \tag{2.17}$$

Using this remark as our starting point we are now in a position to state the main result of this section whose proof can be performed along the same lines as in [6]. In particular, the divergence-free property of E and H follows by taking the projection of D and B on the space $H(\mathrm{div}\,0,\mathbb{R}^3) \oplus H(\mathrm{div}\,0,\mathbb{R}^3)$. Recall that $H(\mathrm{div}\,0,\mathbb{R}^3) = \{V \in L^2(\mathbb{R}^3),\ \mathrm{div}\,V = 0\}$. For the properties of $H(\mathrm{div}\,0,\mathbb{R}^3)$ and related spaces, see, for example, [4].

THEOREM 2.1. *Under the assumptions (A1), (A2), and(A3), Problem I has a unique solution in* $D[N]$.

Problem I in a spatially periodic chiral medium (with rapidly varying physical parameters) has been studied from a rigorous homogenization theory point of view in [2] (where in fact the more general case of bianisotropic media is treated). In that work, it has been shown that the solution of the corresponding problem converges to the solution of

Problem I for an effective spatially homogeneous medium whose (constant) coefficients are determined.

3. The optical response approximation

In the previous section, we treated the full nonlocal set of equations, modelling dispersive chiral media, as far as solvability is concerned. Though the mathematical treatment of the full problem is feasible, in a number of important applications (e.g., in wave propagation or scattering problems), the full nonlocal problem may be cumbersome to handle. Thus, local approximations to the full problem have been proposed, that will keep the general features of chiral media, without the mathematical complications introduced by the nonlocality of the model.

In practice, a very common approximation scheme to the full constitutive relations for the medium is used, where essentially the convolution integrals are truncated to a Taylor series in the derivative of the fields. Using this expansion of the convolution integrals and the Maxwell constitutive relations, we may obtain the so-called DBF constitutive relations for chiral media

$$D = \epsilon(I + \beta\,\mathrm{curl})E, \qquad B = \mu(I + \beta\,\mathrm{curl})H, \tag{3.1}$$

where β is the chirality measure, considered here as a parameter that will be chosen so that a criterion for optimality is satisfied. This approximation is usually called the *optical response approximation*. For such constitutive relations, the equations for the fields become

$$\begin{aligned} \mathrm{curl}\,\tilde{E} &= -\frac{\partial}{\partial t}\{\mu(I + \beta\,\mathrm{curl})\tilde{H}\}, \\ \mathrm{curl}\,\tilde{H} &= \frac{\partial}{\partial t}\{\epsilon(I + \beta\,\mathrm{curl})\tilde{E}\}, \\ \mathrm{div}\,\tilde{E} &= 0, \qquad \mathrm{div}\,\tilde{H} = 0, \end{aligned} \tag{3.2}$$

supplemented with the initial conditions

$$\tilde{E}(x,0) = E_0(x), \qquad \tilde{H}(x,0) = H_0(x), \tag{3.3}$$

and the boundary conditions corresponding to the perfect conductor problem. This problem will be called hereafter Problem II. Its solvability is established in the following theorem.

Theorem 3.1. *Under the assumptions A1 and A2, Problem II has a unique solution in* $D[N]$ *for sufficiently small* β.

In this case the Laplace transformed operator equation is of the form $\Theta L\mathscr{E} = P\mathscr{E} + \mathscr{F}$, where Θ and P are suitably defined matrix operators depending on s and β. It can be shown that Θ is invertible; the rest of the proof is similar to that of Theorem 2.1.

The solution to Problem II is a commonly used approximation to the full solution of Problem I.

A very popular method of treating electromagnetic problems in the frequency domain is through the use of Beltrami fields. This method has been used for the explicit construction of the solution of Problem II in [1].

Another interesting approach to Problem II is through the use of Moses eigenfunctions [13]. These form a complete orthonormal basis for L^2 consisting of eigenfunctions of the curl operator.

Specifically, Moses [13] introduced three-dimensional complex vectors $K(x,p;\lambda)$ with $x, p \in \mathbb{R}^3$ which satisfy

$$\operatorname{curl} K(x,p;\lambda) = \lambda |p| K(x,p;\lambda), \quad \lambda = 0, \pm 1; \tag{3.4}$$

that is, $K(x,p;\lambda)$ are eigenvectors of the curl operator and $\lambda|p|$ are the associated eigenvalues. These fields (that will be called Beltrami-Moses fields) satisfy some interesting orthogonality and completeness relations.

We may now define the fields

$$Q_\pm(x,t) = \{E \pm i\eta H\}(x,t), \quad \eta = \sqrt{\frac{\mu}{\epsilon}}, \tag{3.5}$$

which implies that

$$E(x,t) = \frac{1}{2}\{Q_+ + Q_-\}(x,t), \qquad H(x,t) = \frac{1}{2i\eta}\{Q_+ - Q_-\}(x,t). \tag{3.6}$$

Using these fields, we may proceed formally to rewrite Problem II in the following form:

$$\begin{aligned} \operatorname{curl} Q_\pm &= \pm i\sqrt{\mu\epsilon}\frac{\partial}{\partial t}\{(I + \beta \operatorname{curl})Q_\pm(x,t)\}, \\ \operatorname{div} Q_\pm(x,t) &= 0. \end{aligned} \tag{3.7}$$

The associated initial values are

$$Q_\pm(x,0) = E_0(x) \pm i\eta H_0(x). \tag{3.8}$$

Using these Beltrami-Moses fields as kernels for an integral transform, we may define a generalized Fourier transform for vector functions $\psi(x,t)$, the Beltrami-Moses transform, as follows:

$$\hat{\psi}(p,t;\lambda) = \int \overline{K(x,p;\lambda)}\psi(x,t)\,dx. \tag{3.9}$$

The inverse transform is given by the formula

$$\psi(x,t) = \sum_{\lambda} \int K(x,p;\lambda)\hat{\psi}(p,t;\lambda)dp. \tag{3.10}$$

Expanding the fields $Q_\pm$ in terms of the Moses eigenfunctions and using the property that both of these fields have to be divergence-free, we may reduce Problem II to a set of first-order ordinary differential equations for the field amplitudes corresponding to $\lambda = \pm 1$. The electromagnetic fields may be obtained by inversion of the integral transform. This approach is related to the spectral approach to Problem II.

4. The error of the optical response approximation

Recall that (E,H) and $(\tilde{E},\tilde{H})$ are, respectively, the solutions of Problems I and II. We introduce a third problem, the solution of which will furnish the error of the optical response approximation. So, let

$$w_E = E - \tilde{E}, \qquad w_H = H - \tilde{H}. \tag{4.1}$$

After some elementary manipulations, we find that the error of the optical response approximation satisfies the equations

$$\begin{aligned}
\operatorname{curl} w_E &= -\frac{\partial}{\partial t}\{\mu w_H + \mu_1 \star w_H + \xi \star w_E + \mu_1 \star \tilde{H} + \xi \star \tilde{E} - \mu\beta \operatorname{curl}\tilde{H}\},\\
\operatorname{curl} w_H &= \frac{\partial}{\partial t}\{\epsilon w_E + \epsilon_1 \star w_E + \zeta \star w_H + \epsilon_1 \star \tilde{E} + \zeta \star \tilde{H} - \epsilon\beta \operatorname{curl}\tilde{E}\},\\
\operatorname{curl}\tilde{E} &= -\frac{\partial}{\partial t}\{\mu(I+\beta \operatorname{curl})\tilde{H}\},\\
\operatorname{curl}\tilde{H} &= \frac{\partial}{\partial t}\{\epsilon(I+\beta \operatorname{curl})\tilde{E}\},\\
\operatorname{div} w_E &= \operatorname{div} w_H = \operatorname{div}\tilde{E} = \operatorname{div}\tilde{H} = 0,
\end{aligned} \tag{4.2}$$

supplemented with the initial conditions

$$w_E(x,0) = 0, \qquad w_H(x,0) = 0, \qquad \tilde{E}(x,0) = E_0(x), \qquad \tilde{H}(x,0) = H_0(x). \tag{4.3}$$

This problem will be hereafter called Problem III. The solution of Problem III will furnish the error of the optical response approximation for a given solution $(\tilde{E},\tilde{H})$. Observe that the equations for the approximate fields are decoupled from the equations for the error.

A priori estimates are obtained on the solution of Problem III. This is done by reducing the error equations to the form of a Volterra equation of the second kind. By expanding the solution in Moses eigenfunctions, we may rewrite the original system for the error in the compact form

$$A_1 w = \frac{d}{dt}\{A_2 w + A_3 \star w + S\}, \tag{4.4}$$

where

$$
\begin{aligned}
w &= \begin{pmatrix} w_{E,\lambda} \\ w_{H,\lambda} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \lambda|p| & 0 \\ 0 & \lambda|p| \end{pmatrix}, \\
A_2 &= \begin{pmatrix} 0 & -\mu \\ \epsilon & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} -\xi & -\mu_1(\tau) \\ \epsilon_1(\tau) & \zeta \end{pmatrix}, \\
S &= \begin{pmatrix} S_{1,\lambda} \\ S_{2,\lambda} \end{pmatrix} = \begin{pmatrix} -\mu_1 \star \bar{H}_\lambda - \xi \star \bar{E}_\lambda + \lambda\beta\mu|p|\bar{H}_\lambda \\ \epsilon_1 \star \bar{E}_\lambda + \zeta \star \bar{H}_\lambda - \lambda\beta\epsilon|p|\bar{E}_\lambda \end{pmatrix}.
\end{aligned}
\tag{4.5}
$$

Now integrate once over time to rewrite the equation for the error in the following form:

$$
w = \phi \star w + g, \tag{4.6}
$$

where

$$
\phi = A_2^{-1}(A_1 - A_3), \qquad g = -A_2^{-1}S. \tag{4.7}
$$

For the specific system we study here, we have that

$$
\phi = \begin{pmatrix} -\dfrac{\epsilon_1(\tau)}{\epsilon} & \dfrac{\lambda|p| - \zeta}{\epsilon} \\ -\dfrac{\lambda|p| + \xi}{\mu} & -\dfrac{\mu_1(\tau)}{\mu} \end{pmatrix}, \qquad g = \begin{pmatrix} \dfrac{\epsilon_1}{\epsilon} \star \bar{E}_\lambda + \dfrac{\zeta}{\epsilon} \star \bar{H}_\lambda - \lambda\beta|p|\bar{E}_\lambda \\ \dfrac{\mu_1}{\mu} \star \bar{H}_\lambda + \dfrac{\xi}{\mu} \star \bar{E}_\lambda - \lambda\beta|p|\bar{H}_\lambda \end{pmatrix}. \tag{4.8}
$$

This matrix Volterra equation will be used to obtain a priori estimates for the error of the optical response approximation in terms of the Moses transformed fields. The following two results were proved in [6].

Theorem 4.1. *Let*

$$
\Psi(t) := \left(1 - 2\sup_{i,j} ||\phi_{ij}||_{L_1(0,t)}\right)^{-1} > 0. \tag{4.9}
$$

Then, the solution of (4.6) satisfies the following a priori error bound

$$
\sup_i ||w_i||_{L_p(0,t)} \leq \Psi(t) \sup_i ||g_i||_{L_p(0,t)}. \tag{4.10}
$$

It is interesting to notice that an alternative method of obtaining a priori bounds can be developed using the Gronwall inequality. Indeed, in this manner we can readily obtain the following result.

Theorem 4.2. *Suppose that* $\epsilon > 0$, $\mu > 0$, $\xi > 0$, *and* $\zeta > 0$, *and that* $|p| \leq \min(\xi, \zeta)$. *Assuming that the functions* ϕ_{ij} *are bounded, the following estimate holds:*

$$
\sup_t |w_1(t) + w_2(t)| \leq \sup_t |g_1(t) + g_2(t)|. \tag{4.11}
$$

The above estimate provides us with a way to minimize the error of the optical response approximation. One way to do this is by minimizing the upper bound $\sup_i \|g_i\|_{L_r(0,t)}$. This amounts to choosing the value of β so as to minimize the integrals

$$\begin{aligned} ||g_1||_{L_r(0,t)} &= \left\{ \int_0^t \left| \frac{\epsilon_1}{\epsilon} \star \bar{E}_\lambda + \frac{\zeta}{\epsilon} \star \bar{H}_\lambda - \lambda\beta|p|\bar{E}_\lambda \right|^r dt' \right\}^{1/r}, \\ ||g_2||_{L_r(0,t)} &= \left\{ \int_0^t \left| \frac{\mu_1}{\mu} \star \bar{H}_\lambda + \frac{\xi}{\mu} \star \bar{E}_\lambda - \lambda\beta|p|\bar{H}_\lambda \right|^r dt' \right\}^{1/r}. \end{aligned} \tag{4.12}$$

A series of other results were obtained for each p using the expansion of Problem III in Moses eigenfunctions. This approach allows us to find exact forms for the Laplace transform of the error for specified wavenumbers. Numerical techniques can thus be used for the inversion of the Laplace transform and the retrieval of the time dependence of the error term.

An estimate of the error in the spatial variables rather than in terms of the wavenumbers can be obtained in the following way. Adopting the notation of Section 2, the equation for the error may be written in the form

$$Lw = \frac{\partial}{\partial t}(Aw + K \star w + \Phi), \tag{4.13}$$

where Φ is a source term which is related to the solutions of the optical response equation $\tilde{H}$ and $\tilde{E}$. Multiplying by w, integrating over space, and using the properties of the operator L, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|w\|^2) + \left\langle \frac{d}{dt}(K \star w), w \right\rangle + \langle \Phi, w \rangle. \tag{4.14}$$

Under the assumption that the convolution kernel is such that

$$K_1 Aw \le \frac{d}{dt}(K \star w) \le K_2 Aw, \tag{4.15}$$

we obtain

$$\frac{d}{dt}\|w\|^2 + K_1\|w\|^2 \le |\Phi|\,\|w\|, \tag{4.16}$$

from which by use of the Gronwall inequality we may obtain a priori bounds for the error. Similar bounds may be obtained by slight modification of the conditions on the kernels.

5. Homogenization for spatially periodic chiral media

We will now consider Problem I in a spatially periodic chiral medium, that is, we will consider the parameters of the medium ϵ, ϵ_1, μ, μ_1, ζ, and ξ to be periodic functions of x with a period ε. The period ε will be considered to be a small number, a fact that will correspond to a fast spatially varying medium.

This leads us to considering the problem of homogenization for such media, that is, the approximation of a spatially periodic medium with a homogeneous medium (i.e., a medium with constant parameters) having the same properties as the original medium in the limit as $\varepsilon \to 0$. The homogenization problem for periodic structures is a long standing problem in the mathematical and engineering community that has led to the introduction of interesting mathematical techniques and also to interesting engineering applications.

Though the problem of homogenization for the Maxwell equations has been studied extensively in the past, there has been little progress on this subject as far as bianisotropic or chiral media are concerned. While some papers treat versions of the problem from the engineering point of view, the problem has been left untouched from the rigorous mathematical point of view. This aspect of the problem has been studied in [2] for the more general case of bianisotropic media.

Consider the spatially periodic version of Problem I, which consists of Maxwell's equations

$$\begin{gathered} \partial_t D^\varepsilon = \operatorname{curl} H^\varepsilon + F(x,t), \\ \partial_t B^\varepsilon = -\operatorname{curl} E^\varepsilon + G(x,t), \quad x \in \Omega,\ t > 0, \\ E^\varepsilon(x,0) = 0, \quad H^\varepsilon(x,0) = 0, \quad x \in \Omega, \end{gathered} \tag{5.1}$$

subject to the constitutive laws

$$\begin{aligned} D^\varepsilon &= \epsilon^\varepsilon E^\varepsilon + \zeta^\varepsilon * H^\varepsilon + \epsilon_1^\varepsilon * E^\varepsilon, \\ B^\varepsilon &= \mu^\varepsilon H^\varepsilon + \xi^\varepsilon * E^\varepsilon + \mu_1^\varepsilon * H^\varepsilon. \end{aligned} \tag{5.2}$$

The functions $\epsilon^\varepsilon(x)$ and $\mu^\varepsilon(x)$ as well as the functions $\epsilon_1^\varepsilon(x,t)$, $\mu_1^\varepsilon(x,t)$, $\xi^\varepsilon(x,t)$, and $\zeta^\varepsilon(x,t)$ are periodic in x of period εY. We assume that there exists $c > 0$ such that the block matrix

$$\begin{pmatrix} \epsilon + \hat{\epsilon}_1 & \hat{\zeta} \\ \hat{\xi} & \mu + \hat{\mu}_1 \end{pmatrix} =: A(x,p) \tag{5.3}$$

satisfies

$$\langle A(x,p)U, U\rangle \geq c\|U\|^2, \quad x \in \Omega,\ p \in \mathbb{C}_+,\ U \in \mathbb{R}^6. \tag{5.4}$$

We fix a domain $V \subset \Omega$ and consider the operator

$$L^\varepsilon = \begin{pmatrix} -\operatorname{div}((\epsilon + \hat{\epsilon}_1)\operatorname{grad}) & -\operatorname{div}((\hat{\zeta})\operatorname{grad}) \\ -\operatorname{div}((\hat{\xi})\operatorname{grad}) & -\operatorname{div}((\mu + \hat{\mu}_1)\operatorname{grad}) \end{pmatrix} : H_0^1(V) \longrightarrow H^{-1}(V) \tag{5.5}$$

and the corresponding homogenization limit

$$L^h =: \begin{pmatrix} -\operatorname{div}(\tilde{\epsilon}^h \operatorname{grad}) & -\operatorname{div}(\tilde{\zeta}^h \operatorname{grad}) \\ -\operatorname{div}(\tilde{\xi}^h \operatorname{grad}) & -\operatorname{div}(\tilde{\mu}^h \operatorname{grad}) \end{pmatrix}. \tag{5.6}$$

Note that while the coefficients of L^h are spatially constant, they do depend on $p \in \mathbb{C}_+$. We assume that for fixed $x \in \Omega$, the functions $\tilde{\epsilon}^h$, $\tilde{\xi}^h$, $\tilde{\zeta}^h$, and $\tilde{\mu}^h$ are the Laplace transforms of functions ϵ^h, ξ^h, ζ^h, and μ^h on $(0,\infty)$. We then have the following theorem.

Theorem 5.1. *Assume that the Maxwell system (5.1) and (5.2) is uniquely solvable for all $\varepsilon > 0$ and that $\|E^\varepsilon\|_2, \|H^\varepsilon\|_2 \le c$ for all $\varepsilon, t > 0$. Then the solution $(E^\varepsilon, H^\varepsilon)$ of the above system satisfies*

$$E^\varepsilon \longrightarrow E^*, \quad H^\varepsilon \longrightarrow H^*, \quad *\text{-weakly in } L^\infty((0,\infty), L^2(\Omega)), \tag{5.7}$$

where (E^*, H^*) is the unique solution of the Maxwell system

$$\begin{gathered} \partial_t D^* = \operatorname{curl} H^* + F, \\ \partial_t B^* = -\operatorname{curl} E^* + G, \qquad x \in \Omega, \quad t > 0, \\ E^*(x,0) = 0, \qquad H^*(x,0) = 0, \end{gathered} \tag{5.8}$$

subject to the constitutive laws

$$\begin{aligned} D^* &= \epsilon^h * E^* + \zeta^h * H^*, \\ B^* &= \xi^h * E^* + \mu^h * H^*. \end{aligned} \tag{5.9}$$

We do not provide the proof of the theorem here (for a complete proof see [2]) but simply note that in order to prove the above result, we have to work with the Laplace transform of the original problem which assumes the form of an elliptic partial differential equation with spatially periodic coefficients. The homogenization problem for the latter may be addressed using generalizations of standard homogenization techniques based on the use of the div-curl lemma, thus leading to a spatially homogenized equation in Laplace space. Then, inverting the Laplace transform, we arrive at the announced result. For details see [2].

Remark 5.2. (1) The above theorem gives the homogenized coefficients as inverse Laplace transforms of certain functions. In concrete cases one can use numerical schemes to obtain precise approximations of ϵ^h, ξ^h, ζ^h, and μ^h. The Laplace transforms of the homogenized coefficients may be obtained by a proper averaging of the parameters of the medium weighted by the solution of an appropriately formulated "cell problem." For the definition of the cell problem see [2].

(2) It is clear that the functions F and G can also depend on $\varepsilon > 0$, provided that one makes suitable assumptions on their behaviour as $\varepsilon \to 0$.

For completeness here, we present the expressions for the homogenized coefficients for the medium, in Laplace space.

We let $H^1_{\text{per}}(Y)$ denote the closed subspace of $H^1(Y)$ that consists of periodic functions and define the operator $L_{\text{per}} : H^1_{\text{per}}(Y) \to (H^1_{\text{per}}(Y))^*$ by

$$L_{\text{per}} = \begin{pmatrix} -\operatorname{div}(\epsilon I_3 \operatorname{grad}) & -\operatorname{div}(\zeta I_3 \operatorname{grad}) \\ -\operatorname{div}(\xi I_3 \operatorname{grad}) & -\operatorname{div}(\mu I_3 \operatorname{grad}) \end{pmatrix}. \tag{5.10}$$

This operator may be proved to be invertible modulo constants. In particular, we can define (modulo constants) the functions u_1^j, u_2^j, v_1^j, and v_2^j, $j = 1,2,3$, by the relations

$$L_{\text{per}}\begin{pmatrix} u_1^j \\ u_2^j \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \epsilon_{ij}}{\partial y_i} \\ \dfrac{\partial \xi_{ij}}{\partial y_i} \end{pmatrix}, \qquad L_{\text{per}}\begin{pmatrix} v_1^j \\ v_2^j \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \zeta_{ij}}{\partial y_i} \\ \dfrac{\partial \mu_{ij}}{\partial y_i} \end{pmatrix}. \tag{5.11}$$

We define the *homogenized* constant coefficient matrices ϵ^h, ξ^h, ζ^h, and μ^h by

$$\begin{aligned} \epsilon_{ij}^h &= \langle \epsilon_{ij} + \epsilon_{ik}\partial_{y_k} u_1^j + \zeta_{ik}\partial_{y_k} u_2^j \rangle, \\ \xi_{ij}^h &= \langle \xi_{ij} + \xi_{ik}\partial_{y_k} u_1^j + \mu_{ik}\partial_{y_k} u_2^j \rangle, \\ \zeta_{ij}^h &= \langle \zeta_{ij} + \zeta_{ik}\partial_{y_k} v_2^j + \epsilon_{ik}\partial_{y_k} v_1^j \rangle, \\ \mu_{ij}^h &= \langle \mu_{ij} + \mu_{ik}\partial_{y_k} v_2^j + \xi_{ik}\partial_{y_k} v_1^j \rangle, \end{aligned} \tag{5.12}$$

where $\langle g \rangle := |Y|^{-1} \int_Y g$. It is not obvious but it is easy to prove that the block matrix

$$A^h = \begin{pmatrix} \epsilon^h & \zeta^h \\ \xi^h & \mu^h \end{pmatrix} \tag{5.13}$$

is symmetric and positive definite. We note here that one can also deduce relations (5.12) formally by postulating a double-scale expansion for E^ε and H^ε.

6. Nonlinear chiral media

The results presented so far were results valid for chiral media constitutive relations with linear (local or nonlocal) laws involving the electromagnetic fields. Nevertheless there is a rapidly growing interest in nonlinear chiral media. The study of such systems is still in its initial stages and very little work has been done in this direction (see, e.g., [7, 14]). In this section we will examine the effects of nonlinearity on the constitutive relations for chiral media. In particular, we will present some recent results related to the evolution of electromagnetic fields in chiral media with cubic nonlinearity in the weak-dispersion, low-chirality limit. This limit is quite interesting and has been studied in the linear case (for general mathematical results for time-harmonic fields see, e.g., [3] and the references therein). For cubically nonlinear, weakly dispersive media with low-chirality parameter, we derive a set of four coupled partial differential equations of the NLS type for the evolution of the slowly varying envelopes of the electromagnetic fields. With the use of reductive perturbation theory, we reduce the system to a set of integrable partial differential equations, the Mel'nikov system, and thus show the existence in certain limits of vector solitons of the dark-bright type.

6.1. The field equations in the general case. The starting point for the modelling of a nonlinear chiral medium is the Maxwell postulates, in the absence of sources. We assume furthermore that the medium is of infinite extent. To obtain a description of the fields, the above equations will have to be complemented by the constitutive relations for the medium, that give the connection of D and B on the fields E and H. For a weakly nonlinear, weakly dispersive chiral medium with a cubic nonlinearity, we may assume that the constitutive relations in the time domain are of the form

$$\begin{aligned} D &= \epsilon E + \epsilon_1 \star E + \zeta \star H + \delta\epsilon_2 f_1(|E|^2)E, \\ B &= \mu H + \mu_1 \star H + \xi \star E + \delta\mu_2 f_2(|H|^2)H, \end{aligned} \tag{6.1}$$

where by $\star$ we denote the convolution $(f \star G)(x,t) = \int_{-\infty}^{\infty} f(t-t')G(x,t')h(t-t')dt'$. In the above relation, ϵ and μ are the permittivity and permeability of the medium, respectively, and ξ and ζ are the chirality parameters of the medium. Causality in the linear part is ensured by the appearance of the Heaviside function h whereas the nonlinear part is local (and therefore causal). The assumption of locality for the nonlinear part is consistent with the weak dispersion—the weak nonlinearity case we consider. The parameter δ is a small parameter which is associated with the weak nonlinearity. The fact that we have low-chirality and weak nonlinearity is shown in the above constitutive relations by the fact that the nonlinearity in D depends only on E while the nonlinearity in B depends only on H. We further assume that the chirality effects are weaker than the nonlinearity so as to be able to neglect cross terms in the fields H and E. In this work we use constitutive relations with nonlinearities expressed directly in the fields E and H, and not in the Beltrami fields in which they may be decomposed (see, e.g., [14]).

The well-posedness of the above problem in the general case is an intriguing mathematical problem which is currently under consideration. Here we will study the problem for a special class of fields, that is, fields which in the frequency domain are of the form

$$\begin{aligned} E(z,\omega) &= u_1(z,\omega)e_+ + v_1(z,\omega)e_-, \\ H(z,\omega) &= u_2(z,\omega)e_+ + v_2(z,\omega)e_-, \end{aligned} \tag{6.2}$$

where $e_\pm = (1/\sqrt{2})(\hat{x} \pm i\hat{y})$. This *ansatz* contains the most general dependence of the fields in e_+ and e_- (which is a complete basis in the x,y plane). On the other hand it does not contain the longitudinal component and/or transverse dependence of the fields, nevertheless it is still consistent with the divergence-free property of the fields D and B which is valid in this case.

Substituting this *ansatz* in the Maxwell postulates, we arrive at the following set of nonlinear equations (in the frequency domain):

$$\begin{aligned} -i\frac{\partial u_1}{\partial z} &= -i\omega\left(\mu u_2 + \hat{\mu}_1 u_2 + \hat{\zeta} u_1 + \delta\mu_2 f_2\left(|u_2|^2 + |v_2|^2\right)u_2\right), \\ -i\frac{\partial u_2}{\partial z} &= i\omega\left(\epsilon u_1 + \hat{\epsilon}_1 u_1 + \hat{\xi} u_2 + \delta\epsilon_2 f_1\left(|u_1|^2 + |v_1|^2\right)u_1\right), \\ i\frac{\partial v_1}{\partial z} &= -i\omega\left(\mu v_2 + \hat{\mu}_1 v_2 + \hat{\zeta} v_1 + \delta\mu_2 f_2\left(|u_2|^2 + |v_2|^2\right)v_2\right), \\ i\frac{\partial v_2}{\partial z} &= i\omega\left(\epsilon v_1 + \hat{\epsilon}_1 v_1 + \hat{\xi} v_2 + \delta\epsilon_2 f_1\left(|u_1|^2 + |v_1|^2\right)v_1\right), \end{aligned} \tag{6.3}$$

where by $\hat{\phi}(\omega)$ we denote the Fourier transform of $\phi(t)$ (the ω dependence is dropped for convenience). We note that in the above set of equations in the absence of nonlinearity ($\delta = 0$), the first two equations decouple from the other two, that is, the field components in the $\mathbf{e}_+$ and $\mathbf{e}_-$ directions evolve independently. The components, in the absence of boundary conditions, are coupled only through the nonlinearity.

6.2. Derivation of the amplitude equations. Assuming solutions of the form

$$u_j(z,\omega) = U_j \exp(ik_+ z), \quad v_j(z,\omega) = V_j \exp(ik_- z), \quad j = 1,2, \tag{6.4}$$

which in the time domain correspond to wave solutions, we see that in the weakly nonlinear case they will have to satisfy the dispersion relations

$$\begin{gathered} k_\pm^2 \pm i\omega(\hat{\zeta} - \hat{\xi})k_\pm + \omega^2\{\hat{\xi}\hat{\zeta} - (\epsilon + \hat{\epsilon}_1 + \delta\epsilon_2 f_1)(\mu + \hat{\mu}_1 + \delta\mu_2 f_2)\} = 0, \\ f_i = f_i(|U_i|^2 + |V_i|^2), \quad i = 1,2. \end{gathered} \tag{6.5}$$

In the absence of nonlinearity ($\delta = 0$), these two dispersion relations reduce to the dispersion relations for the right-handed and the left-handed polarized waves that are well known to propagate in linear chiral media.

We will now assume that the nonlinear medium supports wave solutions of the form (6.4), where the Fourier transforms of U_j and V_j are considered to be slowly varying functions of space and time. In other words, U_j and V_j are considered to be the envelopes of the wave fields. Using reductive perturbation theory, we may derive modulation equations for the evolution of the envelopes of the fields. One way of doing that is through the dispersion relation of the weakly nonlinear waves in the following way: we expand the dispersion relations in a Taylor expansion around the point (k_0, ω_0) which is a solution of the linear dispersion relation. For the problem at hand, it is enough to keep terms up to the second order. As a result we obtain a polynomial expression in k and ω, and $I_i = |U_i|^2 + |V_i|^2$, $i = 1,2$, the coefficients of the polynomial, are derivatives of ω calculated at the points $k = k_0$ and $I_i = 0$ (linear case).

In order to obtain modulation equations for the envelopes of the fields, we have to return back to physical space and time. This is done through the substitution $(\omega_\pm - \omega_{0,\pm}) \to i(\partial/\partial t)$, $(k - k_0) \to -i(\partial/\partial z)$, and assuming that these operators act on the envelope of the waves and on the relevant temporal and spatial scales. The modulation equations may be derived in an alternative manner by the use of reductive perturbation theory.

Following the procedure described above, we obtain a set of four evolution equations for the envelopes of the wave fields, in the time domain, of the form

$$\begin{aligned} i\frac{\partial U_j}{\partial t} &= i\frac{\partial \omega_+}{\partial k}\frac{\partial U_j}{\partial z} + \frac{1}{2}\frac{\partial^2 \omega_+}{\partial k^2}\frac{\partial^2 U_j}{\partial z^2} + \left(\frac{\partial \omega_+}{\partial I_1}I_1 + \frac{\partial \omega_+}{\partial I_2}I_2\right)U_j, \\ i\frac{\partial V_j}{\partial t} &= i\frac{\partial \omega_-}{\partial k}\frac{\partial V_j}{\partial z} + \frac{1}{2}\frac{\partial^2 \omega_-}{\partial k^2}\frac{\partial^2 V_j}{\partial z^2} + \left(\frac{\partial \omega_-}{\partial I_1}I_1 + \frac{\partial \omega_-}{\partial I_2}I_2\right)V_j, \end{aligned} \tag{6.6}$$

where $j = 1,2$. These NLS equations are coupled through the nonlinear terms I_1 and I_2. The spatial and temporal coordinates appearing in these equations are scaled variables, relevant to the slow variations of the envelopes of the fields. For more details on the derivation of the modulation equations see [7].

6.3. Reduction of the modulation equations to the Mel'nikov system. The coupled NLS equations which arose as modulation equations for the evolution of the fields in chiral media are not integrable by the use of the inverse scattering transform. However, it is possible, through the use of reductive perturbation theory, to reduce the system in the proper spatial and time scales to an integrable system that approximates the behaviour of the original system. This procedure is a general way of understanding the properties of solutions of nonintegrable systems that has been proven fruitful in a number of similar situations (see [7] and the references therein).

To obtain the reduction to an integrable system, we restrict ourselves to solutions of the type

$$U_2 = \rho_1 U_1, \qquad V_2 = \rho_2 V_1, \tag{6.7}$$

for which the original system of NLS equations reduces to a system of two equations.

We will now look for solutions of the above system satisfying the boundary conditions

$$\begin{aligned} |U_1| &\longrightarrow |u|, \quad \text{as } z \longrightarrow \infty, \\ |V_1| &\longrightarrow 0, \quad \text{as } z \longrightarrow \infty, \end{aligned} \tag{6.8}$$

that is, we will look for solutions of the dark soliton type in the right-handed component u, and solutions of the bright soliton type in the left-handed component v. It is clear that the above boundary conditions may be reversed.

With the use of reductive perturbation theory (for details see [7]), a lengthy procedure leads to a system of the Mel'nikov type [12], which is fully integrable by the inverse scattering transform for special cases of the parameters. The Mel'nikov system has soliton solutions in the form of a dark soliton in the right-handed component and a bright soliton in the left-handed component, that is, a localized nonlinear wave propagating in a dispersive medium, on top of a continuous wave background, keeping its shape undistorted. The bright soliton represents a bulge on top of the continuous wave background, whereas the dark soliton represents a dip. For more details on the definition and the properties of dark and bright solitons see, for example, [8].

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EFFICIENT CRITERIA FOR THE STABILIZATION OF PLANAR LINEAR SYSTEMS BY HYBRID FEEDBACK CONTROLS

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We suggest some criteria for the stabilization of planar linear systems via linear hybrid feedback controls. The results are formulated in terms of the input matrices. For instance, this enables us to work out an algorithm which is directly suitable for a computer realization. At the same time, this algorithm helps to check easily if a given linear 2×2 system can be stabilized (a) by a linear ordinary feedback control or (b) by a linear hybrid feedback control.

1. Introduction

Consider a linear control 2×2 system

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{1.1}$$

on $[0,\infty)$, where $x\in\mathbb{R}^2$ is the state variable of the system, $y\in\mathbb{R}^m$ is the output variable, $u\in\mathbb{R}^\ell$ is the control variable, and B and C are given real matrices of the sizes $2\times\ell$ and $m\times 2$, respectively.

If the pair (A,B) is controllable, or more generally, stabilizable, and $\operatorname{rank} C = 2$ (which describes the case of complete observability of the solutions), then it is always possible (see, e.g., [5, 6]) to achieve exponential stability of the zero solution to the control system (1.1) with an arbitrary matrix A. In such a case, there exists a linear ordinary feedback control of the form $u = Gy$ with an $\ell\times m$ matrix G, which yields exponential stability.

Similarly, if $\operatorname{rank} B = 2$ and the pair (A,C) is observable, or at least detectable, then again a suitable linear feedback control of the form $u = Gy$ solves the stabilization problem for system (1.1).

However, it is known that in practice, neither the condition $\operatorname{rank} B = 2$ nor the complete observability of the solutions (i.e., $\operatorname{rank} C = 2$) can be unavailable. The most interesting situation for applications is, therefore, the case when $\operatorname{rank} B = \operatorname{rank} C = 1$.

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A simple example of such a system is *the harmonic oscillator* with the external force as the control, where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}. \tag{1.2}$$

Here, the displacement variable x_1 is available for measurements, while the controller can only change the velocity variable x_2 (we assume that $x = \left(\begin{smallmatrix} x_1 & x_2 \end{smallmatrix}\right)^\top \in \mathbb{R}^2$). This control system is both controllable and observable, but it cannot be stabilized by ordinary (even nonlinear and discontinuous) output feedback controls of the form $u = f(y)$ (see, e.g., [1]).

However, as it was shown by Artstein [1], there exists a *hybrid feedback control* which provides asymptotical stability of the zero solution to (1.1) with the matrices from (1.2).

A hybrid feedback control includes essentially two features (see Section 3 for the formal definitions): a discrete time controller (an automaton) attached to the given dynamical system (i.e., to (1.1) in our case) via the matrices B and C, and a switching algorithm describing when and how a control u should be changed. Artstein's example shows that such a hybrid feedback control may help even when the ordinary feedback fails to stabilize the system.

In [2, 3], the following result is obtained for B and C being nonzero matrices of rank 1: system (1.1) is stabilizable by a linear hybrid feedback control (LHFC) if and only if for at least one $\alpha \in \mathbb{R}$, the matrix $A + \alpha BC$ does not have nonnegative real eigenvalues. This result gives a necessary and sufficient stabilization condition, and it is straightforward that making use of hybrid feedback controls provides a better stabilization criterion compared to any one we can obtain exploiting ordinary feedback controls.

However, the shortcoming of this criterion is that it does not give any explicit description of how its assumptions can be verified in practice. In other words, it does not suggest any efficient, finite-step algorithm in terms of the given matrices (A,B,C), which would answer the question when system (1.1) admits a stabilizing feedback control.

In contrast to [2, 3], the present paper aims at

(1) finding verifiable criteria for LHFC stabilization of system (1.1),

(2) constructing efficient algorithms (which should also be "computer-friendly"), which can easily test a specific system (1.1) in terms of the input matrices (A,B,C) to find out whether the zero solution to (1.1) can be stabilized by an ordinary feedback linear control or by an LHFC.

2. Notations and relevant facts of control theory

We define by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the sets of all natural, real, and complex numbers, respectively. The set $\mathbb{R}$ will in the sequel be naturally identified with $\{z \mid \operatorname{Im} z = 0\} \subset \mathbb{C}$. By $\langle \cdot, \cdot \rangle$ and $|\cdot|$ we mean the scalar product and the Euclidean norm in $\mathbb{R}^2$, respectively. We write $\operatorname{Span}\{b\}$ for the one-dimensional vector space containing a given vector $b \in \mathbb{R}^2$. We also put $\mathbb{C}_- := \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$, $\mathbb{C}_+ := \mathbb{C} \setminus \mathbb{C}_-$, and $\mathbb{C}_s^2 := \{(\lambda_1,\lambda_2) \in \mathbb{C}^2 \mid \lambda_1 = \bar{\lambda}_2\}$.

Let $M(\ell,m)$ denote the set of all real $\ell \times m$ matrices. Matrices will often be addressed as linear operators in the appropriate vector spaces. In the sequel, I and Θ will stand for the identity 2×2 matrix and the zero 2×2 matrix, respectively. Given a matrix $D \in M(2,2)$, we will denote its spectrum by $\sigma(D)$.

In what follows, we will consider system (1.1) for arbitrary but fixed matrices $A \in M(2,2)$, $B \in M(2,\ell)$, and $C \in M(m,2)$ ($\ell, m \in \mathbb{N}$). We also suppose that $\sigma(A) = \{\lambda_1, \lambda_2\}$. Moreover, if $\sigma(A) \subset \mathbb{R}$, then we suppose, without loss of generality, that $\lambda_1 \leq \lambda_2$ (the case $\lambda_1 = \lambda_2$ is not excluded either).

The characteristic and the minimal polynomials of the matrix A will be denoted by $\pi_A(\lambda)$ and $p_A(\lambda)$, respectively. Clearly, $\pi_A(\lambda) = \lambda^2 - \operatorname{tr} A \cdot \lambda + \det A$. The decomposition $\mathbb{C} = \mathbb{C}_- \sqcup \mathbb{C}_+$ implies also a special factorization of the minimal polynomial $p_A = p_A^- p_A^+$, where the zeros of $p_A^-(\lambda)$ and $p_A^+(\lambda)$ belong to $\mathbb{C}_-$ and $\mathbb{C}_+$, respectively. The notation $\langle A|B\rangle$ is used for the controllability space of the pair (A,B), that is, $\langle A|B\rangle := B(\mathbb{R}^\ell) + AB(\mathbb{R}^\ell)$.

We recall some well-known facts (see, e.g., [5, 6]) from the theory of control linear systems, which are summarized in Definitions 2.1, 2.3, and 2.6 and Lemmas 2.2, 2.4, 2.5, 2.7, 2.8, and 2.9. Although some of the results are quite general, we will formulate them for the case of 2×2 systems, as it is the case of interest in this paper.

Definition 2.1. A matrix A is called stable if $\sigma(A) \subset \mathbb{C}_-$.

Lemma 2.2. *The following conditions are equivalent:*

(1) *A is stable;*
(2) $\operatorname{tr} A < 0$, $\det A > 0$*;*
(3) *the trivial solution to $\dot{x} = Ax$ is asymptotically stable.*

Definition 2.3. The pair (A,B) is controllable if $\langle A|B\rangle = \mathbb{R}^2$. The pair (A,C) is observable if the pair $(A^\top, C^\top)$ is controllable.

Lemma 2.4. (I) *The following conditions are equivalent:*

(1) *the pair (A,B) is controllable;*
(2) $\operatorname{rank}(B\ AB) = 2$*;*
(3) *for all $\Lambda \in \mathbb{C}_s^2$, there exists $F \in M(\ell,2)$ such that $\sigma(A+BF) = \Lambda$.*

(II) *The following conditions are equivalent:*

(1) *the pair (A,C) is observable;*
(2) $\operatorname{rank}\begin{pmatrix} C \\ CA \end{pmatrix} = 2$*;*
(3) *for all $\Lambda \in \mathbb{C}_s^2$, there exists $F \in M(2,m)$ such that $\sigma(A+FC) = \Lambda$.*

Lemma 2.5. *If* $\operatorname{rank} B \geq 2$, *then* (A,B) *is controllable, and if* $\operatorname{rank} C \geq 2$, *then* (A,C) *is observable.*

Definition 2.6. The pair (A,B) is called stabilizable if there exists $F \in M(\ell,2)$ such that the matrix $A+BF$ is stable.

The pair (A,C) is called detectable if there exists $F \in M(2,m)$ such that the matrix $A+FC$ is stable.

Lemma 2.7. *The pair (A,B) is stabilizable if and only if* $\ker p_A^+(A) \subset \langle A|B\rangle$.

Lemma 2.8. *The pair (A,C) is detectable if and only if $(A^\top, C^\top)$ is stabilizable.*

Lemma 2.9. *If the pair (A,B) is controllable, then (A,B) is stabilizable, and if the pair (A,C) is observable, then (A,C) is detectable.*

Remark 2.10. We point out that the converse to Lemma 2.9 is not true in general. Indeed, for the matrices $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $B = (1\ 0)^\top$, the pair (A,B) is stabilizable but not controllable and the pair $(A,B^\top)$ is detectable but not observable.

3. Definitions of linear hybrid feedback controls and hybrid feedback stabilization

Definition 3.1. By a discrete automaton we mean in the sequal a 6-tuple $\Delta = (Q, I, \mathcal{M}, T, j, q_0)$, where

(i) Q is a finite set of all possible automaton states (locations);
(ii) the finite set I contains the input alphabet;
(iii) the transition map $\mathcal{M} : Q \times I \to Q$ indicates the location after a transition time, based on the previous location q and input $i \in I$ at the time of transition;
(iv) $T : Q \to (0,\infty)$ is a mapping which sets a period $T(q)$ between transitions times;
(v) $j : \mathbb{R}^m \to I$ is a function with property $j(\lambda y) = j(y)$, $y \in \mathbb{R}^m$, $\lambda > 0$;
(vi) $q_0 = q(0)$ is the state of the automaton at the initial time.

In [1, 4], a similar definition (without condition (v)) is considered. We add (v) to the standard requirements as we are going to use LHFCs in this particular paper (see Definition 3.2).

Intuitively, the automaton follows the output y and uses this information to determine switching times and the values of the new continuous piece of the control function.

For any automaton Δ satisfying (i)–(vi), we can iteratively define a special feedback operator F_Δ. Given $y : [0,\infty) \to \mathbb{R}^m$, the function $F_\Delta y : [0,\infty) \to Q$ is defined by the following:

(1) $(F_\Delta y)(0) = q_0$, $t_1 = T(q_0)$, $(F_\Delta y)(t) \equiv q_0$, $t \in [0,t_1)$;
(2) $(F_\Delta y)(t_1) = \mathcal{M}(q_0, j(y(t_1))) := q(t_1)$, $t_2 = t_1 + T(q(t_1))$, $(F_\Delta y)(t) \equiv q(t_1)$, $t \in [t_1,t_2)$;
(3) if $t_1,\dots,t_k$ and the values $(F_\Delta y)(t)$ for $t \in [0,t_k)$ are already known, then t_{k+1} and $(F_\Delta y)(t)$ are defined for $t \in [t_k,t_{k+1})$ by the equalities

$$\begin{gathered}(F_\Delta y)(t_k) = \mathcal{M}\big(q(t_{k-1}), j(y(t_k))\big) := q(t_k), \qquad t_{k+1} = t_k + T\big(q(t_k)\big), \\ (F_\Delta y)(t) \equiv q(t_k), \quad t \in [t_k, t_{k+1}).\end{gathered} \tag{3.1}$$

The sequence $\{t_k\}_{k=0}^\infty$ ($t_0 = 0$), constructed in the definition of $F_\Delta y$, determines when the automaton should switch between locations. Note that the sequence $\{t_k\}$ is allowed to depend on the output function $y(\cdot)$.

Definition 3.2. The pair $(\Delta, \{G_q\})$, where Δ is a discrete automaton and $\{G_q \mid q \in Q\} \subset M(\ell, m)$, will be addressed as an LHFC; dependence between the control function $u(\cdot)$ and the output function $y(\cdot)$ is defined by $u(t) = G_{q(t_k)} y(t)$, $t \in [t_k,t_{k+1})$ and $k = 0,1,\dots$, where $\{t_k\}_{k=0}^\infty$ is the corresponding sequence of the switching times.

The set of all LHFCs will in the sequel be denoted by $\mathcal{LH}$, *while* $u = (\Delta, \{G_q\}) \in \mathcal{LH}$ will stand for a specific control. According to Definition 3.2, system (1.1), governed by a control $u = (\Delta, \{G_q\}) \in \mathcal{LH}$ (in short, the u-governed system (1.1)), is equivalent to the nonlinear functional differential equation

$$\dot{x}(t) = \left(A + BG_{(F_\Delta Cx)(t)}C\right)x(t), \quad t \in [0, \infty). \tag{3.2}$$

The dynamics of system (1.1), governed by an LHFC u, is a triple $H(t) = (x(t), q(t), \tau(t))$, where $x(\cdot)$ is a solution to (1.1), $q(t)$ is the automaton's location at instance t, and $\tau(t)$ is the time remaining till the next transition instance (see [1]). The function $H(\cdot) : [0, \infty) \to \mathbb{R}^2 \times Q \times (0, \infty)$ is also called a *hybrid trajectory* of system (1.1).

Typical switching procedures (with examples) for systems with LHFC are described in [1, 4] in detail. In [4], some general properties of hybrid trajectories for linear and nonlinear finite-dimensional systems are discussed. In the same paper, one can find a review of the authors' results on some properties of the hybrid dynamics.

We mention here the main existence result from [4], which has a direct relevance to system (1.1) governed by a hybrid feedback control.

Lemma 3.3. *For any $u \in \mathcal{LH}$ and for any $\alpha \in \mathbb{R}^2$, there exists the unique hybrid trajectory $(x(\cdot), q(\cdot), \tau(\cdot))$ of the u-governed system (1.1) with the property $x(0) = \alpha$ (evidently, $x \equiv 0$ if $\alpha = 0$).*

In the sequel, we define by $\mathcal{LH}_1 \subset \mathcal{LH}$ the class of those LHFCs, for which Q contains only one point. Clearly, the class $\mathcal{LH}_1$ can naturally be identified with the class of ordinary linear feedback controls of the form $u = Gy$ with G being an appropriate matrix.

Definition 3.4 [1]. System (1.1) is said to be stabilizable by a control $u \in \mathcal{LH}$ (u-stab.) if the trivial solution to (1.1) is uniformly asymptotically stable. In other words,

(a) for any $\varepsilon > 0$, there is $\delta > 0$ such that every solution $x(\cdot)$ with the property $|x(0)| < \delta$ satisfies the estimate $|x(t)| < \varepsilon$ for $t \geq 0$;
(b) for every solution $x(\cdot)$, $|x(t)| \to 0$ as $t \to \infty$, the convergence being uniform with respect to initial points $x(0) \in K$ for any bounded $K \subset \mathbb{R}^2$.

Definition 3.5. Let $\mathcal{U} \subset \mathcal{LH}$. System (1.1) is called $\mathcal{U}$-stabilizable ($\mathcal{U}$-stab.) if there exists $u \in \mathcal{U}$ such that (1.1) is u-stab. A matrix triple (A_1, B_1, C_1) is called u-stab. or $\mathcal{U}$-stab. if the corresponding system (1.1) with $A = A_1$, $B = B_1$, and $C = C_1$ is u-stab. or $\mathcal{U}$-stab.

4. Some elementary and well-known facts about hybrid stabilization

Lemma 4.1. *Putting* $\operatorname{rank} B = \ell_1$ *and* $\operatorname{rank} C = m_1$, *let $B_1 \in M(2, \ell_1)$ be a matrix consisting of ℓ_1 linearly independent columns of the matrix B, and $C_1 \in M(m_1, 2)$ a matrix consisting of m_1 linearly independent rows of the matrix C. Then the following statements are valid.*

(1) *For all $G \in M(\ell, m)$, there exists the unique matrix $G_1 \in M(\ell_1, m_1)$ such that*

$$BGC = B_1 G_1 C_1. \tag{4.1}$$

Conversely, for all $G_1 \in M(\ell_1, m_1)$, there exists $G \in M(\ell, m)$ such that (4.1) is valid.

(2) *The triple* (A,B,C) *is* $\mathcal{LH}$*-stab. (resp.,* $\mathcal{LH}_1$*-stab.) if and only if* (A,B_1,C_1) *is* $\mathcal{LH}$*-stab. (resp.,* $\mathcal{LH}_1$*-stab.).*

The first statement of the lemma is just a simple exercise from the matrix algebra, while the second statement is a straightforward corollary from the first if one takes into account the definition of the classes $\mathcal{LH}$ and $\mathcal{LH}_1$ in Section 3.

Lemma 4.2. *The triple* (A,B,C) *is* $\mathcal{LH}_1$*-stab. if and only if there exists* $G \in M(\ell,m)$ *such that the matrix* $A+BGC$ *is stable.*

Corollary 4.3. *Let* $B = (b_1\ b_2{}^\top) \neq 0$ *and* $C = (c_1\ c_2) \neq 0$*. Then the triple* (A,B,C) *is* $\mathcal{LH}_1$*-stab. if and only if there exists* $\alpha \in \mathbb{R} : \sigma(A+\alpha BC) \subset \mathbb{C}_-$*.*

Corollary 4.4. *Assume that one of the following statements is valid:*

(1) *the pair* (A,B) *is stabilizable and* $\operatorname{rank} C = 2$*,*
(2) *the pair* (A,C) *is detectable and* $\operatorname{rank} B = 2$*.*

Then (A,B,C) *is* $\mathcal{LH}_1$*-stab.*

Proof. Suppose that the first statement is valid. By Lemma 4.1, one can then assume that $C \in M(2,2)$, $\det C \neq 0$. By Definition 2.6, there exists $F \in M(\ell,2)$ such that the matrix $A+BF$ is stable. Then $A+BGC$ is stable, where $G = FC^{-1}$. According to Lemma 4.2, (A,B,C) is $\mathcal{LH}_1$-stab. Case (2) can be treated similarly. □

Corollary 4.5. *If* $\operatorname{rank} B \geq 2$ *and* $\operatorname{rank} C \geq 2$*, then* (A,B,C) *is* $\mathcal{LH}_1$*-stab.*

In [2, 3], the following result is proved.

