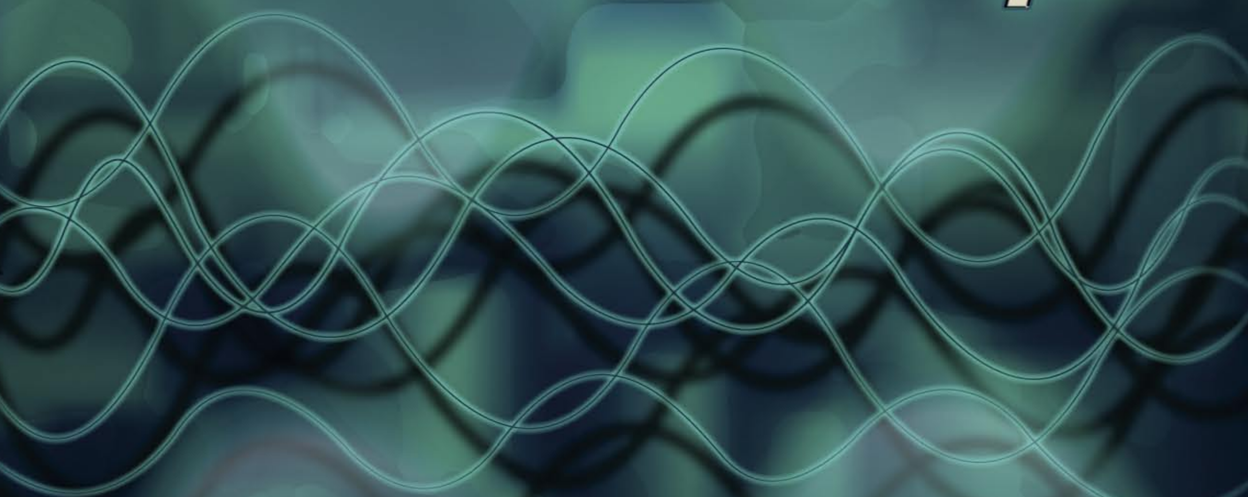
Discrete Oscillation Theory



Ravi P. Agarwal

Martin Bohner

Said R. Grace

Donal O'Regan

Discrete Oscillation Theory

Discrete Oscillation Theory

Ravi P. Agarwal
Martin Bohner
Said R. Grace
Donal O'Regan

Hindawi Publishing Corporation
410 Park Avenue, 15th Floor, #287 pmb, New York, NY 10022, USA
Nasr City Free Zone, Cairo 11816, Egypt
Fax: +1-866-HINDAWI (USA toll-free)

ISBN 977-5945-19-4

Contents

Preface xi

1. Oscillation theory for second-order linear difference equations 1
    - 1.1. Introduction 1
    - 1.2. Basic concepts 3
    - 1.3. Riccati-type transformations 11
    - 1.4. Reid's roundabout theorem and Sturmian theory 15
        - 1.4.1. The disconjugacy characterization theorem 15
        - 1.4.2. Sturmian theory 16
    - 1.5. Disconjugacy and disfocality 17
        - 1.5.1. Dominant and recessive solutions 23
        - 1.5.2. More on disconjugacy characterization 26
        - 1.5.3. Disfocality 31
    - 1.6. Conjugacy criteria 33
    - 1.7. Methods of linear discrete oscillation theory 40
        - 1.7.1. Riccati technique 40
        - 1.7.2. Variational principle 40
    - 1.8. Oscillation and nonoscillation criteria 41
    - 1.9. Comparison theorems 69
    - 1.10. Oscillation, nonoscillation, and monotone solutions 71
    - 1.11. Oscillation criteria by Riccati technique 75
    - 1.12. Oscillation criteria by averaging techniques 87
    - 1.13. Further criteria by averaging techniques 102
    - 1.14. Oscillation criteria for linear damped difference equations 117
        - 1.14.1. Oscillation of a reducible damped equation 118
        - 1.14.2. Oscillation criteria of summation averaging type 121
        - 1.14.3. Oscillation criteria of nonsummation type 127
    - 1.15. Oscillation and nonoscillation criteria for forced equations 131
        - 1.15.1. Implicit-type results 133
        - 1.15.2. Some explicit criteria 137
    - 1.16. Some qualitative properties of solutions 141
        - 1.16.1. Some properties of dominant and recessive solutions 141
        - 1.16.2. Reciprocity principle 148
        - 1.16.3. Behavioral properties of dominant and recessive solutions 148
        - 1.16.4. Further classification of solutions 156
        - 1.16.5. Comparison theorems 161
    - 1.17. More on nonoscillation criteria 163
    - 1.18. Limit point results 168
    - 1.19. Growth and some oscillation criteria 173

1.19.1. Growth of solutions 173
1.19.2. Oscillation criteria 175
1.20. Mixed difference equations 176
1.21. Notes and general discussions 180

2. Oscillation theory for systems of difference equations 183
2.1. Introduction 183
2.2. Discrete variational problems 185
2.2.1. Problem and notation 185
2.2.2. Inhomogeneous discrete quadratic functionals 186
2.2.3. Solution of discrete variational problems 189
2.3. Linear Hamiltonian difference systems 191
2.3.1. Notation and assumptions 191
2.3.2. Special solutions of the Hamiltonian system 194
2.3.3. Riccati matrix difference equations 198
2.4. The discrete Picone formula 203
2.4.1. Discrete quadratic functionals 203
2.4.2. The proof of the discrete Picone identity 207
2.5. Disconjugacy and controllability 212
2.5.1. Focal points 212
2.5.2. Generalized zeros 215
2.5.3. Controllability 220
2.6. Positivity of discrete quadratic functionals 226
2.6.1. The Reid roundabout theorem 226
2.6.2. Sturm's separation and comparison theorems 227
2.6.3. The extended Reid roundabout theorem 231
2.6.4. Separated boundary conditions 235
2.6.5. Solvability of discrete Riccati equations 239
2.7. Sturm-Liouville difference equations 244
2.7.1. Disconjugacy 244
2.7.2. The Reid roundabout theorem 252
2.8. Moore-Penrose inverses 257
2.9. Notes and general discussions 263

3. Oscillation theory for half-linear difference equations 265
3.1. Introduction 265
3.2. Preliminaries and basic concepts 266
3.3. Reid's roundabout theorem and Sturmian theory 273
3.3.1. The disconjugacy characterization theorem 273
3.3.2. Sturmian theory 282
3.4. Nonoscillation and conjugacy criteria 283
3.4.1. Nonoscillation theorems based on the Riccati technique 283
3.4.2. Nonoscillation theorems based on the variational principle 294
3.4.3. Conjugacy criteria 299

3.5. Oscillation criteria 304
3.5.1. Oscillation criteria based on the Riccati technique 304
3.5.2. Further oscillation criteria based on the Riccati technique 319
3.5.3. Oscillation criteria based on the variational principle 336
3.6. Comparison theorems 342
3.6.1. Some results from algebra 342
3.6.2. Some comparison theorems 343
3.6.3. Comparison theorem for generalized Riccati difference equations 347
3.7. More on oscillation, nonoscillation, and positive solutions 347
3.7.1. Preparatory lemmas 348
3.7.2. Sufficient conditions 350
3.7.3. Necessary conditions 353
3.8. Strong oscillation 361
3.8.1. Strong oscillation, nonoscillation, and conditional oscillation 361
3.8.2. Generalized Euler difference equation 364
3.9. Half-linear difference equations with damping term 365
3.10. Half-linear difference equations with forcing term 372
3.11. Notes and general discussions 375

4. Oscillation theory for nonlinear difference equations I 377
4.1. Oscillation criteria 377
4.2. Superlinear oscillation 391
4.3. Sublinear oscillation 399
4.4. Oscillation characterizations 412
4.5. Oscillation of damped nonlinear difference equations 426
4.6. Asymptotic behavior for nonlinear difference equations 435
4.7. Oscillation criteria via Liapunov's second method 443
4.8. Notes and general discussions 451

5. Oscillation theory for nonlinear difference equations II 453
5.1. Oscillation criteria 453
5.1.1. Oscillation criteria—(I_1) 454
5.1.2. Superlinear oscillation 463
5.1.3. Sublinear oscillation 474
5.1.4. Oscillation criteria—(I_2) 479
5.1.5. Oscillation criteria for damped equations 486
5.2. Monotone solutions of nonlinear difference equations 488
5.2.1. Existence of positive monotone solutions of (5.2.2) 488
5.2.2. Existence of monotone solutions of (5.2.1) 491
5.3. Bounded, unbounded, and monotone properties 492
5.3.1. Positive decreasing solutions 492
5.3.2. Positive increasing solutions 503

5.4. Notes and general discussions 514

6. Oscillation theory for difference equations with deviating arguments 515
6.1. Oscillation criteria (I) 516
6.1.1. Oscillation of equation (6.1.1) 516
6.1.2. Comparison theorems 541
6.1.3. Oscillation of equation (6.1.2) 545
6.2. Oscillation criteria (II) 553
6.2.1. Oscillation of equation (6.2.1) 554
6.2.2. Oscillation of equation (6.2.2) 565
6.2.3. Asymptotic behavior 572
6.2.4. Decaying nonoscillatory solutions of equation (6.2.130) 577
6.3. Oscillation via characteristic equations 584
6.3.1. Positive solutions of equation (6.3.1) 585
6.3.2. Oscillation of unbounded solutions of equation (6.3.2) 589
6.4. Oscillation of damped difference equations 594
6.4.1. Almost oscillation 600
6.5. Oscillation of forced difference equations 615
6.6. Notes and general discussions 624

7. Oscillation theory for neutral difference equations 627
7.1. Oscillation criteria via comparison 627
7.1.1. Linearization theorems 635
7.1.2. Comparison results for forced neutral equations 643
7.2. Nonoscillation criteria 648
7.3. Existence of nonoscillatory solutions 656
7.4. Classification of nonoscillatory solutions 667
7.4.1. Nonoscillatory solutions of equation ($7.4.1^+$) 668
7.4.2. Nonoscillatory solutions of equation ($7.4.1^-$) 674
7.4.3. Oscillation and asymptotic behavior 678
7.5. Further oscillation criteria 680
7.6. Oscillation criteria for mixed neutral difference equations 692
7.6.1. Oscillation of equations with constant coefficients 693
7.6.2. Oscillation of equations with periodic coefficients 707
7.6.3. Oscillation via characteristic equations 717
7.7. Notes and general discussions 726

8. Stability and oscillation theory for differential equations with piecewise constant arguments 727
8.1. Oscillation of linear first-order delay equations 727
8.1.1. Oscillation criteria 728
8.1.2. First-order equations of alternately retarded and advanced type 743
8.1.3. Characteristic equations 748
8.2. Oscillation of linear second-order delay equations 751

8.2.1. Existence of solutions 752
8.2.2. The case $\omega = 0$ 756
8.2.3. The case $\omega \neq 0$ 760
8.2.4. Second-order equations of alternately retarded and advanced type 766
8.2.5. Neutral differential equations with piecewise constant argument 770
8.3. Systems of alternately retarded and advanced type 776
8.3.1. The case $A = 0$ 777
8.3.2. The case A nonsingular 781
8.4. Applications 784
8.5. Notes and general discussions 787

9. Miscellaneous topics 791
9.1. Generalized characteristic equations 791
9.2. Difference equations with periodic coefficients 804
9.2.1. Oscillation of equation (9.2.1) 805
9.2.2. Delay difference equations with positive and negative coefficients 820
9.2.3. Oscillation of equation (9.2.3) 825
9.2.4. Oscillation of equation (9.2.4) 832
9.3. Linearized oscillations for difference equations 840
9.3.1. Linearized oscillations for equation (9.3.1) 840
9.3.2. Linearized oscillations for equation (9.3.2) 843
9.4. Oscillation of recursive sequences 849
9.4.1. Oscillation of $x(k+1) = ax(k)/[1+bx(k-\tau)]$ 850
9.4.2. Oscillation of $x(k+1) = [a+bx(k)]/[c+x(k-\tau)]$ 854
9.4.3. Oscillation of the discrete delay logistic equation 863
9.4.4. Oscillation of certain recursive sequences 865
9.5. Global behavior of certain difference equations 871
9.5.1. Global attractivity in a nonlinear second-order difference equation 874
9.5.2. Global asymptotic stability of a second-order difference equation 880
9.5.3. A rational recursive sequence 883
9.5.4. On the recursive sequence $x(k+1) = \alpha + (x(k-1)/x(k))$ 889
9.5.5. Global stability of a certain recursive sequence 894
9.5.6. Global attractivity in a nonlinear delay difference equation 899
9.5.7. On the recursive sequence $x(k+1) = ax(k) + (b + cx(k-1))e^{-x(k)}$ 905
9.5.8. Global attractivity in a differential equation with piecewise constant arguments 914
9.6. Nonlinear difference equations with continuous variable 918
9.7. Oscillation for systems of delay difference equations 928
9.8. Oscillatory behavior of solutions of functional equations 935

9.9. Notes and general discussions 942

Bibliography 947

Index 959

Preface

The theory of difference equations, the methods used in their solutions, and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last ten years, the proliferation of the subject is witnessed by numerous research articles and several monographs, annual international conferences, and new journals devoted to the study of difference equations and their applications. Now even those experts who believe in the universality of differential equations are discovering the sometimes striking divergence between the continuous and the discrete.

This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. Mainly second-order difference equations are treated, but results on first-order and higher-order equations and systems of difference equations are included as well.

There are nine chapters in this book, among which the first one is by far the longest one. Chapter 1 features an in-depth presentation of oscillation theory for second-order linear difference equations. Generalized zeros and the concepts of disconjugacy, disfocality, and oscillation are introduced, and a characterization of disconjugacy in terms of a discrete quadratic functional and a discrete Riccati equation is given. We present oscillation and nonoscillation criteria as well as comparison theorems, and discuss the main tools used in deriving those criteria, namely, the variational principle and the Riccati technique. A discrete version of the well-known Sturmian theory is presented, and the concepts of recessive and dominant solutions are discussed.

In Chapter 2 a generalization of the results given in Chapter 1 is presented, namely, an extension to oscillation theory for symplectic difference systems. Such symplectic difference systems contain linear Hamiltonian difference systems as special cases, which in turn contain self-adjoint vector equations and Sturm-Liouville difference equations of any even order (in particular of second order as discussed in Chapter 1). The main problem here is to appropriately define the concept of a generalized zero of a vector-valued function. The results presented on linear Hamiltonian difference systems are due to Martin Bohner.

Another way of extending the theory of linear second-order difference equations is presented in Chapter 3, namely, a generalization to second-order half-linear difference equations (the equations in Chapter 1 are special cases of the equations in Chapter 3). Again the concepts of generalized zeros and oscillation are introduced and many analogs of the results from Chapter 1, among them numerous oscillation and nonoscillation criteria, are offered in Chapter 3. Most of the results on discrete half-linear equations are due to Pavel Řehák.

Next, Chapters 4 and 5 feature an oscillation theory for second-order nonlinear difference equations. Many necessary and sufficient conditions for oscillation of superlinear and sublinear difference equations are provided. We also derive some oscillation criteria for nonlinear difference equations via Liapunov's second method. Damped nonlinear difference equations are considered as well. Chapter 5 also presents a discussion and classification of positive, bounded, unbounded, increasing, and decreasing solutions of discrete nonlinear equations of second order.

An oscillation theory for difference equations with deviating arguments is presented in Chapter 6. We establish oscillation criteria for certain linear, nonlinear, and half-linear difference equations with deviating (both advanced and retarded) arguments. Oscillation criteria for linear difference equations with deviating arguments via their characteristic equations are given, and many criteria for the oscillation and almost oscillation of linear and nonlinear damped and forced difference equations with deviating arguments are offered.

Chapter 7 consists of an oscillation theory for neutral difference equations. We establish many oscillation criteria for discrete linear and nonlinear neutral second-order equations with and without forcing term. Nonoscillation results for neutral equations with positive and negative coefficients are presented. We also introduce a classification scheme for nonoscillatory solutions of neutral difference equations. Oscillation criteria for neutral equations of mixed type with constant coefficients as well as periodic coefficients are offered, and some of those criteria are derived via associated characteristic equations.

Next, Chapter 8 features an oscillation theory for differential equations with piecewise constant arguments. Such equations have applications in control theory and certain biomedical models. Many criteria for oscillation, nonoscillation, and stability of first- and second-order differential equations with piecewise constant arguments are developed. We also discuss second-order equations and systems of alternately advanced and retarded type as well as neutral differential equations with piecewise constant arguments. As an application to the techniques presented, necessary and sufficient conditions for all positive solutions of the logistic equation with quadratic nonlinearity and piecewise constant arguments to oscillate about its positive equilibrium are obtained.

Finally, in Chapter 9, some miscellaneous topics of interest are discussed. Existence and comparison results of positive solutions of first-order delay difference equations are treated via their generalized characteristic equations. Oscillation criteria for some linear as well as neutral difference equations with periodic coefficients are given. Linearized oscillations for autonomous and nonautonomous delay difference equations are established. A systematic study for the oscillation and global asymptotic stability of various types of recursive sequences is presented. Results on oscillation of second-order nonlinear difference equations with continuous variables are given. Oscillation for systems of delay difference equations is studied. Finally, oscillatory behavior of linear functional equations of second order is discussed.

This book is addressed to a wide audience of specialists such as mathematicians, physicists, engineers, and biologists. It can be used as a textbook at the graduate level and as a reference book for several disciplines. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

Finally, we wish to express our thanks to the staff of *Hindawi Publishing Corporation* for their excellent cooperation during the preparation of this book for publication.

Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan
August 15, 2004

1 Oscillation theory for second-order linear difference equations

1.1. Introduction

In this chapter, we discuss systematically oscillatory and nonoscillatory and various other qualitative properties of solutions of second-order linear difference equations. Since the classical paper of Sturm [258] in which he discussed some qualitative properties of second-order linear differential equations of the type

$$\big(c(t)x'(t)\big)' + q(t)x(t) = 0, \tag{1.1.1}$$

where $c, q \in C([t_0, \infty), \mathbb{R})$ and $c(t) > 0$ for $t \geq t_0$, and from that time, thousands of papers have been published concerning this equation. It is a well-known fact that there is a striking similarity between the qualitative theories of differential equations and difference equations. Moreover, it turns out that it makes sense to study qualitative properties of difference equations (inspite of the fact that they are actually recurrence relations). Therefore, in the last years, a considerable effort has been made to investigate qualitative properties (including very important oscillation theory) of the discrete counterpart of the above equation, namely, the second-order linear difference equation

$$\Delta\big(c(k)\Delta x(k)\big) + q(k)x(k+1) = 0 \tag{1.1.2}$$

or

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k) = 0, \tag{1.1.3}$$

where $\{c(k)\}$ and $\{q(k)\}$ are real-valued sequences defined on $\mathbb{N}$ with $c(k) > 0$. It has been shown that in some aspects these theories are quite analogical, and, on the other hand, that to treat some problems concerning equation (1.1.2), say, one has to use somewhat different methods than those for differential equations. Note that equation (1.1.2) (or (1.1.3)) is frequently considered in the equivalent form of the three-term recurrence relation

$$c(k+1)x(k+2)+p(k)x(k+1)+c(k)x(k)=0, \tag{1.1.4}$$

where $p(k)=q(k)-c(k+1)-c(k)$ (or

$$c(k)x(k+1)+b(k)x(k)+c(k-1)x(k-1)=0, \tag{1.1.5}$$

where $b(k)=q(k)-c(k)-c(k-1)$). We also note that the behavioral properties of equations (1.1.2)–(1.1.5) are the same and hence we will have occasions to discuss each of these equations in appropriate places in the text.

Section 1.2 reviews relevant material and basic concepts of second-order linear difference equations. In Section 1.3 the important concepts of Riccati transformation and Riccati equation are introduced. Section 1.4 contains the statements of some important theorems, namely, the Picone identity, the disconjugacy characterization theorem, and Sturm-type comparison and separation theorems. In Section 1.5, some criteria for disconjugacy as well as necessary and sufficient conditions for disconjugacy and disfocality of second-order linear difference equations are obtained. Several sufficient conditions for second-order linear difference equations to be conjugate on $\mathbb{Z}$ are given in Section 1.6. In Sections 1.7–1.11, we present several types of oscillation and nonoscillation results for second-order linear difference equations. Sections 1.12–1.14 are devoted to the study of the oscillation and nonoscillation of second-order linear difference equations by employing weighted average techniques. Section 1.15 contains several results on oscillation and nonoscillation of second-order forced linear difference equations. In Section 1.16, we present some qualitative properties, namely, boundedness, monotonicity, and zero convergent solutions of second-order difference equations. In Section 1.16 we will also provide some necessary and sufficient conditions for the nonoscillation as well as some comparison results. Sufficiency criteria for the nonoscillation are given in Section 1.17. Limit point results are discussed in Section 1.18. In Section 1.19 we will investigate the growth of solutions as well as some oscillation criteria, and in Section 1.20 we will consider mixed-type difference equations and relate them with usual difference equations used in the literature.

Now we list some notation used in this book: Δ is the usual *forward difference operator*, that is,

$$\Delta x(k)=x(k+1)-x(k), \qquad \Delta^2 x(k)=\Delta\big(\Delta x(k)\big), \tag{1.1.6}$$

and ℓ^∞ is the set of real sequences defined on the set of (positive) integers, where any sequence is bounded with respect to the usual supremum norm. Since the difference calculus may not be as familiar as the differential calculus, we recall here the most frequently applied rules:

$$\Delta\big(u(k)+v(k)\big) = \Delta u(k)+\Delta v(k), \qquad \Delta\big(\alpha u(k)\big) = \alpha\Delta u(k) \quad \text{for } \alpha\in\mathbb{R},$$
$$\Delta\big(u(k)v(k)\big) = v(k+1)\Delta u(k)+u(k)\Delta v(k) = v(k)\Delta u(k)+u(k+1)\Delta v(k),$$
$$\Delta\left(\frac{u(k)}{v(k)}\right) = \frac{v(k)\Delta u(k)-u(k)\Delta v(k)}{v(k)v(k+1)},$$
$$\sum\big(u(k)+v(k)\big) = \sum u(k)+\sum v(k), \qquad \sum\big(\beta u(k)\big) = \beta\sum u(k) \quad \text{for } \beta\in\mathbb{R},$$
$$\sum\big(u(k)\Delta v(k)\big) = u(k)v(k)-\sum v(k+1)\Delta u(k),$$
$$\sum_{j=m}^{k-1} u(j) = v(k)-v(m) \tag{1.1.7}$$

for $v(k)$ such that $\Delta v(k) = u(k)$.

1.2. Basic concepts

In this section we will introduce the second-order self-adjoint difference equation, which is the main topic of this chapter. We will show which second-order linear difference equations can be put in self-adjoint form and establish a number of useful identities.

Consider the second-order linear difference equation

$$\Delta\big(c(k-1)\Delta x(k-1)\big)+q(k)x(k) = 0, \quad k\in\mathbb{N}_0 = \{0,1,2,3,\dots\}, \tag{1.2.1}$$

where Δ is the forward difference operator, that is, $\Delta x(k) = x(k+1)-x(k)$, $\Delta^2 x(k) = \Delta(\Delta x(k))$, and $c(k)$, $q(k)$ are real numbers with $c(k)>0$ for all $k\in\mathbb{N}_0$. Equation (1.2.1) can be viewed as a discrete version of the second-order linear self-adjoint differential equation (1.1.1), where $c(t)>0$ in $[a,b]$ and c,q are continuous on $[a,b]$.

First, we will describe the process of discretization of equation (1.1.1). For small $h = (b-a)/n$, $n\in\mathbb{N} = \{1,2,3,\dots\}$,

$$\begin{aligned} x'(t) &\simeq \frac{x(t)-x(t-h)}{h},\\ \big(c(t)x'(t)\big)' &\simeq \frac{1}{h}\left\{\frac{c(t+h)[x(t+h)-x(t)]}{h}-\frac{c(t)[x(t)-x(t-h)]}{h}\right\}, \end{aligned} \tag{1.2.2}$$

thus

$$\left(c(t)x'(t)\right)' \simeq \frac{1}{h^2}\{c(t+h)x(t+h) - [c(t+h)+c(t)]x(t) + c(t)x(t-h)\}. \tag{1.2.3}$$

Let $t = a+kh$, where k is a discrete variable taking on the integer values $0 \le k \le n$. If x is a solution of equation (1.1.1), then we have

$$\begin{aligned} &c(a+(k+1)h)x(a+(k+1)h) - [c(a+(k+1)h) + c(a+kh)]x(a+kh) \\ &\qquad + c(a+kh)x(a+(k-1)h) + h^2q(a+kh)x(a+kh) \simeq 0. \end{aligned} \tag{1.2.4}$$

If we set $c(k-1) = c(a+kh)$, $q(k) = h^2q(a+kh)$ for $1 \le k \le n$ and $1 \le k \le n-1$, respectively, then

$$c(k)x(k+1) - [c(k) + c(k-1)]x(k) + c(k-1)x(k-1) + q(k)x(k) \simeq 0. \tag{1.2.5}$$

Finally, we can write this equation in the form (1.2.1).

The linear second-order self-adjoint difference equation is then defined to be the equation (1.2.1), where we assume that $c(k)$ is defined and positive on the set of integers $[a, b+1] = \{a, a+1, \dots, b+1\}$ and $q(k)$ is defined on the set of integers $[a+1, b+1]$.

We can also write equation (1.2.1) in the form

$$c(k)x(k+1) + p(k)x(k) + c(k-1)x(k-1) = 0, \tag{1.2.6}$$

where

$$p(k) = q(k) - c(k) - c(k-1) \tag{1.2.7}$$

for $k \in [a+1, b+1]$.

Since equation (1.2.1) is in fact a recurrence relation, given real initial values $x(a)$ and $x(a+1)$ for some $a \in \mathbb{N}$, it is clear that we can inductively obtain the values $x(a+2), x(a+3), \dots$. Hereby, the existence and uniqueness of a solution of the initial value problem (1.2.1), $x(m) = A$, $x(m+1) = B$ is guaranteed. Thus, a sequence of real numbers $x = \{x(k)\}_{k=1}^{\infty}$ is called a solution of equation (1.2.1), if equation (1.2.1) is satisfied for $k \in \mathbb{N}_0$.

Next, we will show that any equation of the form

$$u(k)x(k+1)+v(k)x(k)+w(k)x(k-1)=0, \tag{1.2.8}$$

where $u(k)>0$ on $[a,b+1]$ and $w(k)>0$ on $[a+1,b+1]$ can be written in the self-adjoint form of equation (1.2.1). To see this, multiply both sides of equation (1.2.8) by a positive function $g(k)$ (to be chosen later) to obtain

$$u(k)g(k)x(k+1)+v(k)g(k)x(k)+w(k)g(k)x(k-1)=0. \tag{1.2.9}$$

This would be of the form of equation (1.2.6), which we know we can write in self-adjoint form, provided $u(k)g(k)=c(k)$ and $w(k)g(k)=c(k-1)$. Consequently, we select a function $g(k)>0$ so that $u(k)g(k)=w(k+1)g(k+1)$, or

$$g(k+1)=\frac{u(k)}{w(k+1)}g(k) \tag{1.2.10}$$

for $k\in[a,b]$. Then,

$$g(k)=A\prod_{n=a}^{k-1}\frac{u(n)}{w(n+1)}, \tag{1.2.11}$$

where A is a positive constant. If we choose

$$c(k)=Au(k)\prod_{n=a}^{k-1}\frac{u(n)}{w(n+1)} \tag{1.2.12}$$

and take $q(k)=v(k)g(k)+c(k)+c(k-1)$, then we have that equation (1.2.8) is equivalent to equation (1.2.1).

Let $x(k)$ and $y(k)$ be solutions of equation (1.2.1) in $[a,b+2]$. We define the *Casoratian* or *Wronskian* of x and y by

$$W(k)=W[x(k),y(k)]=\det\begin{pmatrix}x(k) & y(k)\\ x(k+1) & y(k+1)\end{pmatrix}=\det\begin{pmatrix}x(k) & y(k)\\ \Delta x(k) & \Delta y(k)\end{pmatrix}. \tag{1.2.13}$$

Define a linear operator L on the set of functions x defined on $[a, b+2]$ by

$$Lx(k) = \Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k) \quad \text{for } k \in [a+1, b+1]. \tag{1.2.14}$$

Theorem 1.2.1 (Lagrange's identity). *If x and y are defined on $[a, b+2]$, then for $k \in [a+1, b+1]$,*

$$y(k)Lx(k) - x(k)Ly(k) = \Delta\{c(k-1)W[y(k-1), x(k-1)]\}. \tag{1.2.15}$$

Proof. We have

$$\begin{aligned} y(k)Lx(k) &= y(k)\Delta[c(k-1)\Delta x(k-1)] + y(k)q(k)x(k) \\ &= \Delta[y(k-1)c(k-1)\Delta x(k-1)] \\ &\quad - (\Delta y(k-1))c(k-1)\Delta x(k-1) + y(k)q(k)x(k) \\ &= \Delta[y(k-1)c(k-1)\Delta x(k-1) - x(k-1)c(k-1)\Delta y(k-1)] \\ &\quad + x(k)\Delta\big(c(k-1)\Delta y(k-1)\big) + x(k)q(k)y(k) \\ &= \Delta\{c(k-1)W[y(k-1), x(k-1)]\} + x(k)Ly(k) \end{aligned} \tag{1.2.16}$$

for $k \in [a+1, b+1]$. □

By summing both sides of Lagrange's identity from $a+1$ to $b+1$, we obtain the following corollary.

Corollary 1.2.2 (Green's theorem). *Assume that x and y are defined on $[a, b+2]$. Then*

$$\sum_{k=a+1}^{b+1} y(k)Lx(k) - \sum_{k=a+1}^{b+1} x(k)Ly(k) = \{c(k)W[y(k), x(k)]\}_a^{b+1}. \tag{1.2.17}$$

Corollary 1.2.3 (Liouville's formula). *If x and y are solutions of equation (1.2.1), then $W[x(k), y(k)] = d/c(k)$ for $k \in [a, b+1]$, where d is a constant.*

Proof. By the Lagrange identity, we have $\Delta\{c(k-1)W[x(k-1), y(k-1)]\} = 0$ for $k \in [a+1, b+1]$. Hence, $c(k-1)W[x(k-1), y(k-1)] \equiv d$ for $k \in [a+1, b+2]$, where d is a constant. Therefore, $W[x(k), y(k)] = d/c(k)$ for $k \in [a, b+1]$. □

It follows from Corollary 1.2.3 that if x and y are solutions of equation (1.2.1), then either $W[x(k), y(k)] = 0$ for all $k \in [a, b+1]$ (i.e., $x(k)$ and $y(k)$ are linearly dependent on $[a, b+2]$) or $W[x(k), y(k)]$ is of one sign (i.e., $x(k)$ and $y(k)$ are linearly independent on $[a, b+2]$).

Theorem 1.2.4 (Pólya's factorization). *Assume y is a solution of equation (1.2.1) with $y(k) > 0$ on $[a, b+2]$. Then there exist functions g and h with $g(k) > 0$ on $[a, b+2]$ and $h(k) > 0$ on $[a+1, b+2]$ such that for any function x defined on $[a, b+2]$, for $k \in [a+1, b+1]$,*

$$Lx(k) = g(k)\Delta\big[h(k)\Delta\big(g(k-1)x(k-1)\big)\big]. \tag{1.2.18}$$

Proof. Since y is a positive solution of equation (1.2.1), we have by the Lagrange identity that

$$Lx(k) = \frac{1}{y(k)}\Delta\{c(k-1)W[y(k-1), x(k-1)]\} \quad \text{for } k \in [a+1, b+1]. \tag{1.2.19}$$

Now we have

$$\begin{aligned}\Delta\left\{\frac{x(k-1)}{y(k-1)}\right\} &= \frac{y(k-1)\Delta x(k-1) - x(k-1)\Delta y(k-1)}{y(k-1)y(k)} \\ &= \frac{W[y(k-1), x(k-1)]}{y(k-1)y(k)}.\end{aligned} \tag{1.2.20}$$

Thus, it follows that

$$Lx(k) = \frac{1}{y(k)}\Delta\left[c(k-1)y(k-1)y(k)\Delta\left(\frac{x(k-1)}{y(k-1)}\right)\right]. \tag{1.2.21}$$

Finally, to complete the proof, in (1.2.21) we let $g(k) = 1/y(k) > 0$ for $k \in [a, b+2]$ and $h(k) = c(k-1)y(k-1)y(k) > 0$ for $k \in [a+1, b+2]$. □

Definition 1.2.5. The *Cauchy function* $X : [a, b+2] \times [a+1, b+1] \to \mathbb{R}$ is defined as the function that for each fixed $m \in [a+1, b+1]$ is the solution of the initial value problem (1.2.1), $X(m, m) = 0$, $X(m+1, m) = 1/c(m)$.

Theorem 1.2.6. *If y and z are linearly independent solutions of (1.2.1), then the Cauchy function for (1.2.1) is given by*

$$X(k, m) = \frac{\det\begin{pmatrix} y(m) & z(m) \\ y(k) & z(k)\end{pmatrix}}{\det\begin{pmatrix} y(m) & z(m) \\ y(m+1) & z(m+1)\end{pmatrix}}, \quad k \in [a, b+2],\ m \in [a+1, b+1]. \tag{1.2.22}$$

Proof. Since y and z are linearly independent solutions of equation (1.2.1), $W[y(k), z(k)] \neq 0$ for $k \in [a, b+1]$. Hence, equation (1.2.22) is well defined. Note that by expanding $X(k, m)$ in (1.2.22) by the second row in the numerator we have that for each fixed $m \in [a+1, b+1]$, $X(k, m)$ is a linear combination of $y(k)$ and $z(k)$ and so it is a solution of equation (1.2.1). Clearly, $X(m, m) = 0$ and $X(m+1, m) = 1/c(m)$. □

The next theorem shows how the Cauchy function is used to solve initial value problems.

Theorem 1.2.7 (variation of constants formula). *The solution of the initial value problem*

$$\begin{gathered} Lx(k) = e(k) \quad \text{for } k \in [a+1, b+1], \\ x(a) = 0, \qquad x(a+1) = 0 \end{gathered} \tag{1.2.23}$$

is given by

$$x(k) = \sum_{m=a+1}^{k} X(k,m)e(m) \quad \text{for } k \in [a, b+2], \tag{1.2.24}$$

where $X(k,m)$ is the Cauchy function for $Lx(k) = 0$. (Here, if $k = b+2$, then the term $X(b+2, b+2)e(b+2)$ is understood to be zero.)

Proof. Let x be given by equation (1.2.24). By convention $x(a) = 0$. Also,

$$\begin{aligned} x(a+1) &= X(a+1, a+1)e(a+1) = 0, \\ x(a+2) &= X(a+2, a+1)e(a+1) + X(a+2, a+2)e(a+2) \\ &= \frac{e(a+1)}{c(a+1)}. \end{aligned} \tag{1.2.25}$$

It follows that x satisfies $Lx(k) = e(k)$ for $k = a+1$. Now, assume $a+2 \le k \le b+1$. Then, we have

$$\begin{aligned} Lx(k) &= c(k-1)x(k-1) + p(k)x(k) + c(k)x(k+1) \\ &= \sum_{m=a+1}^{k-1} c(k-1)X(k-1,m)e(m) + \sum_{m=a+1}^{k} p(k)X(k,m)e(m) \\ &\quad + \sum_{m=a+1}^{k+1} c(k)X(k+1,m)e(m) \\ &= \sum_{m=a+1}^{k-1} LX(k,m)e(m) + p(k)X(k,k)e(k) \\ &\quad + c(k)X(k+1,k)e(k) + c(k)X(k+1,k+1)e(k+1) \\ &= e(k). \end{aligned} \tag{1.2.26}$$

This finishes the proof. □

Corollary 1.2.8. *The solution of the initial value problem*

$$\begin{aligned} Lx(k) &= e(k) \quad \text{for } k \in [a+1, b+1], \\ x(a) &= A, \qquad x(a+1) = B \end{aligned} \tag{1.2.27}$$

is given by

$$x(k) = y(k) + \sum_{m=a+1}^{k} X(k,m)e(m), \tag{1.2.28}$$

where $X(k,m)$ is the Cauchy function for $Lx(k) = 0$, and y is the solution of the initial value problem $Ly(k) = 0$, $y(a) = A$, $y(a+1) = B$.

Proof. Since y is a solution of $Ly(k) = 0$ and $\sum_{m=a+1}^{k} X(k,m)e(m)$ is a solution of $Lx(k) = e(k)$, we have that

$$x(k) = y(k) + \sum_{m=a+1}^{k} X(k,m)e(m) \tag{1.2.29}$$

solves $Lx(k) = e(k)$. Also, $x(a) = y(a) = A$ and $x(a+1) = y(a+1) = B$. □

Next, instead of the zero of a solution, known for the continuous case, it is necessary to use the following more complicated concept in the corresponding discrete case.

First, we present the following simple lemma which shows that there is no nontrivial solution of equation (1.2.1) with $x(m) = 0$ and $x(m-1)x(m+1) \geq 0$, $m > a$. In some sense this lemma says that nontrivial solutions of equation (1.2.1) can have only "simple" zeros.

Lemma 1.2.9. *If x is a nontrivial solution of (1.2.1) with $x(m) = 0$, $a < m < b+2$, then $x(m-1)x(m+1) < 0$.*

Proof. Since x is a solution of (1.2.1) with $x(m) = 0$, $a < m < b+2$, we obtain from (1.2.6), $c(m)x(m+1) = -c(m-1)x(m-1)$. Since $x(m+1), x(m-1) \neq 0$ and $c(k) > 0$, it follows that $x(m-1)x(m+1) < 0$. □

Because of Lemma 1.2.9 we can define a generalized zero of a solution of equation (1.2.1) as follows

Definition 1.2.10. A solution x of (1.2.1) is said to have a *generalized zero* at m provided $x(m) = 0$ if $m = a$, and if $m > a$ either $x(m) = 0$ or $x(m-1)x(m) < 0$.

Definition 1.2.11. Equation (1.2.1) is called *disconjugate* on $[a, b+1]$ provided no nontrivial solution of equation (1.2.1) has two or more generalized zeros on $[a, b+2]$. Otherwise, equation (1.2.1) is said to be conjugate on $[a, b+2]$.

Of course, in any interval $[a, b+2]$ there is a nontrivial solution with at least one generalized zero.

Example 1.2.12. The difference equation

$$x(k+1) + 2x(k) + 2x(k-1) = 0, \tag{1.2.30}$$

which can be put in self-adjoint form, has

$$x(k) = 2^{k/2}\sin\left(\frac{3\pi k}{4}\right), \qquad y(k) = 2^{k/2}\cos\left(\frac{3\pi k}{4}\right) \tag{1.2.31}$$

as linearly independent solutions. Note that both of these solutions have a generalized zero at $k = 2$.

Example 1.2.13. The difference equation

$$x(k+1) - \sqrt{3}x(k) + x(k-1) = 0 \tag{1.2.32}$$

is disconjugate on any interval of length less than six. This follows from the fact that any solution of this equation is of the form $x(k) = A\sin((\pi k/6) + B)$, where A and B are appropriate constants.

We also see that the difference equation

$$x(k+2) - 7x(k+1) + 12x(k) = 0 \tag{1.2.33}$$

is disconjugate on any interval.

Next, we introduce a discrete quadratic functional.

Definition 1.2.14. (i) Define a class $U = U(a,b)$ of so-called admissible sequences by

$$U(a,b) = \{\xi : [a, b+2] \longrightarrow \mathbb{R} : \xi(a) = \xi(b+1) = 0\}. \tag{1.2.34}$$

(ii) Define the *quadratic functional* $\mathcal{F}$ on $U(a,b)$ by

$$\mathcal{F}(\xi; a, b) = \sum_{k=a}^{b}\left[c(k)\left|\Delta\xi(k)\right|^2 - q(k)\left|\xi(k+1)\right|^2\right]. \tag{1.2.35}$$

(iii) Say that $\mathcal{F}$ is *positive definite* on U provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U(a,b)$, and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

The concept of oscillation and nonoscillation of equation (1.2.1) is defined in the following way.

Definition 1.2.15. Equation (1.2.1) is said to be *nonoscillatory* if there exists $m \in \mathbb{N}$ such that this equation is disconjugate on $[m, n]$ for every $n > m$. In the opposite case, equation (1.2.1) is said to be *oscillatory*.

Oscillation of equation (1.2.1) may be equivalently defined as follows. A nontrivial solution of equation (1.2.1) is called oscillatory if it has infinitely many generalized zeros. In view of the fact that the Sturm-type separation theorem holds (see Theorem 1.4.4 below), we have the following equivalence: any solution of equation (1.2.1) is oscillatory if and only if every solution of equation (1.2.1) is oscillatory. Hence, we can speak about oscillation and nonoscillation of equation (1.2.1).

1.3. Riccati-type transformations

One of the approaches to oscillation and nonoscillation for equation (1.2.1) will be based largely on a discrete version of the *Riccati equation*. If x is a solution of equation (1.2.1) with $x(k)x(k+1) > 0$ for $k \geq m \geq a$, we let

$$w(k) = \frac{c(k-1)\Delta x(k-1)}{x(k-1)} = c(k-1)\left[\frac{x(k)}{x(k-1)} - 1\right] \quad \text{for } k \geq a+1. \tag{1.3.1}$$

Then since

$$w(k) + c(k-1) = c(k-1)\left[\frac{x(k)}{x(k-1)}\right] > 0, \tag{1.3.2}$$

we have

$$\begin{aligned}\Delta w(k) &= \frac{1}{x(k)}\Delta\left[c(k-1)\Delta x(k-1)\right] + c(k-1)\Delta x(k-1)\Delta\left[\frac{1}{x(k-1)}\right] \\ &= -q(k) + w(k)x(k-1)\left[\frac{1}{x(k)} - \frac{1}{x(k-1)}\right] \\ &= -q(k) + w(k)\left[\frac{x(k-1)}{x(k)} - 1\right] \\ &= -q(k) + w(k)\left[\frac{c(k-1)}{w(k)+c(k-1)} - 1\right] \\ &= -q(k) - \frac{w^2(k)}{w(k)+c(k-1)}.\end{aligned} \tag{1.3.3}$$

Hence,

$$\mathcal{R}\left[w(k)\right] = \Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} + q(k) = 0 \quad \text{for } k \geq a+1. \tag{1.3.4}$$

Similarly, we have

$$\mathcal{R}\left[w(k)\right] = \Delta w(k) + \frac{w(k)}{c(k-1)}\left[w(k+1) + q(k)\right] + q(k) = 0. \tag{1.3.5}$$

It should be noted that there are several other discrete versions of the Riccati equation corresponding to equation (1.2.6). Of particular interest, we consider equation (1.2.6) with $b(k) = -p(k)$, that is,

$$c(k)x(k+1) + c(k-1)x(k-1) = b(k)x(k) \quad \text{for } k \geq a+1 \geq 1, \tag{1.3.6}$$

with $c(k) > 0$, $k \geq a$. In fact, equation (1.3.6) is equivalent to the self-adjoint equation

$$-\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k) = 0 \quad \text{for } k \geq a+1, \tag{1.3.7}$$

where $q(k) = b(k) - c(k) - c(k-1)$, that is, $q(k)$ in equation (1.2.1) is replaced by $-q(k)$.

Next, we present the following properties for the solutions of equation (1.3.7).

Lemma 1.3.1. *If equation (1.3.7) is such that any nontrivial solution x can have at most one value $x(\ell) = 0$ with $\ell > a+1$, then any two values $x(m), x(n)$ with $m \neq n$, uniquely determine the solution x.*

Proof. Let u and v be the solutions defined by $u(a) = 0$, $u(a+1) = 1$ and $v(a) = 1$, $v(a+1) = 0$. Then u and v are linearly independent. Consider the system of equations

$$\begin{aligned} x(m) &= \alpha_1 u(m) + \alpha_2 v(m), \\ x(n) &= \alpha_1 u(n) + \alpha_2 v(n). \end{aligned} \tag{1.3.8}$$

If this system does not have a unique solution in terms of α_1 and α_2, then there exist values of α_1 and α_2, not both zero, such that

$$0 = \alpha_1 u(m) + \alpha_2 v(m) = \alpha_1 u(n) + \alpha_2 v(n). \tag{1.3.9}$$

However, this implies that the solution $\alpha_1 u(k) + \alpha_2 v(k)$ assumes the value 0 twice, which is a contradiction. □

Lemma 1.3.2. *Suppose $|b(k)| \geq c(k-1) + c(k)$. If v is a solution of (1.3.6) such that $|v(N+1)| \geq |v(N)|$ for some integer $N \geq a+1$, then $|v(k+1)| \geq |v(k)|$ for all $k \geq N$. If there exists a sequence $\{\eta(k)\}$ of nonnegative numbers such that*

$$|b(k)| \geq [1+\eta(k)]c(k) + c(k-1), \quad \sum_k \eta(k) = \infty, \tag{1.3.10}$$

then $|v(k)| \to \infty$ as $k \to \infty$.

Proof. Use induction and assume that it is true for some $k \geq N+1$, that is, $|v(k)| \geq |v(k-1)|$. Then for all $k \geq N$,

$$\begin{aligned} |v(k+1)| &= \frac{1}{c(k)} |b(k)v(k) - c(k-1)v(k-1)| \\ &\geq \frac{1}{c(k)} (|b(k)| - c(k-1)) |v(k)|. \end{aligned} \tag{1.3.11}$$

If in addition we have (1.3.10), then (1.3.11) becomes $|v(k+1)| \geq [1+\eta(k)]|v(k)|$. However, assuming $k \geq N$, the above inequality implies that

$$v(k+1) \geq v(N) \prod_j [1+\eta(j)] \quad \text{for } N \leq j \leq k, \tag{1.3.12}$$

which means $|v(k)| \to \infty$ as $k \to \infty$. This completes the proof. □

Remark 1.3.3. (i) We always assume $c(k) > 0$, $k \geq a \geq 0$. If we have in addition $b(k) \geq c(k) + c(k-1)$, then the solution v of (1.3.6) satisfying $v(a) = 1$ and $v(a+1) = 1$ must have $v(k+1) \geq v(k) \geq 1$. Clearly, v is nonoscillatory, so that the condition $b(k) - c(k) - c(k-1) \geq 0$ is sufficient for nonoscillation of equation (1.3.6).

(ii) Lemma 1.3.2 implies that the hypothesis of Lemma 1.3.1 is satisfied if $|b(k)| \geq c(k) + c(k-1)$.

(iii) Lemma 1.3.2 is sharp in case of constant coefficients. Assume $b(k) \equiv b$ and $c(k) \equiv 1$ for all k. By solving such a difference equation with constant coefficients, we see that all solutions of equation (1.3.6) are bounded if and only if $|b| < 2$.

The following result is elementary but useful.

Lemma 1.3.4. *If there exists a subsequence $b(k_j) \leq 0$, where $k_j \to \infty$ as $j \to \infty$, then equation (1.3.6) is oscillatory.*

Proof. Suppose not. Then, we may assume the existence of a solution x such that $x(k) > 0$ for all sufficiently large k. However, the left-hand side of equation (1.3.6) will always be positive, while the right-hand side will be nonpositive for all values of k_j, which is a contradiction. □

Because of Lemma 1.3.4, in addition to the assumption that $c(k) > 0$ for all $k \geq a \geq 0$, we assume the following. In equation (1.3.6), $b(k) > 0$ for $k \geq a+1$. It would suffice to assume $b(k) > 0$ for all sufficiently large k, but we assume the condition for all k, for simplicity in the results we will present.

Suppose that $x(k)$, $k \geq a \geq 0$, is a solution of equation (1.3.6) such that $x(k) \neq 0$ for $k \geq N$ for some $N \geq a$. The substitution $r(k) = x(k+1)/x(k)$, $k \geq N$, leads to the nonlinear difference equation

$$c(k)r(k) + \frac{c(k-1)}{r(k-1)} = b(k) \quad \text{for } n > N. \tag{1.3.13}$$

Similarly, if we let $z(k) = c(k)x(k+1)/x(k)$ for $n \geq N$, then z satisfies

$$z(k) + \frac{c^2(k-1)}{z(k-1)} = b(k) \quad \text{for } n > N. \tag{1.3.14}$$

If we let $s(k) = b(k+1)x(k+1)/(c(k)x(k))$ for $n \geq N$, then s satisfies

$$g(k)s(k) + \frac{1}{s(k-1)} = 1 \quad \text{for } n > N, \tag{1.3.15}$$

where $g(k) = c^2(k)/(b(k)b(k+1))$.

We note that equation (1.3.14) may be written in the alternative form

$$\Delta z(k-1) + \frac{1}{b(k)} z(k)z(k-1) - z(k) + \frac{c^2(k-1)}{b(k)} = 0 \quad \text{for } n > N. \tag{1.3.16}$$

The transformation $z(k) = c(k)x(k-1)/x(k)$, which leads to equation (1.3.14), is perhaps the nearest analogue for difference equations to the classical Riccati transformation $z(t) = c(t)x'(t)/x(t)$, which transforms the self-adjoint differential equation (1.1.1) into the Riccati equation

$$z'(t) + \frac{1}{c(t)} z^2(t) + q(t) = 0. \tag{1.3.17}$$

Finally, we prove the following theorem.

Theorem 1.3.5. *The following conditions are equivalent.*

(i) *Equation (1.3.6) is nonoscillatory.*
(ii) *Equation (1.3.13) has a positive solution $r(k)$, $k \geq N$ for some $N \geq a > 0$.*
(iii) *Equation (1.3.14) has a positive solution $z(k)$, $k \geq N$ for some $N \geq a > 0$.*
(iv) *Equation (1.3.15) has a positive solution $s(k)$, $k \geq N$ for some $N \geq a > 0$.*

Proof. If equation (1.3.6) is nonoscillatory and $x(k)$, $k \geq 0$ is any solution of equation (1.3.6), then there exists $N \geq a \geq 0$ such that $x(k)x(k-1) > 0$ for all $k \geq N$. The necessity conditions (ii), (iii), and (iv) then follow immediately from the transformation which leads to equations (1.3.13), (1.3.14), and (1.3.15).

Conversely, if $r(k)$, $k \geq N$ is a positive solution of equation (1.3.13), then we may let $x(N) = 1$ and $x(k-1) = r(k)x(k)$ for all $n \geq N$. This defines a positive solution of equation (1.3.6) for $n \geq N$. Given $x(N)$ and $x(N+1)$, the terms $x(N-1), x(N-2), \ldots, x(0)$ may be constructed directly from equation (1.3.6) to give a nonoscillatory solution of equation (1.3.6) for $n \geq 0$. Similar arguments hold for equations (1.3.14) and (1.3.15), which completes the proof. $\square$

1.4. Reid's roundabout theorem and Sturmian theory

We will consider second-order difference operators of the form

$$L_1x(k) = \Delta\big(c_1(k)\Delta x(k)\big) + q_1(k)x(k+1), \tag{1.4.1}$$

$$L_2y(k) = \Delta\big(c_2(k)\Delta y(k)\big) + q_2(k)y(k+1), \tag{1.4.2}$$

where $k \in [a,b]$ with $a,b \in \mathbb{Z}$, $a \leq b$, and $q_i(k)$, $i \in \{1,2\}$, are real-valued sequences defined on $[a,b]$. The sequences $c_i(k)$, $i \in \{1,2\}$, are positive real valued and defined on $[a,b+1]$.

1.4.1. The disconjugacy characterization theorem

Now, we will formulate the discrete version of the so-called Picone identity [235]. Also, we will state some important results, for example, the disconjugacy characterization theorem, the Sturm-type separation theorem, and the Sturm-type comparison theorem. The proofs of these results are postponed to Chapters 2 and 3, where more general results for systems of difference equations and for half-linear difference equations (which include second-order linear difference equations as a special case) are established.

Lemma 1.4.1 (Picone's identity). *Let x and y be defined on $[a,b+2]$ and assume $y(k) \neq 0$ for $k \in [a,b+1]$. Then for $k \in [a,b]$, the following equality holds:*

$$\begin{aligned}
\Delta&\left\{\frac{x(k)}{z(k)}\big[y(k)c_1(k)\Delta x(k) - x(k)c_2(k)\Delta y(k)\big]\right\}\\
&= \big(q_2(k)-q_1(k)\big)x^2(k+1) + \big(c_1(k)-c_2(k)\big)\big(\Delta x(k)\big)^2\\
&\quad + \frac{x(k+1)}{y(k+1)}\big\{y(k+1)L_1x(k) - x(k+1)L_2y(k)\big\}\\
&\quad + \left\{c_2(k)\big(\Delta x(k)\big)^2 - \frac{c_2(k)\Delta y(k)}{y(k+1)}x^2(k+1) + \frac{c_2(k)\Delta y(k)}{y(k)}x^2(k)\right\}.
\end{aligned} \tag{1.4.3}$$

Next, we consider equation (1.2.1), which we may write in the form

$$\Delta\big(c(k)\Delta x(k)\big) + q(k)x(k+1) = 0 \quad \text{for } k \in [a,b], \tag{1.4.4}$$

where $c(k)$ and $q(k)$ are as in equation (1.2.1).

Now, we state the following disconjugacy characterization theorem, also known as Reid's roundabout theorem.

Theorem 1.4.2 (Reid's roundabout theorem). *All of the following statements are equivalent.*

(i) *Equation (1.4.4) is disconjugate on* $[a, b]$.

(ii) *Equation (1.4.4) has a solution* $x(k)$ *without generalized zeros in the interval* $[a, b + 1]$.

(iii) *The Riccati difference equation associated with (1.4.4), namely,*

$$\mathcal{R}[w(k)] = \Delta w(k) + \frac{w^2(k)}{w(k) + c(k)} + q(k) = 0, \tag{1.4.5}$$

where $w(k) = c(k)\Delta x(k)/x(k)$, *has a solution* $w(k)$ *on* $[a, b]$ *satisfying* $c(k) + w(k) > 0$ *on* $[a, b]$.

(iv) *The functional* $\mathcal{F}$ *defined by (1.2.35) is positive definite on* $U(a, b)$.

1.4.2. Sturmian theory

This subsection is devoted to Sturmian theory. Consider two equations

$$L_1 x(k) = 0, \qquad L_2 y(k) = 0, \tag{1.4.6}$$

where the operators L_1 and L_2 are defined by (1.4.1) and (1.4.2), respectively.

Now, we state the following Sturmian theorems.

Theorem 1.4.3 (Sturm's comparison theorem). *Suppose that* $c_2(k) \geq c_1(k)$ *and* $q_1(k) \geq q_2(k)$ *for* $k \in [a, b]$. *If* $L_1 x(k) = 0$ *is disconjugate on* $[a, b]$, *then* $L_2 y(k) = 0$ *is also disconjugate on* $[a, b]$.

Theorem 1.4.4 (Sturm's separation theorem). *Two linearly independent solutions of equation (1.2.1) cannot have a common zero. If a nontrivial solution of equation (1.2.1) has a zero at* t_1 *and a generalized zero at* $t_2 > t_1$, *then any second linearly independent solution has a generalized zero in* $(t_1, t_2]$. *If a nontrivial solution of equation (1.2.1) has a generalized zero at* t_1 *and a generalized zero at* $t_2 > t_1$, *then any second linearly independent solution has a generalized zero in* $[t_1, t_2]$.

The next example shows that, under the definition of a generalized zero in the sense of Definition 1.2.10, the Sturm separation theorem does not hold for all second-order linear homogeneous difference equations.

Example 1.4.5. Consider the *Fibonacci difference equation*

$$x(k + 1) - x(k) - x(k - 1) = 0. \tag{1.4.7}$$

The characteristic equation is $\lambda^2 - \lambda - 1 = 0$. Hence the characteristic values are $(1 \pm \sqrt{5})/2$. Take $x(k) = [(1 - \sqrt{5})/2]^k$ and $y(k) = [(1 + \sqrt{5})/2]^k$. Note that $x(k)$ has a generalized zero at every integer while $y(k) > 0$. Thus, the conclusions of

Theorem 1.4.4 do not hold for equation (1.4.7), at least not with our current definition of generalized zeros as given in Definition 1.2.10. This of course does not contradict Theorem 1.4.4 because the self-adjoint form of equation (1.4.7) is

$$\Delta\big((-1)^{k-1}\Delta x(k-1)\big) + (-1)^{k-1}x(k) = 0, \tag{1.4.8}$$

where $c(k) = (-1)^k \ngtr 0$ for all $k \geq a \geq 0$.

However, in Chapters 2 and 3 we extend the definition of generalized zeros to equations of the form (1.2.1) with $c(k) \neq 0$ for all $k \in \mathbb{Z}$, that is, c will be allowed to change sign. With such an extended definition, a situation as in the case of the Fibonacci sequence may not occur. Note that c in (1.4.8) is in fact changing sign.

Remark 1.4.6. In Theorem 1.4.4 it was noted that two linearly independent solutions cannot have a common zero. In fact, Example 1.2.13 shows that this is not true for generalized zeros as defined in Definition 1.2.10.

1.5. Disconjugacy and disfocality

In this section we will show that disconjugacy is important in obtaining comparison results for solutions of initial value problems and an existence and uniqueness result for solutions of boundary value problems. Also, we will establish some necessary and sufficient conditions for disconjugacy and disfocality of second-order difference equations.

Using Theorem 1.2.7, we can prove the following comparison theorem.

Theorem 1.5.1. *Assume that*

$$Lx(k) = \Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k) = 0 \tag{1.5.1}$$

is disconjugate on $[a, b+2]$ *and that* u, v *satisfy* $Lu(k) \geq Lv(k)$ *for* $k \in [a+1, b+1]$, $u(a) = v(a)$, *and* $u(a+1) = v(a+1)$. *Then* $u(k) \geq v(k)$ *on* $[a, b+2]$.

Proof. Set $w(k) = u(k) - v(k)$. Then $h(k) = Lw(k) = Lu(k) - Lv(k) \geq 0$ for $k \in [a+1, b+1]$. Hence w solves the initial value problem $Lw(k) = h(k)$, $w(a) = 0$, $w(a+1) = 0$. By the variation of constants formula, that is, Theorem 1.2.7, we have

$$w(k) = \sum_{m=a+1}^{k} X(k, m)h(m), \tag{1.5.2}$$

where $X(k, m)$ is the Cauchy function for equation (1.5.1). Since equation (1.5.1) is disconjugate and $X(m, m) = 0$, $X(m+1, m) = 1/c(m) > 0$, we have $X(k, m) > 0$ for $m+1 \leq k \leq b+2$. It follows that $w(k) \geq 0$ on $[a, b+2]$, which gives us the desired assertion. □

Consider the boundary value problem

$$\begin{gathered} \Delta^2 x(k-1) + 2x(k) = 0, \\ x(0) = A, \qquad x(2) = B, \end{gathered} \tag{1.5.3}$$

where A and B are real numbers. If $A = B = 0$, then this boundary value problem has infinitely many solutions. If $A = 0$ and $B \neq 0$, then it has no solution. We show in the following theorem that with the assumption of disconjugacy this type of boundary value problem has a unique solution.

Theorem 1.5.2. *If equation (1.5.1) is disconjugate on $[a, b+2]$, then the boundary value problem*

$$\begin{gathered} Lx(k) = h(k), \\ x(m_1) = A, \qquad x(m_2) = B, \end{gathered} \tag{1.5.4}$$

where $a \leq m_1 < m_2 \leq b+2$ and A, B are constants, has a unique solution.

Proof. Let x_1 and x_2 be linearly independent solutions of equation (1.5.1) and x_p be a particular solution of equation (1.5.4). Then a general solution of equation (1.5.4) is $x(k) = \alpha_1 x_1(k) + \alpha_2 x_2(k) + x_p(k)$, where α_i, $i \in \{1, 2\}$, are arbitrary constants. Now, the boundary conditions lead to the system of equations

$$\begin{aligned} \alpha_1 x_1(m_1) + \alpha_2 x_2(m_1) &= A - x_p(m_1), \\ \alpha_1 x_1(m_2) + \alpha_2 x_2(m_2) &= B - x_p(m_2). \end{aligned} \tag{1.5.5}$$

This system has a unique solution if and only if

$$D = \det \begin{pmatrix} x_1(m_1) & x_2(m_1) \\ x_1(m_2) & x_2(m_2) \end{pmatrix} \neq 0. \tag{1.5.6}$$

If $D = 0$, then there are constants β_i, $i \in \{1, 2\}$, not both zero such that the nontrivial solution $x(k) = \beta_1 x_1(k) + \beta_2 x_2(k)$ satisfies $x(m_1) = x(m_2) = 0$. This contradicts the disconjugacy of equation (1.5.1) on $[a, b+2]$ and the proof is complete. □

The following results characterize the disconjugacy on a certain interval.

Theorem 1.5.3. *Equation (1.5.1) is disconjugate on $[a, b+2]$ if and only if there is a positive solution of equation (1.5.1) on $[a, b+2]$.*

Proof. Assume that equation (1.5.1) is disconjugate on $[a, b+2]$. Let u and v be solutions of equation (1.5.1) satisfying $u(a) = 0$, $u(a+1) = 1$ and $v(b+1) = 1$, $v(b+2) = 0$. By disconjugacy, $u(k) > 0$ on $[a+1, b+2]$ and $v(k) > 0$ on $[a, b+1]$. It follows that $x = u + v$ is a positive solution of equation (1.5.1).

Conversely, assume that equation (1.5.1) has a positive solution on $[a, b+2]$. It follows from the Sturm separation theorem that no nontrivial solution has two generalized zeros in $[a, b+2]$. □

Corollary 1.5.4. *Equation (1.5.1) is disconjugate on* $[a, b+2]$ *if and only if it has a Pólya factorization on* $[a, b+2]$.

Proof. If equation (1.5.1) is disconjugate, then by Theorem 1.5.3 it has a positive solution. By Theorem 1.2.4, equation (1.5.1) has a Pólya factorization. Conversely, assume that equation (1.5.1) has the Pólya factorization

$$g(k)\Delta\left[h(k)\Delta(h(k-1))x(k-1)\right] = 0, \tag{1.5.7}$$

where $g(k) > 0$ on $[a, b+2]$ and $h(k) > 0$ on $[a+1, b+2]$. Thus $x = 1/g$ is a positive solution. By Theorem 1.5.3, equation (1.5.1) is disconjugate on $[a, b+2]$. □

Now, we denote the determinant of the $m \times m$-tridiagonal matrix

$$\begin{pmatrix} p(k) & c(k) & 0 & 0 & 0 & \cdots & 0 \\ c(k) & p(k+1) & c(k+1) & 0 & 0 & \cdots & 0 \\ 0 & c(k+1) & p(k+2) & c(k+2) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & c(k+m-3) & p(k+m-2) & c(k+m-2) \\ 0 & 0 & \cdots & \cdots & 0 & c(k+m-2) & p(k+m-1) \end{pmatrix} \tag{1.5.8}$$

by $M_m(k)$, $a+1 \le k \le b+1$, $1 \le m \le b-k+2$, where $p(k)$ is given by equation (1.2.7).

Theorem 1.5.5. *If*

$$(-1)^m M_m(a+1) > 0 \quad \textit{for } 1 \le m \le b-a+1, \tag{1.5.9}$$

then equation (1.5.1) is disconjugate on $[a, b+2]$.

Proof. Let y be the solution of (1.5.1) satisfying $y(a) = 0$ and $y(a+1) = 1$. By Theorem 1.4.4 it suffices to show that $y(k) > 0$ on $[a+1, b+2]$. We will show that $y(a+m) > 0$ for $1 \le m \le b-a+2$ by induction on m. For $m = 1$, we have $y(a+1) = 1 > 0$. Assume $1 \le m \le b-a+2$ and that $y(a+m-1) > 0$. Using $Ly(k) = 0$, $a+1 \le k \le a+m-1$, and $y(a) = 0$, we get $m-1$ equations

$$\begin{gathered} p(a+1)y(a+1) + c(a+1)y(a+2) = 0, \\ c(a+1)y(a+1) + p(a+2)y(a+2) + c(a+2)y(a+3) = 0, \\ \vdots \\ c(a+m-3)y(a+m-3)+p(a+m-2)y(a+m-2)+c(a+m-2)y(a+m-1)=0, \\ c(a+m-2)y(a+m-2)+p(a+m-1)y(a+m-1)+c(a+m-1)y(a+m)=0. \end{gathered} \tag{1.5.10}$$

By Cramer's rule (here $M_0(a+1) = 1$),

$$y(a+m-1) = -c(a+m-1)y(a+m)\left(\frac{M_{m-2}(a+1)}{M_{m-1}(a+1)}\right). \tag{1.5.11}$$

It follows that $y(a+m) > 0$, so by induction $y(k) > 0$ in $[a, b+2]$. Hence, equation (1.5.1) is disconjugate on $[a, b+2]$. □

Next, we have the following disconjugacy characterization.

Theorem 1.5.6. *Equation (1.5.1) is disconjugate on* $[a, b+2]$ *if and only if the coefficients of equation (1.5.1) satisfy*

$$(-1)^m M_m(k) > 0 \tag{1.5.12}$$

for $a+1 \le k \le b+1$, $1 \le m \le b-k+2$.

Proof. Assume that equation (1.5.1) is disconjugate on $[a, b+2]$. We will show that (1.5.12) holds for $1 \le m \le b-a+1$, $a+1 \le k \le b-m+2$ by induction on m. For $m = 1$, we now show that

$$-M_1(k) = -p(k) > 0 \quad \text{for } a+1 \le k \le b+1. \tag{1.5.13}$$

To this end fix $k_0 \in [a+1, b+1]$ and let x be the solution of equation (1.5.1) satisfying $x(k_0-1) = 0$ and $x(k_0) = 1$. Since $Lx(k_0) = 0$, we have from equation (1.2.6) that $c(k_0)x(k_0+1) + p(k_0)x(k_0) = 0$, that is, $p(k_0) = -c(k_0)x(k_0+1)$. By the disconjugacy, $x(k_0+1) > 0$, so $p(k_0) < 0$. Since $k_0 \in [a+1, b+1]$ is arbitrary, $p(k) < 0$ for $a+1 \le k \le b+1$, and the first step of the induction is complete. Now assume $1 \le m \le b-a+1$ and

$$(-1)^{m-1} M_{m-1}(k) > 0 \tag{1.5.14}$$

for $a+1 \le k \le b-m+3$. We will use this induction hypothesis to show that (1.5.12) holds. Fix $k_1 \in [a+1, b-m+2]$ and let y be the solution of equation (1.5.1) satisfying $y(k_1-1) = 0$ and $y(k_1+m) = 1$. Using these boundary conditions and equation (1.5.1) for $k_1 \le k \le k_1+m-1$, we arrive at the equations

$$\begin{gathered}
p(k_1)y(k_1) + c(k_1)y(k_1+1) = 0, \\
c(k_1)y(k_1) + p(k_1+1)y(k_1+1) + c(k_1+1)y(k_1+2) = 0, \\
\vdots \\
(cy)(k_1+m-3) + (py)(k_1+m-2) + c(k_1+m-2)y(k_1+m-1) = 0, \\
c(k_1+m-2)y(k_1+m-2) + p(k_1+m-1)y(k_1+m-1) = -c(k_1+m-1).
\end{gathered} \tag{1.5.15}$$

Note that the determinant of the coefficients is $M_m(k_1)$. Now, one can easily see that $M_m(k_1) \neq 0$, and by Cramer's rule we have

$$y(k_1 + m - 1) = -c(k_1 + m - 1)\left(\frac{M_{m-1}(k_1)}{M_m(k_1)}\right). \tag{1.5.16}$$

By disconjugacy, $y(k_1+m-1) > 0$, so by using (1.5.14), we have $(-1)^m M_m(k_1) > 0$. Since $k_1 \in [a+1, b-m+2]$ is arbitrary, (1.5.12) holds for $k \in [a+1, b-m+2]$. The converse statement is a special case of Theorem 1.5.5, and hence its proof is omitted. □

To illustrate Theorem 1.5.6, we give the following example.

Example 1.5.7. Consider the difference equation

$$\Delta^2 x(k-1) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{1.5.17}$$

By Theorem 1.5.6, equation (1.5.17) is disconjugate on $[0, \infty)$ (i.e., no nontrivial solution has two generalized zeros on $[0, \infty)$) if and only if $(-1)^m M_m(k) > 0$ for $k, m \in \mathbb{N}$. Therefore, it suffices to show that $(-1)^m M_m(1) > 0$ for $m \in \mathbb{N}$. Here $c(k) \equiv 1$ and $p(k) \equiv -2$. Thus, $M_1(1) = -2$ and

$$M_2(1) = \det\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = 3. \tag{1.5.18}$$

Expanding $M_{m+2}(1)$ along the first row, we find

$$M_{m+2}(1) = -2M_{m+1}(1) - M_m(1). \tag{1.5.19}$$

By solving the initial value problem

$$\begin{gathered} M_{m+2}(1) + 2M_{m+1}(1) + M_m(1) = 0, \\ M_1(1) = -2, \qquad M_2(1) = 3, \end{gathered} \tag{1.5.20}$$

we find $M_m(1) = (-1)^m(m+1)$. Thus, $(-1)^m M_m(1) = m+1 > 0$ for $m \in \mathbb{N}$, and so equation (1.5.17) is disconjugate on $[0, \infty)$.

Theorem 1.5.8. *If equation (1.5.1) is disconjugate on* $[a, b+2]$*, then there are solutions* u *and* v *such that* $u(k) > 0$, $v(k) > 0$ *on* $[a, b+2]$ *and*

$$\det\begin{pmatrix} u(k_1) & v(k_1) \\ u(k_2) & v(k_2) \end{pmatrix} > 0 \tag{1.5.21}$$

whenever $a \leq k_1 < k_2 \leq b+2$.

Proof. By disconjugacy we have from Theorem 1.5.3 that there is a positive solution u on $[a, b+2]$. Let x be a solution of equation (1.5.1) such that u and x are linearly independent. By Liouville's formula, the Casoratian $W[u(k), x(k)]$ is of one sign on $[a, b+1]$. If necessary, we can replace $x(k)$ by $-x(k)$, so we can assume that $W[u(k), x(k)] > 0$ on $[a, b+1]$. Pick $A > 0$ sufficiently large so that $v(k) = x(k) + Au(k) > 0$ on $[a, b+2]$. Note that

$$W[u(k), v(k)] = W[u(k), x(k)] > 0 \quad \text{on } [a, b+1]. \tag{1.5.22}$$

We will show that (1.5.21) holds. To see this, fix $k_1 \in [a, b+1]$. We will show by induction on m that

$$\det \begin{pmatrix} u(k_1) & v(k_1) \\ u(k_1+m) & v(k_1+m) \end{pmatrix} > 0 \tag{1.5.23}$$

for $1 \le m \le b - k_1 + 2$. For $m = 1$ this is true because of (1.5.22). Now, assume $1 < m < b - k_1 + 2$ and

$$\det \begin{pmatrix} u(k_1) & v(k_1) \\ u(k_1+m-1) & v(k_1+m-1) \end{pmatrix} > 0. \tag{1.5.24}$$

The boundary value problem $Ly(k) = 0$, $y(k_1) = 0$, $y(k_1+m-1) = 1$ has a unique solution y by Theorem 1.5.2. Since $y(k)$ is a linear combination of $u(k)$ and $v(k)$,

$$\det \begin{pmatrix} y(k_1) & u(k_1) & v(k_1) \\ y(k_1+m-1) & u(k_1+m-1) & v(k_1+m-1) \\ y(k_1+m) & u(k_1+m) & v(k_1+m) \end{pmatrix} = 0. \tag{1.5.25}$$

Expanding the first column, we get

$$\det \begin{pmatrix} u(k_1) & v(k_1) \\ u(k_1+m) & v(k_1+m) \end{pmatrix} = y(k_1+m) \det \begin{pmatrix} u(k_1) & v(k_1) \\ u(k_1+m-1) & v(k_1+m-1) \end{pmatrix}. \tag{1.5.26}$$

By disconjugacy, $y(k_1+m) > 0$, so

$$\det \begin{pmatrix} u(k_1) & v(k_1) \\ u(k_1+m) & v(k_1+m) \end{pmatrix} > 0. \tag{1.5.27}$$

It follows that (1.5.21) holds for $a \le k_1 < k_2 \le b+2$. □

1.5.1. Dominant and recessive solutions

Definition 1.5.9. If equation (1.5.1) is disconjugate on $[a,\infty)$, then a solution x is said to be *dominant* or *nonprincipal* if

$$\sum_{n=N}^{\infty}\frac{1}{c(n)x(n)x(n+1)}<\infty \quad \text{for some } N\geq a, \tag{1.5.28}$$

and is said to be *recessive* or *principal* if

$$\sum_{n=N}^{\infty}\frac{1}{c(n)x(n)x(n+1)}=\infty \quad \text{for some } N\geq a. \tag{1.5.29}$$

Theorem 1.5.10. *Assume that equation (1.5.1) is disconjugate on $[a,\infty)$. Then the following hold.*

(i) *There exist dominant solutions.*
(ii) *There exists a recessive solution which is essentially unique (up to a constant multiple).*
(iii) *If x_1 is a recessive solution and x_2 is a dominant solution, then*

$$\lim_{k\to\infty}\frac{x_1(k)}{x_2(k)}=0. \tag{1.5.30}$$

Also, if $w_i(k)=c(k)\Delta x_i(k)/x_i(k)$, $i\in\{1,2\}$, then $w_1(k)\leq w_2(k)$ for all sufficiently large k.

Proof. Let u and v be linearly independent solutions of equation (1.5.1). Since equation (1.5.1) is disconjugate on $[a,\infty)$, there is an integer $N\geq a$ so that $v(k)$ is of one sign for $k\geq N$. Now, for $k\geq N$ consider

$$\Delta\left(\frac{u(k)}{v(k)}\right)=\frac{W[v(k),u(k)]}{v(k)v(k+1)}=\frac{C}{c(k)v(k)v(k+1)}, \tag{1.5.31}$$

where C is a constant, by Liouville's formula. It follows that $u(k)/v(k)$ is either increasing or decreasing for $k\geq N$. Let $\beta=\lim_{k\to\infty}u(k)/v(k)\in[-\infty,\infty]$. If $\beta=\pm\infty$, then by interchanging $u(k)$ and $v(k)$, we get that $\beta=0$. Then we may as well assume $-\infty<\beta<\infty$. If $\beta\neq 0$, then we can replace the solution $u(k)$ by the solution $u(k)-\beta v(k)$ to get

$$\lim_{k\to\infty}\frac{u(k)-\beta v(k)}{v(k)}=\beta-\beta=0. \tag{1.5.32}$$

We may assume $\lim_{k\to\infty} u(k)/v(k) = 0$. Set $x_1 = u$. If x_2 is a second linearly independent solution, then $x_2(k) = \alpha x_1(k) + \gamma v(k)$, $\gamma \neq 0$. We have

$$\lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = \lim_{k\to\infty} \frac{u(k)}{\alpha u(k) + \gamma v(k)} = 0. \tag{1.5.33}$$

Pick $N_1 \in \mathbb{N}$ sufficiently large so that $x_2(k)$ is of the same sign for $k \geq N_1$.

Now consider for $k \geq N_1$,

$$\Delta\left(\frac{x_1(k)}{x_2(k)}\right) = \frac{W[x_2(k), x_1(k)]}{x_2(k)x_2(k+1)} = \frac{d}{c(k)x_2(k)x_2(k+1)}, \tag{1.5.34}$$

where d is a constant, by Liouville's formula. Summing both sides of this equation from N_1 to $k-1$, we obtain

$$\frac{x_1(k)}{x_2(k)} - \frac{x_1(N_1)}{x_2(N_1)} = d \sum_{n=N_1}^{k-1} \frac{1}{c(n)x_2(n)x_2(n+1)}. \tag{1.5.35}$$

It follows that

$$\sum_{n=N_1}^{\infty} \frac{1}{c(n)x_2(n)x_2(n+1)} < \infty. \tag{1.5.36}$$

Pick $N_2 \in \mathbb{N}$ so that $x_1(k)$ is of one sign for $k \geq N_2$. Similar to the above, we have

$$\Delta\left(\frac{x_2(k)}{x_1(k)}\right) = \frac{W[x_1(k), x_2(k)]}{x_1(k)x_1(k+1)} = -\frac{d}{c(k)x_1(k)x_1(k+1)}. \tag{1.5.37}$$

Summing both sides of this equation from N_2 to $k-1$, we obtain

$$\frac{x_2(k)}{x_1(k)} - \frac{x_2(N_2)}{x_1(N_2)} = -d \sum_{n=N_2}^{k-1} \frac{1}{c(n)x_1(n)x_1(n+1)}. \tag{1.5.38}$$

It follows that

$$\sum_{n=N_2}^{\infty} \frac{1}{c(n)x_1(n)x_1(n+1)} = \infty. \tag{1.5.39}$$

To prove the last statement in the theorem, pick $N_3 \in \mathbb{N}$ so that both $x_1(k)$ and $x_2(k)$ are of one sign for $k \geq N_3$. Then with $w_i(k) = c(k)\Delta x_i(k)/x_i(k)$, $i \in \{1,2\}$, $k \geq N_3$, we have

$$\begin{aligned} w_1(k) - w_2(k) &= \frac{c(k)\Delta x_1(k)}{x_1(k)} - \frac{c(k)\Delta x_2(k)}{x_2(k)} \\ &= \frac{c(k)W[x_2(k),x_1(k)]}{x_1(k)x_2(k)} \\ &= \frac{d}{x_1(k)x_2(k)}. \end{aligned} \tag{1.5.40}$$

Since $w_i(k)$ is not changed if we replace $x_i(k)$ by $-x_i(k)$, we can assume $x_i(k) > 0$ for $i \in \{1,2\}$, $k \geq N_3$. Using $\lim_{k\to\infty} x_2(k)/x_1(k) = \infty$ and the equality (1.5.38), we get $d < 0$. It then follows from equality (1.5.40) that $w_1(k) < w_2(k)$ for $k \geq N_3$. □

Example 1.5.11. Consider the difference equation

$$x(k+1) - 6x(k) + 8x(k-1) = 0 \quad \text{for } k \in \mathbb{N}. \tag{1.5.41}$$

The characteristic equation is $(\lambda - 2)(\lambda - 4) = 0$. Take $x_1(k) = 2^k$ and $x_2(k) = 4^k$. Then

$$\lim_{k\to\infty} \frac{x_1(k)}{x_2(k)} = \lim_{k\to\infty} \left(\frac{1}{2}\right)^k = 0. \tag{1.5.42}$$

If we write equation (1.5.41) in self-adjoint form, we obtain $c(k) = (1/8)^k$ and $q(k) = 3(1/8)^k$. Hence

$$\sum_{k=0}^{\infty} \frac{1}{c(k)x_1(k)x_1(k+1)} = \frac{1}{2}\sum_{k=0}^{\infty} 2^k = \infty, \tag{1.5.43}$$

that is, $x_1(k) = 2^k$ is dominant, and

$$\sum_{k=0}^{\infty} \frac{1}{c(k)x_2(k)x_2(k+1)} = \frac{1}{4}\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k < \infty, \tag{1.5.44}$$

that is, $x_2(k) = 4^k$ is recessive. Finally, we note that $w_1(k) = 8^{-k} < w_2(k) = 3 \times 8^{-k}$ for $k \in \mathbb{N}_0$. Thus, the conclusions of Theorem 1.5.10 hold for equation (1.5.41).

1.5.2. More on disconjugacy characterization

We will consider equation (1.4.4) with $k \geq a \geq 0$, which can be rewritten as a three-term recurrence relation of the form

$$Rx(k) = c(k+1)x(k+2) - p(k)x(k+1) + c(k)x(k) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{1.5.45}$$

where $p(k) = c(k) + c(k+1) - q(k)$ for $k \in \mathbb{N}_0$.

In what follows, we employ the following notation. For given $N, n \in \mathbb{N}_0$, denote $I_n = [N, N+n]$ and $I_\infty = [N, \infty)$. Now, equation (1.4.4) is disconjugate on I_{n+2}, $n \in \mathbb{N}_0$, if $x \equiv 0$ is the only solution of equation (1.4.4) having two generalized zeros on I_{n+2}.

If $p(N) \leq 0$ for some $N \in \mathbb{N}_0$, then all the coefficients in equation (1.5.45) are nonnegative but some of them are positive. For any nontrivial solution x of equation (1.4.4), $x(N)$, $x(N+1)$, and $x(N+2)$ cannot be of the same sign, and hence $x(k)$ must have at least one generalized zero on $I_2 = [N, N+2]$. Furthermore, let $\beta_0, \beta_1, \ldots, \beta_n$ be $n+1$ positive numbers and consider the equation

$$\sum_{i=0}^{n} \beta_i Rx(N+i) = 0, \tag{1.5.46}$$

or more clearly, if $n = 0$,

$$\beta_0 c(N+1)x(N+2) - \beta_0 p(N)x(N+1) + \beta_0 c(N)x(N) = 0, \tag{$1.5.46^{N,0}$}$$

if $n = 1$,

$$\begin{aligned} &\beta_1 c(N+2)x(N+3) + \left[\beta_0 c(N+1) - \beta_1 p(N+1)\right]x(N+2) \\ &\quad + \left[\beta_1 c(N+1) - \beta_0 p(N)\right]x(N+1) + \beta_0 c(N)x(N) = 0, \end{aligned} \tag{$1.5.46^{N,1}$}$$

and if $n \geq 2$,

$$\begin{aligned} &\beta_n c(N+n+1)x(N+n-2) + \left[\beta_{n-1}c(N+n) - \beta_n p(N+n)\right]x(N+n+1) \\ &\quad + \left[\beta_{n-2}c(N+n-1) + \beta_n c(N+n) - \beta_{n-1}p(N+n+1)\right]x(N+n) \\ &\quad + \cdots + \left[\beta_0 c(N+1) + \beta_2 c(N+2) - \beta_1 p(N+1)\right]x(N+2) \\ &\quad + \left[\beta_1 c(N+1) - \beta_0 p(N)\right]x(N+1) + \beta_0 c(N)x(N) = 0. \end{aligned} \tag{$1.5.46^{N,n}$}$$

Letting the coefficients in equation (1.5.46) be nonnegative, that is,

$$\sum_{i=0}^{n} \beta_i Rx(N+i) = 0 \quad \text{with } \beta_0, \beta_1, \ldots, \beta_n \geq 0, \tag{1.5.47}$$

produces a system of $n+1$ inequalities in terms of $\beta_0, \beta_1, \dots, \beta_n$, namely, if $n = 0$,

$$\beta_0 p(N) \le 0, \tag{$1.5.47^{N,0}$}$$

if $n = 1$,

$$\begin{gathered} \beta_0 p(N) \le \beta_1 c(N+1), \\ \beta_1 p(N+1) \le \beta_0 c(N+1), \end{gathered} \tag{$1.5.47^{N,1}$}$$

and if $n \ge 2$,

$$\begin{gathered} \beta_0 p(N) \le \beta_1 c(N+1), \\ \beta_1 p(N+1) \le \beta_0 c(N+1) + \beta_2 c(N+2), \\ \vdots \\ \beta_{n-1} p(N+n-1) \le \beta_{n-2} c(N+n-1) + \beta_n c(N+n), \\ \beta_n p(N+n) \le \beta_{n-1} c(N+n). \end{gathered} \tag{$1.5.47^{N,n}$}$$

If there are positive numbers $\beta_0, \beta_1, \dots, \beta_n$ satisfying ($1.5.47^{N,n}$), then from equation (1.5.46) it can be seen that each solution x of equation (1.4.4) has a generalized zero on I_{n+2} and of course has a generalized zero on any interval containing I_{n+2} as a subinterval. In fact, we have the following result.

Lemma 1.5.12. *If there exists a set of positive numbers $\beta_0, \beta_1, \dots, \beta_n$ satisfying ($1.5.47^{N,n}$), then for any $h, i \in \mathbb{Z}$ with $h > n$ and $0 \le i \le h - n$, there exist $h+1$ positive numbers $\alpha_0, \alpha_1, \dots, \alpha_h$ satisfying ($1.5.47^{N-i,h-i}$).*

Proof. We only consider the case when $h = n+1$ and $i = 0$ and prove that there exists $\beta_{n+1} > 0$ such that $\beta_0, \beta_1, \dots, \beta_n, \beta_{n+1}$ satisfy ($1.5.47^{N,n+1}$). The proof of the other cases is similar and hence omitted. The first n inequalities in ($1.5.47^{N,n+1}$) are already satisfied and the last two inequalities

$$\begin{gathered} \beta_n p(N+n) \le \beta_{n-1} c(N+n) + \beta_{n+1} c(N+n+1), \\ \beta_{n+1} p(N+n+1) \le \beta_n c(N+n+1) \end{gathered} \tag{1.5.48}$$

are satisfied provided

$$\beta_{n+1} = \begin{cases} \beta_n \dfrac{c(N+n+1)}{p(N+n+1)} & \text{in case } p(N+n+1) > 0, \\ 1 & \text{in case } p(N+n+1) \le 0. \end{cases} \tag{1.5.49}$$

This completes the proof. ∎

Next, we introduce an important sequence of positive numbers defined inductively as follows:

$$s(N,1)\begin{cases}\text{is not defined} & \text{if } p(N)\le 0 \text{ for some } N\in\mathbb{N}_0,\\ =\dfrac{c^2(N+1)}{p(N)} & \text{if } p(N)>0,\end{cases}$$
$$s(N,2)\begin{cases}\text{is not defined} & \text{if } p(N+1)\le s(N,1),\\ =\dfrac{c^2(N+2)}{p(N+1)-s(N,1)} & \text{if } p(N+1)>s(N,1).\end{cases} \tag{1.5.50}$$

In general, suppose that $s(N,1), s(N,2),\dots,s(N,n-1)$ are defined for $n\ge 2$. Then for $n\ge 2$,

$$s(N,n)\begin{cases}\text{is not defined} & \text{if } p(N+n-1)\le s(N,n-1),\\ =\dfrac{c^2(N+n)}{p(N+n-1)-s(N,n-1)} & \text{if } p(N+n-1)>s(N,n-1).\end{cases} \tag{1.5.51}$$

The number $s(N,n)$ can also be written as a continued fraction

$$s(N,n)=\cfrac{c^2(N+n)}{p(N+n-1)-\cfrac{c^2(N+n-1)}{p(N+n-2)-\cdots-\cfrac{c^2(N+2)}{p(N+1)-\cfrac{c^2(N+1)}{p(N)}}}}. \tag{1.5.52}$$

In the following result, the existence of the sequence $s(N,\cdot)$ indeed violates the existence of positive solutions to the corresponding system (1.5.47) and vice versa.