Theorem 4.6. *Let* $b = (b_1\ b_2)^\top \neq 0$ *and* $c = (c_1\ c_2) \neq 0$*. The triple* (A,B,C) *is* $\mathcal{LH}$*-stab. if and only if there exists* $\alpha \in \mathbb{R} : \sigma(A+\alpha BC) \cap [0,\infty) = \varnothing$ *(in other words,* $A+\alpha BC$ *does not have nonnegative real eigenvalues).*

The results of this section show that if we wish to construct an algorithm which would test whether a given triple (A,B,C) provides $\mathcal{LH}_1$-stabilizability or $\mathcal{LH}$-stabilizability of system (1.1), then we need to do the following:

(1) study the cases when the pair (A,B) is not stabilizable or the pair (A,C) is not detectable,
(2) find efficient algorithms for verifying the assumptions of Corollary 4.3 and Theorem 4.6.

5. The cases where (A,B) is not stabilizable and (A,C) is not detectable

Everywhere in Sections 5, 6, and 7, excluding Theorem 5.8, we assume that $B = (b_1\ b_2)^\top \neq 0$ and $C = (c_1\ c_2) \neq 0$.

Lemma 5.1. *The pair* (A,B) *is controllable if and only if for all* $\lambda \in \sigma(A) \cap \mathbb{R}$*,* $B \notin \ker(A-\lambda I)$*.*

Proof. By Lemma 2.4, the pair (A,B) is not controllable if and only if $\det(B\ AB) = 0$, which implies that $\lambda \in \sigma(A) \cap \mathbb{R}$, $B \in \ker(A-\lambda I)$. □

Corollary 5.2. *If* $\operatorname{Im}\lambda_i \neq 0$, *then* (A,B) *is controllable.*

Lemma 5.3. *For* $\lambda_1 \geq 0$, (A,B) *is controllable if and only if* (A,B) *is stabilizable.*

Proof. For $\lambda_1 \geq 0, [p_A^+(A) = p_A(A) = \Theta] \Rightarrow [\ker p_A^+(A) = \mathbb{R}^2]$. If (A,B) is stabilizable, then the latter implication and Lemma 2.7 give the relation $\langle A|B\rangle = \mathbb{R}^2$. According to Definition 2.3, the pair (A,B) is therefore controllable. The converse statement follows easily from Lemma 2.9. □

Lemma 5.4. *Let* $\lambda_1 < 0 \leq \lambda_2$. *Then the following statements are true:*

(1) $[(A,B)$ *is controllable*$] \Leftrightarrow [B \notin \ker(A-\lambda_i I),\ i=1,2]$;
(2) $[(A,B)$ *is not controllable, but stabilizable*$] \Leftrightarrow [B \in \ker(A-\lambda_2 I) \setminus \ker(A-\lambda_1 I)]$;
(3) $[(A,B)$ *is not stabilizable*$] \Leftrightarrow [B \in \ker(A-\lambda_1 I) \setminus \ker(A-\lambda_2 I)]$.

Proof. The first statement follows from Lemma 5.1. The case $B \in \ker(A-\lambda_i I)$, $i=1,2$, is irrelevant. Indeed, under this assumption, we get $(\lambda_1-\lambda_2)B = 0$, which contradicts the conditions $\lambda_1 \neq \lambda_2$ and $B \neq 0$.

Let $B \in \ker(A-\lambda_1 I) \triangle \ker(A-\lambda_2 I)$. Then $\det(\begin{smallmatrix} B & AB \end{smallmatrix}) = 0$, which implies that $\langle A|B\rangle = \operatorname{Span}\{B\}$. Taking into account that $p_A^+(\lambda) = \lambda - \lambda_2$ and using Lemma 2.7, we obtain that $[(A,B)$ is stabilizable $] \Leftrightarrow [\ker(A-\lambda_2 I) = \operatorname{Span}\{B\}] \Leftrightarrow [B \in \ker(A-\lambda_2 I)]$. □

Lemmas 2.4, 2.8, 5.1, 5.3, and 5.4 yield the following theorem.

Theorem 5.5. *For the matrices A, B, C from (1.1),* $[(A,B)$ *is not stabilizable* $] \Leftrightarrow [(\lambda_1 \geq 0, \det(\begin{smallmatrix} B & AB \end{smallmatrix}) = 0) \vee (\lambda_1 < 0 \leq \lambda_2,\ AB = \lambda_1 B)]$; $[(A,C)$ *is not detectable*$] \Leftrightarrow [(\lambda_1 \geq 0, \det\begin{pmatrix} C \\ CA \end{pmatrix} = 0) \vee (\lambda_1 < 0 \leq \lambda_2,\ CA = \lambda_1 C)]$.

Remark 5.6. The condition $\lambda_1 \geq 0$ is equivalent to $\operatorname{tr} A \geq 0$ and $\operatorname{tr}^2 A \geq 4\det A \geq 0$, and the condition $\lambda_1 < 0 \leq \lambda_2$ is equivalent to $[\det A < 0] \vee [\det A = 0$ and $\operatorname{tr} A < 0]$.

Lemma 5.7. *If the pair* (A,B) *is not controllable, then for all* $F = (\begin{smallmatrix} f_1 & f_2 \end{smallmatrix})$, $\sigma(A+BF) = \{\lambda^*, \lambda_j\}$ *for some* $j \in \{1,2\}$, *where* $\lambda^* = \lambda_i + FB$, $i \neq j$; *in this case* $B \in \ker(A+BF-\lambda^* I)$.

Proof. By virtue of Lemma 5.4, $B \in \ker(A-\lambda_i I)$ for at least one $i \in \{1,2\}$. Let $\lambda^* = \lambda_i + FB$. Then $(A+BF-\lambda^* I)B = (A-\lambda_i I)B = 0$, that is, $\lambda^* \in \sigma(A+BF)$ and $B \in \ker(A+BF-\lambda^* I)$.

Clearly, the zeros λ_1 and λ_2 of the polynomial $\pi_A(\lambda)$ and the zeros λ^* and λ^{**} of the polynomial $\pi_{A+BF}(\lambda)$ are related to each other in the following way: $\lambda_1 + \lambda_2 = \operatorname{tr} A$ and $\lambda^* + \lambda^{**} = \operatorname{tr}(A+BF)$. These equalities imply that $\lambda^* + \lambda^{**} = \operatorname{tr} A + FB = \lambda_1 + \lambda_2 + FB = \lambda^* + \lambda_j$, where $j \neq i$. Thus, $\lambda^{**} = \lambda_j$. □

Theorem 5.8. *Let* $A \in M(2,2)$, $B \in M(2,\ell)$, $C \in M(m,2)$, *and either the pair* (A,B) *is not stabilizable or the pair* (A,C) *is not detectable. Then* (A,B,C) *is not* $\mathscr{LH}$*-stab.*

Proof. Assume that (A,B) is not stabilizable. By virtue of Lemmas 2.5 and 2.9, one has that $\operatorname{rank} B \leq 1$, and according to Lemma 4.1, one can assume, without loss of generality, that $B = (\begin{smallmatrix} b_1 & b_2 \end{smallmatrix})^\top$. If $B = 0$, then the statement of the theorem is evident. Let $B \neq 0$. By Theorem 5.5, only one of the following cases can occur:

(1) $0 \leq \lambda_1 < \lambda_2$;
(2) $\lambda_1 < 0 \leq \lambda_2$;

(3) $\lambda := \lambda_1 = \lambda_2 \geq 0$, $\ker(A - \lambda I) = \mathbb{R}^2$;
(4) $\lambda := \lambda_1 = \lambda_2 \geq 0$, $\dim\ker(A - \lambda I) = 1$.

Using Lemma 5.4, one can easily show that in cases (2) and (3),

$$B \in \ker(A - \lambda_1 I), \qquad \ker(A - \lambda_2 I) \setminus \operatorname{Span}\{B\} \neq \varnothing. \tag{5.1}$$

In case (1), either (5.1) or its counterpart, where λ_1 and λ_2 are interchanged, is true. We assume, with no loss of generality, that (5.1) holds true in case (1).

We prove the theorem for cases (1), (2), and (3) simultaneously by choosing $D \in \mathbb{R}^2$ so that

$$B \in \ker(A - \lambda_1 I), \qquad D \in \ker(A - \lambda_2 I) \setminus \operatorname{Span}\{B\}. \tag{5.2}$$

Let $P : \mathbb{R}^2 \to \mathbb{R}$ be the projector onto the subspace $\operatorname{Span}\{D\}$ along the subspace $\operatorname{Span}\{B\}$ so that $PB = 0$, $PD = D$, $(I - P)B = B$, and $(I - P)D = 0$.

We choose an arbitrary $F \in M(1,2)$ and consider a trajectory $x(\cdot)$ of the equation

$$\dot{x} = (A + BF)x \tag{5.3}$$

such that $Px(t_0) \neq 0$ at some instance $t_0 \geq 0$.

Evidently,

$$\delta(t_0) := \frac{d|Px|}{dt}\bigg|_{t=t_0} = \frac{1}{|Px|}\langle Px, P\dot{x}\rangle\bigg|_{t=t_0} = \frac{1}{|Px|}\langle Px, P(A+BF)x\rangle\bigg|_{t=t_0}. \tag{5.4}$$

Conditions (5.2) imply that for some $\mu \in \mathbb{R}$,

$$\begin{aligned} P(A + BF)x &= PAx = PA(Px + (I - P)x) \\ &= PAPx + \mu PAB = \lambda_2 PX + \mu\lambda_1 PB = \lambda_2 PX. \end{aligned} \tag{5.5}$$

Then (5.4) and $\lambda_2 \geq 0$ yield

$$\delta(t_0) = \frac{1}{|Px|} \cdot \lambda_2 \cdot |Px|^2 \bigg|_{t=t_0} = \lambda_2 |Px(t_0)| \geq 0. \tag{5.6}$$

The last relation can be used to verify the following properties:

$$|Px(\cdot)| \text{ does not decrease on } [t_0, \infty), \quad |x(t)| \geq \beta |Px(t_0)|, \quad t \geq t_0, \tag{5.7}$$

where $\beta > 0$ does not depend on F.

We now fix some $u = (\Delta, \{G_q\}) \in \mathscr{LH}$ and consider an arbitrary trajectory $(x(\cdot), q(\cdot), \tau(\cdot))$ of the u-governed system (1.1) with $\alpha := |Px(0)| \neq 0$. Let $\{t_n\}_{n=0}^{\infty}$ be the corresponding sequence of the automaton's switching instances, $t_0 = 0$. Then for all $[t_n, t_{n+1})$, $n \in \mathbb{N} \cup \{0\}$, the first component $x(\cdot)$ of the hybrid trajectory satisfies (5.3), where $F = G_q C$

for some $q \in Q$. Due to (5.7),

$$|x(t)| \geq \beta|Px(t_n)| \geq \beta|Px(0)| = \alpha\beta > 0, \quad t \in [t_n, t_{n+1}). \tag{5.8}$$

As n is arbitrary, $x(t) \nrightarrow 0$, $t \to \infty$. Thus, the theorem is proved for cases (1), (2), and (3).

The proof of case (4) will be divided into two parts.

(a) If $CB \neq 0$, then there exists $G \in M(1,m)$ such that $GBC \neq 0$. By Lemma 5.7, $\sigma(A_G) = \{\lambda, \lambda^*\}$, where $A_G = A + BGC$ and $\lambda^* = \lambda + GCB \neq \lambda$. Clearly, the pair (A_G, B) is not stabilizable. It has already been proved that in case (1) or (2) the triple (A_G, B, C) is not $\mathscr{L}\mathscr{H}$-stab. According to Definitions 3.2 and 3.5, the triple (A,B,C) is not $\mathscr{L}\mathscr{H}$-stab. either.

(b) If $CB = 0$, then (see Lemma 5.7) for all $G \in M(1,m)$, one has $B \in \ker(A_G - \lambda I)$. Hence, the solution to $\dot{x} = A_G x$, $x(0) = B$ is given by $x(t) = e^{A_G t}B = e^{\lambda t}B$, $t \geq 0$. The last equality does not depend on G, so that it constitutes a solution to the u-governed system (1.1) satisfying $x(0) = B$ for any $u \in \mathscr{L}\mathscr{H}$. Since $\lambda \geq 0$, system (1.1) is not $\mathscr{L}\mathscr{H}$-stab.

The theorem can be proved in a similar manner if (A,C) is not detectable. □

6. Efficient criteria for $\mathscr{L}\mathscr{H}_1$-stabilizability of the triple (A,B,C)

Everywhere in Sections 6 and 7, it is assumed that $A \in M(2,2)$, $B \in M(2,1) \neq 0$, and $C \in M(1,2) \neq 0$. In these two sections, we will use the following notation: $A_\alpha = A + \alpha BC$, $\alpha \in \mathbb{R}$, $\omega = \operatorname{tr} A - CAB/CB$ if $CB \neq 0$ (we assume also that ω is not defined if $CB = 0$).

Lemma 6.1. *Let $a,b,c,d \in \mathbb{R}$. The system of inequalities*

$$a + b \cdot \alpha < 0, \qquad c + d \cdot \alpha > 0 \tag{6.1}$$

is solvable with respect to $\alpha \in \mathbb{R}$ if and only if one of the following conditions holds:

(1) $b = d = 0$, $a < 0$, $c > 0$;
(2) $b = 0$, $d \neq 0$, $a < 0$;
(3) $d/b < 0$;
(4) $d/b \geq 0$, $c > ad/b$.

Theorem 6.2. *The triple (A,B,C) is $\mathscr{L}\mathscr{H}_1$-stab. if and only if one of the following conditions holds:*

(1) $CB = CAB = 0$, $\operatorname{tr} A < 0$, $\det A > 0$;
(2) $CB = 0$, $CAB \neq 0$, $\operatorname{tr} A < 0$;
(3) $\omega < 0$;
(4) $\omega \geq 0$, $\det A > \operatorname{tr} A \cdot \omega$.

Proof. Corollary 4.3 and Lemma 2.2 imply that

$$[(A,B,C) \text{ is } \mathscr{L}\mathscr{H}_1\text{-stab.}] \Longleftrightarrow [\exists \alpha \in \mathbb{R} : \operatorname{tr} A_\alpha < 0 \text{ and } \det A_\alpha > 0]. \tag{6.2}$$

Simple direct calculations yield

$$\operatorname{tr} A_\alpha = \operatorname{tr} A + \alpha CB, \qquad \det A_\alpha = \det A + \alpha(\operatorname{tr} A \cdot CB - CAB). \tag{6.3}$$

Thus, the conditions in the right-hand side of (6.2) are equivalent to the solvability of (6.1) with respect to $\alpha \in \mathbb{R}$, where $a = \operatorname{tr} A$, $b = CB$, $c = \det A$, and $d = \operatorname{tr} A \cdot CB - CAB$. Referring to Lemma 6.1 completes the proof. □

Corollary 6.3. *Assume that the matrix A is not stable. Then the triple (A,B,C) is $\mathcal{LH}_1$-stab. if and only if one of conditions (2), (3), and (4) of Theorem 6.2 is fulfilled.*

7. Efficient criteria for $\mathcal{LH}$-stabilizability in terms of the triple (A,B,C)

The equality $\pi_A(\lambda) = \lambda^2 - \operatorname{tr} A \cdot \lambda + \det \lambda$ implies the following lemma.

Lemma 7.1. *For a 2×2 matrix A, $\sigma(A) \cap [0,\infty) = \varnothing$ if and only if one of the following conditions holds:*

(a) $\operatorname{tr} A < 0$, $\det A > 0$,
(b) $\operatorname{tr}^2 A - 4\det A < 0$.

Lemma 7.1, Theorem 4.6, and relation (6.2) yield the following corollary.

Corollary 7.2. *For the matrices A,B,C from (1.1), $[(A,B,C)$ is $\mathcal{LH}_1$-stab.$]$ $\Leftrightarrow$ $[$there exists $\alpha \in \mathbb{R} : \operatorname{tr} A_\alpha < 0,\ \det A_\alpha > 0]$; $[(A,B,C)$ is $\mathcal{LH}$-stab.$]$ $\Leftrightarrow$ $[$there exists $\alpha \in \mathbb{R} : (\operatorname{tr} A_\alpha < 0,$ $\det A_\alpha > 0) \vee (\operatorname{tr}^2 A_\alpha - 4\det A_\alpha < 0)]$.*

The main result of the paper is the following criterion.

Theorem 7.3. *The triple (A,B,C) is $\mathcal{LH}$-stab. if and only if one of the following conditions is fulfilled:*

(1) $CB = 0$, $CAB = 0$, $\operatorname{tr} A < 0$, $\det A > 0$;
(2) $CB = 0$, $CAB \neq 0$;
(3) $\omega < 0$;
(4) $\omega \geq 0$, $\det A > (CAB/CB) \cdot \omega$.

Proof. By (6.3), we have for all $\alpha \in \mathbb{R}$ that

$$f(\alpha) := \operatorname{tr}^2 A_\alpha - 4\det A_\alpha = (CB)^2\alpha^2 - 2(CB \cdot \operatorname{tr} A - 2CAB)\alpha + \operatorname{tr}^2 A - 4\det A. \tag{7.1}$$

Consider the inequality

$$f(\alpha) < 0 \tag{7.2}$$

in some special situations.

(a) If $CB = 0$ and $CAB \neq 0$, then (7.2) is equivalent to the inequality

$$4CAB \cdot \alpha + \operatorname{tr}^2 A - 4\det A < 0 \tag{7.3}$$

which is clearly solvable with respect to $\alpha \in \mathbb{R}$.

(b) If $CB \neq 0$, then the solvability of (7.2) is equivalent to the positivity of the discriminant $4d$ of the quadratic equation $f(\alpha) = 0$, that is, to the condition

$$d = 4(CAB)^2 - 4CAB \cdot \operatorname{tr} A \cdot CB + 4\det A \cdot (CB)^2 > 0. \tag{7.4}$$

The last inequality, in turn, is equivalent to

$$\det A > \frac{CAB}{CB} \cdot \omega. \tag{7.5}$$

Now we are able to continue the proof of the theorem.

(1) Let $CB = CAB = 0$. Due to Theorem 4.6, Lemma 7.1, and relation (6.3), $\mathscr{LH}$-stabilizability of (A,B,C) is equivalent to the condition $\operatorname{tr} A < 0$, $\det A > 0$.

(2) Let $CB = 0$ and $CAB \neq 0$. Because of (a), the triple (A,B,C) is $\mathscr{LH}$-stab.

(3) Let $\omega < 0$. By Theorem 6.2, (A,B,C) is $\mathscr{LH}$-stab.

(4) Let $\omega \geq 0$. By Corollary 7.2, Theorem 6.2, and (b), $\mathscr{LH}$-stabilizability of the triple (A,B,C) is equivalent either to (7.5) or to

$$\det A \geq \operatorname{tr} A \cdot \omega. \tag{7.6}$$

Moreover, $\omega = \operatorname{tr} A - CAB/CB \geq 0$ guarantees the implication (7.6)⇒(7.5). Thus, $\mathscr{LH}$-stabilizability of (A,B,C) is equivalent to (7.5). □

The next two theorems follow directly from Theorems 6.2 and 7.3 and Corollary 7.2.

THEOREM 7.4. *The triple (A,B,C) is not $\mathscr{LH}_1$-stab. but $\mathscr{LH}$-stab. if and only if one of the following conditions is satisfied:*

(1) $CB = 0$, $CAB \neq 0$, $\operatorname{tr} A \geq 0$;
(2) $\omega > 0$, $CAB/CB < \det A/\omega \leq \operatorname{tr} A$.

THEOREM 7.5. *Assume that (A,B,C) is not $\mathscr{LH}_1$-stab. Then (A,B,C) is $\mathscr{LH}$-stab. if and only if one of the following conditions is satisfied:*

(1) $CB = 0$, $CAB \neq 0$;
(2) $\det A > CAB/CB \cdot \omega$.

8. A detailed algorithm which tests $\mathscr{LH}_1$- and $\mathscr{LH}$-stabilizability of the triple (A,B,C)

First of all, we introduce some new notation:

(i) $\mathbf{1} = \{(A,B,C) \mid A \in M(2,2), B \in M(2,\ell), C \in M(m,2) \text{ for some } \ell, m \in \mathbb{N}\}$,
(ii) $\mathbf{LH_0} = \{\Omega \in \mathbf{1} \mid A \text{ is stable}\}$,
(iii) $\mathbf{LH_1} = \{\Omega \in \mathbf{1} \mid A \text{ is not stable, } \Omega \text{ is } \mathscr{LH}_1\text{-stab.}\}$,
(iv) $\mathbf{LH} = \{\Omega \in \mathbf{1} \mid \Omega \text{ is not } \mathscr{LH}_1\text{-stab., but is } \mathscr{LH}\text{-stab.}\}$,
(v) $\mathbf{LH_-} = \{\Omega \in \mathbf{1} \mid \Omega \text{ is not } \mathscr{LH}\text{-stab.}\}$.

Evidently, $\mathbf{1} = \mathbf{LH_0} \sqcup \mathbf{LH_1} \sqcup \mathbf{LH} \sqcup \mathbf{LH_-}$.

Algorithm 8.1 tests if a given triple $\Omega = (A,B,C) \in \mathbf{1}$ belongs to one of the classes $\mathbf{LH_0}$, $\mathbf{LH_1}$, $\mathbf{LH}$, and $\mathbf{LH_-}$.

Remark 8.1. The items **1**(YES) and **3**(YES) of the algorithm follow from Lemma 2.2 and Corollary 4.5, respectively. The items **5**(YES) and **7**(YES) are implied by Theorems 5.5

1 Condition: $\operatorname{tr} A < 0$, $\det A > 0$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_0} \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
2 Condition: B or C is the zero matrix.
(YES) $\longrightarrow \Omega \in \mathbf{LH_-} \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
3 Condition: $\operatorname{rank} B \geq 2$, $\operatorname{rank} C \geq 2$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
(NO) $\longrightarrow$ find $\sigma(A) = \{\lambda_1, \lambda_2\} \longrightarrow$ CONTINUE
4 Condition: $\operatorname{rank} B = 1$.
(YES) $\longrightarrow$ find a nonzero column b
of the matrix $B \longrightarrow$ CONTINUE
(NO) $\longrightarrow$ find a nonzero row c
of the matrix $C \longrightarrow$ go to **7**
5 Condition: $[\lambda_1 \geq 0, \lambda_2 \geq 0, \det(\, b \;\; Ab\,) = 0] \vee [\lambda_1 < 0 \leq \lambda_2, Ab = \lambda_1 b]$
$\vee [\lambda_2 < 0 \leq \lambda_1, Ab = \lambda_2 b]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_-} \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
6 Condition: $\operatorname{rank} C \geq 2$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
(NO) $\longrightarrow$ find a nonzero row c
of the matrix $C \longrightarrow$ go to **8**
7 Condition: $\left[\lambda_1 \geq 0, \lambda_2 \geq 0, \det\begin{pmatrix} c \\ cA \end{pmatrix} = 0\right] \vee [\lambda_1 < 0 \leq \lambda_2, cA = \lambda_1 c]$
$\vee [\lambda_2 < 0 \leq \lambda_1, cA = \lambda_2 c]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_-} \longrightarrow$ END
(NO) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
8 Condition: $[cb = 0, cAb \neq 0, \operatorname{tr} A < 0] \vee [\operatorname{tr} A - cAb/cb < 0]$
$\vee [\operatorname{tr} A - cAb/cb \geq 0, \det A > \operatorname{tr} A(\operatorname{tr} A - cAb/cb)]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH_1} \longrightarrow$ END
(NO) $\longrightarrow$ CONTINUE
9 Condition: $[cb = 0, cAb \neq 0] \vee [\det A > (cAb/cb)(\operatorname{tr} A - cAb/cb)]$.
(YES) $\longrightarrow \Omega \in \mathbf{LH} \longrightarrow$ END
(NO) $\longrightarrow \Omega \in \mathbf{LH_-} \longrightarrow$ END.

Algorithm 8.1

and 5.8, and the items **6**(YES) and **7**(NO) can be obtained from Corollary 4.4. In the items **4**(YES) and **6**(NO), we use Lemma 4.1 to reduce the stabilization problem for the triple (A,B,C) to that for either (A,b,C) or (A,b,c), where b is a nonzero column of B and c is a nonzero row of C ($B = b$ and $C = c$ if $\ell = m = 1$). For **8**(YES), the statement follows from Corollary 6.3. Finally, **9** is implied by Theorem 7.5.

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WHICH SOLUTIONS OF THE THIRD PROBLEM FOR THE POISSON EQUATION ARE BOUNDED?

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This paper deals with the problem $\Delta u = g$ on G and $\partial u/\partial n + uf = L$ on ∂G. Here, $G \subset \mathbb{R}^m$, $m > 2$, is a bounded domain with Lyapunov boundary, f is a bounded nonnegative function on the boundary of G, L is a bounded linear functional on $W^{1,2}(G)$ representable by a real measure μ on the boundary of G, and $g \in L_2(G) \cap L_p(G)$, $p > m/2$. It is shown that a weak solution of this problem is bounded in G if and only if the Newtonian potential corresponding to the boundary condition μ is bounded in G.

Suppose that $G \subset \mathbb{R}^m$, $m > 2$, is a bounded domain with Lyapunov boundary (i.e., of class $C^{1+\alpha}$). Denote by $n(y)$ the outer unit normal of G at y. If $f,g,h \in C(\partial G)$ and $u \in C^2(\operatorname{cl} G)$ is a classical solution of

$$\begin{aligned} \Delta u &= g \quad \text{on } G, \\ \frac{\partial u}{\partial n} + uf &= h \quad \text{on } \partial G, \end{aligned} \tag{1}$$

then Green's formula yields

$$\int_G \nabla u \cdot \nabla v \, d\mathcal{H}_m + \int_{\partial G} u f v \, d\mathcal{H}_{m-1} = \int_{\partial G} h v \, d\mathcal{H}_{m-1} - \int_G g v \, d\mathcal{H}_m \tag{2}$$

for each $v \in \mathcal{D}$, the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$. Here, ∂G denotes the boundary of G and $\operatorname{cl} G$ is the closure of G; $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. Denote by $\mathcal{D}(G)$ the set of all functions from $\mathcal{D}$ with the support in G.

For an open set $V \subset \mathbb{R}^m$, denote by $W^{1,2}(V)$ the collection of all functions $f \in L_2(V)$, the distributional gradient of which belongs to $[L_2(V)]^m$.

Definition 1. Let $f \in L_\infty(\mathcal{H})$, $g \in L_2(G)$ and let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathcal{D}(G)$. We say that $u \in W^{1,2}(G)$ is a weak solution

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in $W^{1,2}(G)$ of the third problem for the Poisson equation

$$\begin{aligned} \Delta u = g \quad &\text{on } G, \\ \frac{\partial u}{\partial n} + uf = L \quad &\text{on } \partial G, \end{aligned} \tag{3}$$

if

$$\int_G \nabla u \cdot \nabla v \, d\mathcal{H}_m + \int_{\partial G} ufv \, d\mathcal{H} = L(v) - \int_G gv \, d\mathcal{H}_m \tag{4}$$

for each $v \in W^{1,2}(G)$.

Denote by $\mathcal{C}'(\partial G)$ the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm. We say that the bounded linear functional L on $W^{1,2}(G)$ is representable by $\mu \in \mathcal{C}'(\partial G)$ if $L(\varphi) = \int \varphi \, d\mu$ for each $\varphi \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $W^{1,2}(G)$, the operator L is uniquely determined by its representation $\mu \in \mathcal{C}'(\partial G)$.

For $x, y \in \mathbb{R}^m$, denote

$$h_x(y) = \begin{cases} (m-2)^{-1} A^{-1} |x-y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y, \end{cases} \tag{5}$$

where A is the area of the unit sphere in $\mathbb{R}^m$. For the finite real Borel measure ν, denote

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y) \, d\nu(y) \tag{6}$$

the Newtonian potential corresponding to ν, for each x for which this integral has sense.

We denote by $\mathcal{C}'_b(\partial G)$ the set of all $\mu \in \mathcal{C}'(\partial G)$ for which $\mathcal{U}\mu$ is bounded on $\mathbb{R}^m \setminus \partial G$.

Remark that $\mathcal{C}'_b(\partial G)$ is the set of all $\mu \in \mathcal{C}'(\partial G)$ for which there is a polar set M such that $\mathcal{U}\mu(x)$ is meaningful and bounded on $\mathbb{R}^m \setminus M$, because $\mathbb{R}^m \setminus \partial G$ is finely dense in $\mathbb{R}^m$ (see [1, Chapter VII, Sections 2, 6], [7, Theorems 5.10 and 5.11]) and $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set. Remark that $\mathcal{H}_{m-1}(M) = 0$ for each polar set M (see [7, Theorem 3.13]). (For the definition of polar sets, see [4, Chapter 7, Section 1]; for the definition of the fine topology, see [4, Chapter 10].)

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ to ∂G.

Lemma 2. *Let $\mu \in \mathcal{C}'(\partial G)$. Then the following assertions are equivalent:*

(1) $\mu \in \mathcal{C}'_b(\partial G)$,
(2) $\mathcal{U}\mu$ *is bounded in* G,
(3) $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

Proof. (2)⇒(3). Since ∂G is a subset of the fine closure of G by [1, Chapter VII, Sections 2, 6] and [7, Theorems 5.10 and 5.11], $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set M, and $\mathcal{H}_{m-1}(M) = 0$ by [4, Theorem 7.33] and [7, Theorem 3.13], then we obtain that $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

(3)⇒(1). Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. For $z \in G$, denote by μ_z the harmonic measure corresponding to G and z. If $y \in \partial G$ and $z \in G$, then

$$\int_{\partial G} h_y(x) d\mu_z(x) = h_y(z) \tag{7}$$

by [7, pages 264, 299]. Using Fubini's theorem, we get

$$\int \mathcal{U}\mu^+ \, d\mu_z = \int_{\partial G} \int_{\partial G} h_y(x) d\mu_z(x) d\mu^+(y) = \int_{\partial G} h_y(z) d\mu^+(y) = \mathcal{U}\mu^+(z). \tag{8}$$

Similarly, $\int \mathcal{U}\mu^- \, d\mu_z = \mathcal{U}\mu^-(z)$. Since $\mathcal{U}\mu \in L_\infty(\mathcal{H})$, μ_z is a nonnegative measure with the total variation 1 (see [4, Lemma 8.12]) which is absolutely continuous with respect to $\mathcal{H}$ by [2, Theorem 1], then we obtain that $|\mathcal{U}\mu(z)| \le \|\mathcal{U}\mu\|_{L_\infty(\mathcal{H})}$.

If $z \in \mathbb{R}^m \setminus \operatorname{cl} G$, choose a bounded domain V with smooth boundary such that $\operatorname{cl} G \cup \{z\} \subset V$. Repeating the previous reasonings for $V \setminus \operatorname{cl} G$, we get $|\mathcal{U}\mu(z)| \le \|\mathcal{U}\mu\|_{L_\infty(\mathcal{H})}$. □

Lemma 3. *Let $f \in L_\infty(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Then $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$. Moreover, there is a bounded linear functional L on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}'_b(\partial G)$ such that $\mathcal{U}(g\mathcal{H}_m)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation*

$$\Delta u = -g \quad \text{on } G, \qquad \frac{\partial u}{\partial n} + uf = L \quad \text{on } \partial G. \tag{9}$$

Proof. Suppose first that g is nonnegative. Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m)$ by [3, Theorem A.6], the energy $\int g\mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m < \infty$. According to [7, Theorem 1.20], we have

$$\int \left| \nabla \mathcal{U}(g\mathcal{H}_m) \right|^2 d\mathcal{H}_m = \int g\mathcal{U}(g\mathcal{H}_m) \, d\mathcal{H}_m < \infty, \tag{10}$$

and therefore $\mathcal{U}(g\mathcal{H}_m) \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.1.4]).

Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$, $f \in L_\infty(\mathcal{H})$ and the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], then the operator

$$L(\varphi) = \int_G \nabla\varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f \varphi \, d\mathcal{H}_{m-1} - \int_G g\varphi \, d\mathcal{H}_m \tag{11}$$

is a bounded linear functional on $W^{1,2}(G)$.

According to [7, Theorem 4.2], there is a nonnegative $\nu \in \mathcal{C}'(\partial G)$ such that $\mathcal{U}\nu = \mathcal{U}(g\mathcal{H}_m)$ on $\mathbb{R}^m \setminus \operatorname{cl} G$. Choose a bounded domain V with smooth boundary such that $\operatorname{cl} G \subset V$. Since $\mathcal{U}\nu$ is bounded in $V \setminus \operatorname{cl} G \subset \mathbb{R}^m \setminus \operatorname{cl} G$, Lemma 2 yields that $\nu \in \mathcal{C}'_b(\partial(V \setminus \operatorname{cl} G))$. Therefore, $\nu \in \mathcal{C}'_b(\partial G)$. According to [13, Lemma 4], there is $\tilde{\nu} \in \mathcal{C}'_b(\partial G)$ such that

$$\int_{\mathbb{R}^m \setminus \operatorname{cl} G} \nabla\varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m = \int_{\mathbb{R}^m \setminus \operatorname{cl} G} \nabla\varphi \cdot \nabla \mathcal{U}\nu \, d\mathcal{H}_m = \int_{\partial G} \varphi \, d\tilde{\nu} \tag{12}$$

for each $\varphi \in \mathcal{D}$. Let $\mu = \tilde{\nu} - f\mathcal{U}(g\mathcal{H}_m)\mathcal{H}$. Since $\mathcal{U}(f\mathcal{U}(g\mathcal{H}_m)\mathcal{H}) \in \mathcal{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18] and $\mathcal{U}(f\mathcal{U}(g\mathcal{H}_m)\mathcal{H})(x) \to 0$ as $|x| \to \infty$, we have $f\mathcal{U}(g\mathcal{H}_m)\mathcal{H} \in \mathcal{C}'_b(\partial G)$. Therefore, $\mu \in \mathcal{C}'_b(\partial G)$.

If $\varphi \in \mathcal{D}$, then $\varphi = \mathcal{U}((-\Delta\varphi)\mathcal{H}_m)$ by [3, Theorem A.2]. According to [7, Theorem 1.20],

$$\begin{aligned}\int_{\mathbb{R}^m} \nabla\varphi \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m &= \int_{\mathbb{R}^m} \nabla\mathcal{U}((-\Delta\varphi)\mathcal{H}_m) \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m \\ &= \int_{\mathbb{R}^m} g\,\mathcal{U}((-\Delta\varphi)\mathcal{H}_m)\, d\mathcal{H}_m \\ &= \int_{\mathbb{R}^m} g\varphi\, d\mathcal{H}_m.\end{aligned} \tag{13}$$

Since $\mathcal{H}_m(\partial G) = 0$,

$$\begin{aligned}&\int_G \nabla\varphi \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f\varphi\, d\mathcal{H}_{m-1} \\ &\quad = \int_G g\varphi\, d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f\varphi\, d\mathcal{H}_{m-1} \\ &\qquad - \int_{\mathbb{R}^m \setminus \mathrm{cl}\, G} \nabla\varphi \cdot \nabla\mathcal{U}(g\mathcal{H}_m)\, d\mathcal{H}_m \\ &\quad = \int_G g\varphi\, d\mathcal{H}_m + \int_{\partial G} \varphi\, d\mu.\end{aligned} \tag{14}$$

□

Lemma 4. *Let $f \in L_\infty(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Let L be a bounded linear functional on $W^{1,2}(G)$ representable by $\mu \in \mathscr{C}'(\partial G)$. If $u \in L_\infty(G) \cap W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of problem (3), then $\mu \in \mathscr{C}'_b(\partial G)$.*

Proof. Let $w = u - \mathcal{U}(g\mathcal{H}_m)$. According to Lemma 3, there is a bounded linear functional $\tilde{L}$ on $W^{1,2}(G)$ representable by $\nu \in \mathscr{C}'_b(\partial G)$ such that w is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned}\Delta w = 0 &\quad \text{on } G, \\ \frac{\partial w}{\partial n} + wf = L - \tilde{L} &\quad \text{on } \partial G.\end{aligned} \tag{15}$$

Fix $x \in G$. Choose a sequence G_j of open sets with C^∞ boundary such that $\mathrm{cl}\, G_j \subset G_{j+1} \subset G$, $x \in G_1$, and $\cup G_j = G$. Fix $r > 0$ such that $\Omega_{2r}(x) \subset G_1$. Choose an infinitely differentiable function ψ such that $\psi = 0$ on $\Omega_r(x)$ and $\psi = 1$ on $\mathbb{R}^m \setminus \Omega_{2r}(x)$. According to Green's identity,

$$\begin{aligned}w(x) &= \lim_{j\to\infty} \left[\int_{\partial G_j} h_x(y) \frac{\partial w(y)}{\partial n}\, d\mathcal{H}_{m-1}(y) - \int_{\partial G_j} w(y) n(y) \cdot \nabla h_x(y)\, d\mathcal{H}_{m-1}(y) \right] \\ &= \lim_{j\to\infty} \left[\int_{G_j} \nabla w(y) \cdot \nabla (h_x(y)\psi(y))\, d\mathcal{H}_m(y) \right. \\ &\qquad \left. - \int_{G_j} \nabla (w(y)\psi(y)) \cdot \nabla h_x(y)\, d\mathcal{H}_m(y) \right]\end{aligned}$$

$$= \int_G \nabla w(y) \cdot \nabla (h_x(y)\psi(y)) d\mathcal{H}_m(y) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y)$$
$$= \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y). \tag{16}$$

According to [16, Theorem 2.3.2], there is a sequence of infinitely differentiable functions w_n such that $w_n \to w\psi$ in $W^{1,2}(G)$. According to [6, Section 2],

$$\begin{aligned} w(x) &= \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \lim_{n\to\infty} \int_G \nabla w_n(y) \cdot \nabla h_x(y) d\mathcal{H}_m(y) \\ &= \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \lim_{n\to\infty} \int_{\partial G} w_n(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y). \end{aligned} \tag{17}$$

Since the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], we obtain

$$w(x) = \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \int_{\partial G} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y). \tag{18}$$

Since $w \in L_\infty(G)$ by Lemma 3, the trace of w is an element of $L_\infty(\mathcal{H})$. Since

$$\begin{aligned} &\left| \int_{\partial G} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \right| \\ &\quad \le \|w\|_{L_\infty(\mathcal{H})} \int_{\partial G} |n(y) \cdot \nabla h_x(y)| \, d\mathcal{H}_{m-1}(y) \\ &\quad \le \|w\|_{L_\infty(\mathcal{H})} \left[\sup_{z\in\partial G} \int_{\partial G} |n(y) \cdot \nabla h_z(y)| \, d\mathcal{H}_{m-1}(y) + \frac{1}{2} \right] < \infty \end{aligned} \tag{19}$$

by [6, Lemma 2.15 and Theorem 2.16] and the fact that ∂G is of class $C^{1+\alpha}$, the function

$$x \longmapsto \int_{\partial G} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \tag{20}$$

is bounded in G. Since $\mathcal{U}\nu$ is bounded in G and $\mathcal{U}(fw\mathcal{H})$ is bounded in G by [6, Corollary 2.17 and Lemma 2.18], the function $\mathcal{U}\mu$ is bounded in G by (18). Thus, $\mu \in \mathcal{C}_b'(\partial G)$ by Lemma 2. □

Notation 5. Let X be a complex Banach space and T a bounded linear operator on X. We denote by $\operatorname{Ker} T$ the kernel of T, by $\sigma(T)$ the spectrum of T, by $r(T)$ the spectral radius of T, by X' the dual space of X, and by T' the adjoint operator of T. Denote by I the identity operator.

Theorem 6. *Let X be a complex Banach space and K a compact linear operator on X. Let Y be a subspace of X' and T a closed linear operator from Y to X such that $y(Tx) = x(Ty)$ for each $x, y \in Y$. Suppose that $K'(Y) \subset Y$ and $KTy = TK'y$ for each $y \in Y$. Let $\alpha \in C \setminus \{0\}$, $\operatorname{Ker}(K' - \alpha I)^2 = \operatorname{Ker}(K' - \alpha I) \subset Y$, and $\{\beta \in \sigma(K');\ (\beta - \alpha) \cdot \alpha \le 0\} \subset \{\alpha\}$. If $x, y \in X$, $(K' - \alpha I)x = y$, then $x \in Y$ if and only if $y \in Y$.*

Proof. If $x \in Y$, then $y \in Y$. Suppose that $y \in Y$. Since K is a compact operator, the operator K' is a compact operator by [14, Chapter IV, Theorem 4.1]. Suppose first that $\alpha \in \sigma(K')$. Since K' is compact, then α is a pol of the resolvent by [5, Satz 50.4]. Since

$$\operatorname{Ker}(K' - \alpha I)^2 = \operatorname{Ker}(K' - \alpha I), \tag{21}$$

the ascent of $(K' - \alpha I)$ is equal to 1. Since α is a pol of the resolvent and the ascent of $(K' - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space X' is the direct sum of $\operatorname{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X')$ and the descent of $(K' - \alpha I)$ is equal to 1. Since the descent of $(K' - \alpha I)$ is equal to 1, we have

$$(K' - \alpha I)^2(X') = (K' - \alpha I)(X'). \tag{22}$$

Since the space X' is the direct sum of $\operatorname{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X') = (K' - \alpha I)^2(X')$, the operator $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$. If $\alpha \notin \sigma(K')$, then the space X' is the direct sum of $\operatorname{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X')$, and the operator $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$. Therefore, there are $x_1 \in \operatorname{Ker}(K' - \alpha I) \subset Y$ and $x_2 \in (K' - \alpha I)(X')$ such that $x_1 + x_2 = x$. We have $(K' - \alpha I)x_2 = y$.

Denote by Z the closure of Y. Since $K'(Y) \subset Y$, we obtain $K'(Z) \subset Z$. Denote by K'_Z the restriction of K' to Z. Then K'_Z is a compact operator in Z. Since $\operatorname{Ker}(K' - \alpha I)^2 \subset Y$, we have

$$\operatorname{Ker}\left(K'_Z - \alpha I\right)^2 = \operatorname{Ker}(K' - \alpha I)^2 = \operatorname{Ker}(K' - \alpha I) = \operatorname{Ker}\left(K'_Z - \alpha I\right). \tag{23}$$

If $\alpha \notin \sigma(K'_Z)$, then the space Z is the direct sum of $\operatorname{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z)$, and the operator $(K'_Z - \alpha I)$ is invertible on Z. Suppose that $\alpha \in \sigma(K'_Z)$. Since K'_Z is compact, then α is a pol of the resolvent by [5, Satz 50.4]. Since

$$\operatorname{Ker}\left(K'_Z - \alpha I\right)^2 = \operatorname{Ker}\left(K'_Z - \alpha I\right), \tag{24}$$

the ascent of $(K'_Z - \alpha I)$ is equal to 1. Since α is a pol of the resolvent and the ascent of $(K'_Z - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space Z is the direct sum of $\operatorname{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z)$ and the descent of $(K'_Z - \alpha I)$ is equal to 1. Since the descent of $(K'_Z - \alpha I)$ is equal to 1, we have

$$\left(K'_Z - \alpha I\right)^2(Z) = (K' - \alpha I)(Z). \tag{25}$$

Since the space Z is the direct sum of $\operatorname{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z) = (K'_Z - \alpha I)^2(Z)$, the operator $(K'_Z - \alpha I)$ is invertible on $(K'_Z - \alpha I)(Z)$. Since $y \in Y \subset Z$, there are $y_1 \in \operatorname{Ker}(K'_Z - \alpha I)$ and $y_2 \in (K'_Z - \alpha I)(Z)$ such that $y = y_1 + y_2$. Since X' is the direct sum of $\operatorname{Ker}(K' - \alpha I) = \operatorname{Ker}(K'_Z - \alpha I)$ and $(K' - \alpha I)(X') \supset (K'_Z - \alpha I)(Z)$ and $y \in (K' - \alpha I)(X')$, we obtain that $y_1 = 0$ and $y_2 = y$. Thus, $y \in (K'_Z - \alpha I)(Z)$. Since $(K'_Z - \alpha I)$ is invertible on $(K'_Z - \alpha I)(Z)$, there is $z \in (K'_Z - \alpha I)(Z)$ such that $(K'_Z - \alpha I)(z) = y$. Since $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$, we deduce that $x_2 = z \in (K'_Z - \alpha I)(Z) \subset Z$.

Now, let $w \in \operatorname{Ker}(K' - \alpha I)$. Fix a sequence $\{z_k\} \subset Y$ such that $z_k \to z = x_2$. Then

$$\begin{aligned} w(Ty) = y(Tw) &= [(K' - \alpha I)x_2](Tw) = \lim_{k\to\infty}[(K' - \alpha I)z_k](Tw) \\ &= \lim_{k\to\infty} z_k((K - \alpha I)Tw) = \lim_{k\to\infty} z_k(T(K' - \alpha I)w) = \lim_{k\to\infty} z_k(0) = 0. \end{aligned} \tag{26}$$

Since $w(Ty) = 0$ for each $w \in \operatorname{Ker}(K' - \alpha I)$, [15, Chapter 10, Theorem 3] yields $Ty \in (K - \alpha I)(X)$.

Denote by $\tilde{K}'$ the restriction of K' to $(K' - \alpha I)(X)$. If we denote by P the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator K', then $P(X') = (K' - \alpha I)(X')$ by [5, Satz 50.2] and $\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Therefore,

$$\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\} \subset \{\beta;\ (\beta - \alpha) \cdot \alpha > 0\} \subset \bigcup_{t>0} \{\beta;\ |\beta - \alpha - t\alpha| < |t\alpha|\}. \tag{27}$$

Since $\{\beta;\ |\beta - \alpha - t_1\alpha| < |t_1\alpha|\} \subset \{\beta;\ |\beta - \alpha - t_2\alpha| < |t_2\alpha|\}$ for $0 < t_1 < t_2$ and $\sigma(\tilde{K}')$ is a compact set (see [14, Chapter VI, Theorem 1.3, and Lemma 1.5], there is $t > 0$ such that $\sigma(\tilde{K}') \subset \{\beta;\ |\beta - \alpha - t\alpha| < |t\alpha|\}$. Therefore, $r(\tilde{K}' - \alpha I - t\alpha I) < |t\alpha|$. Since we have $r(t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)) < 1$, the series

$$V = \sum_{k=0}^{\infty} (-1)^k [t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)]^k \tag{28}$$

converges. Easy calculation yields that V is the inverse operator of the operator $I + t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I) = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)$. Since $t^{-1}\alpha^{-1}y = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)x_2$, we have $x_2 = t^{-1}\alpha^{-1}Vy$. Denote $z_k = t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)]^k y$. Then

$$x_2 = \sum_{k=0}^{\infty} z_k. \tag{29}$$

Since $K'(Y) \subset Y$, $z_k \in Y$ for each k. Since $KT = TK'$ on Y, we have $Tz_k = t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(K - \alpha I - t\alpha I)]^k Ty$.

Since $(K - \alpha I)$, $(K - \alpha I)^2$, $(K' - \alpha I)$, and $(K' - \alpha I)^2$ are Fredholm operators with index 0 (see [14, Chapter V, Theorem 3.1]), [14, Chapter VII, Theorem 3.2] yields

$$\dim \operatorname{Ker}(K - \alpha I)^2 = \dim \operatorname{Ker}(K' - \alpha I)^2 = \dim \operatorname{Ker}(K' - \alpha I) = \dim \operatorname{Ker}(K - \alpha I), \tag{30}$$

and thus $\operatorname{Ker}(K - \alpha I)^2 = \operatorname{Ker}(K - \alpha I)$. If $\alpha \notin \sigma(K)$, then the space X is the direct sum of $\operatorname{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$, and the operator $(K - \alpha I)$ is invertible on X. Suppose that $\alpha \in \sigma(K)$. Since K is compact, then α is a pol of the resolvent by [5, Satz 50.4]. Since

$$\operatorname{Ker}(K - \alpha I)^2 = \operatorname{Ker}(K - \alpha I), \tag{31}$$

the ascent of $(K - \alpha I)$ is equal to 1. Since α is a pol of the resolvent and the ascent of $(K - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space X is the direct sum of $\operatorname{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$ and the descent of $(K - \alpha I)$ is equal to 1. Since the descent of $(K - \alpha I)$ is equal to 1, we have $(K - \alpha I)^2(X) = (K - \alpha I)(X)$. Since the space X is the direct sum of $\operatorname{Ker}(K-\alpha I)$ and $(K-\alpha I)(X)=(K-\alpha I)^2(X)$, the operator $(K-\alpha I)$ is invertible on $(K-\alpha I)(X)$. Denote by $\hat{K}$ the restriction of K to $(K-\alpha I)(X)$. If we denote by Q the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator K, then $Q(X)=(K-\alpha I)(X)$ by [5, Satz 50.2] and $\sigma(\hat{K})=\sigma(K)\setminus\{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Since $\sigma(K)=\sigma(K')$ by [14, Chapter VI, Theorem 4.6], we obtain $\sigma(\hat{K})\subset\{\beta;\ |\beta-\alpha-t\alpha|<|t\alpha|\}$. Therefore, $r(\hat{K}-\alpha I-t\alpha I)<|t\alpha|$. Since $Ty\in(K-\alpha X)$ and $r(t^{-1}\alpha^{-1}(\hat{K}-\alpha I-t\alpha I))<1$, the series

$$\sum_{k=0}^{\infty} Tz_k=\sum_{k=0}^{\infty} t^{-1}\alpha^{-1}\big[-t^{-1}\alpha^{-1}(\hat{K}-\alpha I-t\alpha I)\big]^k Ty \tag{32}$$

converges. Since T is closed, $x_2=\sum z_k$, and $\sum Tz_k$ converges, then the vector x_2 lies in Y, the domain of T. □

THEOREM 7. *Let $f\in L_\infty(\mathcal{H})$, $f\geq 0$, and $g\in L_2(G)\cap L_p(\mathbb{R}^m)$, where $p>m/2$, $g=0$ on $\mathbb{R}^m\setminus G$. Let L be a bounded linear functional on $W^{1,2}(G)$ representable by $\mu\in\mathscr{C}'(\partial G)$. If u is a weak solution in $W^{1,2}(G)$ of problem (3), then $u\in L_\infty(G)$ if and only if $\mu\in\mathscr{C}'_b(\partial G)$.*

Proof. If $u\in L_\infty(G)$, then $\mu\in\mathscr{C}'_b(\partial G)$ by Lemma 4.