Lemma 1.5.13. *Let $N,n\in\mathbb{N}_0$. Then there exist no positive numbers $\beta_0,\beta_1,\dots,\beta_n$ satisfying (1.5.47^{N,n}) if and only if the sequence of positive numbers $\{s(N,i+1)\}_{i=0}^{n}$ is well defined.*

Proof. Suppose that (1.5.47^{N,n}) has no positive solutions. From Lemma 1.5.12 it follows that (1.5.47^{N,0}) has no positive solutions. As a result $p(N)>0$ and $s(N,1)$ is well defined. Assume that for some $j\in[1,n]$, $s(N,1),\dots,s(N,j)$ are defined, but $s(N,j+1)$ is not defined, that is, $p(N+j)\le s(N,j)$. Let

$$\alpha_i=\begin{cases}\alpha_{i+1}\dfrac{s(N,i+1)}{c(N+i+1)} & \text{for } i=0,1,\dots,j-1,\\ 1 & \text{for } i=j.\end{cases} \tag{1.5.53}$$

Then, $\alpha_0, \alpha_1, \dots, \alpha_j$ are positive numbers satisfying

$$\begin{aligned}
\alpha_0 p(N) &= \alpha_1 c(N+1),\\
\alpha_1 p(N+1) &= \alpha_0 c(N+1) + \alpha_2 c(N+2),\\
&\vdots\\
\alpha_{j-1} p(N+j-1) &= \alpha_{j-2} c(N+j-1) + \alpha_j c(N+j),\\
\alpha_j p(N+j) = p(N+j) &\le s(N,j) = \alpha_{j-1} c(N+j).
\end{aligned} \tag{1.5.54}$$

Hence $\alpha_0, \alpha_1, \dots, \alpha_j$ satisfy $(1.5.47^{N,j})$. This contradicts Lemma 1.5.12 and proves the necessity.

Suppose that $\{s(N,i+1)\}_{i=0}^{n}$ is defined and that $\beta_0, \beta_1, \dots, \beta_n$ are positive numbers satisfying $(1.5.47^{N,n})$. From $(1.5.47^{N,n})$ we have

$$\begin{aligned}
\beta_0 &\le \frac{\beta_1 s(N,1)}{c(N+1)},\\
\beta_1 &\le \frac{\beta_2 s(N,2)}{c(N+2)},\\
&\vdots\\
\beta_{n-1} &\le \frac{\beta_n s(N,n)}{(N+n)},\\
\beta_n p(N+n) &\le \beta_{n-1} c(N+n).
\end{aligned} \tag{1.5.55}$$

From the last two inequalities in (1.5.55) it follows that $p(N+n) \le s(N,n)$. This contradicts the definition of $s(N,n+1)$ and proves the sufficiency. $\square$

Next, we present the following disconjugacy characterization.

Theorem 1.5.14. *Let $N, n \in \mathbb{N}_0$. The following statements are equivalent.*

(I_1) *There exist no positive numbers $\beta_0, \beta_1, \dots, \beta_n$ satisfying $(1.5.47^{N,n})$.*

(I_2) *The sequence $\{s(N,i+1)\}_{i=0}^{n}$ is well defined, that is,*

$$p(N) > 0, \tag{1.5.56}$$

and for $1 \le i \le n$,

$$p(N+i) > \cfrac{c^2(N+i)}{p(N+i-1) - \cfrac{c^2(N+i-1)}{p(N+i-2) - \cdots - \cfrac{c^2(N+2)}{p(N+1) - \cfrac{c^2(N+1)}{p(N)}}}}. \tag{1.5.57}$$

(I_3) *Equation (1.4.4) is disconjugate on I_{n+2}.*

Proof. In view of Lemma 1.5.13, $(\mathrm{I}_1)\Leftrightarrow(\mathrm{I}_2)$.

Now we show $(\mathrm{I}_2)\Rightarrow(\mathrm{I}_3)$. Let $x = \{x(k)\}_{k=N}^{N+n+2}$ be the solution of equation (1.4.4) on I_{n+2} satisfying $x(N) = \varepsilon$, where $\varepsilon > 0$ is a constant, $x(N+1) = 1$. From equation (1.5.45), we obtain

$$\begin{aligned} x(N+2) &= \frac{c(N+1)}{s(N,1)} - \varepsilon\frac{c(N)}{c(N+1)}, \\ x(N+3) &= \frac{c(N+1)}{s(N,1)}\frac{c(N+2)}{s(N,2)} - \varepsilon\frac{c(N)p(N+1)}{c(N+1)c(N+2)}, \\ &\vdots \\ x(N+n+2) &= \prod_{i=1}^{n+1}\frac{c(N+1)}{s(N,i)} - \varepsilon\frac{c(N)p(N+1)\cdots p(N+n)}{c(N+1)\cdots c(N+n+1)}. \end{aligned} \tag{1.5.58}$$

Choose $\varepsilon > 0$ sufficiently small so that x is positive on I_{n+1}. Then there are no nontrivial solutions of equation (1.4.4) having two generalized zeros on I_{n+2}. For otherwise, if a nontrivial solution had two generalized zeros on I_{n+2}, then by Theorem 1.4.4, x would have a generalized zero between the two generalized zeros of the former solution. Therefore, equation (1.4.4) is disconjugate on I_{n+2}.

Finally we show $(\mathrm{I}_3)\Rightarrow(\mathrm{I}_1)$. First, we claim that $(1.5.47^{N,0})$ has no positive solutions. If not, then $p(N) \le 0$. Let $y = \{y(k)\}_{k=N}^{\infty}$ be the solution of equation (1.4.4) satisfying $y(N+2) = p(N)$ and $y(N+1) = 2c(N+1)$. From $(1.5.46^{N,0})$, if $p(N) = 0$, then $y(N) = y(N+2) = 0$ and $y(N+1) = 2c(N+1) > 0$, and if $p(N) < 0$, then $y(N) = c(N+1)p(N)/c(N) < 0$, $y(N+1) > 0$, and $y(N+2) = p(N) < 0$. In either case y has two generalized zeros on I_2. This contradicts (I_3) and proves that $(1.5.47^{N,0})$ has no positive solutions. Next, we suppose that for some integer $j \in [1,n]$, $(1.5.47^{N,0}), \dots, (1.5.47^{N,j-1})$ have no positive solutions but $(1.5.47^{N,j})$ are satisfied by positive numbers $\beta_0,\beta_1,\dots,\beta_j$. Then Lemma 1.5.13 implies that $s(N,1),\dots,s(N,j)$ are defined but $p(N+j) \le s(N,j)$. Thus, the $j+1$ positive numbers $\alpha_0,\alpha_1,\dots,\alpha_j$ defined by (1.5.53) must satisfy (1.5.54) and hence $(1.5.47^{N,j})$. Substituting $\alpha_i = \beta_i$ into $(1.5.46^{N,j})$ and noting (1.5.54), we have

$$c(N+j+1)x(N+j+2) + \big[s(N,j) - p(N+j)\big]x(N+j+1) + \alpha_0 c(N)x(N) = 0. \tag{1.5.59}$$

Let $y = \{y(n)\}_{n=N}^{\infty}$ be the solution of equation (1.4.4) with the restrictions

$$y(N+j+2) = p(N+j) - s(N,j), \qquad y(N+j+1) = 2c(N+j+1). \tag{1.5.60}$$

It follows from (1.5.59) that if $p(N+j) = s(N,j)$, then

$$y(N+j+2) = y(N) = 0, \qquad y(N+j+1) = 2c(N+j+1) > 0, \tag{1.5.61}$$

and if $p(N+j) < s(N,j)$, then

$$y(N+j+2) < 0, \qquad y(N+j+1) > 0, \qquad y(N) = \frac{c(N+j)y(N+j+2)}{\alpha_0 c(N)} < 0. \tag{1.5.62}$$

In any case, y has two generalized zeros on $I_{j+2} \subset I_{n+2}$ which is against (I_3). This contradiction asserts (I_1) and completes the proof. □

Corollary 1.5.15. *Let $N, n \in \mathbb{N}_0$. The following statements are equivalent.*

(i_1) *There exist positive numbers $\beta_0, \beta_1, \ldots, \beta_n$ satisfying ($1.5.47^{N,n}$).*

(i_2) *Either $p(N) \le 0$, or $p(N) > 0$ and for some integer $j \in [1,n]$, $s(N,1), \ldots,$ $s(N,j)$ are well defined but $p(N+j) \le s(N,j)$.*

(i_3) *Each nontrivial solution of equation (1.4.4) has at least one generalized zero on I_{n+2}.*

(i_4) *There exists a nontrivial solution y of equation (1.4.4) having at least two generalized zeros on I_{n+2}.*

Proof. Theorem 1.5.14 implies that (i_1), (i_2), and (i_4) are equivalent. It suffices to show that (i_1)⇒(i_3) and (i_3)⇒(i_2).

First we address (i_1)⇒(i_3). This is obvious, since each coefficient in ($1.5.46^{N,n}$) is nonnegative and some coefficients are positive, and hence a nontrivial solution x satisfying ($1.5.46^{N,n}$) must have a generalized zero on I_{n+2}.

Now we show (i_3)⇒(i_2). Suppose that (i_3) holds but (i_2) does not. Then $s(N,1), \ldots, s(N,n+1)$ are defined. As in the proof of Theorem 1.5.14, we can then construct a positive solution on I_{n+2}, which is a contradiction. This completes the proof. □

1.5.3. Disfocality

Definition 1.5.16. Equation (1.4.4) is called *right disfocal* on I_{n+2}, where $n \in \mathbb{N}_0$, if $x \equiv 0$ is the only solution of equation (1.4.4) such that x has a generalized zero at $N+i$ and Δx has a generalized zero at $N+j$, $0 \le i < j \le n+1$.

We note that by a discrete version of Rolle's theorem, right disfocality of equation (1.4.4) implies disconjugacy of equation (1.4.4) on a certain interval.

The following result is a necessary and sufficient condition for disfocality of equation (1.4.4) on an interval.

Theorem 1.5.17. *Let $N, n \in \mathbb{N}_0$. Then equation (1.4.4) is right disfocal on I_{n+2} if and only if the sequence $\{s(N,i+1)\}_{i=0}^{n}$ is well defined and*

$$c(N+i) > s(N,i) \quad \text{for } i \in \{1,2,\ldots,n+1\}. \tag{1.5.63}$$

Proof. Suppose that equation (1.4.4) is right disfocal on I_{n+2}. Then equation (1.4.4) is disconjugate on this interval and hence $\{s(N, i+1)\}_{i=0}^{n}$ is defined in view of Theorem 1.5.14. Let $x = \{x(k)\}_{k=N}^{N+n+2}$ be the solution of equation (1.4.4) satisfying $x(N) = 0$ and $x(N+1) = 1$. Then

$$x(N+i) = \prod_{j=1}^{i-1} \frac{c(N+j)}{s(N,j)} \quad \text{for } i \in \{2,3,\dots,n+2\}. \tag{1.5.64}$$

On the other hand, $\Delta x(N) = 1$ and

$$\Delta x(N+i) = x(N+i)\frac{c(N+i) - s(N,i)}{s(N,i)} \quad \text{for } i \in \{1,2,\dots,n+1\}. \tag{1.5.65}$$

By the definition of right disfocality we see that $\Delta x(N+i) > 0$ for $0 \le i \le n+1$. Thus, (1.5.63) follows from (1.5.65) immediately.

Suppose that $\{s(N, i+1)\}_{i=0}^{n}$ is defined and (1.5.63) holds. We can then define a solution x by $x(N) = 0$, $x(N+1) = 1$, and (1.5.64). The difference of x satisfies (1.5.65), and hence x is increasing because of (1.5.63). Let $y = \{y(k)\}_{k=N}^{N+n+2}$ be any solution of equation (1.4.4) such that y has a generalized zero at $K = N+i$ and Δy has a generalized zero at $M = N+j$, $0 \le i < j \le n+1$. We claim that $y \equiv 0$. If not, then by Theorem 1.5.14, equation (1.4.4) is disconjugate on I_{n+2} and hence K is the only generalized zero of y on I_{n+2}. Without loss of generality, we may assume that $y(k) > 0$ for $K+1 \le k \le N+n+2$ and M is the only generalized zero of Δy on $[K, M]$. Clearly, $\Delta y(k) > 0$ for $K \le k \le M-1$ and $\Delta y(M) \le 0$. It is well known that

$$y(k)c(k)\Delta x(k) - x(k)c(k)\Delta y(k) = C \tag{1.5.66}$$

(see Corollary 1.2.3), where C is a constant number depending only on the solutions x and y. If $k = M$, then (1.5.66) gives $C > 0$.

There are three possibilities to consider:

(i) if $K = N$, then $y(N) = 0$. Letting $k = N$ in (1.5.66) yields $x(N) < 0$, which is a contradiction;

(ii) if $K > N$ and $y(K) = 0$, then (1.5.66) with $k = K$ implies $x(K) < 0$ which is impossible by virtue of (1.5.64);

(iii) if $K > N$ and $y(K) > 0$, then $y(K-1) < 0$ and $\Delta y(K-1) > 0$ since K is a generalized zero of y. Consequently, if we let $k = K-1$ in (1.5.66), then we have $x(K-1) < 0$ which contradicts the definition of x.

In any case we obtain the desired contradiction which gives $y \equiv 0$, and hence equation (1.4.4) is right disfocal on I_{n+2}. This proves the claim and completes the proof. □

1.6. Conjugacy criteria

In this section we will present some conjugacy criteria for a second-order linear difference equation which is a special case of equation (1.4.4), namely, the equation

$$\Delta^2 x(k) + q(k)x(k+1) = 0, \tag{1.6.1}$$

where $q(k)$, $k \in \mathbb{Z}$, is a real-valued sequence.

In equation (1.4.4), if $c(k) > 0$ in the interval under consideration, say, $[0, N]$, then the transformation

$$x(k) = \rho(k)y(k) \quad \text{with } \rho(0) = 1,\ \rho(k+1) = \frac{1}{c(k)\rho(k)} \tag{1.6.2}$$

transforms equation (1.4.4) into an equation of the form (1.6.1). Indeed, by direct computation one can verify the equality

$$\begin{aligned}
&\rho(k+1)\big[\Delta\big(c(k)\Delta x(k)\big) + q(k)x(k+1)\big] \\
&\quad = \Delta\big(c(k)\rho(k)\rho(k+1)\Delta y(k)\big) + \rho(k+1)\big[\Delta\big(c(k)\Delta\rho(k)\big) + q(k)\rho(k+1)\big]y(k+1),
\end{aligned} \tag{1.6.3}$$

which yields the required transformation.

Here, we find the conditions which guarantee that equation (1.6.1) possesses a nontrivial solution having at least two generalized zeros in a given interval.

To obtain conjugacy criteria for equation (1.6.1), we employ the following two auxiliary lemmas.

Lemma 1.6.1. *Let $x = \{x(k)\}_{k\in\mathbb{Z}}$ and $y = \{y(k)\}_{k\in\mathbb{Z}}$ be any pair of sequences such that $x^2(k)+y^2(k) > 0$ for $k \in \mathbb{Z}$ and let $z(k) = (x(k)+iy(k))/\sqrt{x^2(k)+y^2(k)}$, where $i = \sqrt{-1} \in \mathbb{C}$. Then*

$$z(k+1) = \frac{x(k)x(k+1) + y(k)y(k+1) + iW[x(k), y(k)]}{h(k)h(k+1)} z(k), \tag{1.6.4}$$

where $h(k) = \sqrt{x^2(k)+y^2(k)}$ and $W[x(k), y(k)] = x(k)y(k+1) - x(k+1)y(k)$.

Proof. By a direct computation, we obtain

$$\begin{aligned}
z(k+1)-z(k) &= \frac{x(k+1)+iy(k+1)}{h(k+1)} - \frac{x(k)+iy(k)}{h(k)} \\
&= \left[\frac{(x(k+1)+iy(k+1))h(k)}{(x(k)+iy(k))h(k+1)} - 1\right]\frac{x(k)+iy(k)}{h(k)} \\
&= \left[\frac{(x(k+1)+iy(k+1))h(k)(x(k)-iy(k))}{h^2(k)h(k+1)} - 1\right]z(k) \\
&= \left[\frac{x(k)x(k+1)+y(k)y(k+1)+iW[x(k),y(k)]}{h(k)h(k+1)} - 1\right]z(k).
\end{aligned} \tag{1.6.5}$$

This proves our claim. □

Lemma 1.6.2. *Let x, y, z be as in Lemma 1.6.1 and suppose $W[x(k), y(k)] > 0$. Then*

$$0 < \arg z(k+1) - \arg z(k) < \pi. \tag{1.6.6}$$

Proof. From (1.6.4) it follows that

$$\arg z(k+1) - \arg z(k) = \arg\left(x(k)x(k+1) + y(k)y(k+1) + iW[x(k), y(k)]\right), \tag{1.6.7}$$

and since $W[x(k), y(k)] > 0$, we obtain (1.6.6). □

Next, we present the following conjugacy result for equation (1.6.1).

Theorem 1.6.3. *Suppose that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that*

$$\limsup_{n\to\infty} \sum_{k=0}^{n} \arctan \frac{\varepsilon_1}{2\alpha_k(q,\varepsilon_1)} > \frac{\pi}{4}, \tag{1.6.8}$$

$$\limsup_{n\to-\infty} \sum_{k=n}^{1} \arctan \frac{\varepsilon_2}{2\beta_k(q,\varepsilon_2)} > \frac{\pi}{4}, \tag{1.6.9}$$

where

$$\begin{aligned}
&\alpha_0 = 1+\varepsilon_1, \quad \text{and for} \quad k \geq 1, \\
&\alpha_k = \alpha_k(q,\varepsilon_1) = \left[\varepsilon_1 - \sum_{i=0}^{n} q(i) + 1\right] \prod_{j=0}^{k-1} \left(\varepsilon_1 - \sum_{i=0}^{j-1} q(i) + 1\right)^2, \\
&\beta_1 = 1+\varepsilon_2, \quad \text{and for} \quad k \leq 0, \\
&\beta_k = \beta_k(q,\varepsilon_2) = \left[\varepsilon_2 - \sum_{i=k-1}^{-1} q(i) + 1\right] \prod_{j=k+1}^{1} \left(\varepsilon_2 - \sum_{i=j-1}^{-1} q(i) + 1\right)^2.
\end{aligned} \tag{1.6.10}$$

Then equation (1.6.1) is conjugate on $\mathbb{Z}$.

Proof. In the first part of the proof we consider the interval $[0,\infty)$ and we will show that the solution x of equation (1.6.1) given by the initial conditions $x(0) = 1$ and $x(1) = 1$ has a generalized zero in $[2,\infty)$. Let y be another solution of equation (1.6.1) given by the initial conditions $y(0) = 1$ and $y(1) = 1+\varepsilon_1$, and let z be defined as in Lemmas 1.6.1 and 1.6.2.

Assume by contradiction that x has no positive generalized zero, that is, $x(k) > 0$ for $k \in [2,\infty)$. Then $y(k) > x(k)$ for $k \in \mathbb{N}$. Indeed if $y(m) \le x(m)$ for some $m \in \mathbb{N}$, then the solution $x_1 = y - x$ satisfies $x_1(0) = 0$, $x_1(1) = \varepsilon_1 > 0$, and $x_1(m) \le 0$. This leads to a contradiction, since x is positive throughout $\mathbb{N}$.

Denote $u(k) = x(k)x(k+1) + y(k)y(k+1)$. By Lemma 1.6.1, we have

$$z(k+1) = \frac{u(k)+i\varepsilon_1}{h(k)h(k+1)}z(k) \tag{1.6.11}$$

and hence

$$\arg z(k+1) = \sum_{j=0}^{k} \arg\left(u(j)+i\varepsilon_1\right) + \arg z(0) = \sum_{j=0}^{k} \arctan\frac{\varepsilon_1}{u(j)} + \frac{\pi}{4}. \tag{1.6.12}$$

Further, denote $w(k) = \Delta y(k)/y(k)$. Then $1 + w(k) > 0$ and w satisfies the discrete Riccati equation

$$\Delta w(k) + q(k) + \frac{w^2(k)}{1+w(k)} = 0. \tag{1.6.13}$$

Hence $\Delta w(k) \le -q(k)$ and thus

$$w(k) \le w(0) - \sum_{i=0}^{k-1} q(i) = \varepsilon_1 - \sum_{i=0}^{k-1} q(i). \tag{1.6.14}$$

It follows that

$$y(k) = \prod_{j=0}^{k-1}\left(w(j)+1\right) \le \prod_{j=0}^{k-1}\left(\varepsilon_1 - \sum_{i=0}^{j-1} q(i) + 1\right). \tag{1.6.15}$$

Consequently,

$$\frac{1}{w(k)} \ge \frac{1}{2y(k)y(k+1)} \ge \frac{1}{2\alpha_k}. \tag{1.6.16}$$

Since condition (1.6.8) holds and $\arg z(0) = \pi/4 < \pi/2$, there exists $m \in \mathbb{N}_0$ such that $\arg z(m) < \pi/2$ and

$$\arg z(m+1) \ge \frac{\pi}{4} + \sum_{k=0}^{m} \arctan\frac{\varepsilon_1}{2\alpha_k} \ge \frac{\pi}{2}. \tag{1.6.17}$$

Since $0 < \arg z(m+1) - \arg z(m) < \pi$ by Lemma 1.5.13 and by the fact that $\arg z(m) > -\pi/2$, we have $\arg z(m+1) < 3\pi/2$; hence $x(m) > 0$ and $x(m+1) \leq 0$, which means that x has a generalized zero in $(m, m+1)$, which is a contradiction.

In the second part of this proof, we consider the interval $(-\infty, 1]$. We define the *backward difference operator* $\tilde{\Delta}$ by $\tilde{\Delta} y(k) = y(k-1) - y(k)$. Since we have $\tilde{\Delta}^2 y(k) = \Delta^2 y(k-2)$, equation (1.6.1) takes the form

$$\tilde{\Delta}^2 y(k) + q(k-2) y(k-1) = 0. \tag{1.6.18}$$

Now, let y_1 be the solution of equation (1.6.1) given by the initial conditions $y_1(1) = 1$ and $y_1(0) = 1 + \varepsilon_2$, and let x be the same as in the previous part of the proof. Again, we will show that the assumption $x(k) > 0$ for all $k < 0$ leads to a contradiction. Denote $h_1(k) = \sqrt{x^2(k) + y_1^2(k)}$ and let $z_1(k) = (x(k) + i y_1(k))/h_1(k)$. Similarly as in Lemma 1.6.1, we obtain $z_1(k) = (u_1(k) - i\varepsilon_2)/(h_1(k-1)h_1(k))$ and $u_1(k) = x(k-1)x(k) + y_1(k-1)y_1(k)$, and hence

$$\begin{aligned}
z_1(k-1) &= \frac{h_1(k)h_1(k-1)}{u_1(k) - i\varepsilon_2} z_1(k) \\
&= \frac{h_1(k)h_1(k-1)\left[u_1(k) + i\varepsilon_2\right]}{\left[x(k-1)x(k) + y_1(k-1)y_1(k)\right]^2 + \left[x(k-1)y_1(k) - x(k)y_1(k-1)\right]^2} z_1(k) \\
&= \frac{h_1(k-1)h_1(k)\left(u_1(k) + i\varepsilon_2\right)}{x^2(k)\left[x^2(k-1) + y_1^2(k-1)\right] + y_1^2(k)\left[x^2(k-1) + y_1^2(k-1)\right]} z_1(k) \\
&= \frac{u_1(k) + i\varepsilon_2}{h_1(k-1)h_1(k)} z_1(k).
\end{aligned} \tag{1.6.19}$$

Similarly as in the first part of the proof, the assumption $x(k) > 0$ for $k < 0$ implies $y(k) > x(k) > 0$ for all nonnegative integers. Let $w_1(k) = \Delta y_1(k)/y_1(k)$. By a direct computation, we have

$$\tilde{\Delta} w_1(k) = -q(k-2) - \frac{w_1^2(k)}{1 + w_1(k)} \leq -q(k-2) \tag{1.6.20}$$

and hence for $k \leq 0$,

$$y_1(k) = \prod_{j=k+1}^{1} \left(1 + w_1(j)\right) y_1(1) \leq \prod_{j=k+1}^{1} \left(1 - \sum_{i=j+1}^{1} q(i-2) + \varepsilon_2\right). \tag{1.6.21}$$

This implies, in the same way as for $k \geq 1$,

$$\begin{aligned}
\arg z_1(k-1) &= \sum_{j=k}^{1} \arg\left(u_1(j) + i\varepsilon_2\right) + \arg z_1(1) \\
&= \sum_{j=k}^{1} \arctan \frac{\varepsilon_2}{u_1(j)} + \frac{\pi}{4} \\
&\geq \sum_{j=k}^{1} \arctan \frac{\varepsilon_2}{2\beta_k(q,\varepsilon_2)} + \frac{\pi}{4}.
\end{aligned} \tag{1.6.22}$$

As in the proof of the above case, we see that condition (1.6.9) contradicts $x(k) > 0$ for $k < 0$. Consequently, the solution x has at least two generalized zeros, that is, equation (1.6.1) is conjugate on $\mathbb{Z}$. □

Another alternative of Theorem 1.6.3 is shown as follows.

Theorem 1.6.4. *Suppose that there exist $\varepsilon_1, \varepsilon_2 > 0$ and integers $m = m(\varepsilon_1) \in [0,\infty)$, $n = n(\varepsilon_2) \in (-\infty, 1)$ such that*

$$\sum_{k=0}^{m} \arctan \frac{\varepsilon_1}{2\alpha_k} \geq \frac{\pi}{4}, \qquad \sum_{k=n}^{1} \arctan \frac{\varepsilon_2}{2\beta_k} \geq \frac{\pi}{4}, \tag{1.6.23}$$

where α_k and β_k are defined as in Theorem 1.6.3. Then equation (1.6.1) is conjugate on $\mathbb{Z}$.

Corollary 1.6.5. *Suppose that*

$$\liminf_{n\to\infty} \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=0}^{j-1} q(i) = C_1 > 0, \tag{1.6.24}$$

$$\liminf_{n\to-\infty} \frac{1}{2-n} \sum_{j=n}^{1} \sum_{i=j-1}^{-1} q(i) = C_2 > 0. \tag{1.6.25}$$

Then equation (1.6.1) is conjugate on $\mathbb{Z}$.

Proof. Suppose that condition (1.6.24) holds. There exists $m \in \mathbb{N}$ such that

$$\sum_{j=0}^{n} \sum_{i=0}^{j-1} q(i) > \frac{3}{4} C_1(n+1) \quad \text{whenever } n \in (m,\infty). \tag{1.6.26}$$

Then

$$\sum_{j=0}^{n} \left[\frac{3}{4} C_1 - \sum_{i=0}^{j-1} q(i) \right] < 0. \tag{1.6.27}$$

Hence

$$0 < \exp\left(\sum_{j=0}^{n}\left[\frac{3}{4}C_1 - \sum_{i=0}^{j-1} q(i)\right]\right) < 1, \tag{1.6.28}$$

and consequently

$$0 < \prod_{j=0}^{n} \exp\left(\frac{3}{4}C_1 - \sum_{i=0}^{j-1} q(i)\right) < 1. \tag{1.6.29}$$

Now, let $\varepsilon_1 = C_1/4$ and $\ell \in (m+1, \infty)$. We obtain

$$\begin{aligned}
&\sum_{k=0}^{\ell} \arctan \frac{\varepsilon_1}{2\Big[\varepsilon_1 - \sum_{i=0}^{k-1} q(i) + 1\Big] \prod_{j=0}^{k-1}\Big(\varepsilon_1 - \sum_{i=0}^{j-1} q(i) + 1\Big)^2} \\
&\quad = \sum_{k=0}^{\ell} \arctan \frac{\varepsilon_1}{2\prod_{j=0}^{k}\Big(\varepsilon_1 - \sum_{i=0}^{j-1} q(i) + 1\Big)\prod_{j=0}^{k-1}\Big(\varepsilon_1 - \sum_{i=0}^{j-1} q(i) + 1\Big)} \\
&\quad \geq \sum_{k=m+1}^{\ell} \arctan \Bigg[\frac{\varepsilon_1}{2\prod_{j=0}^{k}\exp\Big(3C_1/4 - \sum_{i=0}^{j-1} q(i) - C_1/2\Big)} \\
&\qquad\qquad \times \frac{1}{\prod_{j=0}^{k-1}\exp\Big(3C_1/4 - \sum_{i=0}^{j-1} q(i) - C_1/2\Big)}\Bigg] + M \\
&\quad > \sum_{k=m+1}^{\ell} \arctan \frac{\varepsilon_1}{2\prod_{j=0}^{k}\exp\big(-(1/2)C_1\big)\prod_{j=0}^{k}\exp\big(-(1/2)C_1\big)} + M \\
&\quad \longrightarrow \infty \quad \text{as } \ell \longrightarrow \infty,
\end{aligned} \tag{1.6.30}$$

where

$$M = \sum_{k=0}^{m} \arctan \frac{\varepsilon_1}{2\Big[\varepsilon_1 - \sum_{i=0}^{k-1} q(i) + 1\Big]\prod_{j=0}^{k-1}\Big(\varepsilon_1 - \sum_{i=0}^{j-1} q(i) + 1\Big)^2}. \tag{1.6.31}$$

Similarly, using (1.6.25), we can prove that

$$\sum_{k=n}^{1} \arctan \frac{\varepsilon_2}{2\Big[\varepsilon_2 - \sum_{i=k-1}^{-1} q(i) + 1\Big]\prod_{j=k+1}^{1}\Big(\varepsilon_2 - \sum_{i=j-1}^{-1} q(i) + 1\Big)^2} \tag{1.6.32}$$

tends to infinity if $n \to -\infty$ and $\varepsilon_2 = C_2/4$. □

Next, we present the following sufficient conditions for conjugacy of equation (1.6.1) in the interval $[n, \infty)$, $n \in \mathbb{N}$.

Theorem 1.6.6. *Suppose that $q(k) \geq 0$ for $k \in \mathbb{N}$. A sufficient condition for conjugacy of equation (1.6.1) on an interval $[n, \infty)$, $n \in \mathbb{N}$, is that there exist integers ℓ, m with $n < \ell < m$ such that*

$$\frac{1}{\ell - n} < \sum_{k=\ell}^{m} q(k). \tag{1.6.33}$$

Proof. We will show that the solution x of equation (1.6.1) given by the initial conditions $x(n) = 0$ and $x(n+1) = 1$ has a generalized zero in (n, ∞). For, suppose it does not. Then without loss of generality we can assume $x(k) > 0$ in (n, ∞) and $\Delta x(k) \geq 0$ in $[n, \infty)$, since if $\Delta x(k) < 0$ at some point in (n, ∞), we would have a generalized zero in (n, ∞) by the condition $q(k) \geq 0$. From equation (1.6.1), we obtain

$$\Delta x(m+1) = \Delta x(\ell) - \sum_{k=\ell}^{m} q(k)x(k+1). \tag{1.6.34}$$

Since $q(k) \geq 0$, using the discrete mean value theorem, we have

$$\frac{x(\ell)}{\ell - n} = \frac{x(\ell) - x(n)}{\ell - n} \geq \Delta x(k) \geq \Delta x(\ell) \tag{1.6.35}$$

for some $k \in [n+1, \ell - 1]$. Thus

$$x(\ell) \geq (\ell - n)\Delta x(\ell). \tag{1.6.36}$$

Hence, by using (1.6.36), we have

$$\begin{aligned}
\Delta x(m+1) &= \Delta x(\ell) - \sum_{k=\ell}^{m} q(k)x(k+1) \\
&\leq \Delta x(\ell) - \sum_{k=\ell}^{m} q(k)x(\ell) \\
&\leq \Delta x(\ell) - (\ell - n)\Delta x(\ell) \sum_{k=\ell}^{m} q(k) \\
&= \Delta x(\ell)\left[1 - (\ell - n) \sum_{k=\ell}^{m} q(k)\right].
\end{aligned} \tag{1.6.37}$$

By condition (1.6.33), the factor between the brackets is negative. If $\Delta x(\ell) > 0$, then $\Delta x(m+1) < 0$, implying a generalized zero in $(m+1, \infty)$. If $\Delta x(\ell) = 0$, then $\Delta x(m) < 0$ since $\sum_{k=0}^{m} q(k)x(k+1) > 0$ (by assumptions). In either case, x has a generalized zero in $(m+1, \infty)$, and so equation (1.6.1) is conjugate on $[n, \infty)$. □

Finally, we state the following simple conjugacy criterion for equation (1.6.1) in $\mathbb{Z}$.

Theorem 1.6.7. *Suppose that* $q(k) \not\equiv 0$ *and* $\sum_{j=-\infty}^{\infty} q(j) = \lim_{k\to\infty} \sum_{j=-k}^{k} q(j)$ *exists as a finite number. If*

$$\sum_{j=-\infty}^{\infty} q(j) \geq 0, \tag{1.6.38}$$

then equation (1.6.1) is conjugate on $\mathbb{Z}$.

1.7. Methods of linear discrete oscillation theory

1.7.1. Riccati technique

As we will see in the next sections, the Riccati technique is very important in oscillation theory. Here, we will provide the following fundamental result.

Lemma 1.7.1. *Equation (1.2.1) is nonoscillatory if and only if there exists a sequence* $\{w(k)\}$ *with* $c(k) + w(k) > 0$ *for large* k *such that*

$$\mathcal{R}[w(k)] \leq 0, \tag{1.7.1}$$

where $\mathcal{R}[w(k)]$ *is as in (1.3.4) or (1.3.5).*

1.7.2. Variational principle

The further method, known from the oscillation theory of equation (1.1.1), is the so-called variational principle. It is based on the equivalence from Theorem 1.4.2(i)–(iv). More precisely, we will use the following lemma.

Lemma 1.7.2. *Equation (1.2.1) is nonoscillatory if and only if there exists* $m \in \mathbb{N}$ *such that*

$$\mathcal{F}(\xi; m, \infty) = \sum_{k=m}^{\infty} \left[c(k) \left| \Delta\xi(k) \right|^2 - q(k) \left| \xi(k+1) \right|^2 \right] > 0 \tag{1.7.2}$$

for every nontrivial $\xi \in U(m)$, *where*

$$U(m) = \left\{ \xi = \{\xi(k)\}_{k=1}^{\infty} : \xi(k) = 0,\ k \leq m,\ \exists\, n > m : \xi(k) = 0,\ k \geq n \right\}. \tag{1.7.3}$$

Remark 1.7.3. It is clear that to prove oscillation of equation (1.2.1) it suffices to find for any $m \in \mathbb{N}$ a (nontrivial) admissible sequence $\xi \in U(m)$ for which $\mathcal{F}(\xi) \leq 0$.

The proofs of Lemmas 1.7.1 and 1.7.2 will be given in Chapters 2 and 3 for more general equations. We note that, in the following sections we will only employ the Riccati technique, while the variational principle will be considered in Chapters 2 and 3.

1.8. Oscillation and nonoscillation criteria

In this section we will present many criteria for the oscillation and nonoscillation of equation (1.3.6). Comparison results of equations of the same type as (1.3.6) are discussed.

To obtain some of the results in this section, we will need the following lemmas.

Lemma 1.8.1. *Let $q(k) \geq p(k) > 0$, $k \in \mathbb{N}$, and let $\{x(k)\}_{k=0}^{\infty}$ be an eventually positive solution of the equation*

$$q(k)x(k) + \frac{1}{x(k-1)} = 1. \tag{1.8.1}$$

Then the equation

$$p(k)y(k) + \frac{1}{y(k-1)} = 1 \tag{1.8.2}$$

has a solution $\{y(k)\}_{k=0}^{\infty}$ satisfying $y(k) \geq x(k) > 1$ for all $k \in \mathbb{N}_0$.

Proof. We note first that any positive solution $\{x(k)\}$ of equation (1.8.1) is readily seen to satisfy $x(k) > 1$ for all $k \in \mathbb{N}_0$. This follows because equation (1.8.1) implies $1/x(k-1) < 1$ and hence $x(k-1) > 1$ for all $k \in \mathbb{N}_0$. Given such a solution of equation (1.8.1), define $y(k)$ for $k \in \mathbb{N}_0$ inductively by choosing $y(0) = x(0)$ and letting $y(k)$ satisfy equation (1.8.2). In order to be assured that $y(k)$ for $k \in \mathbb{N}$ is well defined by equation (1.8.2), we need to know that $y(k) \neq 0$ for $k \in \mathbb{N}$. But if $x(k-1) \leq y(k-1)$ and equation (1.8.2) holds, then equations (1.8.1) and (1.8.2) imply that

$$p(k)y(k) = 1 - \frac{1}{y(k-1)} = q(k)x(k) + \frac{1}{x(k-1)} - \frac{1}{y(k-1)} \geq q(k)x(k), \tag{1.8.3}$$

so

$$y(k) \geq \frac{q(k)x(k)}{p(k)} \geq x(k) > 1. \tag{1.8.4}$$

Also $q(k) \geq p(k) > 0$ by hypothesis. Therefore the sequence $\{y(k)\}$ is well defined and is a solution of equation (1.8.2) by definition. Thus, for all $k \in \mathbb{N}$, $y(k)$ satisfies equation (1.8.2) and the inequality $y(k) \geq x(k)$ for $k \in \mathbb{N}_0$, which completes the proof. $\square$

Assume that $c(k) \equiv 1$ in equation (1.3.6). Then equation (1.3.6) can be written as

$$x(k+1) + x(k-1) = [q(k) + 2]x(k), \tag{1.8.5}$$

where $q(k) = b(k) - 2$ for $k \in \mathbb{N}$, and the alternate form (1.3.7) becomes

$$-\Delta^2 x(k-1) + q(k)x(k) = 0. \tag{1.8.6}$$

By Theorem 1.3.5, equation (1.8.5) is nonoscillatory if and only if the related Riccati equation

$$r(k) + \frac{1}{r(k-1)} = q(k) + 2 \tag{1.8.7}$$

has a solution $r(k)$ defined for all sufficiently large k. In the following lemma we will compare solutions of equation (1.8.7) with solutions of an equation of the same form, in which the coefficients $q(k)$ are replaced by coefficients $\xi(k)$ defined as follows. For any fixed $M \in \mathbb{N}$, let

$$\xi(k) = \begin{cases} q(k) & \text{for } k \leq M - 1, \\ q(M) + q(M+1) & \text{for } k = M, \\ q(k+1) & \text{for } k \geq M + 1. \end{cases} \tag{1.8.8}$$

For such a sequence of coefficients we consider the equation

$$y(k+1) + y(k-1) = [\xi(k) + 2]y(k), \tag{1.8.9}$$

and the related Riccati equation

$$h(k) + \frac{1}{h(k-1)} = \xi(k) + 2. \tag{1.8.10}$$

Now, we present the following result.

Lemma 1.8.2. *Suppose that equation (1.8.5) is nonoscillatory and let x be a solution of equation (1.8.5) such that $x(k) > 0$ for $n \geq N - 1$ for some $N \in \mathbb{N}$. For any fixed $M > N$ define the sequence $\{\xi(k)\}$ as in (1.8.8). Then equation (1.8.9) is nonoscillatory. Moreover, if y is the solution of equation (1.8.9) satisfying the initial conditions $y(M-1) = x(M-1)$ and $y(M) = x(M)$, then $y(k) > 0$ for all $k \geq N - 1$ and the sequence $\{h(k)\}$ defined by $h(k) = y(k+1)/y(k)$ for $k \geq N - 1$ is a solution of equation (1.8.10) satisfying*

$$\begin{gathered} h(k) = r(k) > 0 \quad \text{for } N - 1 \leq k \leq M - 1, \\ h(k) \geq r(k+1) > 0 \quad \text{for } k \geq M, \end{gathered} \tag{1.8.11}$$

where $r(k) = x(k+1)/x(k)$ for $k \geq N - 1$.

Proof. Given a solution x of equation (1.8.5) such that $x(k) > 0$ for $k \geq N-1$, let $r(k) = x(k+1)/x(k)$ for $k \geq N-1$ so that r is a solution of equation (1.8.7). Let y be the solution of equation (1.8.9) as stated. Since $y(M-1) = x(M-1)$, $y(M) = x(M)$, and $\xi(k) = q(k)$ for $k \leq M-1$, it is clear from equations (1.8.5) and (1.8.9) that $y(k) = x(k)$ for $k \leq M$. Thus $y(k) > 0$ for $N-1 \leq k \leq M$, so we may define $h(k) = y(k+1)/y(k)$ for $N-1 \leq k \leq M$. Then $h(k) > 0$ for $N-1 \leq k \leq M$. Also, dividing equation (1.8.9) by $y(k)$, we see that $h(k)$ satisfies equation (1.8.10) for $N \leq k \leq M$. We need to show that $h(M) > 0$ so that equation (1.8.10) can be used to define $h(M+1)$. To show this, we first write equation (1.8.1) for $k = M$ and $k = M+1$ and add the results to obtain

$$r(M+1) = q(M) + q(M+1) + 4 - \left[r(M) + \frac{1}{r(M)}\right] - \frac{1}{r(M-1)}. \tag{1.8.12}$$

Now, $y(k) = x(k)$ for $k \leq M$, so in particular $r(M-1) = h(M-1)$. Furthermore, $r(M) + (1/r(M)) \geq 2$. Thus (1.8.7) and (1.8.12) imply that

$$r(M+1) \leq \xi(M) + 2 - \frac{1}{h(M-1)} = h(M). \tag{1.8.13}$$

Thus

$$r(M+1) \leq h(M), \tag{1.8.14}$$

and since $r(M+1) > 0$, we have $h(M) > 0$.

We may therefore define $h(M+1)$ by equation (1.8.10), that is,

$$h(M+1) = \xi(M+1) + 2 - \frac{1}{h(M)}. \tag{1.8.15}$$

It follows from equations (1.8.9) and (1.8.15), and the definition of $h(M)$ that $h(M+1) = y(M+2)/y(M+1)$. Also, (1.8.7), (1.8.8), (1.8.14), and (1.8.15) imply that

$$\begin{aligned} r(M+2) &= q(M+2) + 2 - \frac{1}{r(M+1)} \leq \xi(M+1) + 2 - \frac{1}{h(M)} \\ &= h(M+1), \end{aligned} \tag{1.8.16}$$

and hence

$$0 < r(M+2) \leq h(M+1). \tag{1.8.17}$$

Proceeding inductively as in steps (1.8.15), (1.8.16), and (1.8.17), we conclude that $h(k)$ is defined for all $k \geq N-1$ and satisfies (1.8.11), which therefore completes the proof. □

Next, we give the following simple comparison result.

Lemma 1.8.3. *If $\{x(k)\}$, $k \geq N > 0$, is a positive solution of*

$$x(k) + \frac{1}{x(k-1)} = \eta_1(k), \tag{1.8.18}$$

and if $\eta(k) \geq \eta_1(k) > 0$ for $k \geq N$, then

$$y(k) + \frac{1}{y(k-1)} = \eta(k) \tag{1.8.19}$$

has a solution with $y(k) \geq x(k)$ for $k \geq N$.

Proof. Given such a sequence $\{x(k)\}$, let $x(N) = y(N)$ and define

$$y(N+1) = \eta(N+1) - \frac{1}{y(N)} \geq \eta_1(N+1) - \frac{1}{x(N)} = x(N+1). \tag{1.8.20}$$

Thus, $y(k)$ satisfies equation (1.8.19) for $k = N+1$ and $y(N+1) \geq x(N+1) > 0$. Proceeding inductively, we construct the required solution $\{y(k)\}$. □

We now use the Riccati difference equations written in the forms (1.3.13), (1.3.14), and (1.3.15) to develop various conditions for oscillation and nonoscillation in terms of the coefficients of equation (1.3.6).

Theorem 1.8.4. *If $b(k) \leq c(k-1)$ for all sufficiently large $k \in \mathbb{N}$ and if*

$$\limsup_{k\to\infty} \frac{c(k)}{c(k-1)} > \frac{1}{2}, \tag{1.8.21}$$

then equation (1.3.6) is oscillatory.

Proof. Assume that equation (1.3.6) is nonoscillatory. Then, by Theorem 1.3.5(ii), equation (1.3.13) has a positive solution $\{r(k)\}$, $k \geq N$, for some $N \in \mathbb{N}$. From the hypotheses, for some $M \geq N$, we have

$$\frac{c(k)}{c(k-1)} r(k) + \frac{1}{r(k-1)} = \frac{b(k)}{c(k-1)} \leq 1 \quad \text{for } k \geq M. \tag{1.8.22}$$

Since condition (1.8.21) holds, for some $\beta > 1/2$ there is a sequence $k_j \to \infty$ with $c(k_j)/c(k_j - 1) > \beta$ for all $j \in \mathbb{N}$. Then (1.8.22) implies

$$\beta r(k_j) + \frac{1}{r(k_j - 1)} < 1 \tag{1.8.23}$$

for all sufficiently large j. Since all terms in (1.8.22) are positive, we can conclude that $c(k)r(k)/c(k-1) < 1$ and $1/r(k-1) < 1$ for all sufficiently large k, hence $r(k) > 1$ and $1/r(k) > c(k)/c(k-1)$ for all sufficiently large k. In particular,

$$\frac{1}{r(k_j)} > \frac{c(k_j)}{c(k_j - 1)} > \beta \tag{1.8.24}$$

and $r(k_j) > 1$, so $\beta r(k_j) > \beta$ for all sufficiently large j. It follows that each term on the left in (1.8.23) is greater than β for all sufficiently large j. Hence $\beta < 1/2$, which is a contradiction, and the proof is complete. □

Example 1.8.5. The difference equation

$$\frac{1}{4}x(k+1) + x(k-1) = x(k) \quad \text{for } k \in \mathbb{N} \tag{1.8.25}$$

has a nonoscillatory solution $x(k) = 2^k$, $k \in \mathbb{N}_0$. Only condition (1.8.21) of Theorem 1.8.4 is violated.

We note that the condition $b(k) \le c(k-1)$ of Theorem 1.8.4 is not itself sufficient to imply that equation (1.3.6) is oscillatory.

As an extension of Theorem 1.8.4, we have the following result. Its proof is omitted as it is similar to that of Theorem 1.8.4.

Theorem 1.8.6. *If for some constant $\gamma > 0$, $b(k) \le \gamma c(k-1)$ for all sufficiently large k and if*

$$\limsup_{k\to\infty} \frac{c(k)}{c(k-1)} > \frac{\gamma^2}{2}, \tag{1.8.26}$$

then equation (1.3.6) is oscillatory.

Consider equation (1.3.6) written in the form (1.3.15), where

$$g(k) = \frac{c^2(k)}{b(k)b(k+1)} \quad \forall k \in \mathbb{N}_0. \tag{1.8.27}$$

Theorem 1.8.7. *If*

$$g(k) \geq \frac{1}{4-\varepsilon} \quad \textit{for some } \varepsilon > 0 \textit{ and for all sufficiently large } k \in \mathbb{N}, \tag{1.8.28}$$

then equation (1.3.6) is oscillatory.

Proof. Without loss of generality, because of Theorem 1.3.5, we assume that $0 < \varepsilon < 4$. Suppose that equation (1.3.6) is nonoscillatory. Then equation (1.3.15) has a positive solution $\{s(k)\}$, $k \geq N$, for some $N \in \mathbb{N}$, that is, $s(k)$ satisfies the equation

$$g(k)s(k) + \frac{1}{s(k-1)} = 1 \quad \text{for } n \geq N, \tag{1.8.29}$$

where g is defined by (1.8.27). Since $g(k) \geq 1/(4-\varepsilon)$ by hypothesis, Lemma 1.8.1 implies that the equation

$$\left(\frac{1}{4-\varepsilon}\right)v(k) + \frac{1}{v(k-1)} = 1 \tag{1.8.30}$$

has a solution $\{v(k)\}$, $k \geq N$, which satisfies $v(k) \geq s(k) > 1$ for all $k \geq N$. We now define a positive sequence $\{x(k)\}$, $k \geq N$, inductively by letting

$$\begin{aligned} x(N) &= 1, \\ x(k+1) &= \frac{1}{\sqrt{4-\varepsilon}}v(k)x(k) \quad \text{for } k \geq N. \end{aligned} \tag{1.8.31}$$

Then $v(k) = \sqrt{4-\varepsilon}(x(k+1)/x(k))$. Substituting this into equation (1.8.30), we find that $\{x(k)\}$ is a positive solution of the equation

$$x(k+1) + x(k-1) = \sqrt{4-\varepsilon}x(k) \quad \text{for } k > N. \tag{1.8.32}$$

But this is impossible because equation (1.8.32) is oscillatory, since it has the solutions $\{\cos k\theta\}$ and $\{\sin k\theta\}$, $k \in \mathbb{N}$, where $\theta = \tan^{-1}(\sqrt{\varepsilon/(4-\varepsilon)})$. Thus we have a contradiction and the theorem follows. □

The following example shows that condition (1.8.28) cannot in general be replaced by the weaker condition

$$g(k) \geq \frac{1}{4-\varepsilon(k)}, \quad \varepsilon(k) > 0, \ \lim_{k\to\infty} \varepsilon(k) = 0. \tag{1.8.28$'$}$$

Example 1.8.8. Consider the equation

$$x(k+1) + x(k-1) = \left(\frac{\sqrt{k+1} + \sqrt{k-1}}{\sqrt{k}}\right) x(k) \quad \text{for } k \in \mathbb{N}. \tag{1.8.33}$$

Here $c(k) \equiv 1$ and $b(k) = (\sqrt{k+1} + \sqrt{k-1})/\sqrt{k}$. It is readily verified that $b(k) < 2$ and $\lim_{k\to\infty} b(k) = 2$, hence $b(k)b(k+1) < 4$ and $\varepsilon = 4 - b(k)b(k+1) \to 0$ as $k \to \infty$. Thus condition (1.8.28′) holds, but equation (1.8.33) is nonoscillatory and one of its solutions is $x(k) = \sqrt{k}$.

We note that the self-adjoint form of equation (1.8.33) becomes

$$\Delta^2 x(k-1) - q(k)x(k) = 0, \tag{1.8.34}$$

where $q(k) = \Delta^2(\sqrt{k-1})/\sqrt{k}$. This may be thought of as a discrete analogue of the well-known *Euler equation*

$$x''(t) + \frac{1}{4t^2} x(t) = 0. \tag{1.8.35}$$

Equation (1.8.35) has $\sqrt{t}$ and $\sqrt{t}\ln t$ as nontrivial solutions, and the coefficient of x is $1/(4t^2) = -(1/\sqrt{t})d^2\sqrt{t}/dt^2$.

We now turn to an nonoscillation criterion which is a companion to the oscillation condition given in Theorem 1.8.7.

Theorem 1.8.9. *If*

$$g(k) \le \frac{1}{4} \quad \textit{for all sufficiently large } k \in \mathbb{N}, \tag{1.8.36}$$

then equation (1.3.6) is nonoscillatory.

Proof. Assume $g(k) \le 1/4$ for all $k \ge N > 0$. Construct a solution $\{s(k)\}$ of equation (1.3.15) inductively by defining

$$\begin{gathered} s(N) = 2, \\ s(k) = \frac{1}{g(k)}\left[1 - \frac{1}{s(k-1)}\right] \quad \text{for } k > N. \end{gathered} \tag{1.8.37}$$

We note that if $s(k-1) \geq 2$ for any $k \geq N$, then

$$g(k)s(k) = 1 - \frac{1}{s(k-1)} \geq \frac{1}{2}, \tag{1.8.38}$$

so $s(k) \geq 1/(2g(k)) \geq 4(1/2) = 2$. Therefore, the sequence $\{s(k)\}$, $k \geq N$ is well defined by (1.8.37), and it is readily verified that $s(k)$ satisfies equation (1.3.15). We thus have a positive solution of equation (1.3.15), so equation (1.3.6) is nonoscillatory by Theorem 1.3.5. This completes the proof. □

Corollary 1.8.10. *If*

$$b(k) \geq \max\{c(k-1), 4c(k)\} \quad \textit{for all sufficiently large } k \in \mathbb{N}, \tag{1.8.39}$$

then equation (1.3.6) is nonoscillatory.

Proof. If condition (1.8.39) holds, then $b(k+1) \geq c(k)$ and $b(k) \geq 4c(k)$. Therefore $g(k) \leq 1/4$ for all sufficiently large k. Hence the corollary follows from Theorem 1.8.9. □

Theorem 1.8.11. *If*

$$g(k_j) \geq 1 \quad \textit{for a sequence } k_j \longrightarrow \infty, \tag{1.8.40}$$

then equation (1.3.6) is oscillatory.

Proof. If equation (1.3.6) is nonoscillatory, then by Theorem 1.3.5, equation (1.3.15) has a positive solution $\{s(k)\}$, $k \geq N$, for some $N \in \mathbb{N}$. From equation (1.3.15), $g(k)s(k) < 1$ and $s(k) > 1$ for all $n > N$. Hence $g(k) < 1$ for $n > N$. Thus if $g(k) \geq 1$ for arbitrarily large values of k, equation (1.3.6) must be oscillatory, which completes the proof. □

The following corollary is immediate.

Corollary 1.8.12. *If*

$$\limsup_{k\to\infty} g(k) > 1, \tag{1.8.41}$$

then equation (1.3.6) is oscillatory.

Next, we present the following corollaries obtained from Theorem 1.8.11.

Corollary 1.8.13. *If*

$$\limsup_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} g(j) > 1, \tag{1.8.42}$$

then equation (1.3.6) is oscillatory.

Proof. If equation (1.3.6) is nonoscillatory, then $g(k) < 1$ for all sufficiently large k, say $k \geq N > 0$, by Theorem 1.8.11. So $\sum_{j=N}^{k} g(j) < k - N + 1$ for $k \geq N$. It follows that

$$\frac{1}{k}\sum_{j=1}^{k} g(j) < 1 + \frac{\gamma}{k} \quad \text{for some constant } \gamma. \tag{1.8.43}$$

This leads to a contradiction with condition (1.8.42), so the corollary follows. □

Corollary 1.8.14. *If*

$$\sum_{j=1}^{\infty} g^{-m}(j) < \infty \quad \textit{for some constant } m > 0, \tag{1.8.44}$$

then equation (1.3.6) is oscillatory.

Proof. Let $\sum_{j=1}^{\infty} g^{-m}(j) = G$, $0 < G < \infty$ for some $m > 0$. For any constant $\alpha > 1$, choose β such that $(1/\alpha) + (1/\beta) = 1$. Using Hölder's inequality, we obtain

$$k = \sum_{j=1}^{k} g^{1/\alpha}(j) g^{-1/\alpha}(j) \leq \left(\sum_{j=1}^{k} \left[g^{1/\alpha}(j)\right]^{\alpha}\right)^{1/\alpha} \left(\sum_{j=1}^{k} \left[g^{-1/\alpha}(j)\right]^{\beta}\right)^{1/\beta}. \tag{1.8.45}$$

Then

$$k^{1/\alpha} k^{1/\beta} = k \leq \left(\sum_{j=1}^{k} g(j)\right)^{1/\alpha} \left(\sum_{j=1}^{k} g^{1-\beta}(j)\right)^{1/\beta}, \tag{1.8.46}$$

from which it follows that

$$\frac{1}{k}\sum_{j=1}^{k} g(j) \geq \left[\frac{k}{\sum_{j=1}^{k} g^{1-\beta}(j)}\right]^{\alpha/\beta}. \tag{1.8.47}$$

In particular, if we choose $\alpha = (1+m)/m$ and $\beta = 1+m$, then inequality (1.8.47) implies that

$$\frac{1}{k}\sum_{j=1}^{k} g(j) \geq \left(\frac{K}{G}\right)^{1/m} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{1.8.48}$$

Therefore, equation (1.3.6) is oscillatory, by Corollary 1.8.13. □

Corollary 1.8.15. *If for some constants $m > 0$, $\varepsilon > 0$, and for all sufficiently large k,*

$$\sum_{j=1}^{\infty} \left[\frac{b(j)}{c(j-1)}\right]^{m} < \infty, \qquad \frac{c(k)}{c(k-1)} \geq \varepsilon, \tag{1.8.49}$$

then equation (1.3.6) is oscillatory.

Proof. If $c(k)/c(k-1) \geq \varepsilon$, then $c(k) \geq \varepsilon c(k-1)$ for all $k \geq N$ for some $N \in \mathbb{N}$. We may also assume that N is large enough so that $[b(k+1)/c(k)]^m \leq 1$ for $k \geq N$, since $\sum^{\infty}[b(j)/c(j-1)]^m < \infty$. Thus

$$\begin{aligned}\sum_{j=N}^{\infty} g^{-m}(j) &= \sum_{j=N}^{\infty}\left[\frac{b(j)b(j+1)}{c^2(j)}\right]^m \\ &\leq \sum_{j=N}^{\infty}\left[\frac{b(j+1)}{c(j)}\right]^m\left[\frac{b(j)}{\varepsilon c(j-1)}\right]^m \\ &\leq \frac{1}{\varepsilon^m}\sum_{j=N}^{\infty}\left[\frac{b(j)}{c(j-1)}\right]^m < \infty.\end{aligned} \tag{1.8.50}$$

Therefore, equation (1.3.6) is oscillatory by Corollary 1.8.14. □

Similarly, we state the following immediate results.

Corollary 1.8.16. *If for some constants $m > 0$, $\varepsilon > 0$ and for all sufficiently large k,*

$$\sum_{j=1}^{\infty}\left[\frac{b(j)}{c(j)}\right]^m < \infty, \qquad \frac{b(k+1)}{b(k)} \leq \varepsilon, \tag{1.8.51}$$

then equation (1.3.6) is oscillatory.

Corollary 1.8.17. *If for some constant $m > 0$ and for all sufficiently large k, either*

$$\sum_{j=1}^{\infty}\left[\frac{b(j)}{c(j-1)}\right]^m < \infty, \qquad b(k) \leq c(k) \tag{1.8.52}$$

or

$$\sum_{j=1}^{\infty}\left[\frac{b(j)}{c(j)}\right]^m < \infty, \qquad b(k+1) \leq c(k), \tag{1.8.53}$$

then equation (1.3.6) is oscillatory.

We turn now to a corollary of Theorem 1.8.7 which is related to Theorem 1.8.4.

Corollary 1.8.18. *If for some $\varepsilon > 0$ and for all sufficiently large k,*

$$b(k) \leq c(k-1), \qquad \frac{c(k)}{c(k-1)} \geq \frac{1}{4-\varepsilon}, \tag{1.8.54}$$

then equation (1.3.6) is oscillatory.

Proof. We have

$$\begin{aligned} g(k) &= \frac{c^2(k)}{b(k)b(k+1)} = \left(\frac{c(k)}{b(k+1)}\right)\left(\frac{c(k-1)}{b(k)}\right)\left(\frac{c(k)}{c(k-1)}\right) \\ &\geq 1 \times 1 \times \frac{1}{4-\varepsilon} = \frac{1}{4-\varepsilon}. \end{aligned} \tag{1.8.55}$$

The result now follows from Theorem 1.8.7. □

A corollary similar to Theorem 1.8.9 is presented as follows.

Corollary 1.8.19. *If for every sufficiently large k,*

$$b(k) \geq c(k-1), \qquad \frac{c(k)}{c(k-1)} \leq \frac{1}{4}, \tag{1.8.56}$$

then equation (1.3.6) is oscillatory.

Remark 1.8.20. Example 1.8.5 shows that the condition $b(k) \leq c(k-1)$ is not itself sufficient for oscillation. Similarly, we note that the condition $c(k)/c(k-1) \leq 1/4$ in Corollary 1.8.19 is not sufficient for nonoscillation. This can be seen by the example $c(k) = 4^{-k}$, $b(k) = c(k-1)/2$. Here $c(k)/c(k-1) = 1/4$, but $g(k) \equiv 1$ for all k. Hence equation (1.3.6) is oscillatory by Theorem 1.8.7.

We note that Theorems 1.8.7 and 1.8.9 together imply that the constant 1/4 is the best possible. Next, we proceed further in this direction and obtain the following necessary condition for nonoscillation.

We define

$$\gamma(k,m) = 4\prod_{j=0}^{m}\left(\frac{1}{4g(k+j)}\right), \tag{1.8.57}$$

where $g(k)$ is defined by (1.8.27), $m \geq 0$ is any constant, and $k \geq N$ for some $N \in \mathbb{N}_0$.

Theorem 1.8.21. *Suppose that equation (1.3.6) is nonoscillatory. Then there exists $N \in \mathbb{N}$ such that for any $n \geq N$ and $m \geq 0$,*

$$\gamma(k,m) > 1. \tag{1.8.58}$$

Proof. Assume that equation (1.3.6) is nonoscillatory and let x be a solution of equation (1.3.6) such that $x(k) \neq 0$ for $k \geq N$. Let $z(k) = c(k)x(k+1)/x(k)$ for $k \geq N$. Then from equation (1.3.14) we may write

$$\begin{aligned} b(k)b(k+1) &= \left[z(k) + \frac{c^2(k-1)}{z(k-1)}\right]\left[z(k+1) + \frac{c^2(k)}{z(k)}\right] \\ &= c^2(k)\left[1 + \frac{1}{\alpha(k-1)}\right][1+\alpha(k)], \end{aligned} \tag{1.8.59}$$

where $\alpha(k) = z(k)z(k+1)/c^2(k) > 0$.

Similarly,

$$b(k+1)b(k+2) = c^2(k+1)\left[1+\frac{1}{\alpha(k)}\right][1+\alpha(k+1)]. \tag{1.8.60}$$

It follows that

$$\begin{aligned} & b(k)b^2(k+1)b(k+2) \\ & \quad = c^2(k)c^2(k+1)\left[1+\frac{1}{\alpha(k-1)}\right][1+\alpha(k)]\left[1+\frac{1}{\alpha(k)}\right][1+\alpha(k+1)] \\ & \quad \geq 4c^2(k)c^2(k+1)\left[1+\frac{1}{\alpha(k-1)}\right][1+\alpha(k+1)]. \end{aligned} \tag{1.8.61}$$

Proceeding inductively, we obtain

$$\begin{aligned} & b(k)b^2(k+1)\cdots b^2(k+m)b(k+m+1) \\ & \quad \geq 4^m c^2(k)\cdots c^2(k+m)\left[1+\frac{1}{\alpha(k-1)}\right][1+\alpha(k+m)] \\ & \quad \geq 4^m c^2(k)\cdots c^2(k+m). \end{aligned} \tag{1.8.62}$$

Since this is equivalent to (1.8.58), the proof is complete. □

The contrapositive of Theorem 1.8.21 says: equation (1.3.6) is oscillatory if for every $N \in \mathbb{N}$ there exists $k \geq N$ such that $\gamma(k,m) \leq 1$ for some constant $m > 0$. Since this statement involves two variables k and m, one can state corollaries in various forms. For example, we have the following corollary.

Corollary 1.8.22. *If for some $m \geq 0$ there exists a sequence $k_j \to \infty$ such that $\gamma(k_j,m) \leq 1$, then equation (1.3.6) is oscillatory.*

The following theorem is also an immediate corollary of Theorem 1.8.21.

Theorem 1.8.23. *If for every $N \in \mathbb{N}$ there exists $K \geq N$ such that*

$$\liminf_{m\to\infty} \gamma(K,m) < 1, \tag{1.8.63}$$

then equation (1.3.6) is oscillatory.

We also note that Theorem 1.8.7 becomes a corollary of Theorem 1.8.21. Specifically, if $g(k) \geq 1/(4-\varepsilon)$ for some $\varepsilon > 0$ and for all $k \geq N$, then

$$g(k)g(k+1)\cdots g(k+m) \geq \frac{1}{(4-\varepsilon)^{m+1}} > \frac{1}{4^m}, \tag{1.8.64}$$

if $k \geq N$ and m is large enough. Thus, equation (1.3.6) is oscillatory.

Another corollary of Theorem 1.8.21 is the following result.

Corollary 1.8.24. *If*

$$\liminf_{k\to\infty} \frac{c(k)}{4^k} = 0, \tag{1.8.65}$$

and

$$\frac{\prod_{j=1}^{k} b(j)}{\prod_{j=1}^{k} c(j)} \quad \text{is bounded, say by } M \text{ as } k \longrightarrow \infty, \tag{1.8.66}$$

then equation (1.3.6) is oscillatory.

Proof. If equation (1.3.6) is nonoscillatory, Theorem 1.8.21 implies that for some $N \in \mathbb{N}$ and all $m \geq 0$,

$$\frac{b(N)\cdots b(N+m)}{c(N)\cdots c(N+m)} \frac{b(N+1)\cdots b(N+m+1)}{c(N)\cdots c(N+m)} \frac{b(N)}{c(N+m+1)} \frac{c(N+m+1)}{b(N)} \geq 4^m. \tag{1.8.67}$$

However, the left-hand side of (1.8.67) is bounded above by $M^2 c(N+m+1)/b(N)$.
Thus, we have

$$M^2 \frac{c(N+m+1)}{b(N)} \geq 4^m \quad \text{for } m \geq 0, \tag{1.8.68}$$

which implies

$$4^{-(N+m+1)} c(N+m+1) \geq 4^{-(N+1)} \frac{b(N)}{M^2} \quad \forall m \geq 0, \tag{1.8.69}$$

which is a contradiction to condition (1.8.65). □

Corollary 1.8.25. *If condition (1.8.65) of Corollary 1.8.24 is replaced by*

$$\sum_{j=1}^{\infty} \frac{1}{c(j)} = \infty, \tag{1.8.70}$$

then the conclusion of Corollary 1.8.24 holds.

Proof. As in the proof of Corollary 1.8.24 we obtain (1.8.68). Now (1.8.68) implies

$$\frac{1}{c(N+m+1)} \leq \frac{1}{4^m}\left(\frac{M^2}{b(N)}\right), \tag{1.8.71}$$

which contradicts condition (1.8.70). □

We note that condition (1.8.58) is a necessary condition for nonoscillation. The following example shows that it is not a sufficient condition.

Example 1.8.26. Consider the difference equation

$$x(k+1)+x(k-1)=b(k)x(k) \quad \forall k \in \mathbb{N}_0, \tag{1.8.72}$$

where

$$b(2k)=\frac{4^{2-k}}{2}, \quad b(2k-1)=4^{k-1} \quad \text{for } k \in \mathbb{N}. \tag{1.8.73}$$

It is easy to see that

$$g(2k-1)=\frac{1}{2}, \quad g(2k)=\frac{1}{8} \quad \text{for } k \in \mathbb{N}. \tag{1.8.74}$$

We also note that for any $k \in \mathbb{N}$ and $m \geq 0$ condition (1.8.58) is satisfied. However, we claim that with this definition of $b(k)$ and $c(k) \equiv 1$, equation (1.8.72) is oscillatory.

Suppose that it is not so. Then equation (1.3.15) has a positive solution $\{s(k)\}$ defined for all k sufficiently large. By equation (1.3.15), we have $1/s(k-1) \leq 1$, that is,

$$s(k-1) \geq 1. \tag{1.8.75}$$

Since g is defined by (1.8.70), equation (1.3.15) implies that

$$s(2k-1)=2\left[1-\frac{1}{s(2k-2)}\right], \tag{1.8.76}$$

$$s(2k)=8\left[1-\frac{1}{s(2k-1)}\right]. \tag{1.8.77}$$

Substituting equation (1.8.76) in equation (1.8.77) yields

$$s(2k)=4\left[\frac{s(2k-2)-2}{s(2k-2)-1}\right]=4-\frac{4}{s(2k-2)-1}. \tag{1.8.78}$$

Since $s(2k-2)>1$ and $s(2k)>1$ by (1.8.75), it follows that (1.8.78) implies $s(2k-2)>2$ and $s(2k)<4$. Thus

$$2<s(j)<4 \quad \text{when } j \text{ is even and sufficiently large.} \tag{1.8.79}$$

It follows from (1.8.78) and (1.8.79) that

$$4 - \frac{4}{s(j) - 1} > 3 \quad \text{for all even } j \text{ sufficiently large.} \tag{1.8.80}$$

Inequality (1.8.80) implies that $s(j) > 5$ which contradicts (1.8.79). Thus equation (1.8.72) must be oscillatory.

We may note that Theorem 1.8.23 fails to apply to equation (1.8.72), since in this case, we have

$$\gamma(K, m) = \begin{cases} 4 & \text{if } m \text{ is odd,} \\ 2 & \text{if } K \text{ is odd and } m \text{ is even,} \\ 8 & \text{if } K \text{ and } m \text{ are both even.} \end{cases} \tag{1.8.81}$$

Thus

$$\liminf_{m \to \infty} \gamma(K, m) = \begin{cases} 2 & \text{if } K \text{ is odd,} \\ 4 & \text{if } K \text{ is even,} \end{cases} \tag{1.8.82}$$

so condition (1.8.63) fails to hold. Therefore, we conclude that if condition (1.8.63) is not satisfied for arbitrarily large K, then equation (1.3.6) may or may not be oscillatory. This leads us to ask what additional conditions are sufficient for equation (1.3.6) to be oscillatory if condition (1.8.63) is not satisfied.

Theorem 1.8.27. *If for some $K \in \mathbb{N}$,*

$$\liminf_{m \to \infty} \gamma(K, m) \neq \limsup_{m \to \infty} \gamma(K, m), \tag{1.8.83}$$

then equation (1.3.6) is oscillatory.

Remark 1.8.28. Note that if condition (1.8.83) holds for some $K \in \mathbb{N}$, then it holds for all $K \in \mathbb{N}$ by definition of $\gamma(K, m)$.

Proof of Theorem 1.8.27. Suppose that equation (1.3.6) is nonoscillatory. Then by Theorem 1.3.5 there is a positive sequence $\{s(k)\}$ which satisfies the Riccati equation (1.3.15)

$$g(j)s(j) + \frac{1}{s(j-1)} = 1 \tag{1.8.84}$$

for all sufficiently large j, say $j \geq K$. By Remark 1.8.28, we may take this to be the same value K as in the hypothesis of the theorem. Multiplying equation (1.3.15) by $1/g(j)$ yields

$$\frac{1}{g(j)} = s(j) + \frac{1}{g(j)s(j-1)} \quad \text{for } j \geq K, \tag{1.8.85}$$

so that

$$\frac{1}{g(j)g(j+1)} = \left[s(j) + \frac{1}{g(j)s(j-1)}\right]\left[s(j+1) + \frac{1}{g(j+1)s(j)}\right]. \tag{1.8.86}$$

From (1.8.86), we obtain

$$\frac{1}{g(j)} = g(j+1)s(j)s(j+1)\left[1 + \frac{1}{g(j)s(j)s(j-1)}\right]\left[1 + \frac{1}{g(j+1)s(j+1)s(j)}\right], \tag{1.8.87}$$

so for $j \geq K$,

$$\begin{aligned}\frac{1}{g(j)} &= \left[1 + \frac{1}{g(j)s(j)s(j-1)}\right][1 + g(j+1)s(j)s(j+1)] \\ &= \left[1 + \frac{1}{\xi(j)}\right][1 + \xi(j+1)],\end{aligned} \tag{1.8.88}$$

where we define $\xi(j) = g(j)s(j)s(j-1)$. Note that $\xi(j) > 0$ for $j \geq K$.

From (1.8.57) and (1.8.88), we have

$$\begin{aligned}\gamma(K,m) &= 4\prod_{j=0}^{m}\frac{1}{4}\left[1 + \frac{1}{\xi(K+j)}\right][1 + \xi(K+j+1)] \\ &= 4\left[1 + \frac{1}{\xi(K)}\right][1 + \xi(K+m+1)]\prod_{i=1}^{m}\frac{1}{4}\left[1 + \frac{1}{\xi(K+i)}\right][1 + \xi(K+i)].\end{aligned} \tag{1.8.89}$$

Now it is easy to see that

$$\gamma(K,m) = 4\left[1 + \frac{1}{\xi(K)}\right][1 + \xi(K+m+1)]\prod_{i=1}^{m}\left[1 + \frac{1}{4\xi(K+i)}(\xi(K+i) - 1)^2\right] \tag{1.8.90}$$

for $m > 0$. Next, we rewrite this last equation as

$$\gamma(K,m) = 4\left[1 + \frac{1}{\xi(K)}\right][1 + \xi(K+m+1)]\prod_{i=1}^{m}(1 + A(i)), \tag{1.8.91}$$

where

$$A(i) = \frac{1}{4\xi(K+i)}[\xi(K+i) - 1]^2 \geq 0 \quad \text{for } i \geq 1. \tag{1.8.92}$$

Now from the hypotheses we must have $\liminf_{m\to\infty} \gamma(K,m) < \infty$. Hence, there exists a finite bound B such that

$$\gamma(K, m(k)) \le B \tag{1.8.93}$$

for some sequence of subscripts $m(k) \to \infty$. Since $\xi(j) > 0$ for all j, (1.8.91) and (1.8.93) imply that $\prod_{i=1}^{m(k)}[1 + A(i)]$ is bounded. Since $m(k) \to \infty$ and $A(i) \ge 0$, it follows that

$$\prod_{i=1}^{\infty}[1 + A(i)] \quad \text{is bounded,} \tag{1.8.94}$$

which implies that

$$\sum_{i=1}^{\infty} A(i) \quad \text{is finite.} \tag{1.8.95}$$

Therefore,

$$\lim_{i\to\infty} A(i) = 0. \tag{1.8.96}$$

Thus, by (1.8.92) we have

$$\lim_{j\to\infty} \xi(j) = 1, \tag{1.8.97}$$

that is,

$$\lim_{j\to\infty} g(j)s(j)s(j-1) = 1. \tag{1.8.98}$$

Since $A(i) \ge 0$, (1.8.91), (1.8.94), and (1.8.97) imply that $\lim_{m\to\infty} \gamma(K,m)$ exists, which contradicts condition (1.8.83). This completes the proof. □

In the next theorem we refer to ℓ^2, the space of square summable sequences. We remark, as in Remark 1.8.28, that if the hypotheses of Theorems 1.8.29 and 1.8.32 below hold for some $K \in \mathbb{N}$, then they hold for all $K \in \mathbb{N}$.

Theorem 1.8.29. *If for some $K \in \mathbb{N}$,*

$$\liminf_{m\to\infty} \gamma(K,m) < \infty, \tag{1.8.99}$$

and if the sequence

$$\left[\frac{1}{g(j)} - 4\right] \notin \ell^2, \tag{1.8.100}$$

then equation (1.3.6) is oscillatory.

Proof. Assume that equation (1.3.6) is nonoscillatory. Because of the condition (1.8.99), we proceed as in the proof of Theorem 1.8.27 to arrive at (1.8.95) and (1.8.97), and thus by (1.8.92), we have

$$\sum_{j=1}^{\infty} \frac{[\xi(j)-1]^2}{4\xi(j)} < \infty. \tag{1.8.101}$$

Since $\lim_{j\to\infty} \xi(j) = 1$, it follows from (1.8.101) that

$$[\xi(j)-1] \in \ell^2. \tag{1.8.102}$$

Expanding the right-hand side of (1.8.88) leads to

$$\frac{1}{g(j)} = 1 + \frac{1}{\xi(j)} + \xi(j+1) + \frac{\xi(j+1)}{\xi(j)}, \tag{1.8.103}$$

so

$$\begin{aligned}\frac{1}{g(j)} - 4 &= \frac{2}{\xi(j)} - 2 + \xi(j+1) - 1 + \frac{\xi(j+1)}{\xi(j)} - \frac{1}{\xi(j)} \\ &= \frac{2}{\xi(j)}[1-\xi(j)] + [\xi(j+1)-1] + \frac{1}{\xi(j)}[\xi(j+1)-1].\end{aligned} \tag{1.8.104}$$

By (1.8.97) and (1.8.102), each term on the right-hand side of (1.8.104) is in ℓ^2, so $[(1/g(j)) - 4] \in \ell^2$. Therefore, if condition (1.8.100) holds, then equation (1.3.6) is oscillatory, as claimed. □

Corollary 1.8.30. *If for some $K \in \mathbb{N}$, condition (1.8.99) holds and*

$$\lim_{j\to\infty} \frac{1}{g(j)} \neq 4, \tag{1.8.105}$$

then equation (1.3.6) is oscillatory.

Proof. The condition (1.8.105) implies the condition (1.8.100). □

Example 1.8.31. The difference equation

$$x(k+1) + x(k-1) = 2x(k) \quad \text{for } k \in \mathbb{N}_0 \tag{1.8.106}$$

has linearly independent solutions $x_1(k) \equiv 1$ and $x_2(k) = k$ for $k \in \mathbb{N}_0$. Here $\lim_{m\to\infty} \gamma(K,m) = 4$. Only condition (1.8.100) of Theorem 1.8.29 (or condition (1.8.105) of Corollary 1.8.30) is violated.

Theorem 1.8.32. *If for some $K \in \mathbb{N}$ the sequence $\{y(K,m)\}$ is eventually monotone increasing in m, then equation (1.3.6) is nonoscillatory.*

Proof. If $y(K,m)$ is eventually monotone increasing in m, then there exists $M > 0$ such that for all $m > M$, $y(K,m) \geq y(K,m-1)$. From (1.8.57) this implies that $(K+m) \leq 1/4$ for all $m \geq M$. Thus equation (1.3.6) is nonoscillatory by Theorem 1.8.9. □

Next, we present the following criterion for oscillation of equation (1.8.5).

Theorem 1.8.33. *Let $\{q(k)\}$, $k \in \mathbb{N}$, be a sequence with the property that for any $N \in \mathbb{N}$ there exist integers $K > N$ and $n \in \mathbb{N}$ such that*

$$\sum_{i=0}^{n} q(K+i) \leq -2. \tag{1.8.107}$$

Then equation (1.8.5) is oscillatory.

Proof. Let $\{q(k)\}$ be such a sequence and suppose that equation (1.8.5) is nonoscillatory. Let x be a solution of equation (1.8.5) with $x(k) > 0$ for $k \geq N-1$ for some $N \in \mathbb{N}$. Then the sequence $\{r(k)\}$ defined by $r(k) = x(k+1)/x(k)$, $k \geq N-1$, satisfies the Riccati equation (1.8.7). By the hypothesis, we may choose $K > N$ and $n \in \mathbb{N}$ such that condition (1.8.107) holds.

For each $j \in \{0,1,\ldots,n\}$ we define a sequence $\{\xi(i,k)\}$, $k \in \mathbb{N}$, by setting $\xi(0,k) = q(k)$, $k \in \mathbb{N}$ and for each $j \in \{1,2,\ldots,n\}$,

$$\xi(j,k) = q(k) \quad \text{for } k \leq K-1, \tag{1.8.108}$$

$$\xi(j,K) = \xi(j-1,K) + \xi(j-1,K+1), \tag{1.8.109}$$

$$\xi(j,k) = \xi(j-1,k+1) \quad \text{for } k \geq K-1. \tag{1.8.110}$$

We consider the difference equations (1.8.9) and (1.8.10) with the coefficients $\xi(k)$ replaced by $\xi(j,k)$, $j \in \{1,2,\ldots,n\}$ as follows:

$$y(k+1) + y(k-1) = [\xi(j,k)+2]y(k), \tag{1.8.111}$$

$$h(k) + \frac{1}{h(k-1)} = \xi(j,k) + 2. \tag{1.8.112}$$

For each $j \in \{1,2,\ldots,n\}$ we let $\{y(j,k)\}$, $k \in \mathbb{N}_0$, be the solution of (1.8.111) satisfying the initial conditions $y(j,K-1) = x(K-1)$ and $y(j,K) = x(K)$. Repeated application of Lemma 1.8.2 shows that for $j \in \{1,2,\ldots,n\}$, equation (1.8.111) is nonoscillatory and the sequence $\{h(j,k)\} = \{y(j,k+1)/y(j,k)\}$ is defined for all

$k \geq N-1$ and is a solution of equation (1.8.112) satisfying

$$\begin{aligned} h(j,k) &= h(j-1,k) > 0 \quad \text{for } N-1 \leq k \leq K-1, \\ h(j,k) &\geq h(j-1,k+1) > 0 \quad \text{for } k \geq K, \end{aligned} \tag{1.8.113}$$

where $h(0,k) = r(k)$ for $k \geq N-1$. It follows that the right-hand side of equation (1.8.112) is positive, hence $\xi(j,k) > -2$ for $j \in \{1,2,\ldots,n\}$ and $k \geq N-1$. However, repeated application of (1.8.109) and (1.8.110) yields

$$\begin{aligned} \xi(n,K) &= \xi(0,K) + \xi(0,K+1) + \cdots + \xi(0,K+n) \\ &= q(K) + q(K+1) + \cdots + q(K+n) \leq -2 \end{aligned} \tag{1.8.114}$$

by hypothesis, which contradicts $\xi(n,K) > -2$ and completes the proof. □

Corollary 1.8.34. *If*

$$\liminf_{k\to\infty} \sum_{j=1}^{k} q(j) = -\infty, \tag{1.8.115}$$

then equation (1.8.5) is oscillatory.

Next, we state the analogue of the well-known Leighton-Wintner criterion for equation (1.3.6).

Theorem 1.8.35 (Leighton-Wintner criterion). *If*

$$\sum^{\infty} \frac{1}{c(j)} = \infty, \tag{1.8.116}$$

$$\sum^{\infty} q(j) = -\infty, \tag{1.8.117}$$

then equation (1.3.6) is oscillatory.

The following example shows that condition (1.8.117) by itself is not sufficient to imply oscillation of equation (1.3.6).

Example 1.8.36. The difference equation

$$k^2 x(k+1) + (k-1)^2 x(k-1) = \frac{k(2k^2-1)}{k+1} x(k) \quad \text{for } k \in \mathbb{N} \tag{1.8.118}$$

has a solution $x(k) = 1/k$. Clearly, (1.8.117) is satisfied since $q(k) = -1/(k+1)$ while condition (1.8.116) is violated since $\sum^{\infty} 1/j^2 < \infty$.

It turns out that the behavior of the solution $x(k) = 1/k$, $k \in \mathbb{N}$, in Example 1.8.36 is typical for this case, as the following theorem indicates.

Theorem 1.8.37. *Let condition (1.8.117) hold and equation (1.3.6) be nonoscillatory. Then not only must every solution of equation (1.3.6) be eventually of one sign, but it must be decreasing in absolute value as well.*

Proof. Suppose that condition (1.8.117) holds and equation (1.3.6) is nonoscillatory. Then any solution x is eventually of one sign, say, $x(k) > 0$ for all $k \geq N$, or $x(k) < 0$ for all $k \geq N$, for some $N \in \mathbb{N}_0$. Let $r(k) = x(k+1)/x(k)$ for $k \geq N$. Then the Riccati equation (1.3.13) can be written as

$$c(k)[r(k)-1] + c(k-1)\left[\frac{1}{r(k-1)} - 1\right] = q(k) \quad \text{for } k \geq N+1. \tag{1.8.119}$$

Summing both sides of (1.8.119) from $N+1$ to k and rearranging terms yields

$$c(k)[1-r(k)] = c(N)\left[\frac{1}{r(N)} - 1\right] + \sum_{j=N+1}^{k-1} \frac{c(j)}{r(j)}[r(k)-1]^2 - \sum_{j=N+1}^{k} q(j). \tag{1.8.120}$$

Using condition (1.8.117), the right-hand side of (1.8.120) is positive for all sufficiently large k, say, $k > K > N+1$. Therefore $1 - r(k) > 0$ for all $k > K$. Hence

$$0 < r(k) < 1 \quad \forall k > N. \tag{1.8.121}$$

Thus $0 < x(k+1)/x(k) < 1$ for all $k > N$. It follows that

$$|x(k+1)| < |x(k)| \quad \forall k > N, \tag{1.8.122}$$

which completes the proof. □

The argument used in Theorem 1.8.37 also yields the following lemma, from which various oscillation criteria readily follow.

Lemma 1.8.38. *Let condition (1.8.117) hold and suppose that for any $N \in \mathbb{N}$ there exists $K > N+1$ such that*

$$b(K) - c(K-1) + \sum_{j=N+1}^{K-1} q(j) \leq 0. \tag{1.8.123}$$

Then equation (1.3.6) is oscillatory.

Proof. Let equation (1.3.6) be nonoscillatory and let condition (1.8.117) hold. We may use $q(k) = b(k) - c(k) - c(k-1)$ to rewrite (1.8.120) as

$$\begin{aligned} c(k) - c(k)r(k) = {} & c(N)\left[\frac{1}{r(N)} - 1\right] + \sum_{j=N+1}^{k-1} \frac{c(j)}{r(j)}[r(k)-1]^2 \\ & - b(k) + c(k) + c(k-1) - \sum_{j=N+1}^{k-1} q(j). \end{aligned} \tag{1.8.124}$$

By Theorem 1.8.37, we may also assume that N is so large that (1.8.121) is satisfied, so that $c(N)[(1/r(N)) - 1] > 0$. Subtracting $c(k)$ from both sides, we obtain

$$\begin{aligned} -c(k)r(k) = c(N)\left[\frac{1}{r(N)} - 1\right] + \sum_{j=N+1}^{k-1} \frac{c(j)}{r(j)}[r(j)-1]^2 \\ - \left[b(k) - c(k-1) + \sum_{j=N+1}^{k-1} q(j)\right]. \end{aligned} \tag{1.8.125}$$

For $k = K > N + 1$ such that (1.8.123) holds, the right-hand side of (1.8.125) is positive, but the left-hand side is negative, a contradiction which completes the proof. □

The following results are immediate consequences of Lemma 1.8.38.

Theorem 1.8.39. *If condition (1.8.117) holds and $b(k) \le c(k-1)$ for all sufficiently large k, then equation (1.3.6) is oscillatory.*

Theorem 1.8.40. *If condition (1.8.117) holds and $\{b(k)\}$ is a bounded sequence, then equation (1.3.6) is oscillatory.*

For the case $c(k) \equiv 1$, we have $q(k) = b(k) - 2$. So Theorem 1.8.40 has the following immediate corollary.

Corollary 1.8.41. *If $\{b(k)\}$ is bounded and*

$$\sum^{\infty} [b(k) - 2] = -\infty, \tag{1.8.126}$$

then equation (1.8.5) is oscillatory.

Another related result to Corollary 1.8.41 is the following theorem due to Hinton and Lewis.

Theorem 1.8.42 (Hinton-Lewis criterion). *If $b(k) \le 2$ for all $k \in \mathbb{N}_0$ and*

$$\sum_{n=0}^{\infty} |b(n) - 2| < \infty, \qquad \liminf_{n\to\infty} \sum_{j=n}^{\infty} [b(j) - 2] < -1, \tag{1.8.127}$$

then equation (1.8.5) is oscillatory.

We now consider some examples in which we can employ Theorem 1.8.40 and at the same time illustrate a technique of determining oscillation or nonoscillation by transforming two distinct equations of the form of equation (1.3.6) to the same Riccati-type equation.

Example 1.8.43. Consider the difference equations

$$c(k)x(k+1) + c(k-1)x(k-1) = \sqrt{2}x(k) \quad \text{for } k \in \mathbb{N}, \tag{1.8.128}$$

where $c(k) \equiv 1$ for k odd, and $1/2$ for k even, and

$$c(k)x(k+1) + c(k-1)c(k-1) = b(k)x(k) \quad \text{for } k \in \mathbb{N}, \tag{1.8.129}$$

where $c(k)$ is the same as in (1.8.128) and $b(k) \equiv 1$ for k odd, and 2 for k even.

Equation (1.8.128) is oscillatory by Theorem 1.8.40 since $b(k) \equiv \sqrt{2}$ and $q(k) = b(k) - c(k) - c(k-1) = \sqrt{2} - 1 - (1/2)$. However, none of the direct oscillation criteria presented here, so far, apply to equation (1.8.129). So we consider the substitution $s(k) = b(k+1)x(k+1)/(c(k)x(k))$, which transforms equation (1.3.6) into the form (1.3.15). If this transformation is applied to equations (1.8.128) and (1.8.129), then we obtain in each case the same equation of the form (1.3.15), since in both cases, we have $g(k) = 1/2$ for odd k and $1/8$ for even k. Thus, by Theorem 1.3.5, equations (1.8.128) and (1.8.129) are both nonoscillatory if and only if the corresponding Riccati equation (1.3.15) has a positive solution defined for all sufficiently large k. But we know that equation (1.8.128) is oscillatory, hence equation (1.8.129) must be oscillatory as well.

We note that equation (1.8.72) in Example 1.8.26 also leads to precisely the same transformed equation (1.3.15) as do equations (1.8.128) and (1.8.129). Hence, we have here a much briefer argument for the oscillation of equation (1.8.72) than we gave in Example 1.8.26.

In the following example, we will employ Theorem 1.8.40 to show that the monotonicity hypothesis of Theorem 1.8.32 cannot be replaced by the condition that $\gamma(K,m) \to \infty$ as $m \to \infty$.

Example 1.8.44. Consider the difference equation

$$c(k)x(k+1) + c(k-1)x(k-1) = x(k) \quad \text{for } k \in \mathbb{N}, \tag{1.8.130}$$

where

$$c(l) = \begin{cases} \dfrac{1}{4\sqrt{\sqrt{2}}} & \text{if } l = 3k, \\ \dfrac{1}{\sqrt{2}} & \text{if } l = 3k+1, \\ \dfrac{1}{\sqrt{2\sqrt{2}}} & \text{if } l = 3k+2. \end{cases} \tag{1.8.131}$$

Then

$$g(l) = \begin{cases} \dfrac{1}{16\sqrt{2}} & \text{if } l = 3k, \\ \dfrac{1}{2} & \text{if } l = 3k+1, \\ \dfrac{1}{2\sqrt{2}} & \text{if } l = 3k+2, \end{cases} \tag{1.8.132}$$

so with $K = 2$,

$$\begin{aligned} \gamma(2, 3m+1) &= \frac{b(3m+3)b(3m+4)}{4c^2(3m+3)} \frac{b(3m+2)b(3m+3)}{4c^2(3m+2)} \frac{b(3m+1)b(3m+2)}{4c^2(3m+1)} \\ &\quad \times \gamma(2, 3m-2) \\ &= \frac{16\sqrt{2}}{4} \frac{2\sqrt{2}}{4} \frac{2}{4} \gamma(2, 3m-2) = 2\gamma(2, 3m-2). \end{aligned} \tag{1.8.133}$$

Thus $\gamma(2, j) \to \infty$ as $j \to \infty$. Also,

$$\gamma(2, 3m+2) = \frac{b(3m+4)b(3m+5)}{4c^2(3m+4)} \gamma(2, 3m+1) = \frac{1}{2} \gamma(2, 3m+1). \tag{1.8.134}$$

These previous two equations imply that $\gamma(2, j) \to \infty$ as $j \to \infty$ but not monotonically.

Now, define $q(k) = b(k) - c(k) - c(k-1)$. Then

$$q(l) = \begin{cases} 1 - \dfrac{1}{4\sqrt{\sqrt{2}}} - \dfrac{1}{\sqrt{2\sqrt{2}}} & \text{if } l = 3k, \\ 1 - \dfrac{1}{\sqrt{2}} - \dfrac{1}{4\sqrt{\sqrt{2}}} & \text{if } l = 3k+1, \\ 1 - \dfrac{1}{2\sqrt{2}} - \dfrac{1}{\sqrt{2}} & \text{if } l = 3k+2. \end{cases} \tag{1.8.135}$$

So, for all k, $q(3k) + q(3k+1) + q(3k+2) \simeq 3 - 3.0238 = -0.238$. Therefore, $\sum^{\infty} q(j) = -\infty$. Since the sequence $\{b(k)\}$ is bounded, equation (1.8.130) must be oscillatory by Theorem 1.8.40.

Next, we will introduce a technique for extending known oscillation criteria for equation (1.3.6) by employing the following result. We assume throughout that

$$g(k) = \frac{c^2(k)}{b(k)b(k+1)} < 1 \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{1.8.136}$$

In this case $\eta(k) = (1/g(k)) - 1 > 0$ for $k \geq N \geq 0$.

Theorem 1.8.45. *Let condition (1.8.136) hold. If the equation*

$$y(k+1) + y(k-1) = \eta(k)y(k) \tag{1.8.137}$$

is oscillatory, then equation (1.3.6) is also oscillatory.

Proof. Assume that equation (1.3.6) is nonoscillatory and let x be a solution with $x(k) > 0$ for $k \geq N \geq 0$. Then by Theorem 1.3.5, $z(k) = c(k)x(k+1)/x(k)$ is a positive solution of equation (1.3.14) for $k > N$. If we take equation (1.3.14) for k and for $k+1$, multiply the corresponding sides and divide the result by $c^2(k)$, we obtain

$$\frac{z(k)z(k+1)}{c^2(k)} + 1 + \frac{[z(k)z(k+1)/c^2(k)]}{[z(k-1)z(k)/c^2(k-1)]} + \frac{c^2(k-1)}{z(k-1)z(k)} = \frac{1}{g(k)}. \tag{1.8.138}$$

We rewrite this equation as

$$r(k) + 1 + \frac{r(k)}{r(k-1)} + \frac{1}{r(k-1)} = \frac{1}{g(k)} \quad \text{for } k > N, \tag{1.8.139}$$

where $r(k) = z(k)z(k+1)/c^2(k) > 0$. Thus, $\{r(k)\}$ is a positive solution of

$$r(k) + \frac{1}{r(k-1)} = \frac{1}{g(k)} - 1 - \frac{r(k)}{r(k-1)} = \eta(k) - \varepsilon(k) \quad \text{for } k > N, \tag{1.8.140}$$

where $\varepsilon(k) = r(k)/r(k-1) > 0$. By applying Lemma 1.8.3 with $\eta_1(k) = \eta(k) - \varepsilon(k)$, we see that there exists a sequence $v(k) \geq r(k) > 0$, $k > N$, satisfying the equation $v(k) + (1/v(k-1)) = \eta(k)$, $k > N$. This equation is of the form (1.3.14) with $c(k) \equiv 1$ and with $b(k)$ replaced by $\eta(k)$. Since $\{v(k)\}$ is a positive solution, we may apply Theorem 1.3.5 to conclude that equation (1.8.137) is nonoscillatory, which is a contradiction. This completes the proof. □

We now apply some of the results of this section to equation (1.8.137) to obtain new oscillation criteria for equation (1.3.6).

Theorem 1.8.46. *Equation (1.3.6) is oscillatory if condition (1.8.136) holds and for some $N \in \mathbb{N}$,*

$$\liminf_{k\to\infty} \sum_{j=N}^{k} \left[\frac{1}{g(j)} - 3\right] = -\infty. \tag{1.8.141}$$

Proof. This result follows by setting $q(k) = \eta(k) - 2 = (1/g(k)) - 3$ in condition (1.8.115) and then applying Corollary 1.8.34. □

Next, we will apply Theorem 1.8.11 to equation (1.8.137) and obtain the following oscillation result for equation (1.3.6).

Theorem 1.8.47. *If condition (1.8.136) holds and*

$$g(k_j) + g(k_j + 1) \geq 1 \quad \textit{for some sequence } k_j \longrightarrow \infty, \tag{1.8.142}$$

then equation (1.3.6) is oscillatory.

Proof. We apply Theorem 1.8.11 to equation (1.8.137), which tells us that equation (1.8.136) is oscillatory if $g_1(k_j) \geq 1$ for some sequence $k_j \to \infty$, where

$$g_1(k) = \frac{1}{\eta(k)\eta(k+1)} = \left[\frac{1}{g(k)} - 1\right]^{-1}\left[\frac{1}{g(k+1)} - 1\right]^{-1} \quad \text{for } k \geq N. \tag{1.8.143}$$

Thus,

$$\left[\frac{1}{g(k_j)} - 1\right]\left[\frac{1}{g(k_j+1)} - 1\right] \leq 1 \quad \text{for some sequence } k_j \longrightarrow \infty. \tag{1.8.144}$$

But, since $0 < g(k) < 1$ for all $k \geq N$, some simple algebra shows that condition (1.8.144) is equivalent to condition (1.8.142). □

Similarly, Theorem 1.8.7 leads to the following result.

Theorem 1.8.48. *If (1.8.136) holds and for some $\varepsilon > 0$ and for all $k \geq N > 0$,*

$$g(k) + g(k+1) + (3 - \varepsilon)g(k)g(k+1) \geq 1, \tag{1.8.145}$$

then equation (1.3.6) is oscillatory.

We will proceed further and extend the above result by applying Theorem 1.8.45 to equation (1.8.137). We will assume that

$$g(k) + g(k+1) < 1 \quad \forall k \geq N \text{ for some } N \in \mathbb{N}. \tag{1.8.146}$$

We let

$$\begin{gathered} g_1(k) = \frac{1}{\eta(k)\eta(k+1)}, \quad \eta(k) = \frac{1}{g(k)} - 1, \\ \eta_1(k) = \frac{1}{g_1(k)} - 1 = \frac{1 - g(k) - g(k+1)}{g(k)g(k+1)} \quad \text{for } k \geq N. \end{gathered} \tag{1.8.147}$$

Now, we have the following result.

Theorem 1.8.49. *If condition (1.8.145) holds and the equation*

$$u(k+1) + u(k-1) = \eta_1(k)u(k) \tag{1.8.148}$$

is oscillatory, then equation (1.8.136) is oscillatory also.

In Theorem 1.8.49, note that $\eta_1(k) > 0$ for $k \geq N$.

As in Theorem 1.8.47 we can apply Theorem 1.8.11 to equation (1.8.148) and obtain the following oscillation result for equation (1.3.6).

Theorem 1.8.50. *If condition (1.8.146) holds and*

$$\left[\frac{1 - g(k_j) - g(k_j+1)}{g(k_j)g(k_j+1)}\right]\left[\frac{1 - g(k_j+1) - g(k_j+2)}{g(k_j+1)g(k_j+2)}\right] \leq 1 \tag{1.8.149}$$

for some sequence $k_j \to \infty$, then equation (1.3.6) is oscillatory.

Another extension is presented below. We assume that condition (1.8.136) holds and let $e(k) = 1/[4g(k)g(k+1)] - 1$ for all sufficiently large k.

Theorem 1.8.51. *If condition (1.8.136) holds and either the equation*

$$y(k+1) + y(k-1) = e(2k)y(k) \quad \text{or} \quad z(k+1) + z(k-1) = e(2k-1)z(k) \tag{1.8.150}$$

is oscillatory, then equation (1.3.6) is oscillatory.

Proof. Assume that equation (1.3.6) is nonoscillatory and let x be a solution with $x(k) > 0$ for $k \geq N > 0$. Proceeding as in the proof of Theorem 1.8.45 we obtain (1.8.139). However, (1.8.139) may be rewritten in the form

$$(1 + r(k))\left(1 + \frac{1}{r(k-1)}\right) = \frac{1}{g(k)} \quad \text{for } k > N. \tag{1.8.151}$$

From (1.8.151), we obtain

$$(1 + r(k))\left(1 + \frac{1}{r(k-1)}\right)(1 + r(k+1))\left(1 + \frac{1}{r(k)}\right) = \frac{1}{g(k)g(k+1)}. \tag{1.8.152}$$

By the inequality $(1 + \gamma)(1 + (1/\gamma)) \geq 4$ for $\gamma > 0$, (1.8.152) implies that

$$4(1 + r(k+1))\left(1 + \frac{1}{r(k-1)}\right) \leq \frac{1}{g(k)g(k+1)} \quad \text{for } k > N. \tag{1.8.153}$$

Then

$$r(k+1)+\frac{1}{r(k-1)} \leq \frac{1}{4g(k)g(k+1)}-1-\frac{r(k+1)}{r(k-1)}=e(k)-\frac{r(k+1)}{r(k-1)}. \tag{1.8.154}$$

Thus $\{r(k)\}$ is a positive solution of an equation of the form

$$r(k+1)+\frac{1}{r(k-1)}=e(k)-\delta(k) \quad \text{for } k>N, \tag{1.8.155}$$

where $\delta(k) \geq r(k+1)/r(k-1)$. Next, consider the related first-order equation

$$u(k)+\frac{1}{u(k-1)}=e(2k)-\delta(2k). \tag{1.8.156}$$

The sequence $u(k) = r(2k+1)$ is a positive solution of equation (1.8.156) for $k > (N-1)/2$. By Lemma 1.8.3, the equation $v(k)+(1/v(k-1)) = e(2k)$ also has a positive solution. We then apply Theorem 1.3.5 to conclude that

$$y(k+1)+y(k-1)=e(2k)y(k) \tag{1.8.157}$$

is nonoscillatory. Thus, if equation (1.8.157) is oscillatory, equation (1.3.6) must be oscillatory also.

Similarly, $u(k) = r(2k)$ is a positive solution of the equation

$$u(k)+\frac{1}{u(k-1)}=e(2k-1)-\delta(2k-1). \tag{1.8.158}$$

By Lemma 1.8.3, the equation $v(k)+(1/v(k-1)) = e(2k-1)$ must have a positive solution $\{v(k)\}$. Again, an application of Theorem 1.3.5 implies that the equation

$$z(k+1)+z(k-1)=e(2k-1)z(k) \tag{1.8.159}$$

is nonoscillatory. Thus, if equation (1.8.159) is oscillatory, then so is equation (1.3.6). □

As in Theorem 1.8.46, we can apply Corollary 1.8.41 to equations (1.8.150) of Theorem 1.8.51 and obtain the following immediate result.

Theorem 1.8.52. *Let condition (1.8.136) hold and*

$$\liminf_{k\to\infty} \sum_{j=N}^{k} \left[e(j) - 3\right] = -\infty, \tag{1.8.160}$$

or, equivalently,

$$\liminf_{k\to\infty} \sum_{j=N}^{k} \left[\gamma(j,1) - 3\right] = -\infty, \tag{1.8.161}$$

where $\gamma(k,m)$ is defined as in (1.8.57). Then equation (1.3.6) is oscillatory.

1.9. Comparison theorems

In this section we will present comparison results for equations of type (1.3.6). In addition to equations (1.3.6) and (1.3.15), we consider the equations

$$c_1(k)y(k+1) + c_1(k-1)y(k-1) = b_1(k)y(k), \tag{1.9.1}$$

$$g_1(k)S(k) + \frac{1}{S(k-1)} = 1, \tag{1.9.2}$$

where $c_1(k) > 0$ and $b_1(k) > 0$ for all $k \geq N > 0$ and $g_1(k) = c_1^2(k)/(b_1(k)b_1(k+1))$.

Now we present the following comparison result.

Theorem 1.9.1. *Suppose that*

$$g_1(k) \geq g(k) \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{1.9.3}$$

If equation (1.9.1) is nonoscillatory, then equation (1.3.6) is nonoscillatory also.

Proof. If equation (1.9.1) is nonoscillatory, Theorem 1.3.5 implies that equation (1.9.2) has a positive solution $\{S(k)\}$ defined for all $k \geq N$ for some $N \in \mathbb{N}_0$. We may assume that $g_1(k) \geq g(k)$ for all $k \geq N$ also. Note that $S(k) > 1$ for all $k \geq N$ since equation (1.9.2) implies that $1/S(k-1) < 1$ for all $k \geq N$. Choose $s(N)$ such that $s(N) \geq S(N) > 1$ and define $s(N+1)$ using (1.3.15). Thus, from equations (1.3.15) and (1.9.2), it follows that

$$\begin{aligned} g(N+1)s(N+1) = 1 - \frac{1}{s(N)} &= g_1(N+1)S(N+1) + \frac{1}{S(N)} - \frac{1}{s(N)} \\ &\geq g_1(N+1)S(N+1), \end{aligned} \tag{1.9.4}$$

hence

$$s(N+1) \geq \left(\frac{g_1(N+1)}{g(N+1)}\right) S(N+1) \geq S(N+1) > 1. \tag{1.9.5}$$

By induction, we may thus obtain a solution $\{s(k)\}$ of equation (1.3.15) for $k \geq N$, satisfying $s(k) > 1$ for all $k \geq N$. Theorem 1.3.5 then implies that equation (1.3.6) is nonoscillatory, which completes the proof. □

Corollary 1.9.2. *Suppose that*

$$c_1(k) \geq c(k), \quad b_1(k) \leq b(k) \quad \textit{for all sufficiently large } k \in \mathbb{N}. \tag{1.9.6}$$

If equation (1.9.1) is nonoscillatory, then equation (1.3.6) is nonoscillatory also.

Proof. By hypothesis

$$\frac{c_1^2(k)}{c^2(k)} \geq 1 \geq \frac{b_1(k)b_1(k+1)}{b(k)b(k+1)} \quad \text{for } k \geq N \geq 0, \tag{1.9.7}$$

which implies that $g_1(k) \geq g(k)$. The result now follows from Theorem 1.9.1. □

Next, by comparing Corollary 1.9.2 with Theorem 1.4.3, we have the following analogue of Sturm's comparison theorem.

Theorem 1.9.3 (Sturm's comparison theorem). *Suppose that*

$$c_1(k) \leq c(k), \qquad b_1(k) - c_1(k) - c_1(k-1) \leq b(k) - c(k) - c(k-1) \tag{1.9.8}$$

for all sufficiently large k. Then if equation (1.9.1) is nonoscillatory, equation (1.3.6) is nonoscillatory also.

Finally, we consider a comparison example in which Theorem 1.9.1 is applicable, but Corollary 1.9.2 and Theorem 1.9.3 are not.

Example 1.9.4. Consider the difference equation

$$y(k+1) + y(k-1) = b_1(k)y(k) \quad \text{for } k \in \mathbb{N}_0, \tag{1.9.9}$$

where

$$b_1(2k) = \frac{2}{3^{k-1}}, \quad b_1(2k-1) = 3^{k-1} \quad \text{for } k \in \mathbb{N}. \tag{1.9.10}$$

We compare the coefficients of equation (1.9.9) with the coefficients in equation (1.8.72). We have $c(k) = c_1(k) = 1$, $b_1(2k-1) \leq b(2k-1)$, and $b_1(2k) \geq b(2k)$

for $k \in \mathbb{N}$. Hence, Corollary 1.9.2 and Theorem 1.9.3 are not applicable. However, $g_1(2k-1) = 1/2$ and $g_1(2k) = 1/6$ for $k \in \mathbb{N}$ in equation (1.9.9), while $g(2k-1) = 1/2$ and $g(2k) = 1/8$ for $k \in \mathbb{N}$ in equation (1.8.72). Thus, Theorem 1.9.1 is applicable and since equation (1.8.72) was shown in Example 1.8.26 to be oscillatory, equation (1.9.9) is oscillatory also.

1.10. Oscillation, nonoscillation, and monotone solutions

Here we will discuss oscillation, nonoscillation, and existence of monotone solutions of equation (1.5.45) (or equation (1.4.4)). First, we will establish necessary and sufficient conditions for the nonoscillation of equation (1.4.4).

Theorem 1.10.1. *The following statements are equivalent.*

(i) *There exists $N \in \mathbb{N}_0$ such that for any $n \in \mathbb{N}_0$, there is no set of positive numbers $\beta_0, \beta_1, \ldots, \beta_n$ satisfying ($1.5.47^{N,n}$).*

(ii) *There exists $N \in \mathbb{N}_0$ such that the sequence $\{s(N,n)\}_{n=1}^{\infty}$ is well defined, that is,*

$$p(N) > 0, \tag{1.10.1}$$

and for $n \in \mathbb{N}$,

$$p(N+n) > \cfrac{c^2(N+n)}{p(N+n-1) - \cfrac{c^2(N+n-1)}{p(N+n-2) - \cdots - \cfrac{c^2(N+2)}{p(N+1) - \cfrac{c^2(N+1)}{p(N)}}}}. \tag{1.10.2}$$

(iii) *Equation (1.4.4) is nonoscillatory.*

Proof. Lemma 1.5.13 implies that (i)⇔(ii). Next, we show that (ii)⇒(iii). It is easy to verify by equation (1.5.45) that the sequence $\{x(k)\}$ defined by $x(N) = 0$, $x(N+1) = 1$, and

$$x(N+n) = \prod_{i=1}^{n-1} \left[\frac{c(N+i)}{s(N,i)} \right] \quad \text{for } n \geq 2 \tag{1.10.3}$$

is a positive solution which is nonoscillatory. Therefore, equation (1.4.4) is nonoscillatory.

Now, we prove that (iii)⇒(i). Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.4.4). Then there is an $N \in \mathbb{N}_0$ such that $x(k)x(k+1) > 0$ for all $k \geq N$, that is, such that x has no generalized zeros on $[N, \infty)$. By the separation theorem, that is, Theorem 1.4.4, equation (1.4.4) is disconjugate on I_{n+2} for any $n \in \mathbb{N}_0$. We conclude that (i) is true by virtue of Theorem 1.5.14, and this completes the proof. □

The contrapositive of Theorem 1.10.1 gives necessary and sufficient conditions for the oscillation of equation (1.4.4). For convenience in applications, we state the following result.

Corollary 1.10.2. *Equation (1.4.4) is oscillatory if and only if for any $k \in \mathbb{N}_0$ there exist $N = N(k) \geq k$ and $n = n(k) \geq 0$ such that when $n = 0$,*

$$p(N) \leq 0, \tag{1.10.4}$$

and when $n \in \mathbb{N}$,

$$p(N+i)\begin{cases} > 0 & \text{if } i = 0, \\ > \dfrac{c^2(N+i)}{p(N+i-1) - \cdots - \dfrac{c^2(N+1)}{p(N)}} & \text{if } 1 \leq i \leq n-1, \\ \leq \dfrac{c^2(N+n)}{p(N+n-1) - \cdots - \dfrac{c^2(N+1)}{p(N)}} & \text{if } i = n. \end{cases} \tag{1.10.5}$$

Some of the known oscillation results presented above can be drawn from Corollary 1.10.2. We have the following examples.

(I_1) If $n = 0$ or $n = 1$ in Corollary 1.10.2, we obtain Theorem 1.8.11, that is, if $p(k) \leq 0$ or $p(k)p(k+1) \leq c^2(k+1)$ for a sequence of indices $k = k_j \to \infty$ then equation (1.5.45) is oscillatory.

(I_2) If $n = 2$ in Corollary 1.10.2, Corollary 1.10.1 reduces to Theorem 1.8.47, that is, if $p(k) > 0$ and $p(k)p(k+1) > c^2(k+1)$ for all sufficiently large k, but $p(k)p(k+1)p(k+2) \leq p(k)c^2(k+2) + p(k+2)c^2(k+1)$ for a sequence of indices $k = k_j \to \infty$, then equation (1.4.4) is oscillatory.

(I_3) If $n = 3$ in Corollary 1.10.2, we get Theorem 1.8.50, that is, if $p(k) > 0$, $p(k)p(k+1)p(k+2) > p(k)c^2(k+2)+p(k+2)c^2(k+1)$ for all sufficiently large k, but the inequality

$$\begin{aligned}&\left[1 - g^*(k+1) - g^*(k+2)\right]\left[1 - g^*(k+2) - g^*(k+3)\right] \\ &\quad \leq g^*(k+3)\left(g^*(k+2)\right)^2 g^*(k+3),\end{aligned} \tag{1.10.6}$$

where $g^*(k) = c^2(k)/(p(k-1)p(k))$ holds for a sequence of indices $k = k_j \to \infty$, then equation (1.5.45) is oscillatory.

Proof of (I_3). By the first two conditions, $p(k)p(k+1) > c^2(k+1)$ for all sufficiently large k. By the sufficiency of Corollary 1.10.2 it suffices to show that if (1.10.6) holds for an index k, then

$$p(k)p(k+1)p(k+2)p(k+3) \leq H(k), \tag{1.10.7}$$

where

$$H(k) = p(k+2)p(k+3)c^2(k+1) + p(k)p(k+3)c^2(k+2) \\ + p(k)p(k+1)c^2(k+3) - c^2(k+1)c^2(k+3). \tag{1.10.8}$$

From (1.10.6) and the definition of $g^*(k)$, after some manipulation, we get

$$\begin{aligned} &p(k)p(k+1)p(k+2)p(k+3) \\ &\quad \le H(k) + p(k)p(k+3)c^2(k+2) \\ &\qquad + c^2(k+1)c^4(k+2)c^2(k+3)\big[p(k+1)p(k+2)\big]^{-2} \\ &\qquad - c^2(k+2)\big[p(k+1)p(k+2)\big]^{-1}\big[H(k) + c^2(k+1)c^2(k+3)\big]. \end{aligned} \tag{1.10.9}$$

Rearranging this and using $c^2(k+2) < p(k+1)p(k+2)$, we have

$$\begin{aligned} &p(k)p(k+1)p(k+2)p(k+3)\left[1 - \frac{c^2(k+2)}{p(k+1)p(k+2)}\right] \\ &\quad \le H(k)\left[1 - \frac{c^2(k+2)}{p(k+1)p(k+2)}\right]. \end{aligned} \tag{1.10.10}$$

From this, (1.10.6) follows and the proof is complete. □

Example 1.10.3. Consider the difference equation

$$\Delta^2 x(k) + q(k)x(k+1) = 0, \tag{1.10.11}$$

or equivalently the three-term recurrence relation

$$x(k+2) + x(k) = p(k)x(k+1), \tag{1.10.12}$$

where $p(k) = 2 - q(k)$. For each $i \ge 0$ let $N = 6i$ and let $q(N) = q(N+4) = 1$, $q(N+1) = q(N+3) = 0$, $q(N+2) = -1/3$, $q(N+5) = -a$, where a is a constant. Then we have $p(N) = p(N+4) = 1$, $p(N+1) = p(N+3) = 2$, $p(N+2) = 7/3$, $p(N+5) = 2+a$. Since the coefficients are periodic, we can set $s(k) = s(N,k)$ for all $N = 6i$. Then, $s(1) = s(2) = 1$, $s(3) = 3/4$, $s(4) = 4/5$, $s(5) = 5$. Now, we consider the following two cases.

(I) If $a > 3$, then $s(6) = 1/(a-3)$ and equation (1.10.11) is disconjugate on $[N, N+7]$ by Theorem 1.5.14.

(II) If $a \le 3$, then $s(6)$ cannot be defined and hence equation (1.10.11) is oscillatory by Corollary 1.10.2.

However, it is interesting to observe that $a > 3$ does not guarantee nonoscillation. Suppose that $a > 3$ and that for some $M = 6n$, $s(M) = u$ is defined, and $u < (a-3)/(5a-6) = u_0$. Thus, $s(M+1), \dots, s(M+6)$ can also be defined, and

$$s(M+6) = \frac{1-5u}{a-3-(5a-6)u} = f(u). \tag{1.10.13}$$

If $u \geq u_0$, then $s(M+6)$ cannot be defined. Note that $f(0) = s(6)$ and $f(u)$ is increasing for $u \in [0, u_0)$. Consider the equation $u = f(u)$. It has two positive real roots $u_1 \leq u_2$ provided $a \geq 9 + \sqrt{53} = a_0$. We claim that $a \geq a_0$ implies nonoscillation. In fact, if $a \geq a_0$, then $s(6) < u_1 < u_0$. Since $f(u)$ is increasing, $f(u_1) = u_1$, and $u < f(u) < u_1$ for $u < u_1$, the sequence $\{s(6n)\}_{n=1}^{\infty}$ defined by $s(6n+6) = f(s(6n))$ is increasing and bounded above by u_1. This means that $s(k)$ can be defined for all $k \in \mathbb{N}$, and equation (1.10.11) is then nonoscillatory by Theorem 1.10.1.

If $3 < a < a_0$, then there is a positive number δ depending only on a such that $f(u) \geq u + \delta$ for $0 < u < u_0$. Consequently, the iteration $s(6n+6) = f(s(6n))$ can be carried out at most a finite number of times until s reaches a value larger than u_0. Thus, $\{s(k)\}$ is a finite sequence. From Corollary 1.10.2 and what was proved for the case $a \leq 3$, it follows that equation (1.10.11) is oscillatory if $a < a_0$.

Next, we investigate the existence of monotone solutions of equation (1.4.4).

Theorem 1.10.4. *Let $N \in \mathbb{N}$. Then the following statements are equivalent.*

(i) *Equation (1.4.4) has a solution which is either nonnegative and increasing, or nonpositive and decreasing on I_∞.*

(ii) *Each nontrivial solution of equation (1.4.4) has at most one generalized zero on I_∞, and each nontrivial solution having a generalized zero on I_∞ is strictly monotone starting from the generalized zero.*

(iii) *$\{s(N,n)\}_{n=1}^{\infty}$ is well defined and*

$$c(N+n) > s(N,n) \quad \forall n \in \mathbb{N}. \tag{1.10.14}$$

Proof. Suppose that (iii) holds. Then the solution of equation (1.4.4) with $x(N) = 0$ and $x(N+1) = 1$ can be defined by

$$x(N+i) = \prod_{j=1}^{i-1}\left[\frac{c(N+j)}{s(N,j)}\right] \quad \text{for } i \geq 2, \tag{1.10.15}$$

and satisfies

$$\Delta x(N+i) = x(N+i)\left[\frac{c(N+i) - s(N,i)}{s(N,i)}\right] \quad \text{for } i \in \mathbb{N}. \tag{1.10.16}$$

It follows from (1.10.14) and (1.10.16) that $\Delta x(N+i) > 0$ for $i \geq 0$, that is, x is a nonpositive and increasing solution. We then conclude (iii)⇒(i).

To prove (i)$\Rightarrow$(ii), let x be a nonnegative, increasing solution of equation (1.4.4) on I_∞. Then $x(N) > 0$, $\Delta x(N) > 0$, and $x(k) > 0$, $\Delta x(k) > 0$ for $k > N$. For any integer $M \geq N + 2$, as in the proof of sufficiency of Theorem 1.5.17, we can similarly prove that equation (1.4.4) is right disfocal on $[N, M]$. Then each nontrivial solution having a generalized zero in $[N, M - 1]$ must be increasing starting from the generalized zero. Since M is an arbitrarily large number, (ii) holds. If x is a nonpositive, decreasing solution, the proof is similar.

Finally, it is obvious that (ii)$\Rightarrow$(iii) by Theorem 1.5.17. This completes the proof. □

Remark 1.10.5. In Theorem 1.10.4, those solutions which have no generalized zeros at all may not be strictly monotone (e.g., the equation $\Delta^2 x(k) = 0$ has nonzero constant solutions); they may even not be monotone (see Example 1.10.6 below).

Example 1.10.6. Consider the equation (1.10.11) (or equation (1.10.12)), where $q(4i) = 0$, $q(4i+1) = q(4i+3) = -2$, $q(4i+2) = 2/3$ for $i \in \mathbb{N}_0$. Then $p(4i) = 2$, $p(4i+1) = p(4i+3) = 4$, $p(4i+2) = 4/3$ for $i \in \mathbb{N}_0$. Set $s(k) = s(0,k)$ for $k \in \mathbb{N}$. A straightforward computation gives $s(1) = 1/2$, $s(2) = 2/7$, $s(3) = 21/22$, and $s(4) = 22/67$. They are all less than 1. Suppose $s(k) < 1$ for $1 \leq k \leq 4i$. We will prove that $s(k) < 1$ for $1 \leq k \leq 4i + 4$. In fact,

$$\begin{aligned}
s(4i+1) &= \frac{1}{p(4i) - s(4i)} < \frac{1}{2-1} = 1, \\
s(4i+2) &= \frac{1}{p(4i+1) - s(4i+1)} < \frac{1}{4-1} = \frac{1}{3}, \\
s(4i+3) &= \frac{1}{p(4i+2) - s(4i+2)} < \frac{1}{(4/3) - (1/3)} = 1, \\
s(4i+4) &= \frac{1}{p(4i+3) - s(4i+3)} < \frac{1}{4-1} = \frac{1}{3}.
\end{aligned} \tag{1.10.17}$$

By induction, we know that $s(k) < c(k) < 1$ for all $k \in \mathbb{N}$. Thus, Theorem 1.10.4(iii) is satisfied, and hence equation (1.4.4) is nonoscillatory and right disfocal on $[0, k]$ for any $k \in \mathbb{N}$. But, it is easy to verify that the sequence

$$x(4i) = x(4i+1) = x(4i+2) = 2, \quad x(4i+3) = 3 \quad \text{for } i \in \mathbb{N}_0 \tag{1.10.18}$$

is a positive solution whose first-order difference is oscillatory.

1.11. Oscillation criteria by Riccati technique

The results of this section are oscillation and nonoscillation criteria for equation (1.4.4) which are based on the Riccati technique. In particular, we consider the case when $\sum^\infty q(j)$ is convergent and $\sum^\infty 1/c(j) = \infty$. We will prove that several various additional assumptions, along with the above conditions, are sufficient for equation (1.4.4) to be oscillatory.

First we give some auxiliary lemmas. We start with a technical result concerning certain behavior of the sequence $\sum^k q(j)$.

Lemma 1.11.1. *Suppose that there exist $n_0, n_1 \in \mathbb{Z}$ with $n_1 \geq n_0$ such that*

$$\sum_{j=k}^{\infty} q(j) \geq 0, \qquad \sum_{j=n_1}^{\infty} q(j) > 0 \tag{1.11.1}$$

for all $k \geq n_0$. Then there exists $m \geq n_0$ such that

$$\sum_{j=m}^{k} q(j) \geq 0 \quad \forall k \geq m. \tag{1.11.2}$$

Proof. If $\sum_{j=n_1}^{k} q(j) \geq 0$ for all $k \geq n_1$, then we are done. If not, then there exists $n_2 > n_1$ such that $\sum_{j=n_1}^{n_2} q(j) < 0$. Hence we may define

$$N = \sup\left\{k > n_1 : \sum_{j=n_1}^{k} q(j) < 0\right\}. \tag{1.11.3}$$

From (1.11.1) it follows that $N < \infty$, from which we have $\sum_{j=N}^{k} q(j) \geq 0$ for all $k \geq N$. This proves the lemma. □

In the sequel, we will assume that

$$\text{condition (1.11.1) holds and } q(k) \not\equiv 0 \text{ eventually} \tag{1.11.4}$$

and

$$\sum^{\infty} \frac{1}{c(j)} = \infty. \tag{1.11.5}$$

The next lemma shows that under certain assumptions, a positive solution of equation (1.4.4) has a positive difference.

Lemma 1.11.2. *Assume that conditions (1.11.4) and (1.11.5) hold and let $x = \{x(k)\}$ be a nonoscillatory solution of equation (1.4.4) such that $x(k) > 0$ for all $k \geq n_0$. Then there exists $N \geq n_0$ such that $\Delta x(k) > 0$ for all $k \geq N$.*

Proof. The proof is by contradiction. We will consider two cases.

Case 1. Suppose that $\Delta x(k) < 0$ for all large k, say $K \geq N \geq n_0$. Without loss of generality, we may suppose that (1.11.2) holds for $k \geq N$ and $q(N) \geq 0$. Define $Q(k) = \sum_{j=N}^{k} q(j)$ for $k \geq N$ and $Q(N-1) = 0$. Thus, we have for $k \geq N$,

$$\begin{aligned}\sum_{j=N}^{k} q(j)x(j+1) &= \sum_{j=N}^{k} \Delta Q(j-1)x(j+1) \\ &= Q(k)x(k+2) - \sum_{j=N}^{k} Q(j)\Delta x(j+1) \geq 0.\end{aligned} \tag{1.11.6}$$

Therefore,

$$\sum_{j=N}^{k} \Delta\big(c(j)\Delta x(j)\big) = c(k+1)\Delta x(k+1) - c(N)\Delta x(N) \le 0, \tag{1.11.7}$$

so that

$$\Delta x(k+1) \le -\frac{a}{c(k+1)} \quad \text{for } k \ge N, \tag{1.11.8}$$

where $a = -c(N)\Delta x(N) > 0$. Summing both sides of (1.11.8) from N to k, we obtain

$$x(k+2) - x(N+1) = \sum_{j=N}^{k} \Delta x(j+1) \le -a \sum_{j=N}^{k} \frac{1}{c(j+1)}. \tag{1.11.9}$$

By (1.11.5) we have $x(k) \to -\infty$ as $k \to \infty$, which is a contradiction.

Case 2. Assume that there exists a subsequence $k_\ell \to \infty$ such that $\Delta x(k_\ell) \le 0$ and $\Delta x(k_\ell + 1) \ge 0$. Setting $v(k) = -c(k)\Delta x(k)/x(k)$, we obtain from equation (1.4.4) that

$$\Delta v(k) = q(k) + \frac{v^2(k)}{c(k) - v(k)}, \quad c(k) - v(k) > 0. \tag{1.11.10}$$

Hence

$$\begin{aligned} 0 \ge \sum_{j=k_\ell}^{k_{\ell+1}} \Delta v(j) &= v(k_{\ell+1}) - v(k_\ell) \\ &= \sum_{j=k_\ell}^{k_{\ell+1}} q(j) + \sum_{j=k_\ell}^{k_{\ell+1}} \frac{v^2(j)}{c(j) - v(j)}. \end{aligned} \tag{1.11.11}$$

Summing this equation as $k_{\ell+1} \to \infty$, we obtain

$$\sum_{j=k_\ell}^{\infty} q(j) < 0. \tag{1.11.12}$$

Since $v(k) \not\equiv 0$, (1.11.12) contradicts condition (1.11.4), and the proof of the lemma is complete. □

In the following lemma, a necessary condition for nonoscillation of equation (1.4.4) is given in the case when $\lim_{k\to\infty} \sum^{k} q(j)$ is convergent and the hypotheses of Lemma 1.11.2 hold.

Lemma 1.11.3. *Let the hypotheses of Lemma 1.11.2 hold and assume further that*

$$\sum_{j=1}^{\infty} q(j) = \lim_{k\to\infty} \sum_{j=1}^{k} q(j) \quad \text{is convergent.} \tag{1.11.13}$$

Let x be a nonoscillatory solution of (1.4.4) with $x(k) > 0$ for all $k \geq N \geq n_0$. Then there exists $n \geq N$ such that

$$w(k) \geq \sum_{j=n}^{\infty} q(j) + \sum_{j=n}^{\infty} \frac{w^2(j)}{w(j)+c(j)} \quad \text{for } k \geq n, \tag{1.11.14}$$

where $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq n$.

Proof. From Lemma 1.11.2, there exists n such that $w(k) > 0$ for $k \geq n$ and $w(k)$ satisfies the Riccati equation (1.4.5) for $k \geq n$. Summing this equation from k to $m > k \geq n$ we obtain

$$w(m+1) - w(k) + \sum_{j=k}^{m} q(j) + \sum_{j=k}^{m} \frac{w^2(j)}{w(j)+c(j)} = 0. \tag{1.11.15}$$

Therefore,

$$0 < w(m+1) \leq w(n) - \sum_{j=k}^{m} q(j) \quad \forall m > n, \tag{1.11.16}$$

and hence

$$w(k) \geq \sum_{j=k}^{m} q(j) + \sum_{j=k}^{m} \frac{w^2(j)}{w(j)+c(j)} \quad \text{for } m > k \geq n. \tag{1.11.17}$$

Letting $m \to \infty$, we obtain (1.11.14). □

Lemma 1.11.4. *Let conditions (1.11.5) and (1.11.13) hold and assume further that $q(k) \geq 0$ (and eventually nontrivial) for all $k \geq N \geq n_0$. Let x be a nonoscillatory solution of (1.4.4) with $x(k) > 0$ for all $k \geq N$. Then $w(k) = c(k)\Delta x(k)/x(k) > 0$ for all $k \geq N$ and $w(k) \to 0$ as $k \to \infty$. Furthermore,*

$$w(k) \leq \left(\sum_{j=N}^{k-1} \frac{1}{c(j)}\right)^{-1} \quad \forall k > N. \tag{1.11.18}$$

Proof. From Lemma 1.11.2 and equation (1.4.5) we have $w(k) > 0$ and

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k)} \leq 0 \quad \text{for } k \geq N. \tag{1.11.19}$$

Hence

$$w(k)w(k+1) + c(k)\Delta w(k) \leq 0 \quad \text{for } k \geq N, \tag{1.11.20}$$

and therefore

$$\Delta\left[-\frac{1}{w(k)} + \sum_{j=1}^{k-1}\frac{1}{c(j)}\right] = \frac{c(k)\Delta w(k) + w(k)w(k+1)}{c(k)w(k)w(k+1)} \leq 0 \quad \text{for } k \geq N. \tag{1.11.21}$$

Summation of the above inequality from N to $k-1$ gives

$$-\frac{1}{w(k)} + \sum_{j=1}^{k-1}\frac{1}{c(j)} \leq \sum_{j=1}^{N}\frac{1}{c(j)} \quad \text{for } k \geq N, \tag{1.11.22}$$

so that

$$\frac{1}{w(k)} \geq \sum_{j=N+1}^{k-1}\frac{1}{c(j)}, \tag{1.11.23}$$

and $w(k) \to 0$ as $k \to \infty$ by (1.11.5). This completes the proof. □

If conditions (1.11.4) and (1.11.13) hold for $k \geq N \geq n_0$, then we define the sequence $Q(k;m)$ as follows:

$$Q(k;m) = \begin{cases} \displaystyle\sum_{j=k}^{\infty} q(j), & m = 0 \text{ for } k \geq N, \\ \displaystyle\sum_{j=k}^{\infty}\frac{Q^2(j;m-1)}{Q(j;m-1)+c(j)} + Q(k;0), & m \geq 1 \text{ for } k \geq N. \end{cases} \tag{1.11.24}$$

Now, we present the following theorem.

Theorem 1.11.5. *Let conditions (1.11.4), (1.11.5), and (1.11.13) hold. Then equation (1.4.4) is oscillatory provided one of the following two conditions hold.*

(I_1) *There exists $m \in \mathbb{N}$ such that $Q(k;0),\dots,Q(k;m-1)$ defined by (1.11.24) satisfy*

$$\sum_{j=k}^{\infty}\frac{Q^2(j;m-1)}{Q(j;m-1)+c(j)} = \infty. \tag{1.11.25}$$

(I_2) *There exists an integer $n \geq N$ such that*

$$\limsup_{m\to\infty} Q(n;m) = \infty. \tag{1.11.26}$$

Proof. Suppose, by contradiction, that equation (1.4.4) is nonoscillatory. Let (I_1) hold. If $m = 1$, then let x be a solution of equation (1.4.4) such that $x(k) > 0$ for $k \geq N \geq n_0$. Let $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq N$. Then by Lemmas 1.11.2 and 1.11.3, we have

$$w(k) \geq \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)} + \sum_{j=k}^{\infty} q(j) \geq Q(k;0) \tag{1.11.27}$$

for $k \geq n \geq N$. Hence

$$\sum_{j=k}^{\infty} \frac{Q^2(j;0)}{Q(j;0)+c(j)} \leq \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)} \quad \text{for } k \geq n \tag{1.11.28}$$

since the function $s^2/(s+c)$ is increasing for $s > 0$ and $c > 0$. But the last inequality contradicts assumption (1.11.25) for $m = 1$. Similarly, if $m > 1$, then from (1.11.14) and (1.11.24) we have that

$$w(k) \geq Q(k;i) \quad \text{for } i \in \{0,1,\dots,m-1\}, \tag{1.11.29}$$

and hence

$$\sum_{j=k}^{\infty} \frac{Q^2(j;m-1)}{Q(j;m-1)+c(j)} \leq \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)} < \infty \quad \text{for } k \geq n, \tag{1.11.30}$$

which again contradicts (1.11.25).

Suppose next that (I_2) holds. Obviously, in view of $w(k) \geq Q(k;m)$, $m \in \mathbb{N}_0$, (1.11.14) yields $\limsup_{m\to\infty} Q(n;m) \leq w(n) < \infty$, which is a contradiction. This completes the proof. □

As applications of Theorem 1.11.5 we give the following results.

Corollary 1.11.6. *Assume that $c(k) \equiv 1$ for $k \geq n_0 \geq 0$ and*

$$\sum_{j=k}^{\infty} q(j) \geq \frac{\alpha_0}{k} \tag{1.11.31}$$

for all sufficiently large k, where $\alpha_0 > 1/4$. Then equation (1.4.4) is oscillatory.

Proof. From (1.11.31) we have that $Q(k;0) \geq \alpha_0/k$, and so (1.11.24) gives

$$Q(k;1) = \sum_{j=k}^{\infty} \frac{Q^2(j;0)}{Q(j;0)+1} + Q(k;0) \geq \sum_{j=k}^{\infty} \frac{\alpha_0^2}{j(\alpha+j)} + \frac{\alpha_0}{k}. \tag{1.11.32}$$

But

$$\sum_{j=k}^{\infty} \frac{\alpha_0^2}{j(\alpha_0+j)} \geq \int_k^{\infty} \frac{\alpha_0^2}{s(\alpha_0+s)} ds = \alpha_0 \ln\left(\frac{\alpha_0+k}{k}\right). \tag{1.11.33}$$

Substituting (1.11.33) into (1.11.32), we obtain

$$Q(k;1) \geq \frac{\alpha_0}{k} + \alpha_0 \ln\left(\frac{\alpha_0+k}{k}\right) = \frac{\alpha_1}{k}, \tag{1.11.34}$$

where $\alpha_1 = \alpha_0 + \alpha_0 \ln[(\alpha_0+k)/k] > \alpha_0$. In general,

$$\begin{aligned} Q(k;m) &\geq \sum_{j=k}^{\infty} \frac{Q^2(k;m-1)}{Q(k;m-1)+1} + Q(k;0) \\ &\geq \sum_{j=k}^{\infty} \frac{\alpha_{m-1}^2}{j(\alpha_{m-1}+j)} + \frac{\alpha_0}{k} \\ &\geq \alpha_{m-1} \ln\left(\frac{\alpha_{m-1}+k}{k}\right) + \frac{\alpha_0}{k} \\ &= \frac{\alpha_m}{k}, \end{aligned} \tag{1.11.35}$$

where

$$\alpha_m = \alpha_0 + k\alpha_{m-1} \ln\left(\frac{\alpha_{m-1}+k}{k}\right). \tag{1.11.36}$$

It is easy to see that $\alpha_m < \alpha_{m+1} < \cdots$ for $m \in \mathbb{N}_0$. We claim that

$$\lim_{m\to\infty} \alpha_m = \infty. \tag{1.11.37}$$

If not, let $L = \lim_{m\to\infty} \alpha_m < \infty$. Then from (1.11.36) we have

$$L = \alpha_0 + kL \ln\left(\frac{L+k}{k}\right). \tag{1.11.38}$$

Letting $k \to \infty$ in (1.11.38) we obtain

$$L = \alpha_0 + L^2. \tag{1.11.39}$$

But equation (1.11.39) has no real solution if $\alpha_0 > 1/4$. Hence we must have (1.11.37). Then from above we have $\lim_{m\to\infty} Q(n;m) = \infty$, that is, (1.11.26) holds, so the conclusion follows from Theorem 1.11.5. □

Corollary 1.11.7. *Assume that $c(k) \equiv 1$ for $k \geq n_0 \geq 0$ and*

$$q(k) \geq \frac{\alpha_0}{k^2} \tag{1.11.40}$$

for all sufficiently large k, where $\alpha_0 > 1/4$ is a constant. Then equation (1.4.4) is oscillatory.

Proof. Since

$$\sum_{j=k}^{\infty} \frac{1}{j^2} \geq \int_k^{\infty} \frac{ds}{s^2} = \frac{1}{k}, \tag{1.11.41}$$

we see that $\sum_{j=k}^{\infty} q(j) \geq \alpha_0/k$, so Corollary 1.11.7 follows from Corollary 1.11.6. □

Theorem 1.11.8. *Let the assumptions of Lemma 1.11.4 hold and $Q(k;m)$ be defined as in (1.11.24). Assume further that*

$$\limsup_{k\to\infty} \left[\sum_{j=N}^{k-1} \frac{1}{c(j)}\right] Q(k;m) > 1 \tag{1.11.42}$$

for some $m \in \mathbb{N}_0$. Then equation (1.4.4) is oscillatory.

Proof. If not, then as in the proof of Theorem 1.11.5 we have $Q(k;m) \leq w(k)$ for $k \geq N \geq n_0$, $m \in \mathbb{N}_0$, and hence by Lemma 1.11.4

$$Q(k;m) \leq \left(\sum_{j=N}^{k-1} \frac{1}{c(j)}\right)^{-1} \quad \text{for } k \geq N,\ m \in \mathbb{N}_0. \tag{1.11.43}$$

But then

$$\limsup_{k\to\infty} \left(\sum_{j=N}^{k-1} \frac{1}{c(j)}\right) Q(k;m) \leq 1, \tag{1.11.44}$$

which contradicts condition (1.11.42). □

Remark 1.11.9. In particular, under the assumptions of Lemma 1.11.4, the condition

$$\limsup_{k\to\infty}\left(\sum_{j=N}^{k-1}\frac{1}{c(j)}\right)\sum_{j=k}^{\infty}q(j)>1 \tag{1.11.45}$$

guarantees oscillation of equation (1.4.4).

Using the estimate (1.11.18) from Lemma 1.11.4, we have the following criterion.

Theorem 1.11.10. *Let the assumptions of Lemma 1.11.4 hold and assume further that there exist two sequences of positive integers $\{m_\ell\}$ and $\{n_\ell\}$ with $N+1 \le m_\ell < n_\ell$ and $m_\ell \to \infty$ as $\ell \to \infty$ such that*

$$\sum_{j=m_\ell}^{n_\ell} q(j) \ge c(n_\ell+1)+\left(\sum_{j=N}^{m_\ell-1}\frac{1}{c(j)}\right)^{-1} \tag{1.11.46}$$

for large ℓ. Then equation (1.4.4) is oscillatory.

Proof. On the contrary, suppose that equation (1.4.4) is nonoscillatory and let x be a solution with $x(k) > 0$ for $k \ge N \ge n_0$. As in the proof of Lemma 1.11.4 we may suppose that $w(k) = c(k)\Delta x(k)/x(k) > 0$ for $k \ge N_1 \ge N$ and $w(k)$ satisfies the Riccati equation (1.4.5). Hence

$$\Delta w(k)+q(k)<0 \quad \text{for } k \ge N, \tag{1.11.47}$$

which yields

$$w(n_\ell+1)-w(m_\ell)+\sum_{j=m_\ell}^{n_\ell}q(j)<0 \quad \text{for } N<m_\ell<n_\ell, \tag{1.11.48}$$

so that

$$\sum_{j=m_\ell}^{n_\ell}q(j)<w(m_\ell)-w(n_\ell+1)<c(n_\ell+1)+w(m_\ell). \tag{1.11.49}$$

Thus, by Lemma 1.11.4, we have

$$\sum_{j=m_\ell}^{n_\ell} q(j) < c(n_\ell+1)+\left(\sum_{j=N}^{m_\ell-1}\frac{1}{c(j)}\right)^{-1}, \tag{1.11.50}$$

which contradicts condition (1.11.46). □

Remark 1.11.11. Under the assumptions of Lemma 1.11.4, a necessary condition for the existence of a nonoscillatory solution of equation (1.4.4) is

$$q(k) < \left(\sum_{j=N}^{k-1} \frac{1}{c(j)}\right)^{-1} \quad \text{for all large } k \in \mathbb{N}. \tag{1.11.51}$$

Indeed the inequality (1.11.47) implies

$$0 < \frac{w(k+1)}{w(k)} < 1 - \frac{q(k)}{w(k)}. \tag{1.11.52}$$

Thus $q(k) < w(k)$, and so Lemma 1.11.4 implies (1.11.51).

As a further use of the estimate (1.11.18) from Lemma 1.11.4, we give a criterion, where a "weighted sequence" appears in the assumptions.

Theorem 1.11.12. *In addition to the assumptions of Lemma 1.11.4, assume that there exists a sequence $\rho = \{\rho(k)\}$ with $\rho(k) > 0$ for all $k \in \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sum_{j=N\geq n_0}^{k} \rho(j)q(j) = \infty, \tag{1.11.53}$$

$$\lim_{k\to\infty} \left(\sum_{j=N}^{k} \rho(j)q(j)\right)^{-1} \sum_{j=N}^{k} \frac{\left[\Delta\rho(j-1)\right]^2}{\rho(j)} \left[c(j) + \left(\sum_{i=N}^{j-1} \frac{1}{c(i)}\right)^{-1}\right] = 0. \tag{1.11.54}$$

Then equation (1.4.4) is oscillatory.

Proof. Suppose the contrary and let $x = \{x(k)\}$ be an eventually positive solution of equation (1.4.4). As in Lemma 1.11.4, we have the Riccati equation (1.4.5) with $w(k) = c(k)\Delta x(k)/x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Multiplying equation (1.4.5) by $\rho(k)$ and summing from N to k we have

$$\sum_{j=N}^{k} \rho(j)q(j) = -\sum_{j=N}^{k} \rho(j)\Delta w(j) - \sum_{j=N}^{k} \rho(j)\frac{w^2(j)}{w(j)+c(j)} \quad \text{for } k \geq N. \tag{1.11.55}$$

Obviously,

$$\begin{aligned}
-\sum_{j=N}^{k} \rho(j)\Delta w(j) &= \rho(N)w(N) - \rho(k+1)w(k+1) + \sum_{j=N}^{k} w(j+1)\Delta\rho(j) \\
&= \sum_{j=N}^{k} w(j)\Delta\rho(j-1) + w(N)\rho(N-1) - w(k+1)\rho(k) \\
&\leq \sum_{j=N}^{k} w(j)\Delta\rho(j-1) + w(N)\rho(N-1).
\end{aligned} \tag{1.11.56}$$

Using this in (1.11.55) we obtain

$$\begin{aligned}
\sum_{j=N}^{k} \rho(j)q(j) &\le w(N)\rho(N-1) + \sum_{j=N}^{k} w(j)\Delta\rho(j-1) - \sum_{j=N}^{k} \frac{\rho(j)w^2(j)}{w(j)+c(j)} \\
&= w(N)\rho(N-1) - \sum_{j=N}^{k}\left[\left(\frac{\rho(j)w^2(j)}{w(j)+c(j)}\right)^{1/2} - \frac{1}{2}\left(\frac{w(j)+c(j)}{\rho(j)w^2(j)}\right)^{1/2} w(j)\Delta\rho(j-1)\right]^2 \\
&\quad + \frac{1}{4}\sum_{j=N}^{k} \frac{(\Delta\rho(j-1))^2}{\rho(j)}[c(j)+w(j)] \\
&\le w(N)\rho(N-1) + \frac{1}{4}\sum_{j=N}^{k} \frac{(\Delta\rho(j-1))^2}{\rho(j)}[c(j)+w(j)].
\end{aligned} \tag{1.11.57}$$

So by Lemma 1.11.4 we find

$$\sum_{j=N}^{k} \rho(j)q(j) \le w(N)\rho(N-1) + \frac{1}{4}\sum_{j=N}^{k} \frac{(\Delta\rho(j-1))^2}{\rho(j)}\left[c(j) + \left(\sum_{i=N}^{j-1}\frac{1}{c(i)}\right)^{-1}\right]. \tag{1.11.58}$$

Combining (1.11.53), (1.11.54), and (1.11.58), we now get the desired contradiction. □

Remark 1.11.13. If we assume that

$$\liminf_{k\to\infty} c(k)\sum_{j=N}^{k-1}\frac{1}{c(j)} > 0, \tag{1.11.59}$$

then condition (1.11.54) reduces to

$$\lim_{k\to\infty}\left(\sum_{j=N}^{k}\rho(j)q(j)\right)^{-1}\sum_{j=N}^{k}\frac{(\Delta\rho(j-1))^2}{\rho(j)}c(j) = 0. \tag{1.11.60}$$

Indeed, if (1.11.59) holds, then

$$\sum_{j=N}^{k-1}\frac{1}{c(j)} > \frac{M}{c(k)}, \tag{1.11.61}$$

where $0 < M < \liminf_{k\to\infty}\sum_{j=N}^{k-1} 1/c(j)$. Hence condition (1.11.54) reduces to (1.11.60).

One can observe that condition (1.11.59) is not a serious restriction. For example $c(k) = k$, $c(k) = 1/k$, $c(k) = e^{-k}$, and $c(k) \equiv 1$ all satisfy (1.11.59).

Next, and by choosing a suitable weighted sequence in Theorem 1.11.12, we get the following oscillation result.

Corollary 1.11.14. *Let the assumptions of Lemma 1.11.4 hold along with condition (1.11.59). Assume further that there exists $\mu \in (0,1)$ such that*

$$\lim_{k\to\infty} \sum_{j=N}^{k} \left(\sum_{i=1}^{j} \frac{1}{c(i)} \right)^{\mu} q(j) = \infty. \tag{1.11.62}$$

Then equation (1.4.4) is oscillatory.

Proof. Define $\rho(k)$ from Theorem 1.11.12 as $\rho(k) = (\sum_{j=1}^{k} 1/c(j))^{\mu}$. We will show that (1.11.60) holds. We have from the mean value theorem

$$0 < \Delta\rho(k-1) \le \frac{\mu}{c(k)} \left(\sum_{j=1}^{k-1} \frac{1}{c(j)} \right)^{\mu-1}, \tag{1.11.63}$$

so that

$$\frac{\left(\Delta\rho(k-1)\right)^2}{\rho(k)} c(k) \le \frac{\mu^2}{c(k)} \frac{\left(\sum_{j=1}^{k-1} 1/c(j) \right)^{2(\mu-1)}}{\left(\sum_{j=1}^{k} 1/c(j) \right)^{\mu}}. \tag{1.11.64}$$

From condition (1.11.59) there exists a positive constant M such that

$$\sum_{j=1}^{k} \frac{1}{c(j)} \le M \sum_{j=1}^{k-1} \frac{1}{c(j)} \quad \text{for } k \ge 2, \tag{1.11.65}$$

and hence (since $\mu^2 < 1$) we have

$$\sum_{j=N+1}^{k} \frac{\left(\Delta\rho(j-1)\right)^2}{\rho(j)} c(j) < M \sum_{j=N+1}^{k} \left(\sum_{i=1}^{j} \frac{1}{c(j)} \right)^{\mu-2} \Delta\left(\sum_{i=1}^{j-1} \frac{1}{c(j)} \right). \tag{1.11.66}$$

Setting $u(k) = \sum_{j=1}^{k-1} 1/c(j)$, the right-hand side of (1.11.66) can be written in the form $\sum_{j=N}^{k-1} u^{\mu-2}(j+1)\Delta u(j)$. Let $f(t) = u(k) + (t-k)\Delta u(k)$ for $k \le t \le k+1$. Then $f'(t) = \Delta u(k)$ and $f(t) \le u(k+1)$ for $k < t < k+1$ and hence we get

$$\frac{\Delta u(k)}{u^{2-\mu}(k+1)} = \int_k^{k+1} \frac{\Delta u(k)}{u^{2-\mu}(k+1)} dt \le \int_k^{k+1} \frac{f'(t)}{f^{2-\mu}(t)} dt. \tag{1.11.67}$$

It follows that

$$\sum_{j=N}^{k-1} \frac{\Delta u(k)}{u^{2-\mu}(k+1)} \leq \int_N^k \frac{f'(t)}{f^{2-\mu}(t)} dt = \frac{1}{1-\mu}\left[\left(\sum_{j=1}^{N} \frac{1}{c(j)}\right)^{\mu-1} - \left(\sum_{j=1}^{k} \frac{1}{c(j)}\right)^{\mu-1}\right]. \tag{1.11.68}$$

Therefore condition (1.11.60) holds, and so the statement follows from Theorem 1.11.12. ☐

The following example illustrates the methods above.

Example 1.11.15. Consider the difference equation

$$\Delta^2 x(k) + \left[\frac{1}{(k+1)^2} + \frac{(-1)^k}{2(k+1)}\right] x(k+1) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{1.11.69}$$

Here

$$q(k) = \frac{1}{(k+1)^2} + \frac{(-1)^k}{2(k+1)}, \tag{1.11.70}$$

which is not of one sign. However, we have

$$\sum_{j=N}^{\infty} q(j) \geq \frac{1}{N+1} - \frac{1}{2(N+1)} = \frac{1}{2(N+1)}. \tag{1.11.71}$$

All conditions of Corollary 1.11.6 are satisfied, and hence equation (1.11.69) is oscillatory.

1.12. Oscillation criteria by averaging techniques

In this section we will be interested in using the Riccati equation (1.4.5) along with certain averaging techniques to obtain some discrete oscillation and nonoscillation criteria for equation (1.4.4). Some criteria are analogues of known oscillation results for the differential equation (1.1.1).

We will need the following conditions which will be imposed in the theorems to follow:

$$\limsup_{k\to\infty} k^{-3/2} \sum_{j=n_0}^{k} c(j) < \infty, \tag{1.12.1}$$

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{j=n_0}^{k} c(j) < \infty, \tag{1.12.2}$$

$$\text{there exists a constant } M > 0 \text{ with } 0 < c(k) \leq M \text{ for all } k \geq n_0. \tag{1.12.3}$$

Clearly (1.12.3)⇒(1.12.2)⇒(1.12.1).

Now, we present the following result when condition (1.12.1) holds and equation (1.4.4) is nonoscillatory.

Theorem 1.12.1. *Assume that (1.12.1) holds. Further suppose that equation (1.4.4) is nonoscillatory. Then the following statements are equivalent.*

(i) *The limit*

$$\lim_{k\to\infty}\frac{1}{k}\sum_{n=n_0}^{k}\sum_{j=n_0}^{n}q(j)\quad \textit{exists (as a finite number).} \tag{1.12.4}$$

(ii) *The limit*

$$\liminf_{k\to\infty}\frac{1}{k}\sum_{n=n_0}^{k}\sum_{j=n_0}^{k}q(j) > -\infty. \tag{1.12.5}$$

(iii) *For any solution x with $x(k)x(k+1)>0$ for $k\ge N$ for some $N\ge n_0$, the sequence $w(k)=c(k)\Delta x(k)/x(k)$, $k\ge N$, satisfies*

$$\sum_{j=N}^{\infty}\frac{w^2(j)}{w(j)+c(j)}<\infty. \tag{1.12.6}$$

Proof. (i)⇒(ii) is obvious. For (ii)⇒(iii) suppose, to the contrary, that there is a nonoscillatory solution x of equation (1.4.4) such that

$$w(k)=\frac{c(k)\Delta x(k)}{x(k)} > -c(k)\quad \forall k\ge N, \tag{1.12.7}$$

$$\sum_{j=N}^{\infty}\frac{w^2(j)}{w(j)+c(j)}=\infty. \tag{1.12.8}$$

From the Riccati equation (1.4.5) we have

$$-w(k+1)=-w(N)+\sum_{j=N}^{k}\frac{w^2(j)}{w(j)+c(j)}+\sum_{j=N}^{k}q(j), \tag{1.12.9}$$

and therefore

$$\begin{aligned}\frac{1}{k}\sum_{j=N}^{k}-w(j+1)&=-\left(\frac{k-N+1}{k}\right)w(N)+\frac{1}{k}\sum_{n=N}^{k}\sum_{j=N}^{n}\frac{w^2(j)}{w(j)+c(j)}\\&\quad+\frac{1}{k}\sum_{n=N}^{k}\sum_{j=N}^{n}q(j)\quad \text{for } k\ge N.\end{aligned} \tag{1.12.10}$$

From (1.12.5), (1.12.8), and (1.12.10) we obtain

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=N}^{k} -w(j+1) = \infty, \tag{1.12.11}$$

and hence

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=N}^{k} |w(j)| = \infty. \tag{1.12.12}$$

Let

$$W(k) = \frac{w^2(k)}{w(k)+c(k)} \quad \text{for } k \geq N. \tag{1.12.13}$$

Then $W(k) \geq 0$, and $W(k) = 0$ if and only if $w(k) = 0$. Let

$$a(k) = \begin{cases} \dfrac{w^2(k)}{W(k)} & \text{if } w(k) \neq 0, \\ 0 & \text{if } w(k) = 0,\ k \geq N. \end{cases} \tag{1.12.14}$$

Then we have $c(k) \geq a(k) - w(k)$ for $k \geq N$ and hence

$$k^{-3/2} \sum_{j=N}^{k} c(j) \geq k^{-3/2} \sum_{j=N}^{k} a(j) + k^{-3/2} \sum_{j=N}^{k} -w(j). \tag{1.12.15}$$

Now from (1.12.1), (1.12.15), and the fact that $a(k) \geq 0$ for $k \geq N$ it follows that

$$\limsup_{k\to\infty} k^{-3/2} \sum_{j=N}^{k} -w(j) < \infty. \tag{1.12.16}$$

Therefore, dividing both sides of (1.12.10) by $k^{1/2}$, from (1.12.16) and condition (1.12.2) we obtain

$$\limsup_{k\to\infty} k^{-3/2} \sum_{n=N}^{k} \sum_{j=N}^{n} w(j) < \infty. \tag{1.12.17}$$

Since

$$\begin{aligned} k^{-1/2} \sum_{j=N}^{k} W(j) &= k^{-3/2} k \sum_{j=N}^{k} W(j) \\ &\leq k^{-3/2} \sum_{n=k}^{2k} \sum_{j=N}^{k} W(j) \\ &\leq 2^{3/2} (2k)^{-3/2} \sum_{n=N}^{2k} \sum_{j=N}^{n} W(j), \end{aligned} \tag{1.12.18}$$

we have

$$\limsup_{k\to\infty} k^{-1/2} \sum_{j=N}^{k} W(j) < \infty. \tag{1.12.19}$$

On the other hand, from (1.12.19) there is a positive constant M such that

$$\begin{aligned}
\left(\sum_{j=N}^{k} |w(j)|\right)^2 &= \left(\sum_{j=N}^{k} [a(j)W(j)]^{1/2}\right)^2 \\
&\leq \left(\sum_{j=N}^{k} a(j)\right)\left(\sum_{j=N}^{k} W(j)\right) \\
&\leq Mk^{1/2} \sum_{j=N}^{k} a(j).
\end{aligned} \tag{1.12.20}$$

Therefore,

$$k^{-3/2} \sum_{j=N}^{k} a(j) \geq \frac{1}{M}\left[\frac{1}{k} \sum_{j=N}^{k} |w(j)|\right]^2, \tag{1.12.21}$$

and so from (1.12.12), (1.12.15), (1.12.16), and (1.12.21) we have

$$\lim_{k\to\infty} k^{-3/2} \sum_{j=N}^{k} c(j) = \infty, \tag{1.12.22}$$

which contradicts condition (1.12.1). Therefore (ii)⇒(iii).

For (iii)⇒(i), let w be the sequence from (iii) and let $A(k) = \sum_{j=N}^{k} |w(j)|$. Then we have

$$\begin{aligned}
\left[\sum_{j=N}^{k} w(j)\right]^2 &\leq A^2(k) = \left[\sum_{j=N}^{k} W(j)[w(j)+c(j)]\right]^2 \\
&\leq \left(\sum_{j=N}^{k} W(j)\right)\left(\sum_{j=N}^{k} [w(j)+c(j)]\right) \\
&\leq B\left(A(k) + \sum_{j=N}^{k} c(j)\right) \\
&\leq 2B \max\left\{A(k), \sum_{j=N}^{k} c(j)\right\},
\end{aligned} \tag{1.12.23}$$

where $B = \sum_{j=N}^{\infty} W(j)$. Hence we have

$$A(k) \le \max\left\{2B, \left(2B \sum_{j=N}^{k} c(j)\right)^{1/2}\right\}. \tag{1.12.24}$$

It follows from (1.12.1) and (1.12.24) that $\lim_{k\to\infty} A(k)/k = 0$ so that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=N}^{k} [-w(j+1)] = 0, \tag{1.12.25}$$

and the result now follows by letting $k \to \infty$ in (1.12.10). The proof is therefore complete. □

Now we are ready to state the following oscillation criteria for equation (1.4.4).

Corollary 1.12.2 (Hartman-Wintner criterion). *Let condition (1.12.1) hold. A sufficient condition for equation (1.4.4) to be oscillatory is that either*

$$-\infty < \liminf_{k\to\infty} \frac{1}{k} \sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j) < \limsup_{k\to\infty} \frac{1}{k} \sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j) \tag{1.12.26}$$

or

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j) = \infty. \tag{1.12.27}$$

Remark 1.12.3. The criteria given in Corollary 1.12.2 are discrete analogues of the known oscillation criteria due to Hartman and Wintner for equation (1.1.1) with $c(t) \equiv 1$.

Theorem 1.12.4. *Let conditions (1.12.1) and (1.12.4) hold.*

(I_1) *If (1.4.4) is nonoscillatory, then there exists a sequence $w(k) > -c(k)$ for $k \ge N$ for some $N \ge n_0 \ge 0$ such that*

$$w(k) = b - \sum_{j=n_0}^{k-1} q(j) + \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)} \quad \textit{for } k \ge N, \tag{1.12.28}$$

where

$$b = \lim_{k\to\infty} \frac{1}{k} \sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j). \tag{1.12.29}$$

(I_2) *If there exist a sequence w with $w(k) > -c(k)$ for $k \geq N \geq n_0 \geq 0$ and a constant b such that*

$$w(k) \geq b - \sum_{j=n_0}^{k-1} q(j) + \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)} \geq 0 \tag{1.12.30}$$

or

$$w(k) \leq b - \sum_{j=n_0}^{k-1} q(j) + \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)} \leq 0, \tag{1.12.31}$$

then equation (1.4.4) is nonoscillatory.

Proof. First we show (I_1). By Theorem 1.12.1 the limit b in (1.12.29) exists (as a finite number). Now since

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{k} \sum_{n=N}^{k} \sum_{j=N}^{n} q(j) &= \lim_{k\to\infty} \left[\sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j) - (k-N) \sum_{j=0}^{N-1} q(j) \right] \\ &= b - \sum_{j=n_0}^{N-1} q(j),\end{aligned} \tag{1.12.32}$$

we obtain (1.12.28) by letting $k \to \infty$ in (1.12.10) and then replacing N by k.

Now we show (I_2). Suppose that conditions (1.12.1) and (1.12.4) hold and there exists a constant b such that (1.12.31) holds. Let

$$y(k) = b - \sum_{j=n_0}^{k-1} q(j) + \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j)+c(j)}. \tag{1.12.33}$$

Then

$$\Delta y(k) = -q(k) - \frac{w^2(k)}{w(k)+c(k)}. \tag{1.12.34}$$

Now since $w(k) \geq y(k) \geq 0$ or $w(k) \leq y(k) \leq 0$, we have

$$\frac{w^2(k)}{w(k)+c(k)} \geq \frac{y^2(k)}{y(k)+c(k)}, \tag{1.12.35}$$

(see the proof of Theorem 1.11.5) so that

$$\Delta y(k) + \frac{y^2(k)}{y(k)+c(k)} + q(k) \leq 0, \quad y(k) > -c(k) \text{ for } k \geq N \geq n_0 \geq 0. \tag{1.12.36}$$

Hence, by Lemma 1.7.1, equation (1.4.4) is nonoscillatory. □

Next, we will investigate the situation when condition (1.12.4) fails to hold and give the following oscillation criterion for equation (1.4.4).

Theorem 1.12.5. *Let condition (1.12.2) hold. If*

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j) = -\infty, \tag{1.12.37}$$

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{n=n_0}^{k} \sum_{j=n_0}^{n} q(j) > -\infty, \tag{1.12.38}$$

then equation (1.4.4) is oscillatory.

Proof. Suppose to the contrary that equation (1.4.4) is nonoscillatory and let x be any nonoscillatory solution of equation (1.4.4). Let $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq N \geq n_0 \geq 0$. Since condition (1.12.1) follows from (1.12.2), Theorem 1.12.1 and (1.12.37) imply that

$$\sum_{j=N}^{\infty} \frac{w^2(j)}{w(j)+c(j)} = \infty. \tag{1.12.39}$$

But from (1.12.10) we have

$$\begin{aligned}\limsup_{k\to\infty} \frac{1}{k} \sum_{j=N}^{k} [-w(j+1)] &\geq \liminf_{k\to\infty} \frac{1}{k} \sum_{n=N}^{k} \sum_{j=N}^{n} \frac{w^2(j)}{w(j)+c(j)} \\ &\quad + \limsup_{k\to\infty} \frac{1}{k} \sum_{n=N}^{k} \sum_{j=N}^{n} q(j) - w(N) = \infty,\end{aligned} \tag{1.12.40}$$

which is impossible from $-w(j+1) < c(j+1)$ and condition (1.12.2). This completes the proof. □

The next two results show that if condition (1.12.3) holds and if equation (1.4.4) is nonoscillatory, then whether condition (1.12.4) does or does not hold is equivalent to rather strong convergence or divergence of the sequence q.

Theorem 1.12.6. *Let condition (1.12.3) hold and assume equation (1.4.4) is nonoscillatory. Then the following statements are equivalent.*

(I) $\lim_{k\to\infty} \sum_{j=n_0\geq 0}^{k} q(j)$ *exists (as a finite number).*
(II) *Condition (1.12.4) is satisfied.*
(III) *Condition (1.12.5) is satisfied.*
(IV) *For any nonoscillatory solution x with $x(k)x(k+1) > 0$ for $k \geq N$ for some $N \geq n_0 \geq 0$, the sequence $w(k) = c(k)\Delta x(k)/x(k)$, $k \geq N$, satisfies (1.12.6).*

Proof. Obviously (I)⇒(II)⇒(III). Theorem 1.12.1 shows that (III) and (IV) are equivalent. Therefore, we only need to show that (IV)⇒(I). But this is immediate by letting $k \to \infty$ in (1.12.9) and observing that (IV) implies that $w(k) \to 0$ as $k \to \infty$. This completes the proof. □

A counterpart to Theorem 1.12.6 is the following result.

Theorem 1.12.7. *Let condition (1.12.3) hold and assume that equation (1.4.4) is nonoscillatory. Then the following statements are equivalent.*

(i) $\sum_{j=n_0\ge 0}^{\infty} q(j) = -\infty$.
(ii) *Condition (1.12.37) is satisfied.*
(iii) *There exists a nonoscillatory solution x of (1.4.4) with $x(k)x(k+1) > 0$, $k \ge N$ for some $N \ge n_0 \ge 0$ such that*

$$\sum_{j=N}^{\infty} \frac{w^2(j)}{w(j)+c(j)} = \infty, \tag{1.12.41}$$

where $w(k) = c(k)\Delta x(k)/x(k) > -c(k)$ for $k \ge N$.

Proof. Obviously (i)⇒(ii). By Theorem 1.12.6, (ii) and (iii) are equivalent. We need only show that (iii)⇒(i). But this is clear since from (1.12.9) we have

$$\begin{aligned} \sum_{j=N}^{k} q(j) &= -\sum_{j=N}^{k} \frac{w^2(j)}{w(j)+c(j)} + w(N) - w(k+1) \\ &\le -\sum_{j=N}^{k} \frac{w^2(j)}{w(j)+c(j)} + w(N) + M \longrightarrow -\infty, \end{aligned} \tag{1.12.42}$$

where the constant M is as in (1.12.3), that is, (i) holds. The proof is therefore complete. □

The following oscillation result is immediate.

Corollary 1.12.8. *Let condition (1.12.3) hold. A sufficient condition for equation (1.4.4) to be oscillatory is that either*

$$\lim_{k\to\infty} \sum_{j=n_0\ge 0}^{k} q(j) = \infty \tag{1.12.43}$$

or

$$\liminf_{k\to\infty} \sum_{j=n_0}^{k} q(j) < \limsup_{k\to\infty} \sum_{j=n_0}^{k} q(j). \tag{1.12.44}$$

Remark 1.12.9. One may observe that the condition of type (1.12.43) cannot be replaced by the condition

$$\lim_{k\to\infty}\sum_{j=n_0}^{k} q(j) = -\infty. \tag{1.12.45}$$

In fact, if condition (1.12.45) holds, both oscillation and nonoscillation for equation (1.4.4) are possible. For example, if $q(k) = -1$ for all $k \geq n_0 \geq 0$, then equation (1.4.4) with $c(k) \equiv 1$ is of course nonoscillatory.

The next theorem enables us to present an example to show that condition (1.12.45) is compatible with oscillation.

Theorem 1.12.10. *If there exist two sequences of integers n_k and m_k with $n_k \geq m_k+1$ such that $m_k \to \infty$ as $k \to \infty$ and*

$$\sum_{j=m_k}^{n_k-1} q(j) \geq c(m_k) + c(n_k), \tag{1.12.46}$$

then equation (1.4.4) is oscillatory.

Proof. Suppose that equation (1.4.4) is nonoscillatory. Then there exists a nonoscillatory solution x with $x(k)x(k+1) > 0$ for $k \geq N$ for some $N \geq n_0 \geq 0$. Let $w(k) = c(k)\Delta x(k)/x(k)$, $k \geq N$. Then w satisfies (1.4.5) and $w(k) > -c(k)$ for $k \geq N$. We will show that

$$\sum_{j=N}^{k-1} q(j) < c(N) + c(k) \tag{1.12.47}$$

holds for all $k \geq N+1$, and then this contradiction will prove the theorem.

From equation (1.4.5) we have

$$\begin{aligned} q(N) &= w(N) - w(N+1) - \frac{w^2(N)}{w(N)+c(N)} < c(N+1) + \frac{w(N)c(N)}{w(N)+c(N)} \\ &= c(N+1) + c(N) - \frac{c^2(N)}{w(N)+c(N)} < c(N+1) + c(N). \end{aligned} \tag{1.12.48}$$

Then (1.12.46) holds for $k = N+1$. For any $k \geq N+2$, from equation (1.4.5) we have

$$\begin{aligned} \sum_{j=N+1}^{k-1} q(j) &= w(N+1) - w(k) - \sum_{j=N+1}^{k-1} \frac{w^2(j)}{w(j)+c(j)} \\ &\leq w(N+1) + c(k). \end{aligned} \tag{1.12.49}$$

Since

$$w(N+1) = c(N)\left[1 - \frac{x(N)}{x(N+1)}\right] - q(N) < c(N) - q(N), \tag{1.12.50}$$

(1.12.46) follows immediately. This completes the proof. □

Example 1.12.11. Let $m_k = 4k$ with $k \in \mathbb{N}$. Define $c(k) \equiv 1$,

$$q(m_k) = q(m_k + 1) = q(m_k + 2) = 1, \qquad q(m_k + 3) = -4 \quad \text{for } k \in \mathbb{N}. \tag{1.12.51}$$

Then $\sum_{j=m_k}^{m_k+3} q(j) = 3 > 2 = c(m_k) + c(m_3 + 3)$ for all $k \in \mathbb{N}$. Equation (1.4.4) is oscillatory by Theorem 1.12.10. It is clear that condition (1.12.45) is satisfied.

If conditions (1.12.3) and (1.12.4) hold, then the nonoscillation of equation (1.4.4) implies the convergence of the sequence q, and so $\sum_{j=n_0}^{\infty} q(j) = b$, where b is defined as in (1.12.29). We may state then the following results involving a Riccati summation equation.

Theorem 1.12.12. *Let conditions (1.12.3) and (1.12.4) hold.*

(I_1) *If equation (1.4.4) is nonoscillatory, then there exists a sequence w with $w(k) > -c(k)$ for $k \geq N$ for some $N \geq n_0 \geq 0$, satisfying*

$$w(k) = Q(k) + \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j) + c(j)} \quad \text{for } k \geq N, \tag{1.12.52}$$

where

$$Q(k) = \sum_{j=k}^{\infty} q(j). \tag{1.12.53}$$

(I_2) *If there exists a sequence w such that $w(k) > -c(k)$ for $k \geq N$ for some $N \geq n_0 \geq 0$, satisfying*

$$w(k) \geq Q(k) + \sum_{j=k}^{\infty} \frac{w^2(j)}{w(j) + c(j)} \geq 0, \tag{1.12.54}$$

then equation (1.4.4) is nonoscillatory. (In (1.12.54), the inequality sign at both places can be changed.)

Proof. The proof is similar to the proof of Theorem 1.12.4 and hence is omitted. □

Theorem 1.12.13. *Suppose that $c(k) \leq 1$ for $k \in \mathbb{N}$. If for all $N \in \mathbb{N}$ there exists $n \geq N$ such that*

$$\lim_{k\to\infty} \sum_{j=n}^{k} q(k) \geq 1, \tag{1.12.55}$$

then equation (1.4.4) is oscillatory.

Proof. Suppose to the contrary that equation (1.4.4) is nonoscillatory. Then there exist $N \in \mathbb{N}$ and a solution of equation (1.4.4) with $x(k)x(k+1) > 0$ for $k \geq N$. Let $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq N$. Then w satisfies equation (1.4.5) and $c(k) + w(k) > 0$ for $k \geq N$. Pick $n \geq N$ such that (1.12.55) holds. Summing equation (1.4.5) from n to k, we get

$$w(k+1) = w(n) - \sum_{j=n}^{k} q(j) - \sum_{j=n}^{k} \frac{w^2(j)}{w(j)+c(j)} \quad \text{for } k \geq n. \tag{1.12.56}$$

Hence

$$w(k+1) = \frac{w(n)c(n)}{w(n)+c(n)} - \sum_{j=n}^{k} q(j) - \sum_{j=n+1}^{k} \frac{w^2(j)}{w(j)+c(j)}. \tag{1.12.57}$$

Now we consider the following two cases.

Case 1. If $\sum_{j=n+1}^{\infty} w^2(j)/(w(j)+c(j)) = \infty$, then we get a contradiction from equation (1.12.57) since $w(k) > -c(k) \geq -1$ for $k \geq N$.

Case 2. Assume $\sum_{j=n+1}^{\infty} w^2(j)/(w(j)+c(j)) < \infty$. Then it is easy to conclude that $w^2(k)/(w(k)+c(k)) \to 0$ as $k \to \infty$ and therefore $w(k) \to 0$ as $k \to \infty$. In view of the nonnegativity of $w^2(k)/(w(k)+c(k))$, we have

$$-w(k+1) \geq -\frac{w(n)c(n)}{w(n)+c(n)} + \sum_{j=n}^{k} q(j). \tag{1.12.58}$$

Hence from (1.12.55), one can easily see the proof

$$0 \geq -\frac{w(n)c(n)}{w(n)+c(n)} + \limsup_{k\to\infty} \sum_{j=n}^{k} q(j) > 1, \tag{1.12.59}$$

which is a contradiction. This completes the proof. □

Next we study the possibility of extending Theorems 1.12.1 and 1.12.5 by using a "weighted averaging" technique similar to that introduced by Coles and Willett for differential equations of type (1.1.1).

To do this we will introduce the sequence $\{\rho(k)\}_{k=n_0\geq 0}^{\infty}$ which satisfies

$$0 \leq \rho(k) \leq 1 \quad \text{for } k \in \mathbb{N}_0, \qquad \sum_{j=n_0}^{\infty} \rho(j) = \infty. \tag{1.12.60}$$

We denote the partial sum of ρ by $B(k;n) = \sum_{j=n}^{k} \rho(j)$ and, furthermore, define $B(k) = B(k;n_0) = \sum_{j=n_0}^{k} \rho(j)$.

Theorem 1.12.14. *If equation (1.4.4) is nonoscillatory and there exists a sequence $\{\rho(k)\}$ satisfying (1.12.60) such that*

$$B^{-3/2}(k) \sum_{k=n_0}^{k} \rho(k)c(k+1) \leq M < \infty \tag{1.12.61}$$

for all sufficiently large k and some constant $M > 0$, then the following statements are equivalent.

(i) $\lim_{k\to\infty}(1/B(k)) \sum_{n=n_0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j) = b$ *exists as a finite number.*
(ii) $\liminf_{k\to\infty}(1/B(k)) \sum_{n=n_0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j) > -\infty$.
(iii) *For any nonoscillatory solution x of (1.4.4), the sequence $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq N$ for some $N \geq n_0 \geq 0$ satisfies (1.12.6).*

Proof. (i)⇒(ii) is obvious. For (ii)⇒(iii) suppose that it is not true, that is, $\sum_{j=N}^{\infty} w^2(j)/\left(w(j)+c(j)\right) = \infty$. From (1.12.9) we have for all sufficiently large n

$$\begin{aligned} B^{-1}(k;N) \sum_{n=N}^{k} \rho(n) \sum_{j=N}^{n} \frac{w^2(j)}{w(j)+c(j)} &+ B^{-1}(k;N) \sum_{n=N}^{k} \rho(n) \sum_{j=N}^{n} q(j) \\ = w(N) + B^{-1}(k;N) \sum_{j=N}^{k} \rho(j)\big[-w(j+1)\big], & \end{aligned} \tag{1.12.62}$$

and since the first term on the left-hand side of (1.12.62) tends to ∞ as $k \to \infty$, condition (ii) implies

$$\lim_{k\to\infty} B^{-1}(k;N) \sum_{j=N}^{k} \rho(j)\left|w(j+1)\right| = \infty. \tag{1.12.63}$$

Let $W(k)$ and $a(k)$ be defined by (1.12.13) and (1.12.14), respectively. We get

$$\begin{aligned} B^{-3/2}(k;N)\sum_{j=N}^{k}\rho(j)c(j+1) \geq B^{-3/2}(k;N)\sum_{j=N}^{k}\rho(j)a(j+1) \\ + B^{-3/2}(k;N)\sum_{j=N}^{k}\rho(j)[-w(j+1)]. \end{aligned} \tag{1.12.64}$$

Since all the terms on the right-hand side of (1.12.64) are positive, condition (1.12.61) implies that they are bounded. By (1.12.60) for any fixed $k \geq N+1$ we can find $m > k$ such that $B(k;N) \leq B(m) - B(k) \leq 2B(k;N)$ and hence

$$B(m;N) \leq 3B(k;N). \tag{1.12.65}$$

We have then

$$\begin{aligned} B^{-1/2}(k;N)\sum_{j=N}^{k}W(j) &= B^{-3/2}(k;N)B(k;N)\sum_{j=N}^{k}W(j) \\ &\leq B^{-3/2}(k;N)[B(m)-B(k)]\sum_{j=N}^{k}W(j) \\ &\leq B^{-3/2}(k;N)\sum_{n=k+1}^{m}\left[\rho(n)\sum_{j=N}^{n}W(j)\right] \\ &\leq B^{-3/2}(k;N)\sum_{n=N}^{m}\left[\rho(n)\sum_{j=N}^{n}W(j)\right] \\ &\leq 3^{3/2}(B^{-3/2}(k;N))\sum_{n=N}^{m}\left[\rho(n)\sum_{j=N}^{n}W(j)\right]. \end{aligned} \tag{1.12.66}$$

But on the other hand

$$\begin{aligned} B^{-3/2}(k;N)\sum_{n=N}^{k}\rho(n)\sum_{j=N}^{n}W(j) = B^{-3/2}(k;N)w(N) \\ + B^{-3/2}(k;N)\sum_{j=N}^{k}\rho(j)[-w(j+1)] \\ - B^{-3/2}(k;N)\sum_{n=N}^{k}\rho(n)\sum_{j=N}^{n}q(j), \end{aligned} \tag{1.12.67}$$

and since the right-hand side of (1.12.67) is bounded above in view of (1.12.64) and condition (ii), so also is the left-hand side and therefore by the above estimate, $B^{-1/2}(k;N)\sum_{j=N}^{k}W(j)$ is also bounded above.