Suppose now that $\mu\in\mathscr{C}'_b(\partial G)$. Let $w=u-\mathcal{U}(g\mathcal{H}_m)$. According to Lemma 3, there is a bounded linear functional $\tilde{L}$ on $W^{1,2}(G)$ representable by $\tilde{\mu}\in\mathscr{C}'_b(\partial G)$ such that w is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{gathered}\Delta w=0\quad\text{on } G,\\ \frac{\partial w}{\partial n}+wf=\tilde{L}\quad\text{on }\partial G.\end{gathered} \tag{33}$$

Define for $\varphi\in L_\infty(\mathcal{H})$ and $x\in\partial G$,

$$T\varphi(x)=\frac{1}{2}\varphi(x)+\int_{\partial G}\varphi(y)\frac{\partial}{n(y)}h_x(y)\,d\mathcal{H}(y)+\mathcal{U}(f\varphi\mathcal{H}). \tag{34}$$

Since $\mathcal{U}(f\mathcal{H})\in\mathscr{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18], the operator T is a bounded linear operator on $L_\infty(\mathcal{H})$ by [11, Proposition 8] and [6, Lemma 2.15]. The operator $T-(1/2)I$ is compact by [12, Theorem 20] and [6, Theorem 4.1 and Corollary 1.11]. According to [10, Theorem 1], there is $\nu\in\mathscr{C}'(\partial G)\subset(L_\infty(\mathcal{H}))'$ such that $T'\nu=\tilde{\mu}$ and

$$\int_G\nabla\mathcal{U}\nu\cdot\nabla v\,d\mathcal{H}_m+\int_{\partial G}\mathcal{U}\nu f v\,d\mathcal{H}=\int v\,d\tilde{\mu}, \tag{35}$$

for each $v\in\mathcal{D}$.

Remark that $\mathscr{C}'(\partial G)$ is a closed subspace of $(L_\infty(\mathcal{H}))'$. According to [11, Proposition 8], we have $T'(\mathscr{C}'(\partial G))\subset\mathscr{C}'(\partial G)$. Denote by τ the restriction of T' to $\mathscr{C}'(\partial G)$. According to [10, Lemma 11] and [14, Chapter VI, Theorem 1.2], we have $\sigma(\tau)\subset\{\beta;\ \beta\geq 0\}$. Since $\sigma(\tau')=\sigma(\tau)$ (see [15, Chapter VIII, Section 6, Theorem 2]), each $\beta\in\sigma(T)$ is an eigenvalue (see [14, Chapter VI, Theorem 1.2]), and T is the restriction of τ' to $L_\infty(\mathcal{H})$, we obtain that $\sigma(T')=\sigma(T)\subset\{\beta;\ \beta\geq 0\}$ by [15, Chapter VIII, Section 6, Theorem 2].

According to [9, Theorem 1.11], we have $\operatorname{Ker} T' \subset \mathscr{C}_b'(\partial G)$. According to [9, Lemma 1.10] and [10, Lemmas 12 and 13], $\operatorname{Ker} T' = \operatorname{Ker}(T')^2$. Denote, for $\rho \in \mathscr{C}_b'(\partial G)$, by $V\rho$ the restriction of $\mathscr{U}\rho$ to ∂G. Then V is a closed operator from $\mathscr{C}_b'(\partial G)$ to $L_\infty(\mathscr{H})$ by [13, Lemma 5]. If $\rho \in \mathscr{C}_b'(\partial G)$, then $VT'\rho = TV\rho$ by [13, Lemma 4]. If $\rho_1, \rho_2 \in \mathscr{C}_b'(\partial G)$, then ρ_1 and ρ_2 have finite energy by [13, Proposition 23], [7, Theorem 1.20], and

$$\int \mathscr{U}\rho_1 \, d\rho_2 = \int_{\mathbb{R}^m} \nabla \mathscr{U}\rho_1 \cdot \nabla \mathscr{U}\rho_2 \, d\mathscr{H}_m = \int \mathscr{U}\rho_2 \, d\rho_1. \tag{36}$$

Since $T'\nu = \tilde{\mu} \in \mathscr{C}_b'(\partial G)$, Theorem 6 yields that $\nu \in \mathscr{C}_b'(\partial G)$. Since ν has finite energy $\int \mathscr{U}\nu \, d\nu$ and $\int \mathscr{U}\nu \, d\nu = \int |\nabla \mathscr{U}\nu|^2 d\mathscr{H}_m$ by [7, Theorem 1.20], we obtain that $\mathscr{U}\nu \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.14]). Since $\mathscr{D}$ is dense in $W^{1,2}(G)$ by [16, Theorem 2.3.2], relation (35) yields that the function $\mathscr{U}\nu$ is a weak solution in $W^{1,2}(G)$ of (33). Since $v = \mathscr{U}\nu - w$ is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{gathered} \Delta v = 0 \quad \text{on } G, \\ \frac{\partial v}{\partial n} + vf = 0 \quad \text{on } \partial G, \end{gathered} \tag{37}$$

and $f \geq 0$, we obtain

$$0 = \int_G \nabla v \cdot \nabla v \, d\mathscr{H}_m + \int_{\partial G} v f v \, d\mathscr{H} \geq \int_G |\nabla v|^2 d\mathscr{H}_m \geq 0. \tag{38}$$

Therefore, $\nabla v = 0$ on G and there is a constant c such that $v(x) = c$ for $\mathscr{H}_m$-a.a. $x \in G$ by [16, Corollary 2.1.9]. Since $\nu \in \mathscr{C}_b'(\partial G)$, the function $\mathscr{U}\nu$ is bounded in G. Since $u(x) = \mathscr{U}(g\mathscr{H}_m)(x) + \mathscr{U}\nu(x) - c$ for $\mathscr{H}_m$-a.a. $x \in G$ and $\mathscr{U}(g\mathscr{H}_m) \in \mathscr{C}(\mathbb{R}^m)$ by Lemma 3, we obtain $u \in L_\infty(G)$. $\square$

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DARBOUX-LAMÉ EQUATION AND ISOMONODROMIC DEFORMATION

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The Darboux-Lamé equation is defined as the double Darboux transformation of the Lamé equation, and is studied from the viewpoint of the isomonodromic deformation theory. It is shown that the second-order ordinary differential equation of Fuchsian type on $\mathbf{P}_1$ corresponding to the second Darboux-Lamé equation is obtained as isomonodromic deformation of some specific Gauss' hypergeometric differential equation.

1. Introduction

We consider the nth Lamé equation

$$\frac{\partial^2}{\partial x^2} f(x) - \big(n(n+1)\wp(x,\tau) - \lambda\big) f(x) = 0, \tag{1.1}$$

where n is a natural number and $\wp(x,\tau)$ is the Weierstrass elliptic function with the fundamental periods 1 and τ such that $\Im\tau > 0$. If the fundamental period τ and the discrete eigenvalue λ satisfy a kind of degenerate condition obtained in [6], we can construct the nth algebro-geometric elliptic potential $u^{**}_{n,\lambda}(x,\xi)$ with the complex parameter ξ by the method of double Darboux transformation. We call the ordinary differential equation

$$\frac{\partial^2}{\partial x^2} f(x) - \big(u^{**}_{n,\lambda}(x,\xi) - \lambda\big) f(x) = 0 \tag{1.2}$$

the nth Darboux-Lamé equation of degenerate type. The purpose of the present work is to clarify the isomonodromic property of equation (1.2) regarding ξ as the deformation parameter. Various authors have formerly clarified the isospectral property of the double Darboux transformation (the double commutation) of the nth algebro-geometric potential. See, for example, [2, 6] and the references therein. However, we could not treat the isomonodromic deformation problems, for $n \geq 3$, in this paper, while the isospectral deformation problem have been almost completely solved for general n.

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2. Preliminaries

In this section, the necessary materials are summarized. We refer the reader to [5, 6] for more precise information.

We consider the second-order linear ordinary differential operator in the complex domain

$$H(u) = -\frac{\partial^2}{\partial x^2} + u(x), \quad x \in \mathbb{C}, \tag{2.1}$$

where $u(x)$ is a meromorphic function. The functions $Z_n(u)$, $n \in \mathbb{N}$, defined by the recursion relation

$$Z_0(u) \equiv 1, \qquad Z_n(u) = \Lambda(u) Z_{n-1}(u), \quad n = 1, 2, \ldots, \tag{2.2}$$

which are the differential polynomials in $u(x)$, are called the *KdV polynomials*, where

$$\Lambda(u) = \left(\frac{\partial}{\partial x}\right)^{-1} \cdot \left(\frac{1}{2}u' + u\frac{\partial}{\partial x} - \frac{1}{4}\frac{\partial^3}{\partial x^3}\right) \tag{2.3}$$

is the Λ-operator associated with the differential operator $H(u)$.

Let $V(u)$ be the linear span of all KdV polynomials over $\mathbb{C}$. If $\dim V(u) = n+1$, then $u(x)$ is called the nth algebro-geometric potential and we write $\operatorname{rank} V(u) = n$. If $u(x)$ is the nth algebro-geometric potential, then there uniquely exist the polynomials $a_j(\lambda)$, $j = 0, 1, \ldots, n$, in the spectral parameter λ of degree $n - j + 1$ such that

$$Z_{n+1}(u - \lambda) = \sum_{j=0}^{n} a_j(\lambda) Z_j(u - \lambda). \tag{2.4}$$

For this fact, see [5, 6]. The *M-function* $M(x, \lambda; u)$ associated with $u(x)$ is the differential polynomial defined by

$$M(x, \lambda; u) = Z_n(u - \lambda) - \sum_{j=1}^{n} a_j(\lambda) Z_{j-1}(u - \lambda). \tag{2.5}$$

The spectral discriminant

$$\Delta(\lambda; u) = M_x(x, \lambda; u)^2 - 2M(x, \lambda; u) M_{xx}(x, \lambda; u) + 4\left(u(x) - \lambda\right) M(x, \lambda; u)^2 \tag{2.6}$$

is the polynomial of degree $2n+1$ in λ with constant coefficients. Let

$$\operatorname{Spec} H(u) = \{\lambda \mid \Delta(\lambda; u) = 0\}, \tag{2.7}$$

which corresponds to the discrete spectrum of the operator $H(u)$. If $\lambda_j \in \operatorname{Spec} H(u)$, then we have

$$\left(H(u) - \lambda_j\right) M(x, \lambda_j; u)^{1/2} = 0. \tag{2.8}$$

We call $M(x, \lambda_j; u)^{1/2}$ the *M-eigenfunction*.

For $f(x) \in \ker(H(u)-\lambda) \setminus \{0\}$, the Darboux transformation is the operator $H(u^*)$ with the potential $u^*(x)$ defined by

$$u^*(x) = u(x) - 2\frac{\partial^2}{\partial x^2}\log f(x). \tag{2.9}$$

We sometimes call the resulted potential $u^*(x)$ itself the Darboux transformation.

Suppose $f(x) \in \ker(H(u)-\lambda) \setminus \{0\}$, then we have

$$\frac{1}{f(x)} \in \ker\left(H(u^*)-\lambda\right) \setminus \{0\}. \tag{2.10}$$

This fact is called Darboux's lemma [1]. The Darboux transformation of the algebro-geometric potential $u(x)$ by the corresponding M-eigenfunction

$$u^*_{\lambda_j}(x) = u(x) - 2\frac{\partial^2}{\partial x^2}\log M(x,\lambda_j;u)^{1/2} = u(x) - \frac{\partial^2}{\partial x^2}\log M(x,\lambda_j;u) \tag{2.11}$$

is called the *algebro-geometric Darboux transformation* (*ADT*). Let

$$\widehat{M}(x,\lambda_j;u) = \int M(x,\lambda_j;u)\,dx \tag{2.12}$$

and fix the integration constant appropriately; then, by Darboux's lemma, mentioned above, it follows that the function $F_{\lambda_j}(x,\xi)$, defined by

$$F_{\lambda_j}(x,\xi) = \frac{\phi_{\lambda_j}(\xi)+\xi\widehat{M}(x,\lambda_j;u)}{M(x,\lambda_j;u)^{1/2}}, \tag{2.13}$$

is the 1-parameter family of the eigenfunction of $H(u^*_{\lambda_j})$ associated with the eigenvalue λ_j, that is,

$$\left(H\left(u^*_{\lambda_j}\right)-\lambda_j\right)F_{\lambda_j}(x,\xi) = 0, \tag{2.14}$$

where $\phi_{\lambda_j}(\xi)$ is an arbitrary function which depends only on ξ. The function $\phi_{\lambda_j}(\xi)$ will be determined exactly so that the ADDT, which is defined below, of the nth Lamé equation is isomonodromic.

The *algebro-geometric double Darboux transformation* (*ADDT*) is defined as the Darboux transformation of $u^*_{\lambda_j}(x)$ by $F_{\lambda_j}(x,\xi)$:

$$\begin{aligned} u^{**}_{\lambda_j}(x,\xi) &= u^*_{\lambda_j}(x) - 2\frac{\partial^2}{\partial x^2}\log F_{\lambda_j}(x,\xi) \\ &= u(x) - 2\frac{\partial^2}{\partial x^2}\log\left(\phi_{\lambda_j}(\xi)+\xi\widehat{M}(x,\lambda_j;u)\right). \end{aligned} \tag{2.15}$$

In what follows, we assume that $\phi_{\lambda_j}(\xi)$ does not identically vanish since $\phi_{\lambda_j}(\xi) \equiv 0$, then the ADDT $u^{**}_{\lambda_j}(x,\xi)$ does not depend on ξ.

Let

$$\mathrm{Spec}_m H(u) = \{\lambda_j \mid \text{the multiple roots of } \Delta(\lambda; u) = 0\} \tag{2.16}$$

which we call the multiple spectrum of $H(u)$. It is shown in [6] that if $u(x)$ is the nth algebro-geometric potential, then $u^*_{\lambda_j}(x)$ is the $(n-1)$th algebro geometric potential if and only if $\mathrm{Spec}_m H(u) \neq \emptyset$ and $\lambda_j \in \mathrm{Spec}_m H(u)$.

If n is a natural number, then the nth Lamé potential $u_n(x,\tau) = n(n+1)\wp(x,\tau)$ is known to be the nth algebro-geometric potential (see, e.g., [5]).

Let $M_n(x,\lambda,\tau)$ be the M-function associated with the nth Lamé potential $u_n(x,\tau)$, that is, $M_n(x,\lambda,\tau) = M(x,\lambda; u_n(x,\tau))$. Let

$$D(\tau; u_n) = R\Big(\Delta(\lambda; u_n), \frac{d}{d\lambda}\Delta(\lambda; u_n)\Big), \tag{2.17}$$

where $R(P,Q)$ is the resultant of polynomials $P(\lambda)$ and $Q(\lambda)$. If $D(\tau_*; u_n) = 0$ for $\tau_* \in H^+$, then there exists $\lambda_* \in \mathrm{Spec}_m H(u_n)$, that is, λ_* is the multiple root of $\Delta(\lambda; u_n) = 0$ and

$$\mathrm{rank} u^*_{n,\lambda_*}(x,\tau_*) = n-1, \tag{2.18}$$

where $u^*_{n,\lambda_*}(x)$ is the Darboux transformation of the nth Lamé potential $u_n(x)$ by the corresponding M-eigenfunction $M_n(x,\lambda_*,\tau_*)^{1/2}$, that is,

$$u^*_{n,\lambda_*}(x,\tau_*) = u_n(x,\tau_*) - \frac{\partial^2}{\partial x^2}\log M_n(x,\lambda_*,\tau_*). \tag{2.19}$$

Let

$$\Theta_n = \{\tau \mid D(\tau; u_n) = 0\} \subset H^+ \tag{2.20}$$

and we call it the *lattice of degenerate periods* associated with the nth Lamé potential $u_n(x)$. One can immediately see that the lattice of degenerate periods Θ_n is the discrete subset of H^+.

Now, we enumerate several examples of the degenerate condition for the Lamé potentials. For this purpose, we must carry out elementary but very complicated computation. Hence, here we explain only the simplest case $n = 1$. See also [3] for another method of computation.

KdV polynomials. We have

$$Z_0(u_n) = 1, \qquad Z_1(u_n) = \frac{1}{2}u_n, \qquad Z_2(u_n) = \frac{3}{8}u_n^2 - \frac{1}{8}u_n''. \tag{2.21}$$

Computation of the M-function $M_2(x,\lambda,\tau)$. Let ρ_0 and ρ_1 be the constants such that

$$\begin{aligned} Z_2(u_1) &= \Big(\frac{3}{8}\Big)(4\wp^2) - \Big(\frac{1}{8}\Big)(2\wp'') \\ &= \rho_0 Z_0(u_1) + \rho_1 Z_1(u_1) = \rho_0 + \rho_1\wp. \end{aligned} \tag{2.22}$$

Hence, $\rho_1 = 0$ and $\rho_0 = (1/8)g_2$ follow immediately. On the other hand, according to [5, Theorem 3, page 414], define the coefficients $\alpha_\nu^{(n)}$, $\nu = 0,1,2,\dots,n$, for $n = 0,1,2$, then we immediately have

$$\alpha_1^{(2)} = \frac{3}{2}, \qquad \alpha_0^{(2)} = 1, \qquad \alpha_0^{(1)} = \frac{1}{2}, \qquad \alpha_1^{(1)} = \alpha_0^{(0)} = 1. \tag{2.23}$$

Moreover, by [5, Lemma 7, page 417], we have

$$Z_2(u_1 - \lambda) = a_1(\lambda)Z_1(u_1 - \lambda) + a_0(\lambda)Z_0(u_1 - \lambda), \tag{2.24}$$

where

$$\begin{aligned} a_0(\lambda) &= -\alpha_0^{(2)}\lambda^2 + \alpha_0^{(1)}\rho_1\lambda + \alpha_0^{(0)}\rho_0 = -\lambda^2 + \frac{1}{8}g_2, \\ a_1(\lambda) &= -\alpha_1^{(2)}\lambda + \alpha_1^{(1)} = -\frac{3}{2}\lambda. \end{aligned} \tag{2.25}$$

Hence, we have

$$M_1(x,\lambda,\tau) = \frac{1}{2}(2\wp - \lambda) - \left(-\frac{3}{2}\lambda\right) = \wp + \lambda. \tag{2.26}$$

Computation of the spectral discriminant $\Delta(\lambda;u_1)$. Using the M-function $M_1(x,\lambda,\tau)$ obtained above, we have

$$\Delta(\lambda;u_1) = \wp'^2 - 2(\wp+\lambda)\wp'' + 4(2\wp - \lambda)(\wp+\lambda)^2 = -4\lambda^3 + g_2\lambda - g_3. \tag{2.27}$$

For the first Lamé potential $u_1(x,\tau)$, since $g_2(\tau)^3 - 27g_3(\tau)^2 \neq 0$ for any $\tau \in H^+$, $\mathrm{Spec}_m H(u_1) = \varnothing$ holds for any $\tau \in H^+$, that is, $\Theta_1 = \varnothing$.

On the other hand, for the second Lamé potential $u_2(x,\tau)$, we can compute the spectral discriminant similarly to the above example, and

$$\Delta(\lambda;u_2) = -4(\lambda^2 - 3g_2(\tau))\left(\lambda^3 - \frac{9}{4}g_2(\tau)\lambda - \frac{27}{4}g_3(\tau)\right) \tag{2.28}$$

follows. Hence $\mathrm{Spec}_m H(u_2) \neq \varnothing$ holds if and only if $g_2(\tau) = 0$. Note that $g_2(\tau) = 0$ holds if and only if $J(\tau) = 0$, where $J(\tau) = g_2(\tau)^3/(g_2(\tau)^3 - 27g_3(\tau)^2)$ is the elliptic modular function. Since $g_2(e^{2\pi i/3}) = 0$, by the modular invariance of $J(\tau)$, we have

$$\begin{aligned} \Theta_2 &= \left\{\tau \mid \tau = \frac{ae^{2\pi i/3} + b}{ce^{2\pi i/3} + d}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\right\} \subset H^+ \\ &= \{\tau \mid \Im\tau > 0\}. \end{aligned} \tag{2.29}$$

3. The second Darboux-Lamé equation

Suppose that $\tau_* \in \Theta_n$ and $\lambda_* \in \mathrm{Spec}_m H(u_n(x,\tau_*))$. Let $M_n(x,\lambda_*,\tau_*)$ be the M-function corresponding to the nth Lamé potential $u_n(x,\tau_*)$. By (2.15), the ADDT of $u_n(x,\tau_*)$ is expressed as

$$u^{**}_{n,\lambda_*}(x,\xi) = u_n(x,\tau_*) - 2\frac{\partial^2}{\partial x^2}\log\left(\phi_{\lambda_*}(\xi) + \xi\widehat{M}_n(x,\lambda_*,\tau_*)\right), \tag{3.1}$$

where $\widehat{M}_n(x,\lambda_*,\tau_*)$ is defined by (2.12). We call the 1-parameter family of the ordinary differential equation (1.2) with the potential $u^{**}_{n,\lambda_*}(x,\xi)$, defined above, *the nth Darboux-Lamé equation of degenerate type*, and $\mathrm{DL}_n(\tau_*,\lambda_*,\xi)$ denotes that 1-parameter family (1.2).

In what follows, we construct exactly the second Darboux-Lamé equation of degenerate type. Suppose that $\tau_* \in \Theta_2$, then, by the direct calculation parallel to that for $M_1(x, \lambda,\tau)$, we have

$$M_2(x,\lambda,\tau_*) = \lambda^2 + 3\wp(x,\tau_*)\lambda + 9\wp(x,\tau_*)^2. \tag{3.2}$$

Since we have shown $g_2(\tau_*) = 0$ in Section 2,

$$\Delta(\lambda;u_2) = -4\lambda^2\left(\lambda^3 - \frac{27}{4}g_3(\tau_*)\right) \tag{3.3}$$

follows. Hence $\mathrm{Spec}_m H(u_2(x,\tau_*)) = \{0\}$ and we have

$$M_2(x,0,\tau_*)^{1/2} = 3\wp(x,\tau_*) \in \ker H(u_2). \tag{3.4}$$

Therefore, the ADT $u^*_{2,0}(x)$ is given by

$$u^*_{2,0}(x) = 2\wp(x,\tau_*) - \frac{2g_3(\tau_*)}{\wp(x,\tau_*)^2} \tag{3.5}$$

and, by Darboux's lemma, we have

$$\frac{1}{M_2(x,0,\tau_*)^{1/2}} = \frac{1}{3\wp(x,\tau_*)} \in \ker H(u^*_{2,0}). \tag{3.6}$$

On the other hand, we have

$$\widehat{M}_2(x,0,\tau_*) = \int M_2(x,0,\tau_*)\,dx = \int 9\wp(x,\tau_*)^2 dx = \frac{3}{2}\wp'(x,\tau_*). \tag{3.7}$$

Hence, by (3.1),

$$u^{**}_{2,0}(x,\tau_*) = u_2(x) - 2\frac{\partial^2}{\partial x^2}\log\left(\phi_0(\xi) + \frac{3}{2}\xi\wp'(x,\tau)\right) \tag{3.8}$$

follows. Thus we have the following lemma.

LEMMA 3.1. *The second Darboux-Lamé equation of degenerate type is explicitly expressed as*

$$\frac{\partial^2}{\partial x^2}f(x) = \frac{6\wp(x,\tau_*)\left(\phi_0(\xi)^2 - 3\xi\phi_0(\xi)\wp'(x,\tau) + (27/4)g_3(\tau_*)\xi^2\right)}{\left(\phi_0(\xi) + (3/2)\xi\wp'(x,\tau_*)\right)^2}f(x). \tag{3.9}$$

Moreover,

$$F_0(x,\xi) = \frac{\phi_0(\xi) + (3/2)\xi\wp'(x,\tau_*)}{3\wp(x,\tau_*)}. \tag{3.10}$$

The isospectral property of the potential $u_{2,0}^{**}(x,\tau_*)$ will be discussed in the forthcoming paper [7].

4. The Fuchsian equation on $\mathbf{P}_1$

Suppose that $\tau_* \in \Theta_2$, that is, $g_2(\tau_*) = 0$. Since $g_2(\tau_*)^3 - 27g_3(\tau_*)^2 \neq 0$, $g_3(\tau_*) \neq 0$ follows. In what follows, we fix one of the square roots of $g_3(\tau_*)$ and denote it by γ, that is, $\gamma^2 = g_3(\tau_*)$. Then, by the variable transformation

$$z = \frac{1}{2i\gamma}\wp'(x,\tau_*) + \frac{1}{2}, \tag{4.1}$$

the second Darboux-Lamé equation $\mathrm{DL}_2(\tau_*,0,\xi)$ is transformed to the second-order ordinary differential equation of Fuchsian type on $\mathbf{P}_1$:

$$z(z-1)\frac{\partial^2}{\partial z^2}\hat{f}(z,\xi) + \left(\frac{4}{3}z - \frac{2}{3}\right)\frac{\partial}{\partial z}\hat{f}(z,\xi) = \Gamma\left(z, \frac{3i\gamma\xi - 2\phi_0(\xi)}{6i\gamma\xi}\right)\hat{f}(z,\xi) \tag{4.2}$$

with the parameter ξ, where

$$\Gamma(x,s) = \frac{2}{3}\frac{(2s-1)z + s(s-2)}{(z-s)^2}, \qquad \hat{f}(z,\xi) = f(x,\xi). \tag{4.3}$$

We denote the 1-parameter family of the ordinary differential equation (4.2) by $\widehat{\mathrm{DL}}_2(\tau_*, 0,\xi)$. The regular singular points of equation (4.2) are

$$z = 0, 1, \infty, \qquad s = s(\xi) = \frac{3i\gamma\xi - 2\phi_0(\xi)}{6i\gamma\xi}. \tag{4.4}$$

In what follows, we assume that $\phi_0(0) \neq 0$. Then it follows that $s(0) = \infty$, and that the differential equation $\widehat{\mathrm{DL}}_2(\tau_*,0,0)$ coincides with the hypergeometric equation

$$z(z-1)\frac{\partial^2}{\partial z^2}\hat{f}(z) + \left(\frac{4}{3}z - \frac{2}{3}\right)\frac{\partial}{\partial z}\hat{f}(z) = \frac{2}{3}\hat{f}(z). \tag{4.5}$$

Now, we construct the fundamental system of solutions of $\widehat{\mathrm{DL}}_2(\tau_*,0,\xi)$ (4.2). Suppose $\xi \neq 0$. Then, by Darboux's lemma and (3.1),

$$\begin{aligned} f_1(x,\xi) &= \frac{1}{F_0(x,\xi)} = \frac{6\wp(x,\tau_*)}{2\phi_0(\xi) + 3\xi\wp'(x,\tau_*)}, \\ f_2(x,\xi) &= f_1(x,\xi)\int\frac{1}{f_1(x,\xi)^2}dx \\ &= \frac{\wp(x,\tau_*)}{6(2\phi_0(\xi) + 3\xi\wp'(x,\tau_*))}\int\frac{(2\phi_0(\xi) + 3\xi\wp'(x,\tau_*))^2}{\wp(x,\tau_*)^2}dx \end{aligned} \tag{4.6}$$

are the fundamental system of solutions of the second Darboux-Lamé equation

$$H\left(u_{2,0}^{**}(x,\xi)\right)f(x,\xi) = -\frac{\partial^2}{\partial x^2}f(x,\xi) + u_{2,0}^{**}(x,\xi)f(x,\xi) = 0 \tag{4.7}$$

such that $W[f_1, f_2] = f_1 f_{2x} - f_{1x} f_2 = 1$. By the variable transformation (4.1), we have

$$\wp(x,\tau_*) = \gamma^{2/3}z^{1/3}(1-z)^{1/3}, \qquad \wp'(x,\tau_*) = i\gamma(2z-1). \tag{4.8}$$

Let $\widehat{f}_j(z,\xi) = f_j(x,\xi)$, $j = 1,2$. We immediately have

$$\widehat{f}_1(z,\xi) = \frac{6\gamma^{2/3}z^{1/3}(1-z)^{1/3}}{2\phi_0(\xi) + 3i\gamma\xi(2z-1)}. \tag{4.9}$$

Similarly, we have

$$\widehat{f}_2(z,\xi) = \frac{iz^{1/3}(1-z)^{1/3}}{18\gamma\left(2\phi_0(\xi) + 3i\gamma\xi(2z-1)\right)}\int \frac{\left(2\phi_0(\xi) + 3i\gamma\xi(2z-1)\right)^2}{z^{4/3}(1-z)^{4/3}}dz. \tag{4.10}$$

5. Isomonodromic property of $\widehat{\mathrm{DL}}_2(\tau_*,0,\xi)$

The following is the well-known criterion for the isomonodromic property.

Lemma 5.1. *Suppose that the second-order ordinary differential equation*

$$\frac{\partial^2}{\partial z^2}f(z,\xi) + p(z,\xi)\frac{\partial}{\partial z}f(z,\xi) + q(z,\xi)f(z,\xi) = 0 \tag{5.1}$$

is of Fuchsian type on $\mathbf{P}_1$ *with the parameter* ξ. *The monodromy group for this equation is independent of the parameter* ξ *if and only if there exist* $a(z,\xi)$ *and* $b(z,\xi)$, *which are rational in* z, *such that*

$$\frac{\partial}{\partial \xi}f(z,\xi) = a(z,\xi)\frac{\partial}{\partial z}f(z,\xi) + b(z,\xi)f(z,\xi). \tag{5.2}$$

By the above general criterion, to show that the monodromy matrix associated with the fundamental system $\widehat{f}_1(z,\xi)$, $\widehat{f}_2(z,\xi)$ is independent of the parameter ξ, it suffices to show that $a(z,\xi)$ and $b(z,\xi)$, defined by

$$a(z,\xi) = \frac{\begin{vmatrix} \widehat{f}_{1\xi}(z,\xi) & \widehat{f}_1(z,\xi) \\ \widehat{f}_{2\xi}(z,\xi) & \widehat{f}_2(z,\xi) \end{vmatrix}}{\begin{vmatrix} \widehat{f}_{1z}(z,\xi) & \widehat{f}_1(z,\xi) \\ \widehat{f}_{2z}(z,\xi) & \widehat{f}_2(z,\xi) \end{vmatrix}}, \qquad b(z,\xi) = \frac{\begin{vmatrix} \widehat{f}_{1z}(z,\xi) & \widehat{f}_{1\xi}(z,\xi) \\ \widehat{f}_{2z}(z,\xi) & \widehat{f}_{2\xi}(z,\xi) \end{vmatrix}}{\begin{vmatrix} \widehat{f}_{1z}(z,\xi) & \widehat{f}_1(z,\xi) \\ \widehat{f}_{2z}(z,\xi) & \widehat{f}_2(z,\xi) \end{vmatrix}}, \tag{5.3}$$

are rational functions of z. Let

$$g(z,\xi) = \frac{\left(2\phi_0(\xi) + 3i\gamma\xi(2z-1)\right)^2}{z^{4/3}(1-z)^{4/3}}, \tag{5.4}$$

then, by (4.9) and (4.10), the expression

$$\widehat{f}_2(z,\xi) = \widehat{f}_1(z,\xi)\int g(z,\xi)dz \tag{5.5}$$

follows. Hence, we have

$$\begin{aligned} \begin{vmatrix} \widehat{f}_{1z}(z,\xi) & \widehat{f}_1(z,\xi) \\ \widehat{f}_{2z}(z,\xi) & \widehat{f}_2(z,\xi) \end{vmatrix} &= -\widehat{f}_1(z,\xi)^2 g(z,\xi), \\ \begin{vmatrix} \widehat{f}_{1\xi}(z,\xi) & \widehat{f}_1(z,\xi) \\ \widehat{f}_{2\xi}(z,\xi) & \widehat{f}_2(z,\xi) \end{vmatrix} &= -\widehat{f}_1(z,\xi)^2 \int g_\xi(z,\xi)dz. \end{aligned} \tag{5.6}$$

Thus

$$a(z,\xi) = \frac{\int g_\xi(z,\xi)dz}{g(z,\xi)} \tag{5.7}$$

follows. On the other hand, we immediately have

$$b(z,\xi) = -\frac{\widehat{f}_{1z}(z,\xi)}{\widehat{f}_1(z,\xi)}a(z,\xi) + \frac{\widehat{f}_{1\xi}(z,\xi)}{\widehat{f}_1(z,\xi)}. \tag{5.8}$$

We have

$$\begin{aligned} \frac{\widehat{f}_{1z}(z,\xi)}{\widehat{f}_1(z,\xi)} &= \frac{\partial}{\partial z}\log \widehat{f}_1(z,\xi) = \frac{1-2z}{3z(1-z)} - \frac{6i\gamma\xi}{2\phi_0(\xi)+3i\gamma\xi(2z-1)}, \\ \frac{\widehat{f}_{1\xi}(z,\xi)}{\widehat{f}_1(z,\xi)} &= \frac{\partial}{\partial \xi}\log \widehat{f}_1(z,\xi) = -\frac{2\phi_0'(\xi)+6i\gamma z}{2\phi_0(\xi)+3i\gamma\xi(2z-1)}. \end{aligned} \tag{5.9}$$

These are both rational functions of z. Hence, if $a(z,\xi)$ is a rational function of z, then $b(z,\xi)$ is also a rational function of z. Therefore, we have the following lemma.

Lemma 5.2. *The family $\widehat{\mathrm{DL}}_2(\tau_*,0,\xi)$ is isomonodromic if and only if the integral constant of the indefinite integral*

$$G(z,\xi) = (z-z^2)^{1/3}\int \frac{4\phi_0(\xi)\phi_0'(\xi) - 9\xi g_3(\tau_*)(1-2z)^2}{(z-z^2)^{4/3}}dz \tag{5.10}$$

is determined so that $G(z,\xi)$ is the rational function of z for all ξ.

Proof. By direct calculation, we have

$$\begin{aligned} a(z,\xi) = \frac{z-z^2}{(2\phi_0 - 3i\gamma\xi(1-2z))^2}\Bigg\{ & 2G(z,\xi) - 12i\gamma(\phi_0(\xi) \\ & +\xi\phi_0'(\xi))(z-z^2)^{1/3}\int \frac{1-2z}{(z-z^2)^{4/3}}dz\Bigg\}. \end{aligned} \tag{5.11}$$

On the other hand, we have

$$\int \frac{1-2z}{(z-z^2)^{4/3}}dz = \left(\frac{3}{z-1} - \frac{3}{z}\right)(z-z^2)^{2/3} + \text{const}. \tag{5.12}$$

This completes the proof. □

Next we have the following lemma.

Lemma 5.3. *The integral constant of the indefinite integral*

$$(z-z^2)^{1/3}\int \frac{z^2+c}{(z-z^2)^{4/3}}dz \tag{5.13}$$

can be determined so that it is the rational function of z if and only if $c=-1$.

Proof. Firstly, suppose $c=-1$, then we immediately have

$$\int \frac{z^2-1}{(z-z^2)^{4/3}}dz = \frac{3(z-z^2)^{2/3}}{z} + \alpha. \tag{5.14}$$

Hence, putting $\alpha = 0$, we have

$$(z-z^2)^{1/3}\int \frac{z^2-1}{(z-z^2)^{4/3}}dz = 3(1-z). \tag{5.15}$$

Secondly, by the above, we have

$$(z-z^2)^{1/3}\int \frac{z^2+c}{(z-z^2)^{4/3}}dz = (c+1)(z-z^2)^{1/3}\int \frac{1}{(z-z^2)^{4/3}}dz + 3(1-z). \tag{5.16}$$

Let

$$p(z) = (z-z^2)^{1/3}\int \frac{1}{(z-z^2)^{4/3}}dz, \tag{5.17}$$

then we have

$$\frac{\partial}{\partial z}\log p(z) = \frac{1-2z}{3(z-z^2)} + \frac{1}{(z-z^2)p(z)}. \tag{5.18}$$

This implies that

$$z(z-1)\frac{\partial}{\partial z}p(z) = \frac{1}{3}(2z-1)p(z) - 1. \tag{5.19}$$

Assume that one can choose the integration constant so that $p(z)$ is the rational function. Then there exist the polynomials $p_1(z)$, $p_2(z)$, and $p_3(z)$ such that $p_1(0) = p_2(0) = 0$ and

$$p(z) = p_1\left(\frac{1}{z}\right) + p_2\left(\frac{1}{z-1}\right) + p_3(z). \tag{5.20}$$

Let

$$p_1\left(\frac{1}{z}\right) = \sum_{j=1}^{l} \frac{\alpha_j}{z^j}, \qquad p_2\left(\frac{1}{z-1}\right) = \sum_{j=1}^{m} \frac{\beta_j}{(z-1)^j}, \qquad p_3(z) = \sum_{j=0}^{n} \gamma_j z^j, \tag{5.21}$$

then we have

$$\begin{aligned}
&z(z-1)\frac{\partial}{\partial z}p_1\left(\frac{1}{z}\right) - \frac{1}{3}(2z-1)p_1\left(\frac{1}{z}\right) \\
&\quad = -\sum_{j=1}^{l} \frac{j\alpha_j}{z^{j-1}} + \sum_{j=1}^{l} \frac{j\alpha_j}{z^j} - \sum_{j=1}^{l} \frac{2}{3}\frac{\alpha_j}{z^{j-1}} + \sum_{j=1}^{l} \frac{1}{3}\frac{\alpha_j}{z^j} = c_1, \\
&z(z-1)\frac{\partial}{\partial z}p_2\left(\frac{1}{z-1}\right) - \frac{1}{3}(2z-1)p_2\left(\frac{1}{z-1}\right) \\
&\quad = \left((z-1)^2 + (z-1)\right)\frac{\partial}{\partial z}p_2\left(\frac{1}{z-1}\right) - \frac{1}{3}\left(2(z-1)+1\right)p_2\left(\frac{1}{z-1}\right) \\
&\quad = -\sum_{j=1}^{m} \frac{j\beta_j}{(z-1)^{j-1}} - \sum_{j=1}^{m} \frac{j\beta_j}{(z-1)^j} - \sum_{j=1}^{l} \frac{2}{3}\frac{\beta_j}{(z-1)^{j-1}} - \sum_{j=1}^{l} \frac{1}{3}\frac{\beta_j}{(z-1)^j} = c_2, \\
&z(z-1)\frac{\partial}{\partial z}p_3(z) - \frac{1}{3}(2z-1)p_3(z) \\
&\quad = \sum_{j=1}^{n} j\gamma_j z^{j+1} - \sum_{j=1}^{m} j\gamma_j z^j - \sum_{j=0}^{m} \frac{2}{3}\gamma_j z^{j+1} + \sum_{j=0}^{m} \frac{1}{3}\gamma_j z^j = c_3,
\end{aligned} \tag{5.22}$$

where $c_1 + c_2 + c_3 = -1$. By these relations, we have

$$l\alpha_l - \frac{1}{3}\alpha_l = 0. \tag{5.23}$$

Hence $\alpha_l = 0$ follows. Moreover, one verifies that

$$\begin{gathered}
-j\alpha_j + (j-1)\alpha_{j-1} - \frac{2}{3}\alpha_j + \frac{1}{3}\alpha_{j-1} = 0, \quad j = 2,3,\dots,l, \\
-\alpha_1 - \frac{2}{3}\alpha_1 = c_1.
\end{gathered} \tag{5.24}$$

These imply that $\alpha_l = \alpha_{l-1} = \cdots = \alpha_1 = 0$, that is, $p_1(z) = 0$ and $c_1 = 0$. Similarly, one can show that $p_2(z) = 0$ and $c_2 = 0$. On the other hand, we have

$$n\gamma_n - \frac{2}{3}\gamma_n = 0. \tag{5.25}$$

Hence $\gamma_n = 0$ follows. Moreover, we have

$$\begin{gathered}
j\gamma_j - (j+1)\gamma_{j+1} - \frac{2}{3}\gamma_j + \frac{1}{3}\gamma_{j+1} = 0, \quad j = 1,2,\dots,n-1, \\
-\gamma_1 - \frac{2}{3}\gamma_0 + \frac{1}{3}\gamma_1 = 0.
\end{gathered} \tag{5.26}$$

These imply that $\gamma_n = \gamma_{n-1} = \cdots = \gamma_0 = 0$. On the other hand, we have

$$-\frac{1}{3}\gamma_0 = c_3. \tag{5.27}$$

Hence $c_3 = 0$ follows. This is contradiction. This completes the proof. □

Let

$$K = -\frac{4\phi_0(\xi)\phi_0'(\xi)}{9\xi g_3(\tau_*)}, \tag{5.28}$$

then we have

$$\begin{aligned} G(z,\xi) &= -9\xi g_3(\tau_*)(z-z^2)^{1/3}\int \frac{(1-2z)^2-K}{(z-z^2)^{4/3}}dz \\ &= -9\xi g_3(\tau_*)(z-z^2)^{1/3}\left(2\int \frac{1-2z}{(z-z^2)^{4/3}}dz + 4\int \frac{z^2-(K+1)/4}{(z-z^2)^{4/3}}dz\right). \end{aligned} \tag{5.29}$$

Hence, we can determine the integral constant so that $G(z,\xi)$ is the rational function of z if and only if $K = 3$. Since we assumed that $\phi_0(0) \neq 0$, we have

$$\phi_0(\xi) = \left(-\frac{27}{4}g_3(\tau_*)\xi^2 + c\right)^{1/2}, \quad c \neq 0. \tag{5.30}$$

Thus, we proved the following theorem.

Theorem 5.4. *Suppose $\tau_* \in \Theta_2$. Let $\phi_0(\xi)$ be defined as in (5.30). Then, the monodromy group for $\widehat{\mathrm{DL}}_2(\tau_*,0,\xi)$ is isomorphic to that for Gauss' hypergeometric differential equation (4.5) for every $\xi \in \mathbf{P}_1$.*

6. Monodromy group of $\widehat{\mathrm{DL}}_2(\tau_*,0,0)$

By Theorem 5.4, to carry out the calculation of the monodromy group of $\widehat{\mathrm{DL}}_2(\tau_*,0,\xi)$, it suffices to do it for $\widehat{\mathrm{DL}}_2(\tau_*,0,0)$.

We denote $D = \mathbf{P}_1 \setminus \{0,1,\infty\}$ and let $\pi_1(D,z_0)$ be the fundamental group of D with the base point $z_0 \in D$. Let

$$y_1(z) = \hat{f}(z), \qquad y_2(z) = z\hat{f}'(z) - \frac{1}{3}\hat{f}(z), \tag{6.1}$$

and $X(z) = {}^t(y_1(z), y_2(z))$. Then the Okubo form [4, page 177] of the Gauss' hypergeometric differential equation (4.5) is expressed as

$$(z-B)\frac{\partial}{\partial z}X = AX, \tag{6.2}$$

where

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1/3 & 1 \\ 4/9 & -2/3 \end{pmatrix}. \tag{6.3}$$

Let $\chi(z,z_0)=(X_1(z),X_2(z))$ be the fundamental system of solutions of the Okubo form (6.2) near z_0. On the other hand, let $\gamma\in\pi_1(D,z_0)$ and let $\chi^\gamma(z,z_0)$ be the analytic continuation along the closed path γ. Then there exists $M_\gamma\in\mathrm{GL}(2,\mathbb{C})$ such that $\chi^\gamma(z,z_0)=\chi(z,z_0)M_\gamma$. The map $\mu_\chi:\pi_1(D,z_0)\to\mathrm{GL}(2,\mathbb{C})$ defined by $\mu_\chi(\gamma)=M_\gamma$ is the linear representation of the fundamental group $\pi_1(D,z_0)$. The image $G=\mu_\chi(\pi_1(D,z_0))$ is called the monodromy group associated with the fundamental system $\chi(z,z_0)$.

Let $F(\alpha,\beta,\gamma;z)$ be the hypergeometric function. According to [4, pages 178-179], define the holomorphic solutions $Y(z,a)$ and the nonholomorphic solutions $X(z,a)$, for $a=0,1$, as follows:

$$\begin{gathered}
Y(z,0)={}^t\Big(-\frac{1}{3}F\Big(1,-\frac{2}{3},\frac{2}{3};z\Big),F\Big(1,-\frac{2}{3},-\frac{1}{3};z\Big)\Big),\\
X(z,0)={}^t\Big(z^{1/3}F\Big(\frac{4}{3},-\frac{1}{3},\frac{4}{3};z\Big),-\frac{1}{3}z^{1/3}F\Big(\frac{7}{3},\frac{2}{3},\frac{7}{3};z\Big)\Big),\\
Y(z,1)={}^t\Big(F\Big(1,-\frac{1}{3},\frac{2}{3};1-z\Big),\frac{9}{4}F\Big(1,-\frac{1}{3},-\frac{1}{3};1-z\Big)\Big),\\
X(z,1)={}^t\Big(3(z-1)^{1/3}F\Big(-\frac{1}{3},\frac{4}{3},\frac{1}{3};1-z\Big),(z-1)^{-2/3}F\Big(-\frac{4}{3},\frac{1}{3},-\frac{2}{3};1-z\Big)\Big).
\end{gathered}\tag{6.4}$$

The matrix functions $(Y(z,0),X(z,0))$ and $(Y(z,1),X(z,1))$ are both the fundamental systems of solutions of the Okubo form (6.2) defined near $z=0$ and $z=1$, respectively. By the method explained precisely in [4, pages 193–199], using these fundamental systems, one can solve the connection problem and finally obtain the generators M_0 and M_1 of the monodromy group G as follows:

$$M_0=\begin{pmatrix}\exp\left(\frac{2}{3}\pi i\right) & \exp\left(\frac{2}{3}\pi i\right)-1\\ 0 & 1\end{pmatrix},\qquad M_1=\begin{pmatrix}1 & 0\\ -\exp\left(-\frac{1}{3}\pi i\right)-1 & 1\end{pmatrix}.\tag{6.5}$$

It is easy to see that the monodromy group of the Okubo form (6.2) coincides with that of Gauss' hypergeometric equation (4.5).

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MULTIVALUED SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NONCONVEX-VALUED RIGHT-HAND SIDE

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We investigate the existence of mild solutions on a compact interval to some classes of semilinear neutral functional differential inclusions. We will rely on a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler and on Schaefer's fixed-point theorem combined with lower semicontinuous multivalued operators with decomposable values.

1. Introduction

This paper is concerned with the existence of mild solutions defined on a compact real interval for first- and second-order semilinear neutral functional differential inclusions (NFDIs).

In Section 3, we consider the following class of semilinear NFDIs:

$$\begin{gathered} \frac{d}{dt}\left[y(t)-f\left(t,y_t\right)\right] \in Ay(t)+F\left(t,y_t\right), \quad \text{a.e. } t \in J:=[0,b], \\ y(t)=\phi(t), \quad t \in [-r,0], \end{gathered} \tag{1.1}$$

where $F : J \times C([-r,0],E) \to \mathcal{P}(E)$ is a multivalued map, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$, $\phi \in C([-r,0],E)$, $f : J \times C([-r,0],E) \to E$, $\mathcal{P}(E)$ is the family of all subsets of E, and E is a real separable Banach space with norm $|\cdot|$.

Section 4 is devoted to the study of the following second-order semilinear NFDIs:

$$\begin{gathered} \frac{d}{dt}\left[y'(t)-f\left(t,y_t\right)\right] \in Ay(t)+F\left(t,y_t\right), \quad t \in J, \\ y(t)=\phi(t), \quad t \in [-r,0],\ y'(0)=\eta, \end{gathered} \tag{1.2}$$

where F, ϕ, f, $\mathcal{P}(E)$, and E are as in problem (1.1), A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, and $\eta \in E$.

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For any continuous function y defined on the interval $[-r,b]$ and any $t \in J$, we denote by y_t the element of $C([-r,0],E)$, defined by

$$y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0]. \tag{1.3}$$

Here, $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

In the last two decades, several authors have paid attention to the problem of existence of mild solutions to initial and boundary value problems for semilinear evolution equations. We refer the interested reader to the monographs by Goldstein [11], Heikkilä and Lakshmikantham [13], and Pazy [19], and to the paper of Heikkilä and Lakshmikantham [14]. In [17, 18], existence theorems of mild solutions for semilinear evolution inclusions are given by Papageorgiou. Recently, by means of a fixed-point argument and the semigroup theory, existence theorems of mild solutions on compact and noncompact intervals for first- and second-order semilinear NFDIs with a convex-valued right-hand side were obtained by Benchohra and Ntouyas in [1, 4]. Similar results for the case $A = 0$ are given by Benchohra and Ntouyas in [2]. Here, we will extend the above results to semilinear NFDIs with a nonconvex-valued right-hand side. The method we are going to use is to reduce the existence of solutions to problems (1.1) and (1.2) to the search for fixed points of a suitable multivalued map on the Banach space $C([-r,b],E)$. For each intial value problem (IVP), we give two results. In the first one, we use a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler [7] (see also Deimling [8]). This method was applied recently by Benchohra and Ntouyas in [3], in the case when $A = 0$ and $f \equiv 0$. In the second one, we use Schaefer's theorem combined with a selection theorem of Bressan and Colombo [5] for lower semicontinuous (l.s.c) multivalued operators with decomposable values.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this paper.

We denote by $\mathcal{P}(E)$ the set of all subsets of E normed by $\|\cdot\|_{\mathcal{P}}$. Let $C([-r,0],E)$ be the Banach space of all continuous functions from $[-r,0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \le \theta \le 0\}. \tag{2.1}$$

By $C([-r,b],E)$ we denote the Banach space of all continuous functions from $[-r,b]$ into E with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r,b]\}. \tag{2.2}$$

A measurable function $y : J \to E$ is Bochner-integrable if and only if $|y|$ is Lebesgue-integrable. (For properties of the Bochner-integral, see, e.g., Yosida [24].) By $L^1(J,E)$ we denote the Banach space of functions $y : J \to E$ which are Bochner-integrable and normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt, \tag{2.3}$$

and by $B(E)$ the Banach space of bounded linear operators from E to E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}. \tag{2.4}$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),
(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s,t \in \mathbb{R}$,
(iii) the map $t \mapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in \mathbb{R}. \tag{2.5}$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y|_{t=0}. \tag{2.6}$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Fattorini [9], Goldstein [11], and to the papers of Travis and Webb [22, 23]. For properties of semigroup theory, we refer the interested reader to the books of Goldstein [11] and Pazy [19].

Let (X,d) be a metric space. We use the following notations:

$$\begin{aligned} P(X) &= \{Y \in \mathcal{P}(X) : Y \neq \emptyset\}, \\ P_{cl}(X) &= \{Y \in P(X) : Y \text{ closed}\}, \\ P_b(X) &= \{Y \in P(X) : Y \text{ bounded}\}, \\ P_{cp}(X) &= \{Y \in P(X) : Y \text{ compact}\}. \end{aligned} \tag{2.7}$$

Consider $H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{2.8}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$.