Now by the Schwarz inequality we have

$$
\begin{aligned}
B^{-2}(k;N)&\bigg(\sum_{j=N}^{k}\rho(j)\,|\,w(j+1)\,|\bigg)^2 \\
&\le \bigg[B^{-3/2}(k;N)\bigg(\sum_{j=N}^{k}\rho(j)a(j+1)\bigg)\bigg]\bigg[B^{-1/2}(k;N)\sum_{j=N}^{k}\rho(j)W(j+1)\bigg] \\
&\le MB^{-1/2}(k;N)\sum_{j=N}^{k}W(j+1)\le M_1<\infty,
\end{aligned}
\tag{1.12.68}
$$

where M_1 is a constant. This contradicts (1.12.63) and hence proves (iii).

For (iii)$\Rightarrow$(i), let w be the sequence in (iii) and let $F(k)=\sum_{j=N}^{k}\rho(j)|w(j+1)|$. Then

$$
\begin{aligned}
F^2(k)&=\bigg[\sum_{j=N}^{k}\rho(j)W^{1/2}(j+1)\big(w(j+1)+c(j+1)\big)^{1/2}\bigg]^2 \\
&\le\bigg[\sum_{j=N}^{k}\rho(j)W(j+1)\bigg]\bigg[\sum_{j=N}^{k}\rho(j)c(j+1)+F(k)\bigg] \\
&\le K\bigg[\sum_{j=N}^{k}\rho(j)c(j+1)+F(k)\bigg] \\
&\le 2K\max\bigg\{F(k),\sum_{j=N}^{k}\rho(j)c(j+1)\bigg\},
\end{aligned}
\tag{1.12.69}
$$

where $K=\sum_{j=N+1}^{\infty}W(j)$. Therefore

$$
F(k)\le\max\bigg\{2K,\bigg[2K\sum_{j=N}^{k}\rho(j)c(j+1)\bigg]^{1/2}\bigg\},
\tag{1.12.70}
$$

$$
\begin{aligned}
&F(k)B^{-1}(k;N) \\
&\quad\le\max\bigg\{2KB^{-1}(k;N),\bigg[2K\bigg(\sum_{j=N}^{k}\rho(j)c(j+1)\bigg)B^{-2}(k;N)\bigg]^{1/2}\bigg\}.
\end{aligned}
\tag{1.12.71}
$$

From (1.12.61), (1.12.71) and (1.12.60), we obtain $F(k)B^{-1}(k;N)\to 0$ as $k\to\infty$, so that (i) follows from (1.12.62). This completes the proof. $\square$

Corollary 1.12.15. *Suppose that there exists a sequence $\{\rho(k)\}$ satisfying (1.12.60) such that condition (1.12.61) holds. A sufficient condition for equation (1.4.4) to be oscillatory is that either*

$$\begin{aligned} -\infty &< \liminf_{k\to\infty} B^{-1}(k) \sum_{n=n_0\geq 0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j) \\ &< \limsup_{k\to\infty} B^{-1}(k) \sum_{n=n_0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j) \end{aligned} \tag{1.12.72}$$

or

$$\lim_{k\to\infty} B^{-1}(k) \sum_{n=n_0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j) = \infty. \tag{1.12.73}$$

Example 1.12.16. Consider (1.4.4) with $q(k) = 1/k$ and $q(0) = 1$, $c(2k+1) = \sqrt{2k+1}$ and $c(2k) = (2k)^2$, and let $\rho(2k) = 0$ and $\rho(2k+1) = 1$ for $k \in \mathbb{N}_0$. It is easy to check that conditions (1.12.60), (1.12.61), and (1.12.73) are satisfied, and hence equation (1.4.4) is oscillatory by Corollary 1.12.15.

We note that condition (1.12.1) is not satisfied and hence Corollary 1.12.2 fails to apply to equation (1.4.4) with $q(k)$ and $c(k)$ as above.

The next result is the analogue of Theorem 1.12.5 in which condition (1.12.2) is replaced by a more general weighted averaging.

Theorem 1.12.17. *If there exists a sequence ρ satisfying (1.12.60) such that*

$$B^{-1}(k) \sum_{j=n_0\geq 0}^{k} \rho(j)c(j+1) \leq M \tag{1.12.74}$$

for all sufficiently large k and some constant $M > 0$, and

$$\begin{aligned} -\infty &= \liminf_{k\to\infty} B^{-1}(k) \sum_{n=n_0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j) \\ &< \limsup_{k\to\infty} B^{-1}(k) \sum_{n=n_0}^{k} \rho(n) \sum_{j=n_0}^{n} q(j), \end{aligned} \tag{1.12.75}$$

then equation (1.4.4) is oscillatory.

Proof. The proof is similar to the proof of Theorem 1.12.5 and hence is omitted. $\square$

1.13. Further criteria by averaging techniques

For the differential equation (1.1.1) with $c(t) \equiv 1$, it is known that the average function $A_m(t)$ defined by

$$A_m(t) = \frac{1}{t^m}\int_{t_0}^{t}(t-s)^m q(s)ds \quad \text{for some } m \geq 1 \tag{1.13.1}$$

plays a crucial rôle in the oscillation of such an equation. In fact, a well-known sufficient condition for the oscillation of equation (1.1.1) with $c(t) \equiv 1$ is that

$$\limsup_{t\to\infty} A_m(t) = \infty \quad \text{for some integer } m > 1. \tag{1.13.2}$$

Also, a special case of condition (1.13.2), namely,

$$\lim_{t\to\infty} A_1(t) = \infty \tag{1.13.3}$$

suffices for the oscillation of equation (1.1.1) with $c(t) \equiv 1$.

In this section we are interested in obtaining discrete analogues of such oscillation criteria, and we will note the similarities as well as the differences which may arise. We will employ the average sum defined by

$$S(k;m) = \frac{1}{k^{(m)}}\sum_{n=N}^{k}\left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right] \tag{1.13.4}$$

for $k \geq n \geq N \geq n_0 \geq 0$, where

$$\begin{aligned}(k-n)^{(m)} &= (k-n)(k-(n-1))\cdots(k-(n-(m-1)))\\ &= (k-n)(k+1-n)^{(m-1)}\\ &= (k-n)^{(m-1)}(k-(n-(m-1))),\end{aligned} \tag{1.13.5}$$

and the sequences q and c are as in equation (1.4.4).

Note that the definition of $(k-n)^{(m)}$ implies that

$$\Delta(k-n)^{(m)} = (k+1-n)^{(m)} - (k-n)^{(m)} = m(k+1-n)^{(m-1)} \quad \text{for } k \geq n. \tag{1.13.6}$$

Before stating and proving the main results, we prove some preparatory lemmas which are interesting in their own right.

Lemma 1.13.1. *For any real sequence $\{x(k)\}$ and any integer $N \geq n_0 \geq 0$,*

$$\sum_{n=N}^{k}(k-n)^{(m)}\Delta x(n) = -(k+1-N)^{(m)}x(N) + m\sum_{n=N}^{k}(k+1-n)^{(m-1)}x(n) \tag{1.13.7}$$

for $k \geq n \geq N$.

Proof. Let $y(n) = (k+1-n)^{(m)}x(n)$ for $k \geq n$. Taking differences with respect to n we obtain

$$\begin{aligned}
\Delta y(n) &= (k-n)^{(m)}\Delta x(n) + \left[(k+1-(n+1))^{(m)} - (k+1-n)^{(m)}\right]x(n) \\
&= (k-n)^{(m)}\Delta x(n) + \left[(k-n)^{(m)} - (k+1-n)^{(m)}\right]x(n) \\
&= (k-n)^{(m)}\Delta x(n) - m(k+1-n)^{(m-1)}x(n).
\end{aligned} \tag{1.13.8}$$

Summing both sides of this equation from N to k we get

$$y(k+1) - y(N) = \sum_{n=N}^{k}(k-n)^{(m)}\Delta x(n) - m\sum_{n=N}^{k}(k+1-n)^{(m-1)}x(n). \tag{1.13.9}$$

Since $y(k+1) = 0$, we obtain

$$\sum_{n=N}^{k}(k-n)^{(m)}\Delta x(n) = -(k+1-N)^{(m)}x(N) + m\sum_{n=N}^{k}(k+1-n)^{(m-1)}x(n). \tag{1.13.10}$$

This completes the proof. □

Next, we state the following result.

Lemma 1.13.2. *Let the sequence* $\{x(k)\} \subset \mathbb{R}$ *be such that* $\liminf_{k\to\infty} \Delta^r x(k) > 0$ *for some* $r \in \mathbb{N}$. *Then* $\lim_{k\to\infty} \Delta^i x(k) = \infty$ *for* $0 \leq i \leq r-1$.

Lemma 1.13.3. *Assume that the sequences* $\{a(k)\}$ *and* $\{b(k)\}$ *are such that the sequence* $\{\Delta^r b(k)\}$ *is increasing for some* $r \in \mathbb{N}_0$ *and* $\Delta^r b(k) \to \infty$ *as* $k \to \infty$. *Then*

$$\limsup_{k\to\infty}\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} \geq \limsup_{k\to\infty}\frac{a(k)}{b(k)} \geq \liminf_{k\to\infty}\frac{a(k)}{b(k)} \geq \liminf_{k\to\infty}\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)}. \tag{1.13.11}$$

Proof. We will only prove

$$\limsup_{k\to\infty}\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} \geq \limsup_{k\to\infty}\frac{a(k)}{b(k)}. \tag{1.13.12}$$

The other remaining parts of (1.13.11) can be proved similarly. Now let

$$\limsup_{k\to\infty}\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} = \ell \in [-\infty, \infty]. \tag{1.13.13}$$

Clearly (1.13.12) holds if $\ell = \infty$. If $\ell \in \mathbb{R}$, then (1.13.13) implies the existence of an integer $N(\varepsilon)$ such that

$$\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} < \ell + \varepsilon \quad \text{for every } \varepsilon > 0,\ k \geq N(\varepsilon). \tag{1.13.14}$$

Since $\Delta^r b(k)$ is increasing, $N(\varepsilon)$ can be chosen so large that $\Delta^{r+1}b(k) > 0$ for all $k \geq N(\varepsilon)$. Therefore (1.13.14) implies that

$$\Delta^{r+1}a(k) < (\ell + \varepsilon)\Delta^{r+1}b(k) \quad \text{for } k \geq N(\varepsilon). \tag{1.13.15}$$

Summing both sides of (1.13.15) from N to k we have

$$\Delta^r a(k+1) - \Delta^r a(N) < (\ell + \varepsilon)\left[\Delta^r b(k+1) - \Delta^r b(N)\right] \quad \text{for } k \geq N(\varepsilon). \tag{1.13.16}$$

Dividing both sides of (1.13.16) by $\Delta^r b(k+1)$ and taking lim sup on both sides as $k \to \infty$, we obtain

$$\limsup_{k\to\infty} \frac{\Delta^r a(k+1)}{\Delta^r b(k+1)} \leq \ell + \varepsilon \quad \text{for every } \varepsilon > 0. \tag{1.13.17}$$

Since ε is arbitrary, we find

$$\limsup_{k\to\infty} \frac{\Delta^r a(k+1)}{\Delta^r b(k+1)} \leq \ell, \tag{1.13.18}$$

so by (1.13.13),

$$\limsup_{k\to\infty} \frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} \geq \limsup_{k\to\infty} \frac{\Delta^r a(k)}{\Delta^r b(k)}. \tag{1.13.19}$$

Since by Lemma 1.13.2 we have $\Delta^i b(k) \to \infty$ as $k \to \infty$ for $0 \leq i \leq r$, we can repeat the same argument above and obtain

$$\limsup_{k\to\infty} \frac{\Delta^{i+1}a(k)}{\Delta^{i+1}b(k)} \geq \limsup_{k\to\infty} \frac{\Delta^i a(k)}{\Delta^i b(k)} \quad \text{for } 0 \leq i \leq r-1. \tag{1.13.20}$$

Combining (1.13.19) and (1.13.20), we obtain (1.13.12).

Finally, if $\ell = -\infty$, then for any constant $\overline{\ell} < 0$ we have

$$\limsup_{k\to\infty} \frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} < \overline{\ell}. \tag{1.13.21}$$

We claim that $\limsup_{k\to\infty} a(k)/b(k) = -\infty$. Suppose not, then

$$\limsup_{k\to\infty} \frac{a(k)}{b(k)} \geq L > -\infty. \tag{1.13.22}$$

Using similar arguments as before, we obtain that $\overline{\ell} > L$ for any $\overline{\ell} < 0$. Thus, L cannot be finite. This contradiction completes the proof. □

Lemma 1.13.4. *Let $U(k) = \sum_{n=N}^{k}(k-n)^{(\overline{m})} f(n)$, where $\{f(n)\}$ is any sequence of real numbers and $\overline{m}, N$ are integers satisfying $\overline{m} \geq 1$ and $N \geq n_0 \geq 0$. Then*

$$\Delta^{r+1}U(k) = \overline{m}(\overline{m}-1)\cdots(\overline{m}-r)\sum_{n=N}^{k}(k+r-1-n)^{(\overline{m}-r-1)} f(n) \tag{1.13.23}$$

for $0 \leq r \leq \overline{m}-1$.

Proof. We have

$$\begin{aligned}
\Delta U(k) &= \sum_{n=N}^{k+1}(k+1-n)^{(\overline{m})} f(n) - \sum_{n=N}^{k}(k-n)^{(\overline{m})} f(n)\\
&= \sum_{n=N}^{k}\left[(k+1-n)^{(\overline{m})} - (k-n)^{(\overline{m})}\right] f(n)\\
&= \overline{m}\sum_{n=N}^{k}(k+1-n)^{(\overline{m}-1)} f(n),
\end{aligned}$$

$$\begin{aligned}
\Delta^2 U(k) &= \Delta U(k+1) - \Delta U(k)\\
&= \overline{m}\sum_{n=N}^{k+1}(k+2-n)^{(\overline{m}-1)} f(n) - \overline{m}\sum_{n=N}^{k}(k+1-n)^{(\overline{m}-1)} f(n)\\
&= \overline{m}\sum_{n=N}^{k+1}\left[(k+2-n)^{(\overline{m}-1)} - (k+1-n)^{(\overline{m}-1)}\right] f(n)\\
&= \overline{m}(\overline{m}-1)\sum_{n=N}^{k+1}(k+2-n)^{(\overline{m}-2)} f(n).
\end{aligned} \tag{1.13.24}$$

Repeating this process $r+1$ times, where $r \leq \overline{m}-1$, we obtain (1.13.23). This completes the proof. □

Lemma 1.13.5. *For any $m \in \mathbb{N}$ and any real sequence $\{f(k)\}$,*

$$\begin{aligned}
\limsup_{k\to\infty} \sum_{n=N}^{k+m-1} f(n) &\geq \limsup_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k-n)^{(m)} f(n) \\
&\geq \liminf_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k-n)^{(m)} f(n) \\
&\geq \liminf_{k\to\infty} \sum_{n=N}^{k+m-1} f(n), \\
\limsup_{k\to\infty} \frac{1}{m(k+m-1)} \sum_{n=N}^{k+m-1} f(n) &\geq \limsup_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k+1-n)^{(m-1)} f(n) \\
&\geq \liminf_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k+1-n)^{(m-1)} f(n) \\
&\geq \liminf_{k\to\infty} \frac{1}{m(k+m-1)} \sum_{n=N}^{k+m-1} f(n), \\
\limsup_{k\to\infty} \frac{f(k+m)}{m} &\geq \limsup_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k+1-n)^{(m-1)} f(n) \\
&\geq \liminf_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k+1-n)^{(m-1)} f(n) \\
&\geq \liminf_{k\to\infty} \frac{f(k+m)}{m}.
\end{aligned} \tag{1.13.25}$$

Proof. The proof is an application of Lemma 1.13.3. We will only prove the first of the above three statements since the other cases can be handled similarly. First, note that $\Delta k^{(m)} = m(k+1)^{(m-1)}$, $\Delta^{m-1}k^{(m)} = m!(k+m-1)$, and hence $\Delta^m k^{(m)} = m!$. Moreover, $\lim_{k\to\infty} \Delta^{m-i}k^{(m)} = \infty$ for $i \in \{0,1,\dots,m-1\}$. Now let

$$a(k) = \sum_{k=N}^{k} (k-n)^{(m)} f(n), \qquad b(k) = k^{(m)}. \tag{1.13.26}$$

Then (1.13.23) with $\overline{m} = m$ and $r = m-1$ yields

$$\frac{\Delta^m a(k)}{\Delta^m b(k)} = \frac{m! \sum_{n=N}^{k+m-1} f(n)}{m!} = \sum_{n=N}^{k+m-1} f(n). \tag{1.13.27}$$

Applying Lemma 1.13.3 with $a(k)$ and $b(k)$ as in (1.13.26), we find

$$\limsup_{k\to\infty}\sum_{n=N}^{k+m-1} f(n) \ge \limsup_{k\to\infty}\frac{a(k)}{b(k)} \ge \liminf_{k\to\infty}\frac{a(k)}{b(k)} \ge \liminf_{k\to\infty}\sum_{n=N}^{k+m-1} f(n). \tag{1.13.28}$$

This completes the proof. □

In the case when $f(k) \ge 0$ for $k \in \mathbb{N}$, we see that both sides of the first statement in Lemma 1.13.5 are equal and equal to $\sum_{n=N}^{\infty} f(n)$. This leads to the following result.

Lemma 1.13.6. *Assume that $m \in \mathbb{N}$ and $\{f(k)\}$ is a sequence of real numbers such that $f(k) \ge 0$ for all $k \ge N \ge n_0 \ge 0$. Then*

$$\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}(k-n)^{(m)}f(n) = \sum_{n=N}^{\infty} f(n). \tag{1.13.29}$$

The following result is an immediate consequence of Lemma 1.13.3.

Corollary 1.13.7. *Let $\{a(k)\}$, $\{b(k)\}$, and r be as in Lemma 1.13.3. If $\lim_{k\to\infty}\Delta^{r+1}a(k)/\Delta^{r+1}b(k) = \gamma \in \mathbb{R}$ exists or $\gamma = \pm\infty$, then $\lim_{k\to\infty} a(k)/b(k) = \gamma$.*

Proof. By Lemma 1.13.3 we see that (1.13.11) holds. But

$$\limsup_{k\to\infty}\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} = \liminf_{k\to\infty}\frac{\Delta^{r+1}a(k)}{\Delta^{r+1}b(k)} = \gamma. \tag{1.13.30}$$

In view of (1.13.11), we get

$$\gamma \ge \limsup_{k\to\infty}\frac{a(k)}{b(k)} \ge \liminf_{k\to\infty}\frac{a(k)}{b(k)} \ge \gamma. \tag{1.13.31}$$

Therefore, $\lim_{k\to\infty} a(k)/b(k) = \gamma$ as required. □

Now we are ready to prove the following oscillation results for equation (1.4.4).

Theorem 1.13.8. *Suppose that there exist $m \in \mathbb{N}$ and a real sequence $\{\phi(k)\}$ such that*

$$\limsup_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=s}^{k}\left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right] \ge \phi(s) \quad \text{for } s \ge 1, \tag{1.13.32}$$

$$\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=n_0\ge 0}^{k}\left[(k+1-n)^{(m-1)}(k-n)^{1/2}\phi^{+}(n)\right] = \infty, \tag{1.13.33}$$

where $\phi^{+}(k) = \max\{\phi(k), 0\}$. Then equation (1.4.4) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (1.4.4), say, $x(k) > 0$ for $k \geq N$ for some $N \geq n_0 > 0$. Define $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq N$. Then $w(k)$ satisfies the Riccati equation (1.4.5). Multiplying both sides of equation (1.4.5) by $(k-n)^{(m)}$, $m \in \mathbb{N}$, and summing from N to k, we get

$$\sum_{n=N}^{k}(k-n)^{(m)}\Delta w(n) = -\sum_{n=N}^{k}(k-n)^{(m)}q(n) - \sum_{n=N}^{k}(k-n)^{(m)}\frac{w^2(n)}{w(n)+c(n)}. \tag{1.13.34}$$

By Lemma 1.13.1, we obtain

$$\sum_{n=N}^{k}(k-n)^{(m)}\Delta w(n) = -(k+1-N)^{(m)}w(N) + m\sum_{n=N}^{k}(k+1-n)^{(m-1)}w(n), \tag{1.13.35}$$

hence

$$\begin{aligned} -(k+1-N)^{(m)}w(N) = &-\sum_{n=N}^{k}(k-n)^{(m)}q(n) \\ &-\sum_{n=N}^{k}(k-n)^{(m)}\frac{w^2(n)}{w(n)+c(n)} \\ &- m\sum_{n=N}^{k}(k+1-n)^{(m-1)}w(n), \end{aligned} \tag{1.13.36}$$

so

$$\begin{aligned} (k+1-N)^{(m)}w(N) = &\sum_{n=N}^{k}\left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right] \\ &+\sum_{n=N}^{k}\Omega(k;n), \end{aligned} \tag{1.13.37}$$

where

$$\Omega(k;n) = (k-n)^{(m)}\frac{w^2(n)}{w(n)+c(n)} + m(k+1-n)^{(m-1)}\left[w(n)+c(n)\right]. \tag{1.13.38}$$

Note that $\Omega(k;n) > 0$ for $k > n \geq N$ since $w(n)+c(n) > 0$. Dividing both sides of (1.13.37) by $k^{(m)}$, we have

$$\frac{(k+1-N)^{(m)}}{k^{(m)}}w(N) = \frac{1}{k^{(m)}}\sum_{n=N}^{k}\left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right] + \frac{1}{k^{(m)}}\sum_{n=N}^{k}\Omega(k;n). \tag{1.13.39}$$

Taking lim sup on both sides of (1.13.39) as $n \to \infty$, we find

$$w(N) \geq \phi(N) + \liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}\Omega(k;n). \tag{1.13.40}$$

Thus

$$w(k) \geq \phi(k) \quad \forall k \geq N, \tag{1.13.41}$$

$$\liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}\Omega(k;n) < \infty. \tag{1.13.42}$$

Using the arithmetic-geometric mean inequality, we obtain for $k \geq n \geq N$,

$$\begin{aligned}\Omega(k;n) &\geq 2(k+1-n)^{(m-1)}\left[m(k-n)w^2(n)\right]^{1/2} \\ &= 2(k+1-n)^{(m-1)}\left[m(k-n)\right]^{1/2}|w(n)|.\end{aligned} \tag{1.13.43}$$

Since (1.13.41) implies that $|w(n)| \geq \phi^+(n)$ for all $n \geq N$, we have

$$\Omega(k;n) \geq 2\left[m(k-n)\right]^{1/2}(k+1-n)^{(m-1)}\phi^+(n) \quad \text{for } k \geq n \geq N. \tag{1.13.44}$$

Summing (1.13.44) from N to k and dividing by $k^{(m)}$, we obtain

$$\frac{1}{k^{(m)}}\sum_{n=N}^{k}\Omega(k;n) \geq \frac{2\sqrt{m}}{k^{(m)}}\sum_{n=N}^{k}\left[(k-n)^{1/2}(k+1-n)^{(m-1)}\phi^+(n)\right] \tag{1.13.45}$$

for $k \geq n \geq N$. From (1.13.33) and (1.13.45) it follows that

$$\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}\Omega(k;n) = \infty, \tag{1.13.46}$$

which contradicts (1.13.42). This completes the proof. □

Corollary 1.13.9. *Suppose that there exists $m \in \mathbb{N}$ such that*

$$\limsup_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=n_0\geq 0}^{k} \left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right] = \infty. \tag{1.13.47}$$

Then equation (1.4.4) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (1.4.4). We define $w(k) = c(k)\Delta x(k)/x(k)$ for $k \geq N \geq n_0 \geq 0$ and proceed as in the proof of Theorem 1.13.8 to obtain (1.13.37). Since $\Omega(k;n)$ is positive for all $k > n \geq N$, we have

$$\frac{1}{k^{(m)}}(k+1-N)^{(m)}w(N) \geq \frac{1}{k^{(m)}} \sum_{n=N}^{k} \left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right]. \tag{1.13.48}$$

Taking lim sup on both sides of (1.13.48) as $k \to \infty$, we obtain

$$w(N) \geq \limsup_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} \left[(k-n)^{(m)}q(n) - m(k+1-n)^{(m-1)}c(n)\right], \tag{1.13.49}$$

which contradicts condition (1.13.47). This completes the proof. □

The following example illustrates the methods above.

Example 1.13.10. Consider the difference equation

$$\Delta\left(\frac{1}{k}\Delta x(k)\right) + (-1)^k k^2 x(k+1) = 0 \quad \text{for } k \geq n_0 \geq 1. \tag{1.13.50}$$

Here, $c(k) = 1/k$ and $q(k) = (-1)^k k^2$ for $k \in \mathbb{N}$. Now $\lim_{k\to\infty}(1/k)\sum_{n=1}^{k} c(n) = 0$ and

$$\begin{aligned} \limsup_{k\to\infty} \frac{1}{k} \sum_{n=1}^{k} (k-n)q(n) &= \limsup_{k\to\infty} \frac{1}{k} \sum_{n=1}^{k} (-1)^n n^2 (k-n) \\ &= \limsup_{k\to\infty} \frac{1}{k} \sum_{n=1}^{k} \sum_{i=1}^{n} (-1)^i i^2 = \infty. \end{aligned} \tag{1.13.51}$$

Thus condition (1.13.47) is satisfied for $m = 1$, and hence equation (1.13.50) is oscillatory by Corollary 1.13.9.

For the special case of equation (1.4.4) when $c(k) \equiv 1$, $k \in \mathbb{N}$, or for equation (1.6.1), Corollary 1.13.9 takes the following form.

Corollary 1.13.11. *If there exists $m \in \mathbb{N}$ such that*

$$\limsup_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=n_0}^{k} (k-n)^{(m)} q(n) = \infty, \tag{1.13.52}$$

then equation (1.6.1) is oscillatory.

Remark 1.13.12. In the above results we do not impose any restriction on $\sum^{\infty} c(n)$ or $\sum^{\infty} 1/c(n)$. Also, we note that when $m = 1$, Corollary 1.13.9 does not have a continuous analogue, since it is well known (due to Hartman) that the condition

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} (t-s) q(s) ds = \infty \tag{1.13.53}$$

by itself is not sufficient for the oscillation of equation (1.1.1) with $c(t) \equiv 1$.

Next, we present the following results.

Theorem 1.13.13. *Let condition (1.13.32) hold and assume for some $m \in \mathbb{N}$,*

$$\lim_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=n_0}^{k} \left[(k-n)^{(m)} \frac{\left(\phi^{+}(n)\right)^2}{\phi^{+}(n)+c(n)} + m(k+1-n)^{(m-1)} \left(\phi(n)+c(n)\right) \right] = \infty. \tag{1.13.54}$$

Then equation (1.4.4) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (1.4.4), say, $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Proceeding as in the proof of Theorem 1.13.8, we obtain (1.13.42). Now

$$\frac{w^2(n)}{w(n)+c(n)} \geq \frac{w^2(n)}{|w(n)|+c(n)} = c(n) \frac{\left(w(n)/c(n)\right)^2}{|w(n)/c(n)|+1} \quad \text{for } n \geq N. \tag{1.13.55}$$

Applying (1.13.41) in (1.13.55), we find

$$\left| \frac{w(n)}{c(n)} \right| \geq \frac{\phi^{+}(n)}{c(n)} \quad \text{for } n \geq N. \tag{1.13.56}$$

Using the fact that the function $s^2/(s+1)$ is increasing for $s > 0$, one can easily find the estimate

$$\frac{w^2(n)}{w(n)+c(n)} \geq \frac{\left(\phi^{+}(n)\right)^2}{\phi^{+}(n)+c(n)} \quad \text{for } n \geq N. \tag{1.13.57}$$

Thus, by using (1.13.41) and (1.13.57) in (1.13.42), it follows that

$$\liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}\left[(k-n)^{(m)}\frac{\left(\phi^{+}(n)\right)^{2}}{\phi^{+}(n)+c(n)}+m(k+1-n)^{(m-1)}\left(\phi(n)+c(n)\right)\right]<\infty, \tag{1.13.58}$$

which contradicts condition (1.13.54). This completes the proof. □

Theorem 1.13.14. *Assume that condition (1.13.32) holds. If either*

$$\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=n_0}^{k}(k+1-n)^{(m-1)}\left(\phi(n)+c(n)\right)=\infty \tag{1.13.59}$$

or

$$\sum_{n=n_0}^{\infty}\frac{\left(\phi^{+}(n)\right)^{2}}{\phi^{+}(n)+c(n)}=\infty, \tag{1.13.60}$$

then equation (1.4.4) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.4.4). Proceeding as in Theorems 1.13.8 and 1.13.13, we obtain (1.13.42), which in view of the fact that $w(k)+c(k)>0$ for $k>N$ implies that

$$\begin{aligned}\infty&>\liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}\Omega(k;n)\\&>\liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}(k-n)^{(m)}\frac{w^{2}(n)}{w(n)+c(n)}.\end{aligned} \tag{1.13.61}$$

Using Lemma 1.13.6 with $f(n)=w^{2}(n)/(w(n)+c(n))$, inequality (1.13.61) implies

$$\infty>\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}(k-n)^{(m)}\frac{w^{2}(n)}{w(n)+c(n)}=\sum_{n=N}^{\infty}\frac{w^{2}(n)}{w(n)+c(n)}. \tag{1.13.62}$$

It follows from (1.13.57) and (1.13.62) that

$$\infty>\sum_{n=N}^{\infty}\frac{w^{2}(n)}{w(n)+c(n)}>\sum_{n=N}^{\infty}\frac{\left(\phi^{+}(n)\right)^{2}}{\phi^{+}(n)+c(n)}, \tag{1.13.63}$$

which contradicts condition (1.13.60).

Next, suppose that (1.13.59) holds. Then it follows that

$$\begin{aligned}\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}&\left[(k-n)^{(m)}\frac{\left(\phi^+(n)\right)^2}{\phi^+(n)+c(n)}+m(k+1-n)^{(m-1)}\left(\phi(n)+c(n)\right)\right]\\ &\geq\lim_{k\to\infty}\frac{m}{k^{(m)}}\sum_{n=N}^{k}(k+1-n)^{(m-1)}\left(\phi(n)+c(n)\right)=\infty.\end{aligned}\tag{1.13.64}$$

Thus condition (1.13.54) holds, and the proof of this case follows by applying Theorem 1.13.13. □

Theorem 1.13.15. *If condition (1.13.32) holds and*

$$\lim_{k\to\infty}\frac{1}{k}\sum_{n=n_0}^{k}c(n)=\infty,\tag{1.13.65}$$

then equation (1.4.4) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.4.4) and proceed as in the proofs of Theorems 1.13.8 and 1.13.13 to obtain (1.13.42), which yields

$$\liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}(k-n)^{(m)}\frac{w^2(n)}{w(n)+c(n)}<\infty,\tag{1.13.66}$$

and hence (1.13.62) implies that

$$\sum_{n=N}^{\infty}\frac{w^2(n)}{w(n)+c(n)}<\infty.\tag{1.13.67}$$

Also, (1.13.42) leads to

$$\liminf_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}(k+1-n)^{(m-1)}\left(w(n)+c(n)\right)<\infty.\tag{1.13.68}$$

By applying Lemma 1.13.5 with $f(n)=w(n)+c(n)$, one can easily find

$$\liminf_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\left[w(n)+c(n)\right]<\infty.\tag{1.13.69}$$

From (1.13.67) we see that

$$\lim_{k\to\infty}\frac{w^2(k)}{w(k)+c(k)}=0,\tag{1.13.70}$$

and hence there exists an integer $N_1 \geq N$ such that

$$\frac{w^2(k)}{w(k)+c(k)} < 1, \tag{1.13.71}$$

so $w^2(k) - w(k) - c(k) < 0$ for all $k \geq N_1$. Completing the square of the left-hand side of this inequality, we obtain

$$-\left[\frac{1}{4}+c(k)\right]^{1/2} < w(k) - \frac{1}{2} < \left[\frac{1}{4}+c(k)\right]^{1/2} \quad \text{for } k \geq N_1. \tag{1.13.72}$$

From (1.13.69) we get

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{n=N_1}^{k} B(n) \leq \liminf_{k\to\infty} \frac{1}{k} \sum_{n=N_1}^{k} [w(n)+c(n)] < \infty, \tag{1.13.73}$$

where

$$B(n) = c(n) + \frac{1}{2} - \left(\frac{1}{4}+c(n)\right)^{1/2} = \left[\left(\frac{1}{4}+c(n)\right)^{1/2} - \frac{1}{2}\right]^2 \quad \text{for } n \geq N_1. \tag{1.13.74}$$

It follows that

$$\liminf_{k\to\infty} \frac{1}{k} \sum_{n=N_1}^{k} B(n) < \infty. \tag{1.13.75}$$

But

$$\begin{aligned} B(n) &= \left[\left(\frac{1}{4}+c(n)\right)^2 - \frac{1}{2}\right]^2 \frac{\left[(1/4+c(n))^{1/2}+1/2\right]^2}{\left[(1/4+c(n))^{1/2}+1/2\right]^2} \\ &= \frac{c^2(n)}{\left[(1/4+c(n))^{1/2}+1/2\right]^2}, \end{aligned} \tag{1.13.76}$$

and $((1/4)+c(n))^{1/2} > 1/2$ for $n \geq N_1$. Thus,

$$B(n) = \frac{c(n)}{4} - \frac{c(n)}{16[1/4+c(n)]} > \frac{c(n)}{4} - \frac{1}{16} \tag{1.13.77}$$

for $n \geq N_2$ for some $N_2 \geq N_1$. Summing up (1.13.77) from n_2 to k and dividing both sides by k, we find

$$\frac{1}{k}\sum_{n=N_2}^{k} B(n) \geq \frac{1}{4k}\sum_{n=N_2}^{k} c(n) - \frac{k-N_2+1}{16k}, \tag{1.13.78}$$

and hence

$$\liminf_{k\to\infty}\frac{1}{k}\sum_{n=N_2}^{k} B(n) \geq \liminf_{k\to\infty}\frac{1}{4k}\sum_{n=N_2}^{k} c(n) - \frac{1}{16}, \tag{1.13.79}$$

which is impossible in view of (1.13.65) and (1.13.75). The proof is therefore complete. □

Theorem 1.13.16. *Suppose that the conditions*

$$\begin{gathered}\limsup_{k\to\infty}\frac{1}{k}\sum_{n=n_0}^{k} c(n) = \infty,\\ \liminf_{k\to\infty}\frac{1}{k}\sum_{n=n_0}^{k}\left[(k-n)q(n)-c(n)\right] > -\infty\end{gathered} \tag{1.13.80}$$

are satisfied. Then equation (1.4.4) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.4.4). Proceeding as in the proof of Theorem 1.13.8, we obtain (1.13.37) for $k \geq N$, which at $m = 1$ takes the form

$$(k+1-N)w(N) = \sum_{n=N}^{k}\left[(k-n)q(n)-c(n)\right] + \sum_{n=N}^{k}\Omega(k;n), \tag{1.13.81}$$

where

$$\Omega(k;n) = (k-n)\frac{w^2(n)}{w(n)+c(n)} + w(n) + c(n) \quad \text{for } k \geq n \geq N. \tag{1.13.82}$$

Dividing both sides of (1.13.81) by k and taking lim sup on both sides as $k \to \infty$, we obtain

$$w(N) \geq \liminf_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\left[(k-n)q(k)-c(k)\right] + \limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\Omega(k;n), \tag{1.13.83}$$

and hence

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\left[(k-n)\frac{w^2(n)}{w(n)+c(n)}+w(k)+c(k)\right]<\infty. \tag{1.13.84}$$

This implies that

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}(k-n)\frac{w^2(n)}{w(n)+c(n)}<\infty. \tag{1.13.85}$$

By applying Lemma 1.13.6 with $f(n)=w^2(n)/(w(n)+c(n))$, we see that (1.13.62) holds with $m=1$, that is,

$$\lim_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}(k-n)\frac{w^2(n)}{w(n)+c(n)}\quad\text{exists.} \tag{1.13.86}$$

Now from (1.13.84) we conclude that

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\left[w(n)+c(n)\right]<\infty. \tag{1.13.87}$$

The rest of the proof can be modelled according to that of Theorem 1.13.15 and hence is omitted. □

Theorem 1.13.17. *Suppose $\{c(k)\}$ is a bounded sequence. If there exist a sequence $\{\psi(k)\}$ and $m\in\mathbb{N}$ with*

$$\limsup_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=s}^{k}(k-n)^{1/2}q(n)\geq\psi(s) \tag{1.13.88}$$

for all sufficiently large $s\geq N\geq n_0\geq 0$ and

$$\sum_{n=N}^{\infty}\left[\psi^{+}(n)\right]^2=\infty, \tag{1.13.89}$$

where $\psi^{+}(n)=\max\{\psi(n),0\}$, then equation (1.4.4) is oscillatory.

Proof. Let x be an eventually positive solution of equation (1.4.4). As in the proofs of Theorems 1.13.8 and 1.13.15, we see that (1.13.70) holds and since $c(k)$ is bounded for $k\geq N$, we have $w(k)\to 0$ as $k\to\infty$. Consequently, the inequality (1.13.26) of Lemma 1.13.5 with $f(n)=w(n)$ implies

$$\lim_{k\to\infty}\frac{1}{k^{(m)}}\sum_{n=N}^{k}(k+1-n)^{(m-1)}w(n)=0. \tag{1.13.90}$$

Dividing both sides of (1.13.36) by $k^{(m)}$, taking the upper limit as $k \to \infty$ and using (1.13.90), we obtain

$$w(N) \geq \psi(N) + \liminf_{k\to\infty} \frac{1}{k^{(m)}} \sum_{n=N}^{k} (k-n)^{(m)} \frac{w^2(n)}{w(n)+c(n)}. \tag{1.13.91}$$

Clearly, (1.13.67) holds and $w(k) \geq \psi(k)$ for all $k \geq N$. Since $c(k)$ is bounded for $k \geq N$ and $w(k) \to 0$ as $k \to \infty$, there exist an integer $N_1 \geq N$ and a constant $b > 0$ such that $w(k) + c(k) < b$ for all $k \geq N_1$. Thus

$$\frac{w^2(n)}{w(n)+c(n)} \geq \frac{w^2(n)}{b} \quad \text{for } n \geq N_1, \tag{1.13.92}$$

and hence

$$\frac{w^2(n)}{w(n)+c(n)} \geq \frac{\left(\psi^+(n)\right)^2}{b} \quad \text{for } n \geq N_1. \tag{1.13.93}$$

Summing both sides of (1.13.93) from N_1 to k, we have

$$\infty > \sum_{n=N}^{k} \frac{w^2(n)}{w(n)+c(n)} \geq \frac{1}{b} \sum_{n=N}^{k} \left[\psi^+(n)\right]^2, \tag{1.13.94}$$

which contradicts condition (1.13.89). This completes the proof. □

1.14. Oscillation criteria for linear damped difference equations

This section is devoted to the study of the oscillatory behavior of second-order damped difference equations of the form

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k)x(k+1) = 0 \quad \text{for } k \in \mathbb{N}, \tag{1.14.1}$$

where $\{c(k)\}$, $\{p(k)\}$, and $\{q(k)\}$ are sequences of real numbers such that $c(k) > 0$ for $k \geq n_0 \geq 0$. We will show that the oscillation of equation (1.14.1), when $c(k) > p(k)$ eventually, can be studied by transforming the equation into an undamped form similar to equation (1.4.4) for which the oscillatory character can be achieved easily. We also present some oscillation results for equation (1.14.1) when $c(k) \geq p(k)$ eventually, by using summation averaging techniques which are different from those given in the previous sections. Finally, we establish some oscillation criteria for equation (1.14.1) which are independent of the summation of its coefficients.

1.14.1. Oscillation of a reducible damped equation

Consider equation (1.14.1) and assume that

$$c(k) > p(k) \quad \forall k \geq n_0 \geq 0. \tag{1.14.2}$$

In order to transform equation (1.14.1) into an undamped form, we define a sequence of real numbers $\{\mu(k)\}$ by

$$\mu(k) = \prod_{i=n_0}^{k-1} \frac{c(i)}{c(i) - p(i)} \quad \text{for } k \geq n_0 + 1. \tag{1.14.3}$$

Since

$$\begin{aligned} \Delta\mu(k) &= \prod_{i=n_0}^{k} \frac{c(i)}{c(i) - p(i)} - \prod_{i=n_0}^{k-1} \frac{c(i)}{c(i) - p(i)} \\ &= \left[\frac{c(k)}{c(k) - p(k)} - 1\right] \prod_{i=n_0}^{k-1} \frac{c(i)}{c(i) - p(i)} = \frac{p(k)}{c(k) - p(k)}\mu(k), \end{aligned} \tag{1.14.4}$$

it follows that for $k \geq n_0$,

$$\begin{aligned} \Delta\big(\mu(k)c(k)\Delta x(k)\big) &= \mu(k+1)\Delta\big(c(k)\Delta x(k)\big) + \big(c(k)\Delta x(k)\big)\Delta\mu(k) \\ &= \mu(k+1)\Delta\big(c(k)\Delta x(k)\big) + \big(c(k)\Delta x(k)\big)\left(\frac{p(k)}{c(k) - p(k)}\right)\mu(k) \\ &= \mu(k+1)\big[\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k)\big]. \end{aligned} \tag{1.14.5}$$

Thus, equation (1.14.1) takes the form

$$\Delta\big(\mu(k)c(k)\Delta x(k)\big) + \mu(k+1)q(k)x(k+1) = 0 \quad \text{for } k \geq n_0. \tag{1.14.6}$$

Therefore $\{x(k)\}$, $k \geq n_0$, is a solution of equation (1.14.1) if and only if it satisfies equation (1.14.6) for all $k \geq n_0$. This proves the following result.

Theorem 1.14.1. *Assume that condition (1.14.2) holds. Then equation (1.14.1) is oscillatory if and only if equation (1.14.6) is oscillatory.*

Now, oscillation and nonoscillation criteria obtained in the previous sections for equation (1.4.4) and/or other related equations can also be applied to equation (1.14.6) and hence are considered as oscillation and nonoscillation results for equation (1.14.1). The formulation of such results are left to the reader. Here, we only show how to apply the known result due to Hinton and Lewis for the nonoscillation of equation (1.6.1) to equation (1.14.1). First, we state their result.

Theorem 1.14.2. *Suppose that*

$$\sum_{n=N\geq n_0}^{\infty} q(n) < \infty, \qquad \limsup_{k\to\infty} k\sum_{n=k}^{\infty} q(n) < \frac{1}{4}. \tag{1.14.7}$$

Then equation (1.6.1) is nonoscillatory.

By applying the Sturm comparison theorem, one can easily extend Theorem 1.14.2 to equation (1.4.4). In fact, we obtain the following result.

Corollary 1.14.3. *Assume that $c(k) \geq 1$ for $k \geq n_0 \geq 0$ and that condition (1.14.7) holds. Then equation (1.4.4) is nonoscillatory.*

Corollary 1.14.3, when applied to equation (1.14.6), gives the following result.

Corollary 1.14.4. *Let condition (1.14.2) hold and let $\mu(k)$ be defined as in (1.14.3). If*

$$\mu(k)c(k) \geq 1, \qquad \sum_{n=N\geq n_0}^{\infty} \mu(n+1)q(n) < \infty, \qquad \limsup_{k\to\infty} k\sum_{n=k}^{\infty} \mu(n+1)q(n) < \frac{1}{4}, \tag{1.14.8}$$

then equation (1.14.1) is nonoscillatory.

We note that the presence of the damping term in equation (1.14.1) may generate or disrupt the oscillatory and nonoscillatory character of the associated undamped equation. These properties are illustrated in the following example.

Example 1.14.5. Consider the difference equation

$$\Delta^2 x(k) + \frac{1}{k+1}\Delta x(k) + \frac{1}{8(k+1)(k+2)}x(k+1) = 0 \quad \text{for } k \in \mathbb{N}. \tag{1.14.9}$$

Here $p(k) = 1/(k+1)$ and hence

$$\mu(k) = \prod_{i=2}^{k-1} \frac{1}{1 - 1/(i+1)} = \frac{k}{2} \quad \text{for } k \in \mathbb{N}. \tag{1.14.10}$$

Now equation (1.14.9) is equivalent to the equation

$$\Delta\big(k\Delta x(k)\big) + \frac{k+1}{8(k+1)(k+2)}x(k+1) = 0 \quad \text{for } k \in \mathbb{N} \setminus \{1\}. \tag{1.14.11}$$

It is easy to check that the hypotheses of Theorem 1.8.35 are satisfied, and hence equation (1.14.11) is oscillatory. Clearly equation (1.14.9) is oscillatory by Theorem 1.14.1. Now the associated undamped equation, namely,

$$\Delta^2 x(k) + \frac{1}{8(k+1)(k+2)} x(k+1) = 0 \quad \text{for } k \in \mathbb{N} \tag{1.14.12}$$

is nonoscillatory by Theorem 1.14.2 since

$$\begin{gathered} \sum_{n=N>0}^{k} \frac{1}{8(n+1)(n+2)} = \frac{1}{8}\left[\frac{1}{N+1} - \frac{1}{k+2}\right], \\ \sum_{n=k}^{\infty} \frac{1}{8(n+1)(n+2)} = \frac{1}{8(k+1)}, \end{gathered} \tag{1.14.13}$$

which yields

$$\limsup_{k\to\infty} k \sum_{n=k}^{\infty} \frac{1}{8(n+1)(n+2)} = \frac{1}{8} < \frac{1}{4}. \tag{1.14.14}$$

Therefore, we conclude that the damping term in equation (1.14.9) generates oscillation.

Next we consider the damped difference equation

$$\Delta\big(k\Delta x(k)\big) - \Delta x(k) + \frac{k+1}{8(k+1)(k+2)} x(k+1) = 0 \quad \text{for } k \in \mathbb{N}. \tag{1.14.15}$$

Clearly the undamped equation associated to equation (1.14.15) is the equation (1.14.11) which is oscillatory, while equation (1.14.15) is equivalent to the nonoscillatory equation (1.14.12). In this case, we observe that the damping term in equation (1.14.15) disrupts oscillation.

Now with some restrictions on the damping term, one can find a relation between the oscillation of equations with damping term and the oscillation of some equations without damping term, as in the following result.

Theorem 1.14.6. *Assume that $p(k) \le 0$ for $k \ge n_0 \ge 0$. If the equation*

$$\Delta\big(c(k)\Delta x(k)\big) + \big[q(k) + p(k)\big]x(k+1) = 0 \quad \text{for } k \ge n_0 \tag{1.14.16}$$

is oscillatory, then equation (1.14.1) is oscillatory.

Proof. Suppose that equation (1.14.1) is nonoscillatory and let $\{x(k)\}$ be a solution of equation (1.14.1) such that $x(k) > 0$ for $k \ge N \ge n_0$. Using the Riccati transformation $w(k) = c(k)\Delta x(k)/x(k)$, $k \ge N$, equation (1.14.1) takes the form

$$\begin{aligned}
\Delta w(k) &= \frac{\Delta\big(c(k)\Delta x(k)\big)}{x(k+1)} - \frac{w^2(k)}{w(k)+c(k)} \\
&= -q(k) - p(k)\frac{\Delta x(k)}{x(k+1)} - \frac{w^2(k)}{w(k)+c(k)} \\
&= -q(k) - p(k) + p(k)\frac{x(k)}{x(k+1)} - \frac{w^2(k)}{w(k)+c(k)} \\
&\le -\big[q(k)+p(k)\big] - \frac{w^2(k)}{w(k)+c(k)}
\end{aligned} \tag{1.14.17}$$

for $k \ge N$. By applying Lemma 1.7.1 it follows that equation (1.14.16) is nonoscillatory, which is a contradiction. Thus equation (1.14.1) must be oscillatory. This completes the proof. ☐

1.14.2. Oscillation criteria of summation averaging type

Here we will investigate, using a summation averaging technique, the oscillatory behavior of equation (1.14.1) when $c(k) \ge p(k)$ for $k \ge n_0 \ge 0$. It will be convenient to employ the notation

$$F(k) = \rho(k+1)q(k) - \Big[\sqrt{c(k)\rho(k)} - \sqrt{\rho(k+1)\big(c(k)-\rho(k)\big)}\Big]^2 \tag{1.14.18}$$

for $k \ge n_0 \ge 0$, where $\{\rho(k)\}$ denotes a positive real sequence, $k \ge n_0 \ge 0$.

The following lemma is needed.

Lemma 1.14.7. *Suppose that there exist a positive sequence $\{\rho(k)\}$ and a subsequence $\{k_n\}$ with $k_n \to \infty$ as $n \to \infty$ such that for any $N_1 \ge n_0$,*

$$\rho(k_n)c(k_n) \le \sum_{j=N_1}^{k_n-1} F(j) \quad \text{for every } k_n \ge N_1 + 1, \tag{1.14.19}$$

where $F(k)$ is defined as in (1.14.18). Then any solution $\{x(k)\}$ of equation (1.14.1) satisfies $x(k)\Delta x(k) \ge 0$ eventually.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.14.1), say, $x(k) > 0$ for $k \ge N \ge n_0$. Define

$$w(k) = \rho(k)c(k)\frac{\Delta x(k)}{x(k)} \quad \text{for } k \ge N. \tag{1.14.20}$$

Then for $k \ge N$ we obtain

$$\begin{aligned}
\Delta w(k) &= \rho(k+1)\Delta\left(c(k)\frac{\Delta x(k)}{x(k)}\right) + c(k)\frac{\Delta x(k)}{x(k)}\Delta\rho(k) \\
&= \rho(k+1)\left[\frac{\Delta(c(k)\Delta x(k))}{x(k+1)} - c(k)\frac{(\Delta x(k))^2}{x(k)x(k+1)}\right] + c(k)\frac{\Delta x(k)}{x(k)}\Delta\rho(k) \\
&= -\rho(k+1)\left[q(k) + p(k)\frac{\Delta x(k)}{x(k+1)}\right] \\
&\quad - c(k)\frac{\Delta x(k)}{x(k)}\left[\rho(k+1)\frac{\Delta x(k)}{x(k+1)} - \Delta\rho(k)\right] \\
&= -\rho(k+1)q(k) + \rho(k+1)(c(k) - p(k)) + c(k)\rho(k) \\
&\quad - \left[\rho(k+1)(c(k) - p(k))\frac{x(k)}{x(k+1)} + c(k)\rho(k)\frac{x(k+1)}{x(k)}\right].
\end{aligned} \tag{1.14.21}$$

Now by employing the arithmetic-geometric mean inequality we get

$$\begin{aligned}
&\rho(k+1)(c(k) - p(k))\frac{x(k)}{x(k+1)} + c(k)\rho(k)\frac{x(k+1)}{x(k)} \\
&\qquad \ge 2\left[c(k)\rho(k)\rho(k+1)(c(k) - p(k))\right]^{1/2},
\end{aligned} \tag{1.14.22}$$

and so

$$\begin{aligned}
\Delta w(k) &\le -\rho(k+1)q(k) + c(k)\rho(k) + \rho(k+1)(c(k) - p(k)) \\
&\qquad - 2\left[c(k)\rho(k)\rho(k+1)(c(k) - p(k))\right]^{1/2} \\
&= -\rho(k+1)q(k) + \left[(c(k)\rho(k))^{1/2} - (\rho(k+1)(c(k) - p(k)))^{1/2}\right]^2 \\
&= -F(k),
\end{aligned} \tag{1.14.23}$$

that is,

$$\Delta w(k) \le -F(k) \quad \forall k \ge N. \tag{1.14.24}$$

We claim that $w(k) \geq 0$ eventually. To this end assume the existence of $N_1 \geq N$ such that (1.14.19) holds and $\Delta x(N_1) < 0$. Then $w(N_1) < 0$. Summing both sides of (1.14.24) from N_1 to $k-1$, we find

$$w(k) - w(N_1) \leq -\sum_{j=N_1}^{k-1} F(j), \tag{1.14.25}$$

so

$$\rho(k)c(k)\frac{x(k+1)}{x(k)} + \sum_{j=N_1}^{k-1} F(j) - \rho(k)c(k) \leq w(N_1) \quad \text{for } k \geq N_1 + 1. \tag{1.14.26}$$

Thus

$$0 < \rho(k_n)c(k_n)\frac{x(k_n+1)}{x(k_n)} + \left[\sum_{j=N_1}^{k_n-1} F(j) - \rho(k_n)c(k_n)\right] \leq w(N_1) \tag{1.14.27}$$

for $k_n \geq N_1 + 1$, which is a contradiction. This completes the proof. $\square$

Now we are ready to prove the following result.

Theorem 1.14.8. *If in addition to condition (1.14.19) it is assumed that*

$$\limsup_{k\to\infty} \sum_{j=n_0\geq 0}^{k} F(j) = \infty, \tag{1.14.28}$$

then equation (1.14.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.14.1), say, $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Define $w(k)$ by (1.14.20) and proceed as in the proof of Lemma 1.14.7 to obtain (1.14.24), and hence we conclude that

$$w(k) - w(N) \leq -\sum_{j=N}^{k-1} F(j) \quad \text{for } k \geq N + 1. \tag{1.14.29}$$

Now condition (1.14.28) implies that $\liminf_{k\to\infty} w(k) = -\infty$, however $w(k) \geq 0$ eventually by Lemma 1.14.7, which is a contradiction. $\square$

The following example illustrates the methods above.

Example 1.14.9. Consider the damped difference equation

$$\Delta(k\Delta x(k)) + k\Delta x(k) + (-1)^k(k+1)^3 x(k) = 0 \quad \text{for } k \in \mathbb{N}. \tag{1.14.30}$$

Here, $F(k) = (-1)^k(k+1)^2 - 1$ with $\rho(k) = 1/k$, $k \in \mathbb{N}$. It is easy to check that all the hypotheses of Theorem 1.14.8 are satisfied, and hence every solution of equation (1.14.30) is oscillatory. One such solution is

$$x(k)=\begin{cases}1 & \text{if } k\in\{0,1\},\\ \displaystyle\prod_{n=1}^{k-1}\left[1-(-1)^n(n+1)^2\right] & \text{if } k\geq 2.\end{cases} \tag{1.14.31}$$

The following corollaries are immediate.

Corollary 1.14.10. *Assume that*

$$\limsup_{k\to\infty}\left\{\sum_{j=n_0\geq 0}^{k-1}F(j)-\rho(k)c(k)\right\}=\infty. \tag{1.14.32}$$

Then equation (1.14.1) is oscillatory.

Corollary 1.14.11. *If there exists a subsequence $\{k_n\}$ with $k_n\to\infty$ as $n\to\infty$ such that*

$$\begin{gathered} c(k_n)\leq\sum_{j=N}^{k_n-1}q(j)\quad\text{for all large } N,\ k_n\geq N+1,\\ \limsup_{k\to\infty}\sum_{j=n_0\geq 0}^{k}q(j)=\infty, \end{gathered} \tag{1.14.33}$$

then equation (1.4.4) is oscillatory.

Theorem 1.14.12. *If there exists a positive sequence $\{\rho(k)\}$ such that*

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\left[\sum_{j=N}^{n}F(j)-\rho(n+1)c(n+1)\right]=\infty\quad\text{with } N\geq n_0\geq 0, \tag{1.14.34}$$

then equation (1.14.1) is oscillatory.

Now we present the following result.

Corollary 1.14.13. *If there exists a positive sequence $\{\rho(k)\}$ such that*

$$\begin{gathered} \limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\sum_{j=N}^{n}\rho(j+1)q(j)=\infty\quad\text{with } N\geq n_0\geq 0,\\ \limsup_{k\to\infty}\frac{\sum_{n=N}^{k}\left[\sum_{j=N}^{n}\left(\sqrt{\rho(j)c(j)}-\sqrt{\rho(j+1)(c(j)-p(j))}\right)^2+\rho(n+1)c(n+1)\right]}{k}\\ <\infty, \end{gathered} \tag{1.14.35}$$

then equation (1.14.1) is oscillatory.

Proof. From the definition of $F(k)$, for $N \geq n_0$, we have

$$\begin{aligned}
&\sum_{n=N}^{k}\left[\sum_{j=N}^{n} F(j) - \rho(n+1)c(k+1)\right] \\
&\quad = \sum_{n=N}^{k}\left[\sum_{j=N}^{n} \rho(j+1)q(j) - \left(\sqrt{\rho(j)c(j)} - \sqrt{\rho(j+1)\left(c(j)-p(j)\right)}\right)^2 \right. \\
&\qquad\qquad \left. - \rho(n+1)c(n+1)\right] \\
&\quad = \sum_{n=N}^{k}\left(\sum_{j=N}^{n} \rho(j+1)q(j)\right) \\
&\qquad - \sum_{n=N}^{k}\left[\sum_{j=N}^{n}\left(\sqrt{\rho(j)c(j)} - \sqrt{\rho(j+1)\left(c(j)-p(j)\right)}\right)^2 + \rho(n+1)c(n+1)\right].
\end{aligned} \tag{1.14.36}$$

Dividing both sides of (1.14.36) by k, taking lim sup on both sides as $k \to \infty$, and applying Theorem 1.14.12, we obtain the desired result. □

Next, we give the following result.

Theorem 1.14.14. *Let condition (1.14.18) hold and suppose that for every $N \geq n_0$, there exists $n \geq N$ such that*

$$q(n) + p(n) - c(n) > 0. \tag{1.14.37}$$

Then equation (1.14.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.14.1), say, $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Define $w(k)$ by (1.14.20). Then equation (1.14.1) can be transformed into the Riccati-type equation

$$w(k+1) = \rho(k+1)\left[c(k) - p(k) - q(k)\right] - \rho(k+1)\rho(k)\frac{c(k)\left[c(k)-p(k)\right]}{w(k)+\rho(k)c(k)} \tag{1.14.38}$$

for $k \geq N$. Now, condition (1.14.37) implies the existence of a subsequence $\{k_n\}$ with $k_n \to \infty$ as $n \to \infty$ such that $w(k_n) < 0$. On the other hand, Lemma 1.14.7 implies that $w(k) \geq 0$ eventually, a contradiction which completes the proof. □

Theorem 1.14.15. *If there exists a positive sequence $\{\rho(k)\}$ such that*

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{n=N}^{k}\left[\sum_{j=N}^{n}F(j)-\rho(n+1)c(n+1)\right]>-\infty \tag{1.14.39}$$

for all large $N \geq n_0$ and condition (1.14.28) holds, then equation (1.14.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.14.1), say, $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Define $w(k)$ by (1.14.20) and proceed as in the proof of Lemma 1.14.7 to obtain (1.14.24). Now, by condition (1.14.28) we see that $\liminf_{k\to\infty} w(k) = -\infty$. We claim that condition (1.14.39) implies that $w(k)$ is bounded below. From (1.14.24) we see that

$$w(k+1)-w(m)\leq-\sum_{j=m}^{k}F(j)\quad\forall k\geq m\geq N. \tag{1.14.40}$$

Thus, for $k \geq m \geq N$ we have

$$\rho(k+1)c(k+1)\frac{x(k+2)}{x(k+1)}+\sum_{j=m}^{k}F(j)-\rho(k+1)c(k+1)\leq w(m), \tag{1.14.41}$$

so

$$\sum_{j=m}^{k}F(j)-\rho(k+1)c(k+1)\leq w(m). \tag{1.14.42}$$

Summing both sides of (1.14.42) from m to k and dividing by k, we get

$$\frac{1}{k}\sum_{n=m}^{k}\left[\sum_{j=m}^{n}F(j)-\rho(n+1)c(n+1)\right]\leq\frac{k-m+1}{k}w(m), \tag{1.14.43}$$

and hence

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{n=m}^{k}\left[\sum_{j=m}^{n}F(j)-\rho(n+1)c(n+1)\right]\leq w(m) \tag{1.14.44}$$

for all $m \geq N$. Using condition (1.14.39), there exists a constant $\lambda > 0$ such that $-\lambda < w(m)$ for all $m \geq N$, that is, $w(m)$ is bounded from below as required, and therefore, contradicts the fact that $\liminf_{k\to\infty} w(k) = -\infty$. This completes the proof. □

1.14.3. Oscillation criteria of nonsummation type

First, we give the following result.

Theorem 1.14.16. *If for a subsequence $\{k_n\} \subset \mathbb{N}$ with $k_n \to \infty$ as $n \to \infty$,*

$$q(k_n) + p(k_n) - c(k_n) - c(k_n + 1) \geq 0, \quad c(k_n) - p(k_n) \geq 0 \quad \textit{for } k \in \mathbb{N}, \tag{1.14.45}$$

then equation (1.14.1) is oscillatory.

Proof. Equation (1.14.1) in the three-term form becomes

$$\begin{aligned} &c(k+1)x(k+2) + [q(k) + p(k) - c(k) - c(k+1)]x(k+1) \\ &\quad + [c(k) - p(k)]x(k) = 0. \end{aligned} \tag{1.14.46}$$

Thus, if $\{x(k)\}$ is any solution of (1.14.1) such that $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$, then equation (1.14.46) implies

$$\begin{aligned} &c(k_n+1)x(k_n+2) + [q(k_n) + p(k_n) - c(k_n) - c(k_n+1)]x(k_n+1) \\ &\quad + [c(k_n) - p(k_n)]x(k_n) = 0, \end{aligned} \tag{1.14.47}$$

where $k_n \geq N$ for $n \in \mathbb{N}$. It follows from (1.14.45) that $c(k_n + 1)x(k_n + 2) \leq 0$ for $k \in \mathbb{N}$, which is a contradiction. This completes the proof. □

Theorem 1.14.17. *Assume that*

$$\begin{gathered} q(k_n) + p(k_n) - c(k_n) \geq 0, \quad c(k_n + i) - p(k_n + i) \geq 0 \quad \textit{for } i \in \mathbb{N}_0, \\ q(k_n + 1) - c(k_n + 2) \geq 0 \quad \textit{for } n \in \mathbb{N} \end{gathered} \tag{1.14.48}$$

for a subsequence $\{k_n\} \subset \mathbb{N}$ with $k_n \to \infty$ as $n \to \infty$. Then equation (1.14.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.14.1), say, $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Define $w(k)$ by (1.14.20) and hence obtain equation (1.14.38). It follows from equation (1.14.38) and condition (1.14.48) with $i = 0$ that

$$\begin{aligned} w(k_n + 1) &= \rho(k_n + 1)[c(k_n) - p(k_n) - q(k_n)] \\ &\quad - \rho(k_n + 1)\rho(k_n)\frac{c(k_n)[c(k_n) - p(k_n)]}{w(k_n) + \rho(k_n)c(k_n)} \leq 0 \end{aligned} \tag{1.14.49}$$

for $k_n \geq N$, and hence we have

$$\Delta x(k_n) \leq 0 \quad \text{for } k_n > N. \tag{1.14.50}$$

From equation (1.14.46) we have

$$\begin{aligned} c(k_n+2)x(k_n+3) + [q(k_n+1) + p(k_n+1) - c(k_n+1) - c(k_n+2)]x(k_n+2) \\ + [c(k_n+1) - p(k_n+1)]x(k_n+1) = 0. \end{aligned} \tag{1.14.51}$$

By (1.14.50), we see that $x(k_n+2) \leq x(k_n+1)$ for $k_n > N$, and therefore

$$c(k_n+2)x(k_n+3) + [q(k_n+1) - c(k_n+2)]x(k_n+2) \leq 0 \quad \text{if } k_n > N. \tag{1.14.52}$$

This is a contradiction in view of condition (1.14.48) and the fact that $x(k) > 0$ for all $k \geq N$. This completes the proof. □

Remark 1.14.18. The importance of Theorems 1.14.16 and 1.14.17 is that the numbers $c(k) - p(k)$ need not be nonnegative for all large k.

If $p(k) \equiv 0$, then Theorem 1.14.17 leads to the following oscillation criterion for equation (1.4.4).

Corollary 1.14.19. *If for a subsequence $\{k_n\}$ with $k_n \to \infty$ as $n \to \infty$, $q(k_n) \geq c(k_n)$ and $q(k_n+1) \geq c(k_n+2)$ for $k \in \mathbb{N}$, then equation (1.4.4) is oscillatory.*

Theorem 1.14.20. *If there exist a positive sequence $\{\rho(k)\}$ and a real number α such that eventually*

$$\frac{[c(k+1) + c(k) - p(k) - q(k)]^2}{4c(k)[c(k) - p(k)]} \frac{\rho(k+1)}{\rho(k)} \leq \alpha < 1, \quad c(k) > p(k), \tag{1.14.53}$$

and if there exist a subsequence $\{k_n\}$ with $k_n \to \infty$ as $n \to \infty$ and a constant $M \geq 0$ such that

$$[c(k_n+1) + c(k_n) - p(k_n) - q(k_n)] \leq M\rho(k_n)c(k_n)[c(k_n) - p(k_n)], \tag{1.14.54}$$

then equation (1.14.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (1.14.1), say, $x(k) > 0$ for $k \geq N \geq n_0 \geq 0$. Define $w(k)$ by (1.14.20) and obtain equation (1.14.38), which takes the form

$$c(k+1)\frac{x(k+2)}{x(k+1)} = c(k+1) + c(k) - p(k) - q(k) - \rho(k)\frac{c(k)[c(k) - p(k)]}{w(k) + \rho(k)c(k)} \tag{1.14.55}$$

for $k \geq N$. Now we let $\xi(k) = w(k) + \rho(k)c(k)$, $k \geq N$. Then for $k \geq N$,

$$\xi(k) = \rho(k)c(k)\frac{\Delta x(k)}{x(k)} + \rho(k)c(k) = \rho(k)c(k)\frac{x(k+1)}{x(k)} > 0. \tag{1.14.56}$$

Thus from the definition of $\xi(k)$ and equality (1.14.55), for $k \geq N$, we have

$$\begin{aligned} &\xi(k+1) + \rho(k+1)\rho(k)c(k)[c(k) - p(k)]\left(\frac{1}{\xi(k)}\right) \\ &\quad = \rho(k+1)[c(k+1) + c(k) - p(k) - q(k)]. \end{aligned} \tag{1.14.57}$$

Applying the arithmetic-geometric mean inequality to the left-hand side of (1.14.57), we get

$$\begin{aligned} &2\left[\frac{\xi(k+1)}{\xi(k)}\rho(k+1)\rho(k)c(k)(c(k) - p(k))\right]^{1/2} \\ &\quad \leq \rho(k+1)[c(k+1) + c(k) - p(k) - q(k)], \end{aligned} \tag{1.14.58}$$

so

$$\frac{\xi(k+1)}{\xi(k)} \leq \frac{[c(k+1) + c(k) - p(k) - q(k)]^2}{4c(k)\rho(k)[c(k) - p(k)]} \quad \text{for } k \geq N. \tag{1.14.59}$$

Now condition (1.14.53) implies that $\xi(k+1)/\xi(k) \leq \alpha < 1$ for $k \geq N$. Then $\xi(k) \to 0$ as $k \to \infty$, and therefore for every constant $M \geq 0$ there exists $N_1 \geq N$ such that

$$\frac{1}{\xi(k)} \leq M \quad \forall k \geq N_1. \tag{1.14.60}$$

Using (1.14.60) in (1.14.55), for $k \geq N_1$, we get

$$c(k+1)\frac{x(k+2)}{x(k+1)} \leq c(k+1) + c(k) - p(k) - q(k) - M\rho(k)c(k)[c(k) - p(k)]. \tag{1.14.61}$$

Using condition (1.14.54), we obtain

$$c(k_n+1)\frac{x(k_n+2)}{x(k_n+1)} \leq 0 \quad \text{for } k_n \geq N_1, \tag{1.14.62}$$

which is a contradiction. This completes the proof. □

The following corollary is immediate.

Corollary 1.14.21. *If there exist a positive sequence $\{\rho(k)\}$ and a constant α such that*

$$\left[\frac{c(k+1) + c(k) - q(k)}{2c(k)}\right]^2 \left(\frac{\rho(k+1)}{\rho(k)}\right) \leq \alpha < 1 \quad \textit{eventually}, \tag{1.14.63}$$

and if there exist a subsequence $\{k_n\}$ with $k_n \to \infty$ as $n \to \infty$ and a constant $M \geq 0$ such that

$$[c(k_n+1) + c(k_n) - q(k_n)] - M\rho(k_n)c^2(k_n) \leq 0 \quad \textit{for } n \in \mathbb{N}, \tag{1.14.64}$$

then equation (1.4.4) is oscillatory.

The following example illustrates the methods above.

Example 1.14.22. Consider the difference equation

$$\Delta^2 x(k) + qx(k+1) = 0 \quad \text{for } k \in \mathbb{N}, \tag{1.14.65}$$

where $q > 0$ is a constant. It is easy to check that the hypotheses of Corollary 1.14.21 are satisfied if we let $\rho(k) \equiv 1$ and $q \in (0,4)$. Next, for the case when $q \geq 4$, we let

$$\rho(k) = \left[\frac{4\alpha}{(2-q)^2}\right]^k \quad \text{for } k \in \mathbb{N}, \tag{1.14.66}$$

where $\alpha \in (0,1)$ is a constant. Clearly, all conditions of Corollary 1.14.21 are satisfied in this case, and hence we conclude that equation (1.14.65) is oscillatory for all $q > 0$. We also note that Theorem 1.14.16 with $p(k) \equiv 0$ is not applicable to equation (1.14.65) when $q \in (0,2)$.

1.15. Oscillation and nonoscillation criteria for forced equations

In this section we consider the forced equation

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k) = f(k), \tag{1.15.1}$$

and the unforced equation

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k) = 0, \tag{1.15.2}$$

where $\{c(k)\}$, $\{f(k)\}$, and $\{q(k)\}$ are sequences of real numbers with $c(k) > 0$ for all $k \in \mathbb{N}_0$, and f is not eventually identically zero.

We will investigate the oscillation and nonoscillation of equation (1.15.1). First, we require the following definition: if u and x are solutions of equations (1.15.1) and (1.15.2), respectively, then we set

$$W(x,u)(k) = c(k)\big[x(k+1)u(k) - x(k)u(k+1)\big]. \tag{1.15.3}$$

Now, we state the following lemma which is interesting in its own right.

Lemma 1.15.1 (Abel's formula). *If x is a nontrivial solution of equation (1.15.2) and u a solution of equation (1.15.1), then for any $k > n$,*

$$W(x,u)(k) = W(x,u)(n) - \sum_{j=n+1}^{k} x(j)f(j). \tag{1.15.4}$$

Theorem 1.15.2. *For some solution x of equation (1.15.2) and some solution u of equation (1.15.1), suppose that $W(x,u)(k)$ is eventually of one sign (positive or negative). Then equation (1.15.2) is nonoscillatory if and only if u is a nonoscillatory solution of equation (1.15.1), which is equivalent to stating that equation (1.15.2) is oscillatory if and only if u is an oscillatory solution of equation (1.15.1).*

Proof. First we show the "only if" part. Suppose that equation (1.15.2) is nonoscillatory and $c(k)[x(k+1)u(k) - x(k)u(k+1)] \geq 0$ for $k \geq N \geq n_0$. We may assume that N is large enough so that $x(k)$ is of one sign, say $x(k) > 0$ for $k \geq \mathbb{N}$. Then

$$c(k)x(k+1)u(k) \geq c(k)x(k)u(k+1), \tag{1.15.5}$$

so

$$\frac{x(k+1)}{x(k)}u(k) \geq u(k+1) \quad \text{for } k \geq N. \tag{1.15.6}$$

Let n_1 be the first integer greater than or equal to N such that $u(n_1) \le 0$, if such an integer exists. Then $u(k) \le 0$ for $k \ge n_1$. If $u(k) \not\equiv 0$ for $k \ge n_1$, then there exists an integer $n_2 \ge n_1$ such that $u(n_2) < 0$, which means $u(k) < 0$ for $k \ge n_2$. If $u(k) = 0$ for $k \ge n_1$, then $f(k) = 0$ for $k \ge n_1$, but we exclude this possibility. If n_1 does not exist, then $u(k) > 0$ for $k \ge \mathbb{N}$. In either case $\{u(k)\}$ is nonoscillatory. We can apply the same arguments when $W(x,u)(k) \le 0$ or $x(k) \le 0$ eventually.

Now we show the "if" part. Assume that $\{u(k)\}$ is nonoscillatory, say, $u(k) > 0$ for $k \ge N \ge n_0$ and assume that $W(x,u)(k)$ is of one sign, say, $W(x,u)(k) \ge 0$ for $k \ge N$. However, suppose that $\{x(k)\}$ is an oscillatory solution of equation (1.15.2). Since $c(k) > 0$ for $k \ge n_0$,

$$x(k+1) \ge x(k)\left(\frac{u(k+1)}{u(k)}\right) \quad \text{for } k \ge N. \tag{1.15.7}$$

Choose a value $n_1 \ge N$ such that $x(n_1) > 0$. One such value must exist, since equation (1.15.2) cannot have a nontrivial oscillatory solution which is nonpositive. (However, equation (1.15.2) can.) Then inequality (1.15.7) implies that $x(k) > 0$ for $k \ge n_1$, which is a contradiction. A similar argument holds if $W(x,u)(k) \le 0$ and $z(k) \le 0$ eventually. This completes the proof. □

The following corollaries are immediate.

Corollary 1.15.3. *If equation (1.15.2) is nonoscillatory and $\{f(k)\}$ is eventually of one sign, then every solution of equation (1.15.1) is nonoscillatory.*

Proof. Let x be any solution of equation (1.15.2) and let u be any solution of equation (1.15.1). We may choose $n \ge n_0$ large enough so that $x(k)f(k)$ is of one sign for all $k \ge n$. Thus, all the terms of (1.15.4) will be of one sign, and the term $W(x,u)(n)$ is merely a constant. This means that eventually $W(x,u)(k)$ will be of one sign, and the result follows from Theorem 1.15.2. □

Corollary 1.15.4. *If equation (1.15.2) is oscillatory (nonoscillatory) and if there exists a solution x of equation (1.15.2) such that*

$$\sum^{\infty} x(j)f(j) = \pm\infty, \tag{1.15.8}$$

then every solution of equation (1.15.1) is oscillatory (nonoscillatory).

Proof. The hypothesis implies that for any solution u of equation (1.15.1), the quantity $W(x,u)(k)$ in (1.15.4) must eventually be of one sign. □

Corollary 1.15.5. *Suppose $\{f(k)\}$ has the form $\{a(k)x(k)\}$, where $\{a(k)\}$ is of one sign and x is a solution of equation (1.15.2). If equation (1.15.2) is oscillatory (nonoscillatory), then every solution of equation (1.15.1) is oscillatory (nonoscillatory).*

Proof. The argument is the same as the one for Corollary 1.15.4. □

Corollary 1.15.6. *For all sufficiently large k, assume $q(k) \geq 2[c(k) + c(k-1)]$ and $f(k) = (-1)^k a(k)$, where $\{a(k)\}$ is eventually of one sign. Then every solution of equation (1.15.1) oscillates.*

Proof. The hypothesis $q(k) \geq 2[c(k)+c(k-1)]$ implies that every solution of (1.15.2) oscillates and eventually alternates in sign. Thus, $x(k)f(k)=(-1)^k x(k)f(k)$ will be eventually of one sign. The proof of Corollary 1.15.4 now applies. □

We may note that the conclusion of Corollary 1.15.3 may no longer be true if $\{f(k)\}$ is allowed to change sign. To illustrate this we consider the following.

Example 1.15.7. The unforced equation

$$\Delta\left(\frac{1}{2}\Delta x(k-1)\right) = 0 \tag{1.15.9}$$

has linearly independent solutions $x_1(k) \equiv 1$ and $x_2(k) = k$, while the forced equation

$$\Delta\left(\frac{1}{2}\Delta y(k-1)\right) = 4(-1)^{k+1} \tag{1.15.10}$$

has the oscillatory solution $y(k) = 1 + 2(-1)^k$. All conditions of Corollary 1.15.3 are satisfied except that $\{f(k)\}$ is not eventually of one sign.

Parts of Corollaries 1.15.4, 1.15.5, and 1.15.14 assume that equation (1.15.2) is oscillatory and state sufficient conditions for every solution of equation (1.15.2) to oscillate. However, equation (1.15.2) being oscillatory does not always imply that every solution of equation (1.15.1) must oscillate.

The following example illustrates the methods presented above.

Example 1.15.8. The unforced equation

$$\Delta^2 x(k-1) + 4x(k) = 0 \tag{1.15.11}$$

has the oscillatory solutions $(-1)^k$ and $(-1)^k k$, while the forced equation

$$\Delta^2 x(k-1) + 4x(k) = \frac{4k^2 - 2}{k(k^2-1)} \quad \text{for } k > 1 \tag{1.15.12}$$

has a unique nonoscillatory solution $x(k) = 1/k$.

1.15.1. Implicit-type results

By implicit results we mean results in which the coefficients $c(k)$, $f(k)$, and $q(k)$ of equation (1.15.1) are not presented explicitly. Thus, Theorem 1.15.2 is of implicit type. Next, we establish some oscillation and nonoscillation criteria of this type for equation (1.15.1) by employing the transformation $u(k) = x(k)y(k)$, where x

is a solution of equation (1.15.1) satisfying $x(k) \neq 0$, for $k \geq N \geq n_0 \geq 0$ and u is a solution of equation (1.15.1).

Lemma 1.15.9. *Suppose that u and x are solutions of equations (1.15.1) and (1.15.2), respectively, and $x(k) \neq 0$ for $k \geq N$ for some $N \geq n_0 \geq 0$. Define y by $u(k) = x(k)y(k)$ for $k \geq N$. Then*

$$\Delta\big(c(k-1)x(k-1)x(k)\Delta y(k-1)\big) = x(k)f(k) \quad \text{for } k \geq N. \tag{1.15.13}$$

Proof. Since $u(k) = x(k)y(k)$, we have

$$\begin{aligned}
&\Delta u(k-1) = x(k-1)\Delta y(k-1) + y(k)\Delta x(k-1),\\
&\Delta\big(c(k-1)\Delta u(k-1)\big)\\
&\qquad = \Delta\big(c(k-1)x(k-1)\Delta y(k-1)\big) + \Delta\big(c(k-1)y(k)\Delta x(k-1)\big)\\
&\qquad = \Delta\big(c(k-1)x(k-1)\Delta y(k-1)\big) + y(k)\Delta\big(c(k-1)\Delta x(k-1)\big)\\
&\qquad\quad + c(k)\Delta x(k)\Delta y(k).
\end{aligned} \tag{1.15.14}$$

Using equation (1.15.1), we have for $k \geq N$,

$$\begin{aligned}
f(k) &= \Delta\big(c(k-1)x(k-1)\Delta y(k-1)\big) + y(k)\Delta\big(c(k-1)\Delta x(k-1)\big)\\
&\quad + c(k)\Delta x(k)\Delta y(k) + q(k)x(k)y(k)\\
&= \Delta\big(c(k-1)x(k-1)\Delta y(k-1)\big) + c(k)\Delta x(k)\Delta y(k)\\
&\quad + \big[\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)x(k)\big]y(k).
\end{aligned} \tag{1.15.15}$$

Since x is a solution of equation (1.15.2), we find for $k \geq N$,

$$\begin{aligned}
f(k) &= \Delta\big(c(k-1)x(k-1)\Delta y(k-1)\big) + c(k)\Delta x(k)\Delta y(k)\\
&= -c(k-1)x(k-1)\Delta y(k-1) + c(k)x(k+1)\Delta y(k).
\end{aligned} \tag{1.15.16}$$

Multiplying both sides of (1.15.16) by $x(k)$, we get for $k \geq N$,

$$\begin{aligned}
x(k)f(k) &= c(k)x(k)x(k+1)\Delta y(k) - c(k-1)x(k-1)x(k)\Delta y(k-1)\\
&= \Delta\big(c(k-1)x(k-1)x(k)\Delta y(k-1)\big).
\end{aligned} \tag{1.15.17}$$

This completes the proof. □

Theorem 1.15.10. *If there exists an eventually positive solution x of equation (1.15.2) such that for all sufficiently large integers $N \geq n_0 \geq 0$ and for some constant $M > 0$, the conditions*

$$\liminf_{k\to\infty} \sum_{n=N}^{k} x(n)f(n) = -\infty, \qquad \limsup_{k\to\infty} \sum_{n=N}^{k} x(n)f(n) = \infty, \tag{1.15.18}$$

$$\left| \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) \right| \leq M \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \tag{1.15.19}$$

for $k \geq N$, and

$$\sum_{n=N}^{\infty} \frac{1}{c(n)x(n)x(n+1)} = \infty \quad (\textit{i.e., } \{x(k)\} \textit{ is recessive}) \tag{1.15.20}$$

are satisfied, then equation (1.15.1) is oscillatory.

Proof. For the sake of contradiction, we assume that u is a nonoscillatory solution of equation (1.15.1). Without loss of generality, assume that u is eventually positive, say, $u(k) > 0$ and $x(k) \neq 0$ for $k \geq N \geq n_0$. As in Lemma 1.15.9, the sequence $\{y(k)\}$ defined by $u(k) = x(k)y(k)$ is a solution of equation (1.15.13). Now, summing equation (1.15.13) from N to k, we get

$$\sum_{j=N}^{k} x(j)f(j) = c(k)x(k)x(k+1)\Delta y(k) - c(N-1)x(N-1)\Delta x(N)\Delta y(N-1). \tag{1.15.21}$$

By condition (1.15.18), we have

$$\liminf_{k\to\infty} c(k)x(k)x(k+1)\Delta y(k) = -\infty. \tag{1.15.22}$$

Choose N so large that

$$c(N-1)x(N-1)x(N)\Delta y(N-1) < -2M. \tag{1.15.23}$$

Dividing both sides of (1.15.21) by $c(k)x(k)x(k+1)$ and summing from N to k, we obtain for $k \geq N$,

$$\begin{aligned}
y(k+1) &= y(N) + \left[c(N-1)x(N-1)x(N)\Delta y(N-1)\right] \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \\
&\quad + \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) \\
&< y(N) - 2M \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \\
&\quad + \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j).
\end{aligned} \tag{1.15.24}$$

From (1.15.19), we obtain

$$y(k) < y(N) - M \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)}. \tag{1.15.25}$$

By (1.15.20), $y(k)$ is eventually negative which implies that $u(k)$ is also eventually negative, which is a contradiction. This completes the proof. □

Theorem 1.15.11. *If there exist a positive solution x of equation (1.15.2) and an integer $N \geq n_0 \geq 0$ such that*

$$\liminf_{k\to\infty} \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) = -\infty, \tag{1.15.26}$$

$$\limsup_{k\to\infty} \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) = \infty, \tag{1.15.27}$$

$$\sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} < \infty \quad (\textit{i.e., } \{x(k)\} \textit{ is dominant}), \tag{1.15.28}$$

then equation (1.15.1) is oscillatory.

Proof. Suppose that u is a nonoscillatory solution of equation (1.15.1). As in the proof of Theorem 1.15.10, assume that $u(k) > 0$ for $k \geq N$ and obtain

$$\begin{aligned}
y(k+1) &= y(N) + \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) \\
&\quad + \left[c(N-1)x(N-1)x(N)\Delta y(N-1)\right] \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)},
\end{aligned} \tag{1.15.29}$$

which in view of (1.15.26) and (1.15.28), implies that $\liminf_{k\to\infty} y(k) = -\infty$, which is a contradiction to the fact that $\{u(k)\}$ is positive. This completes the proof. □

Example 1.15.12. Consider the forced equation

$$\Delta^2 u(k-1) - u(k) = f(k) \quad \text{for } k \in \mathbb{N}, \tag{1.15.30}$$

where $\{f(k)\}$ is a sequence of real numbers. The corresponding unforced equation

$$\Delta^2 x(k-1) - x(k) = 0 \tag{1.15.31}$$

has a positive solution $x(k) = a^k$, $k \in \mathbb{N}_0$, where $a = (3+\sqrt{5})/2$. If

$$f(k) = (-1)^{k+1}\left[a^{k+1}(2k+1) + a^{k-1}(2k-1)\right], \tag{1.15.32}$$

then all the assumptions of Theorem 1.15.11 are satisfied, and hence (1.15.30) is oscillatory. One such solution is $u(k) = (-1)^k a^k$.

By using the transformation $u(k) = x(k)y(k)$ in $W(x,u)(k)$ defined by (1.15.3), we observe that

$$W(x,u)(k) = -c(k)x(k)x(k+1)\Delta y(k), \tag{1.15.33}$$

and in view of the condition (1.15.18) or (1.15.26), (1.15.27), and (1.15.28), the equality (1.15.21) implies that $\{c(k)x(k)x(k+1)\Delta y(k)\}$ is oscillatory, that is, $W(x,u)(k)$ oscillates. Therefore, Theorem 1.15.2 is not applicable in this case.

We note that Theorem 1.15.2 preserves the oscillatory property of equations (1.15.1) and (1.15.2), whereas Theorems 1.15.10 and 1.15.11 generate oscillation in equation (1.15.1).

1.15.2. Some explicit criteria

We will give some criteria for the oscillation and nonoscillation of equation (1.15.1) that depend only on the coefficients $\{c(k)\}$, $\{f(k)\}$, and/or $\{q(k)\}$.

Theorem 1.15.13. *Suppose that the solutions of equation (1.15.2) are bounded and nonoscillatory and*

$$\sum^{\infty} \frac{f^+(k)}{c(k)} = \infty, \qquad \sum^{\infty} f^-(k) > -\infty \tag{1.15.34}$$

or

$$\sum^{\infty} \frac{f^-(k)}{c(k)} = -\infty, \qquad \sum^{\infty} f^+(k) < \infty, \tag{1.15.35}$$

where $f^+(k) = \max\{f(k), 0\}$ *and* $f^-(k) = \min\{f(k), 0\}$. *Then equation (1.15.1) is nonoscillatory.*

Proof. Assume that condition (1.15.34) holds and that x is an eventually positive solution of equation (1.15.2) satisfying (1.15.28). Then there exist an integer $N \geq n_0 \geq 0$ and positive constants M and λ such that $0 < x(k) \leq M$ and $\sum_{m=N}^{k} f^-(n) \geq -\lambda$ for all $k \geq N$. Then we find

$$\begin{aligned}
&\sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) \\
&\quad = \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \left[\sum_{j=N}^{n} x(j)f^+(j) + \sum_{j=N}^{n} x(j)f^-(j) \right] \\
&\quad \geq \sum_{n=N}^{k} \frac{1}{c(n)x(n+1)} f^+(n) + M \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} f^-(j) \\
&\quad \geq \frac{1}{M} \sum_{n=N}^{k} \frac{f^+(n)}{c(n)} - \lambda M \sum_{n=N}^{k} \frac{1}{c(n)x(n)x(n+1)}.
\end{aligned} \tag{1.15.36}$$

Taking now the limit on both sides as $k \to \infty$ and using (1.15.34), we obtain

$$\sum_{n=N}^{\infty} \frac{1}{c(n)x(n)x(n+1)} \sum_{j=N}^{n} x(j)f(j) = \infty. \tag{1.15.37}$$

If u is any solution of (1.15.1), we obtain (1.15.29). In view of (1.15.28) and (1.15.37), we see that $\lim_{k\to\infty} y(k) = \infty$. Thus $y(k) = u(k)/x(k)$ is eventually positive. Since $x(k) > 0$ eventually, we see that u is an eventually positive solution of equation (1.15.1). The proof of the case when condition (1.15.35) holds is similar. This completes the proof. □

Now, we state the following result.

Corollary 1.15.14. *Assume that $q(k) \geq 0$ for all large k. If*

$$\begin{gathered}
\sum^{\infty} \frac{1}{c(k)} < \infty, \\
c(k) + c(k-1) > q(k) \quad \text{for } k \geq N \geq n_0 \geq 0, \\
\frac{c^2(k)}{[c(k)+c(k-1)-q(k)][c(k)+c(k+1)-q(k+1)]} \leq \frac{1}{4},
\end{gathered} \tag{1.15.38}$$

and either condition (1.15.34) or (1.15.35) holds, then equation (1.15.1) is nonoscillatory.

The following example illustrates the theory presented above.

Example 1.15.15. Consider the forced equation

$$\Delta(6^{k-1}\Delta u(k-1)) + 2\cdot 6^{k-1}u(k) = f(k), \tag{1.15.39}$$

where

$$f(k) = \frac{1}{2}k6^k[1+(-1)^k] + \frac{1}{2}\left(\frac{2}{3}\right)^k[(-1)^k - 1] \quad \text{for } k \in \mathbb{N}. \tag{1.15.40}$$

Here

$$\begin{aligned} f^+(k) &= \begin{cases} k6^k & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases} \\ f^-(k) &= \begin{cases} 0 & \text{if } k \text{ is even,} \\ -\left(\dfrac{2}{3}\right)^k & \text{if } k \text{ is odd.} \end{cases} \end{aligned} \tag{1.15.41}$$

All conditions of Corollary 1.15.14 are satisfied, and hence (1.15.39) is nonoscillatory. One such solution is

$$u(k) = (-1)^k\left[-\frac{k}{4} + \frac{5}{48}\right] - \frac{9}{44}\left(-\frac{1}{9}\right)^k - \frac{9}{14}\left(\frac{1}{9}\right)^k + \left(\frac{3}{2}k - \frac{15}{4}\right). \tag{1.15.42}$$

It would be interesting to obtain oscillation criteria for equation (1.15.1) which are independent of the solutions of equation (1.15.1). Next, we present two results of this type.

Theorem 1.15.16. *Let $\{k_n\} \subset \mathbb{N}$ be a sequence with $k_n \to \infty$ as $n \to \infty$. If*

$$c(k_n) + c(k_n - 1) - q(k_n) \le 0 \quad \textit{for all large } k \in \mathbb{N}, \tag{1.15.43}$$

$$\{f(k_n)\} \quad \textit{is oscillatory}, \tag{1.15.44}$$

then equation (1.15.1) is oscillatory.

Proof. Let u be a nonoscillatory solution of equation (1.15.1), say, $u(k) > 0$ for $k \ge N$ for some N sufficiently large with $N \ge n_0 \ge 0$ such that condition (1.15.43) is satisfied for all $k_n \ge N$. Now, equation (1.15.1) is equivalent to

$$\begin{aligned} &c(k)u(k+1) + c(k-1)u(k-1) \\ &\quad = [c(k) + c(k-1) - q(k)]u(k) + f(k) \quad \text{for } k \ge N. \end{aligned} \tag{1.15.45}$$

Hence, for $k_n \geq N$,

$$\begin{aligned} 0 &< c(k_n)u(k_n+1) + c(k_n-1)u(k_n-1) \\ &= [c(k_n) + c(k_n-1) - q(k_n)]u(k_n) + f(k_n). \end{aligned} \tag{1.15.46}$$

Using condition (1.15.43), we get

$$0 < c(k_n)u(k_n+1) + c(k_n-1)u(k_n-1) \leq f(k_n) \quad \forall k_n \geq N, \tag{1.15.47}$$

which contradicts condition (1.15.44). This completes the proof. □

Example 1.15.17. The forced equation

$$\Delta^2 u(k-1) + 2u(k) = 2(-1)^k + 1 \quad \text{for } k \in \mathbb{N} \tag{1.15.48}$$

has an oscillatory solution $u(k) = (1/2) - (-1)^k$. All conditions of Theorem 1.15.16 are satisfied, and hence equation (1.15.48) is oscillatory.

It will be convenient to employ the following notation for $k \geq N \geq n_0 > 0$:

$$C(k) = \sum_{n=N}^{k} \frac{1}{c(n)}, \qquad F(k) = \frac{1}{C(k)} \sum_{n=N}^{k} \frac{1}{c(n)} \sum_{j=N}^{n} f(j). \tag{1.15.49}$$

Theorem 1.15.18. *If $q(k) \geq 0$ eventually and*

$$\liminf_{k\to\infty} F(k) = -\infty, \qquad \limsup_{k\to\infty} F(k) = \infty, \tag{1.15.50}$$

then equation (1.15.1) is oscillatory.

Proof. Let u be an eventually positive solution of equation (1.15.1). Choose $N \geq n_0 \geq 0$ to be large enough so that $u(k) > 0$ and $q(k) \geq 0$ for $k \geq N$. Summing equation (1.15.1) from N to k and dividing by $c(k)$, we get

$$\Delta u(k) - \frac{1}{c(k)}\left(c(N-1)\Delta u(N-1)\right) + \frac{1}{c(k)} \sum_{j=N}^{k} q(j)u(j) = \frac{1}{c(k)} \sum_{j=N}^{k} f(j). \tag{1.15.51}$$

Summing both sides of (1.15.51) from N to k, dividing by $C(k)$, and using the fact that $u(k) > 0$, we obtain

$$-\frac{u(N)}{C(N)} - c(N-1)\Delta u(N-1) \leq F(k), \tag{1.15.52}$$

which implies that $F(k)$ is bounded from below, which is a contradiction. This completes the proof. □

Example 1.15.19. The forced equation

$$\Delta^2 u(k-1) + k^2 u(k) = (-1)^k \left(4 - k^2\right) \quad \text{for } k \in \mathbb{N} \tag{1.15.53}$$

has an oscillatory solution $u(k) = (-1)^k$. In fact, equation (1.15.53) is oscillatory by Theorem 1.15.18.

1.16. Some qualitative properties of solutions

In this section we study asymptotic behavior, boundedness, and monotonicity properties of homogeneous second-order linear difference equations appearing as

$$\Delta\big(c(k)\Delta x(k)\big) = q(k)x(k+1), \tag{1.16.1}$$

where $\{c(k)\}$ and $\{q(k)\}$ are positive real sequences for $k \in \mathbb{N}$. Under certain conditions it is shown that every solution of equation (1.16.1) must eventually be monotonic. A necessary and sufficient condition for all solutions to be bounded is obtained. Necessary and sufficient conditions for the asymptotic behavior of certain types of solutions are given. Comparison theorems are also presented.

1.16.1. Some properties of dominant and recessive solutions

Recall Definition 1.5.9 and Theorem 1.5.10 which are concerned with recessive and dominant solutions of equation (1.16.1). We see that two linearly independent, eventually positive solutions u and v of equation (1.16.1) are called recessive and dominant, respectively, if

$$\frac{u(k)}{v(k)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{1.16.2}$$

and we also see the following result.

Theorem 1.16.1. *If equation (1.16.1) is nonoscillatory, then there exist a recessive solution u and a dominant solution v such that*

$$\sum^{\infty} \frac{1}{c(n)u(n)u(n+1)} = \infty, \qquad \sum^{\infty} \frac{1}{c(n)v(n)v(n+1)} < \infty. \tag{1.16.3}$$

Theorem 1.16.2. *Suppose that equation (1.16.1) is nonoscillatory.*

(I_1) *If v is an eventually positive solution of (1.16.1) such that*

$$\sum^{\infty} \frac{1}{c(n)v(n)v(n+1)} < \infty, \tag{1.16.4}$$

then v is dominant and u defined by

$$u(k) = v(k) \sum_{j=k}^{\infty} \frac{1}{c(j)v(j)v(j+1)} \tag{1.16.5}$$

is recessive.

(I_2) *If u is an eventually positive solution of (1.16.1) such that*

$$\sum^{\infty} \frac{1}{c(n)u(n)u(n+1)} = \infty, \tag{1.16.6}$$

then u is recessive and v defined by

$$v(k) = u(k) \sum_{j=n}^{k-1} \frac{1}{c(j)u(j)u(j+1)} \tag{1.16.7}$$

is a dominant solution, where n is chosen so large that $u(j) \neq 0$ for $j \geq n$.

Proof. Suppose we have v such that $\sum^{\infty} 1/(c(n)v(n)v(n+1)) < \infty$. Define u as in (1.16.5). Then

$$\frac{u(k)}{v(k)} = \sum_{j=k}^{\infty} \frac{1}{c(j)v(j)v(j+1)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{1.16.8}$$

so that u is recessive and v is dominant. A similar argument proves the remaining case. □

As before, equation (1.16.1) is equivalent to the equation

$$c(k+1)x(k+2) + c(k)x(k) = p(k)x(k+1), \tag{1.16.9}$$

where $q(k) = p(k) - c(k) - c(k+1)$. We will need the following theorem which is concerned with the behavior of recessive and dominant solutions.

Theorem 1.16.3. *If $p(k) - c(k) - c(k+1) \geq 0$ for $k \in \mathbb{N}$, then there exist a recessive solution u and a dominant solution v such that $u(k) > 0$, $u(k+1) \leq u(k)$ eventually, and $v(k) > 0$, $v(k+1) \geq v(k)$ eventually.*

Suppose there exists a nonnegative sequence $\{\varepsilon(k)\}$ such that

$$p(k) - \left(1 + \varepsilon(k)\right)c(k+1) - c(k) \geq 0, \qquad \sum^{\infty} \varepsilon(k) = \infty. \tag{1.16.10}$$

Then $v(k) \to \infty$ as $k \to \infty$.

If there exists a nonnegative sequence $\{\gamma(k)\}$ such that

$$p(k) - c(k+1) - \left(1 + \gamma(k)\right)c(k) \geq 0, \qquad \sum^{\infty} \gamma(k) = \infty, \tag{1.16.11}$$

then $u(k) \to \infty$ as $k \to \infty$.

In the following lemma, we will show that any nontrivial solution of equation (1.16.1) is eventually monotone.

Lemma 1.16.4. *Denote the set of nontrivial solutions of equation (1.16.1) by S. Then any $x \in S$ is eventually monotone and belongs to one of the following two classes:*

$$\begin{aligned} \mathcal{M}^{+} &= \left\{x \in S : \exists n_0 \in \mathbb{N} \text{ such that } x(k)\Delta x(k) > 0 \text{ for } k \geq n_0\right\}, \\ \mathcal{M}^{-} &= \left\{x \in S : x(k)\Delta x(k) < 0 \text{ for } k \in \mathbb{N}\right\}. \end{aligned} \tag{1.16.12}$$

Proof. Let $x \in S$ and consider $\{F(k)\}$ given by $F(k) = c(k)x(k)\Delta x(k)$ for $k \in \mathbb{N}$. Then

$$\begin{aligned} \Delta F(k) &= x(k+1)\Delta\left(c(k)\Delta x(k)\right) + \left(c(k)\Delta x(k)\right)\left(\Delta x(k)\right) \\ &= q(k)x^2(k+1) + c(k)\left(\Delta x(k)\right)^2 \geq 0. \end{aligned} \tag{1.16.13}$$

Thus, $\{F(k)\}$ is a nondecreasing sequence. Since $\{x(k)\}$ is not eventually constant, there are only two possibilities:

(i) there exists $n_0 \in \mathbb{N}$ such that $F(k) > 0$ for $k \geq n_0$,

(ii) $F(k) < 0$ for $k \in \mathbb{N}$.

We first consider case (i). Suppose $x(k)\Delta x(k) < 0$ for $k \in \mathbb{N}$. Then $\{x(k)\}$ is eventually monotone.

Now we discuss case (ii). Suppose $x(k)\Delta x(k) < 0$ for $k \in \mathbb{N}$. Without loss of generality assume $x(1) > 0$ and $\Delta x(1) < 0$. We will show that $\{x(k)\}$ is positive decreasing. If $x(2) < 0$, then from equation (1.16.1) we get

$$\Delta x(2) = \frac{c(1)}{c(2)}\Delta x(1) + \frac{q(1)}{c(2)}x(2) < 0, \tag{1.16.14}$$

which is a contradiction. This completes the proof. □

Remark 1.16.5. In general, the condition $x(k)\Delta x(k) < 0$, $k \in \mathbb{N}$, does not ensure that the sequence $\{x(k)\}$ is eventually of one sign, that is, that $\{x(k)\}$ is nonoscillatory. But such a fact is true when $\{x(k)\}$ is a solution of (1.16.1) as follows from the proof of Lemma 1.16.4.

Lemma 1.16.6. *If x is an eventually positive and increasing solution of equation (1.16.1), then x is a dominant solution.*

Proof. Suppose that x is eventually positive and increasing solution of equation (1.16.1). We must show that x is a dominant solution. By Theorem 1.16.3, we may assume there exist a recessive solution u which is nonincreasing, a dominant solution v which is nondecreasing, and constants a and b such that $x = au + bv$. If $b = 0$, then x is a multiple of u, which is a contradiction since u is nonincreasing. Therefore, assume $b \neq 0$. We must show that $u(k)/x(k) \to 0$ as $k \to \infty$. This would imply by (1.16.2) that x is dominant. Now,

$$\frac{x(k)}{v(k)} = a\frac{u(k)}{v(k)} + b, \quad \text{where } b \neq 0. \tag{1.16.15}$$

We may assume that $x(k)$ and $v(k)$ are eventually positive. Since $u(k)/v(k) \to 0$ as $k \to \infty$, we can conclude that b is positive. This means that

$$\frac{x(k)}{v(k)} \geq \frac{b}{2} \quad \text{if } k \in \mathbb{N} \text{ for some } N \in \mathbb{N}_0. \tag{1.16.16}$$

Thus,

$$\frac{1}{x(k)} \leq \frac{2}{bv(k)}, \tag{1.16.17}$$

which implies

$$0 \leq \frac{u(k)}{x(k)} \leq \frac{2}{b}\left(\frac{u(k)}{v(k)}\right). \tag{1.16.18}$$

Since $\lim_{k\to\infty} u(k)/v(k) = 0$, we have that $\lim_{k\to\infty} u(k)/x(k) = 0$, which means that x is a dominant solution. □

If equation (1.16.1) has unbounded solutions, then the converse of Lemma 1.16.6 is true. Thus, we can present the following result.

Theorem 1.16.7. *Suppose that equation (1.16.1) has unbounded solutions, and suppose that x is an eventually positive solution of equation (1.16.1). Then,*

(I) *x is a dominant solution of (1.16.1) if and only if x is eventually increasing,*
(II) *x is a recessive solution of (1.16.1) if and only if x is eventually decreasing.*

Proof. We will prove (I). Clearly Lemma 1.16.6 is the sufficiency part. For the necessity suppose that x is a dominant solution. We must show that x is eventually increasing. If not, then x is eventually nonincreasing.

Since x is dominant, there exists a recessive solution u which is linearly independent from x. Furthermore, we may choose N so large that $u(k) > 0$ and $x(k) > 0$ for $k \geq N$. By Theorem 1.16.3 and the essential uniqueness of the recessive solution, we may also assume $u(k+1) \leq u(k)$.

By assumption, equation (1.16.1) has an unbounded solution y. Since u and x are linearly independent, there exist constants a and b such that $y = au + bx$. This means that an unbounded solution can be written as a linear combination of two eventually positive nonincreasing solutions; a contradiction which completes the proof. □

In view of Theorem 1.16.7 it may be useful to determine when equation (1.16.1) has unbounded solutions, or equivalently, when all solutions are bounded. This will be given in the following result.

Theorem 1.16.8. *Every solution of equation (1.16.1) is bounded if and only if*

$$S_1 := \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{c(n)} \sum_{j=1}^{n} q(j) < \infty. \tag{1.16.19}$$

Proof. First, assume that all solutions are bounded. For $N \geq 0$, define a solution x by setting $x(N) = 1$ and $x(N+1) = 2$. Then $\Delta x(N) = 1 > 0$. From equation (1.16.1),

$$\begin{aligned} c(N+1)\Delta x(N+1) &= c(N)\Delta x(N) + q(N)x(N+1) \\ &\geq q(N)x(N+1) \geq 0, \end{aligned} \tag{1.16.20}$$

which implies $\Delta x(N+1) > 0$. Thus, we conclude that

$$x(n) \geq 1, \quad \Delta x(n) > 0 \quad \forall n \geq N. \tag{1.16.21}$$

Also, from equation (1.16.1), we have

$$\Delta\big(c(k)\Delta x(k)\big) = q(k)x(k+1) \geq q(k) \quad \text{for } k \geq N. \tag{1.16.22}$$

Summing both sides of (1.16.22) from N to $n-1$, we find

$$c(n)\Delta x(n) \geq c(N)\Delta x(N) + \sum_{j=N}^{n-1} q(j), \tag{1.16.23}$$

so

$$\Delta x(n) \geq \frac{c(N)}{c(n)}\Delta x(N) + \frac{1}{c(n)} \sum_{j=N}^{n-1} q(j) \geq \frac{1}{c(n)} \sum_{j=N}^{n-1} q(j). \tag{1.16.24}$$

Summing both sides of (1.16.24) from $N+1$ to ℓ, we obtain

$$x(\ell+1) \geq x(N+1) + \sum_{n=N+1}^{\ell} \frac{1}{c(n)} \sum_{j=N}^{n-1} q(j), \tag{1.16.25}$$

and so $S_1 < \infty$.

To see the converse, suppose that x is an unbounded solution. By Lemma 1.16.4 we may assume that there exists $N \in \mathbb{N}_0$ such that

$$x(n) > 0, \quad \Delta x(n) > 0 \quad \forall n \geq N. \tag{1.16.26}$$

From equation (1.16.1) and (1.16.26), we have

$$q(k) = \frac{\Delta\big(c(k)\Delta x(k)\big)}{x(k+1)} \geq \frac{c(k+1)\Delta x(k+1)}{x(k+1)} - \frac{c(k)\Delta x(k)}{x(k)}. \tag{1.16.27}$$

Summing both sides of (1.16.27) from N to $n-1$, we get

$$\sum_{j=N}^{n-1} q(j) + \frac{c(N)\Delta x(N)}{x(N)} \geq \frac{c(n)\Delta x(n)}{x(n)}. \tag{1.16.28}$$

Dividing by $c(n)$ and summing both sides of (1.16.28) from $N+1$ to m, we find

$$\sum_{n=N+1}^{m} \frac{1}{c(n)} \sum_{j=N}^{n-1} q(j) + \frac{c(N)\Delta x(N)}{x(N)} \sum_{n=N+1}^{m} \frac{1}{c(n)} \geq \sum_{n=N+1}^{m} \frac{\Delta x(n)}{x(n)}. \tag{1.16.29}$$

Let $f(t) = x(n) + (t-n)\Delta x(n)$ for $n \leq t \leq n+1$. Then $f'(t) = \Delta x(n)$ and $f(t) \geq x(n)$ for $n < t < n+1$. Hence

$$\begin{aligned}
\sum_{n=N+1}^{m} \frac{\Delta x(n)}{x(n)} &= \sum_{n=N+1}^{m} \int_{n}^{n+1} \frac{f'(t)}{x(n)} dt \geq \sum_{n=N+1}^{m} \int_{n}^{n+1} \frac{f'(t)}{f(t)} dt \\
&= \sum_{n=N+1}^{m} \big[\ln x(n+1) - \ln x(n)\big] \\
&= \ln x(m+1) - \ln x(N+1).
\end{aligned} \tag{1.16.30}$$

Since condition (1.16.19) implies

$$S_3 := \lim_{m\to\infty} \sum_{n=N+1}^{m} \frac{1}{c(n)} < \infty, \tag{1.16.31}$$

it follows from (1.16.19), (1.16.29), (1.16.30), and (1.16.31) that $\ln x(n)$ is bounded, which is a contradiction. This completes the proof. □

The following corollaries are immediate.

Corollary 1.16.9. *If $\sum_{n=1}^{\infty} 1/c(n) = \infty$, then equation (1.16.1) has unbounded solutions.*

Proof. For $N \geq 0$, define a solution x by setting $x(N) = 1$ and $x(N+1) = 2$. Arguing as in Theorem 1.16.8, we can derive inequality (1.16.24). Summing both sides of (1.16.24) from $N+1$ to ℓ, we have

$$x(\ell+1) \geq x(N+1) + c(N)\Delta x(N) \sum_{n=N+1}^{\ell} \frac{1}{c(n)} + \sum_{n=N+1}^{\ell} \frac{1}{c(n)} \sum_{j=N}^{n-1} q(j). \tag{1.16.32}$$

Using the fact that $\sum^{\infty} 1/c(n) = \infty$, we see that x must be unbounded. $\square$

Corollary 1.16.10. *If $\{\varepsilon(k)\}$ is a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty} \varepsilon(n) = \infty$ and $q(k) \geq \varepsilon(k)c(k)$ for $k \in \mathbb{N}$, then equation (1.16.1) has unbounded solutions.*

Proof. Clearly

$$\sum_{n=1}^{m} \frac{1}{c(n)} \sum_{j=1}^{n-1} q(j) \geq \sum_{n=1}^{m} \frac{q(n)}{c(n)} \geq \sum_{n=1}^{m} \varepsilon(n) \longrightarrow \infty \quad \text{as } m \longrightarrow \infty. \tag{1.16.33}$$

This means $S_1 = \infty$. By Theorem 1.16.8, equation (1.16.1) must have unbounded solutions. $\square$

Remark 1.16.11. Corollary 1.16.10 contains the first part of Theorem 1.16.3.

Corollary 1.16.12. *If*

$$\limsup_{n\to\infty} \frac{1}{c(n)} \sum_{j=1}^{n} q(j) = M > 0, \quad \textit{where } M \textit{ is a constant}, \tag{1.16.34}$$

then equation (1.16.1) has unbounded solutions.

Proof. The hypothesis of the corollary implies that $S_1 = \infty$. $\square$

Corollary 1.16.13. *Suppose $c_1(k) > 0$ and $q_1(k) > 0$ for $k \in \mathbb{N}$. Suppose further*

$$c_1(k) \geq c(k), \qquad \sum_{k=1}^{n} q_1(k) \leq \sum_{k=1}^{n} q(k) \quad \textit{for } n \in \mathbb{N}. \tag{1.16.35}$$

If all solutions of equation (1.16.1) are bounded, then so are all solutions of the equation

$$\Delta\big(c_1(k)\Delta x(k)\big) = q_1(k)x(k+1) \quad \textit{for } k \in \mathbb{N}. \tag{1.16.36}$$

1.16.2. Reciprocity principle

In this subsection, we present a simple property called *reciprocity principle* which links solutions of equation (1.16.1) with those of a certain difference equation of the same form called *reciprocal equation* or *dual equation*.

Let $x = \{x(k)\}$ be a solution of (1.16.1). One can easily see that $z = \{z(k)\}$ defined by $z(k) = c(k)\Delta x(k)$ for $k \in \mathbb{N}$ is a solution of the dual equation

$$\Delta\left(\frac{1}{q(k)}\Delta z(k)\right) = \frac{1}{c(k+1)}z(k+1), \tag{1.16.37}$$

which comes from equation (1.16.1), when $q(k)$ takes the place of $1/c(k)$ and vice versa. Clearly equation (1.16.1) is oscillatory if and only if equation (1.16.37) is oscillatory. Such a property has been used in order to obtain oscillation criteria for equation (1.16.1). However, equation (1.16.37) appears to be useful even for the investigation of qualitative behavior of the solutions in the nonoscillatory case.

Applying Theorem 1.16.8 to equation (1.16.37), we obtain the following result.

Theorem 1.16.14. *For every solution $x = \{x(k)\}$ of equation (1.16.1), the sequence $\{c(k)\Delta x(k)\}$ is bounded if and only if*

$$S_2 := \lim_{m\to\infty} \sum_{n=1}^{m} q(n) \sum_{j=1}^{n} \frac{1}{c(j+1)} < \infty. \tag{1.16.38}$$

1.16.3. Behavioral properties of dominant and recessive solutions

It is known that recessive solutions are unique up to a constant factor. For completeness we prove the following lemma.

Lemma 1.16.15. *Equation (1.16.1) cannot have two linearly independent recessive solutions.*

Proof. Suppose that equation (1.16.1) does have two linearly independent recessive solutions u and w. By definition, if u is recessive, then there exists a linearly independent dominant solution v of equation (1.16.1) such that (1.16.2) holds. Similarly, if w is recessive, then there exists a linearly independent dominant solution y such that

$$\lim_{k\to\infty} \frac{w(k)}{y(k)} = 0. \tag{1.16.39}$$

If the recessive solutions u and w are linearly independent, then there exist constants a and b such that

$$v(k) = au(k) + bw(k). \tag{1.16.40}$$

Since u and v by definition are linearly independent, $b \neq 0$. Dividing (1.16.40) by $v(k)$ implies

$$1 = a\frac{u(k)}{v(k)} + b\frac{w(k)}{v(k)}. \tag{1.16.41}$$

Using (1.16.2), we see from (1.16.41) that

$$\lim_{k\to\infty} \frac{w(k)}{v(k)} = \frac{1}{b}. \tag{1.16.42}$$

We may also write $v(k) = dw(k) + ey(k)$ for some constants d and e, not both zero. Dividing by $w(k)$ we have

$$\frac{v(k)}{w(k)} = d + e\frac{y(k)}{w(k)}. \tag{1.16.43}$$

Using (1.16.39), we have $\lim_{k\to\infty} y(k)/w(k) = \infty$. This together with (1.16.42) will yield a contradiction if we take the limit in (1.16.43) unless $e = 0$. Therefore, we have $v(k) = dw(k)$. Finally, we may write $y(k) = fu(k) + gv(k)$ for some constants f and g, not both zero. Dividing by $y(k)$ yields

$$\begin{aligned} 1 &= f\frac{u(k)}{y(k)} + g\frac{v(k)}{y(k)} = f\frac{u(k)}{v(k)}\frac{v(k)}{y(k)} + g\frac{v(k)}{y(k)} \\ &= f\frac{u(k)}{v(k)}\frac{dw(k)}{y(k)} + g\frac{dw(k)}{y(k)} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{1.16.44}$$

which is a contradiction. □

Next, we observe the following result.

Lemma 1.16.16. *The following statements are true.*

(i) *Suppose all solutions of equation (1.16.1) are bounded. Then the recessive solution must converge to zero.*
(ii) *A dominant solution of equation (1.16.1) cannot converge to zero.*

Proof. First we show (i). Let u and v be recessive and dominant solutions of (1.16.1), respectively. Since v is bounded, (1.16.2) implies that u converges to zero.

Now we address (ii). Let u and v be recessive and dominant solutions of (1.16.1), respectively. Suppose $\lim_{k\to\infty} v(k) = 0$. By (1.16.2), $\lim_{k\to\infty} u(k) = 0$. This means that every solution of (1.16.1) converges to zero, which contradicts the fact that (1.16.1) does have a positive nondecreasing solution by Theorem 1.16.3. □

We are now able to state our next result on zero convergence of the recessive solution.

Lemma 1.16.17. *Assume equation (1.16.1) has unbounded solutions.*

(I_1) *If (1.16.1) has a recessive solution which is zero convergent, then all dominant solutions v of (1.16.1) satisfy* $\lim_{k\to\infty} c(k)\Delta v(k) = \infty$.

(I_2) *If (1.16.1) has a dominant solution v satisfying* $\lim_{k\to\infty} c(k)\Delta v(k) = \infty$, *then (1.16.1) has a zero convergent recessive solution.*

Proof. As in the proof of Theorem 1.16.8, we can define a solution v such that $v(k) > 0$ and $\Delta v(k) > 0$, $k \geq N$, for some $N \in \mathbb{N}$. By Lemma 1.16.6, v is dominant and satisfies $\sum^{\infty} 1/(c(n)v(n)v(n+1)) < \infty$. Define a solution u of (1.16.1) as in (1.16.5), that is, by setting

$$u(k) = v(k)\sum_{j=k}^{\infty}\frac{1}{c(j)v(j)v(j+1)} \quad \text{for } k \geq N+1. \tag{1.16.45}$$

Theorem 1.16.2 implies that u is a recessive solution. We also note that

$$\lim_{k\to\infty} v(k) = \infty. \tag{1.16.46}$$

If not, then u and v are a pair of linearly independent bounded solutions. This means that all solutions of (1.16.1) are bounded, which is a contradiction. From (1.16.5),

$$\begin{aligned} u(k) &= v(k)\sum_{j=k}^{\infty}\frac{1}{c(j)v(j)v(j+1)} \\ &= \frac{v(k)}{c(k)\Delta v(k)}\sum_{j=k}^{\infty}\frac{c(k)\Delta v(k)}{c(j)\Delta v(j)}\frac{\Delta v(j)}{v(j)v(j+1)} \end{aligned} \tag{1.16.47}$$

for $k \geq N+1$. Since

$$\Delta\big(c(j)\Delta v(j)\big) = q(j)v(j+1) \geq 0 \quad \text{for } j \geq N, \tag{1.16.48}$$

we have $c(k)\Delta v(k) \leq c(j)\Delta v(j)$ for $k \leq j$. This means that

$$\begin{aligned} u(k) &\leq \frac{v(k)}{c(k)\Delta v(k)}\sum_{j=k}^{\infty} 1\cdot\left[\frac{1}{v(j)} - \frac{1}{v(j+1)}\right] \\ &= \frac{v(k)}{c(k)\Delta v(k)}\left[\frac{1}{v(k)} - \lim_{j\to\infty}\left(\frac{1}{v(j+1)}\right)\right] = \frac{1}{c(k)\Delta v(k)}. \end{aligned} \tag{1.16.49}$$

On the other hand,

$$\begin{aligned}
u(k) &= v(k)\sum_{j=k}^{\infty}\frac{1}{c(j)v(j)v(j+1)}\\
&= v(k)\sum_{j=k}^{\infty}\frac{1}{c(j)\Delta v(j)}\frac{\Delta v(j)}{v(j)v(j+1)}\\
&= v(k)\sum_{j=k}^{\infty}\frac{1}{c(j)\Delta v(j)}\left[\frac{1}{v(j)}-\frac{1}{v(j+1)}\right].
\end{aligned}\tag{1.16.50}$$

Since $c(j+1)\Delta v(j+1)\geq c(j)\Delta v(j)\geq 0$ and $v(j+1)\geq v(j)\geq 0$, we have

$$\big(c(j)\Delta v(j)\big)v(j+1)\leq\big(c(j+1)\Delta v(j+1)\big)v(j+2).\tag{1.16.51}$$

Clearly, (1.16.46) and (1.16.49) imply that $\lim_{j\to\infty}(c(j)\Delta v(j))v(j+1)=\infty$. Now for $k\geq N+1$,

$$\begin{aligned}
u(k) = v(k)&\sum_{j=k}^{\infty}\frac{1}{c(j)\Delta v(j)}\left[\frac{1}{v(j)}-\frac{1}{v(j+1)}\right]\\
&=\frac{1}{c(k)\Delta v(k)}-\sum_{j=k}^{\infty}\frac{v(k)}{v(j+1)}\left[\frac{1}{c(j)\Delta v(j)}-\frac{1}{c(j+1)\Delta v(j+1)}\right]\\
&\quad-\lim_{j\to\infty}\frac{v(k)}{\big(c(j)\Delta v(j)\big)v(j+1)}\\
&\geq\frac{1}{c(k)\Delta v(k)}-\sum_{j=k}^{\infty}1\cdot\left[\frac{1}{c(j)\Delta v(j)}-\frac{1}{c(j+1)\Delta v(j+1)}\right]-0\\
&=\lim_{j\to\infty}\left(\frac{1}{c(j)\Delta v(j)}\right).
\end{aligned}\tag{1.16.52}$$

From this and (1.16.49) it follows that

$$\lim_{j\to\infty}\frac{1}{c(j)\Delta v(j)}\leq u(k)\leq\frac{1}{c(k)\Delta v(k)}\quad\text{for }k\geq N+1.\tag{1.16.53}$$

For the sufficiency portion of the lemma, since v is a dominant solution, $\lim_{k\to\infty}c(k)\Delta v(k)=\infty$. This implies by (1.16.53) that $\lim_{k\to\infty}u(k)=0$. However, by Lemma 1.16.15, recessive solutions are essentially unique, so that any recessive solution must be zero convergent.

Conversely, suppose $\lim_{k\to\infty} u(k) = 0$. Then inequality (1.16.53) implies that $\lim_{j\to\infty} c(j)\Delta v(j) = \infty$. We must show that $\lim_{j\to\infty} c(j)\Delta w(j) = \infty$ for any nonnegative dominant solution w. Any such w would have the form

$$w(j) = au(j) + bv(j), \quad \text{where } a, b \text{ are constants with } b > 0. \tag{1.16.54}$$

Using (1.16.46) and the fact that $u(j)$ is bounded, there exists $m \in \mathbb{N}$ such that $|a|u(j) \le (b/2)v(j)$ for $j \ge m$, and hence $w(j) \ge (b/2)v(j)$ for $j \ge m$. From equation (1.16.1),

$$c(k)\Delta v(k) = c(m)\Delta v(m) + \sum_{j=m+1}^{\infty} q(j)v(j+1). \tag{1.16.55}$$

Since $\lim_{k\to\infty} c(k)\Delta v(k) = \infty$, it follows that

$$\lim_{k\to\infty} \sum_{j=m+1}^{k} q(j)v(j+1) = \infty. \tag{1.16.56}$$

Similarly,

$$\begin{aligned} c(k)\Delta w(k) &= c(m)\Delta w(m) + \sum_{j=m+1}^{k} q(j)w(j+1) \\ &\ge c(m)\Delta w(m) + \sum_{j=m+1}^{k} q(j)\left(\frac{b}{2}\right)v(j+1) \\ &\ge c(m)\Delta w(m) + \frac{b}{2}\sum_{j=m+1}^{k} q(j)v(j+1). \end{aligned} \tag{1.16.57}$$

Equation (1.16.56) implies that $\lim_{k\to\infty} c(k)\Delta w(k) = \infty$, which completes the proof of the lemma. □

By means of Theorem 1.16.14 and Lemma 1.16.17, we may easily derive the following necessary and sufficient condition for recessive solutions of (1.16.1) to be zero convergent.

Theorem 1.16.18. *Assume equation (1.16.1) has unbounded solutions. Then every recessive solution of (1.16.1) is zero convergent if and only if*

$$S_2 = \infty, \tag{1.16.58}$$

where S_2 is defined in (1.16.38).

Proof. For the sufficiency part, assume that $S_2 = \infty$. Let x be a solution of (1.16.1) as in the proof of Theorem 1.16.8, that is, $x(k) > 0$ and $\Delta x(k) > 0$ for $k \geq N$ for some $N \in \mathbb{N}$. Then x is an eventually positive and increasing solution, and the sequence $\{c(k)\Delta x(k)\}$ is positive and increasing. Also, Theorem 1.16.3 implies that x is dominant. If $\{c(k)\Delta x(k)\}$ is bounded, then we can easily see that $c(k+1)\Delta x(k+1) \geq c(k)\Delta x(k)$. This means that

$$\begin{aligned} \Delta x(n+1) &\geq \frac{c(N)\Delta x(N)}{c(n+1)} \quad \text{for } n \geq N, \\ x(m+1) &\geq x(N+1) + \sum_{n=N}^{m-1} \frac{c(N)\Delta x(N)}{c(n+1)} \quad \text{for } m \geq N. \end{aligned} \tag{1.16.59}$$

Thus,

$$\begin{aligned} \Delta\big(c(m)\Delta x(m)\big) &= q(m)x(m+1) \\ &\geq q(m)x(N+1) + q(m)\sum_{n=N}^{m-1} \frac{c(N)\Delta x(N)}{c(n+1)}, \end{aligned} \tag{1.16.60}$$

and summing again implies that

$$\begin{aligned} c(k+1)\Delta x(k+1) \geq c(N)\Delta x(N) &+ x(N+1)\sum_{m=N+1}^{k} q(m) \\ &+ c(N)\Delta x(N)\sum_{m=N+1}^{k}\sum_{n=N}^{m-1} \frac{q(m)}{c(n+1)}. \end{aligned} \tag{1.16.61}$$

Since $\{c(k)\Delta x(k)\}$ is bounded, we see that $S_2 < \infty$, which is a contradiction. Therefore, the sequence $\{c(k)\Delta x(k)\}$ must be unbounded such that x is a dominant solution. An application of Lemma 1.16.17 implies that recessive solutions must be zero convergent.

Conversely, if recessive solutions are zero convergent, Lemma 1.16.17 implies that there exists a dominant solution x such that $\{c(k)\Delta x(k)\}$ is unbounded. Now, if $S_2 < \infty$, Theorem 1.16.14 would imply that $\{c(k)\Delta x(k)\}$ must be bounded, which it is not. Therefore S_2 must be infinite. This completes the proof. □

Corollary 1.16.19. *If $\{\varepsilon(k)\}$ is a sequence of nonnegative numbers such that $\sum_{n=1}^{\infty}\varepsilon(n) = \infty$ and $q(k) \geq \varepsilon(k)c(k-1)$ for $k \in \mathbb{N}$, then equation (1.16.1) has zero convergent recessive solutions.*

Proof. If all solutions of (1.16.1) are bounded, then the recessive solution must necessarily be zero convergent by Lemma 1.16.16(i). If equation (1.16.1) has unbounded solutions, then the result follows from Theorem 1.16.18. □

Corollary 1.16.20. *Every recessive solution of*

$$\Delta^2 x(k) = q(k)x(k+1) \quad \textit{for } k \in \mathbb{N} \tag{1.16.62}$$

is zero convergent if and only if

$$\sum_{n=1}^{\infty} nq(n) = \infty. \tag{1.16.63}$$

Proof. Corollary 1.16.9 implies that equation (1.16.62) has unbounded solutions. An application of Theorem 1.16.18 yields the result. □

Theorem 1.16.7, Lemma 1.16.17, and Theorem 1.16.18 all contain the assumption that equation (1.16.1) has unbounded solutions. Here, we will construct an example which indicates that none of these results is necessarily valid if this assumption is removed from the respective hypotheses.

Example 1.16.21. Consider equation (1.16.1) with $c(k) = k^2$, $v(1) = 1$ and define

$$\Delta v(k) = \frac{2 - (1/k)}{k^2} \quad \text{for } k \in \mathbb{N}. \tag{1.16.64}$$

Clearly $v(k) \geq 1$ and $\Delta v(k) > 0$ for $k \in \mathbb{N}$. From equation (1.16.1), we find for $k \in \mathbb{N}$,

$$\begin{aligned} q(k) &= \frac{\Delta\big(c(k)\Delta v(k)\big)}{v(k+1)} = \frac{c(k+1)\Delta v(k+1) - c(k)\Delta v(k)}{v(k+1)} \\ &= \frac{1}{v(k+1)}\left[2 - \frac{1}{k+1} - 2 + \frac{1}{k}\right] = \frac{1}{v(k+1)}\frac{1}{k(k+1)} \\ &\leq \frac{1}{k(k+1)}. \end{aligned} \tag{1.16.65}$$

With these $c(k)$ and $q(k)$, Theorem 1.16.8 implies that all solutions of (1.16.1) are bounded. Thus, we have

$$\lim_{k\to\infty} v(k) = V < \infty. \tag{1.16.66}$$

From (1.16.64), we find

$$c(k)\Delta v(k) = 2 - \frac{1}{k} \longrightarrow 2 \quad \text{as } k \longrightarrow \infty. \tag{1.16.67}$$

Since v is positive and increasing, Lemma 1.16.6 implies that v is dominant. This together with (1.16.67) shows that Lemma 1.16.17 is not necessarily true if (1.16.1) has bounded solutions.

Given v, define a solution u as in (1.16.5), that is,

$$u(k) = v(k) \sum_{j=k}^{\infty} \frac{1}{c(j)v(j)v(j+1)} \quad \text{for } k \in \mathbb{N}. \tag{1.16.68}$$

By Theorem 1.16.2, u is well defined and recessive. Let $K = u(1)/v(1)$. That is,

$$K = \sum_{j=1}^{\infty} \frac{1}{c(j)v(j)v(j+1)}. \tag{1.16.69}$$

By (1.16.66) and (1.16.69), we may choose $B > 0$ such that

$$0 < B - K < \frac{1}{4V}. \tag{1.16.70}$$

Next, define a solution w of (1.16.1) as

$$w(k) = v(k)\left[B - \sum_{j=1}^{k-1} \frac{1}{c(j)v(j)v(j+1)}\right]. \tag{1.16.71}$$

Then

$$\begin{aligned} w(k) &= v(k)\left[B - K + \sum_{j=k}^{\infty} \frac{1}{c(j)v(j)v(j+1)}\right] \\ &= (B-K)v(k) + u(k). \end{aligned} \tag{1.16.72}$$

Thus, w is a positive linear combination of u and v. Moreover,

$$\frac{w(k)}{u(k)} = \left[(B-K)\frac{v(k)}{u(k)} + 1\right] \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \tag{1.16.73}$$

since u and v are recessive and dominant, respectively. Thus, u and w satisfy (1.16.2) so that w is also a dominant solution. We will show that w is eventually decreasing. From (1.16.71),

$$\begin{aligned} w(k+1) - w(k) &= B\big(v(k+1) - v(k)\big) - v(k+1)\sum_{j=1}^{k} \frac{1}{c(j)v(j)v(j+1)} \\ &\quad + v(k)\sum_{j=1}^{k-1} \frac{1}{c(j)v(j)v(j+1)} \\ &= \big(v(k+1) - v(k)\big)\left[B - \sum_{j=1}^{k-1} \frac{1}{c(j)v(j)v(j+1)}\right] - \frac{1}{c(k)v(k)}. \end{aligned} \tag{1.16.74}$$

This implies by (1.16.68) and (1.16.69) that

$$\begin{aligned} c(k)\Delta w(k) &= c(k)\Delta v(k)\left[B - K + \sum_{j=k}^{\infty} \frac{1}{c(j)v(j)v(j+1)}\right] - \frac{1}{v(k)} \\ &= c(k)\Delta v(k)[B - K] + \frac{c(k)\Delta v(k)}{v(k)}u(k) - \frac{1}{v(k)}. \end{aligned} \tag{1.16.75}$$

Taking the limit of both sides of (1.16.75) as $k \to \infty$ and using (1.16.67) and (1.16.70), we obtain

$$\begin{aligned} \lim_{k\to\infty} c(k)\Delta w(k) &= 2[B - K] + 0 - \frac{1}{V} \\ &< \frac{2}{4V} - \frac{1}{V} = -\frac{1}{2V} < 0. \end{aligned} \tag{1.16.76}$$

From this, we see that w is eventually decreasing. Since w is also dominant, this shows that the hypothesis of having unbounded solutions cannot be removed from Theorem 1.16.7.

It may be interesting to point out that u and w together form a pair of linearly independent positive, eventually decreasing solutions. Obviously this could not happen if (1.16.1) has unbounded solutions. Note that by Lemma 1.16.16(ii), w cannot be zero convergent.

Finally, $c(k) = k^2$ and $q(k) \le 1/(k(k+1))$ implies $S_2 < \infty$. However, since all solutions are bounded, the recessive solution is zero convergent by Lemma 1.16.16(i). Therefore, Theorem 1.16.18 is not necessarily true if all solutions are bounded.

1.16.4. Further classification of solutions

In view of Lemma 1.16.4, the class $\mathcal{M}^-$ can be divided into the following two subclasses:

$$\begin{aligned} \mathcal{M}_B^- &= \left\{x \in S : x(k)\Delta x(k) < 0 \text{ for } k \in \mathbb{N},\ \lim_{k\to\infty} x(k) = \ell_x \neq 0\right\}, \\ \mathcal{M}_0^- &= \left\{x \in S : x(k)\Delta x(k) < 0 \text{ for } k \in \mathbb{N},\ \lim_{k\to\infty} x(k) = 0\right\}. \end{aligned} \tag{1.16.77}$$

Clearly, $\mathcal{M}^- = \mathcal{M}_B^- \cup \mathcal{M}_0^-$.

Now, for the class $\mathcal{M}^+$, denote with $\mathcal{M}_B^+$ and $\mathcal{M}_\infty^+$ the subsets of $\mathcal{M}^+$ consisting of bounded and unbounded solutions of equation (1.16.1), respectively. A solution $x \in \mathcal{M}_\infty^+$ is said to be strongly increasing if $\lim_{k\to\infty} |c(k)\Delta x(k)| = \infty$ and regularly increasing otherwise. The subset of strongly increasing solutions will be denoted by $\mathcal{M}_\infty^+(S)$ and that for regularly increasing solutions will be denoted by $\mathcal{M}_\infty^+(R)$.

Thus

$$\begin{aligned}\mathcal{M}_B^+ &= \Big\{x \in \mathcal{M}^+ : \lim_{k\to\infty} |x(k)| < \infty\Big\},\\ \mathcal{M}_\infty^+(R) &= \Big\{x \in \mathcal{M}^+ : \lim_{k\to\infty} |x(k)| = \infty,\ \lim_{k\to\infty} |c(k)\Delta x(k)| < \infty\Big\},\\ \mathcal{M}_\infty^+(S) &= \Big\{x \in \mathcal{M}^+ : \lim_{k\to\infty} |x(k)| = \infty,\ \lim_{k\to\infty} |c(k)\Delta x(k)| = \infty\Big\},\\ \mathcal{M}^+ &= \mathcal{M}_B^+ \cup \mathcal{M}_\infty^+ = \mathcal{M}_B^+ \cup \mathcal{M}_\infty^+(R) \cup \mathcal{M}_\infty^+(S).\end{aligned} \tag{1.16.78}$$

As in the previous subsections, we will show that a crucial rôle in our consideration is played by the series S_1, S_2, and S_3, which are defined in (1.16.19), (1.16.38), and (1.16.31), respectively. For S_2 we may equivalently use

$$\lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{c(n)} \sum_{j=n}^{m} q(j), \tag{1.16.79}$$

and we also introduce S_4 by

$$S_4 := \lim_{m\to\infty} \sum_{j=1}^{m} q(j). \tag{1.16.80}$$

Next, we give some relationships between the convergence or divergence of the series S_i for $i \in \{1, 2, 3, 4\}$.

Lemma 1.16.22. *The following hold.*

(a) *If $S_1 < \infty$, then $S_3 < \infty$.*
(b) *If $S_2 < \infty$, then $S_4 < \infty$.*
(c) *If $S_2 = \infty$, then $S_3 = \infty$ or $S_4 = \infty$.*
(d) *If $S_1 = \infty$, then $S_3 = \infty$ or $S_4 = \infty$.*
(e) *$S_1 < \infty$ and $S_2 < \infty$ if and only if $S_3 < \infty$ and $S_4 < \infty$.*

Proof. For claim (a), let $n_1 \in (0, m)$. Since

$$\sum_{n=1}^{m} \frac{1}{c(n)} \sum_{j=1}^{n} q(j) > \sum_{n=1}^{n_1} \frac{1}{c(n)} \sum_{j=1}^{n} q(j) + \left[\sum_{n=1}^{n_1} q(n)\right]\left[\sum_{n=n_1}^{m} \frac{1}{c(n)}\right], \tag{1.16.81}$$

the assertion follows. Claim (b) follows in a similar way. Claims (c) and (d) follow from the inequalities

$$\begin{aligned}\sum_{n=1}^{m} \frac{1}{c(n)} \sum_{j=n}^{m} q(j) &\le \left[\sum_{j=1}^{m} q(j)\right]\left[\sum_{n=1}^{m} \frac{1}{c(n)}\right],\\ \sum_{n=1}^{m} \frac{1}{c(n)} \sum_{j=1}^{n} q(j) &\le \left[\sum_{j=1}^{m} q(j)\right]\left[\sum_{n=1}^{m} \frac{1}{c(n)}\right].\end{aligned} \tag{1.16.82}$$

Finally, claim (e) immediately follows from (a)–(d). ☐

1.16.4.1. Behavior of class $\mathcal{M}^-$ solutions

We now investigate the convergence of nonoscillatory solutions of equation (1.16.1) that belong to the class $\mathcal{M}^-$.

Theorem 1.16.23. *With respect to equation (1.16.1), we have the following.*

(i) $S_2 = \infty$ *if and only if* $\mathcal{M}^- = \mathcal{M}_0^-$, *that is,* $\mathcal{M}_B^- = \emptyset$.
(ii) $S_1 = \infty$ *and* $S_2 < \infty$ *if and only if* $\mathcal{M}^- = \mathcal{M}_B^-$, *that is,* $\mathcal{M}_0^- = \emptyset$.
(iii) $S_1 < \infty$ *and* $S_2 < \infty$ *if and only if* $\mathcal{M}_0^- \neq \emptyset$ *and* $\mathcal{M}_B^- \neq \emptyset$.

Proof. Part (i) was proved in Theorem 1.16.18. To prove part (ii), assume $S_1 = \infty$, $S_2 < \infty$ and suppose that (1.16.1) has a solution $x = \{x(k)\} \in \mathcal{M}_0^-$. By part (i) there exists at least one solution $y = \{y(k)\} \in \mathcal{M}_B^-$. Since x and y are two linearly independent solutions of equation (1.16.1) and are bounded for $k \in \mathbb{N}$, this contradicts Theorem 1.16.8.

Now assume that every solution x of equation (1.16.1) belongs to $\mathcal{M}_B^-$. The assertion follows from the fact that if $S_1 < \infty$, then there always exists a solution y of equation (1.16.1) in $\mathcal{M}_0^-$.

The proof of (iii) follows from (i) and (ii). $\square$

The following examples illustrate the methods presented above.

Example 1.16.24. The difference equation

$$\Delta\left(\frac{3}{2^k}\Delta x(k)\right) = \frac{1}{2^k}x(k+1) \quad \text{for } k \in \mathbb{N} \tag{1.16.83}$$

satisfies the hypotheses of Theorem 1.16.23(i) and has the subclass $\mathcal{M}_0^-$ solution $x(k) = (2/3)^{k+1} \to 0$ as $k \to \infty$.

Example 1.16.25. The difference equation

$$\Delta\big((k+2)\Delta x(k)\big) = \frac{1}{(k+1)(k+3)}x(k+1) \quad \text{for } k \in \mathbb{N} \tag{1.16.84}$$

satisfies the hypotheses of Theorem 1.16.23(ii) and has the subclass $\mathcal{M}_B^-$ solution $x(k) = (k+2)/(k+1) \to 1$ as $k \to \infty$.

Applying Theorem 1.16.23 to equation (1.16.37), we obtain the following result.

Theorem 1.16.26. *Let x be a class $\mathcal{M}^-$ solution of equation (1.16.1). Then*

(I) $\lim_{k\to\infty} c(k)\Delta x(k) = 0$ *if and only if* $S_1 = \infty$,
(II) $\lim_{k\to\infty} c(k)\Delta x(k) \neq 0$ *if and only if* $S_1 < \infty$ *and* $S_2 = \infty$.

Example 1.16.27. Consider the difference equation

$$\Delta\left(2^k \Delta x(k)\right) = \left(\frac{2^{k+1}}{3}\right) x(k+1) \quad \text{for } k \in \mathbb{N}. \tag{1.16.85}$$

This equation has the solution $x(k) = 3^{-k}$ which satisfies the assumptions of Theorem 1.16.26(I).

Example 1.16.28. The difference equation

$$\Delta\left((k+1)(k+3)\Delta x(k)\right) = \frac{1}{k+3} x(k+1) \quad \text{for } k \in \mathbb{N} \tag{1.16.86}$$

satisfies the hypotheses of Theorem 1.16.26(II) and has the class $\mathcal{M}^-$ solution $x(k) = 1/(k+1)$. This solution satisfies

$$c(k)\Delta x(k) = -\frac{k+3}{k+2} \longrightarrow -1 \neq 0 \quad \text{as } k \longrightarrow \infty. \tag{1.16.87}$$

From Theorems 1.16.23 and 1.16.26, we can relate the asymptotic behavior of a class $\mathcal{M}^-$ solution x of equation (1.16.1) to the behavior of the sequence $\{c(k)\Delta x(k)\}$.

Theorem 1.16.29. *Let x be a class $\mathcal{M}^-$ solution of equation (1.16.1). Then*

(i) $\lim_{k\to\infty} x(k) = 0 = \lim_{k\to\infty} c(k)\Delta x(k)$ *if and only if* $S_1 = S_2 = \infty$,
(ii) $0 = \lim_{k\to\infty} x(k) \neq \lim_{k\to\infty} c(k)\Delta x(k)$ *if and only if* $S_1 < \infty$ *and* $S_2 = \infty$,
(iii) $\lim_{k\to\infty} x(k) \neq \lim_{k\to\infty} c(k)\Delta x(k) = 0$ *if and only if* $S_1 = \infty$ *and* $S_2 < \infty$.

Next, we give asymptotic estimates for the solutions of equation (1.16.1) in $\mathcal{M}^-$.

Corollary 1.16.30. *Let x be a class $\mathcal{M}^-$ solution of equation (1.16.1).*

(I) *If $S_1 < \infty$ and $S_2 = \infty$, then $x(k)$ is asymptotically equivalent to the sequence $C(k) = \sum_{j=k}^{\infty} 1/c(j)$, that is, $\lim_{k\to\infty} x(k)/C(k) \neq 0$ exists and is finite.*
(II) *If $S_1 = \infty$ and $S_2 < \infty$, then $x(k) - \ell_x$ is asymptotically equivalent to $\sum_{k=n}^{\infty} q(k) \sum_{j=n}^{k} 1/c(j)$, where $\ell_x = \lim_{k\to\infty} x(k) \neq 0$.*
(III) *If $S_1 < \infty$, $S_2 < \infty$, and $\lim_{k\to\infty} x(k) = 0$, then $x(k)$ is asymptotically equivalent to $C(k)$.*

Proof. The proof follows from Theorem 1.16.29 and an application of L'Hôpital's rule. Here, we omit the details. □

1.16.4.2. Behavior of class $\mathcal{M}^+$ solutions

The results presented in the previous sections which are concerned with class $\mathcal{M}^+$ solutions of equation (1.16.1) can be summarized as follows.

Theorem 1.16.31. *For equation (1.16.1) the following hold.*

(I) $S_1 < \infty$ *if and only if* $\mathcal{M}^+ = \mathcal{M}_B^+$, *that is, every solution of equation (1.16.1) in* $\mathcal{M}^+$ *is bounded.*
(II) $S_1 = \infty$ *and* $S_2 < \infty$ *if and only if* $\mathcal{M}^+ = \mathcal{M}_\infty^+(R)$, *that is, every solution of equation (1.16.1) in* $\mathcal{M}^+$ *is regularly increasing.*
(III) $S_1 = \infty = S_2$ *if and only if* $\mathcal{M}^+ = \mathcal{M}_\infty^+(S)$, *that is, every solution of equation (1.16.1) in* $\mathcal{M}^+$ *is strongly increasing.*

The following examples illustrate Theorem 1.16.31.

Example 1.16.32. The difference equation

$$\Delta\left(k^2(k+1)\Delta x(k)\right) = \left(1+\frac{1}{k}\right)x(k+1) \quad \text{for } k \in \mathbb{N} \tag{1.16.88}$$

satisfies the hypotheses of Theorem 1.16.31(I) and has the subclass $\mathcal{M}_B^+$ solution $x(k) = 1-(1/k) \to 1$ as $k \to \infty$. We note that $c(k)\Delta x(k) = k \to \infty$ as $k \to \infty$.

Example 1.16.33. Consider the difference equation

$$\Delta\left(\left(\frac{k-1}{k}\right)\frac{1}{\sqrt{k+1}-\sqrt{k}}\Delta x(k)\right) = \frac{1}{k\sqrt{k+1}}x(k+1) \quad \text{for } k \in \mathbb{N}. \tag{1.16.89}$$

This equation has a regularly increasing solution $x(k) = \sqrt{k}$ which satisfies the conditions of Theorem 1.16.31(II). Note that

$$\lim_{k\to\infty} x(k) = \infty, \qquad \lim_{k\to\infty} c(k)\Delta x(k) = \lim_{k\to\infty}\frac{k-1}{k} = 1 < \infty. \tag{1.16.90}$$

Example 1.16.34. The difference equation

$$\Delta\left(k\Delta x(k)\right) = \frac{1}{k+1}x(k+1) \quad \text{for } k \in \mathbb{N} \tag{1.16.91}$$

satisfies the assumptions of Theorem 1.16.31(III) and has the solution $x(k) = k$ belonging to class $\mathcal{M}_\infty^+(S)$ which satisfies $c(k)\Delta x(k) = k \to \infty$ as $k \to \infty$.

1.16.5. Comparison theorems

In this section we will compare the behavior of solutions of the inequality

$$\Delta\big(c(k)\Delta y(k)\big) \le q(k)y(k+1), \tag{1.16.92}$$

with those of equation (1.16.1). We also compare the behavior of solutions of equation (1.16.1) with those of the equation

$$\Delta\big(c(k)\Delta y(k)\big) = q_1(k)y(k+1), \tag{1.16.93}$$

where $\{q_1(k)\}$ is a sequence of positive real numbers.

First, we need the following lemma.

Lemma 1.16.35. *Suppose $r(k) \ge 0$ and $s(k) > 0$ for $k \in \mathbb{N}_0$, and suppose that $t(k) \ge 0$ for $k \in \mathbb{N}$. Then the difference equation*

$$-r(k)\Delta x(k) + s(k+1)\Delta x(k+1) = t(k+1)x(k+1) \tag{1.16.94}$$

has a nontrivial solution which is nonnegative and nonincreasing.

Theorem 1.16.36. *Suppose that inequality (1.16.92) has a positive solution y. Then equation (1.16.1) has a nontrivial solution x which satisfies*

$$0 \le x(k) \le \beta y(k) \quad \text{for } k \in \mathbb{N}_0 \tag{1.16.95}$$

for some constant β. Moreover, if $\Delta y(k) \le 0$, then $\Delta x(k) \le 0$ for $k \in \mathbb{N}_0$.

Proof. We seek a solution of (1.16.1) as $x(k) = y(k)z(k)$, where $z(k)$ is to be determined. Substitution of $x(k+1)$, $\Delta x(k) = y(k)\Delta z(k) + z(k+1)\Delta y(k)$, and $\Delta x(k+1) = z(k+1)\Delta y(k+1) + y(k+2)\Delta z(k+1)$ into equation (1.16.1) leads to

$$\begin{aligned} &- c(k)y(k)\Delta z(k) + c(k+1)y(k+2)\Delta z(k+1) \\ &\quad + \big[\Delta\big(c(k)\Delta y(k)\big) - q(k)y(k+1)\big]z(k+1) = 0. \end{aligned} \tag{1.16.96}$$

Applying Lemma 1.16.35, we see that equation (1.16.96) has a nontrivial solution z which is nonnegative and nonincreasing. Hence (1.16.94) is satisfied, where β is a bound for the sequence $\{z(k)\}$. Furthermore, if $\Delta y(k) \le 0$, then we have $\Delta x(k) = z(k)\Delta y(k) + y(k+1)\Delta z(k) \le 0$, as required. □

Corollary 1.16.37. *Suppose $q_1(k) \le q(k)$ for $k \in \mathbb{N}_0$. If equation (1.16.93) has a positive solution y, then equation (1.16.1) has a nontrivial solution x satisfying (1.16.95). In addition, if $\Delta y(k) \le 0$, then $\Delta x(k) \le 0$ for $k \in \mathbb{N}_0$.*

Corollary 1.16.38. *Suppose $q_1(k) \le q(k)$ for $k \in \mathbb{N}_0$. If equation (1.16.93) has a recessive solution y which is zero convergent, then equation (1.16.1) has a recessive solution which is zero convergent.*

Example 1.16.39. The difference equation

$$\Delta^2 y(k) = \frac{1}{2} y(k+1) \tag{1.16.97}$$

has a dominant solution 2^k and a zero convergent recessive solution 2^{-k}. Hence for any sequence $q(k) \geq 1/2$, the equation

$$\Delta^2 x(k) = q(k)x(k+1) \tag{1.16.98}$$

will have a zero convergent recessive solution as well.

Next, we present the following comparison result.

Theorem 1.16.40. *Suppose $q_1(k) \geq q(k)$ for $k \in \mathbb{N}_0$. Let x be a solution of equation (1.16.1) such that $|x(1)| > \gamma$ and $x(1)\Delta x(1) > 0$ for some constant $\gamma > 0$. Then for any solution y of (1.16.93) with $|y(1)| \geq |x(1)|$, $x(1)y(1) > 0$, $|\Delta y(1)| \geq |\Delta x(1)|$,*

$$|y(k)| \geq |x(k)|, \quad |\Delta y(k)| \geq |\Delta x(k)| \quad \textit{for } k \in \mathbb{N}. \tag{1.16.99}$$

Proof. Without loss of generality, we consider a solution x starting with a positive value. In view of Lemma 1.16.4, the sequences $\{x(k)\}$ and $\{y(k)\}$ are increasing, and so $x(k) > \beta$ and $y(k) > \beta$. Define

$$\gamma(k) = y(k) - x(k) \quad \text{for } k \in \mathbb{N}. \tag{1.16.100}$$

Clearly, $\gamma(1) \geq 0$, $\Delta\gamma(1) = \Delta y(1) - \Delta x(1) \geq 0$, that is, $\gamma(2) \geq \gamma(1)$. We claim that the sequence $\{\gamma(k)\}$ is nondecreasing. Assume there exists $n_0 \geq 2$ such that

$$0 \leq \gamma(i) \leq \gamma(i+1) \quad \text{for } 1 \leq i \leq n_0 - 1, \qquad \gamma(n_0) > \gamma(n_0+1). \tag{1.16.101}$$

Let $\{g(k)\}$ be the sequence

$$g(k) = c(k)[\Delta y(k) - \Delta x(k)]. \tag{1.16.102}$$

Then

$$\Delta g(k) = q_1(k)y(k+1) - q(k)x(k+1) \geq q(k)[y(k+1) - x(k+1)]. \tag{1.16.103}$$

Taking into account that $\gamma(n_0) \geq 0$, we get

$$\Delta g(k) \geq 0. \tag{1.16.104}$$

From (1.16.101) it follows that

$$\begin{gathered} \Delta y(n_0) - \Delta x(n_0) = \gamma(n_0+1) - \gamma(n_0) < 0, \\ \Delta y(n_0-1) - \Delta x(n_0-1) = \gamma(n_0) - \gamma(n_0-1) \geq 0. \end{gathered} \tag{1.16.105}$$

Hence $\Delta g(n_0 - 1) = g(n_0) - g(n_0 - 1) < 0$, which contradicts (1.16.104). Thus, the sequence $\{y(k)\}$ is nondecreasing and so $y(k) \geq 0$. Since $\Delta y(k) - \Delta x(k) = y(k+1) - y(k) \geq 0$, we find $\Delta y(k) \geq \Delta x(k)$. This completes the proof. □

Example 1.16.41. As in Example 1.16.39 we see that the equation (1.16.97) has a dominant solution $\{2^k\}$, and by Theorem 1.16.40 we observe that equation (1.16.98) will have a dominant solution provided that $q \geq 1/2$.

We note that more discussions and extensions of the results of this section will be considered in the upcoming chapters.

1.17. More on nonoscillation criteria

In this section we will consider the difference equation

$$\Delta^2 x(k) + q(k)x(k+1) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{1.17.1}$$

where $\{q(k)\}$ is a sequence of real numbers, and obtain a Riccati equation of the second level which allows us to obtain a nonoscillation criterion which involves no individual values of the sequence q. Let

$$\sum_{j=0}^{\infty} q(j) \quad \text{exist as a finite value }, \tag{1.17.2}$$

and $N \in \mathbb{N}_0$ be so large that $|Q(k)| < 1$ for $k \geq N$, where $Q(k) = \sum_{j=k}^{\infty} q(j)$. We define a sequence Q^* by

$$Q^*(N) = 1, \qquad Q^*(k) = \prod_{j=N}^{k-1} \left(\frac{1 - Q^*(j)}{1 + Q^*(j)} \right) \quad \text{for } k \geq N+1. \tag{1.17.3}$$

Theorem 1.17.1. *Let*

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{n=0}^{k} \sum_{j=0}^{n} q(j) > -\infty, \tag{1.17.4}$$

and assume that equation (1.17.1) is nonoscillatory. Then

$$\sum_{j=N}^{\infty} \left[Q^*(j)\right]^2 = \infty. \tag{1.17.5}$$

Proof. Suppose that (1.17.5) does not hold and let x be any nonoscillatory solution of equation (1.17.1). Let $w = \{w(k)\}$ be the solution given by (1.12.52) (see Theorem 1.12.12) with $c(k) \equiv 1$ for all sufficiently large $k \geq N$. Let v be defined by

$$v(k) = \sum_{j=k}^{\infty} \frac{w^2(j)}{1+w(j)} \quad \text{for } k \geq N. \tag{1.17.6}$$

Then one can easily see that

$$\Delta v(k) + Q(k)\big[v(k+1) + v(k)\big] + v(k+1)v(k) + Q^2(k) = 0. \tag{1.17.7}$$

Let $s(k) = v(N)[Q^*(k)]^2$ for $k \geq N$. Since we have $s(N) = v(N)$ and, furthermore, $Q^*(k+1) = [(1-Q(k))/(1+Q(k))]Q^*(k)$, the same calculation in view of

$$\begin{aligned} s(k+1)s(k) + Q^2(k) &= v^2(N)\big[Q^*(k+1)\big]^2\big[Q^*(k)\big]^2 + Q^2(k) \\ &\geq 2Q(k)v(N)Q^*(k+1)Q^*(k) \end{aligned} \tag{1.17.8}$$

gives

$$\begin{aligned} \Delta s(k) &= -Q(k)\big[s(k+1) + s(k)\big] - 2Q(k)v(N)Q^*(k+1)Q^*(k) \\ &\geq -Q(k)\big[s(k+1) + s(k)\big] - s(k+1)s(k) - Q^2(k). \end{aligned} \tag{1.17.9}$$

Now a comparison of this last inequality with equation (1.17.7) yields

$$v(k) \leq s(k) = v(N)\big[Q^*(k)\big]^2. \tag{1.17.10}$$

Hence, since $\sum_{j=N}^{\infty}[Q^*(j)]^2 < \infty$, we have

$$\sum_{j=N}^{\infty} v(j) < \infty. \tag{1.17.11}$$

Now since $\lim_{k\to\infty} v(k) = 0$, it follows from (1.17.11) and (1.17.6) that

$$\sum_{n=N}^{\infty} n w^2(n) < \infty. \tag{1.17.12}$$

Next, by the Schwarz inequality we have

$$\left(\sum_{j=N}^{k} w(j)\right)^2 \leq \left(\sum_{j=N}^{k} j w^2(j)\right)\left(\sum_{j=N}^{k} \frac{1}{j}\right) \leq K \ln k, \tag{1.17.13}$$

where K is a positive constant independent of k, and hence

$$\left| \sum_{j=N}^{k} w(j) \right| \le \frac{1}{2} \ln k \quad \text{for all large } k\text{, say, } k > e^{4K}. \tag{1.17.14}$$

On the other hand, from $w(k) = \Delta x(k)/x(k)$, we have

$$x(k) = x(N) \prod_{j=N}^{k-1} \left[1 + w(j)\right] \quad \text{for } k \ge N+1, \tag{1.17.15}$$

and therefore it follows that

$$\left|x(k)\right| \le \left|x(N)\right| \exp\left(\sum_{j=N}^{k-1} w(j) \right) \le K_1\sqrt{k}. \tag{1.17.16}$$

This implies

$$\sum_{j=N}^{\infty} \left(x(j)x(j+1)\right)^{-1} = \infty, \tag{1.17.17}$$

which contradicts the existence of recessive solutions. The proof is therefore complete. □

Theorem 1.17.2. *Let condition (1.12.5) hold.*

(I) *If equation (1.17.1) is nonoscillatory, then (1.17.2) holds,*

$$\sum_{j=0}^{\infty} \frac{Q^2(j)}{Q^*(j)} \quad \textit{exists as a finite value}, \tag{1.17.18}$$

and there exists a sequence $v = \{v(k)\}$ with $v(k) > 0$ for $k \ge N$, for some $N \in \mathbb{N}_0$, satisfying the equation

$$v(k) = Q_1(k) + \sum_{j=k}^{\infty} f(j;k)v(j)v(j+1) \quad \textit{for } k \ge N, \tag{1.17.19}$$

where

$$\begin{aligned} Q_1(k) &= \sum_{j=k}^{\infty} f(j;k)Q^2(j), \\ f(j;k) &= \frac{Q^*(k)}{Q^*(j+1)} \frac{1}{1+Q(j)} \quad \textit{for } j \ge k \ge N. \end{aligned} \tag{1.17.20}$$

(II) *If (1.17.2) and (1.17.18) hold and there exists a sequence v with $v(k) > 0$, $k \geq N$ for some $N \geq 0$, satisfying the inequality*

$$v(k) \geq Q_1(k) + \sum_{j=k}^{\infty} f(j;k)v(j)v(j+1) \quad \text{for } k \geq N, \tag{1.17.21}$$

then equation (1.17.1) is nonoscillatory.

Proof. First we show (I). Assume that equation (1.17.1) is nonoscillatory. Then Theorem 1.12.6 implies that (1.17.2) holds. From Theorem 1.12.12 we have the existence of a sequence $w = \{w(k)\}$ satisfying (1.12.52) with $c(k) \equiv 1$. Define the sequence v as in (1.17.6) satisfying equation (1.17.7). Set $v(k) = Q^*(k)u(k)$ for $k \geq N$. Then, since

$$\Delta Q^*(k) = -Q(k)\left[Q^*(k+1) + Q^*(k)\right], \tag{1.17.22}$$

some manipulation gives

$$\Delta u(k) = -\frac{Q^*(k)}{1+Q(k)}u(k)u(k+1) - \frac{1}{Q^*(k+1)}\left(\frac{Q^2(k)}{1+Q(k)}\right), \tag{1.17.23}$$

that is,

$$\begin{aligned} u(k) = u(n+1) &+ \sum_{j=k}^{n} \frac{1}{Q^*(j+1)}\left(\frac{v(j+1)v(j)}{1+Q(j)}\right) \\ &+ \sum_{j=k}^{n} \frac{1}{Q^*(j+1)}\left(\frac{Q^2(j)}{1+Q(j)}\right). \end{aligned} \tag{1.17.24}$$

Since the left-hand side of (1.17.24) is independent of n and each term on the right-hand side is nonnegative, by letting $n \to \infty$ in (1.17.24) we obtain (1.17.18) and

$$\begin{aligned} u(k) = a &+ \sum_{j=k}^{\infty} \frac{1}{Q^*(j+1)}\left(\frac{v(j+1)v(j)}{1+Q(j)}\right) \\ &+ \sum_{j=k}^{\infty} \frac{1}{Q^*(j+1)}\left(\frac{Q^2(j)}{1+Q(j)}\right), \end{aligned} \tag{1.17.25}$$

where $a = \lim_{n\to\infty} u(n) \geq 0$. We claim that $a = 0$. If $a > 0$, then $u(j) \geq a$ for $j \geq k$. But then from (1.17.25) we have

$$u(k) \geq a + \sum_{j=k}^{\infty}\left(\frac{Q^*(j)}{1+Q(j)}\right)u(j+1)u(j) \geq a^2 \sum_{j=k}^{\infty} \frac{Q^*(j)}{1+Q(j)}, \tag{1.17.26}$$

and so from (1.17.26) we conclude that $\sum_{j=0}^{\infty} Q^*(j)$ and $\sum_{j=0}^{\infty}[Q^*(j)]^2$ exist and are finite. This is impossible by Theorem 1.17.1. Hence $a = 0$, so (1.17.19) follows from (1.17.26).

Now we show (II). Let

$$\begin{aligned} u(k) &= \sum_{j=k}^{\infty} f(j;k)v(j+1)v(j) = Q^*(k)\sum_{j=k}^{\infty} \frac{1}{Q^*(j+1)}\left(\frac{v(j+1)v(j)}{1+Q(j)}\right), \\ Q_2(k) &= \sum_{j=k}^{\infty} \frac{1}{Q^*(j+1)}\left(\frac{Q^2(j)}{1+Q(j)}\right) \quad \text{so that} \quad Q_1(k) = Q^*(k)Q_2(k). \end{aligned} \tag{1.17.27}$$

Define $w(k) = Q(k) + Q_1(k) + u(k)$ for $k \ge N$. Then $v(k) \ge w(k) - Q(k) \ge 0$ for $k \ge N$, and we have

$$\begin{aligned} \Delta Q_1(k) &= -\frac{Q^2(k)}{1+Q(k)} - Q(k)[Q^*(k+1) + Q^*(k)]Q_2(k), \\ \Delta u(k) &= -\frac{v(k+1)v(k)}{1+Q(k)} - \frac{Q(k)}{Q^*(k)}[Q^*(k+1) + Q^*(k)]u(k). \end{aligned} \tag{1.17.28}$$

Some rearrangement now gives

$$\begin{aligned} \Delta w(k) &= -Q(k)[w(k+1) + w(k)] + Q(k)Q(k+1) - v(k)v(k+1) - q(k) \\ &\le -Q(k)[w(k+1) + w(k)] + Q(k)Q(k+1) \\ &\quad - (w(k) - Q(k))(w(k+1) - Q(k+1)) - q(k) \\ &= -q(k)w(k) - w(k)w(k+1) - q(k), \end{aligned} \tag{1.17.29}$$

and so the nonoscillation follows from Lemma 1.7.1. □

Next, we give the following sufficient condition for nonoscillation of equation (1.17.1).

Theorem 1.17.3. *Let condition (1.12.5) hold. If (1.17.2) and (1.17.18) hold and the sequence Q_1 satisfies*

$$\sum_{j=k}^{\infty} f(j;k)Q_1(j)Q_1(j+1) \le \frac{1}{4}Q_1(k), \quad k \ge N \text{ for some } N \in \mathbb{N}_0, \tag{1.17.30}$$

then equation (1.17.1) is nonoscillatory.

Proof. From (1.17.30) we can show that $v(k) = 2Q_1(k)$ is a solution of (1.17.21), so the result follows immediately from Theorem 1.17.2. □

We note that if

$$\sum_{j=0}^{\infty} |Q(j)| \quad \text{exists (finite) and} \quad \sum_{j=k}^{\infty} jQ^2(j) < \infty, \tag{1.17.31}$$

then one can verify that (1.17.30) holds and so equation (1.17.1) is nonoscillatory. To see this, observe that

$$Q^*(k) = \prod_{j=N}^{k-1} \left(\frac{1-Q(j)}{1+Q(j)}\right) \longrightarrow \beta > 0 \quad \text{as } k \longrightarrow \infty. \tag{1.17.32}$$

Hence, there exist constants a and b with $0 < a < b$ such that $a \le f(j;k) \le b$ for all $j \ge k \ge N$. If we set $Q_3(k) = \sum_{j=k}^{\infty} Q^2(j)$, then we get $aQ_3(k) \le Q_1(k) \le bQ_3(k)$ and hence

$$\begin{aligned}\sum_{j=k}^{\infty} f(j;k)Q_1(j)Q_1(j+1) &\le b^3 \sum_{j=k}^{\infty} Q_3^2(j) \le b^3 Q_3(k) \sum_{j=k}^{\infty}\sum_{i=j}^{\infty} Q^2(i) \\ &\le \frac{b^3}{a} Q_1(k) \sum_{j=k}^{\infty} jQ^2(j).\end{aligned} \tag{1.17.33}$$

If we now choose k sufficiently large so that

$$\frac{b^3}{a} \sum_{j=k}^{\infty} jQ^2(j) \le \frac{1}{4}, \tag{1.17.34}$$

then we get that equation (1.17.1) is nonoscillatory by Theorem 1.17.3.

1.18. Limit point results

Consider the difference equation

$$\Delta\big(c(k)\Delta x(k)\big) - [q(k) + \lambda]x(k+1) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{1.18.1}$$

where $\{c(k)\}$ and $\{q(k)\}$ are sequences of real numbers, $c(k) > 0$ for $k \in \mathbb{N}_0$, and λ is a real constant. Equation (1.18.1) is equivalent to the equation

$$c(k+1)x(k+2) + c(k)x(k) = b(k)x(k+1) + \lambda x(k+1), \tag{1.18.2}$$

where $q(k) = b(k) - c(k) - c(k+1)$.

Equation (1.18.2) is called *limit circle* if all solutions are in ℓ^2, that is, square summable. Otherwise, it is called *limit point*. It is proven in [42] that if equation (1.18.2) is limit circle for one value of λ, then it is limit circle for every value of λ including $\lambda = 0$.

Before we present the results here, we recall Theorem 1.16.1. Based on this theorem, we can conclude that the nonoscillation of (1.16.1) implies the existence of two linearly independent solutions u and v of equation (1.16.1) such that

$$\sum^{\infty} \frac{1}{c(n)\left[u(n)u(n+1)+v(n)v(n+1)\right]} < \infty. \tag{1.18.3}$$

It should be pointed out that the converse of this is not true. That is, one can easily construct examples where the sum in (1.18.3) is finite but equation (1.16.1) is oscillatory. For instance, if $c(k) \equiv 1$ and $q(k) \equiv -4$, then $(-1)^k$ and $(-1)^k k$ are solutions which satisfy (1.18.3). This differs from the continuous case

$$\left(c(t)x'(t)\right)' = q(t)x(t), \tag{1.18.4}$$

where $c, q \in C([t_0,\infty),\mathbb{R})$ with $c(t) > 0$ for $t \geq t_0$, where the convergence of the integral analogue of the sum in (1.18.3) is equivalent to nonoscillation, see [155, page 354].

Theorem 1.18.1. *If equation (1.16.1) is nonoscillatory and*

$$\sum^{\infty} \frac{1}{\sqrt{c(n)}} = \infty, \tag{1.18.5}$$

then equation (1.18.1) is limit point.

Proof. Suppose not, so that equation (1.18.1) and hence equation (1.16.1) are limit circle. Since equation (1.16.1) is nonoscillatory, let u and v be two linearly independent solutions as found in Theorem 1.16.1. Then

$$\sum^{k} \frac{1}{\sqrt{c(n)}} = \sum^{k} \frac{\left[z(n)\right]^{1/2}}{\left[c(n)z(n)\right]^{1/2}} \leq \left[\sum^{k} z(n)\right]^{1/2}\left[\sum^{k} \frac{1}{c(n)z(n)}\right]^{1/2}, \tag{1.18.6}$$

where $z(n) = u(n)u(n+1)+v(n)v(n+1)$. The first term on the right-hand side of (1.18.6) is bounded because u and v are square summable. The second term is bounded by Theorem 1.16.1. Thus $\sum^{\infty} 1/\sqrt{c(n)}$ is summable, which is a contradiction to (1.18.5). □

Theorem 1.18.2. *If*

$$\left|b(k)\right| \geq c(k)+c(k+1) \quad \textit{for } k \in \mathbb{N}_0, \tag{1.18.7}$$

then equation (1.18.2) is limit point.

Proof. Lemma 1.3.2 implies that the solution of equation (1.16.1) defined by $v(0) = 0$ and $v(1) = 1$ is not square summable. Hence, equation (1.16.1) and therefore (1.18.2) are not limit circle. □

Next, we present the following result.

Theorem 1.18.3. *Suppose that*

$$\sum^{\infty} \frac{1}{c(n)} = \infty, \tag{1.18.8}$$

$$\sum^{\infty} |q(j)|^2 < \infty, \qquad \sum^{\infty} \left(\sum_{i=N\geq 1}^{j} \frac{1}{c(i)} \right)^2 |q(j)|^2 < \infty, \tag{1.18.9}$$

$$\sum^{\infty} \left[\sum_{j=k}^{\infty} \frac{1}{c(j)} \left(\sum_{i=j}^{\infty} |q(i)|^2 \right)^{1/2} \right]^2 < \infty. \tag{1.18.10}$$

Then equation (1.16.1) has no nontrivial ℓ^2-solutions, that is, equation (1.16.1) is limit point.

Proof. Let x be a nontrivial ℓ^2-solution of the equation (1.16.1). Summing equation (1.16.1) from N_1 to $N_2 - 1$ with $N_2 - 1 \geq N_1 \geq N \in \mathbb{N}$, we obtain

$$c(N_2)\Delta x(N_2) - c(N_1)\Delta x(N_1) = \sum_{j=N_1}^{N_2-1} q(j)x(j+1), \tag{1.18.11}$$

and hence, by the Schwarz inequality, we find

$$|c(N_2)\Delta x(N_2) - c(N_1)\Delta x(N_1)| \leq \left(\sum_{j=N_1}^{N_2-1} |q(j)|^2 \right)^{1/2} \left(\sum_{j=N_1}^{N_2-1} |x(j+1)|^2 \right)^{1/2}, \tag{1.18.12}$$

which in view of condition (1.18.9) and the fact that x is a square summable solution implies that $\lim_{k\to\infty} c(k)|\Delta x(k)| = \alpha \in \mathbb{R}$. We claim that $\alpha = 0$. Otherwise, if $\alpha \neq 0$, then we obviously have

$$c(k)|\Delta x(k)| \geq \frac{\alpha}{2} \quad \text{for all sufficiently large } k \in \mathbb{N}. \tag{1.18.13}$$

Summing (1.18.13) and using condition (1.18.8), we obtain a contradiction to the fact that x is an ℓ^2-solution, and hence we have $\alpha = 0$. So, from (1.18.11), we have for every $k \geq N$ that

$$c(k)\Delta x(k) = -\sum_{j=k}^{\infty} q(j)x(j+1). \tag{1.18.14}$$

Dividing both sides of (1.18.14) by $c(k)$ and summing from N_1 to N_2-1 with $N_2-1 \geq N_1 \geq N \in \mathbb{N}$, we get

$$x(N_2)-x(N_1)=\sum_{k=N_1}^{N_2-1}\frac{1}{c(k)}\sum_{j=k}^{\infty}q(j)x(j+1) \tag{1.18.15}$$

from which, by summation by parts and the Schwarz inequality, we obtain

$$\begin{aligned}
|x(N_2)-x(N_1)| &= \left|\sum_{k=N_1}^{N_2-1}\left(\Delta\sum_{i=N_1}^{k-1}\frac{1}{c(i)}\right)\sum_{j=k}^{\infty}q(j)x(j+1)\right| \\
&\leq \left(\sum_{i=N_1}^{N_2-1}\frac{1}{c(i)}\right)\left|\sum_{j=k}^{\infty}q(j)x(j+1)\right| \\
&\quad + \left|\sum_{k=N_1}^{N_2-1}\left(\sum_{i=N_1}^{k}\frac{1}{c(i)}\right)q(k)x(k+1)\right| \\
&\leq 2\sum_{k=N_1}^{\infty}\left[\left(\sum_{i=N_1}^{k}\frac{1}{c(i)}\right)^2|q(k)|^2\right]^{1/2}\left[\sum_{k=N_1}^{\infty}|x(k+1)|^2\right]^{1/2}.
\end{aligned} \tag{1.18.16}$$

Using condition (1.18.9) and the fact that x is a square summable solution, we get $\lim_{k\to\infty}|x(k)| = \alpha_0 \in \mathbb{R}$. As before, we can prove that $\alpha_0 = 0$, and hence $\lim_{k\to\infty} x(k) = 0$. Thus,

$$x(k)=\sum_{j=k}^{\infty}\frac{1}{c(j)}\sum_{i=j}^{\infty}q(i)x(i+1). \tag{1.18.17}$$

Squaring both sides of (1.18.17), summing the obtained result from $N_0 \geq N$ to ∞, and applying the Schwarz inequality, we have

$$\begin{aligned}
\sum_{k=N_0}^{\infty}|x(k)|^2 &\leq \sum_{k=N_0}^{\infty}\left[\sum_{j=k}^{\infty}\frac{1}{c(j)}\sum_{i=j}^{\infty}q(i)x(i+1)\right]^2 \\
&\leq \sum_{k=N_0}^{\infty}\left[\sum_{j=k}^{\infty}\frac{1}{c(j)}\left\{\left(\sum_{i=j}^{\infty}|q(i)|^2\right)^{1/2}\left(\sum_{i=j-1}^{\infty}|x(i)|^2\right)^{1/2}\right\}\right]^2 \\
&\leq \left(\sum_{i=N_0}^{\infty}|x(i)|^2\right)^{1/2}\sum_{k=N_0}^{\infty}\left[\sum_{j=k}^{\infty}\frac{1}{c(j)}\left(\sum_{i=j}^{\infty}|q(i)|^2\right)^{1/2}\right]^2,
\end{aligned} \tag{1.18.18}$$

which in view of condition (1.18.10) and the fact that x is a square summable solution, is an immediate contradiction. This completes the proof. □

Consider the nonlinear equation of the form

$$\Delta\big(c(k)\Delta x(k)\big) = q(k)f\big(x(k+1)\big), \tag{1.18.19}$$

where in addition to the above conditions on $c(k)$ and $q(k)$ we suppose that the function f is continuous on $\mathbb{R}$ and such that

(i) $x \neq 0$ implies $f(x) \neq 0$,
(ii) $\liminf_{x\to\infty} f(x) > 0$,
(iii) for every ℓ^2-solution of equation (1.18.19),

$$\limsup_{k\to\infty} \frac{\sum_{j=k}^{\infty} |x(j)|^2}{\sum_{j=k}^{\infty} |f(x(j))|^2} > 0. \tag{1.18.20}$$

Following the same arguments, one can observe that under the conditions (1.18.8), (1.18.9), and (1.18.10), the equation (1.18.19) has no nontrivial solution with the property

$$\sum^{\infty} |f(x(j))|^2 < \infty. \tag{1.18.21}$$

For a special case of equation (1.18.19), namely, the equation

$$\Delta^2 x(k) = q(k)f\big(x(k+1)\big), \tag{1.18.22}$$

where f satisfies a Lipschitz condition uniformly on every bounded interval and $f(x) = \mathrm{O}(x)$ as $x \to \infty$, we see that condition (iii) and

$$\sum^{\infty} |q(j)|^2 < \infty, \qquad \sum^{\infty} j^2 |q(j)|^2 < \infty, \qquad \sum^{\infty}\sum_{j=k}^{\infty}\left(\sum_{i=j}^{\infty} |q(i)|^2\right)^{1/2} < \infty \tag{1.18.23}$$

guarantee that equation (1.18.22) has no nontrivial solution x with the property (1.18.21).

Remark 1.18.4. We remark that, by using the Hölder inequality instead of the Schwarz inequality, we can obtain analogous results for the nonexistence of ℓ^p-solutions of equations (1.16.1) and (1.18.19). The details are left to the reader.