Then $(P_{b,cl}(X),H_d)$ is a metric space and $(P_{cl}(X),H_d)$ is a generalized (complete) metric space (see [16]).

A multivalued map $N : J \to P_{cl}(X)$ is said to be measurable if, for each $x \in X$, the function $Y : J \to \mathbb{R}$, defined by

$$Y(t) = d(x,N(t)) = \inf\{d(x,z) : z \in N(t)\}, \tag{2.9}$$

is measurable.

Definition 2.1. A multivalued operator $N : X \to P_{cl}(X)$ is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \leq \gamma d(x,y), \quad \text{for each } x,y \in X; \tag{2.10}$$

(b) contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

The multivalued operator N has *a fixed point* if there is $x \in X$ such that $x \in N(x)$. The fixed-point set of the multivalued operator N will be denoted by $\operatorname{Fix} N$.

We recall the following fixed-point theorem for contraction multivalued operators given by Covitz and Nadler [7] (see also Deimling [8, Theorem 11.1]).

Theorem 2.2. *Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then* $\operatorname{Fix} N \neq \varnothing$.

Let $\mathcal{A}$ be a subset of $J \times C([-r,0],E)$. The set $\mathcal{A}$ is $\mathcal{L} \otimes \mathcal{B}$ measurable if $\mathcal{A}$ belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times \mathcal{D}$, where $\mathcal{J}$ is Lebesgue-measurable in J and $\mathcal{D}$ is Borel-measurable in $C([-r,0],E)$. A subset B of $L^1(J,E)$ is decomposable if, for all $u,v \in B$ and $\mathcal{J} \subset J$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in B$, where $\chi_{\mathcal{J}}$ denotes the characteristic function for $\mathcal{J}$.

Let E be a Banach space, X a nonempty closed subset of E, and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. The operator G is l.s.c. if the set $\{x \in X : G(x) \cap C \neq \varnothing\}$ is open for any open set C in E. For more details on multivalued maps, we refer to the books of Deimling [8], Górniewicz [12], Hu and Papageorgiou [15], and Tolstonogov [21].

Definition 2.3. Let Y be a separable metric space and let $N : Y \to \mathcal{P}(L^1(J,E))$ be a multivalued operator. The operator N has property (BC) if it satisfies the following conditions:

(1) N is l.s.c.;
(2) N has nonempty, closed, and decomposable values.

Let $F : J \times C([-r,0],E) \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F} : C([-r,b],E) \longrightarrow \mathcal{P}(L^1(J,E)) \tag{2.11}$$

by letting

$$\mathcal{F}(y) = \{w \in L^1(J,E) : w(t) \in F(t,y_t) \text{ for a.e. } t \in J\}. \tag{2.12}$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated with F.

Definition 2.4. Let $F : J \times C([-r,0],E) \to \mathcal{P}(E)$ be a multivalued function with nonempty compact values. We say that F is of l.s.c. type if its associated Niemytzki operator $\mathcal{F}$ is l.s.c. and has nonempty closed and decomposable values.

Next we state a selection theorem due to Bressan and Colombo.

Theorem 2.5 (see [5]). *Let Y be separable metric space and let $N : Y \to \mathcal{P}(L^1(J,E))$ be a multivalued operator which has property (BC). Then N has a continuous selection, that is, there exists a continuous function (single-valued) $g : Y \to L^1(J,E)$ such that $g(y) \in N(y)$ for every $y \in Y$.*

3. First-order semilinear NFDIs

Now, we are able to state and prove our first theorem for the IVP (1.1). Before stating and proving this result, we give the definition of a mild solution of the IVP (1.1).

Definition 3.1. A function $y \in C([-r,b],E)$ is called a mild solution of (1.1) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$, a.e. on J, $y_0 = \phi$, and

$$\begin{aligned} y(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ + \int_0^t T(t-s)v(s)\,ds, \quad t \in J. \end{aligned} \tag{3.1}$$

Theorem 3.2. *Assume that*

(H1) *A is the infinitesimal generator of a semigroup of bounded linear operators $T(t)$ in E such that $\|T(t)\|_{B(E)} \le M_1$, for some $M_1 > 0$ and $\|AT(t)\|_{B(E)} \le M_2$, for each $t > 0$, $M_2 > 0$;*

(H2) *$F : J \times C([-r,0],E) \to P_{cp}(E)$ has the property that $F(\cdot,u) : J \to P_{cp}(E)$ is measurable for each $u \in C([-r,0],E)$;*

(H3) *there exists $l \in L^1(J,\mathbb{R})$ such that*

$$H_d(F(t,u),F(t,\overline{u})) \le l(t)\|u - \overline{u}\|, \tag{3.2}$$

for each $t \in J$ and $u,\overline{u} \in C([-r,0],E)$, and

$$d(0,F(t,0)) \le l(t), \quad \text{for almost each } t \in J; \tag{3.3}$$

(H4) *$|f(t,u) - f(t,\overline{u})| \le c\|u - \overline{u}\|$ for each $t \in J$ and $u,\overline{u} \in C([-r,0],E)$, where c is a nonnegative constant;*

(H5) *$c + M_2cb + M_1\ell^* < 1$, where $\ell^* = \int_0^b l(s)ds$.*

Then the IVP (1.1) has at least one mild solution on $[-r,b]$.

Proof. Transform problem (1.1) into a fixed-point problem. Consider the multivalued operator $N : C([-r,b],E) \to \mathcal{P}(C([-r,b],E))$, defined by

$$N(y) := \{h \in C([-r,b],E)\} \tag{3.4}$$

such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \displaystyle\int_0^t AT(t-s)f(s,y_s)\,ds & \\ +\displaystyle\int_0^t T(t-s)v(s)\,ds, & \text{if } t \in J, \end{cases} \tag{3.5}$$

where

$$v \in S_{F,y} = \{v \in L^1(J,E) : v(t) \in F(t,y_t) \text{ for a.e. } t \in J\}. \tag{3.6}$$

We remark that the fixed points of N are solutions to (1.1). Also, for each $y \in C([-r,b],E)$, the set $S_{F,y}$ is nonempty since, by (H2), F has a measurable selection (see [6, Theorem III.6]).

We will show that N satisfies the assumptions of Theorem 2.2. The proof will be given in two steps.

Step 1. We prove that $N(y) \in P_{cl}(C([-r,b],E))$ for each $y \in C([-r,b],E)$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \to \tilde{y}$ in $C([-r,b],E)$. Then $\tilde{y} \in C([-r,b],E)$ and there exists $g_n \in S_{F,y}$ such that

$$\begin{aligned} y_n(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ + \int_0^t T(t-s)g_n(s)\,ds, \quad t \in J. \end{aligned} \tag{3.7}$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence if necessary to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F,y}$. Then, for each $t \in J$,

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ + \int_0^t T(t-s)g(s)\,ds, \quad t \in J. \end{aligned} \tag{3.8}$$

So, $\tilde{y} \in N(y)$.

Step 2. We prove that $H_d(N(y_1),N(y_2)) \leq \gamma \|y_1 - y_2\|_\infty$ for each $y_1, y_2 \in C([-r,b],E)$, where $\gamma < 1$.

Let $y_1, y_2 \in C([-r,b],E)$ and $h_1 \in N(y_1)$. Then there exists $g_1(t) \in F(t,y_{1t})$ such that

$$\begin{aligned} h_1(t) = T(t)[\phi(0) - f(0,\phi)] + f(t,y_{1t}) + \int_0^t AT(t-s)f(s,y_{1s})\,ds \\ + \int_0^t T(t-s)g_1(s)\,ds, \quad t \in J. \end{aligned} \tag{3.9}$$

From (H3), it follows that

$$H_d(F(t,y_{1t}),F(t,y_{2t})) \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in J. \tag{3.10}$$

Hence, there is $w \in F(t,y_{2t})$ such that

$$|g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|, \quad t \in J. \tag{3.11}$$

Consider $U : J \to \mathcal{P}(E)$ given by

$$U(t) = \{w \in E : |g_1(t) - w| \leq l(t)\|y_{1t} - y_{2t}\|\}. \tag{3.12}$$

Since the multivalued operator $V(t) = U(t) \cap F(t, y_{2t})$ is measurable (see [6, Proposition III.4]), there exists $g_2(t)$, a measurable selection for V. So, $g_2(t) \in F(t, y_{2t})$ and

$$|g_1(t) - g_2(t)| \leq l(t)\|y_{1t} - y_{2t}\|, \quad \text{for each } t \in J. \tag{3.13}$$

We define, for each $t \in J$,

$$\begin{aligned} h_2(t) = T(t)[\phi(0) - f(0,\phi)] + f(t, y_{2t}) + \int_0^t AT(t-s) f(s, y_{2s})\,ds \\ + \int_0^t T(t-s) g_2(s)\,ds. \end{aligned} \tag{3.14}$$

Then we have

$$\begin{aligned} &|h_1(t) - h_2(t)| \\ &\quad \leq |f(t, y_{1t}) - f(t, y_{2t})| + M_2 \int_0^t |f(t, y_{1s}) - f(t, y_{2s})|\,ds + M_1 \int_0^t |g_1(s) - g_2(s)|\,ds \\ &\quad \leq c\|y_{1t} - y_{2t}\| + M_2 c \int_0^t \|y_{1s} - y_{2s}\|\,ds + M_1 \int_0^t l(s)\|y_{1s} - y_{2s}\|\,ds \\ &\quad \leq c\|y_1 - y_2\|_\infty + M_2 cb\|y_1 - y_2\|_\infty + M_1\|y_1 - y_2\|_\infty \int_0^b l(s)\,ds \\ &\quad \leq [c + M_2 cb + M_1 \ell^*]\|y_1 - y_2\|_\infty. \end{aligned} \tag{3.15}$$

Then

$$\|h_1 - h_2\|_\infty \leq [c + M_2 cb + M_1 \ell^*]\|y_1 - y_2\|_\infty. \tag{3.16}$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N(y_1), N(y_2)) \leq [c + M_2 cb + M_1 \ell^*]\|y_1 - y_2\|_\infty. \tag{3.17}$$

Since $\gamma := c + M_2 cb + M_1 \ell^* < 1$, N is a contraction, and thus, by Theorem 2.2, it has a fixed point y which is a mild solution to (1.1). □

Remark 3.3. Recall that, in the proof of Theorem 3.2, we have assumed that $\gamma < 1$. Since this assumption is hard to verify, we would like point out that using the well-known Bielecki's renorming method, it can be simplified. The technical details are omitted here.

By the help of Schaefer's fixed-point theorem, combined with the selection theorem of Bressan and Colombo for l.s.c. maps with decomposable values, we will present the second existence result for problem (1.1). Before this, we introduce the following hypotheses which are assumed hereafter:

(C1) $F : J \times C([-r,0], E) \to \mathcal{P}(E)$ is a nonempty compact-valued multivalued map such that
  (a) $(t,u) \mapsto F(t,u)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable,
  (b) $u \mapsto F(t,u)$ is l.s.c. for a.e. $t \in J$;

(C2) for each $\rho > 0$, there exists a function $h_\rho \in L^1(J,\mathbb{R}^+)$ such that

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \le h_\rho(t) \tag{3.18}$$

for a.e. $t \in J$, $u \in C([-r,0],E)$ with $\|u\| \le \rho$.

In the proof of our following theorem, we will need the next auxiliary result.

Lemma 3.4 [10]. *Let $F : J \times C([-r,0],E) \to \mathcal{P}(E)$ be a multivalued map with nonempty compact values. Assume that (C1) and (C2) hold. Then F is of l.s.c. type.*

Theorem 3.5. *Assume that hypotheses (C1), (C2), and the following ones are satisfied.*

(A0) *A is the infinitesimal generator of a compact semigroup $T(t)$, $t > 0$, such that $\|T(t)\|_{B(E)} \le M_1$, $M_1 > 0$, and $\|AT(t)\|_{B(E)} \le M_2$, for each $t \ge 0$, $M_2 > 0$.*

(A1) *There exist constants $0 \le c_1 < 1$ and $c_2 \ge 0$ such that*

$$|f(t,u)| \le c_1\|u\| + c_2, \quad t \in J,\ u \in C([-r,0],E). \tag{3.19}$$

(A2) *The function f is completely continuous and, for any bounded set $\mathcal{A} \subseteq C([-r,b],E)$, the set $\{t \to f(t,y_t) : y \in \mathcal{A}\}$ is equicontinuous in $C(J,E)$.*

(A3) *There exist $p \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi : \mathbb{R}^+ \to (0,\infty)$ such that*

$$\|F(t,u)\|_{\mathcal{P}} \le p(t)\psi(\|u\|) \tag{3.20}$$

for a.e. $t \in J$ and each $u \in C([-r,0],E)$ with

$$\int_0^b \widehat{M}(s)ds < \int_{c_0}^\infty \frac{du}{u+\psi(u)}, \tag{3.21}$$

where

$$\begin{aligned} c_0 &= \frac{1}{1-c_1}\left[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + bc_2M_2\right], \\ \widehat{M}(t) &= \max\left\{\frac{1}{1-c_1}c_1M_2, \frac{1}{1-c_1}M_1p(t)\right\}. \end{aligned} \tag{3.22}$$

(A4) *For each $t \in J$, the multivalued map $F(t,\cdot) : C([-r,0],E) \mapsto \mathcal{P}(E)$ maps bounded sets into relatively compact sets.*

Then problem (1.1) has at least one solution.

Proof. Hypotheses (C1) and (C2) imply, by Lemma 3.4, that F is of l.s.c. type. Then, from Theorem 2.5, there exists a continuous function $g : C([-r,b],E) \to L^1([0,b],E)$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C([-r,b],E)$.

Consider the operator $N_1 : C([-r,b],E) \to \mathcal{P}(C([-r,b],E))$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) & \\ +\int_0^t AT(t-s)f(s,y_s)\,ds + \int_0^t T(t-s)g(y)(s)\,ds, & \text{if } t \in J. \end{cases} \tag{3.23}$$

We will show that N_1 is completely continuous. The proof will be given in several steps.

Step 1. The operator N_1 sends bounded sets into bounded sets in $C([-r,b],E)$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant l such that, for each $y \in B_q := \{y \in C([-r,b],E) : \|y\|_\infty \le q\}$, one has $\|N_1(y)\|_\infty \le l$. Let $y \in B_q$, then

$$\begin{aligned} N_1(y)(t) = {} & T(t)[\phi(0) - f(0,\phi)] + f(t,y_t) + \int_0^t AT(t-s)f(s,y_s)\,ds \\ & + \int_0^t T(t-s)g(y)(s)\,ds, \quad t \in J. \end{aligned} \tag{3.24}$$

From (A0), (C2), (A1), and (A3), we have, for each $t \in J$,

$$\begin{aligned} & |N_1(y)(t)| \\ & \quad \le M_1[\|\phi\| + |f(0,\phi)|] + |f(t,y_t)| \\ & \qquad + \int_0^t ||AT(t-s)||_{B(E)} |f(s,y_s)|\,ds + \int_0^t ||T(t-s)||_{B(E)} |g(y)(s)|\,ds \\ & \quad \le M_1[\|\phi\| + c_1 q + c_2] + c_1 q + c_2 + bM_2 c_1 q + bM_2 c_2 + M_1 ||h_q||_{L^1}. \end{aligned} \tag{3.25}$$

Then, for each $h \in N(B_q)$, we have

$$||N_1(y)||_\infty \le M_1[\|\phi\| + c_1 q + c_2] + c_1 q + c_2 + bM_2 c_1 q + bM_2 c_2 + M_1 ||h_q||_{L^1} := l. \tag{3.26}$$

Step 2. The operator N_1 sends bounded sets in $C([-r,b],E)$ into equicontinuous sets.

Using (A2), it suffices to show that the operator $N_2 : C([-r,b],E) \to C([-r,b],E)$, defined by

$$N_2(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0]; \\ T(t)\phi(0) + \int_0^t AT(t-s)f(s,y_s)\,ds & \\ +\int_0^t T(t-s)g(y)(s)\,ds, & \text{if } t \in J, \end{cases} \tag{3.27}$$

maps bounded sets into equicontinuous sets of $C([-r,b],E)$. Let $u_1, u_2 \in J$, $u_1 < u_2$, let $B_q := \{y \in C([-r,b],E) : \|y\|_\infty \le q\}$ be a bounded set in $C([-r,b],E)$, and $y \in B_q$. Then we have

$$
\begin{aligned}
&|N_2(y)(u_2)-N_2(y)(u_1)| \\
&\quad\le |T(u_1)\phi(0)-T(u_2)\phi(0)| + (c_1q+c_2)\int_0^{u_1}\|AT(u_2-s)-AT(u_1-s)\|_{B(E)}ds \\
&\qquad + (c_1q+c_2)\int_{u_1}^{u_2}\|AT(u_2-s)\|_{B(E)}ds + \int_0^{u_1}\|T(u_2-s)-T(u_1-s)\|_{B(E)}h_q(s)ds \\
&\qquad + \int_{u_1}^{u_2}\|T(u_2-s)\|_{B(E)}h_q(s)ds.
\end{aligned}
\tag{3.28}
$$

As $u_2 \to u_1$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $u_1 < u_2 \le 0$ and $u_1 \le 0 \le u_2$ is obvious.

Step 3. The operator N_2 is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([-r,b],E)$. Then

$$
\begin{aligned}
&|N_2(y_n)(t)-N_2(y)(t)| \\
&\quad\le M_2\int_0^t |f(s,y_{n,s})-f(s,y_s)|ds + M_1\int_0^b |g(y_n)(s)-g(y)(s)|ds.
\end{aligned}
\tag{3.29}
$$

Since the function g is continuous and f is completely continuous, then

$$
\begin{aligned}
&\|N_2(y_n)-N_2(y)\|_\infty \\
&\quad\le M_2\sup_{t\in J}\int_0^t |f(s,y_{ns})-f(s,y_s)|ds + M_1\|g(y_n)-g(y)\|_{L^1} \longrightarrow 0.
\end{aligned}
\tag{3.30}
$$

As a consequence of Steps 1, 2, and 3 and (A2), (A4), together with the Arzelá-Ascoli theorem, we can conclude that $N_2 : C([-r,b],E) \to C([-r,b],E)$ is completely continuous.

Step 4. The set $\mathscr{E}(N_1) = \{y \in C([-r,b],E) : y = \lambda N_1(y)$ for some $\lambda \in (0,1)\}$ is bounded.

Let $y \in \mathscr{E}(N_1)$. Then $y = \lambda N_1(y)$ for some $0 < \lambda < 1$, and for $t \in [0,b]$, we have

$$
\begin{aligned}
y(t) = \lambda\Big[&T(t)(\phi(0)-f(0,\phi)) + f(t,y_t) \\
&+ \int_0^t AT(t-s)f(s,y_s)ds + \int_0^t T(t-s)g(y)(s)ds\Big].
\end{aligned}
\tag{3.31}
$$

This implies, by (A0), (A1), and (A3), that for each $t \in J$, we have

$$
\begin{aligned}
|y(t)| \le{}& M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\|y_t\| + c_2 + bc_2M_2 + c_1M_2\int_0^t \|y_s\|ds \\
&+ M_1\int_0^t p(s)\psi(\|y_s\|)ds.
\end{aligned}
\tag{3.32}
$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le b. \tag{3.33}$$

Let $t^* \in [-r,b]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by inequality (3.32), we have, for $t \in J$,

$$\begin{aligned}\mu(t) \le {} & M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\mu(t) + c_2 + bc_2M_2 + c_1M_2\int_0^t \mu(s)ds \\ & + M_1\int_0^t p(s)\psi(\mu(s))ds.\end{aligned} \tag{3.34}$$

Thus

$$\begin{aligned}\mu(t) \le \frac{1}{1-c_1}\Bigg[& M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + bc_2M_2 + c_1M_2\int_0^t \mu(s)ds \\ & + M_1\int_0^t p(s)\psi(\mu(s))ds\Bigg], \quad t \in J.\end{aligned} \tag{3.35}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and inequality (3.35) holds. We take the right-hand side of inequality (3.35) as $v(t)$; then we have

$$\begin{gathered} v(0) = \frac{1}{1-c_1}[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 + bc_2M_2], \quad \mu(t) \le v(t),\ t \in J, \\ v'(t) = \frac{1}{1-c_1}\{c_1M_2\mu(t) + M_1p(t)\psi(\mu(t))\}, \quad t \in J.\end{gathered} \tag{3.36}$$

Since ψ is nondecreasing, we have

$$v'(t) \le \widehat{M}(t)\{v(t) + \psi(v(t))\}, \quad t \in J. \tag{3.37}$$

From this inequality, it follows that

$$\int_0^t \frac{v'(s)}{v(s) + \psi(v(s))}ds \le \int_0^t \widehat{M}(s)ds. \tag{3.38}$$

We then have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \le \int_0^t \widehat{M}(s)ds \le \int_0^b \widehat{M}(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}. \tag{3.39}$$

This inequality implies that there exists a constant K_1 such that $v(t) \le K_1$, $t \in J$, and hence $\mu(t) \le K_1$, $t \in J$. Since for every $t \in J$, $\|y_t\| \le \mu(t)$, we have

$$\|y\|_\infty \le K_1' := \max\{\|\phi\|, K_1\}, \tag{3.40}$$

where K_1' depends only on b, M_1, and M_2 and on the functions p and ψ. This shows that $\mathscr{E}(N_1)$ is bounded. As a consequence of Schaefer's theorem (see [20]), we deduce that N_1 has a fixed point y which is a solution to problem (1.1). $\square$

4. Second-order semilinear NFDIs

Definition 4.1. A function $y \in C([-r,b],E)$ is called a mild solution of (1.2) if there exists a function $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, $y_0 = \phi$, $y'(0) = \eta$, and

$$y(t) = C(t)\phi(0) + S(t)\big(\eta - f(0,\phi)\big) + \int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)v(s)\,ds. \tag{4.1}$$

Theorem 4.2. *Assume that hypotheses (H2), (H3), and (H4) and the following ones are satisfied:*

(H6) *A is an infinitesimal generator of a given strongly continuous bounded and compact cosine family $\{C(t) : t > 0\}$ with $\|C(t)\|_{B(E)} \leq \overline{M}$;*
(H7) $\overline{M}b(c + \ell^*) < 1$.

Then the IVP (1.2) has at least one mild solution on $[-r,b]$.

Proof. Transform problem (1.2) into a fixed-point problem. Consider the multivalued operator $N_3 : C([-r,b],E) \to \mathcal{P}(C([-r,b],E))$ defined by

$$N_3(y) := \{h \in C([-r,b],E)\} \tag{4.2}$$

such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\phi(0) + S(t)\big(\eta - f(0,\phi)\big) & \\ +\displaystyle\int_0^t C(t-s)f(s,y_s)\,ds & \\ +\displaystyle\int_0^t S(t-s)g(s)\,ds, & \text{if } t \in J, \end{cases} \tag{4.3}$$

where $g \in S_{F,y}$.

We will show that N_3 satisfies the assumptions of Theorem 2.2. The proof will be given in two steps.

Step 1. We prove that $N_3(y) \in P_{cl}(C([-r,b],E))$ for each $y \in C([-r,b],E)$.

Indeed, let $(y_n)_{n\geq 0} \in N_3(y)$ such that $y_n \to \tilde{y}$ in $C([-r,b],E)$. Then $\tilde{y} \in C([-r,b],E)$ and there exists $g_n \in S_{F,y}$ such that

$$\begin{aligned} y_n(t) = {} & C(t)\phi(0) + S(t)\big(\eta - f(0,\phi)\big) \\ & + \int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)g_n(s)\,ds. \end{aligned} \tag{4.4}$$

Using the fact that F has compact values and from (H3), we may pass to a subsequence, if necessary, to get that g_n converges to g in $L^1(J,E)$ and hence $g \in S_{F,y}$. Then, for each $t \in J$,

$$\begin{aligned} y_n(t) \longrightarrow \tilde{y}(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_s)\,ds \\ + \int_0^t S(t-s)g(s)\,ds. \end{aligned} \tag{4.5}$$

So, $\tilde{y} \in N_3(y)$.

Step 2. We prove that $H_d(N_3(y_1),N_3(y_2)) \le \gamma\|y_1 - y_2\|_\infty$ for each $y_1, y_2 \in C([-r,b],E)$, where $\gamma < 1$.

Let $y_1, y_2 \in C([-r,b],E)$ and $h_1 \in N_1(y_1)$. Then there exists $g_1(t) \in F(t,y_{1t})$ such that

$$\begin{aligned} h_1(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_{1s})\,ds \\ + \int_0^t S(t-s)g_1(s)\,ds, \quad t \in J. \end{aligned} \tag{4.6}$$

From (H3), it follows that $H_d(F(t,y_{1t}),F(t,y_{2t})) \le l(t)\|y_{1t} - y_{2t}\|$, $t \in J$. Hence, there is $w \in F(t,y_{2t})$ such that $|g_1(t) - w| \le l(t)\|y_{1t} - y_{2t}\|$, $t \in J$. Consider $U : J \to \mathcal{P}(E)$ given by $U(t) = \{w \in E : |g_1(t) - w| \le l(t)\|y_{1t} - y_{2t}\|\}$. Since the multivalued operator $V(t) = U(t) \cap F(t,y_{2t})$ is measurable (see [6, Proposition III.4]), there exists $g_2(t)$, a measurable selection for V. So, $g_2(t) \in F(t,y_{2t})$ and $|g_1(t) - g_2(t)| \le l(t)\|y_{1t} - y_{2t}\|$, for each $t \in J$. We define, for each $t \in J$,

$$h_2(t) = C(t)\phi(0) + S(t)(\eta - f(0,\phi)) + \int_0^t C(t-s)f(s,y_{2s})\,ds + \int_0^t S(t-s)g_2(s)\,ds. \tag{4.7}$$

Then we have

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \overline{M}\int_0^t |f(s,y_{1s}) - f(s,y_{2s})|\,ds + \overline{M}b\int_0^t |g_1(s) - g_2(s)|\,ds \\ &\le \overline{M}c\int_0^t ||y_{1s} - y_{2s}||\,ds + \overline{M}b\int_0^t l(s)||y_{1s} - y_{2s}||\,ds \\ &\le \overline{M}cb||y_1 - y_2||_\infty + \overline{M}b||y_1 - y_2||_\infty\int_0^t l(s)\,ds \\ &\le \overline{M}b(c + \ell^*)||y_1 - y_2||_\infty. \end{aligned} \tag{4.8}$$

Then

$$||h_1 - h_2||_\infty \le \overline{M}b(c + \ell^*)||y_1 - y_2||_\infty. \tag{4.9}$$

By the analogous relation, obtained by interchanging the roles of y_1 and y_2, it follows that

$$H_d(N_3(y_1),N_3(y_2)) \le \overline{M}b(c + \ell^*)||y_1 - y_2||_\infty. \tag{4.10}$$

Since $\overline{M}b(c + \ell^*) < 1$, N_1 is a contraction, and thus, by Theorem 2.2, it has a fixed point y which is a mild solution to (1.2). □

Theorem 4.3. *Assume that hypotheses (H6), (C1), (C2), (A1) (with $c_1, c_2 \geq 0$), (A2), and (A4) and the following one are satisfied:*

(A5) *there exist $p \in L^1(J, \mathbb{R}^+)$ and a continuous and nondecreasing function $\psi : \mathbb{R}^+ \to (0, \infty)$ such that, for a.e. $t \in J$ and each $u \in C([-r,0],E)$,*

$$\|F(t,u)\|_{\mathcal{P}} \leq p(t)\psi(\|u\|), \qquad \int_0^b M(s)ds < \int_{\bar{c}}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}, \tag{4.11}$$

where

$$\begin{gathered} \bar{c} = \overline{M}\|\phi\| + b\overline{M}[|\eta| + c_1\|\phi\| + 2c_2], \\ M(t) = \max\{1, c_1\overline{M}, b\overline{M}p(t)\}. \end{gathered} \tag{4.12}$$

Then the IVP (1.2) has at least one solution on $[-r,b]$.

Proof. Hypotheses (C1) and (C2) imply, by Lemma 3.4, that F is of l.s.c. type. Then, from Theorem 2.5, there exists a continuous function $g : C([-r,b],E) \to L^1([0,b],E)$ such that $g(y) \in \mathcal{F}(y)$ for all $y \in C([-r,b],E)$. Consider the operator $N_4 : C([-r,b],E) \to C([-r,b],E)$ defined by

$$N_4(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\phi(0) + S(t)[\eta - f(0,\phi)] & \\ + \int_0^t C(t-s)f(s,y_s)\,ds + \int_0^t S(t-s)g(y)(s)ds, & \text{if } t \in J. \end{cases} \tag{4.13}$$

As in Theorem 3.5, we can show that N_4 is completely continuous.

Now, we only prove that the set

$$\mathcal{E}(N_4) := \{y \in C([-r,b],E) : y = \lambda N_4(y) \text{ for some } 0 < \lambda < 1\} \tag{4.14}$$

is bounded.

Let $y \in \mathcal{E}(N_4)$. Then $y = \lambda N_4(y)$ for some $0 < \lambda < 1$. Thus

$$\begin{aligned} y(t) = \lambda C(t)\phi(0) + \lambda S(t)[\eta - f(0,\phi)] + \lambda \int_0^t C(t-s)f(s,y_s)\,ds \\ + \lambda \int_0^t S(t-s)g(y)(s)ds, \quad t \in J. \end{aligned} \tag{4.15}$$

This implies by (H4), (H6), (A1), and (A5) that, for each $t \in J$,

$$\begin{aligned} |y(t)| \leq \overline{M}\|\phi\| + b\overline{M}(|\eta| + c_1\|\phi\| + c_2) + c_1\overline{M}\int_0^t \|y_s\|ds + bc_2\overline{M} \\ + b\overline{M}\int_0^t p(s)\psi(\|y_s\|)\,ds. \end{aligned} \tag{4.16}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b. \tag{4.17}$$

Let $t^* \in [-r,b]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by (4.16) we have, for $t \in J$,

$$\begin{aligned} \mu(t) \leq \overline{M}\|\phi\| + b\overline{M}(|\eta| + c_1\|\phi\| + 2c_2) + \int_0^t M(s)\mu(s)ds \\ + \int_0^t M(s)\psi(\mu(s))ds. \end{aligned} \tag{4.18}$$

If $t^* \in [-r,0]$, then $\mu(t) = \|\phi\|$ and the previous inequality holds.

We take the right-hand side of the above inequality as $v(t)$; then we have

$$\begin{gathered} v(0) = \overline{M}\|\phi\| + b\overline{M}(|\eta| + c_1\|\phi\| + 2c_2), \quad \mu(t) \leq v(t),\ t \in J, \\ v'(t) = M(t)\mu(t) + M(t)\psi(\mu(t)), \quad t \in J. \end{gathered} \tag{4.19}$$

Using the nondecreasing character of ψ, we get

$$v'(t) \leq M(t)[v(t) + \psi(v(t))], \quad t \in J. \tag{4.20}$$

This implies, for each $t \in J$, that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^b M(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}. \tag{4.21}$$

This inequality implies that there exists a constant K_2 such that $v(t) \leq K_2$, $t \in J$, and hence $\mu(t) \leq K_2$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty \leq K_2' := \max\{\|\phi\|, K_2\}, \tag{4.22}$$

where K_2' depends only on b, $\overline{M}$, and on the functions p and ψ. This shows that $\mathscr{E}(N_4)$ is bounded.

Set $X := C([-r,b],E)$. As a consequence of Schaefer's theorem (see [20]), we deduce that N_4 has a fixed point y which is a solution to problem (1.2). □

Remark 4.4. The reasoning used above can be applied to obtain the existence results for the following first- and second-order semilinear neutral functional integrodifferential inclusions of Volterra type:

$$\begin{gathered} \frac{d}{dt}[y(t) - f(t,y_t)] - Ay \in \int_0^t k(t,s)F(s,y_s)ds, \quad \text{a.e. } t \in J, \\ y(t) = \phi(t), \quad t \in [-r,0], \\ \frac{d}{dt}[y'(t) - f(t,y_t)] - Ay \in \int_0^t k(t,s)F(s,y_s)ds, \quad \text{a.e. } t \in J, \\ y(t) = \phi(t), \quad t \in [-r,0], \\ y'(0) = \eta, \end{gathered} \tag{4.23}$$

where A, F, f, ϕ, and η are as in problems (1.1) and (1.2) and $k : D \to \mathbb{R}$, $D = \{(t,s) \in J \times J : t \geq s\}$.

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CONDITIONS FOR THE OSCILLATION OF SOLUTIONS OF ITERATIVE EQUATIONS

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We give some oscillation criteria for linear iterative functional equations. We compare obtained theorems with known results. We give applications to discrete equations too.

The problem of oscillation of solutions of differential and difference equations has been investigated by many authors since in the literature, there are many oscillation criteria for these equations (see [2, 5]). However, for the iterative functional equations, the situation is different. Our aim is to give some new oscillation criteria for iterative functional equations. We are of the opinion that it is worth considering iterative functional equations because, in particular, they are recurrence equations which have a lot of applications. They can be used to describe processes in many areas such as biology, meteorology, economics, and so on (see [6]). This paper is concerned with the oscillatory solutions of linear iterative functional equations of the form

$$Q_0(t)x(t)+Q_1(t)x\big(g(t)\big)+Q_2(t)x\big(g^2(t)\big)+\cdots+Q_{m+1}(t)x\big(g^{m+1}(t)\big)=0, \quad m\geq 1, \tag{1}$$

where x is an unknown real-valued function and $Q_k : I \to \mathbb{R}$, for $k = 0,1,\ldots,m+1$, and $g : I \to I$ are given functions, such that $\mathbb{R}$ is the set of real numbers and I denotes an unbounded subset of $\mathbb{R}_+ = [0,\infty)$. By g^m we mean the mth iterate of the function g, that is,

$$g^0(t) = t, \quad g^{m+1}(t) = g\big(g^m(t)\big), \quad t \in I,\ m = 0,1,\ldots. \tag{2}$$

By g^{-1} we mean the inverse function of g and $g^{-m-1}(t) = g^{-1}(g^{-m}(t))$. In this paper, upper indices at the sign of a function will denote iterations. In each instance, we have the relation $g^1(t) = g(t)$. Exponents of a power of a function will be written after a bracket containing the whole expression of the function. We also assume that

$$g(t) \neq t, \quad \lim_{t\to\infty} g(t) = \infty, \quad t \in I. \tag{3}$$

Moreover, we assume that g has an inverse function.

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By a solution of (1), we mean a function $x : I \to \mathbb{R}$ such that $\sup\{|x(s)| : s \in I_{t_0} = [t_0,\infty) \cap I\} > 0$ for any $t_0 \in \mathbb{R}_+$ and x satisfies I in (1).

A solution x of (1) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}$, $t_n \in I$, such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n)x(t_{n+1}) \le 0$ for $n = 1,2,\ldots$. Otherwise it is called nonoscillatory.

The purpose of this paper is to obtain new oscillation criteria for (1). The analogous problem has been considered in [1, 7, 9].

In this paper, we will use the following lemma.

Lemma 1 [9]. *Consider the functional inequalities*

$$x(g^s(t)) \ge p(t)x(g^{s-1}(t)) + q(t)x(g^{m+1}(t)), \tag{4}$$

$$x(g^s(t)) \le p(t)x(g^{s-1}(t)) + q(t)x(g^{m+1}(t)), \tag{5}$$

where $m \ge 1$, $s \in \{1,\ldots,m\}$, $p,q : I \to \mathbb{R}_+$, and g satisfies condition (3). If

$$\liminf_{I\ni t\to\infty} \sum_{i=0}^{m-s} q(g^i(t)) \prod_{j=1}^{m-s+1} p(g^{i+j}(t)) > \left(\frac{m-s+1}{m-s+2}\right)^{m-s+2}, \tag{6}$$

then the functional inequality (4) (resp., (5)) does not have positive (resp., negative) solutions for large $t \in I$.

It is easy to notice that the existence of oscillatory solutions of (1) is connected with the sign of the functions Q_i $(i = 0,1,\ldots,m+1)$ on I. That either $Q_i(t) > 0$ or $Q_i(t) < 0$, for $i = 0,1,\ldots,m+1$ and $t \in I$, implies that every solution of (1) oscillates. So, similarly as in our previous considerations (see, e.g., [9]), we will assume that in (1), one of the coefficients of Q_i $(i = 1,2,\ldots,m)$ has the sign opposite to that of others, that is, there exists $s \in \{1,\ldots,m\}$ such that $Q_s(t) < 0$ and $Q_i(t) > 0$, $i \in \{0,1,\ldots,m+1\} - \{s\}$. So, we further assume that for some $s \in \{1,2,\ldots,m\}$,

$$Q_s(t) < 0, \quad Q_i(t) \ge 0, \quad i = 0,1,\ldots,s-1,s+1,\ldots,m+1 \tag{7}$$

with

$$Q_{s-1}(t), Q_{s+1}(t) > 0 \quad \text{for } t \in I. \tag{8}$$

Without loss of generality, we may assume that $Q_s(t) = -1$, $t \in I$. Then (1) takes the form

$$x(g^s(t)) = \sum_{k=0}^{s-1} Q_k(t)x(g^k(t)) + \sum_{k=s+1}^{m+1} Q_k(t)x(g^k(t)), \quad m \ge 1, \tag{9}$$

where $s \in \{1,2,\ldots,m\}$, $Q_i(t) \ge 0$ $(i = 0,1,\ldots,s-1,s+1,\ldots,m+1)$, and $Q_{s-1}(t), Q_{s+1}(t) > 0$ for $t \in I$.

As usual, we take $\sum_{j=k}^{r} a_j = 0$ and $\prod_{j=k}^{r} a_j = 1$, where $r < k$.

We start from the following theorem.

Theorem 2. *Every solution of (9) is oscillatory if one of the following conditions hold:*

$$\liminf_{I\ni t\to\infty} A(g(t))B(t) > \frac{1}{4} \tag{10}$$

or

$$\begin{aligned}\limsup_{I\ni t\to\infty} \{&A(g(t))B(t) + A(g^2(t))B(g(t)) \\ &+ A(g^2(t))A(g^3(t))B(g(t))B(g^2(t))\} > 1,\end{aligned} \tag{11}$$

where

$$\begin{aligned} A(t) &= \sum_{k=0}^{s-1} Q_k(t)\prod_{j=2}^{s-k} Q_{s+1}(g^{-j}(t)), \\ B(t) &= \sum_{k=s+1}^{m+1} Q_k(t)\prod_{j=2}^{k-s} Q_{s-1}(g^{j}(t)). \end{aligned} \tag{12}$$

Proof. Suppose that (9) has a nonoscillatory solution x and let $x(t) > 0$ for $t \in I_{t_1}$, $t_1 \geq 0$. Then also, in view of assumption (3) about function g, $x(g^i(t)) > 0$, $i \in \{1,2,\dots,m+1\}$, and $t \in I_{t_2}$, $t_2 \geq t_1$. Thus, from (9) we get

$$x(g^s(t)) \geq Q_i(t)x(g^i(t)) \quad \text{for } i = 0,1,\dots,s-1,s+1,\dots,m+1. \tag{13}$$

Hence, we have

$$\begin{aligned} x(g^s(t)) &\geq Q_{s+1}(t)x(g^{s+1}(t)), \\ x(g^{s-2}(t)) &\geq Q_{s+1}(g^{-2}(t))x(g^{s-1}(t)). \end{aligned} \tag{14}$$

From above we obtain

$$x(g^{s-3}(t)) \geq Q_{s+1}(g^{-3}(t))x(g^{s-2}(t)) \geq Q_{s+1}(g^{-3}(t))Q_{s+1}(g^{-2}(t))x(g^{s-1}(t)). \tag{15}$$

Thus,

$$x(g^k(t)) \geq x(g^{s-1}(t))\prod_{j=2}^{s-k} Q_{s+1}(g^{-j}(t)), \quad k = 0,1,2,\dots,s-2. \tag{16}$$

Similarly from inequality (13) we get

$$\begin{aligned} x(g^s(t)) &\geq Q_{s-1}(t)x(g^{s-1}(t)), \\ x(g^{s+2}(t)) &\geq Q_{s-1}(g^{2}(t))x(g^{s+1}(t)). \end{aligned} \tag{17}$$

Hence,

$$x(g^{s+3}(t)) \geq Q_{s-1}(g^{3}(t))x(g^{s+2}(t)) \geq Q_{s-1}(g^{3}(t))Q_{s-1}(g^{2}(t))x(g^{s+1}(t)), \tag{18}$$

$$x(g^k(t)) \geq x(g^{s+1}(t))\prod_{j=2}^{k-s} Q_{s-1}(g^{j}(t)), \quad k = s+2,\dots,m+1. \tag{19}$$

Using now (16) and (19) in (9), we obtain

$$x(g^s(t)) \geq A(t)x(g^{s-1}(t)) + B(t)x(g^{s+1}(t)), \tag{20}$$

where A and B are given by (12). Thus, in view of condition (10) and Lemma 1, inequality (20) cannot possess positive solutions. We obtain a contradiction. Now we prove the second part of the theorem. From (20) for $i \in \{0, 1, 2\}$, we have

$$x(g^{s+i}(t)) \geq A(g^i(t))x(g^{s+i-1}(t)) + B(g^i(t))x(g^{s+i+1}(t)), \tag{21}$$

$$x(g^s(t)) \geq A(t)x(g^{s-1}(t)). \tag{22}$$

From above we obtain

$$\begin{aligned} x(g^{s+2}(t)) &\geq A(g^2(t))x(g^{s+1}(t)), \\ x(g^{s+3}(t)) &\geq A(g^3(t))x(g^{s+2}(t)). \end{aligned} \tag{23}$$

Hence,

$$x(g^{s+3}(t)) \geq A(g^2(t))A(g^3(t))x(g^{s+1}(t)). \tag{24}$$

Using the above inequality in (21) for $i = 2$, we get

$$x(g^{s+2}(t)) \geq A(g^2(t))x(g^{s+1}(t)) + A(g^2(t))A(g^3(t))B(g^2(t))x(g^{s+1}(t)). \tag{25}$$

Now applying inequalities (20) and (25) in (21) for $i = 1$, we have

$$\begin{aligned} x(g^{s+1}(t)) \geq{} & A(t)A(g(t))x(g^{s-1}(t)) \\ & + \{A(g(t))B(t) + A(g^2(t))B(g(t)) \\ & \quad + A(g^2(t))A(g^3(t))B(g(t))B(g^2(t))\}x(g^{s+1}(t)), \end{aligned} \tag{26}$$

$$\begin{aligned} x(g^{s+1}(t)) \geq{} & \{A(g(t))B(t) + A(g^2(t))B(g(t)) \\ & + A(g^2(t))A(g^3(t))B(g(t))B(g^2(t))\}x(g^{s+1}(t)). \end{aligned} \tag{27}$$

Dividing both sides of the above inequality by $x(g^{s+1}(t))$, we get a contradiction with (11). This completes the proof. □

Remark 3. In the particular case when $I = \mathbb{N}$ and $g(n) = n + 1$, from iterative functional equations, we obtain recurrence equations. So, results obtained in this paper can be applied to recurrence equations, too. For example, condition (10) applied to the second-order linear difference equation of the form

$$c(n)x(n+1) + c(n-1)x(n-1) = b(n)x(n), \tag{28}$$

where $n \in \mathbb{N}, b, c : \mathbb{N} \to (0,\infty)$, gives the result obtained by Hooker and Patula in [4, Theorem 5]. However, condition (11) applied to (28) improves the result presented in [3, Theorem 2.3]. Namely, this theorem has the following form: if for some sequence $n_k \to \infty$,

$$\frac{[c(n_k)]^2}{b(n_k)b(n_k+1)} + \frac{[c(n_k+1)]^2}{b(n_k+1)b(n_k+2)} \geq 1, \tag{29}$$

then every solution of (28) is oscillatory. On the other hand, condition (11) applied to (28) has the form

$$\limsup_{n\to\infty} \left\{ \frac{[c(n)]^2}{b(n)b(n+1)} + \frac{[c(n+1)]^2}{b(n+1)b(n+2)} + \frac{[c(n+1)]^2}{b(n+1)b(n+2)} \frac{[c(n+2)]^2}{b(n+2)b(n+3)} \right\} > 1. \tag{30}$$

If we consider (9) with $s = 1$, $I = \mathbb{N}$, and $g(n) = n+1$, then from Theorem 2, we obtain conditions of [8, Theorems 5 and 6].

Now we give another condition for the oscillation of all solutions of (9). It can be applied when Theorem 2 is not satisfied.

Theorem 4. *Suppose that*

$$A(g(t))B(t) \geq \delta > 0, \quad \delta < \frac{1}{4} \text{ for } t \in I, \tag{31}$$

$$\begin{aligned} \limsup_{I \ni t\to\infty} \{ & A(g(t))B(t) + A(g^2(t))B(g(t)) \\ & + A(g^2(t))A(g^3(t))B(g(t))B(g^2(t)) \} > 1 - \delta^2, \end{aligned} \tag{32}$$

where A and B are as previously given. Then all solutions of (9) are oscillatory.

Proof. Let $x(t) > 0$, for $t \in I_{t_1}$, $t_1 \geq 0$, be a nonoscillatory solution of (9). Then, as in the proof of Theorem 2 for $t \in I_{t_2}$, $t_2 \geq t_1$, inequalities (16) and (19) hold. So, inequality (20) is also true. Thus, for sufficiently large t, inequalities (21) and (26) are also satisfied. From (21) for $i = 0$, we have

$$\begin{aligned} x(g^s(t)) &\geq B(t)x(g^{s+1}(t)), \\ A(g(t))x(g^s(t)) &\geq A(g(t))B(t)x(g^{s+1}(t)). \end{aligned} \tag{33}$$

Using assumption (31) in the above inequality, we obtain

$$A(g(t))x(g^s(t)) \geq \delta x(g^{s+1}(t)). \tag{34}$$

The last inequality gives

$$\begin{aligned} A(t)x(g^{s-1}(t)) &\geq \delta x(g^s(t)), \\ A(t)A(g(t))x(g^{s-1}(t)) &\geq \delta^2 x(g^{s+1}(t)). \end{aligned} \tag{35}$$

Now applying the last inequality in (26), we have

$$\begin{aligned} x(g^{s+1}(t)) \geq \delta^2 x(g^{s+1}(t)) & \\ + \{A(g(t))B(t) + A(g^2(t))B(g(t)) & \\ + A(g^2(t))A(g^3(t))B(g(t))B(g^2(t))\}x(g^{s+1}(t)). & \end{aligned} \tag{36}$$

Now dividing both sides of the above inequality by $x(g^{s+1}(t))$, we obtain

$$\begin{aligned} 1 - \delta^2 \geq \{A(g(t))B(t) + A(g^2(t))B(g(t)) & \\ + A(g^2(t))A(g^3(t))B(g(t))B(g^2(t))\}. & \end{aligned} \tag{37}$$

The last inequality contradicts assumption (32). Thus, the theorem is proved. □

Remark 5. The theorems given in this paper are analogous to those presented in [9] but conditions given in both papers are independent. For example, from [9, Theorem 1], it follows that every solution of (9) is oscillatory if

$$\liminf_{I \ni t \to \infty} \sum_{i=0}^{m-s} Q(g^i(t)) \prod_{j=1}^{m-s+1} P(g^{i+j}(t)) > \left(\frac{m-s+1}{m-s+2}\right)^{m-s+2}, \tag{38}$$

where

$$\begin{aligned} P(t) &= \sum_{k=0}^{s-2} Q_k(t) \prod_{l=2}^{s-k} Q_{s+1}(g^{-l}(t)) + Q_{s-1}(t), \\ Q(t) &= \sum_{k=s+1}^{m} Q_k(t) Q_{m+s-k+1}(g^{k-s}(t)) + Q_{m+1}(t). \end{aligned} \tag{39}$$

In order to show the independence of conditions (10) and (38), we consider the following iterative functional equation:

$$x(t+2) = \frac{1}{[t]^2}x(t) + \frac{4}{50t}x(t+1) + \frac{15t}{50}x(t+3) + [t]^2 x(t+4), \quad t > 0. \tag{40}$$

In this equation, $m = 3$, $s = 2$, and $g(t) = t + 1$. Thus, condition (10) takes the form

$$\begin{aligned} &\liminf_{t \to \infty} [Q_0(t+1)Q_3(t-1) + Q_1(t+1)][Q_3(t) + Q_4(t)Q_1(t+2)] \\ &\quad = \lim_{t \to \infty} \left[\frac{1}{[t+1]^2}\frac{15(t-1)}{50} + \frac{4}{50(t+1)}\right]\left[\frac{15t}{50} + [t]^2\frac{4}{50(t+2)}\right] = \frac{361}{2500} < \frac{1}{4}, \end{aligned} \tag{41}$$

and is not fulfilled. But the above-mentioned equation has only oscillatory solutions because for this equation, condition (38) has the form

$$\liminf_{t\to\infty}\left[Q(t)P(g(t))P(g^2(t))+Q(g(t))P(g^2(t))P(g^3(t))\right]>\left(\frac{2}{3}\right)^3, \tag{42}$$

where

$$\begin{aligned} P(t)&=Q_1(t)+Q_0(t)Q_3(g^{-2}(t)),\\ Q(t)&=Q_3(t)Q_3(g(t))+Q_4(t), \end{aligned} \tag{43}$$

and is satisfied because

$$\begin{aligned} \lim_{t\to\infty}&\left\{\left[\frac{15t}{50}\frac{15(t+1)}{50}+[t]^2\right]\left[\frac{1}{[t+1]^2}\frac{15(t-1)}{50}+\frac{4}{50(t+1)}\right]\right.\\ &\times\left[\frac{1}{[t+2]^2}\frac{15t}{50}+\frac{4}{50(t+2)}\right]\\ &+\left[\frac{15(t+1)}{50}\frac{15(t+2)}{50}+[t+1]^2\right]\left[\frac{1}{[t+2]^2}\frac{15t}{50}+\frac{4}{50(t+2)}\right]\\ &\left.\times\left[\frac{1}{[t+3]^2}\frac{15(t+1)}{50}+\frac{4}{50(t+3)}\right]\right\}\\ =&\ 0.314792>\left(\frac{2}{3}\right)^3. \end{aligned} \tag{44}$$

Now we consider the iterative functional equation of the form

$$x(t+2)=\frac{1}{5[t]^2}x(t)+\frac{1}{4t}x(t+1)+\frac{3t}{5}x(t+3)+\frac{3[t]^2}{5}x(t+4),\quad t>0. \tag{45}$$

The above-mentioned equation possesses only oscillatory solutions too. For this equation, condition (38) is not true but condition (10) is satisfied.

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ON CERTAIN COMPARISON THEOREMS FOR HALF-LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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We obtain comparison theorems for the second-order half-linear dynamic equation $[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0$, where $\Phi(x) = |x|^{\alpha-1}\operatorname{sgn} x$ with $\alpha > 1$. In particular, it is shown that the nonoscillation of the previous dynamic equation is preserved if we multiply the coefficient $p(t)$ by a suitable function $q(t)$ and lower the exponent α in the nonlinearity Φ, under certain assumptions. Moreover, we give a generalization of Hille-Wintner comparison theorem. In addition to the aspect of unification and extension, our theorems provide some new results even in the continuous and the discrete case.

1. Introduction

In [17], it was shown that the basic results (in particular, the Reid roundabout theorem and, consequently, the Sturmian theory) known from the oscillation theory of the Sturm-Liouville differential equation

$$(r(t)y')' + p(t)y = 0 \tag{1.1}$$

can be extended to the half-linear dynamic equation

$$[r(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0 \tag{1.2}$$

on an arbitrary time scale $\mathbb{T}$ (i.e., a closed subset of $\mathbb{R}$), where $r(t)$ and $p(t)$ are real right-dense continuous (rd-continuous) functions on $\mathbb{T}$ with $r(t) \neq 0$ and $\Phi(x) = |x|^{\alpha-1}\operatorname{sgn} x$ with $\alpha > 1$. Moreover, in the same paper, it was proved that under the assumption of a right-dense continuity of the coefficients $r(t)$ and $p(t)$, the initial value problem involving (1.2) is uniquely solvable. The terminology *half-linear* is justified by the fact that the space of all solutions of (1.2) is homogeneous, but not generally additive. Thus, it has just half of the properties of a linear space. Equation (1.2) covers the half-linear differential equation (when $\mathbb{T} = \mathbb{R}$)

$$[r(t)\Phi(y')]' + p(t)\Phi(y) = 0 \tag{1.3}$$

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and the half-linear difference equation (when $\mathbb{T} = \mathbb{Z}$)

$$\Delta[r_k\Phi(\Delta y_k)] + p_k\Phi(y_{k+1}) = 0. \tag{1.4}$$

Furthermore, (1.1) is a special case of (1.3) (when $\alpha = 2$), and if $\Phi = \mathrm{id}$ (i.e., $\alpha = 2$), then (1.4) reduces to the Sturm-Liouville difference equation

$$\Delta(r_k\Delta y_k) + p_k y_{k+1} = 0. \tag{1.5}$$

Finally, the linear dynamic equation

$$\left(r(t)y^{\Delta}\right)^{\Delta} + p(t)y^{\sigma} = 0, \tag{1.6}$$

which covers (1.1) and (1.5) when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, is a special case of (1.2) (when $\alpha = 2$). It means that the theory of (1.2) unifies and extends the theories of all mentioned equations and also explains some discrepancies between them. Note that the basic results concerning oscillatory properties of (1.1), (1.5), (1.6), (1.3), and (1.4) can be found, for example, in [9, 15, 16, 20], [1, 2, 11], [3], [5, 6, 13], and [17, 18, 19], respectively.

The most important oscillatory properties of (1.2) are described by the so-called Reid roundabout theorem; see [17, Theorem 2]. There are several important consequences of this theorem; two of them—the Riccati technique and the Sturm-type comparison theorem (see the next section)—are used to prove our results.

In this paper, we present two types of comparison theorems. The first one actually contains two statements. First, we give a condition in terms of the inequality between the integrals $\int_t^\infty p(s)\Delta s$ and $\int_t^\infty P(s)\Delta s$ (i.e., we compare the coefficients "on average;" note that in the classical Sturm-type theorem, the coefficients are compared "pointwise"), where $P(t)$ is the corresponding coefficient to $p(t)$ of the equation which is compared with (1.2). This statement unifies and generalizes [12, Theorem 2] and [18, Theorem 4], and for historical reasons, it can be called of Hille-Wintner type. Note that in [12] (this paper concerns (1.3)), the coefficient $p(t)$ is assumed to be nonnegative. Second, we assume the condition in terms of the inequality between the exponents of the power function Φ. This enables, among others, to compare a half-linear equation with a linear one. Note that, in this sense (i.e., the relation between two equations with different nonlinearities), the statement is new even in the continuous case (i.e., when $\mathbb{T} = \mathbb{R}$). In the proof, we combine the Riccati technique with the application of the Schauder fixed-point theorem. Our second type of comparison theorems says that, under certain additional conditions, the (non)oscillation of (1.2) is preserved when multiplying the coefficient $p(t)$ by a suitable function $q(t)$. It extends the result in [7] and its proof is based on the Riccati technique.

The paper is organized as follows. In Section 2, we give basic information concerning the calculus on time scales, some auxiliary statements including the Riccati technique, and a background for an application of the Schauder fixed-point theorem. The main results—comparison theorems—are proved in Section 3, where some comments and an example can also be found.

2. Preliminaries

We start with introducing the following concepts related to the notion of time scales. It was established by Hilger in his Ph.D. dissertation in 1988; see [10]. We refer to [3] for additional details concerning the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed subset of $\mathbb{R}$). We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Because of the character of our result, we suppose that $\sup\mathbb{T} = \infty$. Define the *forward jump operator* $\sigma(t)$ at $t \in \mathbb{T}$ by $\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\}$, and the *backward jump* operator $\rho(t)$ at $t \in \mathbb{T}$ by $\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. We will also use the notation $\mu(t) := \sigma(t) - t$ which is called the *graininess function*. A function $f : \mathbb{T} \to \mathbb{R}$ is called *(delta) differentiable at* $t \in \mathbb{T}$ with *(delta) derivative* $f^\Delta(t) \in \mathbb{R}$ if there exists the (finite) limit

$$f^\Delta(t) := \lim_{s\to t,\sigma(s)\neq t} \frac{f(\sigma(s)) - f(t)}{\sigma(s) - t}. \tag{2.1}$$

We use the notation $f^\sigma(t) = f(\sigma(t))$ for $t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$. The notations $[a,b]$, $[a,b)$, $[a,\infty)$, and so forth denote time scales intervals. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* provided that f is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. We write $f \in C_{\mathrm{rd}}(\mathbb{T})$. The integral of a rd-continuous function f (it indeed exists) is defined by means of the antiderivative F, that is, $\int_a^b f(t)\Delta t = F(b) - F(a)$, where F is such that $F^\Delta = f$.

We say that a solution y of (1.2) has a *generalized zero* at t in case $y(t) = 0$. We say y has a *generalized zero* in $(t,\sigma(t))$ in case $r(t)y(t)y(\sigma(t)) < 0$ and $\mu(t) > 0$. We say that (1.2) is *disconjugate* on the interval $[a,b]$ if there is no nontrivial solution of (1.2) with two (or more) generalized zeros in $[a,b]$.

Equation (1.2) is said to be *nonoscillatory* (on $[a,\infty)$) if there exists $c \in [a,\infty)$ such that this equation is disconjugate on $[c,d]$ for every $d > c$. In the opposite case, (1.2) is said to be *oscillatory* (on $[a,\infty)$). Oscillation of (1.2) may be equivalently defined as follows. A nontrivial solution y of (1.2) is called *oscillatory* if it has infinitely many (isolated) generalized zeros in $[a,\infty)$. By the Sturm-type separation theorem, see [17, Theorem 3], one solution of (1.2) is (non)oscillatory if and only if every solution of (1.2) is (non)oscillatory. Hence, we can speak about *oscillation* or *nonoscillation of* (1.2).

The classical Sturm's result can be generalized as follows, see [17, Theorem 3].

Proposition 2.1 (Sturm (or Sturm-Picone)-type comparison theorem). *Consider the equation*

$$[R(t)\Phi(y^\Delta)]^\Delta + P(t)\Phi(y^\sigma) = 0, \tag{2.2}$$

where R and P satisfy the same assumptions as r and p. Suppose that $r(t) \le R(t)$ and $P(t) \le p(t)$ on $[T,\infty)$ for all large T. Then (1.2) is nonoscillatory implying that (2.2) is nonoscillatory.

Along with (1.2) (defined on a time scale interval of the form $[a,\infty)$), consider the generalized Riccati dynamic equation

$$\mathcal{R}[w] := w^{\Delta} + p(t) + \mathcal{S}[w,r;\alpha](t) = 0, \tag{2.3}$$

where

$$\mathcal{S}[w,r;\alpha] = \lim_{\lambda\to\mu}\frac{w}{\lambda}\left(1-\frac{r}{\Phi\left(\Phi^{-1}(r)+\lambda\Phi^{-1}(w)\right)}\right). \tag{2.4}$$

As we will see, it is related to the original equation by the Riccati-type substitution $w(t) = r(t)\Phi[y^{\Delta}(t)/y(t)]$. Observe that

$$\mathcal{S}[w,r;\alpha](t) = \begin{cases} \left\{\dfrac{(\alpha-1)}{\Phi^{-1}(r)}|w|^{\beta}\right\}(t) & \text{for right-dense } t, \\[2ex] \left\{\dfrac{w}{\mu}\left(1-\dfrac{r}{\Phi\left(\Phi^{-1}(r)+\mu\Phi^{-1}(w)\right)}\right)\right\}(t) & \text{for right-scattered } t, \end{cases} \tag{2.5}$$

where the l'Hôpital's rule is used in the first case, Φ^{-1} denotes the inverse of Φ (i.e., $\Phi^{-1}(x) = |x|^{\beta-1}\operatorname{sgn} x$), and β is the conjugate number of α (i.e., $1/\alpha+1/\beta = 1$).

The proof of the following statement is based on the Reid roundabout theorem and Sturm-type comparison theorem (see [17, Lemma 14]), and it is usually referred to as the Riccati technique.

Proposition 2.2 (Riccati technique). *Equation (1.2) is nonoscillatory if and only if there exists $T \in [a,\infty)$ and a function w satisfying the generalized Riccati dynamic inequality $\mathcal{R}[w](t) \le 0$ with $\{\Phi^{-1}(r)+\mu\Phi^{-1}(w)\}(t) > 0$ for $t \in [T,\infty)$.*

A behavior of the operator $\mathcal{S}$ with respect to its arguments will be described by the properties of the function

$$S(x,y,\alpha) = \lim_{\lambda\to\mu}\frac{x}{\lambda}\left(1-\frac{y}{\Phi\left(\Phi^{-1}(y)+\lambda\Phi^{-1}(x)\right)}\right). \tag{2.6}$$

Note that the function S can be understood as a "half-linear generalization" of the function $x^2/(y+\mu x)$ that corresponds to the operator occurring in the Riccati dynamic equation associated to linear dynamic equation (1.6), and hence a similar behavior of these functions can be expected in a certain sense.

Lemma 2.3. *The function S has the following properties:*

(i) *let $y > 0$, then $x(\partial S/\partial x)(x,y,\alpha) \ge 0$ for $\Phi^{-1}(y)+\mu\Phi^{-1}(x) > 0$, where $(\partial S/\partial x)(x,y,\alpha) = 0$ if and only if $x = 0$;*

(ii) *$S(x,y,\alpha) \ge 0$ for $\Phi^{-1}(y)+\mu\Phi^{-1}(x) > 0$, where the equality holds if and only if $x = 0$;*

(iii) *if $x > 0$, $y > 0$, and*

$$\gamma := \lim_{\lambda\to\mu}\frac{(1+\lambda z)\ln(1+\lambda z)-\lambda z\ln z}{\lambda} \ge 0, \tag{2.7}$$

where $z := (x/y)^{1/(\alpha-1)}$, then $(\partial S/\partial\alpha)(x,y,\alpha) \ge 0$.

Remark 2.4. (i) Using the L'Hôpital's rule, we have $\gamma = z - z\ln z$ when $\mu = 0$.

(ii) It is easy to see that if $\mu \geq 1$, then $\gamma \geq 0$, since

$$\gamma = \frac{\ln(1+\mu z) + \mu z \ln\left((1+\mu z)/z\right)}{\mu}. \tag{2.8}$$

On the other hand, if $\mu \in [0,1)$, then z being small (more precisely, $z \leq 1$, but in fact, the right-hand side may be greater than 1; it depends on μ) is a sufficient condition for γ to be nonnegative. We notice how the graininess function plays the role in the monotone nature of S. Observe that S is not always nondecreasing with respect to α, even when $x, y > 0$.

(iii) In view of the last remark, if, for example, $w(t) > 0$, $r(t) > 0$, $\lim_{t\to\infty} w(t) = 0$, and $\liminf_{t\to\infty} r(t) > 0$, then $\partial\mathcal{S}(w(t), r(t); \alpha)/\partial\alpha \geq 0$ for large t. It is clear that the last two conditions may be dropped when $\mu(t) \geq 1$ eventually.

Proof. For the proof of (i) and (ii), see [17, Lemma 13].

To prove the property (iii), first note that for $\mu > 0$, the function S can be rewritten as

$$S(x, y, \alpha) = \frac{x}{\mu}\left[1 - \left(1 + \mu\left(\frac{x}{y}\right)^{1/(\alpha-1)}\right)^{1-\alpha}\right], \tag{2.9}$$

while for $\mu = 0$ it takes the form $S(x, y, \alpha) = (\alpha - 1)x\,(x/y)^{1/(\alpha-1)}$. Differentiating S with respect to α, using the known rules, we get

$$\frac{\partial S}{\partial\alpha} = \frac{x}{\mu}(1+\mu z)^{-\alpha}\left[(1+\mu z)\ln(1+\mu z) - \mu z \ln z\right] \tag{2.10}$$

in case $\mu > 0$. If $\mu = 0$, then we obtain $\partial S/\partial\alpha = x(z - z\ln z)$. In view of the assumptions of the lemma, Remark 2.4(i), and the equality

$$\begin{aligned}
&\lim_{\lambda\to\mu} \frac{\partial}{\partial\alpha}\left\{\frac{x}{\lambda}\left[1 - \left(1 + \lambda\left(\frac{x}{y}\right)^{1/(\alpha-1)}\right)^{1-\alpha}\right]\right\} \\
&\quad = \frac{\partial}{\partial\alpha}\left\{\lim_{\lambda\to\mu}\frac{x}{\lambda}\left[1 - \left(1 + \lambda\left(\frac{x}{y}\right)^{1/(\alpha-1)}\right)^{1-\alpha}\right]\right\},
\end{aligned} \tag{2.11}$$

we get the statement. □

The next lemma claims that, under certain assumptions, an eventually positive solution of (nonoscillatory) equation (1.2) has an eventually positive delta-derivative, consequently, (2.3) has a positive solution.

Lemma 2.5. *Assume that $r(t) > 0$,*

$$\liminf_{t\to\infty}\int_T^t p(s)\Delta s \geq 0, \qquad \liminf_{t\to\infty}\int_T^t p(s)\Delta s \not\equiv 0, \tag{2.12}$$

for all large T, and

$$\int_a^\infty r^{1-\beta}(s)\Delta s = \infty. \tag{2.13}$$

If y is a solution of (1.2) such that $y(t) > 0$ for $t \in [T,\infty)$, then there exists $T_1 \in [T,\infty)$ such that $y^\Delta(t) > 0$ for $t \in [T_1,\infty)$.

Proof. The proof is by contradiction. We consider the following two cases.
Case 1. Suppose that $y^\Delta(t) < 0$ for $t \in [T,\infty)$. Then also $[\Phi(y)]^\Delta(t) < 0$ for $t \in [T,\infty)$ since

$$[\Phi(y)]^\Delta(t) = \frac{d}{dy}\Phi[y(\xi)]y^\Delta(t) = (\alpha-1)\,|\,y(\xi)\,|^{\alpha-2}y^\Delta(t) < 0 \tag{2.14}$$

by [3, Theorem 1.87], where $t \le \xi \le \sigma(t)$. Another argument for $[\Phi(y)]^\Delta(t) < 0$ is that if y is decreasing, then $\Phi(y)$ is decreasing as well because of the properties of the function Φ. Without loss of generality we may assume that T is such that $\int_T^t p(s)\Delta s \ge 0$, $t \in [T,\infty)$, reasoning as in [7, Proof of Lemma 13]. Define $Q(t,T) = \int_T^t p(s)\Delta s$. The integration by parts gives

$$\begin{aligned}\int_T^t p(s)\Phi[y^\sigma(s)]\Delta s &= \int_T^t Q^\Delta(s,T)\Phi(y^\sigma(s))\Delta s\\ &= Q(t,T)\Phi(y(t)) - \int_T^t Q(s,T)[\Phi(y(s))]^\Delta \Delta s \ge 0.\end{aligned} \tag{2.15}$$

Integrating (1.2), we have, using the last estimate,

$$r(t)\Phi(y^\Delta(t)) - r(T)\Phi(y^\Delta(T)) = \int_T^t [r(s)\Phi(y^\Delta(s))]^\Delta \Delta s \le 0. \tag{2.16}$$

Hence,

$$y^\Delta(t) \le \frac{r^{\beta-1}(T)y^\Delta(T)}{r^{\beta-1}(t)} \tag{2.17}$$

for $t \in [T,\infty)$. Integrating (2.17) for $t \ge T$, we see that $y(t) \to -\infty$ by (2.13), a contradiction. Therefore, $y^\Delta(t) < 0$ cannot hold for all large t.
Case 2. Next, if $y^\Delta(t) \not> 0$ eventually, then for every (large) $T \in [a,\infty)$, there exists $T_0 \in [T,\infty)$ such that $y^\Delta(T_0) \le 0$ and we may suppose that $\liminf_{t\to\infty}\int_T^t p(s)\Delta s \ge 0$. Since $y(t) > 0$ for $t \in [T,\infty)$, the function $w(t) = r(t)\Phi[y^\Delta(t)/y(t)]$ satisfies the generalized Riccati equation (2.3) with $\{\Phi^{-1}(r) + \mu\Phi^{-1}(w)\}(t) > 0$ for $t \in [T,\infty)$. Integrating (2.3) from T_0 to t, $t \ge T_0$, gives

$$w(t) = w(T_0) - \int_{T_0}^t p(s)\Delta s - \int_{T_0}^t \mathcal{S}(w,r;\alpha)(s)\Delta s. \tag{2.18}$$

Therefore, it follows that $\limsup_{t\to\infty} w(t) < 0$, using the facts $w(T_0) \leq 0$, $w(t)$ is eventually nontrivial, and (2.12) holds. Indeed, there is $M > 0$ such that $\int_{T_0}^t \mathcal{S}(w,r;\alpha)(s)\Delta s \geq M$ and $\int_{T_0}^t p(s)\Delta s \geq -M/2$ for all large t. Hence, there exists $T_1 \in [T,\infty)$ such that $w(t) < 0$ for $t \in [T_1,\infty)$ and so $y^\Delta(t) < 0$ for $t \in [T_1,\infty)$, a contradiction to the first part. □

In the next lemma, a necessary condition for the nonoscillation of (1.2) is given in terms of solvability of generalized Riccati integral inequality under certain assumptions. Note that a closer examination of the proof of Theorem 3.1 shows that this condition is also sufficient.

Lemma 2.6. *Let the assumptions of Lemma 2.5 hold and assume further that $\int_a^\infty p(s)\Delta s = \lim_{t\to\infty}\int_a^t p(s)\Delta s$ is convergent. Let $y(t)$ be a solution of (1.2) such that $y(t) > 0$ for $t \in [T,\infty)$. Then there exists $T_1 \in [T,\infty)$ such that*

$$w(t) \geq \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w,r;\alpha)(s)\Delta s \tag{2.19}$$

for $t \in [T_1,\infty)$, where $w(t) = r(t)\Phi[y^\Delta(t)/y(t)] > 0$.

Proof. From Lemma 2.5, there exists $T_1 \in [T,\infty)$ such that $w(t) > 0$ for $t \in [T_1,\infty)$ and w satisfies (2.3) for $t \in [T,\infty)$ (clearly, with $\{\Phi^{-1}(r) + \mu\Phi^{-1}(w)\}(t) > 0$). Integrating (2.3) from t to s, $s \geq t \geq T_1$, we get

$$w(s) - w(t) + \int_t^s p(\xi)\Delta\xi + \int_t^s \mathcal{S}(w,r;\alpha)(\xi)\Delta\xi = 0. \tag{2.20}$$

Therefore,

$$0 < w(s) \leq w(t) - \int_t^s p(\xi)\Delta\xi, \tag{2.21}$$

and hence,

$$w(t) \geq \int_t^s p(\xi)\Delta\xi + \int_t^s \mathcal{S}(w,r;\alpha)(\xi)\Delta\xi \tag{2.22}$$

for $s \geq t \geq T_1$. Letting $s \to \infty$, we obtain (2.19). □

In the last part of this section, we give a background for the application of the Schauder fixed-point theorem. It will be used in the proof of Theorem 3.1. We start by recalling the Schauder theorem that is applicable for our setting in dynamic equations.

Proposition 2.7 (Schauder fixed-point theorem, [8, Theorem 6.44]). *Let $\mathcal{N}$ be a normed space and X be a nonempty, closed, convex subset of $\mathcal{N}$. If $\mathcal{T}$ is a continuous mapping such that $\mathcal{T}(X) \subseteq X$ (i.e., mapping X into itself) and $\mathcal{T}(X)$ is relatively compact, then $\mathcal{T}$ has a fixed point in X.*

Denote with $C_{TS}^B[a,\infty)$ the linear space of all continuous functions $f : [a,\infty) \to \mathbb{R}$ such that $\sup_{t\in[a,\infty)} |f(t)| < \infty$. Define this supremum to be the norm $\|f\| = \sup_{t\in[a,\infty)} |f(t)|$.

The following statement can be understood as a time scale version of the Arzelà-Ascoli theorem. For the discrete analog of this well-known theorem, see [4, Theorem 3.3]. Note that for $\mathbb{T} = \mathbb{N}$ we get $C_{TS}^B = \ell^\infty$, and condition (ii) in the next lemma holds trivially.

Lemma 2.8. *Let X be a subset of $C_{TS}^B[a,\infty)$ having the following properties:*

(i) *X is bounded;*
(ii) *on every compact subinterval J of $[a,\infty)$, there exists, for any $\varepsilon > 0$, $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$ (i.e., the functions of X are locally equicontinuous);*
(iii) *for every $\varepsilon > 0$, there exists $b \in [a,\infty)$ such that $t_1, t_2 \in [b,\infty)$ implies $|f(t_1) - f(t_2)| < \varepsilon$ for all $f \in X$ (in the "discrete terminology," X is said to be uniformly Cauchy).*

Then X is relative compact.

Proof. By [8, Theorem 6.33], it is sufficient to construct a finite ε-net for any ε. Since the proof is more or less obvious, we mention just some of its important points and omit details. In view of the properties (i), (ii), and (iii), it is possible to construct a two-dimensional grid, where the vertical values are the elements $y_1,\dots,y_m \in \mathbb{R}$, $-K = y_1 < y_2 < \cdots < y_m = K$, K being such that $\|f\| \le K$ for all $f \in X$, and sufficiently close to neighbors, that is, $y_{i+1} - y_i$ is a sufficiently small number depending on ε. The horizontal values $x_1,\dots,x_m \in \mathbb{T}$, $a = x_1 < x_2 < \cdots < x_n = b$, are sufficiently close to their neighbors in the sense that if they are close to dense points, the differences of the values of f $(f \in X)$ at these points are small—depend on ε (this is possible thanks to the local equicontinuity)—or they are isolated and sufficiently far from each other; $b \ge a$ being such that $|f(t_1) - f(t_2)|$ is sufficiently small (depends on ε) whenever t_1, $t_2 \in [b,\infty)$ for all $f \in X$. Such b exists thanks to the property (iii). Now, having such grid for any $f \in X$, we can construct a linear fractional function g which approximate f (in fact, $\|f - g\| < \varepsilon$). The number of functions g constructed in this way is finite and thus the set of such functions forms a finite ε-net for X. □

3. Main results

We start with Hille-Wintner-type comparison theorem involving also the condition in terms of the change of the exponents in the power function Φ. Along with (1.2), consider the equation

$$[R(t)\Phi_{\bar{\alpha}}(x^\Delta)]^\Delta + P(t)\Phi_{\bar{\alpha}}(x^\sigma) = 0, \tag{3.1}$$

where R and P satisfy the same assumptions as r and p, and $\Phi_{\bar{\alpha}}(x) = |x|^{\bar{\alpha}-1}\operatorname{sgn} x$ with $\bar{\alpha} > 1$.

Theorem 3.1. *Assume* $0 < R(t) \le r(t)$,

$$0 \le \int_t^\infty p(s)\Delta s \le \int_t^\infty P(s)\Delta s \tag{3.2}$$

for all large t (in particular, these integrals exist as finite numbers),

$$\int_a^\infty R^{1-\bar\beta}(s)\Delta s = \infty \tag{3.3}$$

with $\bar\beta > 1$ being the conjugate number to $\bar\alpha$, and $1 < \alpha \le \bar\alpha$. Further, suppose that $\liminf_{t\to\infty} R(t) > 0$ when $\mu(t) \not\ge 1$ eventually (if $\mu(t) \ge 1$ eventually, then this condition may be dropped—see also Remark 2.4). If (3.1) is nonoscillatory, then so is (1.2).

Proof. By Lemma 2.6, the assumptions of the theorem imply the existence of a function z (actually, $z = R\Phi_{\bar\alpha}(x^\Delta/x)$, x being an eventually positive increasing solution of (3.1)) and $T \in [a,\infty)$ such that

$$z(t) \ge \int_t^\infty P(s)\Delta s + \int_t^\infty \mathcal{S}(z(s),R(s);\bar\alpha)\Delta s =: Z(t) \tag{3.4}$$

with $z(t) > 0$ for $t \ge T$. Without loss of generality, we may assume that (3.2) holds for $t \ge T$. Define the set $\Omega = \{w \in C^B_{TS}[T,\infty) : 0 \le w(t) \le Z(t) \text{ for } t \ge T\}$ and the operator $\mathcal{T} : \Omega \to C^B_{TS}[T,\infty)$ defined by

$$\mathcal{T}(w)(t) = \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w(s),R(s);\alpha)\Delta s \tag{3.5}$$

for $w \in \mathcal{T}$. In view of the assumptions of the theorem and the properties of $\mathcal{S}$, the operator $\mathcal{T}$ is well defined. It is very easy to see that Ω is closed and convex.

We show that $\mathcal{T}$ maps Ω into itself. Suppose that $w \in \Omega$ and define $v(t) = \mathcal{T}(w)(t)$, $t \ge T$. Obviously, $v(t) \ge 0$ for $t \ge T$. We prove that $v(t) \le Z(t)$. First note that since $w \in \Omega$ is small for large t and $\liminf_{t\to\infty} R(t) > 0$ (provided that $\mu(t) \not\ge 1$ eventually), we have $w(t)/R(t) \le 1$ for large t (without loss of generality, we may suppose that T is such that $Z(t)/R(t) \le 1$ for $t \ge T$ in case $\mu(t) \not\ge 1$ eventually), and so the assumptions of Lemma 2.3(iii) are satisfied (see also Remark 2.4). Now we get

$$\begin{aligned} v(t) &= \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w(s),R(s);\alpha)\Delta s \\ &\le \int_t^\infty P(s)\Delta s + \int_t^\infty \mathcal{S}(w(s),R(s);\alpha)\Delta s \\ &\le \int_t^\infty P(s)\Delta s + \int_t^\infty \mathcal{S}(w(s),R(s);\bar\alpha)\Delta s \le Z(t) \end{aligned} \tag{3.6}$$

by the assumptions of the theorem and by Lemma 2.3. Hence $\mathcal{T}(\Omega) \subset \Omega$.

According to Lemma 2.8, to prove the relative compactness of $\mathcal{T}(\Omega)$, it is sufficient to verify that conditions (i), (ii), and (iii) hold for $\mathcal{T}(\Omega)$. Clearly, $\mathcal{T}(\Omega) \subset \Omega$ implies the boundedness of $\mathcal{T}(\Omega)$. In view of the definition of $\mathcal{T}$, for any $w \in \Omega$, we have $0 \le -(\mathcal{T}(w))^\Delta(t) = p(t) + \mathcal{S}(w(t),R(t);\alpha) \le p(t) + \mathcal{S}(z(t),R(t);\bar\alpha)$, which proves the equicontinuity of the elements of $\mathcal{T}(\Omega)$. Finally, we verify that $\mathcal{T}(\Omega)$ is "uniformly Cauchy."

Let $\varepsilon > 0$ be given. We have to show that there exists $t_0 \in [T,\infty)$ such that for any $t_1, t_2 \in [t_0,\infty)$, it holds that $|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)| < \varepsilon$ for any $w \in \Omega$. Without loss of generality, suppose that $t_1 < t_2$. Then we have

$$|\mathcal{T}(w)(t_1) - \mathcal{T}(w)(t_2)| \le \left| \int_{t_1}^{t_2} p(s)\Delta s \right| + \int_{t_1}^{t_2} \mathcal{S}(w(s), R(s); \alpha)\Delta s. \tag{3.7}$$

Since the integrals in (3.7) are convergent, for any $\varepsilon > 0$, one can find $t_0 \in [T,\infty)$ such that

$$\left| \int_{t_1}^{t_2} p(s)\Delta s \right| < \frac{\varepsilon}{2}, \qquad \int_{t_1}^{t_2} \mathcal{S}(w(s), R(s); \alpha)\Delta s < \frac{\varepsilon}{2} \tag{3.8}$$

whenever $t_2 > t_1 \ge t_0$. From here and (3.7), we get the desired inequality. Hence, $\mathcal{T}(\Omega)$ is relatively compact.

The last hypothesis, which has to be verified, is the continuity of $\mathcal{T}$ in Ω. Let $\{w_n\}$, $n \in \mathbb{N}$, be a sequence in Ω which uniformly converges on every compact subinterval of $[T,\infty)$ to $\bar{w} \in \Omega$. Because $\mathcal{T}(\Omega)$ is relatively compact, the sequence $\{\mathcal{T}(w_n)\}$ admits a subsequence $\{\mathcal{T}(w_{n_j})\}$ converging in the topology of $C^B_{TS}[T,\infty)$ to $\bar{v}$. The inequality $\mathcal{S}(w_{n_j}(t), R(t); \alpha) \le \mathcal{S}(z(t), R(t); \bar{\alpha})$ implies that the integral $\int_t^\infty \mathcal{S}(w_{n_j}(s), R(s); \alpha)\Delta s$ is totally convergent. Hence, by the Lebesgue dominated convergence theorem on time scales, see [14], the sequence $\{\mathcal{T}(w_{n_j})\}$ converges to $\mathcal{T}(\bar{w})$. In view of the uniqueness of the limit, $\mathcal{T}(\bar{w}) = \bar{v}$ is the only cluster point of the sequence $\{\mathcal{T}(w_n)\}$ that proves the continuity of $\mathcal{T}$ in Ω.

Therefore, it follows from Proposition 2.7 that there exists an element $w \in \Omega$ such that $\mathcal{T}(w) = w$. In view of how the operator $\mathcal{T}$ is defined, this (positive) function w satisfies the equation

$$w(t) = \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w(s), R(s); \alpha)\Delta s, \tag{3.9}$$

$t \ge T$, and hence, also the equation $w^\Delta + p(t) + \mathcal{S}(w, R; \alpha)(t) = 0$, clearly, with $\Phi^{-1}(R) + \mu\Phi^{-1}(w) > 0$. Consequently, the function y, given by

$$y(T) = A \ne 0, \qquad y^\Delta = \left(\frac{w(t)}{R(t)}\right)^{\beta-1} y, \tag{3.10}$$

$t \ge T$, is a nonoscillatory solution of $[R(t)\Phi(y^\Delta)]^\Delta + p(t)\Phi(y^\sigma) = 0$, and hence, this equation is nonoscillatory. The statement now follows from Proposition 2.1. □

Remark 3.2. (i) A closer examination of the previous proof shows that the necessary condition for nonoscillation of (1.2) in Lemma 2.6 is also sufficient.

(ii) It is not difficult to make the following observation. If (2.13) holds, $p(t) \ge 0$ (and eventually nontrivial) for all large t, $\int_a^\infty p(s)\Delta s$ converges, and (1.2) has a positive solution y, then the nonnegative function $w(t) = r(t)\Phi[y^\Delta/y]$ (in fact, it is a solution of (2.3)) is eventually nonincreasing and converges to zero. Moreover, it satisfies the integral equation

$$w(t) = \int_t^\infty p(s)\Delta s + \int_t^\infty \mathcal{S}(w(s), r(s); \alpha)\Delta s. \tag{3.11}$$

Clearly, the solvability of (3.11) is also a sufficient condition for the nonoscillation of (1.2). In this case, it is possible to prove the comparison theorem involving the condition in terms of the inequality between exponents of the nonlinearities immediately by the Riccati technique without using the Schauder fixed-point theorem. Indeed, we have $0 = w^\Delta(t) + p(t) + \mathcal{S}[w, r; \alpha] \geq w^\Delta(t) + p(t) + \mathcal{S}[w, r; \bar{\alpha}]$ for large t provided that $\alpha \geq \bar{\alpha} > 1$ and $\liminf_{t\to\infty} r(t) > 0$ (when $\mu(t) \not\geq 1$ eventually). The fact that $[r(t)\Phi_{\bar{\alpha}}(y^\Delta)]^\Delta + p(t)\Phi_{\bar{\alpha}}(y^\sigma) = 0$ is nonoscillatory then follows from Proposition 2.2. In view of Remark 2.4, the statement can be proved in this way even under the assumptions of Lemma 2.5, provided that $\mu(t) \geq 1$ eventually, since we do not need a solution of (2.3) to be close to zero. In particular, this is satisfied in the discrete case, that is, $\mathbb{T} = \mathbb{Z}$.

Now, we mention a few background details which serve to motivate our second main result. Along with (1.2), consider the equation

$$[r(t)\Phi(y^\Delta)]^\Delta + \lambda p(t)\Phi(y^\sigma) = 0, \tag{3.12}$$

where λ is a real constant, and assume that $r(t) > 0$. We claim that if (1.2) is nonoscillatory and $0 < \lambda \leq 1$, then (3.12) is also nonoscillatory. If $p(t) \geq 0$, then this statement follows immediately from the Sturm comparison theorem (Proposition 2.1). If $p(t)$ may change sign, then dividing (3.12) by λ, we obtain an equivalent equation which is nonoscillatory again by the Sturm theorem. This can be analogously done for oscillatory counterparts. If the constant λ is replaced by a function $q(t)$, then the situation is not so easy (when $p(t)$ may change sign; otherwise the Sturm theorem can be applied immediately). The following statements give an answer to the question "what are the conditions which guarantee that the (non)oscillation of (1.2) is preserved when multiplying the coefficient $p(t)$ by a function $q(t)$?" They generalize [7, Theorem 7, Corollary 8]. Along with (1.2), consider the equation

$$[R(t)\Phi(x^\Delta)]^\Delta + q(t)P(t)\Phi(x^\sigma) = 0, \tag{3.13}$$

where R and P satisfy the same assumptions as r and p.

Theorem 3.3. *Assume that $q(t) \in C^1_{\mathrm{rd}}[a,\infty)$, $0 < r(t) \leq R(t)$, $P(t) \leq p(t)$, $0 < q(t) \leq 1$, and $q^\Delta(t) \leq 0$. Further, let (2.12) and (2.13) hold. Then (1.2) is nonoscillatory implying that (3.13) is nonoscillatory.*

Proof. The assumptions of the theorem imply that there exists a solution y of (1.2) and $T \in [a,\infty)$ such that $y(t) > 0$ and $y^\Delta(t) > 0$ on $[T,\infty)$ by Lemma 2.5. Therefore, the function $w(t) := r(t)\Phi(y^\Delta(t)/y(t)) > 0$ satisfies (2.3) with $\{\Phi^{-1}(r) + \mu\Phi^{-1}(w)\}(t) > 0$

on $[T,\infty)$. We have

$$\begin{aligned} q\mathcal{S}[w,r;\alpha] &= \lim_{\lambda\to\mu}\frac{wq}{\lambda}\left(1-\frac{rq}{\Phi\left[\Phi^{-1}(q)\Phi^{-1}(r)+\lambda\Phi^{-1}(q)\Phi^{-1}(w)\right]}\right) \\ &= \mathcal{S}[qw,qr;\alpha]. \end{aligned} \tag{3.14}$$

Now, multiplying (2.3) by $q(t)$, we get

$$\begin{aligned} 0 &= w^{\Delta}(t)q(t)+p(t)q(t)+\mathcal{S}[qw,qr;\alpha](t) \\ &\geq w^{\Delta}(t)q(t)+P(t)q(t)+\mathcal{S}[qw,qr;\alpha](t) \\ &\geq w^{\Delta}(t)q(t)+w^{\sigma}(t)q^{\Delta}(t)+P(t)q(t)+\mathcal{S}[qw,qr;\alpha](t) \\ &= (wq)^{\Delta}(t)+P(t)q(t)+\mathcal{S}[qw,qr;\alpha](t) \end{aligned} \tag{3.15}$$

for $t\in[T,\infty)$. Hence, the function $v(t)=w(t)q(t)$ satisfies the generalized Riccati inequality $v^{\Delta}(t)+P(t)q(t)+\mathcal{S}[v,qr;\alpha](t)\leq 0$ with

$$\left\{\Phi^{-1}(qr)+\mu\Phi^{-1}(v)\right\}(t)=\Phi^{-1}(q)\left\{\Phi^{-1}(r)+\mu\Phi^{-1}(w)\right\}(t)>0 \tag{3.16}$$

for $t\in[T,\infty)$. Therefore, the equation

$$\left[q(t)r(t)\Phi\left(x^{\Delta}\right)\right]^{\Delta}+q(t)P(t)\Phi\left(x^{\sigma}\right)=0 \tag{3.17}$$

is nonoscillatory by Proposition 2.2, and so (3.13) is nonoscillatory by Proposition 2.1 since $q(t)r(t)\leq r(t)\leq R(t)$. □

Theorem 3.4. *Assume that $q(t)\in C^{1}_{\mathrm{rd}}[a,\infty)$, $0<R(t)\leq r(t)$, $p(t)\leq P(t)$, $q(t)\geq 1$, and $q^{\Delta}(t)\geq 0$. Further, let*

$$\liminf_{t\to\infty}\int_T^t q(s)P(s)\Delta s\geq 0,\qquad \liminf_{t\to\infty}\int_T^t q(s)P(s)\Delta s\not\equiv 0, \tag{3.18}$$

for all large T, and

$$\int_a^{\infty} R^{1-\beta}(s)\Delta s=\infty. \tag{3.19}$$

Then (1.2) is oscillatory implying that (3.13) is oscillatory.

Proof. Suppose, by a contradiction, that (3.13) is nonoscillatory. Then there exists a solution x of (3.13) and $T\in[a,\infty)$ such that $x(t)>0$ and $x^{\Delta}(t)>0$ on $[T,\infty)$ by Lemma 2.5. Therefore, the function $v(t):=R(t)\Phi(x^{\Delta}(t)/x(t))>0$ satisfies

$$v^{\Delta}(t)+q(t)P(t)+\mathcal{S}[v,R;\alpha](t)=0 \tag{3.20}$$

with $\{\Phi^{-1}(R)+\mu\Phi^{-1}(v)\}(t)>0$ on $[T,\infty)$. We have

$$\frac{v^{\Delta}(t)}{q(t)}\geq\frac{v^{\Delta}(t)q(t)}{q^2(t)}-\frac{v(t)q^{\Delta}(t)}{q^2(t)}=\left(\frac{v(t)}{q(t)}\right)^{\Delta} \tag{3.21}$$

at right-dense t, while

$$\frac{v^{\Delta}(t)}{q(t)}=\frac{v^{\sigma}(t)}{\mu(t)q(t)}-\frac{v(t)}{\mu(t)q(t)}\geq\frac{v^{\sigma}(t)}{\mu(t)q^{\sigma}(t)}-\frac{v(t)}{\mu(t)q(t)}=\left(\frac{v(t)}{q(t)}\right)^{\Delta} \tag{3.22}$$

at right-scattered t. Dividing (3.20) by $q(t)$ and using the above estimates, we get

$$0=\frac{v^{\Delta}(t)}{q(t)}+P(t)+\frac{1}{q(t)}\mathcal{S}[v,R;\alpha](t)\geq\left(\frac{v(t)}{q(t)}\right)^{\Delta}+p(t)+\mathcal{S}\left[\frac{v}{q},\frac{R}{q};\alpha\right](t) \tag{3.23}$$

for $t\in[T,\infty)$. Hence, the function $w(t)=v(t)/q(t)$ satisfies the inequality $w^{\Delta}(t)+p(t)+\mathcal{S}[w,R/q;\alpha](t)\leq 0$ with $\{\Phi^{-1}(R/q)+\Phi^{-1}(w)\}>0$ for $t\in[T,\infty)$. Therefore, the equation

$$\left[\frac{R(t)}{q(t)}\Phi(y^{\Delta})\right]^{\Delta}+p(t)\Phi(y^{\sigma})=0 \tag{3.24}$$

is nonoscillatory by Proposition 2.2. Now, since $R(t)/q(t)\leq R(t)\leq r(t)$, (1.2) is nonoscillatory by Proposition 2.1, a contradiction. □

Remark 3.5. A closer examination of the proofs shows that the last two theorems can be improved in the following way (assuming the same conditions).

Theorem 3.3: (1.2) is nonoscillatory implying that (3.17) is nonoscillatory.

Theorem 3.4: (3.24) is oscillatory implying that (3.13) is oscillatory.

Our theorems then follow from the above by virtue of the Sturm-type comparison theorem.

We conclude the paper by the following application of Theorem 3.3.

Example 3.6. Let $\mathbb{T}=\mathbb{Z}$. Then $\mu(t)\equiv 1$, $f^{\Delta}(t)=\Delta f(t)$, and $\int_a^b f(t)\Delta t=\sum_{t=a}^{b-1}f(t)$. Further, let $r(t)=[(t+1)^{\beta-1}-t^{\beta-1}]^{1-\alpha}$ and

$$p(t)=\frac{\gamma}{t(t+1)}+\frac{\lambda(-1)^t}{t}, \tag{3.25}$$

where γ and λ are real constants. It is easy to see that $p(t)$ changes sign for $\lambda\neq 0$. Moreover,

$$\gamma-\lambda<t\sum_{s=t}^{\infty}p(s)<\gamma+\lambda, \tag{3.26}$$

$$\sum_{s=0}^{t-1}r^{1-\beta}(s)=t^{\beta-1}\longrightarrow\infty \tag{3.27}$$

as $t\to\infty$. In [17], it was proved (on general $\mathbb{T}$) that (1.2) is nonoscillatory provided that

$$\begin{gathered}\lim_{t\to\infty}\frac{\mu(t)r^{1-\beta}(t)}{\int_a^t r^{1-\beta}(s)\Delta s}=0,\\ -\frac{2\alpha-1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}<\liminf_{t\to\infty}\mathcal{A}(t)\leq\limsup_{t\to\infty}\mathcal{A}(t)<\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},\end{gathered} \tag{3.28}$$

where

$$\mathcal{A}(t) := \left(\int_a^t r^{1-\beta}(s)\Delta s \right)^{\alpha-1} \int_t^\infty p(s)\Delta s. \tag{3.29}$$

Hence, if $\gamma \geq \lambda > 0$ and

$$\gamma + \lambda < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.30}$$

then (2.12) holds, (1.2) is nonoscillatory because of (3.26), and

$$\mathcal{A}(t) = t \sum_{s=t}^{\infty} p(s). \tag{3.31}$$

Consequently, equation

$$\left[\left((t+1)^{\beta-1} - t^{\beta-1} \right)^{1-\alpha} \Phi(y^\Delta) \right]^\Delta + \left(\frac{\gamma q(t)}{t(t+1)} + \frac{\lambda(-1)^t q(t)}{t} \right) \Phi(y^\sigma) = 0, \tag{3.32}$$

where $q(t)$ is any nonincreasing sequence between 0 and 1, is also nonoscillatory by Theorem 3.3.

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ON LINEAR SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATIONS IN ONE FUNCTIONAL SPACE

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We use a special space of integrable functions for studying the Cauchy problem for linear functional-differential equations with nonintegrable singularities. We use the ideas developed by Azbelev and his students (1995). We show that by choosing the function ψ generating the space, one can guarantee resolubility and certain behavior of the solution near the point of singularity.

1. Linear Volterra operators in Δ_ψ spaces

We consider the following n-dimensional functional-differential equation:

$$\mathcal{L}x \overset{\text{def}}{=} \dot{x} + (K+S)\dot{x} + Ax(0) = f, \tag{1.1}$$

where

$$(Ky)(t) = \int_0^t K(t,s)y(s)ds, \tag{1.2}$$

$$(Sy)(t) = \begin{cases} B(t)y[g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1]. \end{cases} \tag{1.3}$$

The case where K and S are continuous on $L_p[0,1]$ operators is well studied (see, e.g., [1] and the references therein). Here we suppose that the functions $K(t,s)$ and $B(t)$ may be nonintegrable at $t = 0$. More precisely, we will formulate conditions on operators K and S in Sections 2 and 3. Under such conditions, those operators are not bounded on $L[0,1]$ and one has to choose other functional spaces for studying (1.1). We propose a space of integrable functions on $[0,1]$ and show that it may be useful in such a case.

We call Δ_ψ^p space the space of all measurable functions $y : [0,1] \to \mathbb{R}^n$, for which

$$\|y\|_{\Delta_\psi^p} = \sup_{0<h\le 1} \frac{1}{\psi(h)} \left(\int_0^h |y(s)|^p ds \right)^{1/p} < \infty. \tag{1.4}$$

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We assume everywhere below that ψ is a nondecreasing, absolutely continuous function, $\psi(0) = 0$.

Theorem 1.1. *The space Δ_ψ^p is a Banach space.*

Let $X[a,b]$, $Y[a,b]$ be spaces of functions defined on $[a,b]$.

We will call $V : X[0,1] \to Y[0,1]$ the *Volterra* operator [3] if for every $\xi \in [0,1]$ and for any $x_1, x_2 \in X[0,1]$ such that $x_1(t) = x_2(t)$ on $[0,\xi]$, $(Vx_1)(t) = (Vx_2)(t)$ for $t \in [0,1]$.

It is possible to say that each Volterra operator $V : X[0,1] \to Y[0,1]$ generates a set of operators $V_\xi : X[0,\xi] \to Y[0,\xi]$, where $\xi \in (0,1]$. By y_ξ, we denote the restriction of function y defined on $[0,1]$ onto segment $[0,\xi]$.

Theorem 1.2. *Let $V : L \to L$ be a linear bounded operator. Then V is a linear bounded operator in Δ_ψ^p and $\|V\|_{\Delta_\psi^p} \leq \|V\|_{L^p}$.*

Proof. Let $y \in \Delta_\psi^p$. Then

$$\begin{aligned} ||Vy||_{\Delta_\psi^p} &= \sup_{0<h\leq 1} \frac{1}{\psi(h)} ||(V_\xi y_\xi)||_{L[0,\xi]^p} \\ &\leq \sup_{0<h\leq 1} \frac{1}{\psi(h)} ||V_\xi||_{L[0,\xi]} ||y_\xi||_{L[0,\xi]} \leq \|V\|_L \|y\|_{L^p}. \end{aligned} \tag{1.5}$$

□

Theorem 1.3. *Let $V : \Delta_{\psi_1}^p \to \Delta_{\psi_1}^p$ be linear bounded operator and let*

$$\sup_{t\in[0,1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty. \tag{1.6}$$

Then V is linear and bounded in $\Delta_{\psi_2}^p$ and

$$\|V\|_{\Delta_{\psi_2}^p} \leq \|V\|_{\Delta_{\psi_1}^p} \sup_{\xi\in[0,1]} \sup_{\tau\in[0,\xi]} \frac{\psi_1(\xi)\psi_2(\tau)}{\psi_2(\xi)\psi_1(\tau)}. \tag{1.7}$$

Proof. Let $y \in \Delta_{\psi 2}$. Then

$$\begin{aligned} \|Vy\|_{\Delta_{\psi_2}^p} &\leq \sup_{\xi\in[0,1]} \frac{||Vy_\xi||_{L[0,\xi]}\psi_1(\xi)}{\psi_2(\xi)\psi_1(\xi)} \leq \sup_{\xi\in[0,1]} \frac{||Vy_\xi||_{\Delta_{\psi_1}^p[0,\xi]}\psi_1(\xi)}{\psi_2(\xi)} \\ &\leq \|V\|_M {\psi_1}^p \sup \frac{||y_\psi||_{\Delta_{\psi_1}^p}\psi_1(\xi)}{\psi_2(\xi)} \\ &\leq \|V\|_M {\psi_1}^p \sup_{\xi\in[0,1]} \sup_{\tau\in[0,\psi]} \frac{||y_\tau||_{L[0,\tau]}\psi_1(\xi)\psi_2(\tau)}{\psi_1(\tau)\psi_2(\xi)\psi_2(\tau)} \\ &\leq \|y\|_{\Delta_{\psi_2}^p} \|V\|_{\Delta_{\psi 1}^p} \sup_{\xi\in[0,1]} \sup_{\tau\in[0,\xi]} \frac{\psi_1(\xi)\psi_2(\tau)}{\psi_2(\xi)\psi_1(\tau)}. \end{aligned} \tag{1.8}$$

□

Corollary 1.4. *If $V_1 : \Delta_{\psi_1}^p \to \Delta_{\psi_1}^p$ and $V_2 : \Delta_{\psi_2}^p \to \Delta_{\psi_2}^p$ are linear continuous Volterra operators, then $V = V_1 + V_2$ is continuous on space Δ_ψ^p generated by $\psi(t) = \min(\psi_1(t), \psi_2(t))$ and $\|V\|_{\Delta_\psi^p} \leq \|V_1\|_{\Delta_{\psi_1}^p} + \|V_2\|_{\Delta_{\psi_2}^p}$.*

2. Operator K

In this section, we consider the integral operator (1.2). We will show that under certain conditions on matrix $K(t,s)$, a function ψ may be indicated such that K is bounded on Δ_ψ and its norm is limited by a given number.

We say that matrix $K(t,s)$ satisfies the $\mathcal{N}$ condition if for some p and p_1 such that $1 \le p \le p_1 < \infty$ and for any $\varepsilon \in (0,1]$,

$$||K_\varepsilon(t,\cdot)||_{L[0,t]} \in L_{p'}[\varepsilon,1]. \tag{2.1}$$

Here $K_\varepsilon(t,s)$ is a restriction of $K(t,s)$ onto $[\varepsilon,1]\times[0,t]$, $1/p+1/p'=1$.

The $\mathcal{N}$ condition admits a nonintegrable singularity at point $t=0$.

Lemma 2.1. *Let nonnegative function $\omega:[0,1]\to\mathbb{R}$ be nonincreasing and having a nonintegrable singularity at $t=0$.*

Then $\psi(t)=\exp[\int_1^t\omega(s)ds]$ is absolutely continuous on $[0,1]$, does not decrease, and is a solution of the equation $\int_1^t\omega(s)x(s)ds=x(t)$.