Example 1.18.5. The difference equation $\Delta^2 x(k) = 0$, $k \in \mathbb{N}$, has nontrivial solutions $x(k) \equiv 1$ and $x(k) = k$ which are not ℓ^2-solutions. All conditions of Theorem 1.18.3 are satisfied. Also, we see that the difference equation

$$\Delta\big(k\Delta x(k)\big) = \frac{1}{k+1}x(k+1) \quad \text{for } k \in \mathbb{N} \tag{1.18.24}$$

satisfies the hypotheses of Theorem 1.18.3 and has a nontrivial solution $x(k) = k$ which is not square summable.

1.19. Growth and some oscillation criteria

We consider the difference equation

$$\Delta\big(c(k)\Delta x(k)\big) + q(k)x(k) = f(k), \tag{1.19.1}$$

where $\{c(k)\}$, $\{q(k)\}$, and $\{f(k)\}$ are sequences of real numbers with $c(k) > 0$ for $k \in \mathbb{N}_0$.

We will present some results on the growth of all solutions of equation (1.19.1). Also, we discuss some sufficient conditions for nonoscillation and oscillation of equation (1.19.1).

1.19.1. Growth of solutions

The first result is concerned with the growth of all solutions, whereas the second result gives a growth estimate for nonoscillatory solutions. Let $C(n_0, k) = \sum_{j=n_0\geq 0}^{k-1} 1/c(j)$. We always assume that $\lim_{k\to\infty} C(n_0, k) = \infty$.

We will need the following lemma which is the discrete analogue of the well-known Gronwall inequality, see [4].

Lemma 1.19.1 (Gronwall's inequality). *Let $\{u(k)\}$, $\{p(k)\}$, and $\{q(k)\}$ be sequences of nonnegative real numbers for $k \geq m \in \mathbb{N}$ and let the inequality*

$$u(k) \leq p(k) + q(k)\sum_{\ell=n}^{k-1} f(\ell)u(\ell) \tag{1.19.2}$$

be satisfied. Then for all $k \geq m \in \mathbb{N}$,

$$u(k) \leq p(k) + q(k)\sum_{\ell=m}^{k-1} p(\ell)f(\ell)\prod_{\tau=\ell+1}^{k-1}\big(1 + q(\tau)f(\tau)\big). \tag{1.19.3}$$

In the case when $p(k) = c_0$, $q(k) \equiv 1$, and $c_0 \geq 0$ is a constant, one can easily see that

$$u(k) \leq c_0 \exp\left(\sum_{\ell=m}^{k-1} f(\ell)\right) \quad \text{for } k \geq m \in \mathbb{N}. \tag{1.19.4}$$

Theorem 1.19.2. *If*

$$\sum^{\infty} \left|f(j)\right| < \infty, \qquad \sum^{\infty} q(j)C(n_0, k) < \infty, \tag{1.19.5}$$

then every solution x of equation (1.19.1) satisfies

$$\left|x(k)\right| = \mathrm{O}\left(C(n_0, k)\right) \quad \text{as } k \longrightarrow \infty. \tag{1.19.6}$$

Proof. Let x be a solution of equation (1.19.1). From (1.19.1) by successive summation, we obtain

$$x(k) = x(n_0) + c(n_0)\Delta x(n_0) \sum_{j=n_0}^{k-1} \frac{1}{c(j)} - \sum_{j=n_0}^{k-1} \frac{1}{c(j)} \sum_{i=n_0}^{j-1} q(i)x(i) + \sum_{j=n_0}^{k-1} \frac{1}{c(j)} \sum_{i=n_0}^{j-1} f(i), \tag{1.19.7}$$

and so

$$|x(k)| \le a_1 C(n_0,k) + \sum_{j=n_0}^{k-1} \frac{1}{c(j)} \sum_{i=n_0}^{j-1} q(i)|x(i)| + \sum_{j=n_0}^{k-1} \frac{1}{c(j)} \sum_{i=n_0}^{j-1} |f(i)| \tag{1.19.8}$$

for some constant $a_1 > 0$ and all $k \ge n_1 \ge n_0$. By the first condition in (1.19.5), there is a constant $a_2 > 0$ such that

$$\sum_{j=n_0}^{k-1} \frac{1}{c(j)} \sum_{i=n_0}^{j-1} |f(i)| \le a_2 C(n_0,k) \quad \text{for } k \ge n_1. \tag{1.19.9}$$

Thus for $k \ge n_1$, we have

$$|x(k)| \le aC(n_0,k) + C(n_0,k) \sum_{j=n_0}^{k-1} q(j)|x(j)| \tag{1.19.10}$$

for some constant $a > 0$, and so

$$\left(\frac{|x(k)|}{C(n_0,k)}\right) \le a + \sum_{j=n_0}^{k-1} q(j)C(n_0,j)\left(\frac{x(j)}{C(n_0,j)}\right). \tag{1.19.11}$$

Using the discrete Gronwall inequality, that is, Lemma 1.19.1, we get

$$\frac{|x(k)|}{C(n_0,k)} \le a \exp\left(\sum_{j=k_0}^{k-1} q(j)C(n_0,j)\right), \tag{1.19.12}$$

and the conclusion of the theorem follows from condition (1.19.5). □

Theorem 1.19.3. *If $q(k) \ge 0$ for $k \ge n_0 \ge 0$, then any nonoscillatory solution x of equation (1.19.1) satisfies*

$$|x(k)| = O\left(C(n_0,k) + \sum_{j=n_0}^{k-1} \frac{1}{c(j)} \sum_{i=n_0}^{j-1} |f(i)|\right) \quad \text{as } k \longrightarrow \infty. \tag{1.19.13}$$

Proof. Let x be a nonoscillatory solution of equation (1.19.1). Then $x(k) \geq 0$ or $x(k) \leq 0$ for all $k \geq s \geq n_0$. If $x(k) \geq 0$, then from equation (1.19.1) we have

$$0 \leq x(k) \leq x(s) + c(s)\Delta x(s) \sum_{j=s}^{k-1} \frac{1}{c(j)} + \sum_{j=s}^{k-1} \frac{1}{c(j)} \sum_{i=s}^{k-1} |f(i)|. \tag{1.19.14}$$

If $x(k) \leq 0$, then

$$0 \geq x(k) \geq x(s) + c(s)\Delta x(s) \sum_{j=s}^{k-1} \frac{1}{c(j)} - \sum_{j=s}^{k-1} \frac{1}{c(j)} \sum_{i=s}^{k-1} |f(i)|. \tag{1.19.15}$$

Hence in either case

$$|x(k)| \leq |x(s)| + c(s)|\Delta x(s)| C(s,k) + \sum_{j=s}^{k-1} \frac{1}{c(j)} \sum_{i=s}^{k-1} |f(i)|, \tag{1.19.16}$$

and the conclusion follows. □

1.19.2. Oscillation criteria

Theorem 1.19.4. *If $q(k) \geq 0$, $k \geq n_0 \geq 0$, and for every constant $a > 0$ and all large $s \geq n_0$*

$$\liminf_{k\to\infty} \left[\sum_{j=s}^{k-1} \frac{1}{c(j)} \sum_{i=s}^{j-1} f(i) + aC(s,k) \right] < 0, \tag{1.19.17}$$

$$\limsup_{k\to\infty} \left[\sum_{j=s}^{k-1} \frac{1}{c(j)} \sum_{i=s}^{k-1} f(i) - aC(s,k) \right] > 0, \tag{1.19.18}$$

then equation (1.19.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (1.19.1), say, $x(k) \geq 0$ for $k \geq s \geq n_0$. From equation (1.19.1) we have

$$\Delta\big(c(k)\Delta x(k)\big) = -q(k)x(k) + f(k) \leq f(k) \quad \text{for } k \geq s \geq n_0. \tag{1.19.19}$$

As before, it is easy to check that there exist a constant $a_1 > 0$ and $n_1 \geq s$ such that

$$x(k) \leq a_1 C(s,k) + \sum_{j=s}^{k-1} \frac{1}{c(j)} \sum_{i=s}^{j-1} f(i) \quad \text{for } k \geq n_1. \tag{1.19.20}$$

Taking lim inf on both sides of (1.19.20) as $k \to \infty$ and applying condition (1.19.17), we obtain a contradiction to the fact that $x(k) \geq 0$ for $k \geq s$. The proof of the case when $x(k) \leq 0$ for $k \geq s$ is similar. □

When $q(k) \leq 0$ for $k \geq n_0$, then we have the following result.

Theorem 1.19.5. *If $q(k) \leq 0$ for $k \geq n_0$ and for every constant $a > 0$ and all large $s \geq n_0$,*

$$\liminf_{k\to\infty}\left[\sum_{j=s}^{k-1}\frac{1}{c(j)}\sum_{i=s}^{j-1}f(i)+aC(s,k)\right]=-\infty, \tag{1.19.21}$$

$$\limsup_{k\to\infty}\left[\sum_{j=s}^{k-1}\frac{1}{c(j)}\sum_{i=s}^{j-1}f(i)-aC(s,k)\right]=\infty, \tag{1.19.22}$$

then all bounded solutions of equation (1.19.1) are oscillatory.

Proof. Let x be a bounded nonoscillatory solution of equation (1.19.1), say, $x(k) \geq 0$ for $k \geq s \geq n_0$. The proof of the case when $x(k) \leq 0$, $k \geq s \geq n_0$, is similar. It is easy to see that there exist a constant $a_1 > 0$ and an integer $n_1 \geq s$ such that

$$x(k)\geq\sum_{j=s}^{k-1}\frac{1}{c(j)}\sum_{i=s}^{j-1}f(i)-a_1C(s,k)\quad\text{for } k\geq n_1. \tag{1.19.23}$$

Taking lim sup on both sides of (1.19.23) as $k \to \infty$ and using condition (1.19.22), we obtain a contradiction to the fact that $\{x(k)\}$ is bounded. This completes the proof. □

1.20. Mixed difference equations

For a sequence $\{x(k)\}$ and a fixed constant α, we define $\Delta_\alpha x(k) = x(k+1) - \alpha x(k)$. When $\alpha = 1$, $\Delta_1 x(k) = \Delta x(k)$. In this section we will compare the oscillatory behavior of the mixed difference equations

$$\Delta\big(c(k-1)\Delta_\alpha x(k-1)\big)+q(k)x(k)=0, \tag{1.20.1}$$

$$\Delta_\alpha\big(c(k-1)\Delta_\alpha x(k-1)\big)+q(k)x(k)=0, \tag{1.20.2}$$

with equations of the form

$$\Delta\big(c(k-1)\Delta x(k-1)\big)+q(k)x(k)=0, \tag{1.20.3}$$

where $\{c(k)\}$ and $\{q(k)\}$ are sequences of real numbers, and $c(k) > 0$ for $k \in \mathbb{N}_0$.

Equations (1.20.1) and (1.20.2) are equivalent to

$$c(k)x(k+1)+\big[q(k)-\alpha c(k)-c(k-1)\big]x(k)+\alpha c(k-1)x(k-1)=0, \tag{1.20.4}$$

$$c(k)x(k+1)+\big[q(k)-\alpha\big(c(k)+c(k-1)\big)\big]x(k)+\alpha^2c(k-1)x(k-1)=0, \tag{1.20.5}$$

respectively. As before, if there exists a subsequence $\{k_n\}$ with

$$q(k_n) - [\alpha c(k_n) + c(k_n - 1)] \geq 0, \tag{1.20.6}$$

where $k_n \to \infty$ as $n \to \infty$, then equation (1.20.4) is oscillatory for all $\alpha \geq 0$. Also, if there exists a subsequence $\{k_n\}$ with

$$q(k_n) - \alpha[c(k_n) + c(k_n - 1)] \geq 0, \tag{1.20.7}$$

where $k_n \to \infty$ as $n \to \infty$, then equation (1.20.5) is oscillatory for all α.

Next, we see that equations (1.20.4) and (1.20.5) are equivalent to

$$\begin{gathered}\Delta(c(k-1)\Delta x(k-1)) + [q(k) - (\alpha-1)c(k)]x(k) + (\alpha-1)c(k-1)x(k-1) = 0,\\ \Delta(c(k-1)\Delta x(k-1)) + [q(k) - (\alpha-1)(c(k) + c(k-1))]x(k)\\ +(\alpha^2-1)c(k-1)x(k-1) = 0,\end{gathered} \tag{1.20.8}$$

respectively.

To obtain the main result, we first prove the following result.

Theorem 1.20.1. *Assume that $q(k) > 0$ for $k \in \mathbb{N}_0$ and*

$$\sum^{\infty} \frac{1}{c(j)} = \infty. \tag{1.20.9}$$

If the inequality

$$\Delta(c(k-1)\Delta x(k-1)) + q(k)x(k) \leq 0 \tag{1.20.10}$$

has an eventually nonnegative solution, then equation (1.20.3) also has an eventually nonnegative solution.

Proof. Let x be an eventually nonnegative solution of inequality (1.20.10). It is easy to see that $\Delta x(k) \geq 0$ eventually. Let

$$y(k+1) = c(k)\Delta x(k) \quad \text{for } k \in \mathbb{N}. \tag{1.20.11}$$

Then $\Delta x(k) = y(k+1)/c(k) \geq 0$. Summing this inequality from $n_0 \geq 0$ to $k-1$ we obtain

$$x(k) = x(n_0) + \sum_{j=n_0}^{k-1} \frac{y(j+1)}{c(j)} = Y(c(k), y(k+1)). \tag{1.20.12}$$

By (1.20.10) and (1.20.11), we have

$$\Delta y(k) + q(k)Y(c(k), y(k+1)) \leq 0. \tag{1.20.13}$$

Summing (1.20.13) from k to $m > k$ we find

$$y(m+1) - y(k) \leq -\sum_{j=k}^{m} q(j) Y\big(c(j), y(j+1)\big). \tag{1.20.14}$$

Thus

$$y(k) \geq \sum_{j=k}^{\infty} q(j) Y\big(c(j), y(j+1)\big). \tag{1.20.15}$$

Now, we define a sequence of successive approximations $\{w(k,i)\}$ as follows:

$$\begin{aligned} w(k,0) &= y(k), \\ w(k,i+1) &= \sum_{j=k}^{\infty} q(j) Y\big(c(j), w(j+1,i)\big) \quad \text{for } i \in \mathbb{N}_0. \end{aligned} \tag{1.20.16}$$

Obviously, we can prove that $0 \leq w(k,i) \leq y(i)$ and $w(k,i+1) \leq w(k,i)$ for $i \in \mathbb{N}_0$. Thus the sequence $\{w(k,i)\}$ is nonnegative and nonincreasing in i for each k. So we may define $w(k) = \lim_{i\to\infty} w(k,i) \geq 0$. Since $0 \leq w(k) \leq w(k,i) \leq y(k)$ for all $i \in \mathbb{N}_0$ and since $Y(c(j), w(j+1,i)) \leq Y(c(j), w(j+1))$, the convergence of the series in (1.20.16) is uniform with respect to i. Taking the limit on both sides of (1.20.16), we obtain

$$w(k) = \sum_{j=k}^{\infty} q(j) Y\big(c(j), w(j+1)\big). \tag{1.20.17}$$

Therefore,

$$\Delta w(k) = -q(k) Y\big(c(k), w(k+1)\big). \tag{1.20.18}$$

Denote

$$v(k) = Y\big(c(k), w(k+1)\big). \tag{1.20.19}$$

Then $\Delta v(k) = w(k+1)/c(k)$. Thus

$$w(k+1) = c(k)\Delta v(k). \tag{1.20.20}$$

From (1.20.18), (1.20.19), and (1.20.20), we obtain

$$\Delta\big(c(k-1)\Delta v(k-1)\big) + q(k)v(k) = 0. \tag{1.20.21}$$

We showed that equation (1.20.3) has a nonnegative solution $\{v(k)\}$. This completes the proof. □

Now, the following results are immediate.

Corollary 1.20.2. *Let $\alpha \geq 1$, $q(k) \geq (\alpha - 1)c(k)$ for $k \in \mathbb{N}_0$, and suppose that condition (1.20.9) holds. If equation (1.20.1) has an eventually nonnegative solution, then the equation*

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + \big[q(k) - (\alpha - 1)c(k)\big]x(k) = 0 \tag{1.20.22}$$

also has an eventually nonnegative solution.

Proof. Let $\{x(k)\}$ be an eventually nonnegative solution of equation (1.20.1). The case when $\alpha = 1$ is obvious. Thus we consider the case $\alpha > 1$. From equation (1.20.1) it follows that

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + \big[q(k) - (\alpha - 1)c(k)\big]x(k) \leq 0. \tag{1.20.23}$$

Applying Theorem 1.20.1, we obtain the desired conclusion. □

Corollary 1.20.3. *Let $\alpha \geq 1$, $q(k) \geq (\alpha - 1)[c(k) + c(k-1)]$ for $k \in \mathbb{N}_0$ and suppose that condition (1.20.9) holds. If equation (1.20.2) has an eventually nonnegative solution, then the equation*

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + \big[q(k) - (\alpha - 1)\big(c(k) + c(k-1)\big)\big]x(k) = 0 \tag{1.20.24}$$

also has an eventually nonnegative solution.

From Corollaries 1.20.2 and 1.20.3, we give the following results.

Theorem 1.20.4. *Let $\alpha \geq 1$, $q(k) \geq (\alpha - 1)c(k)$ for $k \in \mathbb{N}_0$, and suppose that condition (1.20.9) holds. If equation (1.20.22) is oscillatory, then equation (1.20.1) is also oscillatory.*

Theorem 1.20.5. *Let $\alpha \geq 1$, $q(k) \geq (\alpha-1)[c(k)+c(k-1)]$ for $k \in \mathbb{N}_0$, and suppose that condition (1.20.9) holds. If equation (1.20.24) oscillates, then equation (1.20.2) also oscillates.*

Equations (1.20.22) and (1.20.24) are of the form of equation (1.20.3). Such equations are studied extensively in this chapter. Therefore, one can easily formulate many criteria for the oscillation of equations (1.20.1) and (1.20.2). The statements and formulations of such results are left to the reader.

A result as that in Theorem 1.20.1 for the inequality

$$\Delta\big(c(k-1)\Delta x(k-1)\big) - q(k)x(k) \geq 0, \tag{1.20.25}$$

and the equation

$$\Delta\big(c(k-1)\Delta x(k-1)\big) - q(k)x(k) = 0, \tag{1.20.26}$$

where $\{c(k)\}$ and $\{q(k)\}$ are sequences of positive numbers, is stated as follows.

Theorem 1.20.6. *Suppose that condition (1.20.9) holds. If inequality (1.20.25) has an eventually nonnegative bounded solution, then equation (1.20.26) also has an eventually nonnegative bounded solution.*

As an application of Theorem 1.20.6, we state the following result.

Theorem 1.20.7. *Let* $0 < \alpha \leq 1$, *suppose that condition (1.20.9) holds, and assume*

$$s(k) = q(k) + (\alpha - 1)\big[c(k) + c(k-1)\big] > 0 \quad \text{for } k \in \mathbb{N}_0. \tag{1.20.27}$$

If every bounded solution of the equation

$$\Delta\big(c(k-1)\Delta x(k-1)\big) - s(k)x(k) = 0 \tag{1.20.28}$$

is oscillatory, then every bounded solution of equation (1.20.3) is also oscillatory.

1.21. Notes and general discussions

(1) The results of Section 1.2 are taken from Agarwal [4], Atkinson [42], and Kelley and Peterson [173]. Lemmas 1.2.9 and 1.3.1–1.3.4 are given by Patula [216, 217], and Theorem 1.3.5 is due to Hooker and Patula [162].
(2) The results of Section 1.4 are special cases of those that will be presented in Chapter 2 and Section 3.3. The proofs of these results will be postponed.
(3) Theorems 1.5.1–1.5.10 are taken from Kelley and Peterson [173] while the rest of the results in this section is due to Chen [82].
(4) The results of Section 1.6 on conjugacy criteria for second-order linear difference equations are due to Došlý and Řehák [111].
(5) Lemmas 1.7.1 and 1.7.2 will be considered in more detail in Section 3.4.
(6) Lemma 1.8.1 is taken from Hooker and Patula [162], Lemma 1.8.2 is due to Kwong et al. [189], and Lemma 1.8.3 is given by Hooker et al. [161]. Theorem 1.8.4–Corollary 1.8.19 are due to Hooker and Patula [162], while Theorem 1.8.21–Corollary 1.8.25 and Theorems 1.8.33–1.8.42 are taken from Kwong et al. [189]. Theorems 1.8.29, 1.8.32, and 1.8.45 towards the end of this section are due to Hooker et al. [161].
(7) The results of Section 1.9 are taken from Kwong et al. [189].
(8) The results presented in Section 1.10 are due to Chen [82].
(9) The results of Section 1.11 are due to Erbe and Zhang [125].
(10) The results of Section 1.12 are taken from Chen and Erbe [84]. We note that it is still an open problem to have a "full discrete version" of the Hartman-Wintner theorem for linear difference equations (1.2.1) with the discrete counterpart of the condition $\int^{\infty} ds/c(s) = \infty$. For more results in this direction we refer to Erbe and Yan [122].
(11) The results of Section 1.13 are taken from Grace et al. [141], and for some related results we refer the reader to Wong and Agarwal [285].

(12) The results of Section 1.14 are due to Grace et al. [140]; see also [143].
(13) Lemma 1.15.1–Corollary 1.15.6 are taken from Patula [217], while the rest of the results of this section is due to Grace and El-Morshedy [142].
(14) The results presented in Sections 1.16.1–1.16.3 and Theorem 1.16.40 are taken from Cheng et al. [91], while Theorems 1.16.29–1.16.36 are due to Thandapani et al. [268]. The rest of the results of this section is new.
(15) The results of Section 1.17 are due to Chen and Erbe [84].
(16) Theorems 1.18.1 and 1.18.2 are taken from Patula [216], while Theorem 1.18.3 is new.
(17) The results of Section 1.19 are extracted from Szmanda [262].
(18) The results of Section 1.20 are new and are based on the concept of mixed difference equations introduced by Popenda [236].
(19) We note that more results with different techniques on the disconjugacy of linear difference equations can be added to those in Sections 1.4 and 1.5. Using equivalent identities of Wronskian determinants for difference equations, Hartman [156, Theorem 5.2] obtained a necessary and sufficient condition for disconjugacy of an nth-order linear difference equation on a certain interval. But this condition is not easy to verify. For example, if the length of the interval is $m+k$ with $m>0$, then the sign of $\binom{m+k}{k}$ determinants of mth order must be calculated. For the right disfocality of an nth-order linear difference equation, Eloe [117] gives several equivalent statements in terms of the existence of a set of solutions whose Wronskian has certain positive subdeterminants. Van Doorn [272] applies the theory of orthogonal polynomials to the study of oscillation of equation (1.2.1) and proves that the question of whether equation (1.2.1) is nonoscillatory is equivalent to the question of whether a sequence defined by the coefficients in (1.2.1) is a chain sequence (a sequence of the form $\{(1-g(k-1))g(k)\}_{k=1}^{\infty}$ with $0 \le g(0) \le 1$ and $0 < g(k) < 1$ for $k \in \mathbb{N}$). Therefore, his condition needs the availability of criteria for chain sequences.
(20) For results concerning oscillation and nonoscillation of linear differential equations, we refer the reader to the monographs [8, 19, 20, 99, 150, 155, 250, 251, 259]. For more results on oscillation and nonoscillation of linear difference equations, we refer the reader to the monographs [4, 19, 36, 42, 150, 173, 205]. We also note that the problem of finding necessary and sufficient conditions for oscillation of equations (1.2.1) and (1.1.1), given purely in terms of the coefficients of these equations, is even harder and still seems to remain open.
(21) The concept of strong oscillation introduced by Nehari [207] and considered in Cheng and Li [90] will be considered in full detail in Section 3.8. Therefore we postpone such a discussion.

2 Oscillation theory for systems of difference equations

2.1. Introduction

It is a well-known fact in discrete variational calculus that the quadratic functional

$$\sum_{k=0}^{N} \left\{ p_k (\Delta y_k)^2 + q_k y_{k+1}^2 \right\}, \tag{2.1.1}$$

(with real p_k, q_k, and $p_k > 0$) is positive definite, that is, positive for all nontrivial real sequences $y = (y_k)_{0 \le k \le N+1}$ with $y_0 = y_{N+1} = 0$, if and only if the difference equation

$$\Delta\{p_k \Delta y_k\} = q_k y_{k+1} \tag{2.1.2}$$

is disconjugate on $[0, N+1] \cap \mathbb{Z}$, that is, has no nontrivial solution y with $y_0 = 0$ which satisfies at (at least) one point $k \in [0, N] \cap \mathbb{Z}$ either $y_{k+1} = 0$ or $y_k y_{k+1} < 0$ (see [173, Theorem 8.10]). The generally known corresponding result from classical variational calculus (see, e.g., [128, Theorem 26.1]) already holds, when y does not have any "regular" zeros, that is, points t with $y(t) = 0$, in the real interval under consideration, while in the present discrete case we have to deal with somehow "generalized" zeros.

A similar problem results when wishing to define generalized zeros of solutions of difference equations of higher order

$$\sum_{\nu=0}^{n} (-\Delta)^\nu \{ r_k^{(\nu)} \Delta^\nu y_{k+1-\nu} \} = 0 \quad \text{for } 0 \le k \le N - n \tag{2.1.3}$$

(with real $r_k^{(\nu)}$ and $r_k^{(n)} > 0$). The starting point of the study of disconjugacy for (2.1.3) was the fundamental article on difference equations by Hartman [156] in 1978. Since then it has been an open problem to define disconjugacy for (2.1.3) appropriately and to show that it is equivalent to the positive definiteness of the functional

$$\mathcal{F}_0(y) = \sum_{k=0}^{N} \sum_{\nu=0}^{n} r_k^{(\nu)} \{\Delta^\nu y_{k+1-\nu}\}^2, \tag{2.1.4}$$

that is, to $\mathcal{F}_0(y) > 0$ for all $y \neq 0$ with $y_{1-n+i} = y_{N+1-i} = 0$, $0 \leq i \leq n-1$. In this chapter we present an answer to this question by considering general linear Hamiltonian difference systems of the form

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k \quad \text{for } 0 \leq k \leq N \tag{2.1.5}$$

(with $n \times n$-matrices A_k, B_k, C_k) and by treating (2.1.3) as a special case of system (2.1.5).

We will define generalized zeros of vector-valued solutions and disconjugacy for general systems (2.1.5). Then it is possible to prove the main result on positive definiteness for discrete quadratic functionals

$$\mathcal{F}_0(x, u) = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}, \tag{2.1.6}$$

where $\mathcal{F}_0$ is called positive definite in case of $\mathcal{F}_0(x, u) > 0$ for all (x, u) with $x \neq 0$, $x_0 = x_{N+1} = 0$, and $\Delta x_k = A_k x_{k+1} + B_k u_k$, $0 \leq k \leq N$. A result which lists conditions that are equivalent to the positive definiteness of $\mathcal{F}$ is called a "Reid roundabout theorem." This terminology is due to Ahlbrandt (see [26, 27, 28, 31, 32, 34]), and it is in honor of the corresponding results of Reid in the "continuous case;" compare [249, Theorem VII.5.1] and [251, Theorem V.6.3] (see also [99, Chapter 2] as well as [206]). Our characterization of the positive definiteness of $\mathcal{F}_0$ contains besides the disconjugacy of (2.1.5), that is, the discrete version of Jacobi's condition, also a connection to Riccati matrix difference equations of the form

$$Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k). \tag{2.1.7}$$

The main result of this chapter is a Reid roundabout theorem for general Hamiltonian systems (2.1.5) with an extended functional and extended boundary conditions that are presented for the continuous case in the book "Quadratic Functionals in Variational Analysis and Control Theory" by Kratz (see especially [182, Theorem 2.4.1]). Here as well as there the central tool for the proof of the Reid roundabout theorem is a "Picone identity" (see [43, Proposition 6.1], [182, Theorem 1.2.1], and [183, 235]), and we will prove a discrete version of Picone's identity in this chapter.

2.2. Discrete variational problems

2.2.1. Problem and notation

In this section we will (with close reference to [182, Section 2.3]) explain in a motivating way why it does make sense to deal with discrete quadratic functionals and their positive definiteness. To this end it is necessary to introduce some terminology. Throughout we let $N \in \mathbb{N}$ and use the notation

$$J = [0, N] \cap \mathbb{Z}, \qquad J^* = [0, N+1] \cap \mathbb{Z}. \tag{2.2.1}$$

For functions x defined on J^* we define the shift x^σ on J by $x_k^\sigma = x_{k+1}$ for all $k \in J$. Let be given real matrix-valued and vector-valued functions on J

$$Q, \tilde{Q}, P, \mathcal{A}, \mathcal{B}, q, p, c \tag{2.2.2}$$

of type $n \times n$, $n \times m$, $m \times m$, $n \times n$, $n \times m$, $n \times 1$, $m \times 1$, $n \times 1$ as well as real (constant) matrices and vectors

$$S, S^*, s, s^* \tag{2.2.3}$$

of type $2n \times 2n$, $2n \times 2n$, $2n \times 1$, $2n \times 1$, which are supposed to satisfy the assumptions

$$\begin{gathered} Q, P \quad \text{symmetric on } J, \\ P > 0, \quad \tilde{\mathcal{B}} := \mathcal{B}^T \mathcal{B} > 0 \quad \text{on } J, \\ S \quad \text{symmetric}, \ s^* \in \operatorname{Im} S^*. \end{gathered} \tag{2.2.4}$$

Furthermore, for $x = (x_k)_{k \in J^*}$ and $v = (v_k)_{k \in J}$ we define functions and functionals by

$$\begin{aligned} \tilde{\Omega}(x, v) &= (x^\sigma)^T Q x^\sigma + 2(x^\sigma)^T \tilde{Q} v + v^T P v + 2(x^\sigma)^T q + 2p^T v, \\ T(x) &= \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \left\{ S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + 2s \right\}, \\ \mathcal{F}(x, v) &= \sum_{k \in J} \tilde{\Omega}_k(x, v) + T(x). \end{aligned} \tag{2.2.5}$$

We say that (x, v) is admissible if $\Delta x = \mathcal{A} x^\sigma + \mathcal{B} v + c$ holds on J and write $x \in \tilde{\mathcal{R}}$ in case of $S^* \binom{-x_0}{x_{N+1}} = s^*$.

Subject to all this notation we now want to deal with the *inhomogeneous discrete variational problem*

$$\begin{gathered} \mathcal{F}(x, v) \longrightarrow \min, \\ (x, v) \quad \text{admissible}, \ x \in \tilde{\mathcal{R}}. \end{gathered} \tag{2.2.6}$$

We start our examinations of problem (2.2.6) in the next subsection with some auxiliary results. Theorem 2.2.5 in Section 2.2.3 then is the end of these examinations and at the same time the starting point for all subsequent sections of this chapter.

We need some more notation. First of all, we define on J

$$\begin{gathered} A = \mathcal{A} - \mathcal{B}P^{-1}\tilde{Q}^T, \qquad B = \mathcal{B}P^{-1}\mathcal{B}^T, \qquad C = Q - \tilde{Q}P^{-1}\tilde{Q}^T, \\ a = c - \mathcal{B}P^{-1}p, \qquad b = q - \tilde{Q}P^{-1}p, \\ \Omega(x,u) = (x^\sigma)^T C x^\sigma + u^T B u + 2b^T x^\sigma - p^T P^{-1} p, \\ \Omega^h(\eta,\xi) = (\eta^\sigma)^T C \eta^\sigma + \xi^T B \xi, \end{gathered} \tag{2.2.7}$$

as well as finally

$$T^h(\eta) = \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T S \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}. \tag{2.2.8}$$

Note that B and C are symmetric as defined above due to the assumptions on P and Q.

2.2.2. Inhomogeneous discrete quadratic functionals

For the remainder of this section we use the notation and assumptions introduced in Section 2.2.1.

Lemma 2.2.1. *If $v = P^{-1}\{\mathcal{B}^T u - \tilde{Q}^T x^\sigma - p\}$ holds, then*

$$\Omega(x,u) = \tilde{\Omega}(x,v), \qquad \mathcal{A}x^\sigma + \mathcal{B}v + c = Ax^\sigma + Bu + a. \tag{2.2.9}$$

Proof. Let $v = P^{-1}\{\mathcal{B}^T u - \tilde{Q}^T x^\sigma - p\}$. Then

$$\begin{aligned} \mathcal{A}x^\sigma + \mathcal{B}v + c &= \{\mathcal{A} - \mathcal{B}P^{-1}\tilde{Q}^T\}x^\sigma + \mathcal{B}P^{-1}\mathcal{B}^T u + c - \mathcal{B}P^{-1}p \\ &= Ax^\sigma + Bu + a, \\ \tilde{\Omega}(x,v) &= (x^\sigma)^T Q x^\sigma + 2\left((x^\sigma)^T\tilde{Q} + p^T\right)P^{-1}(\mathcal{B}^T u - \tilde{Q}^T x^\sigma - p) \\ &\quad + 2(x^\sigma)^T q + \left(u^T\mathcal{B} - (x^\sigma)^T\tilde{Q} - p^T\right)P^{-1}(\mathcal{B}^T u - \tilde{Q}^T x^\sigma - p) \\ &= (x^\sigma)^T\{Q - 2\tilde{Q}P^{-1}\tilde{Q}^T + \tilde{Q}P^{-1}\tilde{Q}^T\}x^\sigma + u^T\mathcal{B}P^{-1}\mathcal{B}^T u \\ &\quad + p^T P^{-1}(\mathcal{B}^T u - \tilde{Q}^T x^\sigma - p) + 2(x^\sigma)^T q - \left(u^T\mathcal{B} + (x^\sigma)^T\tilde{Q}\right)P^{-1}p \\ &= \Omega(x,u). \end{aligned} \tag{2.2.10}$$

The proof is complete. □

Remark 2.2.2 (admissibility). In view of the preceding auxiliary result we call (x,u) for convenience (also) *admissible* if $\Delta x = Ax^\sigma + Bu + a$ holds. Moreover, we say that (η,ζ) (resp., (η,ξ)) is *h-admissible*, whenever $\Delta\eta = \mathcal{A}\eta^\sigma + \mathcal{B}\zeta$ (resp., $\Delta\eta = A\eta^\sigma + B\xi$) holds.

Lemma 2.2.3 (first and second variation). *Let (x,u) be admissible and (η,ξ) be h-admissible. We put*

$$\begin{aligned}
\mathcal{F}_1(x,u,\eta,\xi) &= \sum_{k\in J}\eta_{k+1}^T\{C_kx_{k+1} - A_k^Tu_k + b_k - \Delta u_k\}\\
&\quad + \begin{pmatrix}-\eta_0\\ \eta_{N+1}\end{pmatrix}^T\left\{\begin{pmatrix}u_0\\ u_{N+1}\end{pmatrix} + S\begin{pmatrix}-x_0\\ x_{N+1}\end{pmatrix} + s\right\},\\
\mathcal{F}_2(\eta,\xi) &= \sum_{k\in J}\Omega_k^h(\eta,\xi) + T^h(\eta).
\end{aligned} \tag{2.2.11}$$

If $v = P^{-1}\{\mathcal{B}^Tu - \tilde{Q}^Tx^\sigma - p\}$ and $\zeta = P^{-1}\{\mathcal{B}^T\xi - \tilde{Q}^T\eta^\sigma\}$ holds, then

$$\mathcal{F}(x+\eta, v+\zeta) - \mathcal{F}(x,v) = 2\mathcal{F}_1(x,u,\eta,\xi) + \mathcal{F}_2(\eta,\xi). \tag{2.2.12}$$

Proof. Let (x,u) and (η,ξ) be admissible and h-admissible, respectively. From

$$v = P^{-1}\{\mathcal{B}^Tu - \tilde{Q}^Tx^\sigma - p\}, \qquad \zeta = P^{-1}\{\mathcal{B}^T\xi - \tilde{Q}^T\eta^\sigma\}, \tag{2.2.13}$$

it follows that

$$v+\zeta = P^{-1}\{\mathcal{B}^T(u+\xi) - \tilde{Q}^T(x+\eta)^\sigma - p\} \tag{2.2.14}$$

holds. Therefore, Lemma 2.2.1 yields

$$\begin{aligned}
\tilde{\Omega}(x+\eta, v+\zeta) - \tilde{\Omega}(x,v) &= \Omega(x+\eta, u+\xi) - \Omega(x,u)\\
&= \left((x+\eta)^\sigma\right)^TC(x+\eta)^\sigma + (u+\xi)^TB(u+\xi)\\
&\quad + 2b^T(x+\eta)^\sigma - p^TP^{-1}p\\
&\quad - \left(x^\sigma\right)^TCx^\sigma - u^TBu - 2b^Tx^\sigma + p^TP^{-1}p\\
&= \Omega^h(\eta,\xi) + 2\left(\eta^\sigma\right)^T\{Cx^\sigma + b\} + 2u^TB\xi\\
&= \Omega^h(\eta,\xi) + 2\left(\eta^\sigma\right)^T\{Cx^\sigma - A^Tu + b\} + 2u^T\Delta\eta.
\end{aligned} \tag{2.2.15}$$

Using summation by parts we find

$$\begin{aligned}
&\mathcal{F}(x+\eta, v+\zeta) - \mathcal{F}(x,v) \\
&\quad = 2\sum_{k\in J} \eta_{k+1}^T\{C_k x_{k+1} - A_k^T u_k + b_k - \Delta u_k\} + 2\sum_{k=0}^{N} \Delta(\eta_k^T u_k) + \sum_{k\in J} \Omega_k^h(\eta,\xi) \\
&\qquad + \begin{pmatrix} -x_0-\eta_0 \\ x_{N+1}+\eta_{N+1}\end{pmatrix}^T \left\{ S\begin{pmatrix} -x_0-\eta_0 \\ x_{N+1}+\eta_{N+1}\end{pmatrix} + 2s\right\} - \begin{pmatrix} -x_0 \\ x_{N+1}\end{pmatrix}^T \left\{ S\begin{pmatrix} -x_0 \\ x_{N+1}\end{pmatrix} + 2s\right\} \\
&\quad = 2\sum_{k\in J} \eta_{k+1}^T\{C_k x_{k+1} - A_k^T u_k + b_k - \Delta u_k\} + 2\begin{pmatrix} -\eta_0 \\ \eta_{N+1}\end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1}\end{pmatrix} \\
&\qquad + \sum_{k\in J} \Omega_k^h(\eta,\xi) + 2\begin{pmatrix} -\eta_0 \\ \eta_{N+1}\end{pmatrix}^T \left\{ S\begin{pmatrix} -x_0 \\ x_{N+1}\end{pmatrix} + s\right\} + \begin{pmatrix} -\eta_0 \\ \eta_{N+1}\end{pmatrix}^T S\begin{pmatrix} -\eta_0 \\ \eta_{N+1}\end{pmatrix} \\
&\quad = 2\mathcal{F}_1(x,u,\eta,\xi) + \mathcal{F}_2(\eta,\xi).
\end{aligned} \tag{2.2.16}$$

These calculations prove our auxiliary result. □

Remark 2.2.4 (boundary value problem). The functional $\mathcal{F}_1$ from Lemma 2.2.3 is called the *first variation*, while $\mathcal{F}_2$ is the *second variation*, and the difference equation $\Delta u = Cx^\sigma - A^T u + b$ is referred to as being the *Euler-Lagrange equation* of the problem (2.2.6); $\Delta x = Ax^\sigma + Bu + a$ is called the *equation of motion*. In what follows we consider the *boundary value problem*

$$\begin{gathered}
\Delta x = Ax^\sigma + Bu + a, \qquad \Delta u = Cx^\sigma - A^T u + b, \\
(x,u) \in \mathcal{R}.
\end{gathered} \tag{2.2.17}$$

Here we write $(x,u)\in\mathcal{R}$ if (x,u) satisfies the *boundary condition*

$$R^*\begin{pmatrix} -x_0 \\ x_{N+1}\end{pmatrix} + R\begin{pmatrix} u_0 \\ u_{N+1}\end{pmatrix} = s^* - Rs, \tag{2.2.18}$$

where we put $R^* := RS + S^*$ with a matrix R satisfying $\operatorname{rank}\begin{pmatrix} R & S^*\end{pmatrix} = 2n$ and $\operatorname{Im} R^T = \operatorname{Ker} S^*$ (for the existence of such an R see [182, Corollary 3.1.3]). Besides the already introduced notation $x \in \tilde{\mathcal{R}}$ for $S^*\binom{-x_0}{x_{N+1}} = s^*$, we also use $\eta \in \tilde{\mathcal{R}}^h$ if $S^*\binom{-\eta_0}{\eta_{N+1}} = 0$ holds. Finally, an (x,u) satisfies the *natural boundary conditions* (we write $(x,u)\in\mathcal{R}_n$) whenever

$$R\left\{\begin{pmatrix} u_0 \\ u_{N+1}\end{pmatrix} + S\begin{pmatrix} -x_0 \\ x_{N+1}\end{pmatrix} + s\right\} = 0 \tag{2.2.19}$$

holds. Then it is a known fact (see [182, Lemma 2.3.2]) that the statement

$$(x,u)\in\mathcal{R}\iff x\in\tilde{\mathcal{R}},\ (x,u)\in\mathcal{R}_n \tag{2.2.20}$$

is true.

2.2.3. Solution of discrete variational problems

In the last subsection of this motivating section we now prove the main theorem on the solution of inhomogeneous discrete variational problems (2.2.6), which shows a connection between the variational problem (2.2.6), the boundary value problem (2.2.17), and the first and second variation. This theorem gives necessary as well as sufficient conditions for solvability of (2.2.6).

Theorem 2.2.5 (discrete variational problem). (i) *Let $(\hat{x},\hat{u})$ solve (2.2.17), and let*

$$\mathcal{F}_2(\eta,\xi)\geq 0\quad \forall\ h\text{-admissible }(\eta,\xi)\text{ with }\eta\in\tilde{\mathcal{R}}^h. \tag{2.2.21}$$

Then $(\hat{x},\hat{v})$ with $\hat{v}:=P^{-1}\{\mathcal{B}^T\hat{u}-\tilde{Q}^T\hat{x}^\sigma-p\}$ solves problem (2.2.6).

(ii) *Let $(\hat{x},\hat{v})$ solve (2.2.6). Then*

$$\mathcal{F}_1(\hat{x},\hat{u},\eta,\xi)=0,\quad \mathcal{F}_2(\eta,\xi)\geq 0\quad \forall\ h\text{-admissible }(\eta,\xi)\text{ with }\eta\in\tilde{\mathcal{R}}^h, \tag{2.2.22}$$

where we put

$$\begin{aligned}\hat{u}&:=\mathcal{B}\tilde{\mathcal{B}}^{-1}\{P\hat{v}+\tilde{Q}^T\hat{x}^\sigma+p\}\quad \text{on } J,\\ \hat{u}_{N+1}&:=(I-\mathcal{A}_{N+1}^T)\mathcal{B}_N\tilde{\mathcal{B}}_N^{-1}\{P_N\hat{v}_N+\tilde{Q}_N^T\hat{x}_N^\sigma+p\}+Q_N\hat{x}_{N+1}+\tilde{Q}_N\hat{v}_N+q_N.\end{aligned} \tag{2.2.23}$$

Proof. First of all, suppose $(\hat{x},\hat{u})$ solves the boundary value problem (2.2.17). Then we have $(\hat{x},\hat{u})\in\mathcal{R}$, that is, $\hat{x}\in\tilde{\mathcal{R}}$ and $(\hat{x},\hat{u})\in\mathcal{R}_n$ because of Remark 2.2.4, as well as

$$\Delta\hat{x}=A\hat{x}^\sigma+B\hat{u}+a,\qquad \Delta\hat{u}=C\hat{x}^\sigma-A^T\hat{u}+b. \tag{2.2.24}$$

We put $\hat{v}=P^{-1}\{\mathcal{B}^T\hat{u}-\tilde{Q}^T\hat{x}^\sigma-p\}$, and then it follows from (2.2.6) (see also Remark 2.2.2) that $(\hat{x},\hat{v})$ is admissible. Now let (x,v) be an arbitrary admissible pair with $x\in\tilde{\mathcal{R}}$. We define u as above in (ii) with x and v instead of $\hat{x}$ and $\hat{v}$. Then we have $v=P^{-1}\{\mathcal{B}^Tu-\tilde{Q}^Tx^\sigma-p\}$, and because of Lemma 2.2.1 (x,u) is also admissible. We proceed by putting

$$\eta:=x-\hat{x},\qquad \xi:=u-\hat{u},\qquad \zeta:=v-\hat{v}. \tag{2.2.25}$$

Obviously, (η, ξ) is h-admissible, and because of $\zeta = P^{-1}\{\mathcal{B}^T\xi - \tilde{Q}^T\eta^\sigma\}$, we may apply Lemma 2.2.3 to see that

$$\mathcal{F}(x,v) - \mathcal{F}(\hat{x},\hat{v}) = 2\mathcal{F}_1(\hat{x},\hat{u},\eta,\xi) + \mathcal{F}_2(\eta,\xi) \tag{2.2.26}$$

holds. Because of

$$S^*\begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix} = S^*\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} - S^*\begin{pmatrix} -\hat{x}_0 \\ \hat{x}_{N+1} \end{pmatrix} = s^* - s^* = 0, \tag{2.2.27}$$

we have $\eta \in \tilde{\mathcal{R}}^h$, and this yields $\mathcal{F}_2(\eta, \xi) \geq 0$ because of our assumptions. Finally, $(\hat{x}, \hat{u}) \in \mathcal{R}_n$ means that

$$R\left\{\begin{pmatrix} \hat{u}_0 \\ \hat{u}_{N+1} \end{pmatrix} + S\begin{pmatrix} -\hat{x}_0 \\ \hat{x}_{N+1} \end{pmatrix} + s\right\} = 0 \tag{2.2.28}$$

holds, and because we also have $\binom{-\eta_0}{\eta_{N+1}} \in \operatorname{Ker} S^* = \operatorname{Im} R^T$, it follows that

$$\begin{aligned} \mathcal{F}_1(\hat{x},\hat{u},\eta,\xi) &= \sum_{k\in J} \eta_{k+1}^T \{C_k\hat{x}_{k+1} - A_k^T\hat{u}_k + b_k - \Delta\hat{u}_k\} \\ &\quad + \begin{pmatrix} -\eta_0 \\ \eta_{N+1} \end{pmatrix}^T \left\{\begin{pmatrix} \hat{u}_0 \\ \hat{u}_{N+1} \end{pmatrix} + S\begin{pmatrix} -\hat{x}_0 \\ \hat{x}_{N+1} \end{pmatrix} + s\right\} \\ &= 0 \end{aligned} \tag{2.2.29}$$

holds. Altogether, this means

$$\mathcal{F}(x,v) - \mathcal{F}(\hat{x},\hat{v}) = \mathcal{F}_2(\eta,\xi) \geq 0, \tag{2.2.30}$$

that is, $\mathcal{F}(\hat{x},\hat{v}) \leq \mathcal{F}(x,v)$. Thus it is clear that $(\hat{x},\hat{v})$ solves problem (2.2.6) as stated.

To show (ii), let $(\hat{x},\hat{v})$ be a solution of (2.2.6). Then we define $\hat{u}$ as in (ii). Thus we again have $\hat{v} = P^{-1}\{\mathcal{B}^T\hat{u} - \tilde{Q}^T\hat{x}^\sigma - p\}$, and Lemma 2.2.1 yields the admissibility of $(\hat{x},\hat{u})$. Let (η,ξ) be h-admissible with $\eta \in \tilde{\mathcal{R}}^h$. For $\alpha \in \mathbb{R}$ we put

$$\eta^\alpha := \alpha\eta, \qquad \xi^\alpha := \alpha\xi, \qquad \zeta^\alpha := P^{-1}\left\{\mathcal{B}^T\xi^\alpha - \tilde{Q}^T(\eta^\sigma)^\alpha\right\}. \tag{2.2.31}$$

Hence (η^α,ξ^α) is h-admissible for each $\alpha \in \mathbb{R}$ with $\eta^\alpha \in \tilde{\mathcal{R}}^h$, and therefore we have $\hat{x} + \eta^\alpha \in \tilde{\mathcal{R}}$ as well as (with Lemma 2.2.1) the admissibility of $(\hat{x}+\eta^\alpha, \hat{v}+\zeta^\alpha)$ for all $\alpha \in \mathbb{R}$. Again we may apply Lemma 2.2.3 to obtain

$$\mathcal{F}(\hat{x}+\eta^\alpha,\hat{v}+\zeta^\alpha) - \mathcal{F}(\hat{x},\hat{v}) = 2\mathcal{F}_1(\hat{x},\hat{u},\eta^\alpha,\xi^\alpha) + \mathcal{F}_2(\eta^\alpha,\xi^\alpha) \tag{2.2.32}$$

for all $\alpha \in \mathbb{R}$. Since $(\hat{x}, \hat{v})$ solves problem (2.2.6), we have

$$0 \leq 2\mathcal{F}_1(\hat{x}, \hat{u}, \eta^\alpha, \xi^\alpha) + \mathcal{F}_2(\eta^\alpha, \xi^\alpha) = 2\alpha\mathcal{F}_1(\hat{x}, \hat{u}, \eta, \xi) + \alpha^2\mathcal{F}_2(\eta, \xi) \tag{2.2.33}$$

for all $\alpha \in \mathbb{R}$, that is,

$$\mathcal{F}_2(\eta, \xi) + \frac{2\mathcal{F}_1(\hat{x}, \hat{u}, \eta, \xi)}{\alpha} \geq 0 \quad \forall \alpha \in \mathbb{R} \setminus \{0\}. \tag{2.2.34}$$

This immediately yields $\mathcal{F}_1(\hat{x}, \hat{u}, \eta, \xi) = 0$ and therefore $\mathcal{F}_2(\eta, \xi) \geq 0$. □

Remark 2.2.6 (definiteness). The preceding theorem reduces solvability of an inhomogeneous discrete variational problem to the question in which case

$$\mathcal{F}_2(\eta, \xi) \geq 0 \quad \forall\ h\text{-admissible } (\eta, \xi) \text{ with } \eta \in \tilde{\mathcal{R}}^h \tag{2.2.35}$$

holds. Because of their importance, discrete quadratic functionals satisfying this condition will be called *positive semidefinite*, and we write $\mathcal{F}_2 \geq 0$. Note that $\mathcal{F}_2$ is a homogeneous functional and that both the equation of motion under consideration and the boundary condition are homogeneous as well.

An $\mathcal{F}_2$ with

$$\mathcal{F}_2(\eta, \xi) > 0 \quad \forall\ h\text{-admissible } (\eta, \xi) \text{ with } \eta \in \tilde{\mathcal{R}}^h,\ \eta \neq 0 \tag{2.2.36}$$

is said to be *positive definite*, and we write $\mathcal{F}_2 > 0$. If $(\hat{x}, \hat{u})$ solves problem (2.2.17) and if $\mathcal{F}_2 > 0$, then the solution $(\hat{x}, \hat{v})$ of (2.2.6) from Theorem 2.2.5(i) is unique in the sense that

$$\mathcal{F}(\hat{x}, \hat{v}) < \mathcal{F}(x, v) \quad \forall \text{ admissible } (x, v) \text{ with } x \in \tilde{\mathcal{R}},\ x \neq \hat{x} \tag{2.2.37}$$

holds. Conversely, the existence of a solution of (2.2.6) which is unique in the above sense implies the positive definiteness of $\mathcal{F}_2$.

2.3. Linear Hamiltonian difference systems

2.3.1. Notation and assumptions

The examinations and results of the preceding section now lead us to the study of the system (2.1.5), that is,

$$\Delta\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}\begin{pmatrix} x^\sigma \\ u \end{pmatrix} \quad \text{on } J, \tag{2.3.1}$$

which we call a *linear Hamiltonian difference system*. On J, the real $n \times n$-matrix-valued functions A, B, and C are supposed to satisfy throughout our general assumptions

$$\begin{aligned} \tilde{A} &:= (I - A)^{-1} \quad \text{exists on } J, \\ B, C & \quad \text{are symmetric on } J. \end{aligned} \tag{2.3.2}$$

While we use small letters for vector-valued solutions $x, u \in \mathbb{R}^n$ of (2.1.5) (i.e., for which $\Delta x = Ax^\sigma + Bu$ and $\Delta u = Cx^\sigma - A^T u$ hold), we abbreviate $n \times n$-matrix-valued X, U which satisfy

$$\Delta X = AX^\sigma + BU, \qquad \Delta U = CX^\sigma - A^T U \tag{2.3.3}$$

with capital letters.

Now we shortly discuss the assumptions on A, B, and C. First of all the symmetry assumption on B and C is very natural (also in view of Section 2.2) and it is not a big restriction at all. Note also that we neither assume $B \geq 0$ like in the continuous case (see [99, 182, 249, 251]) nor B nonsingular like in literature on the discrete case (see [28, 120, 121, 222]). Exactly by avoiding this nonsingularity assumption we will be allowed to give results for the well-discussed case of Sturm-Liouville difference equations of higher order (see [36, 156, 221]) in Section 2.7. Now we turn our attention to the assumptions on A. Obviously we may solve the equation $\Delta x_k = A_k x_{k+1} + B_k u_k$ for x_{k+1} whenever $I - A_k$ is invertible, and then we have

$$x_{k+1} = \tilde{A}_k x_k + \tilde{A}_k B_k u_k, \tag{2.3.4}$$

if we put as agreed $\tilde{A}_k = (I - A_k)^{-1}$. Then we may write system (2.1.5) equivalently as

$$\begin{pmatrix} x \\ u \end{pmatrix}^\sigma = \tilde{S} \begin{pmatrix} x \\ u \end{pmatrix} \quad \text{with } \tilde{S} = \begin{pmatrix} \tilde{A} & \tilde{A}B \\ C\tilde{A} & C\tilde{A}B + (\tilde{A}^{-1})^T \end{pmatrix}. \tag{2.3.5}$$

Here, $\tilde{S}$ is a *symplectic* $2n \times 2n$-matrix-valued function on J, that is,

$$\tilde{S}^T \mathcal{J} \tilde{S} = \mathcal{J} \quad \text{with the } 2n \times 2n\text{-matrix } \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{2.3.6}$$

and this is checked easily. Finally the system

$$\begin{pmatrix} x \\ u \end{pmatrix} = \tilde{S}^{-1} \begin{pmatrix} x \\ u \end{pmatrix}^\sigma \quad \text{with } \tilde{S}^{-1} = \begin{pmatrix} B\tilde{A}^T C + \tilde{A}^{-1} & -B\tilde{A}^T \\ -\tilde{A}^T C & \tilde{A}^T \end{pmatrix} \tag{2.3.7}$$

is also equivalent with (2.1.5). All these examinations now prove that, subject to our assumptions, each of the initial value problems

$$\begin{gathered}\Delta x = Ax^\sigma + Bu, \quad \Delta u = Cx^\sigma - A^T u \quad \text{on } J,\\ x_m = m_x, \qquad u_m = m_u\end{gathered} \tag{2.3.8}$$

with $m_x, m_u \in \mathbb{R}^n$, $m \in J^*$, and

$$\begin{gathered}\Delta X = AX^\sigma + BU, \quad \Delta U = CX^\sigma - A^T U \quad \text{on } J,\\ X_m = M_x, \qquad U_m = M_u\end{gathered} \tag{2.3.9}$$

with $n \times n$-matrices M_x, M_u, and $m \in J^*$ is uniquely solvable. The solution may be conveniently computed by recursion using the matrices $\tilde{S}_k$, $k \in J$. For example, the solution of (2.3.9) in case of $m = 0$ is given by

$$\begin{pmatrix} X_k \\ U_k \end{pmatrix} = \tilde{S}_{k-1}\tilde{S}_{k-2}\cdots\tilde{S}_0\begin{pmatrix} M_x \\ M_u \end{pmatrix}, \quad k \in J^*. \tag{2.3.10}$$

To end this introductory subsection we now define *transition matrices* for $k, m \in J^*$ with $k \geq m$,

$$\Phi_{km} := \begin{cases} \tilde{A}_{k-1}\tilde{A}_{k-2}\cdots\tilde{A}_m & \text{if } k > m,\\ I & \text{if } k = m,\end{cases} \tag{2.3.11}$$

and for $k \in J^*$,

$$\Phi_k := \Phi_{k0} = \begin{cases} \tilde{A}_{k-1}\tilde{A}_{k-2}\cdots\tilde{A}_0 & \text{if } k \neq 0,\\ I & \text{if } k = 0,\end{cases} \tag{2.3.12}$$

as well as *controllability matrices* for $k \in J^*$ and $m \in J$ with $m + k \in J^*$,

$$G_k^{(m)} := \begin{cases} \begin{pmatrix} \Phi_{m+k,m}B_m & \Phi_{m+k,m+1}B_{m+1} & \cdots & \Phi_{m+k,m+k-1}B_{m+k-1}\end{pmatrix}, & k \neq 0,\\ 0, & k = 0,\end{cases} \tag{2.3.13}$$

and for $k \in J^*$,

$$G_k := G_k^{(0)} = \begin{cases} \begin{pmatrix} \Phi_{k0}B_0 & \Phi_{k1} & \Phi_{k2}B_2 & \cdots & \Phi_{k,k-1}B_{k-1}\end{pmatrix}, & k \neq 0,\\ 0, & k = 0.\end{cases} \tag{2.3.14}$$

Using this notation we can give the following result.

Lemma 2.3.1. *Let (x, u) be a solution of $\Delta x = Ax^\sigma + Bu$ on J. Then*

$$x_k = \Phi_k x_0 + G_k \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \end{pmatrix} \quad \forall k \in J^*, \tag{2.3.15}$$

$$x_{k+m} = \Phi_{k+m,m} x_m + G_k^{(m)} \begin{pmatrix} u_m \\ \vdots \\ u_{m+k-1} \end{pmatrix} \quad \forall k, m \in J^* \text{ with } m+k \in J^*. \tag{2.3.16}$$

Proof. In order to obtain (2.3.15) (i.e., (2.3.16) for $m = 0$) by induction, note that

$$\tilde{A}_k \Phi_k = \Phi_{k+1}, \quad \begin{pmatrix} \tilde{A}_k G_k & \tilde{A}_k B_k \end{pmatrix} = G_{k+1} \quad \forall k \in J. \tag{2.3.17}$$

Because of

$$\begin{pmatrix} \tilde{A}_{k+m} G_k^{(m)} & \tilde{A}_{k+m} B_{k+m} \end{pmatrix} = G_{k+1}^{(m)} \quad \forall k, m \in J \text{ with } m+k \in J, \tag{2.3.18}$$

statement (2.3.16) follows as well using induction. □

2.3.2. Special solutions of the Hamiltonian system

Some solutions of system (2.1.5) with special properties are introduced in this subsection. The starting point for doing this is the following auxiliary result which we call as in the continuous case *Wronskian identity*, and this result holds of course as well for vector-valued solutions of (2.1.5).

Lemma 2.3.2 (Wronskian identity). *Let (X, U) and $(\tilde{X}, \tilde{U})$ be two $n \times n$-matrix-valued solutions of (2.1.5). Then*

$$X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv W \quad \text{on } J^* \tag{2.3.19}$$

with a (constant) matrix W.

Proof. If (X, U) and $(\tilde{X}, \tilde{U})$ are solutions of (2.1.5), then it follows, using the discrete product rule, that on J

$$\begin{aligned} \Delta\{X^T \tilde{U} - U^T \tilde{X}\} &= (X^\sigma)^T \Delta \tilde{U} + (\Delta X^T)\tilde{U} - \{(\Delta U^T)\tilde{X}^\sigma + U^T \Delta \tilde{X}\} \\ &= (X^\sigma)^T \{C\tilde{X}^\sigma - A^T \tilde{U}\} + \Big\{(X^\sigma)^T A^T + U^T B\Big\}\tilde{U} \\ &\quad - \Big\{(X^\sigma)^T C - U^T A\Big\}\tilde{X}^\sigma - U^T\{A\tilde{X}^\sigma + B\tilde{U}\} \\ &= 0 \end{aligned} \tag{2.3.20}$$

holds. Thus, $X^T \tilde{U} - U^T \tilde{X}$ is constant on J^*. □

Definition 2.3.3 (bases of the Hamiltonian system). (i) A solution (X, U) of (2.1.5) is called a *conjoined basis* of (2.1.5) whenever

$$X_k^T U_k \equiv U_k^T X_k, \qquad \operatorname{rank}\begin{pmatrix} X_k^T & U_k^T \end{pmatrix} \equiv n \tag{2.3.21}$$

holds on J^*.

(ii) Two conjoined bases (X, U), $(\tilde{X}, \tilde{U})$ of (2.1.5) are called *normalized conjoined bases* of (2.1.5) if the relation

$$X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv I \tag{2.3.22}$$

holds on J^*.

(iii) For $m \in J^*$ let (X, U) and $(\tilde{X}, \tilde{U})$ be those solutions of (2.1.5) with

$$X_m = 0, \qquad U_m = I, \qquad \tilde{X}_m = -I, \qquad \tilde{U}_m = 0. \tag{2.3.23}$$

Then (X, U) is referred to as the *principal solution* of (2.1.5) at m, while $(\tilde{X}, \tilde{U})$ is called the *associated solution* of (2.1.5) at m, and (X, U), $(\tilde{X}, \tilde{U})$ are said to be the *special normalized conjoined bases* of (2.1.5) at m.

Remark 2.3.4 (rules for normalized conjoined bases). Because of the Wronskian identity, it suffices to check the conditions $X_k^T U_k = U_k^T X_k$ from Definition 2.3.3(i) and $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k = I$ from (ii) at only one point $k \in J^*$. Since the examinations of the preceding subsection yield

$$\begin{pmatrix} X_{k+1} \\ U_{k+1} \end{pmatrix} = \tilde{S}_k \begin{pmatrix} X_k \\ U_k \end{pmatrix} \tag{2.3.24}$$

with an invertible matrix $\tilde{S}_k$ for each $k \in J$, we have

$$\operatorname{rank}\begin{pmatrix} X_{k+1}^T & U_{k+1}^T \end{pmatrix} = \operatorname{rank}\begin{pmatrix} X_k^T & U_k^T \end{pmatrix} \quad \forall k \in J, \tag{2.3.25}$$

such that the condition $\operatorname{rank}\begin{pmatrix} X_k^T & U_k^T \end{pmatrix} = n$ from Definition 2.3.3(i) has to be checked only at a single point $k \in J^*$, too. Hence, both the principal solution and the associated solution of (2.1.5) at m from Definition 2.3.3(iii) are conjoined bases of (2.1.5), and the special normalized conjoined bases of (2.1.5) at m are in fact normalized conjoined bases of (2.1.5).

Moreover note that for any conjoined basis (X, U) there exists another conjoined basis $(\tilde{X}, \tilde{U})$ such that (X, U), $(\tilde{X}, \tilde{U})$ are normalized conjoined bases. For this purpose one only needs to employ the solution $(\tilde{X}, \tilde{U})$ of problem (2.1.5) satisfying the initial conditions for some $m \in J^*$:

$$\begin{pmatrix} \tilde{X}_m \\ \tilde{U}_m \end{pmatrix} = \begin{pmatrix} -U_m \\ X_m \end{pmatrix} \left\{ X_m^T X_m + U_m^T U_m \right\}^{-1}. \tag{2.3.26}$$

Finally we emphasize (see also [182, Proposition 1.1.5]) that two solutions (X,U), $(\tilde{X},\tilde{U})$ of (2.1.5) are normalized conjoined bases of (2.1.5) if and only if the matrix

$$\begin{pmatrix} X_0 & \tilde{X}_0 \\ U_0 & \tilde{U}_0 \end{pmatrix} \tag{2.3.27}$$

is symplectic (cf. (2.3.6)). In this case all the matrices $\left(\begin{smallmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{smallmatrix}\right)$, $k \in J^*$, are symplectic, and we have

$$\begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{U}_k^T & -\tilde{X}_k^T \\ -U_k^T & X_k^T \end{pmatrix} \quad \forall k \in J^*. \tag{2.3.28}$$

Because we apply the resulting rules for normalized conjoined bases so many times in this chapter, we list them now separately: the formulas

$$\begin{gathered} X\tilde{U}^T - \tilde{X}U^T = \tilde{U}X^T - U\tilde{X}^T = X^T\tilde{U} - U^T\tilde{X} = \tilde{U}^TX - \tilde{X}^TU = I, \\ X^TU - U^TX = \tilde{X}^T\tilde{U} - \tilde{U}^T\tilde{X} = X\tilde{X}^T - \tilde{X}X^T = U\tilde{U}^T - \tilde{U}U^T = 0 \end{gathered} \tag{2.3.29}$$

hold on J^*, and on J the rules

$$\begin{gathered} \tilde{X}^\sigma X^T - X^\sigma \tilde{X}^T = \tilde{A}B, \qquad \tilde{U}^\sigma X^T - U^\sigma \tilde{X}^T = C\tilde{A}B + (\tilde{A}^{-1})^T, \\ X^\sigma \tilde{U}^T - \tilde{X}^\sigma U^T = \tilde{A}, \qquad U^\sigma \tilde{U}^T - \tilde{U}^\sigma U^T = C\tilde{A} \end{gathered} \tag{2.3.30}$$

are valid.

Lemma 2.3.5. *Let be given a conjoined basis (X,U) of (2.1.5). Let*

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \tag{2.3.31}$$

hold for some $k \in J$. Then

$$\operatorname{Ker} X_{k+1}^T \subset \operatorname{Ker} B_k\tilde{A}_k^T, \tag{2.3.32}$$

$$x_k \in \operatorname{Im} X_k, \qquad \Delta x_k = A_k x_{k+1} + B_k u_k \quad \textit{imply} \quad x_{k+1} \in \operatorname{Im} X_{k+1}. \tag{2.3.33}$$

Proof. Let (X, U), $(\tilde{X}, \tilde{U})$ be normalized conjoined bases of (2.1.5) (for the construction of $(\tilde{X}, \tilde{U})$ we refer to (2.3.26)). Let $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ hold for some $k \in J$. With $c \in \operatorname{Ker} X_{k+1}^T$ it follows from (2.3.29) that

$$X_{k+1}\tilde{X}_{k+1}^T c = \tilde{X}_{k+1}X_{k+1}^T c = 0, \tag{2.3.34}$$

that is, $\tilde{X}_{k+1}^T c \in \operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ holds. Using (2.3.30), this yields

$$0 = X_k\tilde{X}_{k+1}^T c = \{\tilde{X}_k X_{k+1}^T + B_k\tilde{A}_k^T\}c = B_k\tilde{A}_k^T c. \tag{2.3.35}$$

Thus we have $\operatorname{Ker} X_{k+1}^T \subset \operatorname{Ker} B_k\tilde{A}_k^T$, and (2.3.32) is already shown.

Now let

$$x_k = X_k c \in \operatorname{Im} X_k \quad \text{with } \Delta x_k = A_k x_{k+1} + B_k u_k. \tag{2.3.36}$$

Then

$$\begin{aligned} x_{k+1} &= \tilde{A}_k x_k + \tilde{A}_k B_k u_k \\ &= \tilde{A}_k X_k c + \tilde{A}_k B_k u_k \\ &= X_{k+1}c + \tilde{A}_k B_k (u_k - U_k c) \in \operatorname{Im} X_{k+1}, \end{aligned} \tag{2.3.37}$$

since (2.3.32) yields $\operatorname{Im} \tilde{A}_k B_k \subset \operatorname{Im} X_{k+1}$. Hence the proof of statement (2.3.33) is complete. □

The following auxiliary result (cf. also [182, Proposition 1.1.6]) gives a further characterization of normalized conjoined bases.

Lemma 2.3.6 ("big" Hamiltonian system). *Define $2n \times 2n$-matrix-valued functions A^*, B^*, and C^* by*

$$A^* := \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \qquad B^* := \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \qquad C^* := \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix}, \tag{2.3.38}$$

and moreover for $n \times n$-matrix-valued X, $\tilde{X}$, U, $\tilde{U}$,

$$X^* := \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix}, \qquad U^* := \begin{pmatrix} I & 0 \\ U & \tilde{U} \end{pmatrix}. \tag{2.3.39}$$

Then (X, U), $(\tilde{X}, \tilde{U})$ are normalized conjoined bases of (2.1.5) if and only if (X^*, U^*) constitutes a conjoined basis of the ("big") system

$$\Delta\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A^* & B^* \\ C^* & -A^{*T} \end{pmatrix}\begin{pmatrix} x^\sigma \\ u \end{pmatrix} \quad \text{on } J. \tag{2.3.40}$$

Proof. The relations

$$\begin{aligned} X^{*^T}U^* &= \begin{pmatrix} X^TU & X^T\tilde{U} \\ I+\tilde{X}^TU & \tilde{X}^T\tilde{U} \end{pmatrix}, \qquad \operatorname{rank}\begin{pmatrix} X^{*^T} & U^{*^T} \end{pmatrix} = 2n, \\ \tilde{A}^*(X^*+B^*U^*) &= \begin{pmatrix} 0 & I \\ \tilde{A}(X+BU) & \tilde{A}(\tilde{X}+B\tilde{U}) \end{pmatrix}, \\ C^*(X^*)^\sigma + (I-A^{*^T})U^* &= \begin{pmatrix} I & 0 \\ CX^\sigma + (I-A^T)U & C\tilde{X}^\sigma + (I-A^T)\tilde{U} \end{pmatrix}, \end{aligned} \tag{2.3.41}$$

immediately prove the statement. □

2.3.3. Riccati matrix difference equations

As in the continuous case (see [182, Proposition 1.1.2]) there is an intimate connection between solutions of the linear Hamiltonian difference system and Riccati matrix difference equations in the present discrete case also. First of all, for a symmetric $n\times n$-matrix-valued Q on J^* we define a *Riccati operator* by

$$(R[Q]_k)_{k\in J} = R[Q] := \tilde{A}^T(Q^\sigma - C)\tilde{A}(I+BQ) - Q, \tag{2.3.42}$$

and show the following first auxiliary result.

Lemma 2.3.7. *Let Q be symmetric on J^*. Let*

$$\Delta X_k = A_kX_{k+1} + B_kU_k, \qquad \operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \tag{2.3.43}$$

hold for some $k\in J$.

(i) *If $QX = UX^\dagger X$ holds on J^*, then*

$$R[Q]_kX_k = \tilde{A}_k^T\{\Delta U_k - C_kX_{k+1} + A_k^TU_k\}X_k^\dagger X_k. \tag{2.3.44}$$

(ii) *If $\Delta U_k = C_kX_{k+1} - A_k^TU_k$ and $X^TQX = X^TU$ on J^*, then*

$$D_k := X_kX_{k+1}^\dagger \tilde{A}_kB_k \quad \text{is symmetric.} \tag{2.3.45}$$

More precisely,

$$D_k = B_k - B_k\tilde{A}_k^T(Q_{k+1}-C_k)\tilde{A}_kB_k. \tag{2.3.46}$$

In this case

$$X_k^TR[Q]_kX_k = B_kR[Q]_kX_k = 0 \tag{2.3.47}$$

is also valid. If $QX = UX^\dagger X$ on J^* is assumed, then

$$R[Q]_kX_k = 0. \tag{2.3.48}$$

Proof. First of all we assume $\Delta X_k = A_k X_{k+1} + B_k U_k$ and $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$. Because of (2.3.32) from Lemma 2.3.5 and Lemma 2.8.6 from Section 2.8 on Moore-Penrose inverses, the three formulas

$$X_k = X_k X_{k+1}^\dagger X_{k+1}, \qquad X_{k+1}^\dagger X_{k+1} X_k^\dagger = X_k^\dagger, \qquad \tilde{A}_k B_k = X_{k+1} X_{k+1}^\dagger \tilde{A}_k B_k \tag{2.3.49}$$

hold. By putting $Z := U - QX$, the calculation

$$\begin{aligned} R[Q]_k X_k &= \tilde{A}_k^T (Q_{k+1} - C_k)(\tilde{A}_k X_k + \tilde{A}_k B_k Q_k X_k) X_k^\dagger X_k - Q_k X_k \\ &= \tilde{A}_k^T (Q_{k+1} - C_k)(X_{k+1} - \tilde{A}_k B_k Z_k) X_k^\dagger X_k - Q_k X_k \\ &= \tilde{A}_k^T (Q_{k+1} X_{k+1} - C_k X_{k+1}) X_{k+1}^\dagger (X_{k+1} - \tilde{A}_k B_k Z_k) X_k^\dagger X_k - Q_k X_k \\ &= \tilde{A}_k^T (\Delta U_k - C_k X_{k+1} + A_k^T U_k - Z_{k+1}) X_{k+1}^\dagger (X_{k+1} - \tilde{A}_k B_k Z_k) X_k^\dagger X_k \\ &\quad + U_k X_{k+1}^\dagger (X_{k+1} - \tilde{A}_k B_k Z_k) X_k^\dagger X_k - Q_k X_k X_k^\dagger X_k \\ &= \tilde{A}_k^T (\Delta U_k - C_k X_{k+1} + A_k^T U_k - Z_{k+1}) X_{k+1}^\dagger (X_{k+1} - \tilde{A}_k B_k Z_k) X_k^\dagger X_k \\ &\quad + Z_k X_k^\dagger X_k - U_k X_{k+1}^\dagger \tilde{A}_k B_k Z_k X_k^\dagger X_k \end{aligned} \tag{2.3.50}$$

shows that in case of $QX = UX^\dagger X$, that is, of $ZX^\dagger X = 0$, the equation

$$R[Q]_k X_k = \tilde{A}_k^T \{\Delta U_k - C_k X_{k+1} + A_k^T U_k\} X_k^\dagger X_k \tag{2.3.51}$$

holds.

We now assume additionally $\Delta U_k = C_k X_{k+1} - A_k^T U_k$. This obviously then implies $R[Q]_k X_k = 0$. We now want to replace the assumption $QX = UX^\dagger X$ by the weaker assumption $X^T Q X = X^T U$. Then (2.3.7) yields

$$\begin{aligned} D_k &= X_k X_{k+1}^\dagger \tilde{A}_k B_k \\ &= \{(B_k \tilde{A}_k^T C_k + \tilde{A}_k^{-1}) X_{k+1} - B_k \tilde{A}_k^T U_{k+1}\} X_{k+1}^\dagger \tilde{A}_k B_k \\ &= B_k \tilde{A}_k^T C_k \tilde{A}_k B_k + B_k - B_k \tilde{A}_k^T (X_{k+1}^\dagger)^T X_{k+1}^T U_{k+1} X_{k+1}^\dagger \tilde{A}_k B_k \\ &= B_k \tilde{A}_k^T C_k \tilde{A}_k B_k + B_k - B_k \tilde{A}_k^T (X_{k+1}^\dagger)^T X_{k+1}^T Q_{k+1} X_{k+1} X_{k+1}^\dagger \tilde{A}_k B_k \\ &= B_k - B_k \tilde{A}_k^T (Q_{k+1} - C_k) \tilde{A}_k B_k. \end{aligned} \tag{2.3.52}$$

Hence D_k is symmetric. Using

$$\tilde{A}_k X_k = X_{k+1} - \tilde{A}_k B_k U_k = X_{k+1} X_{k+1}^\dagger (X_{k+1} - \tilde{A}_k B_k U_k) = X_{k+1} X_{k+1}^\dagger \tilde{A}_k X_k, \tag{2.3.53}$$

$X^T Z = 0$, and (2.3.50), we have

$$\begin{aligned}
X_k^T R[Q]_k X_k &= X_k^T U_k X_{k+1}^\dagger \tilde{A}_k B_k Z_k X_k^\dagger X_k \\
&= U_k^T D_k Z_k X_k^\dagger X_k \\
&= U_k^T D_k^T Z_k X_k^\dagger X_k \\
&= U_k^T B_k \tilde{A}_k^T (X_{k+1}^\dagger)^T X_k^T Z_k X_k^\dagger X_k \\
&= 0, \qquad (2.3.54) \\
B_k R[Q]_k X_k &= B_k Z_k X_k^\dagger X_k - \{\tilde{A}_k^{-1} X_{k+1} - X_k\} X_{k+1}^\dagger \tilde{A}_k B_k Z_k X_k^\dagger X_k \\
&= D_k Z_k X_k^\dagger X_k \\
&= D_k^T Z_k X_k^\dagger X_k \\
&= 0,
\end{aligned}$$

so that the proof of this technical auxiliary result is complete. □

The next result gives information on the existence of some Q with the properties required in Lemma 2.3.7.

Lemma 2.3.8. *Let be given a symplectic $2n \times 2n$-matrix $\begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix}$. Then*

$$Q := UX^\dagger + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X)U^T \tag{2.3.55}$$

is symmetric, and $QX = UX^\dagger X$ holds. Furthermore

$$Q^* := \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X & X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X)U^T \\ \{X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X)U^T\}^T & UX^\dagger + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X)U^T \end{pmatrix} \tag{2.3.56}$$

is symmetric and satisfies $Q^* X^* = U^* X^{*\dagger} X^*$ (with the notation (2.3.39)).

Proof. Using (2.3.29), we obtain

$$(I - X^\dagger X)U^T X = (I - X^\dagger X)X^T U = \{X(I - X^\dagger X)\}^T U = 0, \tag{2.3.57}$$

and thus $QX = UX^\dagger X$ is shown. Now the calculation

$$\begin{aligned}
Q &= UX^\dagger + UX^\dagger \tilde{X} U^T - \tilde{U} U^T - UX^\dagger \tilde{X} X^\dagger X U^T + \tilde{U} X^\dagger X U^T \\
&= UX^\dagger X \tilde{U}^T - \tilde{U} U^T - UX^\dagger \tilde{X} X^\dagger X U^T + \tilde{U} X^\dagger X U^T
\end{aligned} \tag{2.3.58}$$

(note $X^\dagger \tilde{X} X^\dagger X = X^\dagger \tilde{X} X^T (X^\dagger)^T$) proves the symmetry of Q.

From what we have shown so far the symmetry of Q^* follows, and by applying (2.3.29), we find

$$
\begin{aligned}
Q^*X^* &= Q^* \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix} \\
&= \begin{pmatrix} X^\dagger X & -X^\dagger \tilde{X} X^\dagger X + X^\dagger \tilde{X} + X^\dagger \tilde{X}(I - X^\dagger X)U^T \tilde{X} \\ UX^\dagger X & M \end{pmatrix} \\
&= \begin{pmatrix} X^\dagger X & -X^\dagger \tilde{X} X^\dagger X + X^\dagger \tilde{X} - X^\dagger \tilde{X}(I - X^\dagger X) \\ UX^\dagger X & M \end{pmatrix} \\
&= \begin{pmatrix} X^\dagger X & 0 \\ UX^\dagger X & \tilde{U} \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ U & \tilde{U} \end{pmatrix} \begin{pmatrix} X^\dagger X & 0 \\ 0 & I \end{pmatrix} \\
&= U^* X^{*\dagger} X^*,
\end{aligned}
\tag{2.3.59}
$$

where we put

$$
\begin{aligned}
M &:= (X^\dagger)^T + U(I - X^\dagger X)\tilde{X}^T (X^\dagger)^T + UX^\dagger \tilde{X} + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X)U^T \tilde{X} \\
&= (I + U\tilde{X}^T)(X^\dagger)^T - UX^\dagger \tilde{X} X^\dagger X + UX^\dagger \tilde{X} - (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X) \\
&= \tilde{U}.
\end{aligned}
\tag{2.3.60}
$$

Hence all statements are proved. In (2.3.59), we made use of

$$
X^{*\dagger} X^* = \begin{pmatrix} X^\dagger X & 0 \\ 0 & I \end{pmatrix}, \tag{2.3.61}
$$

and the validity of (2.3.61) is ensured by Lemma 2.8.7 from Section 2.8 on Moore-Penrose inverses. □

Remark 2.3.9. It would be nice to solve the equation $R[Q]_k X_k = 0$ for Q_{k+1} in order to compute a solution Q by recursion. This is possible whenever both X_k and $I + B_k Q_k$ are invertible, and then we have

$$
\begin{aligned}
Q_{k+1} &= C_k + (I - A_k^T)Q_k(I + B_k Q_k)^{-1}(I - A_k) \\
&= C_k + (I - A_k^T)Q_k\left\{(I + B_k Q_k)^{-1} - I\right\}(I - A_k) + (I - A_k^T)Q_k(I - A_k) \\
&= C_k - (I - A_k^T)Q_k(I + B_k Q_k)^{-1} B_k Q_k(I - A_k) + (I - A_k^T)Q_k(I - A_k).
\end{aligned}
\tag{2.3.62}
$$

Hence Q is a solution of the difference equation

$$\begin{aligned}\Delta Q_k &= C_k - A_k^T Q_k - Q_k A_k + A_k^T Q_k A_k \\ &\quad - (I - A_k^T) Q_k \left\{ (I + B_k Q_k)^{-1} B_k \right\} Q_k (I - A_k),\end{aligned} \tag{2.3.63}$$

which is called a *Riccati matrix difference equation.*

The last result of this preliminary section now gives a criterion for the existence (and this includes the invertibility of $I + BQ$) of a symmetric solution Q of the Riccati matrix difference equation (cf. also [124, Theorem 6]).

Lemma 2.3.10 ("equivalence" with a Riccati equation). *Let $\mu, \nu \in J^*$ with $\mu < \nu$. There exist symmetric matrices Q_k, $\mu \le k \le \nu$, with (2.1.7), that is,*

$$Q_{k+1} = C_k + (I - A_k^T) Q_k (I + B_k Q_k)^{-1} (I - A_k), \quad \mu \le k \le \nu - 1, \tag{2.3.64}$$

if and only if there is a conjoined basis (X, U) of (2.1.5) with invertible matrices X_k, $\mu \le k \le \nu$. In this case, $Q_k := U_k X_k^{-1}$, $\mu \le k \le \nu$, are then symmetric matrices satisfying (2.1.7) on $[\mu, \nu - 1] \cap \mathbb{Z}$, and

$$X_k X_{k+1}^{-1} \tilde{A}_k B_k = (I + B_k Q_k)^{-1} B_k \quad \forall \mu \le k \le \nu - 1. \tag{2.3.65}$$

Proof. Let $\tilde{J} = [\mu, \nu - 1] \cap \mathbb{Z}$ and $\tilde{J}^* = [\mu, \nu] \cap \mathbb{Z}$. If there exists a conjoined basis (X, U) of (2.1.5) with invertible matrices X_k, $k \in \tilde{J}^*$, then

$$Q = UX^{-1} \quad \text{on } \tilde{J}^* \tag{2.3.66}$$

is symmetric and satisfies $X^T Q X = X^T U$. Note that (2.3.29) yields $X^T U = U^T X$ and thus

$$UX^{-1} = (X^{-1})^T U^T = (UX^{-1})^T \quad \text{on } \tilde{J}^*. \tag{2.3.67}$$

Because of Lemma 2.3.7(ii), we then have $R[Q]X = 0$ on $\tilde{J}$ and hence $R[Q] = 0$, and Remark 2.3.9 shows that Q satisfies (2.1.7) on $\tilde{J}$ because

$$I + B_k Q_k = (X_k + B_k U_k) X_k^{-1} = (I - A_k) X_{k+1} X_k^{-1} \tag{2.3.68}$$

is invertible for $k \in \tilde{J}$.

Conversely we assume Q_k, $k \in \tilde{J}^*$, to be symmetric and to solve equation (2.1.7). We put

$$\begin{aligned} X_\mu = I, \qquad X_{k+1} &= \tilde{A}_k (I + B_k Q_k) X_k, \quad k \in \tilde{J}, \\ U_k &= Q_k X_k, \quad k \in \tilde{J}^*. \end{aligned} \tag{2.3.69}$$

Then X_k are invertible for $k \in \tilde{J}^*$, and for $k \in \tilde{J}$,

$$\begin{aligned} A_k X_{k+1} + B_k U_k &= \{I - (I - A_k)\} \tilde{A}_k (I + B_k Q_k) X_k + B_k Q_k X_k \\ &= X_{k+1} - X_k \\ &= \Delta X_k. \end{aligned} \tag{2.3.70}$$

Since $QX = U$ on $\tilde{J}^*$, Lemma 2.3.7(i) yields

$$0 = R[Q]_k X_k = \tilde{A}_k^T \{\Delta U_k - C_k X_{k+1} + A_k^T U_k\}, \tag{2.3.71}$$

and thus $\Delta U_k = C_k X_{k+1} + A_k^T U_k$ for $k \in \tilde{J}$. By extending to a solution (X, U) of (2.1.5) on the whole interval J^*, we obtain because of

$$X_\mu^T U_\mu = Q_\mu, \qquad \operatorname{rank} \begin{pmatrix} X_\mu^T & U_\mu^T \end{pmatrix} = n, \tag{2.3.72}$$

a conjoined basis (X, U) of (2.1.5) with invertible matrices X_k, $k \in \tilde{J}^*$. $\square$

2.4. The discrete Picone formula

2.4.1. Discrete quadratic functionals

The contents of Section 2.2 serve to motivate the following terminologies and definitions. Again we are given (as in Section 2.2) a symmetric $2n \times 2n$-matrix S. The present chapter mainly deals with the discrete quadratic functional

$$\mathcal{F}(x, u) := \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}. \tag{2.4.1}$$

The following definitions (cf. also Remark 2.2.6) are basic. We fix an arbitrary $2n \times 2n$-matrix R for declaring certain boundary conditions. In addition we put for convenience

$$\mathcal{F}_0(x, u) := \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}. \tag{2.4.2}$$

Definition 2.4.1 (positive definiteness). (i) A pair (x,u) with $x,u : J^* \to \mathbb{R}^n$ is called *admissible* (with respect to (2.1.5)) if it satisfies the equation of motion

$$\Delta x = Ax^\sigma + Bu \quad \text{on } J. \tag{2.4.3}$$

Moreover an $x : J^* \to \mathbb{R}^n$ is said to be admissible, if there exists $u : J^* \to \mathbb{R}^n$ such that (x,u) is admissible.

(ii) An $x : J^* \to \mathbb{R}^n$ satisfies the boundary condition in case of

$$\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} R^T, \tag{2.4.4}$$

and then this is abbreviated by $x \in \tilde{\mathcal{R}}$.

(iii) The functional $\mathcal{F}$ is called *positive definite* (write $\mathcal{F} > 0$), if $\mathcal{F}(x,u)$ is positive for each admissible (x,u) with $x \in \tilde{\mathcal{R}}$ and $x \neq 0$. Finally, say $\mathcal{F}_0$ is positive definite and write $\mathcal{F}_0 > 0$, whenever $\mathcal{F}_0(x,u) > 0$ holds for each admissible (x,u) with $x \neq 0$ and $x_0 = x_{N+1} = 0$.

Remark 2.4.2. Note that the present functional $\mathcal{F}$ is exactly the second variation $\mathcal{F}_2$ from Section 2.2.

In case of $R = 0$ we obviously have $x \in \tilde{\mathcal{R}}$ if and only if both x_0 and x_{N+1} vanish. Thus $\mathcal{F}_0$ is positive definite if and only if $\mathcal{F} > 0$ with $R = S = 0$. We will first deal with this case $R = S = 0$ but will then remove this restricting assumption with the aid of Lemma 2.3.6.

Lemma 2.4.3. *For admissible* (x,u)

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} x_{k+1}^T\{C_k x_{k+1} - A_k^T u_k - \Delta u_k\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix}. \tag{2.4.5}$$

Proof. Let (x,u) be admissible. Then the computation

$$\begin{aligned}
&\sum_{k=0}^{N} x_{k+1}^T\{C_k x_{k+1} - A_k^T u_k - \Delta u_k\} \\
&\quad = \sum_{k=0}^{N} \{x_{k+1}^T(C_k x_{k+1} - A_k^T u_k) + (\Delta x_k^T)u_k - \Delta(x_k^T u_k)\} \\
&\quad = \sum_{k=0}^{N} \Big\{x_{k+1}^T C_k x_{k+1} + (\Delta x_k - A_k x_{k+1})^T u_k\Big\} - \sum_{k=0}^{N} \Delta\{x_k^T u_k\} \\
&\quad = \sum_{k=0}^{N} \Big\{x_{k+1}^T C_k x_{k+1} + (B_k u_k)^T u_k\Big\} - \{x_{N+1}^T u_{N+1} - x_0^T u_0\} \\
&\quad = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} - \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix}
\end{aligned} \tag{2.4.6}$$

shows the validity of our claim. □

Corollary 2.4.4. *If (x,u) is a solution of system (2.1.5), then*

$$\mathcal{F}_0(x,u) = \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix}. \tag{2.4.7}$$

Proof. Since (x,u) is a solution of (2.1.5), it is not only admissible but also satisfies

$$\Delta u_k = C_k x_{k+1} - A_k^T u_k \quad \forall k \in J. \tag{2.4.8}$$

Thus the preceding Lemma 2.4.3 yields the claim. □

Lemma 2.4.5. *Let (X,U) be a conjoined basis of system (2.1.5). For $m \in J$, set $D_m = X_m X_{m+1}^{\dagger} \tilde{A}_m B_m$. Then the following holds.*

(i) *If* $\operatorname{Ker} X_{m+1} \not\subset \operatorname{Ker} X_m$*, then there exists an admissible (x,u) with*

$$x_0 = X_0 d \in \operatorname{Im} X_0, \qquad x_{N+1} = 0, \qquad x \neq 0, \qquad \mathcal{F}_0(x,u) = -d^T X_0^T U_0 d. \tag{2.4.9}$$

(ii) *If* $\operatorname{Ker} X_{m+1} \subset \operatorname{Ker} X_m$*, then for each $c \in \mathbb{R}^n$ there is an admissible (x,u) with*

$$x_0 = X_0 d \in \operatorname{Im} X_0, \qquad x_{N+1} = 0, \qquad \mathcal{F}_0(x,u) = c^T D_m c - d^T X_0^T U_0 d. \tag{2.4.10}$$

Proof. First of all let the assumptions from part (i) be satisfied. Then there exists $d \in \operatorname{Ker} X_{m+1} \setminus \operatorname{Ker} X_m$. We now put

$$\begin{aligned} x_k &:= \begin{cases} X_k d & \text{for } 0 \le k \le m, \\ 0 & \text{for } m+1 \le k \le N+1, \end{cases} \\ u_k &:= \begin{cases} U_k d & \text{for } 0 \le k \le m, \\ 0 & \text{for } m+1 \le k \le N+1. \end{cases} \end{aligned} \tag{2.4.11}$$

Hence $x_0 = X_0 d \in \operatorname{Im} X_0$, $x_{N+1} = 0$, and $x_m \neq 0$. The admissibility of (x,u) follows from

$$\tilde{A}_m (x_m + B_m u_m) = \tilde{A}_m (X_m d + B_m U_m d) = X_{m+1} d = 0 = x_{m+1}. \tag{2.4.12}$$

Formula (2.4.5) from Lemma 2.4.3 now yields

$$\begin{aligned} \mathcal{F}_0(x,u) &= \sum_{k=0}^{N} x_{k+1}^T \{C_k x_{k+1} - A_k^T u_k - \Delta u_k\} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} \\ &= \sum_{k=0}^{m-1} x_{k+1}^T \{C_k x_{k+1} - A_k^T u_k - \Delta u_k\} - x_0^T u_0 \\ &= -d^T X_0^T U_0 d. \end{aligned} \tag{2.4.13}$$

Now we suppose that the assumptions from part (ii) are satisfied. Then D_m is symmetric because of Lemma 2.3.7(ii). Let $c \in \mathbb{R}^n$, $d := -X_{m+1}^{\dagger}\tilde{A}_m B_m c$, and put

$$x_k := \begin{cases} X_k d & \text{for } 0 \le k \le m, \\ 0 & \text{for } m+1 \le k \le N+1, \end{cases}$$
$$u_k := \begin{cases} U_k d & \text{for } 0 \le k \le m-1, \\ \tilde{A}_m^T (X_{m+1}^{\dagger})^T X_m^T c & \text{for } k = m, \\ 0 & \text{for } m+1 \le k \le N+1. \end{cases} \tag{2.4.14}$$

Then we have as before $x_0 = X_0 d \in \operatorname{Im} X_0$ and $x_{N+1} = 0$. From

$$\begin{aligned} \tilde{A}_m(x_m + B_m u_m) &= \tilde{A}_m\Big\{ - D_m c + B_m \tilde{A}_m^T (X_{m+1}^{\dagger})^T X_m^T c\Big\} \\ &= \tilde{A}_m(- D_m + D_m^T)c \\ &= 0 \\ &= x_{m+1}, \end{aligned} \tag{2.4.15}$$

the admissibility of (x, u) follows, and (2.4.5) shows the desired statement

$$\begin{aligned} \mathcal{F}_0(x,u) + d^T X_0^T U_0 d &= \sum_{k=0}^{m-1} x_{k+1}^T \{C_k x_{k+1} + (I - A_k^T)u_k - u_{k+1}\} \\ &= x_m^T\{C_{m-1}x_m + (I - A_{m-1}^T)u_{m-1} - u_m\} \\ &= d^T X_m^T\Big\{C_{m-1}X_m d + (I - A_{m-1}^T)U_{m-1}d - \tilde{A}_m^T (X_{m+1}^{\dagger})^T X_m^T c\Big\} \\ &= d^T X_m^T U_m d - d^T\{X_{m+1}^T - U_m^T B_m \tilde{A}_m^T\}(X_{m+1}^{\dagger})^T X_m^T c \\ &= d^T X_m^T U_m d - d^T X_m^T c + d^T U_m^T D_m c \\ &= d^T X_m^T U_m d + c^T B_m \tilde{A}_m^T (X_{m+1}^{\dagger})^T X_m^T c - d^T U_m^T X_m d \\ &= c^T D_m^T c \\ &= c^T D_m c, \end{aligned} \tag{2.4.16}$$

so that the proof of our auxiliary result is complete. □

The goal of the next subsection is to "complete the square" with the functional $\mathcal{F}_0$ in a suitable way suggested by the following auxiliary result.

Lemma 2.4.6. *If there exist symmetric $n \times n$-matrices Q_k, $z_k \in \mathbb{R}^n$, $k \in J^*$, and nonnegative definite $n \times n$-matrices D_k, $k \in J$, with*

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} z_k^T D_k z_k, \quad x + Dz = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma \quad \text{on } J \\ \text{for all admissible } (x,u) \text{ with } x_0 = x_{N+1} = 0, \tag{2.4.17}$$

then $\mathcal{F}_0 > 0$.

Proof. Let (x,u) be admissible with $x_0 = x_{N+1} = 0$. Then the above assumptions immediately yield $\mathcal{F}_0(x,u) \geq 0$. Now we suppose that $\mathcal{F}_0(x,u)$ vanishes. Then

$$D_k z_k = 0 \quad \forall 0 \leq k \leq N, \tag{2.4.18}$$

because of $D \geq 0$ on J. It follows that

$$x = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma \tag{2.4.19}$$

holds on J. Since $x_{N+1} = 0$, this relation now implies

$$x_{N+1} = x_N = x_{N-1} = \cdots = x_1 = x_0 = 0. \tag{2.4.20}$$

Hence we have $\mathcal{F}_0(x,u) > 0$ for all admissible (x,u) with $x \neq 0$ and $x_0 = x_{N+1} = 0$, that is, $\mathcal{F}_0 > 0$. ☐

2.4.2. The proof of the discrete Picone identity

The following lemma, in which we again as in (2.3.42) put

$$R[Q] = \tilde{A}^T(Q^\sigma - C)\tilde{A}(I + BQ) - Q, \tag{2.4.21}$$

is the foundation of our examinations in this subsection.

Lemma 2.4.7. *Let (x,u) be admissible and Q symmetric on J^*. Set*

$$z := u - Qx, \qquad D := B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B. \tag{2.4.22}$$

Then

$$\Delta\{x^T Q x\} - (x^\sigma)^T C x^\sigma - u^T B u + z^T D z = x^T\{R^T[Q] - QBR[Q]\}x + 2u^T BR[Q]x, \\ x + Dz - BR[Q]x = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma. \tag{2.4.23}$$

Proof. In order to shorten our calculations we first of all put

$$\tilde{Q} := \tilde{A}^T(Q^\sigma - C)\tilde{A}. \tag{2.4.24}$$

Then we have

$$\begin{aligned}
&\Delta\{x^TQx\} - (x^\sigma)^T Cx^\sigma - u^TBu + z^TDz\\
&\quad = (x+Bu)^T\tilde{A}^TQ^\sigma\tilde{A}(x+Bu) - x^TQx - (x+Bu)^T\tilde{A}^TC\tilde{A}(x+Bu)\\
&\qquad - u^TBu + (u-Qx)^T\{B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B\}(u-Qx)\\
&\quad = (x+Bu)^T\tilde{Q}(x+Bu) - x^TQx - u^TBu + (u-Qx)^T\{B - B\tilde{Q}B\}(u-Qx)\\
&\quad = x^T\{\tilde{Q} - Q + Q(B - B\tilde{Q}B)Q\}x + u^T\{B\tilde{Q}B - B + (B - B\tilde{Q}B)\}u\\
&\qquad + 2u^T\{B\tilde{Q} - (B - B\tilde{Q}B)Q\}x\\
&\quad = x^T\{\tilde{Q} + QB\tilde{Q} - Q - QB\tilde{Q} - QB\tilde{Q}BQ + QBQ\}x\\
&\qquad + 2u^T\{B\tilde{Q} + B\tilde{Q}BQ - BQ\}x\\
&\quad = x^T\{(I+QB)\tilde{Q} - Q - QB(\tilde{Q}(I+BQ) - Q)\}x\\
&\qquad + 2u^TB\{\tilde{Q}(I+BQ) - Q\}x\\
&\quad = x^T\{R^T[Q] - QBR[Q]\}x + 2u^TBR[Q]x.
\end{aligned} \tag{2.4.25}$$

Moreover,

$$\begin{aligned}
\{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma &= x + Bu - B\tilde{Q}(x+Bu)\\
&= x + (B - B\tilde{Q}B)u - B\tilde{Q}x - (B - B\tilde{Q}B)Qx\\
&\quad - \{B\tilde{Q}BQ - BQ\}x\\
&= x + (B - B\tilde{Q}B)(u - Qx) - \{B\tilde{Q}(I+BQ) - BQ\}x\\
&= x + Dz - BR[Q]x,
\end{aligned} \tag{2.4.26}$$

and this shows all required statements. □

Proposition 2.4.8 (Picone's identity). *Let (X, U) be a conjoined basis of (2.1.5) with* $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$. *In addition, let (x, u) be admissible with $x_0 \in \operatorname{Im} X_0$. Then* $x \in \operatorname{Im} X$,

$$\sum_{k=0}^{N}\{x_{k+1}^TC_kx_{k+1} + u_k^TB_ku_k\} = x_{N+1}^TQ_{N+1}x_{N+1} - x_0^TQ_0x_0 + \sum_{k=0}^{N} z_k^TD_kz_k, \tag{2.4.27}$$

$$x + Dz = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma \quad \text{on } J. \tag{2.4.28}$$

Here, $Q := XX^\dagger UX^\dagger$, $z := u - Qx = u - (UX^\dagger)^Tx$, and

$$D := B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B = X(X^\sigma)^\dagger\tilde{A}B. \tag{2.4.29}$$

Proof. Because of $x_0 \in \operatorname{Im} X_0$, our assumptions obviously imply with the aid of (2.3.33) from Lemma 2.3.5

$$x_k \in \operatorname{Im} X_k \quad \forall\, 0 \leq k \leq N+1. \tag{2.4.30}$$

Since $Q = XX^\dagger UX^\dagger = (X^\dagger)^T X^T UX^\dagger$ is symmetric and satisfies $X^T QX = X^T U$, Lemma 2.3.7(ii) yields

$$X^T R[Q]X = BR[Q]X = 0. \tag{2.4.31}$$

An application of Lemma 2.4.7 shows

$$(x^\sigma)^T Cx^\sigma + u^T Bu = \Delta\{x^T Qx\} + z^T Dz, \tag{2.4.32}$$

as well as (2.4.28). It follows from (2.4.32) by summation that

$$\begin{aligned}\mathcal{F}_0(x,u) &= \sum_{k=0}^{N} \Delta\{x_k^T Q_k x_k\} + \sum_{k=0}^{N} z_k^T D_k z_k \\ &= x_{N+1}^T Q_{N+1} x_{N+1} - x_0^T Q_0 x_0 + \sum_{k=0}^{N} z_k^T D_k z_k,\end{aligned} \tag{2.4.33}$$

and this is exactly identity (2.4.27). Finally we have for $x = Xc \in \operatorname{Im} X$,

$$\begin{aligned}Qx &= XX^\dagger UX^\dagger x = XX^\dagger UX^\dagger Xc = (X^\dagger)^T X^T UX^\dagger Xc \\ &= (X^\dagger)^T U^T XX^\dagger Xc = (X^\dagger)^T U^T Xc = (UX^\dagger)^T x,\end{aligned} \tag{2.4.34}$$

and $D = X(X^\sigma)^\dagger \tilde{A}B$ follows from Lemma 2.3.7(ii). □

Remark 2.4.9 (Picone's identity). Instead of $Q = XX^\dagger UX^\dagger$ in the above Picone identity we could also have used any symmetric Q with $QX = UX^\dagger X$ because of Lemma 2.3.7(ii) and Lemma 2.4.7, especially

$$Q = UX^\dagger + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X)U^T, \tag{2.4.35}$$

where $(\tilde{X}, \tilde{U})$ is a conjoined basis of (2.1.5) which "complements" (X,U) to normalized conjoined bases (see (2.3.26) in Section 2.3.1).

In order to arrive at a "big" Picone identity which will be of use in the case of the more general boundary conditions suggested by Definition 2.4.1(ii), we now apply Picone's identity from Proposition 2.4.8 to the "big" Hamiltonian system (2.3.40) from Lemma 2.3.6. While Proposition 2.4.8 works with a single conjoined basis of (2.1.5), we now need two normalized conjoined bases (X, U), $(\tilde{X}, \tilde{U})$ of (2.1.5). Lemma 2.3.6 then states that (X^*, U^*) with

$$X^* = \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix}, \qquad U^* = \begin{pmatrix} I & 0 \\ U & \tilde{U} \end{pmatrix} \tag{2.4.36}$$

is a conjoined basis of (2.3.40).

We have $\operatorname{Ker} X^*_{k+1} \subset \operatorname{Ker} X^*_k$ for some $k \in J$ if and only if $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ is true. In this case we have with Q^* given by (2.3.56) and with A^*, B^*, and C^* given by (2.3.38) that

$$\begin{aligned} D^*_k &:= X^*_k X^{*\dagger}_{k+1} \tilde{A}^*_k B^*_k \\ &= B^*_k - B^*_k \tilde{A}^{*T}_k (Q^*_{k+1} - C^*_k) \tilde{A}^*_k B^*_k \\ &= \begin{pmatrix} 0 & 0 \\ 0 & B_k - B_k \tilde{A}^T_k (Q_{k+1} - C_k) \tilde{A}_k B_k \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & D_k \end{pmatrix}, \end{aligned} \tag{2.4.37}$$

when we again put $D_k = X_k X^\dagger_{k+1} \tilde{A}_k B_k$ as before.

Moreover, for (constant) $\alpha \in \mathbb{R}^n$, we have

$$\Delta \begin{pmatrix} \alpha \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta x \end{pmatrix}, \qquad A^* \begin{pmatrix} \alpha \\ x \end{pmatrix}^\sigma + B^* \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ Ax^\sigma + Bu \end{pmatrix}, \tag{2.4.38}$$

so that (x, u) turns out to be admissible (with respect to (2.1.5)) exactly in the case of admissibility of (x^*, u^*) with

$$x^* := \begin{pmatrix} \alpha \\ x \end{pmatrix}, \qquad u^* := \begin{pmatrix} 0 \\ u \end{pmatrix} \tag{2.4.39}$$

(with respect to (2.3.40)). Then we have $x^* \in \operatorname{Im} X^*$ if and only if there exists $c \in \mathbb{R}^n$ with $x = Xc + \tilde{X}\alpha$. This is true if and only if there are $c, d \in \mathbb{R}^n$ with

$$\begin{pmatrix} x \\ d \end{pmatrix} = \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix} \begin{pmatrix} c \\ \alpha \end{pmatrix}, \tag{2.4.40}$$

and hence

$$\begin{pmatrix} c \\ \alpha \end{pmatrix} = \begin{pmatrix} \tilde{U}^T & -\tilde{X}^T \\ -U^T & X^T \end{pmatrix} \begin{pmatrix} x \\ d \end{pmatrix} \tag{2.4.41}$$

(see (2.3.28) from Remark 2.3.4). This in turn is equivalent to the existence of $d \in \mathbb{R}^n$ with $\alpha + U^T x = X^T d$, that is, to $\alpha + U^T x \in \operatorname{Im} X^T$.

After all this preliminary work we now may prove the generalization of Picone's identity.

Theorem 2.4.10 (extension of Picone's identity). *Let (X, U), $(\tilde{X}, \tilde{U})$ be normalized conjoined bases of system (2.1.5). Let* $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$ *hold on J. In addition, let (x, u) be admissible and let $\alpha \in \mathbb{R}^n$ be a constant with $\alpha + U_0^T x_0 \in \operatorname{Im} X_0^T$. Then $\alpha + U^T x \in \operatorname{Im} X^T$,*

$$\begin{aligned} &\sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} \\ &\qquad = \sum_{k=0}^{N} z_k^T D_k z_k + \begin{pmatrix} \alpha \\ x_{N+1} \end{pmatrix}^T Q_{N+1}^* \begin{pmatrix} \alpha \\ x_{N+1} \end{pmatrix} - \begin{pmatrix} \alpha \\ x_0 \end{pmatrix}^T Q_0^* \begin{pmatrix} \alpha \\ x_0 \end{pmatrix}, \end{aligned} \tag{2.4.42}$$

$$x + Dz = \{\tilde{A}^{-1} - B\tilde{A}^T (Q^\sigma - C)\} x^\sigma - B\tilde{A}^T (\tilde{Q}^\sigma)^T \alpha \quad \text{on } J. \tag{2.4.43}$$

Here, $Q^* = \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X & \tilde{Q} \\ \tilde{Q}^T & Q \end{pmatrix}$ *with*

$$\begin{aligned} \tilde{Q} = X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X) U^T, &\qquad Q = UX^\dagger + (UX^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X) U^T, \\ z := u - Qx - \tilde{Q}^T \alpha, &\qquad D := B - B\tilde{A}^T (Q^\sigma - C)\tilde{A}B = X(X^\sigma)^\dagger \tilde{A}B. \end{aligned} \tag{2.4.44}$$

Proof. Since because of the preceding remark admissibility of (x, u) with respect to (2.1.5) is equivalent with admissibility of $\left(x^* = \binom{\alpha}{x},\ u^* = \binom{0}{u}\right)$ with respect to (2.3.40), we may apply Proposition 2.4.8 to the big system (2.3.40) because of $x_0^* \in \operatorname{Im} X_0^*$. Then Remark 2.4.9 yields

$$x_k^* \in \operatorname{Im} X_k^* \quad (\text{i.e., } \alpha + U_k^T x_k \in \operatorname{Im} X_k \ \forall k \in J^*). \tag{2.4.45}$$

With

$$z^* := u^* - Q^* x^* = \begin{pmatrix} 0 \\ u \end{pmatrix} - \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X\alpha + \tilde{Q}x \\ \tilde{Q}^T \alpha + Qx \end{pmatrix} = \begin{pmatrix} X^\dagger \tilde{X} X^\dagger X\alpha - \tilde{Q}x \\ z \end{pmatrix}, \tag{2.4.46}$$

an application of Picone's identity (2.4.27) yields

$$\begin{aligned}
&\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}\\
&\quad = \sum_{k=0}^{N}\{x_{k+1}^{*^T} C_k^* x_{k+1}^* + u_k^{*^T} B_k^* u_k^*\}\\
&\quad = x_{N+1}^{*^T} Q_{N+1}^* x_{N+1}^* - x_0^{*^T} Q_0^* x_0^* + \sum_{k=0}^{N} z_k^{*^T} D_k^* z_k^*\\
&\quad = \begin{pmatrix}\alpha\\ x_{N+1}\end{pmatrix}^T Q_{N+1}^* \begin{pmatrix}\alpha\\ x_{N+1}\end{pmatrix} - \begin{pmatrix}\alpha\\ x_0\end{pmatrix}^T Q_0^* \begin{pmatrix}\alpha\\ x_0\end{pmatrix} + \sum_{k=0}^{N} z_k^T D_k z_k.
\end{aligned} \tag{2.4.47}$$

Observe also that we have as noted in Remark 2.4.9

$$D^* = \begin{pmatrix}0 & 0\\ 0 & D\end{pmatrix}. \tag{2.4.48}$$

Finally it follows from (2.4.28) that

$$\begin{aligned}
\begin{pmatrix}\alpha\\ x+Dz\end{pmatrix} &= x^* + D^* z^*\\
&= \left\{\tilde{A}^{*^{-1}} - B^*\tilde{A}^{*^T}\left((Q^*)^\sigma - C^*\right)\right\}(x^*)^\sigma\\
&= \left\{\begin{pmatrix}I & 0\\ 0 & \tilde{A}^{-1}\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & B\tilde{A}^T\end{pmatrix}\begin{pmatrix}-(X^\sigma)^\dagger \tilde{X}^\sigma (X^\sigma)^\dagger X^\sigma & \tilde{Q}^\sigma\\ (\tilde{Q}^\sigma)^T & Q^\sigma - C\end{pmatrix}\right\}\begin{pmatrix}\alpha\\ x^\sigma\end{pmatrix}\\
&= \begin{pmatrix}\alpha\\ \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma - B\tilde{A}^T(\tilde{Q}^\sigma)^T\alpha\end{pmatrix},
\end{aligned} \tag{2.4.49}$$

and this shows all of our claims. □

2.5. Disconjugacy and controllability

2.5.1. Focal points

After the calculations from the last section and in view of Picone's identities from Proposition 2.4.8 and Theorem 2.4.10, it is now clear how to introduce the following central definition.

Definition 2.5.1 (focal points). Let (X, U) be a conjoined basis of (2.1.5), and let $k \in J$. Then, call $k+1$ a *focal point* of X (or, of (X, U)), whenever

$$\operatorname{Ker} X_{k+1} \not\subset \operatorname{Ker} X_k. \tag{2.5.1}$$

Moreover, say that X (or, (X, U)) has a focal point in the interval $(k, k+1)$ if

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \qquad D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k \not\geq 0. \tag{2.5.2}$$

Remark 2.5.2. Obviously X has no focal point in $(k, k+1]$, $k \in J$, if and only if

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \qquad D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k \geq 0. \tag{2.5.3}$$

Of special importance for our theory are conjoined bases (X, U) of (2.1.5) that do not have any focal points throughout the interval $(0, N+1]$, that is, for those, the kernels of X are slowly—not necessarily strictly—"decreasing" and there is no "change of signs" when passing from k to $k+1$. Note that this condition itself does not guarantee nonsingularity of at least one matrix X_m. However, if such a matrix exists at all, then each of the subsequent matrices X_k, $m < k \leq N+1$, are nonsingular as well.

The concept of focal points is the most central one of this chapter. We now give an illustrative example (see also Chapter 1). Another important example may be found in Section 2.7.

Example 2.5.3 (Sturm-Liouville difference equations of second order). Let be given the Sturm-Liouville difference equation of second order

$$\Delta^2 x_k = c_k x_{k+1}. \tag{2.5.4}$$

This equation may be rewritten as a Hamiltonian system in the form

$$\Delta x_k = u_k, \qquad \Delta u_k = c_k x_{k+1}. \tag{2.5.5}$$

Using our notation, we thus have $n = 1$, $A_k \equiv 0$, $B_k \equiv 1$, and $C_k = c_k$. Each $(x, \Delta x)$ with $x \neq 0$ is a conjoined basis. If $x_{k+1} = 0$, then either x vanishes identically or $x_k \neq 0$ holds, and then $k+1$ is a focal point of x due to Definition 2.5.1 because of $\operatorname{Ker} x_{k+1} \not\subset \operatorname{Ker} x_k$. If $x_{k+1} \neq 0$, then $\operatorname{Ker} x_{k+1} \subset \operatorname{Ker} x_k$, and

$$x_k x_{k+1}^{\dagger} = \frac{x_k}{x_{k+1}} \tag{2.5.6}$$

is negative if and only if x has a change of signs from k to $k+1$. Then x has a focal point in the interval $(k, k+1)$ according to Definition 2.5.1.

The following result, which contains a characterization of positive definiteness of $\mathcal{F}_0$ from (2.4.2) via focal points, also supports our definition of focal points.

Proposition 2.5.4. *The functional $\mathcal{F}_0$ is positive definite, that is,*

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} > 0 \tag{2.5.7}$$

for all admissible (x,u) with $x_0 = x_{N+1} = 0$ and $x \neq 0$ if and only if the principal solution (X,U) of (2.1.5) at 0 has no focal points on $(0, N+1]$, that is, satisfies

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad D_k := X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0 \quad \forall k \in J. \tag{2.5.8}$$

More precisely, the following hold.

(i) *If* $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *and* $D_k \geq 0$ *is true for all* $k \in J$, *then* $\mathcal{F}_0 > 0$.

(ii) *If* $\operatorname{Ker} X_{m+1} \not\subset \operatorname{Ker} X_m$ *holds for some* $m \in J$, *then there exists an admissible* (x,u) *with* $x_0 = x_{N+1} = 0$, $x \neq 0$, *and* $\mathcal{F}_0(x,u) = 0$.

(iii) *If* $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *is true for all* $k \in J$, *but* $D_m \not\geq 0$ *holds for some* $m \in J$, *then there exists an admissible* (x,u) *with* $x_0 = x_{N+1} = 0$, $x \neq 0$, *and* $\mathcal{F}_0(x,u) < 0$.

Proof. Statement (i) follows immediately by Lemma 2.4.6 and Picone's identity, Proposition 2.4.8, while (ii) and (iii) follow from Lemma 2.4.5. □

The contents of the following corollary is well known. We refer, for example, to [28, Theorem 8], [34, Theorem 3.1], [106, Proposition 1], or [120, Theorem 2.5].

Corollary 2.5.5 (regular case). *Let B_k be invertible for all $k \in J$. Then, $\mathcal{F}_0 > 0$ if and only if the principal solution (X,U) of (2.1.5) at 0 satisfies*

$$X_k^T \{B_k^{-1}(I - A_k)\} X_{k+1} > 0 \quad \forall k \in J \setminus \{0\}. \tag{2.5.9}$$

Proof. Since $X_1 = \tilde{A}_0 X_0 + \tilde{A}_0 B_0 U_0 = \tilde{A}_0 B_0$ is invertible, the condition $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ for all $k \in J$ is in this case equivalent to

$$X_k \quad \text{invertible } \forall 1 \leq k \leq N+1. \tag{2.5.10}$$

Then $X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0$ for all $k \in J$ is equivalent to $X_k X_{k+1}^{-1} \tilde{A}_k B_k > 0$ for all $k \in J \setminus \{0\}$, respectively, with

$$\begin{aligned} 0 &< \{B_k^{-1}(I - A_k) X_{k+1}\}^T \left\{B_k \tilde{A}_k^T (X_{k+1}^{-1})^T X_k^T\right\} \{B_k^{-1}(I - A_k) X_{k+1}\} \\ &= X_k^T B_k^{-1}(I - A_k) X_{k+1} \end{aligned} \tag{2.5.11}$$

for all $k \in J \setminus \{0\}$. Hence the claim follows from Proposition 2.5.4. □

Remark 2.5.6. Let B_0 be invertible. Then it follows from the proof of the above Corollary 2.5.5 that $\mathcal{F}_0 > 0$ holds if and only if

$$X_{k+1} \quad \text{is invertible and} \quad X_k X_{k+1}^{-1} \tilde{A}_k B_k \geq 0 \quad \forall k \in J, \tag{2.5.12}$$

where (X, U) is the principal solution of (2.1.5) at 0.

2.5.2. Generalized zeros

After introducing focal points of conjoined bases, that is, of special matrix-valued solutions (X, U) of (2.1.5) in the preceding section, we now are going to deal with vector-valued solutions (x, u) of (2.1.5). We start our examinations with the following observation.

Lemma 2.5.7. *Let (X, U) be a conjoined basis of (2.1.5) with* $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *for some $k \in J$. Set $D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k$. Then*

$$X_{k+1} d = \tilde{A}_k B_k c, \tag{2.5.13}$$

always implies

$$X_k d = D_k c, \qquad d^T X_k^T B_k^{\dagger} (I - A_k) X_{k+1} d = c^T D_k c. \tag{2.5.14}$$

Proof. Let (X, U) be a conjoined basis of (2.1.5). Let $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ hold for some $k \in J$, and this implies $X_k = X_k X_{k+1}^{\dagger} X_{k+1}$ by Lemma 2.8.6. Now let $X_{k+1} d = \tilde{A}_k B_k c$. Then we have

$$X_k d = X_k X_{k+1}^{\dagger} X_{k+1} d = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k c = D_k c, \tag{2.5.15}$$

as well as

$$\begin{aligned} d^T X_k^T B_k^{\dagger} (I - A_k) X_{k+1} d &= c^T D_k^T B_k^{\dagger} B_k c = c^T D_k B_k^{\dagger} B_k c \\ &= c^T X_k X_{k+1}^{\dagger} \tilde{A}_k B_k B_k^{\dagger} B_k c \\ &= c^T X_k X_{k+1}^{\dagger} \tilde{A}_k B_k c \\ &= c^T D_k c, \end{aligned} \tag{2.5.16}$$

where we have used the symmetry of D_k which has been established in Lemma 2.3.7(ii). ☐

Proposition 2.5.8. *The principal solution of (2.1.5) at* 0 *has no focal points in* $(0, N+1]$ *if and only if the condition*

$$\begin{gathered} \textit{for all solutions } (x, u) \textit{ of (2.1.5) with } x_0 = 0 \textit{ and all } k \in J, \\ x_k \neq 0, \quad x_{k+1} \in \operatorname{Im} \tilde{A}_k B_k \quad \textit{always imply} \quad x_k^T B_k^{\dagger} (I - A_k) x_{k+1} > 0 \end{gathered} \tag{2.5.17}$$

is satisfied.

Proof. Let (X, U) be the principal solution of (2.1.5) at 0. First of all we suppose $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$ and $D = X(X^\sigma)^\dagger \tilde{A}B \geq 0$ on J. Now let be given a solution (x, u) of (2.1.5) with $x_0 = 0$. From the unique solvability of initial value problems (2.3.8) (with $m_x = 0$ and $m_u = u_0$) it follows that $x = Xu_0$. Let $x_k \neq 0$ and $x_{k+1} = \tilde{A}_k B_k c \in \operatorname{Im} \tilde{A}_k B_k$ hold for some $k \in J$. Hence

$$X_{k+1} u_0 = x_{k+1} = \tilde{A}_k B_k c, \tag{2.5.18}$$

so that we may apply Lemma 2.5.7. Thus we have both

$$\begin{gathered} x_k = X_k u_0 = D_k c, \\ x_k^T B_k^\dagger (I - A_k) x_{k+1} = u_0^T X_k^T B_k^\dagger (I - A_k) X_{k+1} u_0 = c^T D_k c. \end{gathered} \tag{2.5.19}$$

Altogether this yields $x_k^T B_k^\dagger (I - A_k) x_{k+1} > 0$ because of $D_k \geq 0$, and this proves the validity of (2.5.17).

Conversely we assume (2.5.17). Let $k \in J$. First we take $d \in \operatorname{Ker} X_{k+1}$ and define

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X \\ U \end{pmatrix} d. \tag{2.5.20}$$

Then (x, u) is a solution of (2.1.5) with $x_0 = 0$. Since $x_{k+1} = X_{k+1} d = 0 \in \operatorname{Im} \tilde{A}_k B_k$ holds, the assumption $x_k \neq 0$ leads with condition (2.5.17) to the contradiction $x_k^T B_k^\dagger (I - A_k) x_{k+1} > 0$ and thus has to be wrong. Hence we have $X_k d = x_k = 0$, that is, $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$. Finally let $c \in \mathbb{R}^n$ be arbitrary and

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X \\ U \end{pmatrix} d, \qquad d := X_{k+1}^\dagger \tilde{A}_k B_k c. \tag{2.5.21}$$

Again (x, u) solves (2.1.5) with $x_0 = 0$. Moreover, (2.3.32) from Lemma 2.3.5 and Lemma 2.8.6 yield

$$x_{k+1} = X_{k+1} d = X_{k+1} X_{k+1}^\dagger \tilde{A}_k B_k c = \tilde{A}_k B_k c \in \operatorname{Im} \tilde{A}_k B_k, \tag{2.5.22}$$

and this implies

$$\begin{gathered} x_k = X_k d = D_k c, \\ x_k^T B_k^\dagger (I - A_k) x_{k+1} = d^T X_k^T B_k^\dagger (I - A_k) X_{k+1} d = c^T D_k c \end{gathered} \tag{2.5.23}$$

by Lemma 2.5.7. If x_k does not vanish, then (2.5.17) guarantees $c^T D_k c > 0$. At the same time $x_k = 0$ implies $c^T D_k c = 0$. Altogether, D_k turns out to be positive semidefinite. This proves that the principal solution of (2.1.5) at 0 has no focal points on $(0, N + 1]$ in this case. □

Definition 2.5.9 (disconjugacy). (i) Let (x, u) be a solution of (2.1.5) and $k \in J$. Then $k + 1$ is called a *generalized zero* of x (or, of (x, u)) whenever

$$x_k \neq 0, \qquad x_{k+1} \in \operatorname{Im} \tilde{A}_k B_k, \qquad x_k^T B_k^\dagger (I - A_k) x_{k+1} = 0. \tag{2.5.24}$$

Moreover, say that x (or, (x, u)) has a generalized zero in the interval $(k, k + 1)$ if

$$x_k \neq 0, \qquad x_{k+1} \in \operatorname{Im} \tilde{A}_k B_k, \qquad x_k^T B_k^\dagger (I - A_k) x_{k+1} < 0. \tag{2.5.25}$$

(ii) System (2.1.5) is said to be *disconjugate* on a subset $\tilde{J} = [\mu, \nu] \cap J^*$ with $\mu, \nu \in J^*$ and $\mu < \nu$ if no solution of (2.1.5) has more than one generalized zero on $\tilde{J} \setminus \{\mu\}$ and if no solution (x, u) of (2.1.5) with $x_\mu = 0$ has at least one generalized zero on $\tilde{J} \setminus \{\mu\}$.

Remark 2.5.10 (alternative definition of disconjugacy). One could as well have had disconjugacy defined in the following way, and this is essentially done, for example, in [36]: on the interval $\tilde{J}$ under consideration the left endpoint μ is a generalized zero of x only if $x_\mu = 0$. Besides that, generalized zeros are defined as in Definition 2.5.9. Then (2.1.5) is disconjugate on $\tilde{J}$ whenever no solution of (2.1.5) has more than one generalized zero on $\tilde{J}$. We did choose the way from the preceding Definition 2.5.9, although the definition of disconjugacy then is a bit more "complicated." The difference is that the notion of a generalized zero does not depend on the interval under consideration $\tilde{J}$ when using our definition, while the notion of disconjugacy does depend on this interval.

Lemma 2.5.11. (i) *Let $k \in J$. The interval $(k, k + 1]$ contains a generalized zero of a solution (x, u) of (2.1.5) if and only if there exist $c \in \mathbb{R}^n$ with*

$$x_k \neq 0, \qquad x_{k+1} = \tilde{A}_k B_k c, \qquad x_k^T c \leq 0. \tag{2.5.26}$$

(ii) *System (2.1.5) is not disconjugate on J^* if and only if there exist a solution (x, u) of (2.1.5) and vectors $c_m, c_p \in \mathbb{R}^n$ with $m, p \in J$ and $m < p$ such that*

$$\begin{gathered} x_{m+1} = \tilde{A}_m B_m c_m, \qquad x_{p+1} = \tilde{A}_p B_p c_p, \qquad x_p \neq 0, \\ x_m^T c_m \leq 0, \qquad x_p^T c_p \leq 0. \end{gathered} \tag{2.5.27}$$

Proof. Let (x,u) be a solution of (2.1.5). Suppose $x_{k+1} = \tilde{A}_k B_k c$ holds for some $k \in J$ and some $c \in \mathbb{R}^n$. Hence

$$x_k = (I - A_k)x_{k+1} - B_k u_k = B_k c - B_k u_k = B_k(c - u_k), \tag{2.5.28}$$

and this implies

$$\begin{aligned} x_k^T B_k^\dagger (I - A_k)x_{k+1} &= x_k^T B_k^\dagger B_k c \\ &= (c - u_k)^T B_k B_k^\dagger B_k c \\ &= (c - u_k)^T B_k c \\ &= x_k^T c. \end{aligned} \tag{2.5.29}$$

Together with Definition 2.5.9(i) this yields (i) of our auxiliary result.

Now we turn our attention to (ii). First of all, suppose (2.1.5) is not disconjugate on J^*. Then there exists an "exceptional solution" (x,u) of (2.1.5) with one of the following properties. Either we have $x_0 \neq 0$, and then there are (at least) two different generalized zeros of x in the interval $(0, N+1]$. Since any interval $(k, k+1]$, $k \in J$, may contain at most one generalized zero of x (observe Definition 2.5.9(i)), there exist in this case $m, p \in J$ with $m < p$ such that there is one generalized zero in $(m, m+1]$ as well as one in $(p, p+1]$, and then (2.5.27) follows from (i). Or we have $x_0 = 0$, and then there exists (at least) one generalized zero of x in the interval $(0, N+1]$. Let this zero be, say, in the interval $(p, p+1]$ with $p \in J \setminus \{0\}$. By (i) there exists some $c_p \in \mathbb{R}^n$ with $x_p \neq 0$, $x_{p+1} = \tilde{A}_p B_p c_p$, and $x_p^T c_p \leq 0$. Moreover $x_1 = \tilde{A}_0 B_0 u_0$ with $x_0^T u_0 = 0$. Again, (2.5.27) follows. Conversely let be given a solution (x,u) of (2.1.5), $c_m, c_p \in \mathbb{R}^n$ with $m, p \in J$, $m < p$, and (2.5.27). Then $p > 0$, and because of (i) there is one generalized zero of x in $(p, p+1]$. If $x_0 = 0$ holds, then it follows immediately that (2.1.5) is not disconjugate on J^*. In the case of $x_0 \neq 0$, there obviously exists a second generalized zero in $(0, m+1] \cap J$, so that again (2.1.5) is not disconjugate on J^*. Thus, all statements of this auxiliary result are shown. □

Proposition 2.5.12. *If (2.1.5) is not disconjugate on J^*, then there exists an admissible (x,u) with $x_0 = x_{N+1} = 0$, $x \neq 0$, and*

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} \left\{ x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k \right\} \leq 0. \tag{2.5.30}$$

Proof. Suppose that (2.1.5) is not disconjugate on J^*. Hence Lemma 2.5.11(ii) gives a solution $(\tilde{x}, \tilde{u})$ of (2.1.5), two points $m, p \in J$ with $m < p$, and two vectors $c_m, c_p \in \mathbb{R}^n$ such that

$$\tilde{x}_{m+1} = \tilde{A}_m B_m c_m, \qquad \tilde{x}_m^T c_m \leq 0, \qquad \tilde{x}_p \neq 0, \qquad \tilde{x}_{p+1} = \tilde{A}_p B_p c_p, \qquad \tilde{x}_p^T c_p \leq 0. \tag{2.5.31}$$

We now define for $k \in J^*$

$$x_k := \begin{cases} \tilde{x}_k & \text{for } m+1 \le k \le p, \\ 0 & \text{otherwise}, \end{cases}$$
$$u_k := \begin{cases} c_m & \text{for } k = m, \\ \tilde{u}_k & \text{for } m+1 \le k \le p-1, \\ \tilde{u}_p - c_p & \text{for } k = p, \\ 0 & \text{otherwise}. \end{cases} \tag{2.5.32}$$

Observe $0 \le m < m+1 \le p \le N < N+1$ to see $x_0 = x_{N+1} = 0$. Now $x_p = \tilde{x}_p \neq 0$ so that x is not trivial. With

$$\begin{gathered} \tilde{A}_m x_m + \tilde{A}_m B_m u_m = \tilde{A}_m B_m c_m = \tilde{x}_{m+1} = x_{m+1}, \\ \tilde{A}_p x_p + \tilde{A}_p B_p u_p = \tilde{A}_p \tilde{x}_p + \tilde{A}_p B_p (\tilde{u}_p - c_p) = \tilde{x}_{p+1} - \tilde{A}_p B_p c_p = 0 = x_{p+1}, \end{gathered} \tag{2.5.33}$$

we ensure the admissibility at the two "problem crossings" from m to $m+1$ and from p to $p+1$. Since

$$\tilde{A}_k x_k + \tilde{A}_k B_k u_k = x_{m+1} \quad \forall k \in J \setminus \{m, p\} \tag{2.5.34}$$

is clearly satisfied, admissibility of (x, u) follows. An application of (2.4.5) from Lemma 2.4.3 now yields

$$\begin{aligned} \mathcal{F}_0(x,u) &= \sum_{k=m}^{p-1} x_{k+1}^T \{ C_k x_{k+1} - A_k^T u_k - \Delta u_k \} \\ &= x_{m+1}^T \{ C_m x_{m+1} + (I - A_m^T) u_m - u_{m+1} \} \\ &\quad + x_p^T \{ C_{p-1} x_p + (I - A_{p-1}^T) u_{p-1} - u_p \} \\ &= \tilde{x}_{m+1}^T \{ C_m \tilde{x}_{m+1} + (I - A_m^T) c_m - \tilde{u}_{m+1} \} \\ &\quad + \tilde{x}_p^T \{ C_{p-1} x_p + (I - A_{p-1}^T) \tilde{u}_{p-1} - (\tilde{u}_p - c_p) \} \\ &= \tilde{x}_{m+1}^T (I - A_m^T)(- \tilde{u}_m + c_m) + \tilde{x}_p^T c_p \\ &= c_m^T B_m (c_m - \tilde{u}_m) + \tilde{x}_p^T c_p \\ &= c_m^T \tilde{x}_m + \tilde{x}_p^T c_p \\ &= \tilde{x}_m^T c_m + \tilde{x}_p^T c_p \le 0, \end{aligned} \tag{2.5.35}$$

where we used (2.5.28) for the solution $(\tilde{x}, \tilde{u})$ of (2.1.5). □

At this point all the notions and concepts are available to prove the Reid roundabout theorem for $R = S = 0$ in Section 2.6.1 which discusses the main results. However, before starting this proof, we first turn our attention to another important concept, namely, the concept of controllability of discrete systems.

2.5.3. Controllability

Definition 2.5.13 (controllability). The system (2.1.5) is said to be *controllable* (or, *identically normal*) on J^* whenever there exists $k \in J^*$ such that for any solution (x, u) of (2.1.5) and for any $m \in J^*$ with $m + k \in J^*$,

$$x_m = x_{m+1} = \cdots = x_{m+k} = 0, \tag{2.5.36}$$

always implies $x = u = 0$ on J^*. In this case the minimal number $\kappa \in J^*$ with this property is said to be the *controllability index* of the system (2.1.5).

In other words, the vanishing of x on a "$\kappa + 1$-long interval" implies already the triviality of x and u on all of J^*. We now give two characterizations of controllability. While the first result gives a method on how to check controllability of a special system in a convenient way, the second characterization serves mainly to illustrate the concept. We remark that, in case of controllability of (2.1.5) on J^*, $1 \le \kappa \le N + 1$ obviously is always true.

Lemma 2.5.14. *The system (2.1.5) is controllable on J^* if and only if there exists $k \in J^*$ such that*

$$\operatorname{rank} G_k^{(m)} = n \quad \forall m \in J^* \text{ with } m + k \in J^*. \tag{2.5.37}$$

In this case

$$\kappa = \min\left\{k \in J^* : \operatorname{rank} G_k^{(m)} = n \text{ for all } m \in J^* \text{ with } m + k \in J^*\right\} \tag{2.5.38}$$

is the controllability index of (2.1.5). Here, the controllability matrices $G_k^{(m)}$ are defined by (2.3.13).

Proof. Let $\alpha \in \mathbb{R}^n$ and $k \in J^*$. Moreover, let $m \in J^*$ with $m + k \in J^*$. Suppose that $x_{m+k} = 0$ and $u_{m+k} = \alpha$ hold for some solution (x, u) of (2.1.5). We now prove that (with Φ from (2.3.11))

$$x_{m+\nu} = 0, \quad u_{m+\nu} = \Phi_{m+k,m+\nu}^T \alpha \quad \forall 0 \le \nu \le k \tag{2.5.39}$$

if and only if $\alpha \in \operatorname{Ker} G_k^{(m)^T}$. First of all observe

$$\bigcap_{\nu=0}^{k-1} \operatorname{Ker}\left(B_{m+\nu} \Phi_{m+k,m+\nu}^T\right) = \operatorname{Ker} G_k^{(m)^T}. \tag{2.5.40}$$

If (2.5.39) holds, then we have for all $0 \le \nu \le k-1$

$$B_{m+\nu}\Phi^T_{m+k,m+\nu}\alpha = B_{m+\nu}u_{m+\nu} = \Delta x_{m+\nu} - A_{m+\nu}x_{m+\nu+1} = 0, \tag{2.5.41}$$

that is, $\alpha \in \operatorname{Ker} G_k^{(m)^T}$. Conversely we now assume $\alpha \in \operatorname{Ker} G_k^{(m)^T}$. Since (2.5.39) is true for $\nu = k$, we now assume inductively that (2.5.39) already holds for $0 < \nu \le k$. Hence

$$\begin{aligned} u_{m+\nu-1} &= \tilde{A}^T_{m+\nu-1}\{u_{m+\nu} - C_{m+\nu-1}x_{m+\nu}\} \\ &= \tilde{A}^T_{m+\nu-1}\Phi^T_{m+k,m+\nu}\alpha \\ &= \Phi^T_{m+k,m+\nu-1}\alpha, \end{aligned} \tag{2.5.42}$$

and therefore

$$\begin{aligned} x_{m+\nu-1} &= \tilde{A}^{-1}_{m+\nu-1}x_{m+\nu} - B_{m+\nu-1}u_{m+\nu-1} \\ &= -B_{m+\nu-1}\Phi^T_{m+k,m+\nu-1}\alpha \\ &= 0. \end{aligned} \tag{2.5.43}$$

This proves the above claim. In case of rank $G_k^{(m)} < n$ we now know that there is a solution (x,u) of (2.1.5) with

$$x_m = x_{m+1} = \cdots = x_{m+k} = 0, \qquad u_{m+k} \ne 0. \tag{2.5.44}$$

Hence $x_m = x_{m+1} = \cdots = x_{m+k} = 0$ does not imply $x = u = 0$, so that (2.1.5) is not controllable on J^*.

Conversely $x_m = x_{m+1} = \cdots = x_{m+k} = 0$ implies

$$u = \tilde{A}^T u^\sigma \quad \text{on } [m, m+k-1] \cap \mathbb{Z}, \tag{2.5.45}$$

so that (2.5.39) and thus $u_{m+k} \in \operatorname{Ker} G_k^{(m)^T} = \{0\}$ in case of rank $G_k^{(m)} = n$ follows. However, there is only one solution of (2.1.5) with $x_{m+k} = u_{m+k} = 0$ (see (2.3.8) with $m_x = u_x = 0$), and therefore both x and u turn out to be trivial so that (2.1.5) is controllable on J^*. □

Lemma 2.5.15. *The system (2.1.5) is controllable on J^* if and only if there exists $k \in J^*$ such that the condition*

$$\begin{gathered} \textit{for each } m \in J^* \textit{ with } m+k \in J^* \textit{ and all } a, b \in \mathbb{R}^n, \\ \textit{there exists an admissible } (x,u) \textit{ with } x_m = a \textit{ and } x_{m+k} = b \end{gathered} \tag{2.5.46}$$

is satisfied.

Proof. We show that, for $k \in J^*$, (2.5.46) holds exactly in the case of (2.5.37). Then the claim follows by Lemma 2.5.14. First, if there exists $k \in J^*$ with (2.5.46), then for $b \in \mathbb{R}^n$ and $m \in J^*$ with $m+k \in J^*$ there exists an admissible (x,u) with $x_m = 0$ and $x_{m+k} = b$. Then (2.3.16) yields for this (x,u)

$$b = G_k^{(m)} \begin{pmatrix} u_m \\ \vdots \\ u_{m+k-1} \end{pmatrix}, \tag{2.5.47}$$

so that $\operatorname{Im} G_k^{(m)} = \mathbb{R}^n$, that is, $\operatorname{rank} G_k^{(m)} = n$.

Conversely assume the existence of $k \in J^*$ with (2.5.37). Let be given $m \in J^*$ with $m+k \in J^*$ and two vectors $a, b \in \mathbb{R}^n$. Since $G_k^{(m)}$ has full rank, the matrix $G_k^{(m)} G_k^{(m)^T}$ is invertible, and we may choose an admissible (x,u) with $x_m = a$ and with the prescribed controls

$$\begin{pmatrix} u_m \\ \vdots \\ u_{m+k-1} \end{pmatrix} = G_k^{(m)^T} \left\{ G_k^{(m)} G_k^{(m)^T} \right\}^{-1} \{ b - \Phi_{m+k,m} a \}. \tag{2.5.48}$$

Then (2.3.16) yields

$$x_{m+k} = \Phi_{m+k,m} a + G_k^{(m)} G_k^{(m)^T} \left\{ G_k^{(m)} G_k^{(m)^T} \right\}^{-1} \{ b - \Phi_{m+k,m} a \} = b. \tag{2.5.49}$$

Therefore (2.5.46) holds. □

Example 2.5.16 (regular case). Let B_k be invertible for all $k \in J$. Then we have

$$\operatorname{rank} G_1^{(m)} = \operatorname{rank} (\tilde{A}_m B_m) = n \quad \forall m \in J. \tag{2.5.50}$$

The corresponding system (2.1.5) is controllable on J^* by Lemma 2.5.14 and has controllability index $\kappa = 1 \in J^*$.

Remark 2.5.17. We have $G_k = G_k^{(0)}$ for all $k \in J^*$ (see (2.3.14)). If the condition

$$A_k \equiv A, \qquad B_k = B\tilde{B}_k, \quad \tilde{B}_k \text{ invertible on } J \tag{2.5.51}$$

holds with two (constant) $n \times n$-matrices A and B, then Lemma 2.5.14 shows that controllability of (2.1.5) on J^* is equivalent to the existence of $k \in J^*$ with

$$\operatorname{rank} \tilde{G}_k = n, \tag{2.5.52}$$

where we put $\tilde{G}_k = (\tilde{A}^{k-1}B \quad \tilde{A}^{k-2}B \quad \cdots \quad \tilde{A}B \quad B)$ with $\tilde{A} = (I - A)^{-1}$. This follows because for all $m \in J^*$ with $m + k \in J^*$ we have

$$\begin{aligned}
\operatorname{rank} G_k^{(m)} &= \operatorname{rank}\left(\tilde{A}^k B\tilde{B}_m \quad \tilde{A}^{k-1}B\tilde{B}_{m+1} \quad \cdots \quad \tilde{A}^2 B\tilde{B}_{m+k-2} \quad \tilde{A}B\tilde{B}_{m+k-1}\right) \\
&= \operatorname{rank}\left(\tilde{A}^k B \quad \tilde{A}^{k-1}B \quad \cdots \quad \tilde{A}^2 B \quad \tilde{A}B\right) \\
&= \operatorname{rank}\tilde{G}_k.
\end{aligned} \tag{2.5.53}$$

In the case of controllability of (2.1.5) on J^*,

$$\kappa = \min\{k \in J^* : \operatorname{rank}\tilde{G}_k = n\} \tag{2.5.54}$$

then is the controllability index of (2.1.5).

Example 2.5.18 (Sturm-Liouville difference equations). An important example of a controllable system, separately treated in Section 2.7, is described by the $n \times n$-matrices

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix}. \tag{2.5.55}$$

With $b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$ we calculate

$$\begin{aligned}
\operatorname{rank} G_k &= \operatorname{rank}\left(\tilde{A}^{k-1}B \quad \tilde{A}^{k-2}B \quad \cdots \quad B\right) \\
&= \operatorname{rank}\left(b \quad \tilde{A}b \quad \tilde{A}^2 b \quad \cdots \quad \tilde{A}^{k-1}b\right) \\
&= \operatorname{rank}\left(b \quad (\tilde{A}-I)b \quad (\tilde{A}-I)^2 b \quad \cdots \quad (\tilde{A}-I)^{k-1}b\right) \\
&= \min\{k, n\}
\end{aligned} \tag{2.5.56}$$

for all $k \in J^*$ because of

$$\tilde{A} = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad \tilde{A} - I = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{2.5.57}$$

Thus the resulting system (2.1.5) (with arbitrary symmetric matrices C_k, $k \in J$) is in case of $N + 1 \geq n$ controllable on J^* because of Lemma 2.5.14 (or, Remark 2.5.17) with controllability index $\kappa = n \in J^*$.

We conclude this section with some results that give a connection between the two central concepts "disconjugate" and "controllable."

Lemma 2.5.19. *Let* (X, U) *be a conjoined basis of (2.1.5) with* $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$ *on* J*. Then, using the notation from (2.3.12) and (2.3.14), the following statements hold.*

(i) $\operatorname{Im} X = \operatorname{Im} (\Phi X_0 \quad G)$.

(ii) *If (2.1.5) is controllable on* J^* *with controllability index* $\kappa \in J^*$*, then*

$$X_k \quad \text{is invertible } \forall k \in [\kappa, N+1] \cap J^*. \tag{2.5.58}$$

(iii) *If* (X, U), $(\tilde{X}, \tilde{U})$ *are normalized conjoined bases of (2.1.5), then*

$$\operatorname{Im} \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix} = \operatorname{Im} \begin{pmatrix} 0 & I & 0 \\ \Phi X_0 & \Phi \tilde{X}_0 & G \end{pmatrix}. \tag{2.5.59}$$

Proof. Obviously (ii) is an immediate consequence of (i) by using Lemma 2.5.14. Alternatively, (ii) can be shown using the definition of controllability (see Definition 2.5.13) as follows: Let $c \in \operatorname{Ker} X_\kappa$. Then $(x := Xc,\ u := Uc)$ solves the system (2.1.5), and

$$\operatorname{Ker} X_\kappa \subset \operatorname{Ker} X_{\kappa-1} \subset \cdots \subset \operatorname{Ker} X_0 \tag{2.5.60}$$

implies $x_0 = x_1 = \cdots = x_\kappa = 0$. Controllability now requires $x = u = 0$ on J^*, so that (2.3.29) yields

$$c = \{\tilde{U}_0^T X_0 - \tilde{X}_0^T U_0\} c = \tilde{U}_0^T x_0 - \tilde{X}_0^T u_0 = 0, \tag{2.5.61}$$

and thus $\operatorname{Ker} X_\kappa = \{0\}$, that is, X_κ is invertible.

Now we prove (i). First of all it follows with (2.3.15) from Lemma 2.3.1 that

$$X_k = \Phi_k X_0 + G_k \begin{pmatrix} U_0 \\ \vdots \\ U_{k-1} \end{pmatrix} \quad \forall k \in J^*, \tag{2.5.62}$$

so that $\operatorname{Im} X \subset \operatorname{Im} (\Phi X_0 \quad G)$. To show the other direction, we let $k \in J^*$ and

$$\Phi_k X_0 \alpha + G_k \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} \in \operatorname{Im} \left(\Phi_k X_0 \quad G_k\right). \tag{2.5.63}$$

Let

$$x_0 := X_0\alpha, \qquad u_\nu := \begin{cases} \alpha_\nu & \text{for } 0 \le \nu \le k-1, \\ 0 & \text{for } k \le \nu \le N+1, \end{cases} \tag{2.5.64}$$
$$x_{\nu+1} := \tilde{A}_\nu x_\nu + \tilde{A}_\nu B_\nu u_\nu \quad \text{for } \nu \in J.$$

Then (x, u) constructed as above is admissible with $x_0 = X_0\alpha \in \operatorname{Im} X_0$. Because of (2.3.33) from Lemma 2.3.5 we have

$$x_\nu \in \operatorname{Im} X_\nu \quad \forall \nu \in J^*. \tag{2.5.65}$$

With another application of Lemma 2.3.1 this leads to

$$\Phi_k X_0 \alpha + G_k \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{k-1} \end{pmatrix} = \Phi_k x_0 + G_k \begin{pmatrix} u_0 \\ \vdots \\ u_{k-1} \end{pmatrix} = x_k \in \operatorname{Im} X_k. \tag{2.5.66}$$

Therefore $\operatorname{Im}(\Phi X_0 \quad G) \subset \operatorname{Im} X$, so that everything together confirms our claim $\operatorname{Im} X = \operatorname{Im}(\Phi X_0 \quad G)$.

Now, by Lemma 2.3.6, a conjoined basis of the system (2.3.40) is given by (X^*, U^*) with (2.3.39) which satisfies $\operatorname{Ker}(X^*)^\sigma \subset \operatorname{Ker} X^*$ due to our assumptions. Thus (i) shows the relation

$$\begin{aligned} \operatorname{Im} X^* &= \operatorname{Im}\begin{pmatrix} \Phi^* X_0^* & G^* \end{pmatrix} \\ &= \operatorname{Im}\left(\begin{pmatrix} I & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} 0 & I \\ X_0 & \tilde{X}_0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix} \right) \\ &= \operatorname{Im}\begin{pmatrix} 0 & I & 0 & 0 \\ \Phi X_0 & \Phi\tilde{X}_0 & 0 & G \end{pmatrix} \\ &= \operatorname{Im}\begin{pmatrix} 0 & I & 0 \\ \Phi X_0 & \Phi\tilde{X}_0 & G \end{pmatrix}, \end{aligned} \tag{2.5.67}$$

where Φ^* and G^* are defined by (2.3.12) and (2.3.14) (with A^* and B^* from (2.3.38) instead of A and B). □

Corollary 2.5.20. *Let (X, U), $(\tilde{X}, \tilde{U})$ be the special normalized conjoined bases of (2.1.5) at* 0. *Let $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$ hold on J. Then the following holds.*

(i) $\operatorname{Im} X = \operatorname{Im} G$ *and* $\operatorname{Im} X^* = \operatorname{Im}\left(\begin{smallmatrix} I & 0 \\ -\Phi & G \end{smallmatrix}\right)$*s.*

(ii) *If (2.5.51) is satisfied for the system (2.1.5), then (2.1.5) is controllable on J^* if and only if there exists $k \in J^*$ such that X_k is invertible.*

Proof. This follows from the above Lemma 2.5.19 with $X_0 = 0$ and $\tilde{X}_0 = -I$ and from Remark 2.5.17. □

2.6. Positivity of discrete quadratic functionals

2.6.1. The Reid roundabout theorem

Our first main result deals with the discrete quadratic functional (see (2.4.2))

$$\mathcal{F}_0(x,u) = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} \tag{2.6.1}$$

and with the boundary conditions $x_0 = x_{N+1} = 0$, that is, with the situation $R = S = 0$ from Definition 2.4.1. For convenience we restate all notation and definitions in the following theorem.

Theorem 2.6.1 (Reid roundabout theorem). *All of the following statements are equivalent:*

(i) $\mathcal{F}_0 > 0$, *that is,* $\mathcal{F}_0(x,u) = \sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} > 0$ *for all admissible* (x,u) *(i.e., with* $\Delta x_k = A_k x_{k+1} + B_k u_k$ *for all* $k \in J$*) with* $x_0 = x_{N+1} = 0$ *and* $x \neq 0$*;*

(ii) *system (2.1.5) is disconjugate on* J^**, that is, no solution of (2.1.5) has more than one generalized zero and no solution* (x,u) *of (2.1.5) with* $x_0 = 0$ *has at least one generalized zero in* $(0, N+1]$*, where a solution* (x,u) *of (2.1.5) has a generalized zero in* $(k, k+1]$*,* $k \in J$*, in case of*

$$x_k \neq 0, \qquad x_{k+1} \in \operatorname{Im} \tilde{A}_k B_k, \qquad x_k^T B_k^\dagger (I - A_k) x_{k+1} \leq 0; \tag{2.6.2}$$

(iii) *every solution* (x,u) *of system (2.1.5) with* $x_0 = 0$ *has no generalized zero in* $(0, N+1]$*;*

(iv) *the principal solution* (X,U) *of system (2.1.5) at* 0 *has no focal points in* $(0, N+1]$*, that is,*

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0 \quad \forall k \in J, \tag{2.6.3}$$

where (X,U) *is the solution of (2.1.5) with* $X_0 = 0$ *and* $U_0 = I$*.*

Proof. If (ii) does not hold, then Proposition 2.5.12 yields that (i) does not hold either, and thus the step from (i) to (ii) is done. While (iii) follows trivially from (ii), (iii) implies (iv) because of Proposition 2.5.8 (observe that (iii) is the same as (2.5.17)). Finally, (i) follows again from (iv) by Proposition 2.5.4. □

Remark 2.6.2. The most useful characterization of the positive definiteness of $\mathcal{F}_0$ for numerical purposes is without doubt the statement of Theorem 2.6.1(iv). This is because it is easy (see (2.3.10) in Section 2.3.1) to compute the sequence of matrices $X_0, X_1, \dots, X_{N+1}$ recursively according to the formula

$$X_k = \begin{pmatrix} I & 0 \end{pmatrix} \tilde{S}_{k-1}\tilde{S}_{k-2}\cdots\tilde{S}_0 \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad k \in J^*, \tag{2.6.4}$$

with the (symplectic) matrices

$$\tilde{S}_k = \begin{pmatrix} \tilde{A}_k & \tilde{A}_k B_k \\ C_k\tilde{A}_k & C_k\tilde{A}_k B_k + (\tilde{A}_k^{-1})^T \end{pmatrix}, \quad k \in J. \tag{2.6.5}$$

If system (2.1.5) is *time invariant*, that is, if $A_k \equiv A$, $B_k \equiv B$, and $C_k \equiv C$ hold on J with (constant) $n \times n$-matrices A, B, and C, then this computation simplifies (with $\tilde{A} = (I - A)^{-1}$) even more to

$$X_k = \begin{pmatrix} I & 0 \end{pmatrix} \tilde{S}^k \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad k \in J^* \text{ with } \tilde{S} = \begin{pmatrix} \tilde{A} & \tilde{A}B \\ C\tilde{A} & C\tilde{A}B + (\tilde{A}^{-1})^T \end{pmatrix}. \tag{2.6.6}$$

It is easy to do this using a computer, and then it is also no problem to take the obtained sequence of matrices $X_1, X_2, \dots, X_{N+1}$ and check the conditions

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad X_k X_{k+1}^{\dagger}\tilde{A}_k B_k \geq 0, \quad k \in J \setminus \{0\}. \tag{2.6.7}$$

If all these $2N$ conditions are satisfied, then $\mathcal{F}_0$ turns out to be positive definite due to Theorem 2.6.1. Conversely, if one of the conditions is not satisfied, then the computer may cease, and $\mathcal{F}_0$ is not positive definite in this case.

While Theorem 2.6.1(iv) is very easy to check, Theorem 2.6.1(i) is not at all. Here one needs to take any possible candidate (x, u), that is, any admissible (x, u) with $x_0 = x_{N+1} = 0$ and $x \neq 0$, plug it into $\mathcal{F}_0$, and check whether the result is in fact positive or not. Condition Theorem 2.6.1(ii) will be of theoretical use when examining Sturm-Liouville difference equations of higher order in the subsequent Section 2.7.

Finally we remark that there will be more characterizations of $\mathcal{F}_0$ given later. However, for completing this result (see Theorems 2.6.16 and 2.6.18) we need the results of both of the next subsections.

2.6.2. Sturm's separation and comparison theorems

Theorem 2.6.3 (Sturm's separation theorem). *If (at least) one conjoined basis of (2.1.5) has no focal point in* $(0, N+1]$, *then the principal solution of (2.1.5) at* 0 *has no focal point in* $(0, N+1]$ *either.*

Proof. Let (X, U) be a conjoined basis of (2.1.5) without focal points in $(0, N+1]$, that is,

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k \geq 0 \quad \forall k \in J. \tag{2.6.8}$$

We put

$$Q = XX^{\dagger} U X^{\dagger}, \tag{2.6.9}$$

and then Picone's identity (2.4.27) from Proposition 2.4.8 yields for admissible (x, u) with $x_0 = x_{N+1} = 0$ and $z = u - Qx$ that

$$\begin{gathered} \mathcal{F}_0(x, u) = \sum_{k=0}^{N} z_k^T D_k z_k, \\ x + Dz = \{\tilde{A}^{-1} - B\tilde{A}^T (Q^{\sigma} - C)\} x^{\sigma} \end{gathered} \tag{2.6.10}$$

hold. With Lemma 2.4.6 we now have $\mathcal{F}_0 > 0$, and hence it follows with the Reid roundabout theorem of the preceding subsection that the principal solution of (2.1.5) at 0 has no focal point in $(0, N+1]$. □

Remark 2.6.4. Let be given $\mu, \nu \in J^*$ with $\mu < \nu$. Then it follows as in the proof of Theorem 2.6.3 that the principal solution of (2.1.5) at μ has no focal points in $(0, N+1]$ whenever there exists a conjoined basis of (2.1.5) without focal points in $(\mu, \nu]$. It is also possible to read this result the other way around: whenever the principal solution of (2.1.5) at μ has a focal point in $(\mu, \nu]$, then *any* conjoined basis of (2.1.5) has a focal point in $(\mu, \nu]$ as well. This statement is a generalization of Sturm's separation theorem (see Theorem 1.4.4 and [173, Theorem 6.5]) for the Sturm-Liouville difference equations of second order (see Example 2.5.3).

The following theorem, for which we need an easy auxiliary result, is a generalization of Sturm's comparison theorem (see Theorem 1.4.3 and [173, Theorems 6.19 and 8.12]). Observe also that our comparison result is an extension of the corresponding result for invertible B from [123, Theorem 3].

Lemma 2.6.5. *For $k \in J$, let $H_k := \begin{pmatrix} -C_k - A_k^T B_k^{\dagger} A_k & A_k^T B_k^{\dagger} \\ B_k^{\dagger} A_k & -B_k^{\dagger} \end{pmatrix}$. Then for any admissible (x, u),*

$$x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k = -\begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix}^T H_k \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix} \quad \forall k \in J. \tag{2.6.11}$$

Proof. Let (x, u) be admissible, that is, $\Delta x = Ax^\sigma + Bu$ holds. Then on J we have

$$\begin{aligned}
&-\begin{pmatrix} x^\sigma \\ \Delta x \end{pmatrix}^T H \begin{pmatrix} x^\sigma \\ \Delta x \end{pmatrix} - (x^\sigma)^T C x^\sigma \\
&\quad = (x^\sigma)^T A^T B^\dagger A x^\sigma - (x^\sigma)^T A^T B^\dagger \Delta x - \Delta x^T B^\dagger A x^\sigma + \Delta x^T B^\dagger \Delta x \\
&\quad = \{\Delta x - Ax^\sigma\}^T B^\dagger \{\Delta x - Ax^\sigma\} \\
&\quad = u^T B B^\dagger B u \\
&\quad = u^T B u.
\end{aligned} \tag{2.6.12}$$

This ensures the validity of (2.6.11). □

Theorem 2.6.6 (Sturm's comparison theorem). *Consider the two systems (2.1.5), that is,*

$$\Delta x = Ax^\sigma + Bu, \quad \Delta u = Cx^\sigma - A^T u \quad \text{on } J, \tag{2.6.13}$$

$$\Delta x = \underset{\sim}{A} x^\sigma + \underset{\sim}{B} u, \quad \Delta u = \underset{\sim}{C} x^\sigma - \underset{\sim}{A}^T u \quad \text{on } J, \tag{2.6.14}$$

as well as the $2n \times 2n$*-matrix-valued functions on* J

$$H = \begin{pmatrix} -C - A^T B^\dagger A & A^T B^\dagger \\ B^\dagger A & -B^\dagger \end{pmatrix}, \qquad \underset{\sim}{H} = \begin{pmatrix} -\underset{\sim}{C} - \underset{\sim}{A}^T \underset{\sim}{B}^\dagger \underset{\sim}{A} & \underset{\sim}{A}^T \underset{\sim}{B}^\dagger \\ \underset{\sim}{B}^\dagger \underset{\sim}{A} & -\underset{\sim}{B}^\dagger \end{pmatrix}. \tag{2.6.15}$$

If the principal solution of (2.6.14) at 0 *has a focal point in* $(0, N+1]$ *and if*

$$\underset{\sim}{H} \leq H, \qquad \operatorname{Im}\begin{pmatrix} \underset{\sim}{A} - A & \underset{\sim}{B} \end{pmatrix} \subset \operatorname{Im} B \tag{2.6.16}$$

holds, then any conjoined basis of (2.1.5) has a focal point in $(0, N+1]$.

Proof. We suppose that there exists a conjoined basis of (2.1.5) without focal points in $(0, N+1]$. Then the principal solution of (2.1.5) at 0 has no focal points in $(0, N+1]$ either because of Theorem 2.6.3, and therefore Theorem 2.6.1 implies

$$\sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} > 0 \tag{2.6.17}$$

for each (with respect to (2.1.5)) admissible (x, u) with $x_0 = x_{N+1} = 0$ and $x \neq 0$.

Now we let be given an arbitrary (with respect to (2.6.14)) admissible $(x, \underset{\sim}{u})$ with $x_0 = x_{N+1} = 0$ and $x \neq 0$. We thus have $\Delta x = \underset{\sim}{A} x^\sigma + \underset{\sim}{B}\,\underset{\sim}{u}$. Using (2.6.16) there now exists u on J^* with

$$Bu = \{\underset{\sim}{A} - A\} x^\sigma + \underset{\sim}{B}\,\underset{\sim}{u} = \underset{\sim}{A} x^\sigma + \underset{\sim}{B}\,\underset{\sim}{u} - Ax^\sigma = \Delta x - Ax^\sigma \quad \text{on } J, \tag{2.6.18}$$

and thus (x, u) is admissible with respect to (2.1.5). Since $x_0 = x_{N+1} = 0$ and $x \neq 0$, we know that $\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}$ is positive definite. Applying (2.6.11) from Lemma 2.6.5 twice, we find

$$\begin{aligned}
\sum_{k=0}^{N}\{x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + \underset{\sim}{u}_k^T \underset{\sim}{B}_k \underset{\sim}{u}_k\} &= -\sum_{k=0}^{N}\begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix}^T \underset{\sim}{H}_k \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix} \\
&\geq -\sum_{k=0}^{N}\begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix}^T H_k \begin{pmatrix} x_{k+1} \\ \Delta x_k \end{pmatrix} \\
&= \sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} \\
&> 0.
\end{aligned} \tag{2.6.19}$$

Therefore

$$\sum_{k=0}^{N}\{x_{k+1}^T \underset{\sim}{C}_k x_{k+1} + u_k^T \underset{\sim}{B}_k u_k\} > 0 \tag{2.6.20}$$

for all (with respect to (2.6.14)) admissible (x, u) with $x_0 = x_{N+1} = 0$ and $x \neq 0$.

Another application of Theorem 2.6.1 now yields that the principal solution of (2.6.14) at 0 has no focal point in $(0, N + 1]$. □

Corollary 2.6.7. *Let the assumption (2.6.16) be satisfied. If system (2.1.5) is disconjugate on J^*, then system (2.6.14) is disconjugate on J^* also.*

Proof. This statement follows directly from Theorem 2.6.6 by also taking into account Theorem 2.6.1. □

Remark 2.6.8. (i) If B is invertible, then the second part of assumption (2.6.16) is not needed anymore. For this case we also want to refer to the corresponding result in [26, Theorem 6].

(ii) The relation

$$H = \begin{pmatrix} I & -A^T \\ 0 & I \end{pmatrix}\begin{pmatrix} -C & 0 \\ 0 & -B^\dagger \end{pmatrix}\begin{pmatrix} I & 0 \\ -A & I \end{pmatrix} \tag{2.6.21}$$

obviously holds. Thus, if $A = \underset{\sim}{A}$, then the conditions

$$\underset{\sim}{B}^\dagger \geq B^\dagger, \qquad \underset{\sim}{C} \geq C, \qquad \operatorname{Im} \underset{\sim}{B} \subset \operatorname{Im} B, \tag{2.6.22}$$

already imply the statement of Sturm's comparison theorem.

(iii) For the case $B \geq 0$ we may also employ the condition from [182, Theorem 3.1.11].

2.6.3. The extended Reid roundabout theorem

In this subsection we now prove the main result of the present chapter. To begin with, let be given (see also Definition 2.4.1) two $2n \times 2n$-matrices R and S. While S has to be symmetric, the choice of R may be completely arbitrary. We wish to recall and emphasize again what has been shown in Lemma 2.5.19(ii) for the principal solution (X, U) of (2.1.5) at 0, namely, that $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$ on J together with controllability of the system (2.1.5) on J^* implies the invertibility of X_{N+1}. With this preliminary remark our central result then reads as follows.

Theorem 2.6.9 (extended Reid roundabout theorem). *Let the system (2.1.5) be controllable on J^*. Then the following statements are equivalent:*

(i) $\mathcal{F}(x,u) = \sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} + \binom{-x_0}{x_{N+1}}^T S \binom{-x_0}{x_{N+1}} > 0$ *for all admissible (x,u) with $\binom{-x_0}{x_{N+1}} \in \operatorname{Im} R^T$ and $x \neq 0$;*

(ii) *system (2.1.5) is disconjugate on J^*, and*

$$\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} > 0 \tag{2.6.23}$$

for all solutions (x,u) of (2.1.5) with $\binom{-x_0}{x_{N+1}} \in \operatorname{Im} R^T \setminus \{0\}$;

(iii) *the special normalized conjoined bases (X, U), $(\tilde{X}, \tilde{U})$ of (2.1.5) at* 0 *satisfy: X has no focal points in $(0, N+1]$, and*

$$M := R\left\{S + \begin{pmatrix} -X_{N+1}^{-1}\tilde{X}_{N+1} & X_{N+1}^{-1} \\ (X_{N+1}^{-1})^T & U_{N+1}X_{N+1}^{-1} \end{pmatrix}\right\} R^T > 0 \quad \text{on } \operatorname{Im} R. \tag{2.6.24}$$

Proof. First of all we assume that statement (i) holds. Then

$$\sum_{k=0}^{N}\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} > 0 \quad \text{for all admissible } (x,u) \text{ with } x_0 = x_{N+1} = 0 \text{ and } x \neq 0. \tag{2.6.25}$$

From Theorem 2.6.1, the disconjugacy of (2.1.5) on J^* follows. Now let (x,u) be a solution of (2.1.5) with $\binom{-x_0}{x_{N+1}} \in \operatorname{Im} R^T \setminus \{0\}$. Then it follows with (2.4.7) from Corollary 2.4.4 that

$$0 < \mathcal{F}(x,u) = \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \tag{2.6.26}$$

holds, and this proves that statement (ii) is true.

Now suppose condition (ii) holds. Since (2.1.5) is disconjugate on J^*, the principal solution (X, U) of (2.1.5) at 0 has no focal points in $(0, N+1]$ according to Theorem 2.6.1. We let $c \in \operatorname{Im} R \setminus \{0\}$ and define

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix} X_{N+1}^{*^{-1}} R^T c, \tag{2.6.27}$$

where $(\tilde{X}, \tilde{U})$ is the associated solution of (2.1.5) at 0 (see Definition 2.3.3(iii)) and where $X^* = \begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix}$ as usual. Obviously (x, u) is a solution of (2.1.5). We have both

$$\begin{aligned} \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} &= \begin{pmatrix} U_0 & \tilde{U}_0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} X_{N+1}^{*^{-1}} R^T c = U_{N+1}^* X_{N+1}^{*^{-1}} R^T c, \\ \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} &= \begin{pmatrix} -X_0 & -\tilde{X}_0 \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix} X_{N+1}^{*^{-1}} R^T c = R^T c, \end{aligned} \tag{2.6.28}$$

where we put again $U^* = \begin{pmatrix} I & 0 \\ U & \tilde{U} \end{pmatrix}$. Since $c = R\tilde{c} \in \operatorname{Im} R \setminus \{0\}$, $R^T c \neq 0$ follows, because $R^T c = 0$ would imply

$$0 = \tilde{c}^T R^T c = \tilde{c}^T R^T R \tilde{c} = \|R\tilde{c}\|^2 = \|c\|^2. \tag{2.6.29}$$

Therefore $\begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \in \operatorname{Im} R^T \setminus \{0\}$ holds. Condition (ii) guarantees

$$\begin{aligned} 0 &< \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} \\ &= c^T R S R^T c + c^T R U_{N+1}^* X_{N+1}^{*^{-1}} R^T c \\ &= c^T R \left\{ S + \begin{pmatrix} I & 0 \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} \begin{pmatrix} -X_{N+1}^{-1} \tilde{X}_{N+1} & X_{N+1}^{-1} \\ I & 0 \end{pmatrix} \right\} R^T c \\ &= c^T M c \end{aligned} \tag{2.6.30}$$

(observe also Lemma 2.8.7). Hence M is positive definite on the image of R, and condition (iii) is satisfied.

Finally we assume (iii). With (see (2.3.56))

$$Q^* = \begin{pmatrix} -X^\dagger \tilde{X} X^\dagger X & X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X) U^T \\ \{X^\dagger + X^\dagger \tilde{X}(I - X^\dagger X) U^T\}^T & U X^\dagger + (U X^\dagger \tilde{X} - \tilde{U})(I - X^\dagger X) U^T \end{pmatrix}, \tag{2.6.31}$$

we have $M = R(S + Q^*_{N+1})R^T$. Let (x, u) be admissible with $\binom{-x_0}{x_{N+1}} = R^T c \in \operatorname{Im} R^T$. Then it follows from the extension of Picone's identity (2.4.42) from Theorem 2.4.10 for the choice of $\alpha = -x_0$ that

$$\begin{aligned}
\mathcal{F}(x,u) &= \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T Q^*_{N+1} \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} - \begin{pmatrix} -x_0 \\ x_0 \end{pmatrix}^T Q^*_0 \begin{pmatrix} -x_0 \\ x_0 \end{pmatrix} \\
&\quad + \sum_{k=0}^{N} z_k^T D_k z_k + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} \\
&= c^T M c + \sum_{k=0}^{N} z_k^T D_k z_k \\
&\geq 0
\end{aligned} \tag{2.6.32}$$

holds since we may assume $c \in \operatorname{Im} R$ without loss of generality. If now $\mathcal{F}(x, u) = 0$, then

$$D_k z_k = 0 \quad \forall 0 \leq k \leq N \tag{2.6.33}$$

and $c = 0$, that is, $x_0 = x_{N+1} = 0$ follows. This implies together with (2.4.43)

$$x = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma \quad \text{on } J, \tag{2.6.34}$$

so that $x_{N+1} = x_N = \cdots = x_1 = x_0 = 0$ follows. Hence $\mathcal{F} > 0$. □

Remark 2.6.10. The most useful of the above conditions for numerical purposes is surely condition Theorem 2.6.9(iii). Besides checking that the principal solution of (2.1.5) at 0 has no focal points (what we already discussed in Remark 2.6.2) we only need to check the additional condition

$$R\left\{S + \begin{pmatrix} -X_{N+1}^{-1}\tilde{X}_{N+1} & X_{N+1}^{-1} \\ (X_{N+1}^{-1})^T & U_{N+1}X_{N+1}^{-1} \end{pmatrix}\right\} R^T > 0 \quad \text{on } \operatorname{Im} R, \tag{2.6.35}$$

if $R \neq 0$ holds. If $R = 0$, then this condition is removed, and we may apply Theorem 2.6.1 in this case anyway, in which we do not require controllability of (2.1.5) on J^*. Since R and S are known, we only need to find the matrices $\tilde{X}_{N+1}$ and U_{N+1}, and this is most easily done using the formula (for the notation see Remark 2.6.2)

$$\begin{pmatrix} X_{N+1} & \tilde{X}_{N+1} \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} = \tilde{S}_N \tilde{S}_{N-1} \cdots \tilde{S}_0 \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{2.6.36}$$

or in the time-invariant case with

$$\begin{pmatrix} X_{N+1} & \tilde{X}_{N+1} \\ U_{N+1} & \tilde{U}_{N+1} \end{pmatrix} = \tilde{S}^{N+1} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{2.6.37}$$

We now discuss at which points in the proof, controllability of (2.1.5) on J^* (see Definition 2.5.13) was needed and how it is possible to remove this assumption. First of all, Theorem 2.6.9(ii) always follows from Theorem 2.6.9(i), even if we do not have controllability of (2.1.5) on J^*. Now let condition Theorem 2.6.9(ii) hold. As before we let $c \in \operatorname{Im} R \setminus \{0\}$ and define now

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X & \tilde{X} \\ U & \tilde{U} \end{pmatrix} X_{N+1}^{*\dagger} R^T c. \tag{2.6.38}$$

Controllability of (2.1.5) on J^* is now not needed anymore if we require instead

$$\operatorname{Im} R^T \subset \operatorname{Im} X_{N+1}^*. \tag{2.6.39}$$

Because now we have as before with $R^T c = X_{N+1}^* d$

$$\begin{aligned} \begin{pmatrix} u_0 \\ u_{N+1} \end{pmatrix} &= U_{N+1}^* X_{N+1}^{*\dagger} R^T c = U_{N+1}^* X_{N+1}^{*\dagger} X_{N+1}^* d = Q_{N+1}^* X_{N+1}^* d = Q_{N+1}^* R^T c, \\ \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} &= X_{N+1}^* X_{N+1}^{*\dagger} R^T c = X_{N+1}^* X_{N+1}^{*\dagger} X_{N+1}^* d = X_{N+1}^* d = R^T c \neq 0, \end{aligned} \tag{2.6.40}$$

where we now work with Q^* given by (2.3.56) from Lemma 2.3.8. Thus, because of Theorem 2.6.9(ii) and $\left(\begin{smallmatrix} -x_0 \\ x_{N+1} \end{smallmatrix}\right) \in \operatorname{Im} R^T \setminus \{0\}$, we have

$$0 < c^T R S R^T c + c^T R Q_{N+1}^* R^T c = c^T R (S + Q_{N+1}^*) R^T c. \tag{2.6.41}$$

To go back to Theorem 2.6.9(i) starting from this (modified) condition Theorem 2.6.9(iii), one may proceed as in the proof to Theorem 2.6.9 without using controllability of (2.1.5) on J^*.

We thus have shown the following result.

Theorem 2.6.11. *Let (X, U), $(\tilde{X}, \tilde{U})$ be the special normalized conjoined bases of (2.1.5) at* 0.

(i) *If X has no focal points in $(0, N+1]$ and*

$$R\{S + Q_{N+1}^*\} R^T > 0 \quad \textit{on } \operatorname{Im} R, \tag{2.6.42}$$

then $\mathcal{F}$ is positive definite (where Q^* is given by (2.3.56)).

(ii) *In the case of*

$$\operatorname{Im} R^T \subset \operatorname{Im} \begin{pmatrix} 0 & I \\ X_{N+1} & \tilde{X}_{N+1} \end{pmatrix}, \tag{2.6.43}$$

the converse of the implication from (i) holds.

2.6.4. Separated boundary conditions

In this subsection we treat the case of *separated* boundary conditions. That is, we give a characterization of positive definiteness of $\mathcal{F}$ if R and S are of the special form

$$R = \begin{pmatrix} R_0 & 0 \\ 0 & R_{N+1} \end{pmatrix}, \qquad S = \begin{pmatrix} -S_0 & 0 \\ 0 & S_{N+1} \end{pmatrix} \tag{2.6.44}$$

with $n \times n$-matrices R_0, R_{N+1}, S_0, and S_{N+1}. As usual S is symmetric, that is, S_0 and S_{N+1} are symmetric. Then (2.4.1) yields

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0, \tag{2.6.45}$$

and $x \in \tilde{\mathcal{R}}$ if and only if

$$x_0 \in \operatorname{Im} R_0^T, \qquad x_{N+1} \in \operatorname{Im} R_{N+1}^T. \tag{2.6.46}$$

Of course Theorem 2.6.9 applies to this special case. However, we want to give another characterization of positive definiteness of $\mathcal{F}$ which has the advantage that only one single conjoined basis of (2.1.5) is needed, while Theorem 2.6.9(iii) needs two of them. Two start with, we now choose an $n \times n$-matrix S_0^* (see also Remark 2.2.4) with

$$\operatorname{rank} \begin{pmatrix} S_0^* & R_0 \end{pmatrix} = n, \qquad \operatorname{Im} R_0^T = \operatorname{Ker} S_0^*, \tag{2.6.47}$$

and define $R_0^* := R_0 S_0 + S_0^*$. Then we obtain the formulas

$$\begin{gathered} R_0 S_0^{*T} = S_0^* R_0^T = 0, \qquad R_0 R_0^{*T} = R_0 S_0 R_0^T = R_0^* R_0^T, \\ \operatorname{rank}\begin{pmatrix} R_0 & R_0^* \end{pmatrix} = \operatorname{rank}\begin{pmatrix} R_0 & R_0^* - R_0 S_0 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} R_0 & S_0^* \end{pmatrix} = n. \end{gathered} \tag{2.6.48}$$

Using the above notation, our main result concerning separated boundary conditions reads as follows.

Theorem 2.6.12 (separated boundary conditions). *Let system (2.1.5) be controllable on J^*, and let (X,U) be the conjoined basis of (2.1.5) satisfying the initial conditions*

$$X_0 = -R_0^T, \qquad U_0 = R_0^{*T}. \tag{2.6.49}$$

Then $\mathcal{F}$ is positive definite if and only if both of the following conditions are satisfied:

(i) *X has no focal points on $(0, N+1]$;*

(ii) *$M := R_{N+1}\{S_{N+1} + U_{N+1}X_{N+1}^{-1}\}R_{N+1}^T$ is positive definite on* $\operatorname{Im} R_{N+1}$.