Denote

$$\psi(t)=\exp\left[\frac{1}{C}\int_1^t \operatorname*{vrai\,sup}_{s\in[0,\tau]}||K(\tau,s)||\,d\tau\right]. \tag{2.2}$$

Theorem 2.2. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition with $p=1$ and let C be some positive constant. Then operator K is bounded in Δ_ψ with function ψ defined by the equality (2.2) and $\|K\|_{\Delta_\psi}\le C$.*

Proof. Let $x\in\Delta_\psi$ and $y=Kx$. From the $\mathcal{N}$ condition it follows that for almost all $t\in[0,1]$, $K(\cdot,s)\in L_\infty$. Let $\omega(t)=\operatorname{vrai\,sup}_{s\in[0,\tau]}\|K(\tau,s)\|d\tau$. Then

$$\begin{aligned}\left(\int_0^t||y(s)||ds\right) &\le \left[\int_0^t\left(\int_0^\tau ||K(\tau,s)||||x(s)||ds\right)d\tau\right]\\ &\le \int_0^t\left(\operatorname*{vrai\,sup}_{s\in[0,\tau]}||K(\tau,s)||\right)\left(\int_0^\tau||x(s)||ds\right)d\tau\\ &\le \|x\|_{\Delta_\psi}\int_0^t\omega(\tau)\psi(\tau)d\tau.\end{aligned} \tag{2.3}$$

According to Lemma 2.1, $\psi(t)=\exp[(1/C)\int_1^t\omega(s)ds]$ is a solution of the equation $\int_1^t\omega(s)\psi(s)ds=C\psi(t)$, does not decrease, is absolutely continuous, and $\psi(0)=0$. That implies

$$\left(\int_0^t||y(s)||ds\right)\le C\|x\|_{\Delta_\psi}\psi(t). \tag{2.4}$$

□

Remark 2.3. If $K(\cdot,s)$ has bounded variation on s, it is possible to indicate a "wider" space Δ_ψ for which conditions of Theorem 2.2 are satisfied by defining function ψ as

$$\psi(t) = \exp\left[\frac{1}{C}\int_1^t \left(||K(\tau,\tau)||d\tau + \int_0^\tau d_s \operatorname*{var}_{s\in[0,\tau]} ||K(\tau,s)||\right)d\tau\right]. \tag{2.5}$$

Theorem 2.4. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition with $1 < p < \infty$ and let C be some positive constant. Then operator K is bounded in space Δ_ψ^p generated by*

$$\psi(t) = \exp\left[\frac{1}{pC}\int_1^t \left(\int_0^\tau ||K(\tau,s)||^{p'} ds\right)^{p_1/p'} d\tau\right] \tag{2.6}$$

and $\|K\|_{\Delta_\psi^p} \le C$.

Theorem 2.4 can be proved in a way similar to proof of Theorem 2.2.

Lemma 2.5. *Let $K : \Delta_\psi^p \to \Delta_\psi^p$ $(1 < p < \infty)$ be a bounded operator and let its matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition. Then $K : \Delta_\psi^p \to L_p$ is a compact operator.*

Proof. For every $t \in [0,1]$, $(Ky)(t)$ is a linear bounded functional on L_p. Let $\{y_i\}$ be a sequence weakly converging to y_0 in L_p. If $\{y_i\} \subset \Delta_\psi^p$ and $\|y_i\|_{\Delta_\psi^p} \le 1$, then $\|y_0\|_{\Delta_\psi^p} \le 1$. Indeed, if for some $t_1 \in [0,1]$, $((1/\psi(t_1))\int_0^{t_1} \|y(s)\|^p ds)^{1/p} > 1$, then the sequence $ly_i = \int_0^1 l(s)y_i(s)ds$ does not converge to ly_0, where

$$l(s) = \begin{cases} 1, & \text{if } s \le t_1, \\ 0, & \text{if } s > t_1. \end{cases} \tag{2.7}$$

Hence, for almost all $t \in [0,1]$, $\{(Ky_i)(t)\}$ converges and the set Ky is compact in measure. Thus, for the operator $K : \Delta_\psi^p \to L_p$ to be compact, it is necessary and sufficient that the norms of Ky are equicontinuous for $\|y\|_{\Delta_\psi^p} \le M$. Let $\delta \in (0,1)$. As $K : \Delta_\psi^p \to \Delta_\psi^p$ is a bounded operator,

$$\left(\frac{1}{\psi(\delta)}\int_0^\delta ||(Ky)(s)||^p ds\right)^{1/p} \le \Delta_0. \tag{2.8}$$

This implies that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $\delta < \delta_1$, then $(\int_0^\delta \|(Ky)(s)\|^p ds)^{1/p} \le \varepsilon/2$.

Then, from the $\mathcal{N}$ condition, there exists δ_2 such that if $\operatorname{mes} e \le \delta_2$ for some $e \subset [\delta,1]$, then $(\int_e \|(Ky)(s)\|^p ds)^{1/p} \le \varepsilon/2$.

Finally, for $e_1 \subset [\delta,1]$ such that $\operatorname{mes} e_1 \le \min\{\delta_1,\delta_2\}$,

$$\left(\int_{e_1} ||(Ky)(s)||^p ds\right)^{1/p} \le \left(\int_0^\delta ||(Ky)(s)||^p ds\right)^{1/p} + \left(\int_\delta^1 ||(Ky)(s)||^p ds\right)^{1/p} \le \varepsilon. \tag{2.9}$$

□

Lemma 2.6. *Let $\{y_i\} \to y_0$ in L_p $(1 < p < \infty)$ and let the sequence $\{(1/u)y_i\}$ be bounded in Δ_ψ^p for some continuous increasing function u, $u(0) = 0$. Then $\{y_i\} \to y_0$ in Δ_ψ^p.*

Proof. We have

$$\left(\int_0^t \|y_i(s)\|^p ds\right)^{1/p} \le u(t)\left(\int_0^t \left\|\frac{y_i(s)}{u(s)}\right\|^p ds\right)^{1/p} \le Mu(t)\psi(t). \tag{2.10}$$

Thus, $y_i \in \Delta_\psi^p$. Beginning with some N for any $t \in [0,1]$ and for any given $\varepsilon > 0$,

$$\left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} \le \varepsilon. \tag{2.11}$$

Hence,

$$\begin{aligned}\left(\int_0^t \|y_0(s)\|^p ds\right)^{1/p} &\le \left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} + \left(\int_0^t \|y_i(s)\|^p ds\right)^{1/p} \\ &\le \varepsilon + Mu(t)\psi(t) \le Mu(t)\psi(t), \\ \left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} &\le 2Mu(t)\psi(t),\end{aligned} \tag{2.12}$$

beginning with some N_δ for any $\delta > 0$, $\|y_0 - y_i\|_{\Delta_\psi^p} < \delta$. Indeed, Lemma 2.5 guarantees the existence of $\tau \in (0,1]$ such that for all $t \in [0,\tau]$,

$$\left(\int_0^t \|y_i(s) - y_0(s)\|^p ds\right)^{1/p} \le \delta\psi(t). \tag{2.13}$$

Let $t \in [\tau,1]$. Then for $\varepsilon = \delta\psi(\tau)$, (2.11) yields (2.13) for all $t \in [0,1]$. □

Let $u : [0,1] \to \mathbb{R}$ be a continuous increasing function, $u(0) = 0$. Denote

$$\psi(t) = \exp\left[\int_1^t \frac{1}{u(\tau)}\left(\int_0^\tau \|K(\tau,s)\|^{p'} ds\right)^{p/p'} d\tau\right]. \tag{2.14}$$

Lemmas 2.5 and 2.6 imply the following theorem.

THEOREM 2.7. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition with $1 < p < \infty$. And let ψ be defined by (2.14). Then $K : \Delta_\psi^p \to \Delta_\psi^p$ is a compact operator and its spectral radius is equal to zero.*

3. Operator S

Denote

$$\begin{aligned}(S_g y)(t) &= \begin{cases} y[g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1], \end{cases} \\ (Sy)(t) &= B(t)(S_g)(t).\end{aligned} \tag{3.1}$$

In [2], it is shown that S_g is bounded in L_p if $r = (\sup(\operatorname{mes} g^{-1}(E)/\operatorname{mes} E))^{1/p} < \infty$ and $\|S_g\|_{L_p} = r$, where sup is taken on all measurable sets from $[0,1]$.

Let Ω_m be a set of points from $[0,1]$ for which $g(t) \geq mt$, $\beta(t)$ is a nonincreasing majorant of function $\|B(t)\|$, and

$$\varphi(t) = \lim_{\operatorname{mes} e \to 0} \frac{\operatorname{mes} g^{-1}(e)}{\operatorname{mes} e}, \tag{3.2}$$

where e is a closed interval containing t.

We say that operator S_g satisfies the $\mathcal{M}$ condition if $\operatorname{vrai\,sup}_{t\in[\varepsilon,1]} \varphi(t) < \infty$ for any

$$\varepsilon \in (0,1] \operatorname*{vrai\,sup}_{t\in[\varepsilon,1]} ||B(t)|| < \infty, \tag{3.3}$$

and there exists $m \in [0,1)$ such that

$$\mu_m = \operatorname*{vrai\,sup}_{t\in g(\Omega_m)} \left(\beta(t)^p \varphi(t)\right) < \infty. \tag{3.4}$$

Lemma 3.1. *There exists nonincreasing function $u : (0,1] \to \mathbb{R}$ such that $\beta(t)^p\varphi(t) \leq u(t)$ and the function*

$$\psi(t) = \begin{cases} t^{u(t)} & \text{if } t \in (0,1], \\ 0 & \text{if } t = 0, \end{cases} \tag{3.5}$$

is absolutely continuous on $[0,1]$.

Proof. Let $\{t_i\}$ be a decreasing sequence, $t_1 = 1$, $t_i \to 0$. Denote

$$n_i = \operatorname*{vrai\,sup}_{t\in(t_{i+1},t_i)} \left(\beta(t)^p\varphi(t)\right), \quad u(t) = \frac{n_{i+1} - n_i}{t_{i+1} - t_i}(t - t_i) + n_i, \tag{3.6}$$

where $t \in (t_{i+1}, t_i)$. Then $\beta(t)^p\varphi(t) \leq u(t)$, u increases and is absolutely continuous on $[0,1]$. $\square$

Let

$$\nu_m = m^{u(1)}\left[u(1) - \frac{1}{\ln m}\right]. \tag{3.7}$$

Theorem 3.2. *Let operator S_g satisfy the $\mathcal{M}$ condition and let function u satisfy conditions of Lemma 3.1. Then S_g is bonded in Δ_ψ^p with $\psi(t) = t^{u(t)}$ and*

$$||S_g||_{\Delta_\psi^p} \leq \left(\nu_m + \mu_m\right)^{1/p}. \tag{3.8}$$

Proof. Let $y \in \Delta_\psi^p$, $\|y\|_{\Delta_\psi^p} = 1$, and $\delta \in (0,1)$. Denote measures λ and μ on $[\delta,1]$ by $\lambda(e) = \int_e \beta(s)^p ds$ and $\mu(e) = \int_{g^{-1}(e)} \beta(s)^p ds$. Then by the Radon-Nikodym [2] theorem, we have

$$\begin{aligned} \left|\left|\int_\delta^t |(S_g y)(t)|^p ds\right|\right| &\leq \int_{g^{-1}([0,t])\cap[\delta,1]} ||y[g(s)]||^p d\lambda(s) \\ &= \int_{g^{-1}([0,t])\cap[\delta,1]} ||y(s)||^p \frac{d\mu}{d\lambda}(s) d\lambda(s). \end{aligned} \tag{3.9}$$

Then as $g(t) \le t$,

$$\frac{d\mu}{d\lambda}(s) = \lim_{\operatorname{mes} e \to 0} \frac{\int_{g^{-1}(e)} \beta(s)^p ds}{\int_e \beta(s)^p ds} \le \lim_{\operatorname{mes} e \to 0} \frac{\operatorname{vrai\,sup}_{g^{-1}(e)} \beta(s)^p ds}{\operatorname{vrai\,sup}_e \beta(s)^p} \varphi(s) = \varphi(s) \tag{3.10}$$

or

$$\begin{aligned} \left\| \int_\delta^t |(S_g y)(t)|^p ds \right\| &\le \int_{g^{-1}([0,t]\setminus\Omega_m)\cap[\delta,1]} \beta(s)^p \|y(s)\|^p \varphi(s) ds \\ &\quad + \int_{g^{-1}(\Omega_m)\cap[\delta,1]} \beta(s)^p \|y(s)\|^p \varphi(s) ds \\ &\le \int_0^{mt} \beta(s)^p \|y(s)\|^p \varphi(s) ds + \int_0^t \|y(s)\|^p \mu_m ds \\ &\le \int_0^{mt} \|y(s)\|^p u(s) ds + \mu_m \psi(t)^p. \end{aligned} \tag{3.11}$$

We denote function $u_k : (0,1] \to \mathbb{R}$ by $u_k(t) = u(t_i)$, where $t_i = (2^k - i)/2^k$, $i = 0,1,2,\ldots,$ $2^k - 1$. From $u_k \to u$, it follows that

$$\int_0^{mt} \|y(s)\|^p u(s) ds = \lim_{k\to 0} \int_0^{mt} \|y(s)\|^p u_k(s) ds. \tag{3.12}$$

We write function u_k in the form

$$u_k(t) = \begin{cases} u(t_0), & \text{if } t \in (t_1, t_0], \\ u(t_0) + [u(t_1) - u(t_0)], & \text{if } t \in (t_2, t_1], \\ \vdots & \vdots \\ u(t_{k-2}) + [u(t_{k-1}) - u(t_{k-2})], & \text{if } t \in (t_k, t_{k-1}]. \end{cases} \tag{3.13}$$

The condition $t < t_i$ implies that $\int_0^{mt} \|y(s)\|^p ds \le \psi^p(mt) = (mt)^{u(mt)} \le m^{pu(t_i)} \psi^p(t)$ and

$$\begin{aligned} \int_0^{mt} \|y(s)\|^p u(s) ds &\le \sum_{i=1}^{2^k} m^{pu(t_i)} [u(t_i) - u(t_{i-1})] \psi^p(t) + u(1) m^{pu(1)} \psi^p(t) \\ &\le \psi^p(t) \left[\int_{u(1)}^\infty m^s ds + m^{u(1)} u(1) \right] \\ &\le \psi^p(t) m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right], \end{aligned} \tag{3.14}$$

simultaneously for all k. Finally,

$$\begin{aligned} \left\| \int_0^t |(S_g y)(s)|^p ds \right\| &= \lim_{\delta\to 0} \left\| \int_\delta^t |(S_g y)(s)|^p ds \right\| \\ &\le \psi^p(t) m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right] + \psi^p(t) \mu_m \\ &\le \psi^p(t) (\nu_m + \mu_m) \end{aligned} \tag{3.15}$$

which proves the theorem. □

Remark 3.3. From (3.7) and (3.8), it follows that if $\lim_{m\to 1} < 1$, then there exists function ψ such that the norm of operator $S_g : \Delta_\psi^p \to \Delta_\psi^p$ is less than 1.

In some particular cases, it is possible to give less strict conditions on function ψ generating the space Δ_ψ^p. Direct calculations prove the following theorem.

Theorem 3.4. *Let $B(t) \leq C_1/t^\alpha$ and $g(t) = C_2 t^\beta$ with $\beta > 1$. Then $\|S_g\|_{\Delta_\psi^p} \leq C_1/C_2$, where $\psi(t) = t^\gamma$, $\gamma \geq (\alpha p + \beta - 1)/p(\beta - 1)$. If $\gamma > (\alpha p + \beta - 1)/p(\beta - 1)$, then the spectral radius of S_g is equal to zero.*

4. The Cauchy problem

We consider the Cauchy problem for (1.1):

$$(\mathcal{L}x)(t) = f(t), \qquad x(0) = \alpha. \tag{4.1}$$

The theorems of this section are immediate corollaries of Theorems 2.2, 2.4, 2.7, 3.2, and 3.4.

Theorem 4.1. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition and let operator S_g satisfy the $\mathcal{M}$ condition. Let also $\operatorname{vraisup}_{t\in[0,1]} u(t) = \infty$, $(\mu_m)^{1/p} \leq q < 1$, and let the function ψ_1 be given by (2.14). Then if $C < 1 - q$, the Cauchy problem (4.1) has a unique solution in Δ_ψ^p with $\psi(t) = \min\{\psi_1(t), t^{u(t)}\}$ for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.*

Let ω be a solution of the equation

$$m^\omega\left(\omega - \frac{1}{\ln m}\right) \leq C_1^p - q, \quad \gamma = \sup_{t\in[0,1]}\{u(t), \omega\}, \tag{4.2}$$

where $0 \leq q \leq C_1^p < 1$, and u satisfies conditions of Lemma 3.1.

Theorem 4.2. *Let matrix $K(t,s)$ and operator S_g satisfy the $\mathcal{N}$ and $\mathcal{M}$ conditions, respectively. Let $\operatorname{vraisup}_{t\in[0,1]} u(t) < \infty$ and $(\mu_m)^{1/p} \leq q < 1$. Then if $q < C_1$, $(C_1 + C_2) < 1$, then the Cauchy problem (4.1) has a unique solution in Δ_ψ^p with $\psi(t) = \min\{\psi_1(t), t^\gamma\}$ for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.*

Theorem 4.3. *Let matrix $K(t,s)$ satisfy the $\mathcal{N}$ condition, $B(t) \leq C_1/t^\alpha$, $g(t) = C_2 t^\beta$ $(\beta > 1)$, and $\gamma > (\alpha p + \beta - 1)/p(\beta - 1)$. Let also $C < 1$ and $\psi(t) = \min\{\psi_1(t), t^\gamma\}$. Then the Cauchy problem (4.1) has a unique solution for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.*

Example 4.4. The Cauchy problem

$$\begin{gathered}\dot{x}(t) + p(t)\frac{x[h(t)]}{t^k} + q(t)\dot{x}(t^2) = f(t), \quad t \in [0,1], \\ x(\xi) = 0, \quad \text{if } h(\xi) \leq 0,\end{gathered} \tag{4.3}$$

where $h(t) \leq t$, $k > 1$, and p and q are bounded functions, has a solution if $\int_0^t |f(s)|ds \leq M\exp(-t^{1-k})$. If $(t - h(t)) \geq \tau > 0$, then it has a solution if $\int_0^t |f(s)|ds \leq Mt^\gamma$ for $\gamma > 1$.

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NONMONOTONE IMPULSE EFFECTS IN SECOND-ORDER PERIODIC BOUNDARY VALUE PROBLEMS

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We deal with the nonlinear impulsive periodic boundary value problem $u'' = f(t,u,u')$, $u(t_i+) = \mathrm{J}_i(u(t_i))$, $u'(t_i+) = \mathrm{M}_i(u'(t_i))$, $i = 1,2,\dots,m$, $u(0) = u(T)$, $u'(0) = u'(T)$. We establish the existence results which rely on the presence of a well-ordered pair (σ_1,σ_2) of lower/upper functions ($\sigma_1 \le \sigma_2$ on $[0,T]$) associated with the problem. In contrast to previous papers investigating such problems, the monotonicity of the impulse functions J_i, M_i is not required here.

1. Introduction

In recent years, the theory of impulsive differential equations has become a well-respected branch of mathematics. This is because of its characteristic features which provide many interesting problems that cannot be solved by applying standard methods from the theory of ordinary differential equations. It can also give a natural description of many real models from applied sciences (see the examples mentioned in [1, 2]).

In particular, starting with [7], periodic boundary value problems for nonlinear second-order impulsive differential equations of the form (2.1), (2.2), and (2.3) have received considerable attention; see, e.g., [1, 3, 5, 6, 8, 9, 14], where the existence results in terms of lower and upper functions can also be found. However, all impose certain monotonicity requirements on the impulse functions. In contrast to these papers, we provide existence results using weaker conditions (2.10) and (2.11) instead of monotonicity.

Throughout the paper, we keep the following notation and conventions. For a real valued function u defined a.e. on $[0,T]$, we put

$$\|u\|_\infty = \sup_{t\in[0,T]} \operatorname{ess} |u(t)|, \qquad \|u\|_1 = \int_0^T |u(s)|\,ds. \tag{1.1}$$

For a given interval $J \subset \mathbb{R}$, let $\mathbb{C}(J)$ denote the set of real-valued functions which are continuous on J. Furthermore, let $\mathbb{C}^1(J)$ be the set of functions having continuous first derivatives on J, and $\mathbb{L}(J)$ the set of functions which are Lebesgue integrable on J.

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Let $m \in \mathbb{N}$ and let

$$0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T \tag{1.2}$$

be a division of the interval $[0,T]$. We denote

$$\mathrm{D} = \{t_1, t_2, \ldots, t_m\} \tag{1.3}$$

and define $\mathbb{C}^1_{\mathrm{D}}[0,T]$ as the set of functions $u : [0,T] \mapsto \mathbb{R}$,

$$u(t) = \begin{cases} u_{[0]}(t) & \text{if } t \in [0,t_1], \\ u_{[1]}(t) & \text{if } t \in (t_1,t_2], \\ \vdots & \vdots \\ u_{[m]}(t) & \text{if } t \in (t_m,T], \end{cases} \tag{1.4}$$

where $u_{[i]} \in \mathbb{C}^1[t_i,t_{i+1}]$ for $i = 0,1,\ldots,m$. Moreover, $\mathbb{AC}^1_{\mathrm{D}}[0,T]$ stands for the set of functions $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ having first derivatives absolutely continuous on each subinterval (t_i,t_{i+1}), $i = 0,1,\ldots,m$. For $u \in \mathbb{C}^1_{\mathrm{D}}[0,T]$ and $i = 1,2,\ldots,m+1$, we write

$$u'(t_i) = u'(t_i-) = \lim_{t \to t_i-} u'(t), \qquad u'(0) = u'(0+) = \lim_{t \to 0+} u'(t), \tag{1.5}$$

$$\|u\|_{\mathrm{D}} = \|u\|_\infty + \|u'\|_\infty. \tag{1.6}$$

Note that the set $\mathbb{C}^1_{\mathrm{D}}[0,T]$ becomes a Banach space when equipped with the norm $\|\cdot\|_{\mathrm{D}}$ and with the usual algebraic operations.

We say that $f : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$ satisfies the *Carathéodory conditions* on $[0,T] \times \mathbb{R}^2$ if

(i) for each $x \in \mathbb{R}$ and $y \in \mathbb{R}$, the function $f(\cdot,x,y)$ is measurable on $[0,T]$;

(ii) for a.e. $t \in [0,T]$, the function $f(t,\cdot,\cdot)$ is continuous on $\mathbb{R}^2$;

(iii) for each compact set $K \subset \mathbb{R}^2$, there is a function $m_K(t) \in \mathbb{L}[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ holds for a.e. $t \in [0,T]$ and all $(x,y) \in K$.

The set of functions satisfying the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$ will be denoted by $\mathrm{Car}([0,T] \times \mathbb{R}^2)$.

Given a Banach space $\mathbb{X}$ and its subset M, let $\mathrm{cl}(M)$ and ∂M denote the closure and the boundary of M, respectively.

Let Ω be an open bounded subset of $\mathbb{X}$. Assume that the operator $\mathrm{F} : \mathrm{cl}(\Omega) \mapsto \mathbb{X}$ is completely continuous and $\mathrm{F}u \neq u$ for all $u \in \partial\Omega$. Then $\deg(\mathrm{I} - \mathrm{F}, \Omega)$ denotes the *Leray-Schauder topological degree* of $\mathrm{I} - \mathrm{F}$ with respect to Ω, where I is the identity operator on $\mathbb{X}$. For a definition and properties of the degree, see, for example, [4] or [10].

2. Formulation of the problem and main assumptions

Here we study the existence of solutions to the following problem:

$$u'' = f(t,u,u'), \tag{2.1}$$

$$u(t_i+) = \mathrm{J}_i(u(t_i)), \quad u'(t_i+) = \mathrm{M}_i(u'(t_i)), \quad i = 1,2,\ldots,m, \tag{2.2}$$

$$u(0) = u(T), \qquad u'(0) = u'(T), \tag{2.3}$$

where $u'(t_i)$ are understood in the sense of (1.5), $f \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$, $\mathrm{J}_i \in \mathbb{C}(\mathbb{R})$, and $\mathrm{M}_i \in \mathbb{C}(\mathbb{R})$.

Definition 2.1. A *solution of the problem* (2.1), (2.2), and (2.3) is a function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ which satisfies the impulsive conditions (2.2), the periodic conditions (2.3), and for a.e. $t \in [0,T]$ fulfils the equation $u''(t) = f(t,u(t),u'(t))$.

Definition 2.2. A function $\sigma_1 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is called a *lower function of problem* (2.1), (2.2), and (2.3) if

$$\begin{gathered}\sigma_1''(t) \ge f\left(t,\sigma_1(t),\sigma_1'(t)\right) \quad \text{for a.e. } t \in [0,T],\\ \sigma_1\left(t_i+\right) = \mathrm{J}_i\left(\sigma_1\left(t_i\right)\right), \quad \sigma_1'\left(t_i+\right) \ge \mathrm{M}_i\left(\sigma_1'\left(t_i\right)\right), \quad i = 1,2,\dots,m,\\ \sigma_1(0) = \sigma_1(T), \qquad \sigma_1'(0) \ge \sigma_1'(T).\end{gathered} \tag{2.4}$$

Similarly, a function $\sigma_2 \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ is an *upper function of problem* (2.1), (2.2), and (2.3) if

$$\sigma_2''(t) \le f\left(t,\sigma_2(t),\sigma_2'(t)\right) \quad \text{for a.e. } t \in [0,T], \tag{2.5}$$

$$\sigma_2\left(t_i+\right) = \mathrm{J}_i\left(\sigma_2\left(t_i\right)\right), \quad \sigma_2'\left(t_i+\right) \le \mathrm{M}_i\left(\sigma_2'\left(t_i\right)\right), \quad i = 1,2,\dots,m, \tag{2.6}$$

$$\sigma_2(0) = \sigma_2(T), \qquad \sigma_2'(0) \le \sigma_2'(T). \tag{2.7}$$

Throughout the paper we assume

$$\begin{gathered}0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T < \infty, \qquad \mathrm{D} = \{t_1,t_2,\dots,t_m\},\\ f \in \mathrm{Car}\left([0,T]\times\mathbb{R}^2\right), \qquad \mathrm{J}_i \in \mathbb{C}(\mathbb{R}), \qquad \mathrm{M}_i \in \mathbb{C}(\mathbb{R}), \quad i = 1,2,\dots,m;\end{gathered} \tag{2.8}$$

$$\begin{gathered}\sigma_1 \text{ and } \sigma_2 \text{ are, respectively, lower and upper functions of (2.1), (2.2), and (2.3),}\\ \sigma_1 \le \sigma_2 \text{ on } [0,T];\end{gathered} \tag{2.9}$$

$$\sigma_1\left(t_i\right) \le x \le \sigma_2\left(t_i\right) \implies \mathrm{J}_i\left(\sigma_1\left(t_i\right)\right) \le \mathrm{J}_i(x) \le \mathrm{J}_i\left(\sigma_2\left(t_i\right)\right), \quad i = 1,2,\dots,m; \tag{2.10}$$

$$\begin{gathered}y \le \sigma_1'\left(t_i\right) \implies \mathrm{M}_i(y) \le \mathrm{M}_i\left(\sigma_1'\left(t_i\right)\right),\\ y \ge \sigma_2'\left(t_i\right) \implies \mathrm{M}_i(y) \ge \mathrm{M}_i\left(\sigma_2'\left(t_i\right)\right), \quad i = 1,2,\dots,m.\end{gathered} \tag{2.11}$$

Remark 2.3. If $\mathrm{M}_i(0) = 0$ for $i = 1,2,\dots,m$ and $r_1 \in \mathbb{R}$ is such that $\mathrm{J}_i(r_1) = r_1$ for $i = 1,2,\dots,m$ and

$$f\left(t,r_1,0\right) \le 0 \quad \text{for a.e. } t \in [0,T], \tag{2.12}$$

then $\sigma_1(t) \equiv r_1$ on $[0,T]$ is a lower function of problem (2.1), (2.2), and (2.3). Similarly, if $r_2 \in \mathbb{R}$ is such that $\mathrm{J}_i(r_2) = r_2$ for all $i = 1,2,\dots,m$ and

$$f\left(t,r_2,0\right) \ge 0 \quad \text{for a.e. } t \in [0,T], \tag{2.13}$$

then $\sigma_2(t) \equiv r_2$ is an upper function of problem (2.1), (2.2), and (2.3).

3. A priori estimates

At the beginning of this section, we introduce a class of auxiliary problems and prove uniform a priori estimates for their solutions.

Take $d \in \mathbb{R}$, $\tilde{f} \in \mathrm{Car}([0,T]\times\mathbb{R}^2)$, $\tilde{\mathrm{J}}_i \in \mathbb{C}(\mathbb{R})$, and $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\ldots,m$, such that

$$\begin{aligned} &\tilde{f}(t,x,y) < f(t,\sigma_1(t),\sigma_1'(t)) \quad \text{for a.e. } t \in [0,T], \text{ all } x \in (-\infty,\sigma_1(t)),\\ &\qquad\qquad \text{and all } y \in \mathbb{R} \text{ such that } |y-\sigma_1'(t)| \le \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1};\\ &\tilde{f}(t,x,y) > f(t,\sigma_2(t),\sigma_2'(t)) \quad \text{for a.e. } t \in [0,T], \text{ all } x \in (\sigma_2(t),\infty),\\ &\qquad\qquad \text{and all } y \in \mathbb{R} \text{ such that } |y-\sigma_2'(t)| \le \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}; \end{aligned} \tag{3.1}$$

$$\begin{aligned} &\tilde{\mathrm{J}}_i(x) < \mathrm{J}_i(\sigma_1(t_i)) && \text{if } x < \sigma_1(t_i),\\ &\tilde{\mathrm{J}}_i(x) = \mathrm{J}_i(x) && \text{if } x \in [\sigma_1(t_i),\sigma_2(t_i)], \quad (i=1,2,\ldots,m);\\ &\tilde{\mathrm{J}}_i(x) > \mathrm{J}_i(\sigma_2(t_i)) && \text{if } x > \sigma_2(t_i), \end{aligned} \tag{3.2}$$

$$\left.\begin{aligned} &\widetilde{\mathrm{M}}_i(y) \le \mathrm{M}_i(\sigma_1'(t_i)) && \text{if } y \le \sigma_1'(t_i),\\ &\widetilde{\mathrm{M}}_i(y) \ge \mathrm{M}_i(\sigma_2'(t_i)) && \text{if } y \ge \sigma_2'(t_i), \end{aligned}\right. \quad (i=1,2,\ldots,m); \tag{3.3}$$

$$\sigma_1(0) \le d \le \sigma_2(0), \tag{3.4}$$

and consider an auxiliary Dirichlet problem

$$u'' = \tilde{f}(t,u,u'), \tag{3.5}$$

$$u(t_i+) = \tilde{\mathrm{J}}_i(u(t_i)), \quad u'(t_i+) = \widetilde{\mathrm{M}}_i(u'(t_i)), \quad i = 1,2,\ldots,m, \tag{3.6}$$

$$u(0) = u(T) = d. \tag{3.7}$$

Lemma 3.1. *Let (2.8), (2.9), and (2.10) and (3.1), (3.2), (3.3), and (3.4) hold. Then every solution u of (3.5), (3.6), and (3.7) satisfies*

$$\sigma_1 \le u \le \sigma_2 \quad \text{on } [0,T]. \tag{3.8}$$

Proof. Let u be a solution of (3.5), (3.6), and (3.7). Put $v(t) = u(t) - \sigma_2(t)$ for $t \in [0,T]$. Then, by (3.4), we have

$$v(0) = v(T) \le 0. \tag{3.9}$$

So, it remains to prove that $v \le 0$ on $(0,T)$.

Part (i). First, we show that v does not have a positive local maximum at any point of $(0,T)\setminus \mathrm{D}$. Assume, on the contrary, that there is $\alpha \in (0,T)\setminus \mathrm{D}$ such that v has a positive local maximum at α, that is,

$$v(\alpha) > 0, \qquad v'(\alpha) = 0. \tag{3.10}$$

This guarantees the existence of β such that $[\alpha,\beta] \subset (0,T) \setminus \mathrm{D}$ and

$$v(t) > 0, \qquad |v'(t)| < \frac{v(t)}{v(t)+1} < 1 \tag{3.11}$$

for $t \in [\alpha,\beta]$. Using (2.5), (3.1), and (3.11), we get

$$\begin{aligned} v''(t) = u''(t) - \sigma_2''(t) &= \widetilde{f}(t,u(t),u'(t)) - \sigma_2''(t) \\ &> f(t,\sigma_2(t),\sigma_2'(t)) - \sigma_2''(t) \geq 0 \end{aligned} \tag{3.12}$$

for a.e. $t \in [\alpha,\beta]$. Hence,

$$0 < \int_\alpha^t v''(s)ds = v'(t) \tag{3.13}$$

for all $t \in (\alpha,\beta]$. This contradicts that v has a local maximum at α.

Part (ii). Now, assume that there is $t_j \in \mathrm{D}$ such that

$$\max_{t\in(t_{j-1},t_j]} v(t) = v(t_j) > 0. \tag{3.14}$$

Then $v'(t_j) \geq 0$. By (3.2) and (3.3), we get

$$\widetilde{\mathrm{J}}_j(u(t_j)) > \mathrm{J}_j(\sigma_2(t_j)), \qquad \widetilde{\mathrm{M}}_j(u'(t_j)) \geq \mathrm{M}_j(\sigma_2'(t_j)). \tag{3.15}$$

By (3.6) and (2.6), the relations

$$v(t_j+) > 0, \qquad v'(t_j+) \geq 0 \tag{3.16}$$

follow. If $v'(t_j+) > 0$, then there is $\beta \in (t_j,t_{j+1})$ such that

$$v'(t) > 0 \quad \text{on } (t_j,\beta]. \tag{3.17}$$

If $v'(t_j+) = 0$, then we can find β such that $(t_j,\beta] \subset (0,T) \setminus \mathrm{D}$ and (3.11) is satisfied on $(t_j,\beta]$. Consequently, (3.17) is valid in this case as well. As by Part (i) v' cannot change its sign on (t_j,t_{j+1}), in both these cases we have

$$v'(t) \geq 0 \quad \text{on } (t_j,t_{j+1}). \tag{3.18}$$

Now, by (3.16),(3.17), and (3.18), we get

$$\max_{t\in(t_j,t_{j+1}]} v(t) = v(t_{j+1}) > 0. \tag{3.19}$$

Continuing inductively, we get $v(T) > 0$ contrary to (3.9).

Part (iii). Finally, assume that

$$\sup_{t\in(t_j,t_{j+1}]} v(t) = v(t_j+) > 0 \tag{3.20}$$

for some $t_j \in \mathrm{D}$. In view of (3.2), this is possible only if

$$\widetilde{\mathrm{J}}_j(u(t_j)) > \mathrm{J}_j(\sigma_2(t_j)). \tag{3.21}$$

If $u(t_j) \in [\sigma_1(t_j), \sigma_2(t_j)]$, then by (3.2) and (2.10), we have

$$\widetilde{\mathrm{J}}_j(u(t_j)) = \mathrm{J}_j(u(t_j)) \le \mathrm{J}_j(\sigma_2(t_j)), \tag{3.22}$$

contrary to (3.21). If $u(t_j) < \sigma_1(t_j)$, then by (3.2), (2.9), and (2.10), we get

$$\widetilde{\mathrm{J}}_j(u(t_j)) < \mathrm{J}_j(\sigma_1(t_j)) \le \mathrm{J}_j(\sigma_2(t_j)), \tag{3.23}$$

again a contradiction to (3.21). Therefore, $u(t_j) > \sigma_2(t_j)$, that is, $v(t_j) > 0$. Further, (3.20) gives $v'(t_j+) \le 0$. If $v'(t_j+) = 0$, then, as in Part (ii), we get (3.17), which contradicts (3.20). Therefore, $v'(t_j+) < 0$, which yields, with (3.3), that $v'(t_j) < 0$. Thus, in view of Part (i), we deduce that $v' \le 0$ on (t_{j-1}, t_j), that is, $\sup_{t\in(t_{j-1},t_j]} v(t) = v(t_{j-1}+) > 0$. Continuing inductively, we get $v(0) > 0$, contradicting (3.9).

To summarize, we have proved that $v \le 0$ on $[0,T]$ which means that $u \le \sigma_2$ on $[0,T]$. If we put $v = \sigma_1 - u$ on $[0,T]$ and use the properties of σ_1 instead of σ_2, we can prove $\sigma_1 \le u$ on $[0,T]$ by an analogous argument. □

In the proof of Theorem 4.1, we need a priori estimates for derivatives of solutions. To this aim we prove the following lemma.

Lemma 3.2. *Assume that $r \in (0,\infty)$ and that*

$$h \in \mathbb{L}[0,T] \quad \textit{is nonnegative a.e. on } [0,T], \tag{3.24}$$

$$\omega \in \mathbb{C}([1,\infty)) \quad \textit{is positive on } [1,\infty), \qquad \int_1^\infty \frac{ds}{\omega(s)} = \infty. \tag{3.25}$$

Then there exists $r^* \in (1,\infty)$ such that the estimate

$$\|u'\|_\infty \le r^* \tag{3.26}$$

holds for each function $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ satisfying $\|u\|_\infty \le r$ and

$$|u''(t)| \le \omega(|u'(t)|)(|u'(t)| + h(t)) \quad \textit{for a.e. } t \in [0,T], \textit{ for } |u'(t)| > 1. \tag{3.27}$$

Proof. Let $u \in \mathbb{AC}^1_{\mathrm{D}}[0,T]$ satisfy (3.27) and let $\|u\|_\infty \le r$. The mean value theorem implies that there are $\xi_i \in (t_i, t_{i+1})$ such that

$$|u'(\xi_i)| < \frac{2r}{\Delta} + 1, \quad i = 1,2,\dots,m, \tag{3.28}$$

where

$$\Delta = \min_{i=0,1,\dots,m} (t_{i+1} - t_i). \tag{3.29}$$

Put

$$c_0 = \frac{2r}{\Delta} + 1, \qquad \rho = \|u'\|_\infty. \tag{3.30}$$

By replacing u by $-u$ if necessary, we may assume that $\rho > c_0$ and

$$\rho = \sup_{t \in (t_j, t_{j+1}]} u'(t) \quad \text{for some } j \in \{0,1,\dots,m\}. \tag{3.31}$$

Thus we have

$$\rho = u'(\alpha) \quad \text{for some } \alpha \in (t_j, t_{j+1}] \tag{3.32}$$

or

$$\rho = u'(\alpha+) \quad \text{with} \quad \alpha = t_j. \tag{3.33}$$

By (3.28), there is $\beta \in (t_j, t_{j+1})$, $\beta \neq \alpha$, such that $u'(\beta) = c_0$ and $u'(t) \geq c_0$ for all t lying between α and β. Assume that (3.32) occurs. There are two cases to consider: $t_j < \beta < \alpha \leq t_{j+1}$ or $t_j < \alpha < \beta < t_{j+1}$.

Case 1. Let $t_j < \beta < \alpha \leq t_{j+1}$. Since $u'(t) > 1$ on $[\beta, \alpha]$, (3.27) gives

$$u''(t) \leq \omega(u'(t))(u'(t) + h(t)) \quad \text{for a.e. } t \in [\beta, \alpha], \tag{3.34}$$

and hence

$$\int_{c_0}^{\rho} \frac{ds}{\omega(s)} = \int_\beta^\alpha \frac{u''(t)}{\omega(u'(t))} dt \leq \int_\beta^\alpha u'(t) dt + \|h\|_1 \leq 2r + \|h\|_1. \tag{3.35}$$

On the other hand, by (3.25), there is $r^* > c_0$ such that

$$\int_{c_0}^{r^*} \frac{ds}{\omega(s)} > 2r + \|h\|_1, \tag{3.36}$$

which is possible only if $\rho < r^*$, that is, if (3.26) holds.

Case 2. Let $t_j < \alpha < \beta < t_{j+1}$. By (3.27), we get

$$\begin{gathered} -u''(t) \leq \omega(u'(t))(u'(t) + h(t)) \quad \text{for a.e. } t \in [\alpha, \beta], \\ \int_{c_0}^{\rho} \frac{ds}{\omega(s)} = -\int_\alpha^\beta \frac{u''(t)}{\omega(u'(t))} dt \leq 2r + \|h\|_1, \end{gathered} \tag{3.37}$$

so the inequality (3.26) follows.

If (3.33) occurs, a similar argument to that in Case 2 applies and gives (3.26) as well. □

Remark 3.3. Notice that the condition

$$\int_1^\infty \frac{ds}{\omega(s)} = \infty \tag{3.38}$$

in (3.25) can be weakened. In particular, the estimate (3.26) holds whenever $r^* \in (0,\infty)$ is such that

$$\int_{c_0}^{r^*} \frac{ds}{\omega(s)} > 2r + \|h\|_1. \tag{3.39}$$

4. Main results

The main existence result for problem (2.1), (2.2), and (2.3) is provided by the following theorem.

Theorem 4.1. *Assume that (2.8), (2.9), (2.10), and (2.11) hold. Further, let*

$$|f(t,x,y)| \le \omega(|y|)(|y|+h(t)) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \in [\sigma_1(t),\sigma_2(t)], |y| > 1, \tag{4.1}$$

where h and ω fulfil (3.24) and (3.25). Then the problem (2.1), (2.2), and (2.3) has a solution u satisfying (3.8).

Before proving this theorem, we prove the next key proposition where we restrict ourselves to the case that f is bounded by a Lebesgue integrable function.

Proposition 4.2. *Assume that (2.8), (2.9), (2.10), and (2.11) hold. Further, let $m \in \mathbb{L}[0,T]$ be such that*

$$|f(t,x,y)| \le m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in [\sigma_1(t),\sigma_2(t)] \times \mathbb{R}. \tag{4.2}$$

Then the problem (2.1), (2.2), and (2.3) has a solution u fulfilling (3.8).

Proof

Step 1. We construct a proper auxiliary problem.

Let Δ be given by (3.29). Put

$$c = \|m\|_1 + \frac{||\sigma_1||_\infty + ||\sigma_2||_\infty}{\Delta} + ||\sigma_1'||_\infty + ||\sigma_2'||_\infty, \tag{4.3}$$

and for $t \in [0,T]$ and $(x,y) \in \mathbb{R}^2$, define

$$\alpha(t,x) = \begin{cases} \sigma_1(t) & \text{if } x < \sigma_1(t), \\ x & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{if } x > \sigma_2(t), \end{cases} \tag{4.4}$$

$$\beta(y) = \begin{cases} y & \text{if } |y| \le c, \\ c \operatorname{sgn} y & \text{if } |y| > c. \end{cases} \tag{4.5}$$

For a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$, $\varepsilon \in [0,1]$, define functions

$$\omega_k(t,\varepsilon) = \sup_{y \in [\sigma_k'(t)-\varepsilon, \sigma_k'(t)+\varepsilon]} |f(t,\sigma_k(t),\sigma_k'(t)) - f(t,\sigma_k(t),y)|, \quad k = 1,2, \tag{4.6}$$

$$\begin{aligned} \widetilde{\mathrm{J}}_i(x) &= x + \mathrm{J}_i(\alpha(t_i,x)) - \alpha(t_i,x), \\ \widetilde{\mathrm{M}}_i(y) &= y + \mathrm{M}_i(\beta(y)) - \beta(y), \end{aligned} \qquad i = 1,2,\dots,m, \tag{4.7}$$

$$\widetilde{f}(t,x,y) = \begin{cases} f(t,\sigma_1(t),y) - \omega_1\left(t, \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\right) - \dfrac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \text{if } x < \sigma_1(t), \\ f(t,x,y) & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t,\sigma_2(t),y) + \omega_2\left(t, \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\right) + \dfrac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \text{if } x > \sigma_2(t). \end{cases} \tag{4.8}$$

We see that $\omega_k \in \mathrm{Car}([0,T] \times [0,1])$ are nonnegative and nondecreasing in the second variable and $\omega_k(0) = 0$ for $k = 1,2$. Consequently, $\widetilde{f} \in \mathrm{Car}([0,T] \times \mathbb{R}^2)$. Furthermore, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i \in \mathbb{C}(\mathbb{R})$, $i = 1,2,\dots,m$. The auxiliary problem is (3.5), (3.6), and

$$u(0) = u(T) = \alpha(0, u(0) + u'(0) - u'(T)). \tag{4.9}$$

Step 2. We prove that problem (3.5), (3.6), (4.9) *is solvable.*

Let

$$\begin{aligned} G(t,s) &= \begin{cases} \dfrac{t(s-T)}{T} & \text{if } 0 \le t \le s \le T, \\ \dfrac{s(t-T)}{T} & \text{if } 0 \le s < t \le T, \end{cases} \\ G_1(t,s) &= \begin{cases} -\dfrac{t}{T} & \text{if } 0 \le t \le s \le T, \\ \dfrac{T-t}{T} & \text{if } 0 \le s < t \le T. \end{cases} \end{aligned} \tag{4.10}$$

Define an operator $\widetilde{\mathrm{F}} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$\begin{aligned} (\widetilde{\mathrm{F}}u)(t) &= \alpha(0, u(0) + u'(0) - u'(T)) + \int_0^T G(t,s)\widetilde{f}(s,u(s),u'(s))\,ds \\ &\quad + \sum_{i=1}^m G_1(t,t_i)(\widetilde{\mathrm{J}}_i(u(t_i)) - u(t_i)) + \sum_{i=1}^m G(t,t_i)(\widetilde{\mathrm{M}}_i(u'(t_i)) - u'(t_i)). \end{aligned} \tag{4.11}$$

As in [13, Lemma 3.1], we get that $\widetilde{\mathrm{F}}$ is completely continuous and u is a solution of (3.5), (3.6), (4.9) if and only if u is a fixed point of $\widetilde{\mathrm{F}}$.

Denote by I the identity operator on $\mathbb{C}^1_{\mathrm{D}}[0,T]$ and consider the parameter system of operator equations

$$(\mathrm{I} - \lambda\widetilde{\mathrm{F}})u = 0, \quad \lambda \in [0,1]. \tag{4.12}$$

For $R \in (0,\infty)$, define $\mathrm{B}(R) = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u\|_{\mathrm{D}} < R\}$. By (4.2), (4.4), (4.5), (4.6), (4.7), (4.8), and (4.11), we can find $R_0 \in (0,\infty)$ such that $u \in \mathrm{B}(R_0)$ for each $\lambda \in [0,1]$

and each solution u of (4.12). So, for each $R \geq R_0$, the operator $\mathrm{I}-\lambda\widetilde{\mathrm{F}}$ is a homotopy on $\mathrm{cl}(\mathrm{B}(R)) \times [0,1]$ and its Leray-Schauder degree $\deg(\mathrm{I}-\lambda\widetilde{\mathrm{F}}, \mathrm{B}(R))$ has the same value for each $\lambda \in [0,1]$. Since $\deg(\mathrm{I}, \mathrm{B}(R)) = 1$, we conclude that

$$\deg\left(\mathrm{I}-\widetilde{\mathrm{F}}, \mathrm{B}(R)\right) = 1 \quad \text{for } R \in \left[R_0, \infty\right). \tag{4.13}$$

By (4.13), there is at least one fixed point of $\widetilde{\mathrm{F}}$ in $\mathrm{B}(R)$. Hence there exists a solution of the auxiliary problem (3.5), (3.6), (4.9).

Step 3. We find estimates for solutions of the auxiliary problem.

Let u be a solution of (3.5), (3.6), (4.9). We derive an estimate for $\|u\|_\infty$. By (4.7), (4.8), and (2.11), we obtain that $\tilde{f}$, $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i$, $i = 1, 2, \ldots, m$, satisfy (3.1), (3.2), and (3.3). Moreover, in view of (4.4), we have

$$\sigma_1(0) \leq \alpha\left(0, u(0) + u'(0) - u'(T)\right) \leq \sigma_2(0). \tag{4.14}$$

Thus u satisfies (3.8) by Lemma 3.1.

We find an estimate for $\|u'\|_\infty$. By the mean value theorem and (3.8), there are $\xi_i \in (t_i, t_{i+1})$ such that

$$\left|u'(\xi_i)\right| \leq \frac{\|\sigma_1\|_\infty + \|\sigma_2\|_\infty}{\Delta}, \quad i = 1, 2, \ldots, m. \tag{4.15}$$

Moreover, by (3.8) and (4.8), u satisfies (2.1) for a.e. $t \in [0, T]$. Therefore, integrating (2.1) and using (4.2), (4.3), and (4.15), we obtain

$$\|u'\|_\infty \leq \left|u'(\xi_i)\right| + \|m\|_1 < c. \tag{4.16}$$

Hence, by (4.7) and (4.9), we see that u fulfils (2.2) and $u(0) = u(T)$ (i.e., the first condition from (2.3) is satisfied).

Step 4. We verify that u fulfils the second condition in (2.3).

We must prove that $u'(0) = u'(T)$. By (4.9), this is equivalent to

$$\sigma_1(0) \leq u(0) + u'(0) - u'(T) \leq \sigma_2(0). \tag{4.17}$$

Suppose, on the contrary, that (4.17) is not satisfied. Let, for example,

$$u(0) + u'(0) - u'(T) > \sigma_2(0). \tag{4.18}$$

Then, by (4.4), we have $\alpha(0, u(0) + u'(0) - u'(T)) = \sigma_2(0)$. Together with (2.7) and (4.9), this yields

$$u(0) = u(T) = \sigma_2(0) = \sigma_2(T). \tag{4.19}$$

Inserting (4.19) into (4.18), we get

$$u'(0) > u'(T). \tag{4.20}$$

On the other hand, (4.19) together with (3.8) and (4.20) implies that

$$\sigma_2'(0) \geq u'(0) > u'(T) \geq \sigma_2'(T), \tag{4.21}$$

a contradiction to (2.7).

If we assume that $u(0) + u'(0) - u'(T) < \sigma_1(0)$, we can argue similarly and again derive a contradiction to (2.7).

So, we have proved that (4.17) is valid, which means that $u'(0) = u'(T)$. Consequently, u is a solution of (2.1), (2.2), and (2.3) satisfying (3.8). □

Proof of Theorem 4.1. Put

$$c = r^* + ||\sigma_1'||_\infty + ||\sigma_2'||_\infty, \tag{4.22}$$

where $r^* \in (0, \infty)$ is given by Lemma 3.2 for $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. For a.e. $t \in [0, T]$ and all $(x, y) \in \mathbb{R}^2$, define a function

$$g(t,x,y) = \begin{cases} f(t,x,y) & \text{if } |y| \leq c, \\ \left(2 - \dfrac{|y|}{c}\right) f(t,x,y) & \text{if } c < |y| < 2c, \\ 0 & \text{if } |y| \geq 2c. \end{cases} \tag{4.23}$$

Then σ_1 and σ_2 are, respectively, lower and upper functions of the auxiliary problem (2.2), (2.3), and

$$u'' = g(t,u,u'). \tag{4.24}$$

There exists a function $m^* \in \mathbb{L}[0, T]$ such that

$$|f(t,x,y)| \leq m^*(t) \tag{4.25}$$

for a.e. $t \in [0, T]$ and all $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times [-2c, 2c]$. Hence

$$|g(t,x,y)| \leq m^*(t) \quad \text{for a.e. } t \in [0,T], \text{ all } (x,y) \in [\sigma_1(t),\sigma_2(t)] \times \mathbb{R}. \tag{4.26}$$

Since $g \in \operatorname{Car}([0, T] \times \mathbb{R}^2)$, we can apply Proposition 4.2 to problem (4.24), (2.2), (2.3) and get that this problem has a solution u fulfilling (3.8). Hence $\|u\|_\infty \leq r$. Moreover, by (4.1), u satisfies (3.27). Therefore, by Lemma 3.2, $\|u'\|_\infty \leq r^* \leq c$. This implies that u is a solution of (2.1), (2.2), and (2.3). □

The next simple existence criterion, which follows from Theorem 4.1 and Remark 2.3, extends both [5, Theorem 4] and [13, Corollary 3.4].

Corollary 4.3. *Let (2.8) hold. Furthermore, assume that*

(i) $\mathrm{M}_i(0) = 0$ *and* $y\mathrm{M}_i(y) \geq 0$ *for* $y \in \mathbb{R}$ *and* $i = 1,2,\dots,m$;
(ii) *there are* $r_1, r_2 \in \mathbb{R}$ *such that* $r_1 < r_2$, $f(t,r_1,0) \leq 0 \leq f(t,r_2,0)$ *for a.e.* $t \in [0,T]$, $\mathrm{J}_i(r_1) = r_1$, $\mathrm{J}_i(x) \in [r_1,r_2]$ *if* $x \in [r_1,r_2]$, $\mathrm{J}_i(r_2) = r_2$, $i = 1,2,\dots,m$;
(iii) *there are* h *and* ω *satisfying (3.24) and (3.25) with* $\sigma_1(t) \equiv r_1$ *and* $\sigma_2(t) \equiv r_2$ *and such that (4.1) holds.*

Then the problem (2.1), (2.2), and (2.3) has a solution u *fulfilling* $r_1 \leq u \leq r_2$ *on* $[0,T]$.

Remark 4.4. Let $\sigma_1 < \sigma_2$ on $[0,T]$ and $\sigma_1(t_i+) < \sigma_2(t_i+)$ for $i = 1,2,\dots,m$. Having G and G_1 from the proof of Proposition 4.2, we define an operator $\mathrm{F} : \mathbb{C}^1_{\mathrm{D}}[0,T] \mapsto \mathbb{C}^1_{\mathrm{D}}[0,T]$ by

$$\begin{aligned}(\mathrm{F}u)(t) = {} & u(0) + u'(0) - u'(T) + \int_0^T G(t,s) f(s,u(s),u'(s))\,ds \\ & + \sum_{i=1}^m G_1(t,t_i)\big(\mathrm{J}_i(u(t_i)) - u(t_i)\big) + \sum_{i=1}^m G(t,t_i)\big(\mathrm{M}_i(u'(t_i)) - u'(t_i)\big).\end{aligned} \tag{4.27}$$

Let r^* be given by Lemma 3.2 for $r = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$. Define a set

$$\begin{aligned}\Omega = \{u \in \mathbb{C}^1_{\mathrm{D}}[0,T] : \|u'\|_\infty < r^*,\ & \sigma_1(t) < u(t) < \sigma_2(t) \text{ for } t \in [0,T], \\ & \sigma_1(t_i+) < u(t_i+) < \sigma_2(t_i+) \text{ for } i = 1,2,\dots,m\}.\end{aligned} \tag{4.28}$$

As in [13, Lemma 3.1], we get that F is completely continuous and u is a solution of (2.1), (2.2), and (2.3) if and only if u is a fixed point of F. The proofs of Theorem 4.1 and Proposition 4.2 yield the following result about the Leray-Schauder degree of the operator $\mathrm{I} - \mathrm{F}$ with respect to Ω.

Corollary 4.5. *Let* $\sigma_1 < \sigma_2$ *on* $[0,T]$ *and* $\sigma_1(t_i+) < \sigma_2(t_i+)$ *for* $i = 1,2,\dots,m$, *and let the assumptions of Theorem 4.1 be satisfied. Further, assume that* F *and* Ω *are defined by (4.27) and (4.28), respectively. If* $\mathrm{F}u \neq u$ *for each* $u \in \partial\Omega$, *then*

$$\deg(\mathrm{I} - \mathrm{F}, \Omega) = 1. \tag{4.29}$$

Proof. Consider c and g from the proof of Theorem 4.1 and define $\widetilde{\mathrm{J}}_i$, $\widetilde{\mathrm{M}}_i$, $i = 1,2,\dots,m$, and $\tilde{f}$ by (4.7) and (4.8), where we put g instead of f. Define $\widetilde{\mathrm{F}}$ by (4.11) and put $\Omega_1 = \{u \in \Omega : \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\}$. Suppose that $\mathrm{F}u \neq u$ for each $u \in \partial\Omega$. We have

$$\mathrm{F} = \widetilde{\mathrm{F}} \quad \text{on } \mathrm{cl}(\Omega_1). \tag{4.30}$$

and

$$(\mathrm{F}u = u,\ u \in \Omega) \Longrightarrow u \in \Omega_1. \tag{4.31}$$

By the proof of Proposition 4.2, we have that each fixed point u of $\widetilde{\mathrm{F}}$ satisfies (3.8) and consequently $\|u\|_\infty \le r$. Hence, by (4.1), (4.8), and (4.23),

$$|u''(t)| = |g(t,u(t),u'(t))| \le \omega(|u'(t)|)(|u'(t)| + h(t)) \tag{4.32}$$

for a.e. $t \in [0,T]$ and for $|u'(t)| > 1$. Therefore Lemma 3.2 implies that $\|u'\|_\infty \le r^*$. So, by (2.3), $u \in \mathrm{cl}(\Omega_1)$. Now, choose R in (4.13) such that $\mathrm{B}(R) \supset \Omega$. Then, by the excision property of the degree, we get

$$\deg(\mathrm{I}-\mathrm{F},\Omega) = \deg(\mathrm{I}-\mathrm{F},\Omega_1) = \deg(\mathrm{I}-\widetilde{\mathrm{F}},\Omega_1) = \deg(\mathrm{I}-\widetilde{\mathrm{F}},\mathrm{B}(R)) = 1, \tag{4.33}$$

wherefrom, taking into account (4.30), we obtain (4.29). □

Remark 4.6. Following the ideas from [12, 13], the evaluation of $\deg(\mathrm{I}-\mathrm{F},\Omega)$ enables us to prove the existence of solutions to problem (2.1), (2.2), and (2.3) also for nonordered lower/upper functions. This will be included in our next paper [11].

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NONUNIQUENESS THEOREM FOR A SINGULAR CAUCHY-NICOLETTI PROBLEM

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The problem of nonuniqueness for a singular Cauchy-Nicoletti boundary value problem is studied. The general nonuniqueness theorem ensuring the existence of two different solutions is given such that the estimating expressions are nonlinear, in general, and depend on suitable Lyapunov functions. The applicability of results is illustrated by several examples.

1. Introduction

The nonuniqueness of a regular or singular Cauchy problem for ordinary differential equations is studied in several papers such as [3, 4, 5, 13, 14, 15, 16, 17]. Most of these results can also be found in the monograph [1]. The uniqueness of solutions of Cauchy initial value problem for ordinary differential equations with singularity is investigated in [7, 8, 9, 12]. The topological structure of solution sets to a large class of boundary value problems for ordinary differential equations is studied in [2]. First results on the nonuniqueness for a singular Cauchy-Nicoletti boundary value problem are given in [10, 11, 12] by Kiguradze, where sufficient conditions for the nonuniqueness are written in the form of one-sided inequalities for the components in the right-hand side $f(t,x_1,\dots,x_n)$ of the corresponding equation. An expression for the estimation of the jth component $f_j(t,x_1,\dots,x_n)$ of f depends on t and x_j and is linear in $|x_j|$.

In [6], we studied the nonuniqueness for a singular Cauchy problem. Our criteria involve vector Lyapunov functions and the estimations need not be linear. The present paper deals with the nonuniqueness of the singular Cauchy-Nicoletti problem and extends the results of [6] to this more general problem.

Supposing $-\infty \le a < A \le \infty$, $b > 0$, we will use the following notations throughout the paper: $\mathbb{R}^k$ and $\mathbb{R}^+$ denote k-dimensional real Euclidean space and the interval $[0,\infty)$, respectively. $|\cdot|$ is used for the notation of Hölder's 1-norm (the sum of the absolute values of components). $x = (x_1,\dots,x_n)$ denotes a variable vector from $\mathbb{R}^n$ with components $x_1,\dots,x_n$, while $x_0 = (x_{01},\dots,x_{0n})$ stands for a fixed vector from $\mathbb{R}^n$ with components $x_{01},\dots,x_{0n}$. N is equal to the set $\{1,\dots,n\}$. l denotes a fixed number from the set $\{1,\dots,n\}$.

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$i_1, i_2, \dots, i_l$ are fixed integers such that $1 \le i_1 < i_2 < \cdots < i_l \le n$. I is set to be equal to $\{i_1, \dots, i_l\}$. $\Pr x$ denotes a projection of x such that $\Pr x = (x_{i_1}, \dots, x_{i_l})$, while $\Pr^* x$ denotes a complementary projection to $\Pr x$. Clearly, $\Pr^* x = (x_{j_1}, \dots, x_{j_{n-l}})$, where $1 \le j_1 < \cdots < j_{n-l} \le n$, $\{i_1, \dots, i_l\} \cap \{j_1, \dots, j_{n-l}\} = \emptyset$. $R^k_{\alpha,\beta;b}(x_0)$ and $\tilde{R}^k_{a,A}$ are used for the notation of the set $\{(t,x) \in \mathbb{R}^{k+1} : \alpha < t < \beta, |x - x_0| \le b\}$ and the set $\{(t,x) \in \mathbb{R}^{k+1} : a < t < A, x \in \mathbb{R}^k\}$, respectively. The symbol $\hat{R}^n_{a,A}$ will be used for the set $\{(t,x) \in \mathbb{R}^{n+1} : a \le t \le A, x \in \mathbb{R}^n\}$. $\Delta(\alpha,\beta)$ denotes the interval $(\min(\alpha,\beta), \max(\alpha,\beta))$.

The notation $C[\Gamma, \Omega]$ is used for the notation of the class of all continuous mappings $\Gamma \to \Omega$. $AC[[a,A], \mathbb{R}^k]$ and $\widetilde{AC}[[a,A], \mathbb{R}^k]$ denote the class of all absolutely continuous mappings $[a,A] \to \mathbb{R}^k$ and the class of all mappings from $C[[a,A], \mathbb{R}^k]$ which are absolutely continuous on any interval $[\alpha,\beta]$, where $a < \alpha < \beta < A$, respectively. The class of all Lebesgue-integrable mappings $[a,A] \to \mathbb{R}^+$ is denoted by $L[[a,A], \mathbb{R}^+]$. $\mathcal{L}_\tau[\hat{R}^n_{a,A}, \mathbb{R}^{+k}]$ stands for the class of all functions $V(t,x) : \hat{R}^n_{a,A} \to \mathbb{R}^{+k}$ with the following property: $V(t,\cdot)$ is uniformly continuous, and if $a < \alpha < \beta < A$, $\tau \notin [\alpha,\beta]$, then $V(t,x(t))$ is absolutely continuous on $[\alpha,\beta]$ for any absolutely continuous function $x : [\alpha,\beta] \to \mathbb{R}^n$. $K_{\sigma_1,\dots,\sigma_p}[\hat{R}^k_{a,A}, \mathbb{R}^m]$ denotes the class of all mappings $\hat{R}^k_{a,A} \to \mathbb{R}^m$ which satisfy Carathéodory conditions on $R^k_{\alpha,\beta;\varrho}(0)$ for any $\alpha,\beta, a \le \alpha < \beta \le A$, $\sigma_j \notin [\alpha,\beta] (j = 1,\dots,p)$, $\varrho \in (0,\infty)$, $\sigma_1,\dots,\sigma_p$ being numbers from $[a,A]$. $N_0(a,A;\tau_1,\dots,\tau_n)$ is used for the notation of the class $\{\Lambda = (\lambda_{ij}(t))^n_{i,j=1} : \lambda_{ij} \in L[[a,A], \mathbb{R}^+]\}$ such that the system of differential inequalities $|x_i'(t)| \le \sum_{j=1}^n \lambda_{ij}(t)|x_j(t)|$, $t \in [a,A]$, $i \in N$, possesses no nontrivial solution $x(t) = (x_1(t),\dots,x_n(t)) \in AC[[a,A], \mathbb{R}^n]$ satisfying $x_i(\tau_i) = 0$ $(i = 1,\dots,n)$.

The fundamental role in the proof of our main theorem will be played by the following theorem by Kiguradze, which is adapted from [12] (see also [10]) in a simplified form.

Kiguradze Theorem . *Let $a \le \tau_i \le A$, $\hat{x}_{0i} \in \mathbb{R}$ for $i = 1,\dots,n$. Suppose that $f \in K_{\sigma_1,\dots,\sigma_p}[\hat{R}^n_{a,A}, \mathbb{R}^n]$. Assume that the components f_i of f satisfy*

$$f_i(t,x)\operatorname{sgn}\left[(t-\tau_i)(x_i - \hat{x}_{0i})\right] \le \sum_{j=1}^n \lambda_{ij}(t)|x_j| + \mu_i(t) \quad (i = 1,\dots,n) \tag{1.1}$$

for $(t,x) = (t,x_1,\dots,x_n) \in \tilde{R}^n_{a,A}$, where $\hat{x}_{0i} = 0$ if $\tau_i \in \{\sigma_1,\dots,\sigma_p\}$. Suppose that $\Lambda(t) = (\lambda_{ij}(t))^n_{i,j=1} \in N_0(a,A;\tau_1,\dots,\tau_n)$, $\mu_i \in L[[a,A], \mathbb{R}^+]$. Then the Cauchy-Nicoletti problem

$$x' = f(t,x), \qquad x_i(\tau_i) = 0 \quad (i = 1,\dots,n) \tag{1.2}$$

has at least one solution $x(t) = (x_1(t),\dots,x_n(t)) \in AC[[a,A], \mathbb{R}^n]$.

2. Results

Consider a Cauchy-Nicoletti boundary value problem

$$x' = f(t,x), \qquad x_i(t_i) = x_{0i} \quad (i = 1,\dots,n), \tag{2.1}$$

where $f(t,x) = (f_1(t,x_1,\dots,x_n),\dots,f_n(t,x_1,\dots,x_n))$, $f \in K_{\sigma_1,\dots,\sigma_p}[\hat{R}^n_{a,A}, \mathbb{R}^n]$, $x_{0i} \in \mathbb{R}$, and $t_i \in [a,A]$ $(i \in N)$.

Theorem 2.1. *Suppose that there are numbers $c_i \in \mathbb{R}$ $(i \in N)$, $B_i \in [a,A] \setminus \{t_i, \sigma_1, \dots, \sigma_p\}$ $(i \in I)$, a matrix function $\Lambda = (\lambda_{ij})_{i,j=1}^n \in N_0(a,A;\tau_1,\dots,\tau_n)$ and functions $\mu_i \in L[[a,A],\mathbb{R}^+]$ $(i \in N)$ such that $c_i = x_{0i}$ for $i \in N \setminus I$ and*

$$f_i(t,x)[\operatorname{sgn}(t-\tau_i)(x_i - c_i)] \le \sum_{j=1}^n \lambda_{ij}(t)|x_j| + \mu_i(t) \quad (i \in N) \tag{2.2}$$

holds for $(t,x) = (t,x_1,\dots,x_n) \in \tilde{\mathbb{R}}_{a,A}^n$, where $\tau_i = t_i$ or $\tau_i = B_i$ whenever $i \in N \setminus I$ or $i \in I$, respectively.

Assume that

(i) *there exist vector functions $g_i = (g_{i1},\dots,g_{ik_i}) \in K_{a,A,t_i,B_i}[\hat{\mathbb{R}}_{a,A}^{k_i}, \mathbb{R}^{k_i}]$ $(i \in I)$ such that $\operatorname{sgn}(t-t_i) g_{ij}(t,u_1,\dots,u_{j-1},\cdot,u_j,\dots,u_{k_i})$ is nondecreasing for $j = 1,\dots,k_i$ and there is a solution $\varphi_i(t) = (\varphi_{i1}(t),\dots,\varphi_{ik_i}(t))$ of*

$$u_i' = g_i(t,u_1,\dots,u_{k_i}) \tag{2.3}$$

satisfying

$$\varphi_i(t) > 0 \quad \text{for } t \in \Delta(t_i,B_i), \qquad \lim_{t \to t_i} \varphi_i(t) = 0, \qquad \liminf_{t \to B_i} \varphi_i(t) > 0 \tag{2.4}$$

for $i \in I$;

(ii) *$V_i(t,x) = (V_{i1}(t,x),\dots,V_{ik_i}(t,x)) \in \mathcal{L}_{t_i}[\hat{R}_{a,A}^n, {\mathbb{R}^+}^{k_i}]$ $(i \in I)$ are such that there exists $y_0 \in \mathbb{R}^l$ with the property*

$$\sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \Pr y = y_0\} < \liminf_{t \to B_i} \varphi_{ij}(t) \quad (j = 1,\dots,k_i) \tag{2.5}$$

$$|V_i(t,x)| \ge \Psi_i(|x_i - z_i(t)|) \quad \text{for } t \in \Delta(t_i,B_i), \tag{2.6}$$

where $\Psi_i \in C[\mathbb{R}^+,\mathbb{R}^+]$, $z_i \in C[(a,A),\mathbb{R}]$ are such that

$$\Psi_i(0) = 0, \qquad \Psi_i(u) > 0 \quad \text{for } u > 0, \qquad \lim_{t \to t_i} z_i(t) = x_{0i} \tag{2.7}$$

for $i \in I$;

(iii) *there exist positive functions $\varepsilon_{ik} \in C[(a,A),\mathbb{R}^+]$ $(i \in I; k = 1,\dots,k_i)$ such that*

$$\begin{aligned} &\operatorname{sgn}(B_i - t_i) V_{ij}'(t,x(t)) \\ &\quad \ge \operatorname{sgn}(B_i - t_i) g_{ij}(t,\varphi_{i1}(t),\dots,\varphi_{i,j-1}(t),V_{ij}(t,x(t)),\varphi_{i,j+1}(t),\dots,\varphi_{ik_i}(t)) \end{aligned} \tag{2.8}$$

holds for $i \in I$, $j = 1,\dots,k_i$, and for any solution $x(t)$ of (2.1) a.e. on any interval $(\alpha_{i1},\alpha_{i2}) \subseteq \Delta(t_i,B_i)$ for which

$$\begin{aligned} V_{ik}(t,x(t)) &< \varphi_{ik}(t) + \varepsilon_{ik}(t) \quad \text{on } (\alpha_{i1},\alpha_{i2}) \ (k = 1,\dots,k_i), \\ V_{ij}(t,x(t)) &> \varphi_{ij}(t) \quad \text{on } (\alpha_{i1},\alpha_{i2}). \end{aligned} \tag{2.9}$$

Then the Cauchy-Nicoletti boundary value problem (2.1) has at least two different solutions on $[a,A]$, either of which satisfies $V_i(t,x(t)) \le \varphi_i(t)$ for $t \in \Delta(t_i,B_i)$ and $i \in I$.

Proof. Without loss of generality, it can be assumed that $I = \{1,\dots,l\}$,

$$\operatorname{Pr} x = (x_1,\dots,x_l), \qquad \operatorname{Pr}^* x = (x_{l+1},\dots,x_n). \tag{2.10}$$

For any $i \in I$ and $j \in \{1,\dots,k_i\}$, denote

$$L_{ij} = \liminf_{t\to B_i} \varphi_{ij}(t), \qquad S_{ij} = \sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \operatorname{Pr} y = y_0\}. \tag{2.11}$$

According to (2.5) and to the uniform continuity of $V_{ij}(B_i,\cdot)$, we have a relation

$$\begin{aligned} V_{ij}(B_i,y^*) &\le V_{ij}(B_i,y) + V_{ij}(B_i,y^*) - V_{ij}(B_i,y) \\ &\le \frac{1}{2}(L_{ij}+S_{ij}) + \frac{1}{4}(L_{ij}-S_{ij}) = \frac{3}{4}L_{ij} + \frac{1}{4}S_{ij} < L_{ij} \end{aligned} \tag{2.12}$$

for $y \in \mathbb{R}^n$, $\operatorname{Pr} y = y_0$, and for $y^* \in \mathbb{R}^n$ sufficiently close to y. Hence it can be supposed without loss of generality that $y_0 \ne \operatorname{Pr} x_0$.

Further, the uniform continuity of $V_{ij}(B_i,\cdot)$ implies that the inequality

$$\sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \operatorname{Pr} y = y_0 - \lambda(y_0 - \operatorname{Pr} x_0)\} < \liminf_{t\to B_i} \varphi_{ij}(t) \quad (i \in I; j = 1,\dots,k_i) \tag{2.13}$$

holds provided that $\lambda > 0$ is sufficiently small. Therefore, we can choose $\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^l$, $\tilde{x}_1 \ne \tilde{x}_2$, such that

$$\max_{k=1,2}\left[\sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \operatorname{Pr} y = \tilde{x}_k\}\right] < \liminf_{t\to B_i} \varphi_{ij}(t) \quad (i = 1,\dots,l; j = 1,\dots,k_i). \tag{2.14}$$

Choose $\tilde{\xi} \in \{\tilde{x}_1, \tilde{x}_2\}$ arbitrary. Put $\xi = x_0 - (\tilde{\xi}, \operatorname{Pr}^* x_0)$, $X = x - x_0 + \xi$, and $f^*(t,X) = f(t, x_0 + X - \xi)$ for $(t,X) = (t,X_1,\dots,X_n) \in \hat{R}^n_{a,A}$.

Clearly $f^* \in K_{\sigma_1,\dots,\sigma_p}[\hat{R}^n_{a,A}, \mathbb{R}^n]$. By using (2.2), we obtain

$$\begin{aligned} f_i^*(t,X)\operatorname{sgn}[(t-\tau_i)(X_i+\tilde{\xi}_i-c_i)] &\le \sum_{j=1}^{l} \lambda_{ij}(t)|X_j+\tilde{\xi}_j| + \sum_{j=l+1}^{n} \lambda_{ij}(t)|X_j+x_{0j}| + \mu_i(t) \\ &\le \sum_{j=1}^{n} \lambda_{ij}(t)|X_j| + \tilde{\mu}_i(t) \end{aligned} \tag{2.15}$$

for $(t,X) \in \tilde{R}^n_{a,A}$, $i = 1,\dots,l$, and

$$\begin{aligned} f_i^*(t,X)\operatorname{sgn}[(t-\tau_i)X_i] &\le \sum_{j=1}^{l} \lambda_{ij}(t)|X_j+\tilde{\xi}_j| + \sum_{j=l+1}^{n} \lambda_{ij}(t)|X_j+x_{0j}| + \mu_i(t) \\ &\le \sum_{j=1}^{n} \lambda_{ij}(t)|X_j| + \tilde{\mu}_i(t) \end{aligned} \tag{2.16}$$

for $(t,X) \in \tilde{R}^n_{a,A}$, $i = l+1,\dots,n$, where

$$\tilde{\mu}_i(t) = \sum_{j=1}^{l} \lambda_{ij}(t) \,|\, \tilde{\xi}_j \,| + \sum_{j=l+1}^{n} \lambda_{ij}(t) \,|\, x_{0j} \,| + \mu_i(t) \tag{2.17}$$

for $i = 1,2,\dots,n$. As $\tilde{\mu}_i \in L[[a,A],\mathbb{R}^+]$ holds, Kiguradze theorem implies that the boundary value problem

$$X' = f^*(t,X), \qquad X_i(\tau_i) = 0 \quad (i = 1,\dots,n) \tag{2.18}$$

has at least one solution $X(t) \in AC[[a,A],\mathbb{R}^n]$. Hence $x(t) = X(t) + x_0 - \xi$ is a solution of

$$\begin{gathered} x' = f(t,x), \qquad x_i(\tau_i) = \tilde{\xi}_i \quad (i = 1,\dots,l), \\ x_i(\tau_i) = x_{0i} \quad (i = l+1,\dots,n). \end{gathered} \tag{2.19}$$

Now we will prove that $\lim_{t\to t_i} x_i(t) = x_{0i}$ for $i = 1,\dots,l$. Put $m_i(t) = V_i(t,x(t))$, $m_{ij}(t) = V_{ij}(t,x(t))$ for $i = 1,\dots,l$ and $j = 1,\dots,k_i$. In view of (2.14), the inequality

$$m_i(t) < \varphi_i(t) \tag{2.20}$$

holds for $t \in (a,A)$ sufficiently close to B_i. Suppose for definiteness that $t_i < B_i$, that is, $\Delta(t_i,B_i) = (t_i,B_i)$ for some $i \in \{1,\dots,l\}$. We will show that $m_i(t) \le \varphi_i(t)$ for $t \in (t_i,B_i)$. Assume on the contrary that there is a $\tau \in (t_i,B_i)$ such that $m_i(\tau) \le \varphi_i(\tau)$ is not true. Since $x(t)$ is continuous and (2.20) holds for $t \in (a,A)$ sufficiently close to B_i, there exist $j \in \{1,\dots,k_i\}$ and an interval $J_i = (\tau_{i1},\tau_{i2})$ such that $\tau < \tau_{i1} < \tau_{i2} < B_i$,

$$\begin{gathered} m_{ij}(\tau_{i2}) = \varphi_{ij}(\tau_{i2}), \\ \varphi_{ij}(s) < m_{ij}(s) < \varphi_{ij}(s) + \varepsilon_{ij}(s) \quad \text{for } s \in J_i, \\ m_{ik}(s) < \varphi_{ik}(s) + \varepsilon_{ik}(s) \quad \text{for } s \in J_i,\ k = 1,\dots,k_i. \end{gathered} \tag{2.21}$$

Using (2.8), we get

$$m'_{ij}(s) \ge g_{ij}\big(s,\varphi_{i1}(s),\dots,\varphi_{i,j-1}(s),m_{ij}(s),\varphi_{i,j+1}(s),\dots,\varphi_{ik_i}(s)\big) \tag{2.22}$$

a.e. on J_i. As $g_{ij}(t,u_1,\dots,u_{j-1},\cdot,u_{j+1},\dots,u_n(s))$ is nondecreasing, we have

$$m'_{ij}(s) \ge g_{ij}(s,\varphi_{i1}(s),\dots,\varphi_{ik_i}(s)) = \varphi'_{ij}(s) \tag{2.23}$$

a.e. on J_i. Therefore, the function $m_{ij}(t) - \varphi_{ij}(t)$ is nondecreasing on J_i, which is a contradiction to $m_{ij}(\tau_{i2}) = \varphi_{ij}(\tau_{i2})$. Thus

$$0 \le m_i(t) \le \varphi_i(t) \quad \text{for } t \in (t_i,B_i). \tag{2.24}$$

Now the condition $\lim_{t\to t_i+} \varphi_i(t) = 0$ implies $\lim_{t\to t_i+} m_i(t) = 0$. With respect to the continuity of $x_i(t)$ on $[a,A]$, we have $x_i(t_i) = \lim_{t\to t_i} x_i(t) = x_{0i}$. The inequality (2.24) implies $V_i(t,x(t)) \le \varphi_i(t)$ for $t \in \Delta(t_i,B_i)$. $\square$

Corollary 2.2. *Let $c_i \in \mathbb{R}$ $(i \in N)$, $B_i \in [a,A] \setminus \{t_i, \sigma_1, \dots, \sigma_p\}$ $(i \in I)$, a matrix function $\Lambda = (\lambda_{ij})_{i,j=1}^n \in N_0(a,A;\tau_1,\dots,\tau_n)$, and functions $\mu_i \in L[[a,A],\mathbb{R}^+]$ $(i \in N)$ be such that $c_i = x_{0i}$ for $i \in N \setminus I$ and condition (2.2) is fulfilled, where $\tau_i = t_i$ or $\tau_i = B_i$ whenever $i \in N \setminus I$ or $i \in I$, respectively.*

Assume that

(i) *there exist functions $g_i \in K_{a,A,t_i,B_i}[\hat{R}^1_{a,A},\mathbb{R}]$ $(i \in I)$ such that $\operatorname{sgn}(t - t_i) g_i(t,\cdot)$ are nondecreasing and there are solutions $\varphi_i(t)$ of*

$$u_i' = g_i(t,u_i) \tag{2.25}$$

satisfying (2.4);

(ii) *there are $z_i \in \widetilde{AC}[[a,A],\mathbb{R}]$ and $\varepsilon = (\varepsilon_{i_1},\dots,\varepsilon_{i_l}) \in C[(a,A),\mathbb{R}^{+l}]$ such that $z_i(t_i) = x_{0i}$ $(i \in I)$ and the estimation*

$$\operatorname{sgn}(B_i - t_i)\operatorname{sgn}\big(x_i - z_i(t)\big)\big(f_i(t,x) - z_i'(t)\big) \ge \operatorname{sgn}(B_i - t_i) g_i\big(t, |x_i - z_i(t)|\big)(i \in I) \tag{2.26}$$

is fulfilled on $\hat{\Omega} = \{(t,x) : \varphi_i(t) < |x_i - z_i(t)| < \varphi_i(t) + \varepsilon_i(t), t \in \Delta(t_i,B_i)\}$ for almost all $t \in \Delta(t_i,B_i)$. Then the Cauchy-Nicoletti boundary value problem (2.1) has at least two different solutions on $[a,A]$, either of which satisfies $|x_i(t) - z_i(t)| \le \varphi_i(t)$ for $t \in \Delta(t_i,B_i)$ and $i \in I$.

Proof. Without loss of generality, it can be supposed that $I = \{1,\dots,l\}$ and $\operatorname{Pr} x = (x_1,\dots, x_l)$. Put $k_i = 1$ and $V_i(t,x(t)) = V_{i1}(t,x) = |x_i - z_i(t)|$ for $i = 1,\dots,l$. Then

$$\begin{aligned}
\operatorname{sgn}(B_i - t_i) V_{i1}'(t,x(t)) &\ge \operatorname{sgn}(B_i - t_i)\big(f_i(t,x(t)) - z_i'(t)\big)\operatorname{sgn}\big(x_i(t) - z_i(t)\big)\\
&\ge \operatorname{sgn}(B_i - t_i) g_i\big(t, |x_i(t) - z_i(t)|\big)\\
&= \operatorname{sgn}(B_i - t_i) g_i\big(t, V_{i1}(t,x(t))\big)
\end{aligned} \tag{2.27}$$

holds for any solution $x(t)$ of (2.1) a. e. on any interval $(\alpha_{i1},\alpha_{i2}) \subseteq \Delta(t_i,B_i)$ for which $\varphi_i(t) < V_i(t,x(t)) < \varphi_i(t) + \varepsilon_i(t)$ on $(\alpha_{i1},\alpha_{i2})$. The assumptions of Theorem 2.1 are satisfied. □

Example 2.3. Let $f_1,\dots,f_n \in K_0[\hat{R}^n_{0,1},\mathbb{R}]$ be such that

$$\begin{aligned}
f_1(t,x_1,\dots,x_n)\operatorname{sgn} x_1 &\ge \delta(t)|x_1|^\gamma,\\
-f_j(t,x_1,\dots,x_n)\operatorname{sgn} x_j &\le \sum_{k=1}^{j} \lambda_{jk}(t)|x_k| + \mu_j(t) \quad (j = 2,\dots,n)
\end{aligned} \tag{2.28}$$

for $(t,x_1,\dots,x_n) \in \tilde{R}^n_{0,1}$, where $\gamma \in (0,1)$ and $\delta,\lambda_{jk},\mu_j \in L[[0,1],\mathbb{R}^+]$, δ being a positive function. Consider the boundary value problem

$$\begin{aligned}
x_1' &= f_1(t,x_1,\dots,x_n), \quad x_1(0) = 0,\\
x_2' &= f_2(t,x_1,\dots,x_n), \quad x_2(1) = 0,\\
&\vdots\\
x_n' &= f_n(t,x_1,\dots,x_n), \quad x_n(1) = 0.
\end{aligned} \tag{2.29}$$

Put $t_1 = 0, t_2 = t_3 = \cdots = t_n = 1$,

$$g_1(t,u) = \begin{cases} \delta(t)u^\gamma & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases} \tag{2.30}$$

$\lambda_{1k}(t) \equiv 0$ $(k = 1,\dots,n)$, $\lambda_{jk}(t) \equiv 0$ $(j = 2,\dots,n; k = j+1,\dots,n)$, and $\mu_1(t) \equiv 0$. Let $B_1 = 1$. Then $\tau_1 = \tau_2 = \cdots = \tau_n = 1$,

$$\begin{aligned} f_1(t,x_1,\dots,x_n)\operatorname{sgn}[(t-B_1)x_1] &\leq 0, \\ f_j(t,x_1,\dots,x_n)\operatorname{sgn}[(t-1)x_j] &\leq \sum_{k=1}^{n} \lambda_{jk}(t)|x_k| + \mu_j(t) \quad (j = 2,\dots,n), \end{aligned} \tag{2.31}$$

and the equation $u_1' = g_1(t,u)$ has a positive solution

$$\varphi_1(t) = \left[(1-\gamma)\int_0^t \delta(s)\,ds\right]^{1/(1-\gamma)} \tag{2.32}$$

in $(0,1]$ such that $\lim_{t\to 0}\varphi_1(t) = 0$. The assumptions of Corollary 2.2 are fulfilled with $I = \{1\}$, $c_1 = 0$, and $z(t) = z_1(t) \equiv 0$. Therefore, the considered boundary value problem has at least two different solutions on $[a,A]$. Moreover, the first component $x_1(t)$ of these solutions satisfies $|x_1(t)| \leq \varphi_1(t)$ for $t \in (0,1]$.

Corollary 2.4. *Suppose that* $-\infty < a < A < \infty$, $c \in \mathbb{R}$, $\lambda \in L[[a,A],\mathbb{R}^+]$, *and* $\mu \in L[[a,A],\mathbb{R}^+]$. *Let* $B \in [a,A] \setminus \{t_n,\sigma_1,\dots,\sigma_p\}$ *be such that*

$$\tilde{f}(t,x_1,\dots,x_n)\operatorname{sgn}[(t-B)(x_n-c)] \leq \lambda(t)|x_n| + \mu(t) \tag{2.33}$$

for $(t,x) \in \tilde{R}^n_{a,A}$. *Assume that*

(i) *there exists a function* $q \in K_{a,A,t_n,B}[\hat{R}^1_{a,A},\mathbb{R}]$ *such that* $\operatorname{sgn}(t-t_n)q(t,\cdot)$ *is nondecreasing and there is a solution* $\varphi(t)$ *of*

$$u' = q(t,u) \tag{2.34}$$

satisfying

$$\varphi(t) > 0 \quad \text{for } t \in \Delta(t_n,B), \qquad \lim_{t\to t_n}\varphi(t) = 0, \qquad \liminf_{t\to B}\varphi(t) > 0; \tag{2.35}$$

(ii) *there are* $z \in \widetilde{AC}[[a,A],\mathbb{R}]$ *and* $\varepsilon \in C[(a,A),\mathbb{R}^+]$ *such that* $z(t_n) = x_{0n}$ *and*

$$\operatorname{sgn}(B-t_n)\operatorname{sgn}(x_n-z(t))(\tilde{f}(t,x_1,\dots,x_n)-z'(t)) \geq \operatorname{sgn}(B-t_n)q(t,|x_n-z(t)|) \tag{2.36}$$

holds on $\hat{\Omega} = \{(t,x_1,\dots,x_n) : \varphi(t) < |x_n - z(t)| < \varphi(t) + \varepsilon(t), t \in \Delta(t_n,B)\}$ *for almost all* $t \in \Delta(t_n,B)$. *Then the boundary value problem*

$$\begin{gathered} v^{(n)} = \tilde{f}(t,v,v',\dots,v^{(n-1)}), \\ v(t_1) = x_{01}, \quad v'(t_2) = x_{02},\dots, \quad v^{(n-1)}(t_n) = x_{0n} \end{gathered} \tag{2.37}$$

has at least two different solutions on $[a,A]$.

Proof. Put $I = \{n\}$, $k_1 = 1$, $\Pr x = x_n$, $c_n = c$, $g_n(t,u) = q(t,u)$, $\varphi_n(t) = \varphi(t)$, $c_i = x_{0i}$ for $i = 1,\dots,n-1$, $\mu_i(t) = 0$ for $i = 1,\dots,n-1$, $\mu_n(t) = \mu(t)$, $B_n = B$, and

$$\lambda_{ij}(t) = \begin{cases} 1 & \text{for } 1 \le i = j-1 \le n-1, \\ \lambda(t) & \text{for } i = j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.38}$$

Considering the system

$$\begin{aligned} x_1' &= x_2, & x_1(t_1) &= x_{01}, \\ x_2' &= x_3, & x_2(t_2) &= x_{02}, \\ &\vdots & &\vdots \\ x_{n-1}' &= x_n, & x_{n-1}(t_{n-1}) &= x_{0n-1}, \\ x_n' &= \tilde{f}(t,x_1,x_2,\dots,x_n), & x_n(t_n) &= x_{0n}, \end{aligned} \tag{2.39}$$

and applying Corollary 2.2, we get

$$\begin{aligned} f_n(t,x_1,\dots,x_n)\operatorname{sgn}[(t-B_n)(x_n-c_n)] &\le \sum_{j=1}^{n} \lambda_{nj}(t)|x_j| + \mu_n(t), \\ f_i(t,x_1,\dots,x_n)\operatorname{sgn}[(t-t_i)(x_i-c_i)] \le |x_{i+1}| &\le \lambda_{i,i+1}|x_{i+1}| \\ &= \sum_{j=1}^{n} \lambda_{ij}(t)|x_j| + \mu_i(t) \end{aligned} \tag{2.40}$$

for $i = 1,\dots,n-1$. The result follows from Corollary 2.2. □

Example 2.5. Let $\gamma \in (0,1)$. Consider the boundary value problem

$$v'' = p_1(t,v)|v'|^{\gamma}\operatorname{sgn} v' + p_2(t,v,v'), \qquad v(0) = 0, \qquad v'(1) = 0, \tag{2.41}$$

where $p_1 \in K_1[\hat{R}^1_{0,1},\mathbb{R}]$ and $p_2 \in K_1[\hat{R}^2_{0,1},\mathbb{R}]$ are such that

$$\begin{gathered} x_2 p_2(t,x_1,x_2) \le 0 \quad \text{for } (t,x_1,x_2) \in (0,1)\times\mathbb{R}^2, \\ p_1(t,x_1) \le -\delta(t) \quad \text{for } (t,x_1) \in (0,1)\times\mathbb{R}, \end{gathered} \tag{2.42}$$

$\delta \in L[[0,1],\mathbb{R}]$ being a positive function. Since

$$-p_1(t,x_1)|x_2|^{\gamma} - p_2(t,x_1,x_2)\operatorname{sgn} x_2 \ge \delta(t)|x_2|^{\gamma}, \tag{2.43}$$

the assumptions of Corollary 2.4 are fulfilled with $n = 2$, $a = 0$, $A = 1$, $t_1 = 0$, $t_2 = 1$, $c = 0$, $B = 0$, $z(t) \equiv 0$, $\lambda(t) \equiv 0$, $\mu(t) \equiv 0$, and

$$q(t,u) = \begin{cases} -\delta(t)u^\gamma & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases} \qquad \varphi(t) = \left[(1-\gamma)\int_t^1 \delta(s)\,ds\right]^{1/1-\gamma}. \tag{2.44}$$

Therefore, problem (2.41) has at least two different solutions on $[0,1]$.

Corollary 2.6. *Let the assumptions of Corollary 2.2 be fulfilled with the exception that the conditions (i), (ii) are replaced by (i′), (ii′):*

(i′) *there exist functions $h_i, q_i \in K_{a,A,t_i,B_i}[\hat{R}^1_{a,A}, \mathbb{R}]$ $(i \in I)$ such that functions $\operatorname{sgn}(t - t_i)h_i(t,\cdot)$ and $\operatorname{sgn}(t-t_i)q_i(t,\cdot)$ are nondecreasing for $i \in I$ and there are solutions $\varphi_i(t)$, $\psi_i(t)$ of $u_i' = h_i(t,u_i)$ and $v_i' = q_i(t,v_i)$, respectively, satisfying*

$$\begin{aligned} &\varphi_i(t) > 0 \quad \text{for } t \in \Delta(t_i,B_i), \qquad \lim_{t\to t_i}\varphi(t) = 0, \qquad \liminf_{t\to B_i}\varphi(t) > 0, \\ &\psi_i(t) > 0 \quad \text{for } t \in \Delta(t_i,B_i), \qquad \lim_{t\to t_i}\psi(t) = 0, \qquad \liminf_{t\to B_i}\psi(t) > 0 \end{aligned} \tag{2.45}$$

for $t \in I$;

(ii′) *there are $z_i \in \widetilde{AC}[[a,A],\mathbb{R}]$ and $\varepsilon = (\varepsilon_{i_1},\dots,\varepsilon_{i_l}) \in C[(a,A),\mathbb{R}^{+l}]$ such that $z_i(t_i) = x_{0i}$ and the inequalities*

$$\begin{aligned} \operatorname{sgn}(B_j - t_j)\big[\big(f_j(t,x) - z_j'(t)\big) - h_j\big(t,(x_j - z_j(t))_+\big)\big] \geq 0 \quad (j \in I) \\ \operatorname{sgn}(B_j - t_j)\big[-\big(f_j(t,x) - z_j'(t)\big) - q_j\big(t,(x_j - z_j(t))_-\big)\big] \geq 0 \quad (j \in I) \end{aligned} \tag{2.46}$$

are fulfilled on $\hat{\Omega} = \{(t,x) : \varphi_j(t) < x_j - z_j(t) < \varphi_j(t) + \varepsilon_j(t), t \in \Delta(t_j,B_j)\}$ and $\hat{\hat{\Omega}} = \{(t,x) : \psi_j(t) < z_j(t) - x_j < \psi_j(t) + \varepsilon_j(t), t \in \Delta(t_j,B_j)\}$, respectively, for almost all $t \in \Delta(t_j,B_j)$. Then the Cauchy-Nicoletti boundary value problem (2.1) has at least two different solutions on $[a,A]$.

Proof. Without loss of generality, it can again be assumed that $I = \{1,\dots,l\}$ and $\operatorname{Pr} x = (x_1,\dots,x_l)$. Put $k_i = 2$, $g_{i1}(t,u) = h_i(t,u)$, $g_{i2}(t,v) = q_i(t,v)$, $\varphi_{i1}(t) = \varphi_i(t)$, $\varphi_{i2}(t) = \psi_i(t)$, $V_{i1}(t,x) = (x_i - z_i(t))_+$, $V_{i2}(t,x) = (x_i - z_i(t))_-$, and $V_i(t,x) = (V_{i1}(t,x), V_{i2}(t,x))$ for $i \in I$. Then we have

$$\begin{aligned} \operatorname{sgn}(B_i - t_i)V_{i1}'(t,x(t)) &\geq \operatorname{sgn}(B_i - t_i)\big(f_i(t,x(t)) - z_i'(t)\big) \\ &\geq \operatorname{sgn}(B_i - t_i)g_{i1}\big(t, V_{i1}(t,x(t))\big), \\ \operatorname{sgn}(B_i - t_i)V_{i2}'(t,x(t)) &\geq -\operatorname{sgn}(B_i - t_i)\big(f_i(t,x(t)) - z_i'(t)\big) \\ &\geq \operatorname{sgn}(B_i - t_i)g_{i2}\big(t, V_{i2}(t,x(t))\big) \end{aligned} \tag{2.47}$$

for any solution $x = x(t)$ of (2.1) a.e. on any interval $(\alpha_{i1},\alpha_{i2}) \subseteq \Delta(t_i,B_i)$ for which

$$V_{i1}(t,x(t)) < \varphi_i(t) + \varepsilon_i(t), \qquad V_{i2}(t,x(t)) < \psi_i(t) + \varepsilon_i(t) \tag{2.48}$$

on $(\alpha_{i1},\alpha_{i2})$, $i=1,\dots,l$, and

$$V_{i1}(t,x(t))>\varphi_i(t) \quad \text{or} \quad V_{i2}(t,x(t))>\psi_i(t) \tag{2.49}$$

on $(\alpha_{i1},\alpha_{i2})$, respectively. The statement follows from Theorem 2.1. □

Corollary 2.7. *Let the assumptions of Corollary 2.4 be fulfilled with the exception that conditions (i), (ii) are replaced by the following:*

(i′) *there exist functions* $h\in K_{a,A,t_n,B}[\hat{R}^1_{a,A},\mathbb{R}]$ *and* $q\in K_{a,A,t_n,B}[\hat{R}^1_{a,A},\mathbb{R}]$ *such that* $\operatorname{sgn}(t-t_n)h(t,\cdot)$ *and* $\operatorname{sgn}(t-t_n)q(t,\cdot)$ *are nondecreasing and there are solutions* $\varphi(t)$, $\psi(t)$ *of* $u'=h(t,u)$ *and* $v'=q(t,v)$, *respectively, satisfying*

$$\begin{gathered}\varphi(t)>0, \quad \psi(t)>0 \quad \text{for } t\in\Delta(t_n,B), \quad \lim_{t\to t_n}\varphi(t)=\lim_{t\to t_n}\psi(t)=0,\\ \liminf_{t\to B}\varphi(t)>0, \qquad \liminf_{t\to B}\psi(t)>0;\end{gathered} \tag{2.50}$$

(ii′) *there are* $z\in\widetilde{AC}[[a,A],\mathbb{R}]$ *and* $\varepsilon\in C[(a,A),\mathbb{R}^+]$ *such that* $z(t_n)=x_{0n}$ *and*

$$\begin{gathered}\operatorname{sgn}(B-t_n)\big[\tilde{f}(t,x_1,\dots,x_n)-z'(t)-h(t,(x_n-z(t))_+)\big]\geq 0,\\ \operatorname{sgn}(B-t_n)\big[-\tilde{f}(t,x_1,\dots,x_n)+z'(t)-q(t,(x_n-z(t))_-)\big]\geq 0\end{gathered} \tag{2.51}$$

hold on $\hat{\Omega}=\{(t,x_1,\dots,x_n):\varphi(t)<x_n-z(t)<\varphi(t)+\varepsilon(t),\ t\in\Delta(t_n,B)\}$ *and* $\hat{\hat{\Omega}}=\{(t,x_1,\dots,x_n):\psi(t)<z(t)-x_n<\psi(t)+\varepsilon(t),\ t\in\Delta(t_n,B)\}$, *respectively, for almost all* $t\in\Delta(t_n,B)$. *Then the Cauchy-Nicoletti boundary value problem (2.37) has at least two different solutions on* $[a,A]$.

Proof. Corollary 2.7 follows from Corollary 2.6 in the same way as Corollary 2.4 follows from Corollary 2.2. □

Example 2.8. Let $p_1\in K_1[\hat{R}^2_{0,1},\mathbb{R}]$ and $p_2\in K_1[\hat{R}^2_{0,1},\mathbb{R}]$ be such that

$$\begin{gathered}p_1(t,x_1,x_2)\leq -\delta_1(t)\vartheta_1(x_2) \quad \text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}\times(0,\infty),\\ p_1(t,x_1,x_2)\geq \delta_2(t)\vartheta_2(|x_2|) \quad \text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}\times(-\infty,0),\\ x_2p_2(t,x_1,x_2)\leq 0 \quad \text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}^2,\end{gathered} \tag{2.52}$$

where δ_1, δ_2 are positive functions such that $\delta_j\in L[[0,1],\mathbb{R}]$ and $\vartheta_j\in C[[0,\infty),\mathbb{R}^+]$ ($j=1,2$) are nondecreasing and positive on $(0,\infty)$ and satisfying $\vartheta_1(0)=\vartheta_2(0)=0$, $\int_0^1\delta_1(s)\,ds<\int_0^\infty 1/\vartheta_1(s)\,ds<\infty$, and $\int_0^1\delta_2(s)\,ds<\int_0^\infty 1/\vartheta_2(s)\,ds<\infty$.

Consider the boundary value problem

$$w''=p_1(t,w,w')+p_2(t,w,w'), \qquad w(0)=0, \qquad w'(1)=0. \tag{2.53}$$

It holds that

$$\begin{aligned} &-\left[p_1(t,x_1,x_2)+p_2(t,x_1,x_2)+\delta_1(t)\vartheta_1(x_2)\right]\geq 0 \\ &\qquad \text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}\times(0,\infty), \\ &-\left[-p_1(t,x_1,x_2)-p_2(t,x_1,x_2)+\delta_2(t)\vartheta_2(-x_2)\right]\geq 0 \\ &\qquad \text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}\times(-\infty,0). \end{aligned} \tag{2.54}$$

The problems

$$\begin{aligned} u' &= -\delta_1(t)\vartheta_1(u), \qquad u(1)=0, \\ v' &= -\delta_2(t)\vartheta_2(v), \qquad v(1)=0 \end{aligned} \tag{2.55}$$

have positive solutions on $[0,1)$ and condition (2.54) implies

$$\left[p_1(t,x_1,x_2)+p_2(t,x_1,x_2)\right]\operatorname{sgn}x_2\leq 0. \tag{2.56}$$

Therefore, the assumptions of Corollary 2.7 are fulfilled with $a=0$, $A=1$, $c=0$, $z(t)\equiv 0$, $B=0$, $t_1=0$, $t_2=1$, $\lambda(t)\equiv 0$, $\mu(t)\equiv 0$, and

$$\begin{aligned} h(t,u) &= \begin{cases} -\delta_1(t)\vartheta_1(u) & \text{for } (t,u)\in(0,1)\times(0,\infty), \\ 0 & \text{for } (t,u)\in(0,1)\times(-\infty,0], \end{cases} \\ q(t,v) &= \begin{cases} -\delta_2(t)\vartheta_2(v) & \text{for } (t,v)\in(0,1)\times(0,\infty), \\ 0 & \text{for } (t,v)\in(0,1)\times(-\infty,0]. \end{cases} \end{aligned} \tag{2.57}$$

Hence problem (2.53) has at least two solutions on $[0,1]$.

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ACCURATE SOLUTION ESTIMATES FOR NONLINEAR NONAUTONOMOUS VECTOR DIFFERENCE EQUATIONS

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The paper deals with the vector discrete dynamical system $x_{k+1} = A_k x_k + f_k(x_k)$. The well-known result by Perron states that this system is asymptotically stable if $A_k \equiv A = \text{const}$ is stable and $f_k(x) \equiv \tilde{f}(x) = o(\|x\|)$. Perron's result gives no information about the size of the region of asymptotic stability and norms of solutions. In this paper, accurate estimates for the norms of solutions are derived. They give us stability conditions for (1.1) and bounds for the region of attraction of the stationary solution. Our approach is based on the "freezing" method for difference equations and on recent estimates for the powers of a constant matrix. We also discuss applications of our main result to partial reaction-diffusion difference equations.

1. Introduction and notation

Let $\mathbb{C}^n$ be the set of n-complex vectors endowed with the Euclidean norm $\|\cdot\|$. Consider in $\mathbb{C}^n$ the equation

$$x_{k+1} = A_k x_k + f_k(x_k) \quad (k = 0, 1, 2, \dots), \tag{1.1}$$

where A_k $(k = 0, 1, 2, \dots)$ are $n \times n$-complex matrices and $f_k : \mathbb{C}^n \to \mathbb{C}^n$ are given functions. A well-known result of Perron which dates back to 1929 (see [11, page 270], [8, Theorem 9.14], and [6]) states that (1.1) is asymptotically stable if $A_k \equiv A = \text{const}$ is stable (i.e., the spectral radius $r_s(A)$ of A is less than 1) and $f_k(x) = \tilde{f}(x) = o(\|x\|)$. Clearly, this result is purely local. It gives no information about the size of the region of asymptotic stability and norms of solutions.

In this paper, we derive accurate estimates for the norms of solutions. Our approach is based on the "freezing" method for difference equations and on recent estimates for the powers of a constant matrix. Note that the "freezing" method for difference equations was developed in [5]. It is based on the relevant ideas for differential equations (cf. [2, 3, 7]).

Firstly, we consider the linear difference equation

$$x_{k+1} = A_k x_k \quad (k = 0, 1, 2, \dots). \tag{1.2}$$

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As it is well known, the fundamental matrix $X(k)$ of (1.2) can be expressed as

$$X(m) = A_m A_{m-1} \cdots A_0. \tag{1.3}$$

But such a representation does not yield much information about the asymptotic value of solutions, except in the case of constant coefficients $A_k = A$ ($k = 0,1,2,\ldots$), when $X(k) = A^k$ and the Jordan canonical form of A determines the asymptotic behavior of the solutions. To prove the stability of (1.2) is equivalent to proving the boundedness of the sequence $\{\|X(m)\|\}_1^\infty$. This problem is easy to solve under the condition $\sup_k \|A_k\| \le 1$. But it is rather restrictive. The "freezing" method allows us to avoid this condition in the case

$$||A_k - A_j|| \le q_{k-j} \quad (q_k = q_{-k} = \text{const} \ge 0,\ q_0 = 0;\ j,k = 0,1,2,\ldots). \tag{1.4}$$

For example, if $A_k = \sin(ck)B$ ($c = \text{const} > 0$), where B is a constant matrix, then condition (1.4) holds with $q_k = 2\|B\||\sin(ck/2)|$, since

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{1.5}$$

for real α, β.

For an $n \times n$-matrix A, denote

$$g(A) = \left[N^2(A) - \sum_{j=1}^{n} |\lambda_j(A)|^2\right]^{1/2}, \tag{1.6}$$

where $N(A)$ is the Frobenius (Hilbert-Schmidt) norm of a matrix $A : N^2(A) = \operatorname{Trace}(AA^*)$, and $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ are the eigenvalues of A including their multiplicities. The relations

$$\begin{gathered} g(A) \le \left[N^2(A) - |\operatorname{Trace}(A^2)|\right]^{1/2}, \\ g(A) \le \sqrt{\frac{1}{2}} N(A^* - A) \end{gathered} \tag{1.7}$$

are true [3, Section 2.1]. Here A^* is the adjoint matrix. If A is a normal matrix: $A^*A = AA^*$, then $g(A) = 0$. If $A = (a_{ij})$ is a triangular matrix such that $a_{ij} = 0$ for $1 \le j \le i \le n$, then

$$g^2(A) = \sum_{1 \le i \le j \le n} |a_{ij}|^2. \tag{1.8}$$

For a natural number $n > 1$, introduce the numbers

$$\gamma_{n,p} = \sqrt{\frac{\mathbb{C}_{n-1}^p}{(n-1)^p}} \tag{1.9}$$

for $p = 1,2,\ldots,n-1$ and $\gamma_{n,0} = 1$. Here and below,

$$\mathbb{C}_m^k = \frac{m!}{(m-k)!k!} \quad (k = 0,1,2,\ldots,m;\ m = 1,2,\ldots) \tag{1.10}$$

are the binomial coefficients. Evidently, for $n > 2$,

$$\gamma_{n,p}^2 = \frac{(n-2)(n-3)\cdots(n-p)}{(n-1)^{p-1}p!} \le \frac{1}{p!}. \tag{1.11}$$

Due to [4, Theorem 1.2.1], for any $n \times n$-matrix A, the inequality

$$\begin{aligned} ||A^m|| &\le \sum_{k=0}^{m_1} \frac{m! r_s^{m-k}(A) g^k(A) \gamma_{n,k}}{(m-k)!k!} \\ &= \sum_{k=0}^{m_1} \mathbb{C}_m^k r_s^{m-k}(A) g^k(A) \gamma_{n,k} \quad (m_1 = \min\{n-1, m\}) \end{aligned} \tag{1.12}$$

holds for every integer m, where $r_s(A)$ is the spectral radius of A.

2. Preliminary facts

Firstly, we recall a boundedness result for (1.2) which is proven in [5, Lemma 1.1], namely, we recall the following lemma.

Lemma 2.1. *Under condition (1.4), let*

$$\zeta_0 \equiv \sum_{k=1}^{\infty} q_k \sup_{l=1,2,\ldots} ||A_l^k|| < 1. \tag{2.1}$$

Then, every solution $\{x_k\}$ of (1.2) satisfies the inequality

$$\sup_{k=1,2,\ldots} ||x_k|| \le \beta_0 ||x_0|| (1-\zeta_0)^{-1}, \tag{2.2}$$

where

$$\beta_0 = \sup_{k,l=0,1,2,\ldots} ||A_l^k||. \tag{2.3}$$

As a consequence, it is possible to establish the next corollaries.

Corollary 2.2. *Let the conditions*

$$||A_k - A_{k+1}|| \le \tilde{q} \quad (k = 1,2,\ldots;\ \tilde{q} = \text{const} > 0), \tag{2.4}$$

$$\theta_0 \equiv \sum_{k=1}^{\infty} \sup_{l=1,2,\ldots} ||A_l^k|| k < \tilde{q}^{-1} \tag{2.5}$$

hold. Then, every solution $\{x_k\}$ of (1.2) satisfies the inequality

$$||x_k|| \le \beta_0 ||x_0|| (1 - \tilde{q}\theta_0)^{-1} \quad (k = 1,2,\ldots). \tag{2.6}$$

Indeed, under condition (2.4), we have

$$\|A_k - A_j\| \le \tilde{q}|k-j| \quad (j,k = 0,1,2,\dots). \tag{2.7}$$

So $\zeta_0 \le \tilde{q}\theta_0$. Now, the required result follows from Lemma 2.1.

Corollary 2.3. *Let condition (2.4) hold. In addition, for a constant $v > 0$, let*

$$\theta(v) \equiv \sum_{k=1}^{\infty} v^{-k-1} \sup_{l=1,2,\dots} \|A_l^k\| k < \tilde{q}^{-1}. \tag{2.8}$$

Then, every solution $\{x_k\}$ of (1.2) satisfies the inequality

$$\|x_k\| \le v^k m(v) \|x_0\| \left(1 - \tilde{q}\theta(v)\right)^{-1} \quad (k = 1,2,\dots), \tag{2.9}$$

where

$$m(v) \equiv \sup_{l,k=0,1,2,\dots} v^{-k} \|A_l^k\|. \tag{2.10}$$

Indeed, due to condition (2.8), $m(v) < \infty$. Putting $x_k = v^k z_k$ in (1.2) , we get

$$z_{k+1} = v^{-1} A_k z_k. \tag{2.11}$$

Corollary 2.2 and condition (2.8) imply

$$\sup_{k=1,2,\dots} \|z_k\| \le m(v) \|z_0\| \left(1 - \tilde{q}\theta(v)\right)^{-1} \quad (k = 1,2,\dots). \tag{2.12}$$

Hence, the required estimate follows. Recall also the following result from [5].

Theorem 2.4. *Under condition (1.4), let*

$$\rho_0 \equiv \sup_{l=1,2,\dots} r_s(A_l) < 1, \qquad v_0 \equiv \sup_{l=0,1,2,\dots} g(A_l) < \infty, \tag{2.13}$$

$$\xi \equiv \sum_{m=1}^{\infty} \sum_{k=0}^{n-1} \mathbb{C}_m^k \rho_0^{m-k} v_0^k \gamma_{n,k} q_m < 1. \tag{2.14}$$

Then, every solution $\{x_k\}$ of (1.2) is bounded. Moreover,

$$\sup_{k=1,2,\dots} \|x_k\| \le M_0 \|x_0\| (1-\xi)^{-1}, \tag{2.15}$$

where

$$M_0 = \sum_{k=0}^{n-1} v_0^k \gamma_{n,k} \left(\psi_k + k\right)^k \rho_0^{\psi_k} \tag{2.16}$$

with $\psi_k = \max\{0, -k(1+\log\rho_0)/\log\rho_0\}$.

3. The main result

The previous estimates give us a possibility to investigate (1.1) as a nonlinear perturbation of (1.2). For a positive $r \leq \infty$, denote the ball

$$B_r = \{x \in \mathbb{C}^n : \|x\| \leq r\} \tag{3.1}$$

and assume that there are constants $\mu, \nu \geq 0$, such that

$$||f_k(x)|| \leq \nu\|x\| + \mu \quad (x \in B_r;\ k = 0,1,2,\dots). \tag{3.2}$$

Recall that the quantities ρ_0, ν_0, and M_0 are defined by (2.13) and (2.16). Let

$$\psi(A) \equiv \sum_{k=0}^{\infty}\sum_{j=0}^{n-1} \mathbb{C}_k^j \rho_0^{k-j} \nu_0^j \gamma_{n,j}. \tag{3.3}$$

Now we are in a position to formulate the main result of the paper.

Theorem 3.1. *Under the conditions (1.4), (2.13), and (3.2), let*

$$S(f;A) \equiv \sum_{k=0}^{\infty}\sum_{j=0}^{n-1} \mathbb{C}_k^j \rho_0^{k-j} \nu_0^j \gamma_{n,j}(q_k + \nu) < 1. \tag{3.4}$$

Then, any solution $\{x_k\}_{=0}^{\infty}$ of (1.1) satisfies the inequality

$$\sup_{k=1,2,\dots} ||x_k|| \leq \frac{M_0||x_0|| + \mu\psi(A)}{1 - S(f;A)}, \tag{3.5}$$

provided that

$$\frac{M_0||x_0|| + \mu\psi(A)}{1 - S(f;A)} \leq r. \tag{3.6}$$

The proof of this theorem is given afterwards.
Recall that

$$\beta_0 = \sup_{k,l=0,1,\dots} ||A_l^k|| \tag{3.7}$$

and let

$$\theta_1 \equiv \sum_{k=0}^{\infty} \sup_{l=0,1,\dots} ||A_l^k||. \tag{3.8}$$

Lemma 3.2. *Under conditions (1.4) and (3.2), let*

$$S_0 \equiv \sum_{k=0}^{\infty} (q_k + \nu) \sup_{l=0,1,2,\dots} ||A_l^k|| < 1. \tag{3.9}$$

Then, every solution $\{x_k\}$ of (1.1) satisfies the inequality

$$\|x_k\| \le [\beta_0\|x_0\| + \theta_1\mu](1 - S_0)^{-1} \quad (k = 1, 2, \dots), \tag{3.10}$$

provided that

$$[\beta_0\|x_0\| + \theta_1\mu](1 - S_0)^{-1} \le r. \tag{3.11}$$

Proof. Rewrite (1.1) as

$$x_{k+1} - A_l x_k = (A_k - A_l)x_k + f_k(x_k) \tag{3.12}$$

with a fixed integer l. The variation of parameters formula yields

$$x_{l+1} = A_l^{l+1}x_0 + \sum_{j=0}^{l} A_l^{l-j}[(A_j - A_l)x_j + f_j(x_j)]. \tag{3.13}$$

There are two cases to consider: $r = \infty$ and $r < \infty$. First, assume that (3.2) is valid with $r = \infty$, then, by (1.4),

$$\begin{aligned}
\|x_{l+1}\| &\le \beta_0\|x_0\| + \sum_{j=0}^{l} \|A_l^{l-j}\|[q_{l-j}\|x_j\| + \nu\|x_j\| + \mu] \\
&\le \beta_0\|x_0\| + \sum_{j=0}^{l} \|A_l^{l-j}\|(q_{l-j} + \nu)\|x_j\| + \theta_1\mu \\
&\le \beta_0\|x_0\| + \max_{k=0,\dots,l}\|x_k\| \sum_{k=0}^{l} \|A_l^k\|(q_k + \nu) + \mu\theta_1 \\
&\le \beta_0\|x_0\| + \max_{k=1,\dots,l}\|x_k\| \left(\sum_{k=0}^{\infty}(q_k + \nu) \sup_{l=0,1,2,\dots}\|A_l^k\|\right) + \mu\theta_1.
\end{aligned} \tag{3.14}$$

Consequently,

$$\max_{k=1,2,\dots,l+1}\|x_k\| \le \beta_0\|x_0\| + S_0 \max_{k=0,1,\dots,l+1}\|x_k\| + \mu\theta_1. \tag{3.15}$$

But $\beta_0 \ge 1$. So

$$\max_{k=0,1,2,\dots,l+1}\|x_k\| \le \beta_0\|x_0\| + S_0 \max_{k=0,1,\dots,l+1}\|x_k\| + \mu\theta_1. \tag{3.16}$$

Hence,

$$\sup_{k=0,1,2,\dots}\|x_k\| \le \frac{\beta_0\|x_0\| + \mu\theta_1}{1 - S_0}. \tag{3.17}$$

Let now $r < \infty$. Define the function

$$\tilde{f}_k(x) = \begin{cases} f_k(x), & \|x\| \le r, \\ 0, & \|x\| > r. \end{cases} \tag{3.18}$$

Since

$$\left\|\widetilde{f_k}(x)\right\| \le \nu\|x\| + \mu, \quad k = 0,1,\dots;\ x \in B_\infty, \tag{3.19}$$

then the sequence $\{\widetilde{x_k}\}_{=0}^{\infty}$ defined by

$$\widetilde{x_0} = x_0, \qquad \widetilde{x_{k+1}} = A_k\widetilde{x_k} + \widetilde{f_k}(\widetilde{x_k}), \quad k = 0,1,\dots, \tag{3.20}$$

satisfies the inequality

$$\sup_{k=0,1,\dots} ||\widetilde{x_k}|| \le \frac{\beta_0||x_0|| + \mu\theta_1}{1 - S_0} \le r \tag{3.21}$$

according to the above arguments and condition (3.11). But f and $\widetilde{f_k}(x)$ coincide on B_r. So $x_k = \widetilde{x_k}$ for $k = 0,1,2,\dots$. Therefore, (3.10) is satisfied, thus concluding the proof. □

proof of Theorem 3.1. As it was proved in [5, Lemma 2.2], $\beta_0 \le M_0$. Moreover, due to (1.12), we have $\theta_1 \le \psi(A)$ and $S_0 \le S(f : A)$. Now the result is due to Lemma 3.2. □

Remarks 3.3. (a) Under (3.2) with $\mu = 0$, $f_k(0) = 0$ so that $\{0\}$ is a solution of (1.1). Under condition $S(f,A) < 1$, Theorem 3.1 asserts that the trivial solution is stable, and that any initial vector $x_0 \in B_r$, satisfying the condition

$$||x_0|| \le \frac{(1 - S(f,A))r}{M_0}, \tag{3.22}$$

belongs to the region of attraction.

(b) If $\nu \equiv 0$, then every solution of (1.1) with the initial vector x_0 satisfying

$$||x_0||M_0 + \mu\psi(A) \le (1 - \xi)r \tag{3.23}$$

is bounded.

4. Applications

In this section, we will illustrate our main results by considering a partial difference equation. We consider a simple three-level discrete reaction-diffusion equation of the form

$$u_i^{(j+1)} = a_j u_{i-1}^{(j)} + b_j u_i^{(j)} + c_j u_{i+1}^{(j)} + g_i^{(j)} + f_j\left(u_i^{(j)}\right), \tag{4.1}$$

defined on $\Omega = \{(i,j) : i = 0,1,\dots,n+1;\ j = 0,1,\dots\}$, where $\{a_j\}$, $\{b_j\}$, and $\{c_j\}$ are real sequences, $g = \{g_i^{(j)}\}$ is a complex function defined on Ω, and $f_j : \mathbb{C} \to \mathbb{C}$ $(j = 0,1,\dots)$ are given functions. Assume that the side conditions

$$u_0^{(j)} = \delta_j \in \mathbb{C}, \quad j = 0,1,\dots, \tag{4.2}$$

$$u_{n+1}^{(j)} = \gamma_j \in \mathbb{C}, \quad j = 0,1,\dots, \tag{4.3}$$

$$u_i^{(0)} = \tau_j \in \mathbb{C}, \quad i = 1,2,\dots,n, \tag{4.4}$$

are imposed, where $\tau = (\tau_1, \tau_2, \dots, \tau_n) \in \mathbb{C}^n$. A solution of problem (4.1), (4.2), (4.3), and (4.4) is a discrete function $u = \{u_i^{(j)}\}_{(i,j)\in\Omega}$ which satisfies relations (4.1), (4.2), (4.3), and (4.4). The existence and uniqueness of solutions to that problem is obvious, provided that f_j is one-one valued. With the notation

$$u^{(j)} = \left(u_1^{(j)}, u_2^{(j)}, \dots, u_n^{(j)}\right), \tag{4.5}$$

the sequence $\{u^{(j)}\}_{j=0}^{\infty}$ satisfies the vector equation

$$u^{(j+1)} = A_j u^{(j)} + G_j + F_j\left(u^{(j)}\right), \quad j = 0, 1, \dots, \tag{4.6}$$

and the initial condition

$$u^{(0)} = \tau, \tag{4.7}$$

where

$$\begin{gathered} A_j = \begin{bmatrix} b_j & c_j & 0 & \cdots & \cdots & 0 \\ a_j & b_j & c_j & 0 & \cdots & 0 \\ 0 & a_j & b_j & c_j & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & a_j & b_j \end{bmatrix}, \quad j = 0, 1, 2, \dots, \\ G_j = \left(g_1^{(j)}, \dots, g_n^{(j)}\right) + (a_j\delta_j, 0, \dots, 0, c_j\gamma_j), \\ F_j(x) = (f_j(x_1), \dots, f_j(x_n)), \quad x = (x_1, x_2, \dots, x_n). \end{gathered} \tag{4.8}$$

Thus, we can write problem (4.1), (4.2), (4.3), and (4.4) as (1.1) with

$$f_j(x) = F_j(x) + G_j. \tag{4.9}$$

Assume that there are nonnegative constants μ_1 and ν such that

$$\|F_j(x)\| \le \nu\|x\| + \mu_1 \quad (x \in B_r;\ j = 1, 2, \dots). \tag{4.10}$$

In addition,

$$\mu_2 \equiv \sum_{j=0}^{\infty} \|G_j\| < \infty. \tag{4.11}$$

So condition (3.2) holds with $\mu_0 = \mu_1 + \mu_2$. As a direct consequence of Theorem 3.1, we get the following theorem.

Theorem 4.1. *Let conditions (1.4), (4.2), (4.10), and (4.11) hold with $\mu = \mu_1 + \mu_2$ and $x_0 = \tau$. Then, the unique solution $x_j = \{u_i^{(j)}\}_{(i,j)\in\Omega}$ of problem (4.1), (4.2), (4.3), and (4.4) satisfies inequality (3.5).*

Remarks 4.2. Comparing Theorem 4.1 with [10, Theorems 1 and 2], we point out that the hypotheses of Theorem 4.1 can be checked more easily. In this paper, we have used a different approach. Our results do not overlap with those from [9, 10]. Other related works can be found in [1, pages 237–245].

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GENERALIZATIONS OF THE BERNOULLI AND APPELL POLYNOMIALS

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We first introduce a generalization of the Bernoulli polynomials, and consequently of the Bernoulli numbers, starting from suitable generating functions related to a class of Mittag-Leffler functions. Furthermore, multidimensional extensions of the Bernoulli and Appell polynomials are derived generalizing the relevant generating functions, and using the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials. The main properties of these polynomial sets are shown. In particular, the differential equations can be constructed by means of the factorization method.

1. Introduction

A recent paper [7] deals with generalized forms of Bernoulli polynomials, used in order to derive explicit summation formulas, generalizing well-known classical results. Furthermore, the Appell polynomials were applied for the construction of quadrature rules involving Appell instead of Bernoulli polynomials [4, 6]. In our opinion, the technique introduced in [7] using the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials in order to extend to several variables many classical univariable formulas could be exploited in order to find further generalizations of the above results, permitting the construction of new summation formulas and multidimensional quadrature rules.

A preliminary analysis of the main properties of generalized Bernoulli or Appell polynomials is included in this paper.

2. Recalling Bernoulli and Appell polynomials

The Bernoulli polynomials $B_n(x)$ can be defined by means of the generating function

$$G(x,t) := \frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi. \tag{2.1}$$

By putting $x = 0$, we obtain the Bernoulli numbers $B_n := B_n(0)$ and the relevant generating function

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$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{2.2}$$

It is well known that

$$\begin{aligned} &B_n(0) = B_n(1) = B_n, \quad n \neq 1, \\ &B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}, \\ &B_n'(x) = nB_{n-1}(x). \end{aligned} \tag{2.3}$$

The Bernoulli numbers enter into many mathematical formulas, such as

(i) the Taylor expansion in a neighborhood of the origin of the circular and hyperbolic tangent and cotangent functions,
(ii) the sums of powers of natural numbers,
(iii) the remainder term of the Euler-MacLaurin quadrature rule.

The Bernoulli polynomials, first studied by Euler, are employed in the integral representation of differentiable periodic functions, since they are employed for approximating such functions in terms of polynomials. They are also used for representing the remainder term of the composite Euler-MacLaurin quadrature rule (see [22]).

The Appell polynomials [2] are defined by the generating function

$$G_A(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{2.4}$$

where

$$A(t) = \sum_{k=0}^{\infty} \frac{\mathcal{R}_k}{k!} t^k, \quad A(0) \neq 0, \tag{2.5}$$

is an analytic function at $t = 0$, and $\mathcal{R}_k := R_k(0)$.

It is easy to see that for any $A(t)$, the derivatives of $R_n(x)$ satisfy

$$R_n'(x) = nR_{n-1}(x), \tag{2.6}$$

and furthermore,

(i) if $A(t) = t/(e^t - 1)$, then $R_n(x) = B_n(x)$,
(ii) if $A(t) = 2/(e^t + 1)$, then $R_n(x) = E_n(x)$, the Euler polynomials,
(iii) if $A(t) = \alpha_1 \cdots \alpha_m t^m [(e^{\alpha_1 t} - 1) \cdots (e^{\alpha_m t} - 1)]^{-1}$, then the $R_n(x)$ are the Bernoulli polynomials of order m (see [11]),
(iv) if $A(t) = 2^m [(e^{\alpha_1 t} + 1) \cdots (e^{\alpha_m t} + 1)]^{-1}$, then the $R_n(x)$ are the Euler polynomials of order m (see [11]),
(v) if $A(t) = e^{\xi_0 + \xi_1 t + \cdots + \xi_{d+1} t^{d+1}}$, $\xi_{d+1} \neq 0$, then the $R_n(x)$ are the generalized Gould-Hopper polynomials (see [10]), including the Hermite polynomials when $d = 1$ and classical 2-orthogonal polynomials when $d = 2$.

3. Generalizations of the Bernoulli polynomials

Some generalized forms of the Bernoulli polynomials and numbers already appeared in literature:

(i) the generalized Bernoulli polynomials $B_n^{\alpha}(x)$ defined by the generating function

$$\frac{t^{\alpha}e^{xt}}{(e^t-1)^{\alpha}}=\sum_{n=0}^{\infty}B_n^{\alpha}(x)\frac{t^n}{n!},\quad |t|<2\pi, \tag{3.1}$$

by means of which Tricomi and Erdélyi [23] gave an asymptotic expansion of the ratio of gamma functions;

(ii) the polynomials of Nath [19], defined by the generating function

$$\frac{(ht)^{\alpha}(1+wt)^{x/w}}{\left[(1+wt)^{h/w}-1\right]^{\alpha}}=\sum_{n=0}^{\infty}B_{n;h,w}^{\alpha}(x)\frac{t^n}{n!},\quad |t|<\left|\frac{1}{w}\right|; \tag{3.2}$$

(iii) the polynomials of Frappier [12], defined by the generating function

$$\frac{(iz)^{\alpha}e^{(x-1/2)z}}{2^{2\alpha}\Gamma(\alpha+1)J_{\alpha}(iz/2)}=\sum_{n=0}^{\infty}B_{n,\alpha}(x)\frac{z^n}{n!},\quad |z|<2\,|\,j_1\,|, \tag{3.3}$$

where J_{α} is the Bessel function of the first kind of order α and $j_1=j_1(\alpha)$ is the first zero of J_{α}.

4. A new class of generalized Bernoulli polynomials: $B_n^{[m-1]}(x)$, $m\geq 1$

In this section, we introduce a countable set of polynomials $B_n^{[m-1]}(x)$ generalizing the Bernoulli ones (which can be recovered assuming that $m=1$), introduced by Natalini and Bernardini [18].

To this end, we consider a class of Appell polynomials, defined by using a generating function linked to the so-called Mittag-Leffler function,

$$E_{1,m+1}(t):=\frac{t^m}{e^t-\sum_{h=0}^{m-1}(t^h/h!)} \tag{4.1}$$

considered in the general form by Agarwal [1].

The generalized Bernoulli polynomials $B_n^{[m-1]}(x)$, $m\geq 1$, are defined by the generating function

$$G^{[m-1]}(x,t):=\frac{t^m e^{xt}}{e^t-\sum_{h=0}^{m-1}(t^h/h!)}=\sum_{n=0}^{\infty}B_n^{[m-1]}(x)\frac{t^n}{n!}. \tag{4.2}$$

For $m = 1$, we obtain, from the above equation, the generating function $G^{(0)}(x,t) = te^{xt}/(e^t - 1)$ of the classical Bernoulli polynomials $B_n^{(0)}(x)$.

Since $G^{[m-1]}(x,t) = A(t)e^{xt}$, the generalized Bernoulli polynomial belong to the class of Appell polynomials.

It is possible to define the generalized Bernoulli numbers assuming that

$$B_n^{[m-1]} = B_n^{[m-1]}(0). \tag{4.3}$$

The following properties are proved in the above-mentioned paper [18].

(i) Explicit forms:

$$x^n = \sum_{h=0}^{n} \binom{n}{h} \frac{h!}{(h+m)!} B_{n-h}^{[m-1]}(x). \tag{4.4}$$

Inverting this equation, it is possible to find explicit expressions for the polynomials $B_n^{[m-1]}(x)$. The first ones are given by

$$\begin{aligned} B_0^{[m-1]}(x) &= m!, \\ B_1^{[m-1]}(x) &= m!\left(x - \frac{1}{m+1}\right), \\ B_2^{[m-1]}(x) &= m!\left(x^2 - \frac{2}{m+1}x + \frac{2}{(m+1)^2(m+2)}\right), \end{aligned} \tag{4.5}$$

and, consequently, the first generalized Bernoulli numbers are

$$\begin{aligned} B_0^{[m-1]} &= m!, \qquad B_1^{[m-1]} = -\frac{m!}{m+1}, \\ B_2^{[m-1]} &= \frac{2m!}{(m+1)^2(m+2)}. \end{aligned} \tag{4.6}$$

(ii) Recurrence relation for the $B_n^{[m-1]}$ polynomials:

$$B_n^{[m-1]}(x) = \left(x - \frac{1}{m+1}\right) B_{n-1}^{[m-1]}(x) - \frac{1}{n(m-1)!} \sum_{k=0}^{n-2} \binom{n}{k} B_{n-k}^{[m-1]} B_k^{[m-1]}(x). \tag{4.7}$$

This relation, starting from $n = 1$, and taking into account the initial value $B_0^{[m-1]}(x) = m!$, allows a recursive computation for this class of generalized Bernoulli polynomials.

(iii) Differential equation for the $B_n^{[m-1]}$ polynomials:

$$\begin{aligned} &\frac{B_n^{[m-1]}}{n!} y^{(n)} + \frac{B_{n-1}^{[m-1]}}{(n-1)!} y^{(n-1)} + \cdots + \frac{B_2^{[m-1]}}{2!} y'' \\ &\quad + (m-1)!\left(\frac{1}{m+1} - x\right) y' + n(m-1)! y = 0. \end{aligned} \tag{4.8}$$

This is an equation of order n so that all the considered families of polynomials can be viewed as solutions of differential operators of infinite order.

Remark 4.1. Note that the generating function could be written in the form

$$G^{[m-1]}(x,t) := \frac{t^m e^{xt}}{e^t - \sum_{h=0}^{m-1} (t^h/h!)} = m! \sum_{n=0}^{\infty} \tilde{B}_n^{[m-1]}(x) \frac{t^n}{n!} \tag{4.9}$$

so that, putting

$$B_n^{[m-1]}(x) = m! \tilde{B}_n^{[m-1]}(x), \tag{4.10}$$

we obtain the explicit form of the generalized Bernoulli polynomial $\tilde{B}_n^{[m-1]}$ from the preceding one simply by dividing by $m!$, and so decreasing the relevant numerical values.

5. 2D extensions of the Bernoulli and Appell polynomials

The Hermite-Kampé de Fériet [3] (or Gould-Hopper) polynomials [13, 21] have been used recently in order to construct addition formulas for different classes of generalized Gegenbauer polynomials [9].

They are defined by the generating function

$$e^{xt+yt^j} = \sum_{n=0}^{\infty} H_n^{(j)}(x,y) \frac{t^n}{n!} \tag{5.1}$$

or by the explicit form

$$H_n^{(j)}(x,y) = n! \sum_{s=0}^{[n/j]} \frac{x^{n-js} y^s}{(n-js)!s!}, \tag{5.2}$$

where $j \geq 2$ is an integer. The case when $j = 1$ is not considered since the corresponding 2D polynomials are simply expressed by the Newton binomial formula.

It is worth recalling that the polynomials $H_n^{(j)}(x,y)$ are a natural solution of the generalized heat equation

$$\frac{\partial}{\partial y} F(x,y) = \frac{\partial^j}{\partial x^j} F(x,y), \qquad F(x,0) = x^n. \tag{5.3}$$

The case when $j = 2$ is then particularly important (see Widder [24]); it was recently used in order to define 2D extensions of the Bernoulli and Euler polynomials [7].

Further generalizations including the $H_n^{(j)}(x,y)$ polynomials as a particular case are given by

$$e^{x_1 t + x_2 t^2 + \cdots + x_r t^r} = \sum_{n=0}^{\infty} H_n(x_1, x_2, \ldots, x_r) \frac{t^n}{n!}. \tag{5.4}$$

Note that the generating function of the last equation can be written in the form

$$\begin{aligned} e^{x_1 t + x_2 t^2 + \cdots + x_r t^r} &= \sum_{k=0}^{\infty} \frac{(x_1 t + x_2 t^2 + \cdots + x_r t^r)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{k_1+k_2+\cdots+k_r=k} \frac{k!}{k_1!k_2!\cdots k_r!} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r} t^{k_1+2k_2+\cdots+rk_r} \\ &= \sum_{n=0}^{\infty} \left(\sum_{\pi_k(n|r)} n! \frac{x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}}{k_1!k_2!\cdots k_r!} \right) \frac{t^n}{n!}, \end{aligned} \tag{5.5}$$

where $k := k_1 + k_2 + \cdots + k_r$, $n := k_1 + 2k_2 + \cdots + rk_r$, and the sum runs over all the restricted partitions $\pi_k(n|r)$ (containing at most r sizes) of the integer n, with k denoting the number of parts of the partition and k_i the number of parts of size i. Note that, using the ordinary notation for the partitions of n, that is, $n = k_1 + 2k_2 + \cdots + nk_n$, we have to assume $k_{r+1} = k_{r+2} = \cdots = k_n = 0$.

Consequently, the explicit form of the multidimensional Hermite-Kampé de Fériet polynomials

$$H_n(x_1, x_2, \ldots, x_r) = \sum_{\pi_k(n|r)} n! \frac{x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}}{k_1!k_2!\cdots k_r!} \tag{5.6}$$

follows.

Furthermore, they satisfy for every n the *isobaric property* (of weight n)

$$H_n(tx_1, t^2 x_2, \ldots, t^r x_r) = t^n H_n(x_1, x_2, \ldots, x_r), \tag{5.7}$$

and consequently, they are solutions of the first-order partial differential equation

$$x_1 \frac{\partial H_n}{\partial x_1} + 2x_2 \frac{\partial H_n}{\partial x_2} + \cdots + rx_r \frac{\partial H_n}{\partial x_r} = nH_n. \tag{5.8}$$

The multivariate Hermite-Kampé de Fériet polynomials appear as an interesting tool for introducing and studying multidimensional generalizations of the Appell polynomials too, including the Bernoulli and Euler ones, starting from the corresponding generating functions. A first approach in this direction was given in [8].

In the following, we recall some results of Bretti and Ricci [5], presenting some properties of the generalized 2D Appell polynomials, but considering first the case of the 2D Bernoulli polynomials, in order to introduce the subject in a more friendly way. The relevant extensions to the multidimensional Bernoulli and Appell case can be derived almost straightforwardly, but the relevant equations are rather involved.

We will show that for every integer $j \geq 2$, it is possible to define a class of 2D Bernoulli polynomials denoted by $B_n^{(j)}(x, y)$ generalizing the classical Bernoulli polynomials.

Furthermore, the bivariate Appell polynomials $R_n^{(j)}(x,y)$ are introduced by means of the generating function

$$A(t)e^{xt+yt^j} = \sum_{n=0}^{\infty} R_n^{(j)}(x,y)\frac{t^n}{n!}. \tag{5.9}$$

Exploiting the factorization method (see [14, 16]), we show how to derive the differential equations satisfied by these 2D polynomials. The differential equation for the classical Appell polynomials was first obtained by Sheffer [20], and was recently recovered in [15].

Remark 5.1. It is worth noting that recently Professor Ismail [17], avoiding the use of the factorization method, was able to prove that the differential equation of infinite order satisfied by the Appell polynomials is nothing special since it can be stated for a general polynomial family.

Further generalizations are given by the multiindex polynomials defined by means of the generating functions

$$A(t,\tau)e^{xt^l+y\tau^j} = \sum_{n,m=0}^{\infty} R_{n,m}^{(l,j)}(x,y)\frac{t^n}{n!}\frac{\tau^m}{m!} \tag{5.10}$$

or, more generally,

$$A(t_1,\dots,t_r)e^{x_1t_1^{j_1}+\cdots+x_rt_r^{j_r}} = \sum_{n_1,\dots,n_r=0}^{\infty} R_{n_1,\dots,n_r}^{(j_1,\dots,j_r)}(x_1,\dots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}, \tag{5.11}$$

which belong to the set of multidimensional special functions recently introduced by Dattoli and his group.

6. The 2D Bernoulli polynomials $B_n^{(j)}(x,y)$

Starting from the Hermite-Kampé de Fériet (or Gould-Hopper) polynomials $H_n^{(j)}(x,y)$, we define the 2D Bernoulli polynomials $B_n^{(j)}(x,y)$ by means of the generating function

$$G^{(j)}(x,y;t) := \frac{t}{e^t-1}e^{xt+yt^j} = \sum_{n=0}^{\infty} B_n^{(j)}(x,y)\frac{t^n}{n!}. \tag{6.1}$$

It is worth noting that the polynomial $H_n^{(j)}(x,y)$, being *isobaric of weight n*, cannot contain the variable y, for every $n=0,1,\dots,j-1$.

The following results for the $B_n^{(j)}(x,y)$ polynomials can be derived.

(i) Explicit forms of the polynomials $B_n^{(j)}$ in terms of the Hermite-Kampé de Fériet polynomials $H_n^{(j)}$ and vice versa:

$$B_n^{(j)}(x,y) = \sum_{h=0}^{n}\binom{n}{h}B_{n-h}H_n^{(j)}(x,y) = n!\sum_{h=0}^{n}\frac{B_{n-h}}{(n-h)!}\sum_{r=0}^{[h/j]}\frac{x^{h-jr}y^r}{(h-jr)!r!}, \tag{6.2}$$

where B_k denote the Bernoulli numbers;

$$H_n^{(j)}(x,y) = \sum_{h=0}^{n} \binom{n}{h} \frac{1}{n-h+1} B_h^{(j)}(x,y). \tag{6.3}$$

(ii) Recurrence relation:

$$\begin{aligned} B_0^{(j)}(x,y) &= 1, \\ B_n^{(j)}(x,y) &= -\frac{1}{n}\sum_{k=0}^{n-2}\binom{n}{k} B_{n-k}B_k^{(j)}(x,y) \\ &\quad + \left(x-\frac{1}{2}\right)B_{n-1}^{(j)}(x,y) + jy\frac{(n-1)!}{(n-j)!}B_{n-j}^{(j)}(x,y). \end{aligned} \tag{6.4}$$

(iii) Shift operators:

$$\begin{aligned} L_n^- &:= \frac{1}{n}D_x, \qquad L_n^+ := \left(x-\frac{1}{2}\right) - \sum_{k=0}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{n-k} + jyD_x^{j-1}, \\ \mathcal{L}_n^- &:= \frac{1}{n}D_x^{-(j-1)}D_y, \\ \mathcal{L}_n^+ &:= \left(x-\frac{1}{2}\right) + jyD_x^{-(j-1)^2}D_y^{j-1} - \sum_{k=0}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{-(j-1)(n-k)}D_y^{n-k}. \end{aligned} \tag{6.5}$$

(iv) Differential or integrodifferential equations:

$$\begin{aligned} &\left[\frac{B_n}{n!}D_x^n + \cdots + \frac{B_{j+1}}{(j+1)!}D_x^{j+1} + \left(\frac{B_j}{j!} - jy\right)D_x^j \right. \\ &\left. \quad + \frac{B_{j-1}}{(j-1)!}D_x^{j-1} + \cdots + \left(\frac{1}{2}-x\right)D_x + n\right]B_n^{(j)}(x,y) = 0, \end{aligned} \tag{6.6}$$

$$\begin{aligned} &\left[\left(x-\frac{1}{2}\right)D_y + jD_x^{-(j-1)^2}D_y^{j-1} + jyD_x^{-(j-1)^2}D_y^j \right. \\ &\left. \quad - \sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{-(j-1)(n-k)}D_y^{n-k+1} - (n+1)D_x^{(j-1)}\right]B_n^{(j)}(x,y) = 0, \end{aligned} \tag{6.7}$$

$$\begin{aligned} &\left[\left(x-\frac{1}{2}\right)D_x^{(j-1)(n-1)}D_y + (j-1)(n-1)D_x^{(j-1)(n-1)-1}D_y \right. \\ &\quad + jD_x^{(j-1)(n-j)}\left(D_y^{j-1} + yD_y^j\right) - \sum_{k=1}^{n-1}\frac{B_{n-k+1}}{(n-k+1)!}D_x^{(j-1)(k-1)}D_y^{n-k+1} \\ &\left. \quad - (n+1)D_x^{(j-1)n}\right]B_n^{(j)}(x,y) = 0, \quad n \geq j. \end{aligned} \tag{6.8}$$

Note that the last equation can easily be derived by differentiating $(j-1)(n-1)$ times with respect to x both sides of the preceding one, and does not contain antiderivatives for $n \geq j$.

7. The 2D Appell polynomials $R_n^{(j)}(x, y)$

For any $j \geq 2$, the 2D Appell polynomials $R_n^{(j)}(x, y)$ are defined by means of the generating function

$$G_A^{(j)}(x, y; t) := A(t)e^{xt+yt^j} = \sum_{n=0}^{\infty} R_n^{(j)}(x, y)\frac{t^n}{n!}. \tag{7.1}$$

Even in this general case, the polynomial $R_n^{(j)}(x, y)$ is *isobaric of weight n* so that it does not contain the variable y, for every $n = 0, 1, \ldots, j-1$.

(i) Explicit forms of the polynomials $R_n^{(j)}$ in terms of the Hermite-Kampé de Fériet polynomials $H_n^{(j)}$ and vice versa:

$$\begin{aligned} R_n^{(j)}(x, y) &= \sum_{h=0}^{n} \binom{n}{h} \mathcal{R}_{n-h} H_n^{(j)}(x, y) \\ &= n! \sum_{h=0}^{n} \frac{\mathcal{R}_{n-h}}{(n-h)!} \sum_{r=0}^{[h/j]} \frac{x^{h-jr} y^r}{(h-jr)! r!}, \end{aligned} \tag{7.2}$$

where the $\mathcal{R}_k$ are the Appell numbers appearing in the definition (2.5);

$$H_n^{(j)}(x, y) = \sum_{k=0}^{n} \binom{n}{k} Q_{n-k} R_k^{(j)}(x, y), \tag{7.3}$$

where the Q_k are the coefficients of the Taylor expansion in a neighborhood of the origin of the reciprocal function $1/A(t)$.

(ii) Recurrence relation: it is useful to introduce the coefficients of the Taylor expansion

$$\frac{A'(t)}{A(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}. \tag{7.4}$$

The following linear homogeneous recurrence relation for the generalized Appell polynomials $R_n^{(j)}(x, y)$ holds:

$$\begin{aligned} R_0^{(j)}(x, y) &= 1, \\ R_n^{(j)}(x, y) &= (x + \alpha_0) R_{n-1}^{(j)}(x, y) + \binom{n-1}{j-1} j y R_{n-j}^{(j)}(x, y) \\ &\quad + \sum_{k=0}^{n-2} \binom{n-1}{k} \alpha_{n-k-1} R_k^{(j)}(x, y). \end{aligned} \tag{7.5}$$

(iii) Shift operators:

$$
\begin{gathered}
L_n^- := \frac{1}{n} D_x, \qquad L_n^+ := (x+\alpha_0) + \frac{j}{(j-1)!} y D_x^{j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{n-k}, \\
\mathcal{L}_n^- := \frac{1}{n} D_x^{-(j-1)} D_y, \\
\mathcal{L}_n^+ := (x+\alpha_0) + \frac{j}{(j-1)!} y D_x^{-(j-1)^2} D_y^{j-1} + \sum_{k=0}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{-(j-1)(n-k)} D_y^{n-k}.
\end{gathered}
\tag{7.6}
$$

(iv) Differential or integrodifferential equations:

$$
\begin{aligned}
&\left[\frac{\alpha_{n-1}}{(n-1)!} D_x^n + \cdots + \frac{\alpha_j}{j!} D_x^{j+1} + \left(\frac{\alpha_{j-1} + jy}{(j-1)!} \right) D_x^j \right. \\
&\quad \left. + \frac{\alpha_{j-2}}{(j-2)!} D_x^{j-1} + \cdots + (x+\alpha_0) D_x - n \right] R_n^{(j)}(x,y) = 0,
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
&\left[(x+\alpha_0) D_y + \frac{j}{(j-1)!} D_x^{-(j-1)^2} \left(y D_y^j + D_y^{j-1} \right) \right. \\
&\quad \left. + \sum_{k=1}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{-(j-1)(n-k)} D_y^{n-k+1} - (n+1) D_x^{j-1} \right] R_n^{(j)}(x,y) = 0,
\end{aligned}
\tag{7.8}
$$

$$
\begin{aligned}
&\left[(x+\alpha_0) D_x^{(j-1)(n-1)} D_y + (j-1)(n-1) D_x^{(j-1)(n-1)-1} D_y \right. \\
&\quad + \frac{j}{(j-1)!} D_x^{(j-1)(n-j)} \left(y D_y^j + D_y^{j-1} \right) + \sum_{k=1}^{n-1} \frac{\alpha_{n-k}}{(n-k)!} D_x^{(j-1)(k-1)} D_y^{n-k+1} \\
&\quad \left. - (n+1) D_x^{n(j-1)} \right] R_n^{(j)}(x,y) = 0, \quad n \geq j.
\end{aligned}
\tag{7.9}
$$

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LIST OF LECTURES

The following is a list of the lectures that have been presented at the ICDDEA conference.

1. Welcome lecture

P. D. Siafarikas, Welcome-Evangelos K. Ifantis and his work.

2. Opening lecture

N. Artemiadis, The mathematician's share in the general human condition.

3. Plenary lectures

(1) R. P. Agarwal, Recent trends in discrete boundary value problems.
(2) A. Fokas, Differential forms, spectral theory and boundary value problems.
(3) A. Kartsatos, Some recent developments in topological degree theory in Banach spaces and its applications to partial differential equations.
(4) G. Ladas, Open problems and conjectures in difference equations.

4. Invited lectures

(1) N. Alikakos, Continuum limits of particles interacting via diffusion.
(2) I. Antoniou, Statistical analysis of evolution equations
(3) G. Barbatis, Improved L^p Hardy inequalities.
(4) G. Dassios, Anisotropic effects in magnetoenchephalography.
(5) D. Dimitrov, Electrostatics, Lame differential equation and zeros of orthogonal polynomials.
(6) Z. Došlá, Some aspects of nonlinear second order differential and difference equations.
(7) O. Došlý, Symplectic and Hamiltonian systems: Transformations and oscillation theory.
(8) A. Georgescu, Dynamics and bifurcation in biological and economical dynamics.
(9) M. Grammaticopoulos, 3D - Protter's problem for the wave equation: singularities of the solutions.
(10) I. Gyori, Stability results in population model equations.
(11) A. Himonas, Analyticity of the Cauchy problem for an integrable evolution equation.
(12) K. Karanikas, Limit behavior of measurement unitary evolution - Zeno type effect on certain wavelet subspaces.
(13) D. Kravaritis, Nonlinear monotone operators and evolution inclusions in Hilbert spaces.

(14) W. Kryszewski, Solution sets to constrained differential equations in Banach spaces with applications.
(15) A. Laforgia, Means of special functions.
(16) A. Lomtatidze, On bounded solutions of functional differential equations.
(17) S. Ntouyas, Neutral functional differential inclusions: A survey of recent developments.
(18) V. Papanikolaou, The Periodic Euler-Bernoulli Equation.
(19) G. Papaschinopoulos, Boundedeness and asymptotic behavior of the solutions of a fuzzy difference equation.
(20) P. E. Ricci, Generalizations of the Bernoulli and Appell polynomials.
(21) P. Papadopoulos, Global Existence, Blow-Up and Stability Results for Quasilinear Nonlocal Kirchoff Strings on $\mathbb{R}^N$.
(22) J. Stavroulakis, Oscillation criteria for functional differential equations.
(23) I. Stratis, Propagation of electromagnetic waves in complex media: a time-domain analysis.
(24) A. Tertikas, Optimizing Hardy inequalities.
(25) A. Tzavaras, Diffusive N-waves and metastability in Burgers equation.
(26) E. van Doorn, On orthogonal polynomials and infinite systems of first order linear differential equations.
(27) J. Vosmanský, Zeros and related quantities of some Sturm-Liouville functions.

5. Research seminars

(1) M. Bartusek, On noncontinuable solutions.
(2) S. Belmehdi, Integral representation of the solutions to Heun biconfluent equation.
(3) E. Camouzis, Asymptotic behavior and existence of solutions of $x_{n+1} = p + x_{n-1}/x_n$.
(4) L. Čelechovská, Mathematical model of the human liver and the quasilinearization method.
(5) J. Čermák, The pantograph equation and its generalization.
(6) C. Cesarano, Miscellaneous results on Chebyshev polynomials.
(7) A. Cwiszewski, Equilibria for perturbations of m-accretive operators on closed sets.
(8) J. Diblík, Anti-Lyapunov method for systems of discrete equations.
(9) A. Domoshnitsky, On oscillation and asymptotic properties of functional PDE.
(10) V. Ďurikovič, On the solutions of nonlinear initial-boundary value problems.
(11) D. Ellinas, Group theory of quasi probability volumes of Wigner function.
(12) R. Hakl, On the solvability and unique solvability of a boundary value problem for first order functional differential equations.
(13) A. Ivanov, Periodic solutions of a three-dimensional system with delay.
(14) J. Kalas, Nonuniqueness theorem for a singular Cauchy-Nicoletti problem.
(15) C. G. Kokologiannaki, Monotonicity results on the zeros of q-associated polynomials.

(16) R. Korhonen, The value distribution of the derivatives of the Painlevé transcendents
(17) A. Mamourian, On a degenerate elliptic Lavrentiev type equations and BVP.
(18) R. Medina, Accurate estimates for the solutions of difference equations.
(19) D. Medková, Which solutions of the third problem are bounded?
(20) M. Miyake, Structure of formal solutions to first order nonlinear singular partial differential equations in complex domain.
(21) M. Ohmiya, Darboux-Lame equation and isomonodromic deformation.
(22) B. Palumbo. On some inequalities for the product of the gamma functions.
(23) E. N. Petropoulou, On non-linear ordinary difference equations arising in numerical schemes.
(24) A. Ponossov, Stabilizing differential equations by hybrid feedback controls.
(25) A. Poulkou, Sampling and interpolation theories and boundary value problems.
(26) A. Prykarpatsky, Homology structure of ergodic measures related with nonautonomous Hamiltonian systems.
(27) I. Rachunkova, Existence results for higher order singular boundary value problems.
(28) P. Rehak, Comparison theorems for linear dynamic equations on time scales.
(29) A. Reinfelds, Equivalence of nonautonomous differential equations in a Banach space.
(30) C. Schinas, On the system of two difference equations $x_{n+1} = \sum_{i=0}^{k-1} A_i/y_{n-i}^{p_i} + 1/y_{n-k}^{p_k}$, $y_{n+1} = \sum_{i=0}^{k-1} B_i/x_{n-i}^{q_i} + 1/x_{n-k}^{q_k}$
(31) A. Shindiapin, On resolvability of systems of singular functional-differential equations in Martsenkevich-type spaces.
(32) B. Slezak, The relation between the properties of the local and global solutions.
(33) J. Šremr, On a two point boundary value problem for first order linear differential equations with deviating argument.
(34) S. Stanek, On higher order singular boundary value problems.
(35) G. Stefanidou, Some results concerning two fuzzy difference equations.
(36) A. Szawiola, Influence of deviated arguments on oscillation of solutions of differential equations.
(37) M. Tvrdý, Impulsive periodic boundary value problem and topological degree.
(38) E. Tzirtzilakis, Biomagnetic fluid flow in a 3D channel.
(39) P. Vafeas, Comparison of differential representations for radialy symmetric Stokes flow.
(40) L. Velazquez, Differential equations for classical matrix orthogonal polynomials.
(41) K. Vlachou, Exact solutions of the semi-infinite Toda lattice with applications to the inverse spectral problem.
(42) J. Werbowksi, Conditions for the oscillation of solutions of iterative equations.

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