Proof. First assume $\operatorname{Ker} X_{m+1} \not\subset \operatorname{Ker} X_m$ for some $m \in J$. Then, according to Lemma 2.4.5(i), there exists an admissible (x, u) with

$$x_0 = X_0 d \in \operatorname{Im} X_0, \qquad x_{N+1} = 0, \qquad x \neq 0, \qquad \mathcal{F}_0(x, u) = -d^T X_0^T U_0 d. \tag{2.6.50}$$

For this (x, u) we have (because of $X_0 = -R_0^T$) obviously $x \in \tilde{\mathcal{R}}$ and

$$\begin{aligned}
\mathcal{F}(x, u) &= \mathcal{F}_0(x, u) + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0 \\
&= -d^T X_0^T U_0 d - d^T X_0^T S_0 X_0 d \\
&= d^T \{R_0 R_0^{*T} - R_0 S_0 R_0^T\} d \\
&= 0,
\end{aligned} \tag{2.6.51}$$

so that $\mathcal{F}$ is not positive definite in this case.

Now assume $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$ and that $D_m = X_m X_{m+1}^\dagger \tilde{A}_m B_m \geq 0$ does not hold for some $m \in J$. Then there exists $c \in \mathbb{R}^n$ with $c^T D_m c < 0$. From Lemma 2.4.5(ii) it follows that there exists an admissible (x, u) with

$$x_0 = X_0 d \in \operatorname{Im} X_0, \qquad x_{N+1} = 0, \qquad \mathcal{F}_0(x, u) = c^T D_m c - d^T X_0^T U_0 d. \tag{2.6.52}$$

Hence $x \in \tilde{\mathcal{R}}$, $x \neq 0$, and

$$\begin{aligned}
\mathcal{F}(x, u) &= \mathcal{F}_0(x, u) - x_0^T S_0 x_0 \\
&= c^T D_m c - d^T X_0^T U_0 d - d^T X_0^T S_0 X_0 d \\
&= c^T D_m c + d^T \{R_0 R_0^{*T} - R_0 S_0 R_0^T\} d \\
&= c^T D_m c \\
&< 0.
\end{aligned} \tag{2.6.53}$$

Therefore $\mathcal{F}$ cannot be positive definite.

In a further step we now suppose that X has no focal points on $(0, N+1]$ but that M is not positive definite on $\operatorname{Im} R_{N+1}$. Then there exists

$$c \in \operatorname{Im} R_{N+1} \setminus \{0\} \quad \text{with} \quad c^T M c \leq 0. \tag{2.6.54}$$

For $d := R_{N+1}^T c$ we put

$$\begin{pmatrix} x \\ u \end{pmatrix} := \begin{pmatrix} X \\ U \end{pmatrix} X_{N+1}^{-1} d. \tag{2.6.55}$$

Hence we have

$$\begin{aligned}
x_0 &= X_0 X_{N+1}^{-1} d = -R_0^T X_{N+1}^{-1} d \in \operatorname{Im} R_0^T, \\
x_{N+1} &= X_{N+1} X_{N+1}^{-1} d = d = R_{N+1}^T c \in \operatorname{Im} R_{N+1}^T,
\end{aligned} \tag{2.6.56}$$

and thus also $x_{N+1} \neq 0$ because of $c \in \operatorname{Im} R_{N+1} \setminus \{0\}$. Therefore $x \in \tilde{\mathcal{R}}$ and $x \neq 0$. Obviously (x,u) is also admissible. It even solves the system (2.1.5), and thus (2.4.7) from Corollary 2.4.4 yields

$$\begin{aligned}
\mathcal{F}(x,u) &= x_{N+1}^T u_{N+1} - x_0^T u_0 + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0 \\
&= d^T U_{N+1} X_{N+1}^{-1} d + d^T \left(X_{N+1}^{-1}\right)^T R_0 U_0 X_{N+1}^{-1} d \\
&\quad + d^T S_{N+1} d - d^T \left(X_{N+1}^{-1}\right)^T R_0 S_0 R_0^T X_{N+1}^{-1} d \\
&= c^T R_{N+1} \{S_{N+1} + U_{N+1} X_{N+1}^{-1}\} R_{N+1}^T c \\
&\quad + d^T \left(X_{N+1}^{-1}\right)^T \{R_0 R_0^{*^T} - R_0 S_0 R_0^T\} X_{N+1}^{-1} d \\
&= c^T M c \\
&\le 0.
\end{aligned} \tag{2.6.57}$$

Again $\mathcal{F}$ is not positive definite.

It remains to show that $\mathcal{F}$ is positive definite whenever conditions (i) and (ii) hold. To do so we suppose that X has no focal points on $(0, N+1]$ and that $M > 0$ on $\operatorname{Im} R_{N+1}$. Let (x,u) be admissible with $x \in \tilde{\mathcal{R}}$. Then there exist $c_0, c_{N+1} \in \mathbb{R}^n$ with

$$x_0 = R_0^T c_0, \qquad x_{N+1} = R_{N+1}^T c_{N+1}. \tag{2.6.58}$$

Because of $x_0 \in \operatorname{Im} X_0$, we may now apply Picone's identity from Proposition 2.4.8, and because of

$$x_0^T X_0 X_0^\dagger U_0 X_0^\dagger x_0 = -c_0^T R_0 R_0^T \left(R_0^\dagger\right)^T R_0^{*^T} \left(R_0^\dagger\right)^T R_0^T c_0 = -c_0^T R_0^* R_0^T c_0, \tag{2.6.59}$$

by (2.4.27) we find

$$\begin{aligned}
\mathcal{F}(x,u) &= x_{N+1}^T U_{N+1} X_{N+1}^{-1} x_{N+1} + c_0^T R_0^* R_0^T c_0 + \sum_{k=0}^{N} z_k^T D_k z_k \\
&\quad + x_{N+1}^T S_{N+1} x_{N+1} - x_0^T S_0 x_0 \\
&= c_{N+1}^T R_{N+1} \{S_{N+1} + U_{N+1} X_{N+1}^{-1}\} R_{N+1}^T c_{N+1} \\
&\quad + c_0^T \{R_0^* R_0^T - R_0 S_0 R_0^T\} c_0 + \sum_{k=0}^{N} z_k^T D_k z_k \\
&= c_{N+1}^T M c_{N+1} + \sum_{k=0}^{N} z_k^T D_k z_k \\
&\ge 0.
\end{aligned} \tag{2.6.60}$$

If $\mathcal{F}(x,u)$ vanishes, then $c_{N+1} = 0$ and $D_k z_k = 0$ for all $k \in J$. Thus we find $x_{N+1} = R_{N+1}^T c_{N+1} = 0$. With (2.4.28) it follows that x is trivial. Altogether we have $\mathcal{F} > 0$ in this last case. $\square$

Remark 2.6.13. The condition on controllability of (2.1.5) on J^* was only needed (also in Theorem 2.6.12) to ensure invertibility of X_{N+1} if $\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X$. While these conditions are under certain assumptions equivalent in case of the principal solution (X,U) of (2.1.5) at 0 (see Corollary 2.5.20(ii)), we could have derived the above theorem without controllability of (2.1.5) on J^* in case of invertibility of X_0, that is, of R_0.

Furthermore we can give a statement without the assumption of controllability even when R_0 is singular. As in Remark 2.6.10 one may prove the following result.

Theorem 2.6.14. *Let (X,U) be the conjoined basis of (2.1.5) with (2.6.49).*

(i) *If X has no focal points on $(0,N+1]$ and if*

$$R_{N+1}\{S_{N+1}+X_{N+1}X_{N+1}^{\dagger}U_{N+1}X_{N+1}^{\dagger}\}R_{N+1}^T>0 \quad \text{on } \operatorname{Im} R_{N+1}, \tag{2.6.61}$$

then $\mathcal{F}$ is positive definite.

(ii) *In the case of*

$$\operatorname{Im} R_{N+1}^T \subset \operatorname{Im} X_{N+1}, \tag{2.6.62}$$

the converse of the implication from (i) holds.

Remark 2.6.15 (C-disfocality). Finally we shortly discuss a special situation with separated boundary conditions which has been called "C-disfocality" (see [218, 222, 227] by Peil, Peterson, and Ridenhour). Here, a system (2.1.5) is called *C-disfocal* on J^* whenever

$$\begin{gathered}\sum_{k=0}^{N}\{x_{k+1}^TC_kx_{k+1}+u_k^TB_ku_k\}-x_{N+1}^TC_Nx_{N+1}>0\\ \text{for all admissible } (x,u) \text{ with } x_0=0 \text{ and } x\neq 0.\end{gathered} \tag{2.6.63}$$

With our notation we then have

$$R_0=S_0=0,\qquad R_{N+1}=I,\qquad S_{N+1}=-C_N,\qquad R_0^*=S_0^*=I. \tag{2.6.64}$$

The conjoined basis (X,U) of (2.1.5) from Theorem 2.6.12 satisfies the conditions

$$X_0=-R_0^T=0,\qquad U_0=R_0^{*T}=I. \tag{2.6.65}$$

Therefore, (X,U) is the principal solution of (2.1.5) at 0. If (2.1.5) is controllable on J^* (and this is the case, e.g., in [222, Theorem 2]), then (2.1.5) is C-disfocal on J^* according to Theorem 2.6.12 if and only if X has no focal points on $(0,N+1]$ and if

$$U_{N+1}X_{N+1}^{-1}>C_N, \tag{2.6.66}$$

where the last condition is equivalent to

$$U_N X_N^{-1}\left(I + B_N U_N X_N^{-1}\right) > 0 \tag{2.6.67}$$

if X_N is invertible.

2.6.5. Solvability of discrete Riccati equations

As described by Ahlbrandt and Heifetz [32], discrete Riccati matrix difference equations of the form (2.1.7), that is,

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k \left(I + B_k Q_k\right)^{-1}\left(I - A_k\right) \tag{2.6.68}$$

show up in applications including discrete Kalman filtering, robust control, and H^∞ control, and there is a need to answer the question of solvability of these equations by a sequence of symmetric matrices $Q_0, Q_1, \dots, Q_{N+1}$ with

$$D_k = \left(I + B_k Q_k\right)^{-1} B_k \geq 0 \quad \forall 0 \leq k \leq N \tag{2.6.69}$$

(see also [33, 35, 151, 165, 214]). Suitable application of the results presented in this chapter enable us to prove the following central result which confirms a conjecture of Erbe and Yan from [124].

Theorem 2.6.16 (solvability of discrete Riccati equations). *There exists a symmetric solution Q of the Riccati matrix difference equation (2.1.7), that is,*

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k \left(I + B_k Q_k\right)^{-1}\left(I - A_k\right), \tag{2.6.70}$$

with $(I + B_k Q_k)^{-1} B_k \geq 0$ for all $0 \leq k \leq N$, if and only if the principal solution of (2.1.5) at 0 has no focal points on $(0, N+1]$.

Proof. We show that the following two conditions are equivalent:

(i) the principal solution (X, U) of (2.1.5) at 0 has no focal points on $(0, N+1]$, that is,

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k \geq 0 \quad \forall 0 \leq k \leq N, \tag{2.6.71}$$

(ii) there exists a conjoined basis $(\hat{X}, \hat{U})$ of (2.1.5) with invertible $\hat{X}_0$ and without focal points on $(0, N+1]$, that is,

$$\begin{gathered} \hat{X}_k \text{ is invertible} \quad \forall 0 \leq k \leq N+1, \\ \hat{D}_k = \hat{X}_k \hat{X}_{k+1}^{-1} \tilde{A}_k B_k \geq 0 \quad \forall 0 \leq k \leq N. \end{gathered} \tag{2.6.72}$$

Lemma 2.3.10 then yields our claim.

First of all our Sturm separation theorem, Theorem 2.6.3, shows that condition (ii) implies (i). Now suppose that condition (i) holds, and let (X, U), $(\tilde{X}, \tilde{U})$ be the special normalized conjoined bases of (2.1.5) at 0. For the symmetric matrix $\hat{Q} = -X_{N+1}^{\dagger}\tilde{X}_{N+1}X_{N+1}^{\dagger}X_{N+1}$ there exists an orthogonal P with

$$P\hat{Q}P^T = \hat{P} := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n), \tag{2.6.73}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\hat{Q}$. Let

$$\varepsilon := \frac{1}{1 + |\lambda|} > 0 \quad \text{with} \quad \lambda := \min_{1 \le i \le n} \lambda_i. \tag{2.6.74}$$

Then we have for $x \in \mathbb{R}^n \setminus \{0\}$ and $y = Px$,

$$\begin{aligned}
x^T(I + \varepsilon\hat{Q})x &= \|x\|^2 + \varepsilon x^T P^T \hat{P} P x \\
&= \|y\|^2 + \varepsilon y^T \hat{P} y \\
&= \|y\|^2 + \varepsilon \sum_{i=1}^{n} \lambda_i y_i^2 \\
&\ge \|y\|^2 + \varepsilon\lambda \sum_{i=1}^{n} y_i^2 \\
&= \|y\|^2 \left\{ 1 + \frac{\lambda}{1 + |\lambda|} \right\} \\
&\ge \|y\|^2 \left\{ 1 - \frac{|\lambda|}{1 + |\lambda|} \right\} \\
&> 0,
\end{aligned} \tag{2.6.75}$$

so that $I + \varepsilon\hat{Q} > 0$. We now put

$$R = \begin{pmatrix} -\varepsilon I & 0 \\ 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} \frac{1}{\varepsilon} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.6.76}$$

Then it follows with Q given by (2.3.56) that

$$R\{S + Q_{N+1}^*\}R^T = \begin{pmatrix} \varepsilon^2 \left(\frac{1}{\varepsilon} I + \hat{Q} \right) & 0 \\ 0 & 0 \end{pmatrix} > 0 \quad \text{on } \operatorname{Im} R, \tag{2.6.77}$$

and our Reid roundabout theorem, Theorem 2.6.11(i), now yields

$$\begin{gathered}
\sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} + \frac{1}{\varepsilon} ||x_0||^2 > 0 \\
\text{for all admissible } (x, u) \text{ with } x_{N+1} = 0 \text{ and } x \neq 0.
\end{gathered} \tag{2.6.78}$$

Since the present boundary conditions are separated, we may as well apply our Reid roundabout theorem for separated boundary conditions, Theorem 2.6.12. With the terminology from there we have

$$R_0 = -\varepsilon I, \qquad S_0 = -\frac{1}{\varepsilon} I, \qquad S_0^* = 0, \qquad R_0^* = I, \qquad R_{N+1} = S_{N+1} = 0. \tag{2.6.79}$$

Since R_0 is invertible (see Remark 2.6.13) and since condition Theorem 2.6.12(ii) drops because of $R_{N+1} = 0$, it follows from Theorem 2.6.12 that the conjoined basis $(\hat{X}, \hat{U})$ of (2.1.5) with

$$\hat{X}_0 = -R_0^T = \varepsilon I, \qquad \hat{U}_0 = R_0^{*T} = I \tag{2.6.80}$$

has no focal points on $(0, N+1]$ and thus satisfies

$$\begin{aligned} &\hat{X}_k \text{ is invertible} \quad \forall k \in J^*, \\ &\hat{X}_k \hat{X}_{k+1}^{-1} \tilde{A}_k B_k \geq 0 \quad \forall k \in J. \end{aligned} \tag{2.6.81}$$

Hence condition (ii) holds. □

Remark 2.6.17. The solution Q constructed in the proof of Theorem 2.6.16 satisfies the initial condition

$$Q_0 = \{1 + |\lambda|\} I, \tag{2.6.82}$$

where λ is the smallest eigenvalue of

$$-X_{N+1}^{\dagger} \tilde{X}_{N+1} X_{N+1}^{\dagger} X_{N+1}, \tag{2.6.83}$$

and where (X, U), $(\tilde{X}, \tilde{U})$ are the special normalized conjoined bases of (2.1.5) at 0. Therefore it is not necessarily easy to compute a solution Q since λ may not be found in a convenient way.

After the examinations in Section 2.3.3 it is clear that with

$$Q = XX^{\dagger} U X^{\dagger} \quad \text{or} \quad Q = UX^{\dagger} + (UX^{\dagger} \tilde{X} - \tilde{U})(I - X^{\dagger} X) U^T, \tag{2.6.84}$$

we already have symmetric matrices Q that are easier to compute than the above Q and that solve again certain "implicit Riccati equations." By putting again

$$\begin{aligned} R[Q] &= \tilde{A}^T (Q^{\sigma} - C) \tilde{A} (I + BQ) - Q, \\ G_k &= \begin{pmatrix} \tilde{A}_{k-1} \tilde{A}_{k-2} \cdots \tilde{A}_0 B_0 & \tilde{A}_{k-1} \tilde{A}_{k-2} \cdots \tilde{A}_1 B_1 & \cdots & \tilde{A}_{k-1} B_{k-1} \end{pmatrix} \end{aligned} \tag{2.6.85}$$

as in (2.3.42) and (2.3.14), we may extend Theorem 2.6.1 in this direction.

Theorem 2.6.18 (Reid roundabout theorem). *All of the following conditions are equivalent:*

(i) $\mathcal{F}_0 > 0$;
(ii) *system (2.1.5) is disconjugate on* J^*;
(iii) *the principal solution of (2.1.5) at* 0 *has no focal points on* $(0, N+1]$;
(iv) *there exists a conjoined basis* (X, U) *of (2.1.5) with invertible* X_0 *and without focal points on* $(0, N+1]$;
(v) *the Riccati equation*

$$Q^\sigma = C + (I - A^T)Q(I + BQ)^{-1}(I - A) \tag{2.6.86}$$

has a symmetric solution Q *on* J *with* $(I + BQ)^{-1}B \geq 0$;
(vi) *the "implicit Riccati equation"* $R[Q]G = 0$ *has a symmetric solution* Q *on* J *with* $Q_0 = 0$ *and* $B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B \geq 0$;
(vii) *the "implicit Riccati equation"* $(B \quad G^T)R[Q]G = 0$ *has a symmetric solution* Q *on* J *with* $Q_0 = 0$ *and* $B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B \geq 0$.

Proof. The equivalence of statements (i)–(v) is already clear with Theorems 2.6.1 and 2.6.16. We now show that (vi) is equivalent to these statements. The equivalence of (vii) to these statements then follows similarly.

First assume (iii). By (X, U), $(\tilde{X}, \tilde{U})$ we denote the special normalized conjoined bases of (2.1.5) at 0. Then we have

$$\operatorname{Ker} X^\sigma \subset \operatorname{Ker} X, \quad X(X^\sigma)^\dagger \tilde{A}B \geq 0 \quad \text{on } J. \tag{2.6.87}$$

As in (2.3.55) we put

$$Q = UX^\dagger + (UX^\dagger\tilde{X} - \tilde{U})(I - X^\dagger X)U^T \quad \text{on } J. \tag{2.6.88}$$

Then Q is symmetric because of Lemma 2.3.8 and satisfies $QX = UX^\dagger X$ on J. Lemma 2.3.7(ii) then yields $R[Q]X = 0$ on J, and this implies together with Corollary 2.5.20(i) that

$$R[Q]G = 0 \quad \text{on } J. \tag{2.6.89}$$

Moreover, $Q_0 = 0$ and with Lemma 2.3.7(ii)

$$B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B = X(X^\sigma)^\dagger\tilde{A}B \geq 0 \quad \text{on } J, \tag{2.6.90}$$

so that statement (vi) follows.

Conversely assume (vi). Let (x, u) be admissible with $x_0 = x_{N+1} = 0$. Then $x \in \operatorname{Im} G$ on J by (2.3.15) from Lemma 2.5.7, hence

$$R[Q]x = 0 \quad \text{on } J. \tag{2.6.91}$$

From Lemma 2.4.7 it now follows for $z = u - Qx$ and $D = B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B \geq 0$ that

$$\mathcal{F}_0(x, u) = \sum_{k=0}^{N} z_k^T D_k z_k, \quad x + Dz = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma \quad \text{on } J. \tag{2.6.92}$$

Therefore the assumptions of Lemma 2.4.6 are satisfied and $\mathcal{F}_0 > 0$ follows. □

Remark 2.6.19. If the system (2.1.5) is controllable on J^* with controllability index $\kappa \in J^*$, then $R[Q]_k G_k = 0$ (resp., $G_k^T R[Q]_k G_k = 0$) is equivalent to $R[Q]_k = 0$ for all $k \in [\kappa, N+1] \cap \mathbb{Z}$. However, in the interval $[0, \kappa] \cap \mathbb{Z}$, instead of knowing the initial values of the solution Q in Theorem 2.6.18(vi) (and Theorem 2.6.18(vii)), we now have the equation $R[Q]_k G_k = 0$ that is not very easy to deal with.

By applying the proof of the preceding theorem to the big system (2.3.40), we may extend our main result Theorem 2.6.9 by using Lemma 2.3.6 and Corollary 2.5.20(i) and adding Riccati conditions as shown in the following example. The equation

$$R^*[Q^*]\begin{pmatrix} I & 0 \\ -\Phi & G \end{pmatrix} = 0 \tag{2.6.93}$$

has a symmetric solution Q on J given by $Q^* = \begin{pmatrix} \hat{Q} & \tilde{Q} \\ \tilde{Q}^T & Q \end{pmatrix}$ with

$$Q_0^* = 0, \qquad B - B\tilde{A}^T(Q^\sigma - C)\tilde{A}B \geq 0, \tag{2.6.94}$$

where we put

$$R^*[Q^*] = \tilde{A}^{*^T}\left((Q^*)^\sigma - C^*\right)\tilde{A}^*(I + B^*Q^*) - Q^*, \tag{2.6.95}$$

as in (2.3.38) from Lemma 2.3.6 and

$$\Phi_k = \tilde{A}_{k-1}\tilde{A}_{k-2}\cdots\tilde{A}_0 \quad \text{for } k \in J^*, \tag{2.6.96}$$

as in (2.3.12).

2.7. Sturm-Liouville difference equations

2.7.1. Disconjugacy

In this section we assume for convenience (except for Remark 2.7.10)

$$N \geq n \quad \text{with} \quad N, n \in \mathbb{N}. \tag{2.7.1}$$

Moreover, let be given real numbers

$$r_k^{(\nu)} \in \mathbb{R} \quad \text{with} \quad 0 \leq \nu \leq n,\ k \in \mathbb{N}_0, \tag{2.7.2}$$

and let the assumption

$$r_k^{(n)} \neq 0 \quad \forall k \in \mathbb{N}_0 \tag{2.7.3}$$

be satisfied. Then we examine in this section special discrete quadratic functionals of the form

$$\mathcal{F}_0(y) = \sum_{k=0}^{N} \sum_{\nu=0}^{n} r_k^{(\nu)} \{\Delta^\nu y_{k+1-\nu}\}^2, \tag{2.7.4}$$

with real-valued functions

$$y : \bar{J} \longrightarrow \mathbb{R} \quad \text{with} \quad \bar{J} := [1-n, N+1] \cap \mathbb{Z}. \tag{2.7.5}$$

The question concerned with when $\mathcal{F}_0$ is positive definite (in a sense which will be described soon) is strongly connected with the Sturm-Liouville difference equation of order $2n$

$$L(y)_k := \sum_{\mu=0}^{n} (-\Delta)^\mu \{r_k^{(\mu)} \Delta^\mu y_{k+1-\mu}\} = 0 \quad \text{for } 0 \leq k \leq N-n. \tag{2.7.6}$$

Starting with the fundamental work [156] by Hartman, there were several attempts to define "disconjugacy" for equations (2.7.6) in a suitable way and to show that disconjugacy of (2.7.6) is equivalent with positive definiteness of $\mathcal{F}_0$ (see [36, Theorem 1]). Contrary to the methods that have been used to solve this question in the previous literature, we approach this problem in a different manner. For it is well known that any equation (2.7.6) is in a certain sense equivalent to a linear Hamiltonian difference system, and we will make this statement more precise. Then we may apply our results from the previous sections to this system and then in turn get results for (2.7.6) itself. The main result of this section is a Reid roundabout theorem which fits exactly to this special situation (see Theorem 2.7.9 of the next subsection). We now use the notation

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix},$$
$$A_k \equiv A, \quad B_k = \frac{1}{r_k^{(n)}} B, \quad C_k = \begin{pmatrix} r_k^{(0)} & & & \\ & r_k^{(1)} & & \\ & & \ddots & \\ & & & r_k^{(n-1)} \end{pmatrix}, \quad k \in \mathbb{N}_0. \tag{2.7.7}$$

Therefore $I - A$ is invertible, and with $\tilde{A} = (I - A)^{-1}$,

$$I - A = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}, \qquad \tilde{A} = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.7.8}$$

The matrices B_k and C_k are symmetric, so that the system

$$\Delta \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \begin{pmatrix} x^\sigma \\ u \end{pmatrix} \quad \text{on } J \tag{2.7.9}$$

is a Hamiltonian system subject to our usual assumptions (2.3.2). For the remainder of this subsection we use for $z \in \mathbb{R}^n$ the notation

$$z = \begin{pmatrix} z^{(0)} \\ \vdots \\ z^{(n-1)} \end{pmatrix} \quad \text{with} \quad z^{(0)}, \dots, z^{(n-1)} \in \mathbb{R}. \tag{2.7.10}$$

Then a pair (x, u) with $x, u : J^* \to \mathbb{R}^n$ is admissible for (2.7.9) if and only if

$$\begin{pmatrix} \Delta x^{(0)} \\ \vdots \\ \Delta x^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \begin{pmatrix} \left(x^{(0)}\right)^{\sigma} \\ \vdots \\ \left(x^{(n-1)}\right)^{\sigma} \end{pmatrix} + \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \dfrac{1}{r^{(n)}} \end{pmatrix} \begin{pmatrix} u^{(0)} \\ \vdots \\ u^{(n-1)} \end{pmatrix} \tag{2.7.11}$$

holds on J, that is, if

$$\Delta x^{(\nu)} = \left(x^{(\nu+1)}\right)^{\sigma}, \quad 0 \le \nu \le n-2; \qquad \Delta x^{(n-1)} = \frac{1}{r^{(n)}} u^{(n-1)} \quad \text{on } J. \tag{2.7.12}$$

Thus $x : J^* \to \mathbb{R}^n$ is admissible for (2.7.9) if and only if

$$\Delta x^{(\nu)} = \left(x^{(\nu+1)}\right)^{\sigma} \quad \text{on } J\ \forall 0 \le \nu \le n-2. \tag{2.7.13}$$

Finally an admissible (x, u) for (2.7.9) is a solution of (2.7.9) if and only if

$$\begin{pmatrix} \Delta u^{(0)} \\ \vdots \\ \Delta u^{(n-1)} \end{pmatrix} = \begin{pmatrix} r^{(0)} & & & \\ & r^{(1)} & & \\ & & \ddots & \\ & & & r^{(n-1)} \end{pmatrix} \begin{pmatrix} \left(x^{(0)}\right)^{\sigma} \\ \vdots \\ \left(x^{(n-1)}\right)^{\sigma} \end{pmatrix} - \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} u^{(0)} \\ \vdots \\ u^{(n-1)} \end{pmatrix} \tag{2.7.14}$$

is true on J, that is, in case of

$$\Delta u^{(\nu)} = r^{(\nu)} \left(x^{(\nu)}\right)^{\sigma} - u^{(\nu-1)}, \quad 1 \le \nu \le n-1; \qquad \Delta u^{(0)} = r^{(0)} \left(x^{(0)}\right)^{\sigma} \quad \text{on } J. \tag{2.7.15}$$

Lemma 2.7.1 (admissibility). *An $x : J^* \to \mathbb{R}^n$ is admissible for (2.7.9) if and only if there exist $y : \bar{J} \to \mathbb{R}$ with*

$$x_k = \begin{pmatrix} y_k \\ \Delta y_{k-1} \\ \Delta^2 y_{k-2} \\ \vdots \\ \Delta^{n-1} y_{k+1-n} \end{pmatrix} \quad \forall k \in J^*. \tag{2.7.16}$$

In this case for admissible (x, u),

$$u_k^{(n-1)} = r_k^{(n)} \Delta^n y_{k+1-n} \quad \forall k \in J, \tag{2.7.17}$$

as well as $\mathcal{F}_0(y) = \mathcal{F}_0(x, u)$, that is,

$$\sum_{k=0}^{N} \sum_{\nu=0}^{n} r_k^{(\nu)} \{\Delta^\nu y_{k+1-\nu}\}^2 = \sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\}. \tag{2.7.18}$$

Proof. Assume (2.7.16), that is, $x^{(\nu)} = (\sigma^{-1}\Delta)^\nu y$ on J^* for all $0 \le \nu \le n-1$. Then

$$\Delta x^{(\nu)} = \left\{(\sigma^{-1}\Delta)^{\nu+1} y\right\}\sigma = \left(x^{(\nu+1)}\right)^\sigma \quad \text{on } J \tag{2.7.19}$$

follows for each $0 \le \nu \le n-2$, that is, (2.7.15) holds. Thus x is admissible.

Conversely, now let (2.7.13) hold. Then we define $y : \bar{J} \to \mathbb{R}$ by

$$x_0 = \begin{pmatrix} y_0 \\ \Delta y_{-1} \\ \vdots \\ \Delta^{n-1} y_{1-n} \end{pmatrix}, \quad y = x^{(0)} \quad \text{on } J^*. \tag{2.7.20}$$

We have $x_0^{(\nu)} = \Delta^\nu y_{-\nu}$ for all $0 \le \nu \le n-1$. Now let $k \in J$. Then we first have $x_{k+1}^{(0)} = \Delta^0 y_{k+1-0}$, and in case of

$$x_k^{(\nu)} = \Delta^\nu y_{k-\nu} \quad \forall 0 \le \nu \le n-1, \qquad x_{k+1}^{(\mu)} = \Delta^\mu y_{k+1-\mu} \tag{2.7.21}$$

for some $0 \le \mu \le n-2$, we have

$$x_{k+1}^{(\mu+1)} = \Delta x_k^{(\mu)} = x_{k+1}^{(\mu)} - x_k^{(\mu)} = \Delta^\mu y_{k+1-\mu} - \Delta^\mu y_{k-\mu} = \Delta^{\mu+1} y_{k+1-(\mu+1)}, \tag{2.7.22}$$

so that (2.7.16) follows by induction.

Finally let (x, u) be admissible. Then there exists $y : \bar{J} \to \mathbb{R}$ with (2.7.16) and

$$u^{(n-1)} = r^{(n)} \Delta x^{(n-1)} = r^{(n)} \left\{ (\sigma^{-1}\Delta)^n y \right\}^\sigma \quad \text{on } J, \tag{2.7.23}$$

because of (2.7.12). The computation

$$\begin{aligned}
&\sum_{k=0}^{N} \{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\} \\
&\quad = \sum_{k=0}^{N} \left\{ \sum_{\nu=0}^{n-1} r_k^{(\nu)} (x_{k+1}^{(\nu)})^2 + \frac{1}{r_k^{(n)}} (u_k^{(n-1)})^2 \right\} \\
&\quad = \sum_{k=0}^{N} \left\{ \sum_{\nu=0}^{n-1} r_k^{(\nu)} (\Delta^\nu y_{k+1-\nu})^2 + \frac{1}{r_k^{(n)}} (r_k^{(n)} \Delta^n y_{k+1-n})^2 \right\} \\
&\quad = \sum_{k=0}^{N} \sum_{\nu=0}^{n} r_k^{(\nu)} \{\Delta^\nu y_{k+1-\nu}\}^2
\end{aligned} \tag{2.7.24}$$

shows $\mathcal{F}_0(y) = \mathcal{F}_0(x, u)$. □

Now the following fundamental result holds (see, e.g., [36, 124] and for the continuous case [182, Lemma 6.2.1]).

Proposition 2.7.2 (equivalence of (2.7.6) and (2.7.9)). *Let $x, u : J^* \to \mathbb{R}^n$ be given. Then (x, u) is a solution of (2.7.9) if and only if there exists a solution $y : \bar{J} \to \mathbb{R}$ of (2.7.6) with*

$$x_k^{(\nu)} = \Delta^\nu y_{k-\nu} \quad \forall 0 \le \nu \le n-1 \text{ and all } k \in J^*. \tag{2.7.25}$$

In this case there exist $y_{N+2}, \dots, y_{N+n+1} \in \mathbb{R}$ with $L(y)_k = 0$ for all $k \in J$ and

$$u_k^{(\nu)} = \sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu-1} \{r_k^{(\mu)} \Delta^\mu y_{k+1-\mu}\} \quad \forall 0 \le \nu \le n-1 \text{ and all } k \in J^*. \tag{2.7.26}$$

Proof. Let $x, u : J^* \to \mathbb{R}^n$. First let $y : \bar{J} \to \mathbb{R}$ be a solution of (2.7.6) with (2.7.16), that is, with $x^{(\nu)} = (\sigma^{-1}\Delta)^\nu y$ on J^* for all $0 \le \nu \le n-1$. Because of Lemma 2.7.1, x is admissible for (2.7.9). We pick $y_{N+2}, \dots, y_{N+n+1} \in \mathbb{R}$ with $L(y)_k = 0$ on all of J and define $u : J^* \to \mathbb{R}^n$ by

$$u^{(\nu)} := \sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu-1}\Big\{r^{(\mu)}\Big(\big(\sigma^{-1}\Delta\big)^{\mu}\Big)^{\sigma} y\Big\}, \quad 0 \le \nu \le n-1. \tag{2.7.27}$$

Thus it follows for $1 \le \nu \le n-1$ and on J

$$\begin{aligned}
\Delta u^{(\nu)} &= -\sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu}\Big\{r^{(\mu)}\Big(\big(\sigma^{-1}\Delta\big)^{\mu}\Big)^{\sigma} y\Big\} \\
&= r^{(\nu)}\Big\{\big(\sigma^{-1}\Delta\big)^{\nu} y\Big\}^{\sigma} - \sum_{\mu=\nu}^{n}(-\Delta)^{\mu-\nu}\Big\{r^{(\mu)}\Big(\big(\sigma^{-1}\Delta\big)^{\mu}\Big)^{\sigma} y\Big\} \\
&= r^{(\nu)}\big(x^{(\nu)}\big)^{\sigma} - u^{(\nu-1)},
\end{aligned} \tag{2.7.28}$$

as well as finally, again on J,

$$\Delta u^{(0)} = -\sum_{\mu=1}^{n}(-\Delta)^{\mu}\Big\{r^{(\mu)}\Big(\big(\sigma^{-1}\Delta\big)^{\mu}\Big)^{\sigma} y\Big\} = r^{(0)} y^{\sigma} - L(y) = r^{(0)}\big(x^{(0)}\big)^{\sigma}. \tag{2.7.29}$$

Hence (2.7.15) holds and thus (x, u) is a solution of system (2.7.9).

Conversely we suppose that (x, u) is a solution of (2.7.9) on $J = [0, N] \cap \mathbb{Z}$, which we extend to a solution of (2.7.9) on $[0, N+n] \cap \mathbb{Z}$. Because of Lemma 2.7.1 (with $N+n$ instead of N), there exists $y : [1-n, N+n+1] \cap \mathbb{Z} \to \mathbb{R}$ with

$$x^{(\nu)} = \big(\sigma^{-1}\Delta\big)^{\nu} y \quad \text{on } [0, N+n+1] \cap \mathbb{Z}, \tag{2.7.30}$$

and with

$$u^{(n-1)} = r^{(n)}\Big\{\big(\sigma^{-1}\Delta\big)^{n} y\Big\}^{\sigma} \quad \text{on } [0, N+n] \cap \mathbb{Z}. \tag{2.7.31}$$

Thus the statement

$$u^{(\nu)} = \sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu-1}\Big\{r^{(\mu)}\Big(\big(\sigma^{-1}\Delta\big)^{\mu}\Big)^{\sigma} y\Big\} \quad \text{on } [0, N+1+\nu] \cap \mathbb{Z} \tag{2.7.32}$$

is true at least for $\nu = n-1$. If (2.7.32) already holds for $1 \le \nu \le n-1$, then (2.7.15) yields

$$\begin{aligned} u^{(\nu-1)} &= r^{(\nu)}\left(x^{(\nu)}\right)^{\sigma} - \Delta u^{(\nu)} \\ &= r^{(\nu)}\Big\{\left(\sigma^{-1}\Delta\right)^{\nu} y\Big\}^{\sigma} + \sum_{\mu=\nu+1}^{n} (-\Delta)^{\mu-\nu}\Big\{r^{(\mu)}\left(\left(\sigma^{-1}\Delta\right)^{\mu}\right)^{\sigma} y\Big\} \\ &= \sum_{\mu=\nu}^{n} (-\Delta)^{\mu-\nu}\Big\{r^{(\mu)}\left(\left(\sigma^{-1}\Delta\right)^{\mu}\right)^{\sigma} y\Big\} \quad \text{on } [0, N+\nu] \cap \mathbb{Z}. \end{aligned} \tag{2.7.33}$$

Hence (2.7.32) holds for all $0 \le \nu \le n-1$ by induction. Moreover it follows by (2.7.15) that

$$r^{(0)} y^{\sigma} = r^{(0)}\left(x^{(0)}\right)^{\sigma} = \Delta u^{(0)} = \sum_{\mu=1}^{n} (-\Delta)^{\mu}\Big\{r^{(\mu)}\left(\sigma^{-1}\Delta\right)^{\mu} y^{\sigma}\Big\} \tag{2.7.34}$$

holds on $[0, N] \cap \mathbb{Z} = J$, that is, $L(y)_k = 0$ holds for all $k \in J$. In particular y is a solution of (2.7.6). □

Remark 2.7.3 (generalized zeros). We now define disconjugacy for the equation (2.7.6). Of course this definition should be "consistent" with our previous one, that is, equivalent with disconjugacy for the system (2.7.9). First of all we deal with how a generalized zero of a solution y of (2.7.6) could be defined in the sense of Definition 2.5.9(i). Clearly there is a generalized zero in the interval $(k, k+1]$, $k \in J$, of the pair (x, u) from Proposition 2.7.2, whenever

$$x_k \ne 0, \qquad x_{k+1} \in \operatorname{Im} \tilde{A} B_k, \qquad x_k^T B_k^{\dagger}(I - A) x_{k+1} \le 0. \tag{2.7.35}$$

We now turn our attention to the important matrices $\tilde{A}B_k$ and $B_k^{\dagger}(I-A)$ and to the vectors x_k and x_{k+1}. We have

$$\tilde{A} B_k = \frac{1}{r_k^{(n)}} \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad B_k^{\dagger}(I - A) = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & r_k^{(n)} \end{pmatrix}. \tag{2.7.36}$$

Because of (2.7.16) from Lemma 2.7.1,

$$x_k = \begin{pmatrix} y_k \\ \Delta y_{k-1} \\ \vdots \\ \Delta^{n-1} y_{k-n+1} \end{pmatrix}, \qquad x_{k+1} = \begin{pmatrix} y_{k+1} \\ \Delta y_k \\ \vdots \\ \Delta^{n-1} y_{k-n+2} \end{pmatrix} \tag{2.7.37}$$

follows. It is immediately clear that only a vector with equal entries may be an element of $\operatorname{Im}\tilde{A}B_k$.

Thus we have $x_{k+1} \in \operatorname{Im}\tilde{A}B_k$ if and only if

$$y_{k-n+2} = y_{k-n+3} = \cdots = y_k = 0. \tag{2.7.38}$$

Because of

$$(\sigma^{-1}\Delta)^{n-1} = \{\sigma^{-1}(\sigma - I)\}^{n-1} = (I - \sigma^{-1})^{n-1} = \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} (-1)^{\nu} \sigma^{-\nu}, \tag{2.7.39}$$

we have in this case

$$x_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{n-1} y_{k-n+1} \end{pmatrix}, \tag{2.7.40}$$

and $x_k^T B_k^{\dagger}(I - A)x_{k+1} = (-1)^{n-1} r_k^{(n)} y_{k-n+1} y_{k+1}$.

We thus define disconjugacy as follows (see also Definition 2.5.9).

Definition 2.7.4 (disconjugacy for (2.7.6)). (i) Let $y : [1-n, N+1] \cap \mathbb{Z} \to \mathbb{R}$ be a solution of equation (2.7.6) and $k \in J$. Then $k+1$ is called a *generalized zero* of y whenever

$$y_{k-n+1} \neq 0, \qquad y_{k-n+2} = y_{k-n+3} = \cdots = y_k = y_{k+1} = 0. \tag{2.7.41}$$

Moreover, y is said to have a generalized zero in the interval $(k, k+1)$ if

$$y_{k-n+2} = \cdots = y_k = 0, \qquad (-1)^{n-1} r_k^{(n)} y_{k-n+1} y_{k+1} < 0. \tag{2.7.42}$$

(ii) Equation (2.7.6) is called *disconjugate* on J^* if no solution of equation (2.7.6) has more than one generalized zero and if no solution y of equation (2.7.6) with $y_{1-n} = y_{2-n} = \cdots = y_0 = 0$ has at least one generalized zero on $(0, N+1]$.

Corollary 2.7.5 (Jacobi's condition). *Equation (2.7.6) is disconjugate on J^* if and only if $\mathcal{F}_0(y) > 0$ for each $y \neq 0$ with $y_{1-n+\nu} = y_{N+1-\nu} = 0$ for all $0 \leq \nu \leq n-1$.*

Proof. This is an immediate consequence of the examinations in Remark 2.7.3 and of Theorem 2.6.1. □

Remark 2.7.6 (singular case). Obviously the above result is now very clear by using our theory presented in the previous sections of this chapter. We emphasize again that this only worked because we have allowed the matrices B_k in our theory to be singular. For the relevant matrices B_k in this Sturm-Liouville case we have

$$\operatorname{rank} B_k = \operatorname{rank} B = 1, \tag{2.7.43}$$

and thus we have a nonsingular "matrix" only in the case $n = 1$. However, for all larger n the matrices B_k are singular. This also explains why the case $n = 1$ is well examined and developed while there are only some results for Sturm-Liouville difference equations of higher order.

2.7.2. The Reid roundabout theorem

In this subsection we prove a characterization for positive definiteness of $\mathcal{F}_0$ which is more useful for numerical purposes than the above Corollary 2.7.5. Of course we already have such a characterization with Theorem 2.6.1. However it will turn out that the special form of system (2.7.9) already implies

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad X_k X_{k+1}^{\dagger} \tilde{A} B_k = 0 \quad \forall 0 \le k \le n-1 \tag{2.7.44}$$

for the principal solution (X, U) of (2.7.9) at 0 (see Proposition 2.7.8), and thus condition Theorem 2.6.1(iv) "simplifies" significantly. We need the following auxiliary result.

Lemma 2.7.7. *System (2.7.9) is controllable on J^* with controllability index $n \in J^*$. Furthermore the following hold.*

(i) $\operatorname{rank} G_k = \min\{k, n\}$ *for all* $k \in J^*$,

(ii) $0 \le k < n$ *and* $\begin{pmatrix} c_1 \\ \vdots \\ c_k \\ c_{k+1} \end{pmatrix} \in \operatorname{Ker} G_{k+1}$ *imply* $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \operatorname{Ker} G_k$.

Proof. Controllability as well as part (i) are already clear because of Remark 2.5.17 and Example 2.5.18.

Now let $0 \le k < n$ and $\begin{pmatrix} c_1 \\ \vdots \\ c_k \\ c_{k+1} \end{pmatrix} \in \operatorname{Ker} G_{k+1}$. We put $c_\nu = \begin{pmatrix} c_\nu^{(1)} \\ \vdots \\ c_\nu^{(n)} \end{pmatrix} \in \mathbb{R}^n$ for $1 \le \nu \le k+1$. Then we have with $b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$ that

$$0 = \begin{pmatrix} \tilde{A}^{k+1} B & \tilde{A}^k B & \cdots & \tilde{A} B \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{k+1} \end{pmatrix} = \sum_{\nu=1}^{k+1} c_{k+2-\nu}^{(n)} \tilde{A}^\nu b. \tag{2.7.45}$$

The columns $\tilde{A}b, \tilde{A}^2 b, \dots, \tilde{A}^{k+1}b$ are linearly dependent because of (i), and thus we have

$$c_1^{(n)} = c_2^{(n)} = \cdots = c_k^{(n)} = c_{k+1}^{(n)} = 0, \tag{2.7.46}$$

and therefore

$$\begin{pmatrix} \tilde{A}^k B & \tilde{A}^{k-1}B & \cdots & \tilde{A}B \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \sum_{\nu=1}^{k} \tilde{A}^\nu b c_{k+1-\nu}^{(n)} = 0. \tag{2.7.47}$$

Hence we have $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \operatorname{Ker} G_k$. □

We now give the following key result which has been announced previously.

Proposition 2.7.8. *The principal solution (X, U) of (2.7.9) at 0 satisfies:*

(i) $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$ *for all* $0 \le k \le n-1$,
(ii) $\operatorname{rank} X_k = k$ *for all* $0 \le k \le n$,
(iii) $D_k = X_k X_{k+1}^\dagger \tilde{A} B_k = 0$ *for all* $0 \le k \le n-1$.

Proof. Let $0 \le k \le n-1$ and $c \in \operatorname{Ker} X_{k+1}$. Then (2.3.15) from Lemma 2.3.1 yields

$$0 = X_{k+1}c = G_{k+1} \begin{pmatrix} U_0 c \\ \vdots \\ U_{k-1} c \\ U_k c \end{pmatrix}. \tag{2.7.48}$$

Lemma 2.7.7(ii) implies

$$0 = G_k \begin{pmatrix} U_0 c \\ \vdots \\ U_{k-1} c \end{pmatrix} = X_k c, \tag{2.7.49}$$

where we used Lemma 2.3.1 again. Thus $c \in \operatorname{Ker} X_k$ and hence $\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k$. Therefore $\operatorname{Ker} X_{k+1}^T \subset \operatorname{Ker} B_k \tilde{A}^T$ follows with (2.3.32) from Lemma 2.3.5, which according to Lemma 2.8.6 is equivalent to

$$X_{k+1} X_{k+1}^\dagger \tilde{A} B_k = \tilde{A} B_k. \tag{2.7.50}$$

Thus we now have

$$
\begin{aligned}
0 &= X_{k+1}X_{k+1}^{\dagger}\tilde{A}B_k - \tilde{A}B_k = G_{k+1}\begin{pmatrix} U_0 \\ \vdots \\ U_{k-1} \\ U_k \end{pmatrix} X_{k+1}^{\dagger}\tilde{A}B_k - \tilde{A}B_k \\
&= \begin{pmatrix} \tilde{A}^{k+1}B_0 & \tilde{A}^{k}B_1 & \cdots & \tilde{A}^{2}B_{k-1} & \tilde{A}B_k \end{pmatrix} \begin{pmatrix} U_0X_{k+1}^{\dagger}\tilde{A}B_k \\ \vdots \\ U_{k-1}X_{k+1}^{\dagger}\tilde{A}B_k \\ U_kX_{k+1}^{\dagger}\tilde{A}B_k - I \end{pmatrix} \\
&= G_{k+1}\begin{pmatrix} U_0X_{k+1}^{\dagger}\tilde{A}B_k \\ \vdots \\ U_{k-1}X_{k+1}^{\dagger}\tilde{A}B_k \\ U_kX_{k+1}^{\dagger}\tilde{A}B_k - I \end{pmatrix},
\end{aligned}
\tag{2.7.51}
$$

and another application of Lemma 2.7.7(ii) yields

$$
0 = G_k\begin{pmatrix} U_0X_{k+1}^{\dagger}\tilde{A}B_k \\ \vdots \\ U_{k-1}X_{k+1}^{\dagger}\tilde{A}B_k \end{pmatrix} = G_k\begin{pmatrix} U_0 \\ \vdots \\ U_{k-1} \end{pmatrix} X_{k+1}^{\dagger}\tilde{A}B_k = X_kX_{k+1}^{\dagger}\tilde{A}B_k. \tag{2.7.52}
$$

Thus (i) and (iii) are shown. Moreover we have

$$
\operatorname{Ker} X_n \subset \operatorname{Ker} X_{n-1} \subset \cdots \subset \operatorname{Ker} X_1 \subset \operatorname{Ker} X_0, \tag{2.7.53}
$$

and this implies

$$
\operatorname{Im} X_k = \operatorname{Im} G_k \quad \forall 0 \le k \le n, \tag{2.7.54}
$$

because of Corollary 2.5.20(i). Finally Lemma 2.7.7(i) yields

$$
\operatorname{rank} X_k = \operatorname{rank} G_k = k \quad \forall 0 \le k \le n, \tag{2.7.55}
$$

so that claim (ii) is also shown. □

Theorem 2.7.9 (Reid roundabout theorem for (2.7.6)). *Let (X, U) be the principal solution of (2.7.9) at 0 and $N \geq n$. Then the following statements are equivalent:*

(i) $\mathcal{F}_0(y) > 0$ *for each* $y \neq 0$ *with* $y_{-\nu} = y_{N+1-\nu} = 0$ *for all* $0 \leq \nu \leq n-1$;

(ii) *system (2.7.6) is disconjugate on* J^*;

(iii) X_{k+1} *is invertible and* $X_k X_{k+1}^{-1} \tilde{A} B_k \geq 0$ *for all* $k \in [n, N] \cap \mathbb{Z}$;

(iv) *the Riccati equation*

$$Q_{k+1} = C_k + (I - A^T) Q_k (I + B_k Q_k)^{-1} (I - A) \tag{2.7.56}$$

has a symmetric solution Q_k, $k \in [n, N+1] \cap \mathbb{Z}$, *with* $Q_n = U_n X_n^{-1}$ *and* $(I + B_k Q_k)^{-1} B_k \geq 0$ *for all* $k \in [n, N] \cap \mathbb{Z}$.

Proof. The equivalence of (i) and (ii) are already part of Corollary 2.7.5. If $\mathcal{F}_0$ is positive definite, then the principal solution (X, U) of (2.7.9) at 0 has no focal points on $(0, N+1]$ according to Theorem 2.6.1. Since X_n is invertible because of Proposition 2.7.8(ii), statement (iii) follows. If conversely (iii) holds, that is, if X has no focal points on $(0, N+1]$, then X satisfies (see Proposition 2.7.8(i) and (iii))

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad X_k X_{k+1}^{\dagger} \tilde{A} B_k = 0 \quad \forall 0 \leq k \leq n-1, \tag{2.7.57}$$

and thus has no focal points on $(0, n]$ and therefore on all of $(0, N+1]$, which implies the positive definiteness of $\mathcal{F}_0$ by Theorem 2.6.1.

Furthermore (iv) follows from (iii) when choosing

$$Q_k = U_k X_k^{-1} \quad \text{for } n \leq k \leq N+1, \tag{2.7.58}$$

as in Lemma 2.3.10 and when observing

$$(I + B_k Q_k)^{-1} B_k = X_k X_{k+1}^{-1} \tilde{A} B_k \quad \forall n \leq k \leq N. \tag{2.7.59}$$

If (iv) holds, then the equation $R[Q]G = 0$ has the symmetric solution $Q = UX^{-1}$ on $[n, N] \cap \mathbb{Z}$. Moreover,

$$Q = UX^{\dagger} + (UX^{\dagger}\tilde{X} - \tilde{U})(I - X^{\dagger}X) U^T \tag{2.7.60}$$

solves the equation $R[Q]G = 0$ on $[0, n] \cap \mathbb{Z}$ because of Proposition 2.7.8(i) and Lemmas 2.3.7 and 2.5.19(i), where $(\tilde{X}, \tilde{U})$ is the associated solution of (2.7.9) at 0. Furthermore Proposition 2.7.8(iii) (see also Lemma 2.3.7(ii)) implies

$$B_k - B_k \tilde{A}^T (Q_{k+1} - C_k) \tilde{A} B_k = 0 \quad \forall 0 \leq k \leq n-1. \tag{2.7.61}$$

Since the solutions on $(0, n]$ and $[n, N+1]$ have the same value at n, namely, $U_n X_n^{-1}$ it follows by Theorem 2.6.18 that $\mathcal{F}_0$ is positive definite in this case. Thus the statements (i)–(iv) are equivalent. □

Remark 2.7.10. It is obvious that condition Theorem 2.7.9(iii) reads "nicer" than the corresponding condition Theorem 2.6.18(iii). The same holds for conditions Theorem 2.6.18(vi) (resp., Theorem 2.6.18(vii)) and for the condition Theorem 2.7.9(iv): while there are kernel conditions, Moore-Penrose inverses, controllability matrices, and "implicit" Riccati equations in Theorem 2.6.18, in the above theorem we only have "normal" inverses and explicit Riccati equations that allow to compute solutions using a known initial condition. The key to this "nicer" theorem was Proposition 2.7.8 which is based only on Lemma 2.7.7. Thus one may establish that a theorem of the above form may be proved for a special Hamiltonian system (2.1.5) with controllability matrices G_k, $k \in J^*$, whenever we have

(i) system (2.1.5) is controllable on J^* with controllability index κ,

(ii) $0 \le k < \kappa$ and $\begin{pmatrix} c_1 \\ \vdots \\ c_k \\ c_{k+1} \end{pmatrix} \in \operatorname{Ker} G_{k+1}$ imply $\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \in \operatorname{Ker} G_k$.

Now we want to deal with the case of general boundary conditions. Let be given $2n \times 2n$-matrices R and S with S symmetric as usual. Theorem 2.7.9 shows that the principal solution (X, U) of (2.7.9) at 0 has no focal points on $(0, N+1]$ if and only if the condition

$$X_{k+1} \quad \text{is invertible and} \quad X_k X_{k+1}^{-1} \tilde{A} B_k \ge 0 \quad \forall n \le k \le N \tag{2.7.62}$$

holds. Let (X, U), $(\tilde{X}, \tilde{U})$ be the special normalized conjoined bases of (2.7.9) at 0. If $N \ge n$ and

$$x_0 = \begin{pmatrix} y_0 \\ \Delta y_{-1} \\ \vdots \\ \Delta^{n-1} y_{1-n} \end{pmatrix}, \qquad x_{N+1} = \begin{pmatrix} y_{N+1} \\ \Delta y_N \\ \vdots \\ \Delta^{n-1} y_{N-n+2} \end{pmatrix}, \tag{2.7.63}$$

then Theorem 2.6.6 implies

$$\mathcal{F}(y) := \sum_{k=0}^{N} \sum_{\nu=0}^{n} r_k^{(\nu)} \{\Delta^\nu y_{k+1-\nu}\}^2 + \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix}^T S \begin{pmatrix} -x_0 \\ x_{N+1} \end{pmatrix} > 0 \tag{2.7.64}$$

for all $y \ne 0$ with $\binom{-x_0}{x_{N+1}} \in \operatorname{Im} R^T$ if and only if the following two conditions hold:

(i) X_{k+1} is invertible and $X_k X_{k+1}^{-1} \tilde{A} B_k \ge 0$ for all $k \in [n, N] \cap \mathbb{Z}$,

(ii) $R\left\{S + \begin{pmatrix} -X_{N+1}^{-1} \tilde{X}_{N+1} & X_{N+1}^{-1} \\ (X_{N+1}^{-1})^T & U_{N+1} X_{N+1}^{-1} \end{pmatrix}\right\} R^T > 0$ on $\operatorname{Im} R$.

Finally we discuss the case $0 \le N < n$. It follows from Proposition 2.7.8(i) and Proposition 2.7.8(iii) that the principal solution (X, U) of (2.7.9) at 0 satisfies

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k, \quad D_k = X_k X_{k+1}^{\dagger} \tilde{A}_k B_k = 0 \quad \forall k \in J, \tag{2.7.65}$$

that is, (X, U) has no focal points on $(0, N+1]$. Then $\mathcal{F}_0 > 0$ by Theorem 2.6.1. More precisely, any admissible (x, u) with $x_0 = x_{N+1} = 0$ is trivial because of (2.4.28) from Proposition 2.4.8. Thus the statement $\mathcal{F}_0$ has only a formal meaning in the sense that there does not exist an admissible (x, u) with $x_0 = x_{N+1} = 0$ and $x \neq 0$. Now let be given $2n \times 2n$-matrices R and S (with S symmetric). By Theorem 2.6.11(i) we have $\mathcal{F} > 0$ whenever $R\{S + Q_{N+1}^*\}R^T > 0$ on $\operatorname{Im} R$. More precisely, for admissible (x, u) with $\binom{-x_0}{x_{N+1}} = R^T c \in \operatorname{Im} R^T$ we have

$$\begin{gathered} \mathcal{F}(x, u) = c^T R\{S + Q_{N+1}^*\} R^T c, \\ x = \{\tilde{A}^{-1} - B\tilde{A}^T(Q^\sigma - C)\}x^\sigma + B\tilde{A}^T(\tilde{Q}^\sigma)^T x_0 \quad \text{on } J \end{gathered} \tag{2.7.66}$$

(with the notation from Theorem 2.4.10). This follows by putting $D = 0$ on J into the formulas (2.4.42) and (2.4.43) from our extended Picone identity.

2.8. Moore-Penrose inverses

Moore-Penrose inverses are one of the most important tools of the theory presented in this chapter. For this reason we offer here a short overview of these inverses in the present section. The definition and well-known results can be found in an introductory part (see also [44]). The second half of this section contains several auxiliary results which are used in this chapter.

Definition 2.8.1 (Moore-Penrose inverse). Let A be an $m \times n$-matrix. The unique $n \times m$-matrix X with

$$\begin{aligned} XAX &= X, \\ AXA &= A, \\ (XA)^T &= XA, \\ (AX)^T &= AX \end{aligned} \tag{2.8.1}$$

is called the *Moore-Penrose inverse* of A and is denoted by $A^\dagger$.

Of course we first have to make sure that Moore-Penrose inverses are well defined with this definition. This is done in the following result (see also [44, Exercise 1.1 and Theorem 1.5]).

Lemma 2.8.2 (existence and uniqueness). *Let A be an $m \times n$-matrix. Then the following hold.*

(i) *There is at most one matrix X satisfying (2.8.1).*
(ii) *There is at least one matrix X satisfying (2.8.1).*

Proof. First, if both X and Y (instead of X) satisfy (2.8.1), then

$$\begin{aligned} X &= XAX = XAYAX = (XA)^T(YA)^TX = A^TX^TA^TY^TX \\ &= (AXA)^TY^TX = A^TY^TX = (YA)^TX = YAX \\ &= Y(AX)^T = YX^TA^T = YX^T(AYA)^T = YX^TA^TY^TA^T \\ &= Y(AX)^T(AY)^T = YAXAY = YAY = Y \end{aligned} \tag{2.8.2}$$

follows. Now let $\operatorname{rank} A = r$ and $\{u_1, u_2, \ldots, u_r\}$ be a basis of $\operatorname{Im} A$. We define an $m \times r$-matrix A_1 by $A_1 := (u_1 \quad u_2 \quad \cdots \quad u_r)$. Then $\operatorname{rank} A_1 = r$ and $A_1^TA_1$ is invertible. Since any element of $\operatorname{Im} A$ is representable in a unique way as a linear combination of the u_i, $1 \le i \le r$, $A = A_1A_2$ defines a unique $r \times n$-matrix A_2. Because of $\operatorname{rank} A_2 \ge \operatorname{rank} A_1A_2 = \operatorname{rank} A = r$, we have $\operatorname{rank} A_2 = r$ and $A_2A_2^T$ is invertible. Let

$$X := A_2^T\left(A_2A_2^T\right)^{-1}\left(A_1^TA_1\right)^{-1}A_1^T. \tag{2.8.3}$$

Therefore

$$AX = A_1\left(A_1^TA_1\right)^{-1}A_1^T, \qquad XA = A_2^T\left(A_2A_2^T\right)^{-1}A_2, \tag{2.8.4}$$

and hence $XAX = X$ and $AXA = A$. ☐

Thus (2.8.1) describes a well-defined object. Lemma 2.8.2 furthermore yields a formula for $A^\dagger$. If we are given a "full rank factorization" of A (see [44, Lemma 1.5]) by

$$A = A_1A_2 \quad \text{with} \quad \operatorname{rank} A_1 = \operatorname{rank} A_2 = \operatorname{rank} A, \tag{2.8.5}$$

then

$$A^\dagger = A_2^T\left(A_1^TAA_2^T\right)^{-1}A_1^T. \tag{2.8.6}$$

Another representation of $A^\dagger$ is shown as follows.

Lemma 2.8.3 (formula for $A^\dagger$). *The formula*

$$A^\dagger = \lim_{t \to 0^+}\left\{\left(A^TA + tI\right)^{-1}A^T\right\} \tag{2.8.7}$$

holds. In particular the limit exists.

Proof. Again let A be an $m \times n$-matrix with $\operatorname{rank} A = r$. Let

$$s_1, s_2, \dots, s_r > 0 \quad \text{be the positive eigenvalues of } A^T A. \tag{2.8.8}$$

Then there exist linearly independent $u_1, u_2, \dots, u_r \in \mathbb{R}^n$ with $A^T A u_i = s_i u_i$ for all $1 \le i \le r$. Let $y \in \mathbb{R}^n$ be arbitrary. Then there is $y_1 \in \mathbb{R}^n$ and $y_2 \in \operatorname{Ker} AA^T$ with $y = AA^T y_1 + y_2$. From $AA^T y_2 = 0$ it follows that $A^T y_2 = 0$ and thus (see Lemma 2.8.4) $A^\dagger y_2 = 0$. Since $\{u_1, u_2, \dots, u_r\}$ is a basis of $\operatorname{Im} A^T$, there exist unique $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{R}$ with $A^T y_1 = \sum_{i=1}^r \alpha_i u_i$. Now we have for $t \in (0, \min_{1 \le i \le r} s_i)$

$$\begin{aligned}
(A^T A + tI)^{-1} A^T y &= (A^T A + tI)^{-1} A^T \{AA^T y_1 + y_2\} \\
&= (A^T A + tI)^{-1} A^T A \sum_{i=1}^r \alpha_i u_i \\
&= \sum_{i=1}^r (A^T A + tI)^{-1} \alpha_i s_i u_i \\
&= \sum_{i=1}^r \frac{\alpha_i s_i}{s_i + t} u_i.
\end{aligned} \tag{2.8.9}$$

Thus $\lim_{t \to 0^+} \{(A^T A + tI)^{-1} A^T y\}$ exists and

$$\lim_{t \to 0^+} \left\{ (A^T A + tI)^{-1} A^T y \right\} = \sum_{i=1}^r \frac{\alpha_i s_i}{s_i} u_i = \sum_{i=1}^r \alpha_i u_i = A^T y_1 = A^\dagger y. \tag{2.8.10}$$

Since $y \in \mathbb{R}^n$ was arbitrary, the assertion follows. □

The s_i from the above proof are called the *singular values* of A. In general it is possible to prove (see, e.g., [164, Theorem 7.3.5]) that for any $m \times n$-matrix A there exists a "singular value decomposition" of the form $A = V \Sigma W^{-1}$ with unitary $m \times m$ (resp., $n \times n$) matrices V and W and with an $m \times n$-matrix $\Sigma = (s_{ij})$ with

$$s_{ij} = \begin{cases} s_i & \text{if } 1 \le i \le r,\ i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8.11}$$

By defining an $n \times m$-matrix $\Sigma^* = (s_{ij}^*)$ by

$$s_{ij}^* = \begin{cases} \dfrac{1}{s_i} & \text{if } 1 \le i \le r,\ i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.8.12}$$

we have (see [164, Problem 7.3.7]) $A^\dagger = W \Sigma^* V^{-1}$.

Alternative proofs of Lemma 2.8.3 can be found, for example, in [44, Examples 3.20 and 4.40], [164, Problem 7.3.9], and [182, Remark 3.3.2].

Lemma 2.8.4. *For any $m \times n$-matrix A,*

$$\begin{gathered} (A^T)^\dagger = (A^\dagger)^T, \qquad (A^\dagger)^\dagger = A, \\ \operatorname{Ker}(A^\dagger)^T = \operatorname{Ker} A. \end{gathered} \tag{2.8.13}$$

If $\operatorname{rank} A = n$, *then*

$$(AX)^\dagger AX = X^\dagger X \quad \textit{for each } n \times m\textit{-matrix } X. \tag{2.8.14}$$

Proof. The first two statements follow by checking (2.8.1) with A^T and $A^\dagger$ instead of A (see [44, Exercise 1.17(a), (c)]). By using the calculations

$$\begin{gathered} (A^\dagger)^T A^\dagger A = (A^\dagger)^T (A^\dagger A)^T = (A^\dagger A A^\dagger)^T = (A^\dagger)^T, \\ A A^T (A^\dagger)^T = A(A^\dagger A)^T = A A^\dagger A = A, \end{gathered} \tag{2.8.15}$$

$\operatorname{Ker}(A^\dagger)^T = \operatorname{Ker} A$ follows. From

$$\begin{aligned} (AX)^\dagger AX &= (AX)^\dagger AXX^\dagger X = X^T A^T (X^T A^T)^\dagger X^\dagger X \\ &= X^T A^T (X^T A^T)^\dagger X^T A^T A (A^T A)^{-1} (X^\dagger)^T \\ &= X^T A^T A (A^T A)^{-1} (X^\dagger)^T = X^T (X^\dagger)^T = X^\dagger X, \end{aligned} \tag{2.8.16}$$

the last assertion can be seen to be valid. □

Lemma 2.8.5. *Let be given matrices V and F with*

$$VFV = V, \qquad (VF)^T = VF. \tag{2.8.17}$$

Then for any matrix W,

$$\operatorname{Ker} V \subset \operatorname{Ker} W \Longleftrightarrow W = WFV. \tag{2.8.18}$$

Proof. If $W = WFV$, then $\operatorname{Ker} V \subset \operatorname{Ker} W$. Assume conversely $\operatorname{Ker} V \subset \operatorname{Ker} W$. Then $\operatorname{Im} W^T \subset \operatorname{Im} V^T$ holds also. For $W^T c \in \operatorname{Im} W^T$ there exists $V^T d \in \operatorname{Im} V^T$ such that $W^T c = V^T d$. We pick $V d_1 \in \operatorname{Im} V$ and $d_2 \in \operatorname{Ker} V^T$ with

$$d = V d_1 + d_2. \tag{2.8.19}$$

Then the calculation

$$\begin{aligned} W^T c = V^T d = V^T \{V d_1 + d_2\} &= V^T V d_1 \\ &= V^T V F V d_1 = V^T (VF)^T V d_1 \\ &= V^T F^T V^T V d_1 = V^T F^T W^T c \\ &= (WFV)^T c \end{aligned} \tag{2.8.20}$$

shows the truth of the relation $W = WFV$. □

The following consequence of Lemmas 2.8.4 and 2.8.5 is frequently used in the present chapter. It is the "reason" for the appearance of Moore-Penrose inverses in our theory.

Lemma 2.8.6 (characterization of $\operatorname{Ker} V \subset \operatorname{Ker} W$**).** *For two matrices V and W the following statements are equivalent:*

(i) $\operatorname{Ker} V \subset \operatorname{Ker} W$,
(ii) $W = WV^{\dagger}V$,
(iii) $W^{\dagger} = V^{\dagger}VW^{\dagger}$.

Proof. The equivalence of (i) and (ii) follows with $F = V^{\dagger}$ from Lemma 2.8.5. According to the second part of Lemma 2.8.4,

$$\operatorname{Ker}(V^{\dagger})^{T} \subset \operatorname{Ker}(W^{\dagger})^{T} \tag{2.8.21}$$

is equivalent to (i), and this is equivalent to

$$(W^{\dagger})^{T} = (W^{\dagger})^{T}\left\{(V^{\dagger})^{T}\right\}^{\dagger}(V^{\dagger})^{T} \tag{2.8.22}$$

by what we have already shown. The first two parts from Lemma 2.8.4 imply

$$(W^{\dagger})^{T}\left\{(V^{\dagger})^{T}\right\}^{\dagger}(V^{\dagger})^{T} = (W^{\dagger})^{T}V^{T}(V^{\dagger})^{T} = (V^{\dagger}VW^{\dagger})^{T}. \tag{2.8.23}$$

Thus the equivalence of (iii) to (i) and (ii) follows. □

In the proof of the extended Picone identity (see Remark 2.4.9) we need the following result on the Moore-Penrose inverse of a special partitioned matrix.

Lemma 2.8.7 (formula for $X^{*\dagger}$**).** *Let X and $\tilde{X}$ be two $n \times n$-matrices. Then*

$$\begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix}^{\dagger}\begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix} = \begin{pmatrix} X^{\dagger}X & 0 \\ 0 & I \end{pmatrix}. \tag{2.8.24}$$

If $S = I + \tilde{X}\tilde{X}^{T}$, then $\left(\begin{smallmatrix} 0 & I \\ X & \tilde{X} \end{smallmatrix}\right)^{\dagger}$ is equal to

$$\begin{pmatrix} -(S^{-1/2}X)^{\dagger}S^{-1/2}\tilde{X} & (S^{-1/2}X)^{\dagger}S^{-1/2} \\ I - \tilde{X}^{T}S^{-1}\left\{I - X(S^{-1/2}X)^{\dagger}S^{-1/2}\right\}\tilde{X} & \tilde{X}^{T}S^{-1}\left\{I - X(S^{-1/2}X)^{\dagger}S^{-1/2}\right\} \end{pmatrix}. \tag{2.8.25}$$

Proof. The first assertion holds because of Lemma 2.8.4 and

$$\begin{pmatrix} 0 & I \\ X & \tilde{X} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & \tilde{X} \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix}. \tag{2.8.26}$$

It is possible to derive the stated formula with some calculation using Lemma 2.8.3. For convenience we check here the conditions (2.8.1). It is possible to see (by denoting the matrix in (2.8.25) by M) that (observe Lemma 2.8.4)

$$M\begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix} = \begin{pmatrix}(S^{-1/2}X)^{\dagger}S^{-1/2}X & 0\\ 0 & I\end{pmatrix} = \begin{pmatrix}X^{\dagger}X & 0\\ 0 & I\end{pmatrix} \tag{2.8.27}$$

is symmetric. Thus we have

$$\begin{gathered}\begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix} M\begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix} = \begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix}\begin{pmatrix}X^{\dagger}X & 0\\ 0 & I\end{pmatrix} = \begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix},\\ M\begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix} M = \begin{pmatrix}X^{\dagger}X & 0\\ 0 & I\end{pmatrix} M = M,\end{gathered} \tag{2.8.28}$$

since Lemma 2.8.4 yields

$$(S^{-1/2}X)^{\dagger} = (S^{-1/2}X)^{\dagger}S^{-1/2}X(S^{-1/2}X)^{\dagger} = X^{\dagger}X(S^{-1/2}X)^{\dagger}. \tag{2.8.29}$$

Finally,

$$\begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix} M =: \begin{pmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix} \tag{2.8.30}$$

is also symmetric because of

$$\begin{aligned} S_{11} &= I - \tilde{X}^{T}S^{-1}\tilde{X} + \tilde{X}^{T}S^{-1/2}(S^{-1/2}X)(S^{-1/2}X)^{\dagger}S^{-1/2}\tilde{X},\\ S_{22} &= X(S^{-1/2}X)^{\dagger}S^{-1/2} + \tilde{X}\tilde{X}^{T}S^{-1}\left\{I - X(S^{-1/2}X)^{\dagger}S^{-1/2}\right\}\\ &= I - S^{-1} + S^{-1/2}(S^{-1/2}X)(S^{-1/2}X)^{\dagger}S^{-1/2}, \end{aligned} \tag{2.8.31}$$

(observe $\tilde{X}\tilde{X}^{T}S^{-1} = I - S^{-1}$) and

$$\begin{aligned} S_{21}^{T} &= \left[-X(S^{-1/2}X)^{\dagger}S^{-1/2}\tilde{X} + \tilde{X} - \tilde{X}\tilde{X}^{T}S^{-1}\left\{I - X(S^{-1/2}X)^{\dagger}S^{-1/2}\right\}\tilde{X}\right]^{T}\\ &= \left[S^{-1}\tilde{X} - S^{-1}X(S^{-1/2}X)^{\dagger}S^{-1/2}\tilde{X}\right]^{T}\\ &= \tilde{X}^{T}S^{-1}\left\{I - S^{1/2}(X^{T}S^{-1/2})^{\dagger}X^{T}S^{-1/2}S^{-1/2}\right\}\\ &= \tilde{X}^{T}S^{-1}\left\{I - S^{1/2}S^{-1/2}X(S^{-1/2}X)^{\dagger}S^{-1/2}\right\}\\ &= \tilde{X}^{T}S^{-1}\left\{I - X(S^{-1/2}X)^{\dagger}S^{-1/2}\right\}\\ &= S_{12}, \end{aligned} \tag{2.8.32}$$

so that altogether $\begin{pmatrix}0 & I\\ X & \tilde{X}\end{pmatrix}^{\dagger} = M$ follows. □

2.9. Notes and general discussions

(1) The results in this chapter are taken from [46, 47]. For related results, we refer to [45, 48, 49, 50, 51, 52, 53, 54].

(2) The simplest example of a system (2.1.5) is the mentioned Sturm-Liouville difference equation of order two

$$\Delta\{p_k \Delta y_k\} = q_k y_{k+1}, \tag{2.9.1}$$

which results from (2.1.5) upon the choice $n = 1$, $A_k \equiv 0$, $B_k = 1/p_k$, and $C_k = q_k$, and this case is very well studied in the literature (see also Chapter 1). Representatively we mention the papers by Chen, Erbe, Hinton, Hooker, Kwong, Lewis, Patula, Popenda, and Zhang [82, 84, 125, 160, 161, 162, 236, 289].

(3) Another well-discussed example is the equation

$$\Delta\{P_k \Delta y_k\} = Q_k y_{k+1} \tag{2.9.2}$$

(with $n \times n$-matrices P_k and Q_k such that P_k are positive definite), which results from (2.1.5) when choosing $A_k \equiv 0$, $B_k = P_k^{-1}$, and $C_k = Q_k$. Ahlbrandt, Chen, Clark, Erbe, Hooker, Patula, Peil, Peterson, and Ridenhour, dealt with this special equation in [27, 31, 34, 83, 218, 223, 224, 225].

(4) Erbe and Yan introduced systems of the form (2.1.5) with nonsingular matrices B_k in 1992 and studied them in a series of four publications [120, 121, 123, 124].

(5) We also mention the papers on system (2.1.5) and equation (2.1.7) with nonsingular B_k [28, 32, 36, 106, 222] by Ahlbrandt, Došlý, Heifetz, Peterson. Altogether one may see that, in particular, the examples of systems (2.1.5) with *nonsingular* matrices B_k are very well studied in the literature. It is not surprising that there are not too many results known about Sturm-Liouville difference equation (2.7.6) of higher order (see [4, 36, 118, 152, 219, 221, 222, 226, 227] by Agarwal, Ahlbrandt, Eloe, Hankerson, Henderson, Peil, Peterson, and Ridenhour), since the corresponding matrices B_k are singular except the case of $n = 1$. In order to apply results from system (2.1.5) to equation (2.7.6), it is therefore essential to allow the matrices B_k also to be *singular* (see also [45, 47, 48, 50] as well as a generalization of this theory to so-called symplectic systems in [58]).

Thus, the main result given in this chapter is a solution to an open problem, and Theorem 2.6.18 in Section 2.6 is also an answer to both of the following statements. We change the quotations to match them with our present terminology.

(a) "Conjecture. If $B_k \geq 0$ and $\mathcal{F}_0$ positive definite, then there exists a Hermitian solution Q_k of (2.1.7) such that $(I + B_k Q_k)^{-1} B_k \geq 0$" (in [124] by Erbe and Yan).

(b) "An open question is that of existence of a Reid roundabout theorem for systems which allows B_k to be singular" (in [28, page 515] by Ahlbrandt).

(6) Generalizations of some of the results presented in this chapter to the so-called *dynamic equations* case can be found in [5, 6, 12, 29, 30, 74, 77].

3 Oscillation theory for half-linear difference equations

3.1. Introduction

In this chapter we will present oscillation and nonoscillation criteria for second-order half-linear difference equations. In recent years these equations have received considerable attention. This is largely due to the fact that half-linear difference equations occur in a variety of real world problems such as in the study of biological models, in the formulation and analysis of discrete-time systems, in discretization methods for differential equations, in the study of deterministic chaos, and so forth. Moreover, these are the natural extensions of second-order linear difference equations.

In Section 3.2 we will provide some preliminaries for the study of half-linear difference equations and define the basic concepts of half-linear discrete oscillation theory. Section 3.3 contains the Picone-type identity which plays a crucial rôle in the proof of the main result, namely, the disconjugacy characterization theorem (or the so-called Reid roundabout theorem). As we see in this section, the Sturm-type comparison and separation theorems can be easily obtained from the roundabout theorem. In Section 3.4 we will discuss the terms "Riccati technique" and the "variational principle," and employ these techniques to obtain some nonoscillation criteria for half-linear difference equations. We will see that the use of different methods to extend certain types of nonoscillation criteria gives different results. This section is concluded by some conjugacy criteria. Oscillation criteria which are based on the Riccati technique as well as on the variational principle are discussed in Section 3.5. Various comparison theorems for second-order half-linear difference equations of some types other than the classical Sturm types are included in Section 3.6. Further results on oscillation, nonoscillation, and existence of positive nondecreasing solutions of half-linear difference equations are investigated in Section 3.7. The concept of strong oscillation and nonoscillation, conditional oscillation, and some oscillation results as well as investigation of some oscillation properties of discrete generalized Euler equations are given in Section 3.8. Sections 3.9 and 3.10 are devoted to the study of oscillation of second-order half-linear difference equations with a damping term and with a forcing term, respectively.

3.2. Preliminaries and basic concepts

One of the important and more general equations in applied mathematics is the second-order half-linear differential equation

$$\left(c(t)\left|x'(t)\right|^{\alpha-1}\operatorname{sgn}x'(t)\right)' + q(t)\left|x(t)\right|^{\alpha-1}\operatorname{sgn}x(t) = 0, \tag{3.2.1}$$

where we assume that c, q are continuous on $[a,b]$ with $c(t) > 0$ for all $t \in [a,b]$, and $\alpha > 1$ is a constant.

Equation (3.2.1) has been intensively studied in the literature, and it was shown that the basic oscillatory properties of this equation are essentially the same as those of linear equations, which are a special case of equation (3.2.1) for $\alpha = 2$. The terminology "half-linear" is due to the fact that, if a function x is a solution of equation (3.2.1), then for any real constant d the function dx is a solution of the same equation. This means that the space of all solutions of equation (3.2.1) is homogeneous but not generally additive, and thus, it has only half of properties of a linear space. Sometimes half-linear equations are called "homogeneous (of degree $\alpha - 1$)."

Another equivalent form of equation (3.2.1) is

$$(c(t)x'(t))' + q(t)\left|x(t)\right|^{\alpha-1}\operatorname{sgn}x(t)\left|c(t)x'(t)\right|^{2-\alpha} = 0, \tag{3.2.2}$$

or in a slightly more general form

$$(c(t)x'(t))' + q(t)f(x(t), c(t)x'(t)) = 0, \tag{3.2.3}$$

where suitable restrictions on the function f make this equation half linear.

We describe the process of discretization of equation (3.2.1). Consider the second-order half-linear differential equation

$$(\tilde{c}(t)\Psi(y'(t)))' + \tilde{q}(t)\Psi(y(t)) = 0, \tag{3.2.4}$$

where $\Psi(y) = |y|^{\alpha-1}\operatorname{sgn}y$ and the functions $\tilde{c}$, $\tilde{q}$ are continuous on $[a,b]$. For small $h = (b-a)/n$, $n \in \mathbb{N}$, we have $y'(t) \approx (y(t) - y(t-h))/h$ and

$$(\tilde{c}(t)\Psi(y'(t)))' \approx \frac{1}{h}\left\{\frac{\tilde{c}(t+h)\Psi[y(t+h) - y(t)]}{h} - \frac{\tilde{c}(t)\Psi[y(t) - y(t-h)]}{h}\right\}. \tag{3.2.5}$$

Let $t = a+kh$, where k is a discrete variable taking on the integer values $0 \le k \le n$. If y is a solution of equation (3.2.1) on $[a,b]$, then we have

$$\begin{aligned} &\tilde{c}(a+(k+1)h)\Psi[y(a+(k+1)h) - y(a+kh)] \\ &\quad - \tilde{c}(a+kh)\Psi[y(a+kh) - y(a+(k-1)h)] + h^2\tilde{q}(a+kh)\Psi(y(a+kh)) \approx 0. \end{aligned} \tag{3.2.6}$$

Now we set $x(k+1) = y(a+kh)$, $c(k) = \tilde{c}(a+kh)$, and $q(k) = h^2\tilde{q}(a+kh)$. Hence, we get

$$c(k+1)\Psi(x(k+2) - x(k+1)) - c(k)\Psi(x(k+1) - x(k)) + q(k)\Psi(x(k+1)) \approx 0, \tag{3.2.7}$$

and thus

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi(x(k+1)) \approx 0 \tag{3.2.8}$$

for $0 \le k \le n-2$. Note that $x(k)$ is defined for $0 \le k \le n$.

In this section we will define some basic concepts of half-linear discrete oscillation theory. We investigate the second-order half-linear difference equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi(x(k+1)) = 0, \tag{3.2.9}$$

where $c(k)$ and $q(k)$ are real-valued sequences defined on $\mathbb{N}$ (or $\mathbb{Z}$) with $c(k) \ne 0$ and

$$\Psi(x) = |x|^{\alpha-1}\operatorname{sgn} x = |x|^{\alpha-2}x \quad \text{with some } \alpha > 1. \tag{3.2.10}$$

Sometimes we will deal with a special case of equation (3.2.9), namely,

$$\Delta\big(\Psi(\Delta x(k))\big) + q(k)\Psi(x(k+1)) = 0. \tag{3.2.11}$$

Since (3.2.9) is in fact a recurrence relation, given real initial values $x(m), x(m+1)$ for some $m \in \mathbb{N}$, it is clear that we can inductively obtain $x(m+2), x(m+3), \ldots$. Hence, the existence and uniqueness of a solution of the initial value problem (3.2.9), $x(m) = A$, $x(m+1) = B$ is guaranteed.

As in the linear difference equations in Chapter 1, the definitions of the generalized zero of a solution, disconjugacy, and so forth, remain valid for equation (3.2.9) provided $c(k) > 0$. But, since $c(k)$ is only assumed to be nonzero, the definition given below is similar to the one given in Chapter 2 and more general than the one presented in Chapter 1.

Definition 3.2.1. An interval $(m, m+1]$ is said to contain a *generalized zero* of a solution x of equation (3.2.9), if $x(m) \neq 0$ and $c(m)x(m)x(m+1) \leq 0$.

Definition 3.2.2. Equation (3.2.9) is said to be *disconjugate* on an interval $[m, n]$ provided any solution of (3.2.9) has at most one generalized zero on $(m, n+1]$ and the solution $\tilde{x}$ satisfying $\tilde{x}(m) = 0$ has no generalized zeros on $(m, n+1]$. Otherwise, equation (3.2.9) is said to be *conjugate* on $[m, n]$.

The concept of oscillation and nonoscillation of equation (3.2.9) is defined in the following way.

Definition 3.2.3. Equation (3.2.9) is called *nonoscillatory* if there exists $m \in \mathbb{N}$ such that this equation is disconjugate on $[m, n]$ for every $n > m$. Otherwise, equation (3.2.9) is said to be *oscillatory*. That is, a nontrivial solution of equation (3.2.9) is called oscillatory if it has infinitely many generalized zeros.

In view of the fact that the Sturm-type separation theorem extends to equation (3.2.9), we have the following equivalence: any solution of equation (3.2.9) is oscillatory if and only if every solution of equation (3.2.9) is oscillatory. Hence we can speak about oscillation or nonoscillation of equation (3.2.9).

Next, we introduce the so-called discrete α-degree functional.

Definition 3.2.4. Define a class $U = U(m, n)$ of admissible sequences by

$$U(m,n) = \{\xi : [m, n+2] \longrightarrow \mathbb{R} : \xi(m) = \xi(n+1) = 0\}. \tag{3.2.12}$$

Also, define an *α-degree* functional $\mathcal{F}$ on $U(m, n)$ by

$$\mathcal{F}(\xi; m, n) = \sum_{k=m}^{n} \Big[c(k)\,|\,\Delta\xi(k)\,|^{\alpha} - q(k)\,|\,\xi(k+1)\,|^{\alpha}\Big]. \tag{3.2.13}$$

$\mathcal{F}$ is positive definite on U provided $\mathcal{F}(\xi) \geq 0$ for all $\xi \in U(m, n)$, and $\mathcal{F}(\xi) = 0$ if and only if $\xi = 0$.

Next, we will present a discrete variational problem.

Consider the functional $\mathcal{F}(\xi; m, n)$ on the set of admissible sequences

$$\tilde{U} = \{\xi : [m, n+2] \longrightarrow \mathbb{R} : \xi(m) = A \text{ and } \xi(n+1) = B\}. \tag{3.2.14}$$

We are interested in extremizing $\mathcal{F}$ subject to $\xi \in \tilde{U}$. For this purpose denote the summand of $\mathcal{F}$ by

$$f(k, x(k+1), \Delta\xi(k)) = c(k)|\Delta\xi(k)|^{\alpha} - q(k)|\xi(k+1)|^{\alpha}. \tag{3.2.15}$$

Next, we give a necessary condition for $\mathcal{F}$ to have a local extremum at $\tilde{\xi}$ on $\tilde{U}$.

Proposition 3.2.5. *If $\mathcal{F}$ has a local extremum at $\tilde{\xi}$ on $\tilde{U}$, then $\tilde{\xi}(k)$ satisfies equation (3.2.9) on $[m, n]$.*

The proof of this proposition is based on the fact that the assumptions imply that $\tilde{\xi}$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial s}(k, \xi(k+1), \Delta\xi(k)) - \Delta\frac{\partial f}{\partial t}(k, \xi(k+1), \Delta\xi(k)) = 0 \tag{3.2.16}$$

(recall that f is a function of three variables, i.e., $f = f(c, s, t)$), and a simple calculation shows that $\tilde{\xi}(k)$ is a solution of equation (3.2.9) for $k \in [m, n]$.

Thus, we see that equation (3.2.9) is in fact the Euler-Lagrange equation associated to the above variational problem concerning the extremizing of the α-degree functional.

Along with equation (3.2.9) we will consider the so-called generalized *Riccati difference equation*

$$\mathcal{R}[w(k)] := \Delta w(k) + q(k) + \Phi(w(k), c(k)) = 0, \tag{3.2.17}$$

or equivalently,

$$w(k+1) = -q(k) + \tilde{\Phi}(w(k), c(k)), \tag{3.2.18}$$

where

$$\begin{aligned} \Phi(w(k), c(k)) &= w(k)\left[1 - \frac{\Psi(x(k))}{\Psi(x(k+1))}\right] \\ &= w(k)\left(1 - \frac{c(k)}{\Psi[\Psi^{-1}(c(k)) + \Psi^{-1}(w(k))]}\right), \end{aligned} \tag{3.2.19}$$

$$\tilde{\Phi}(w(k), c(k)) = w(k) - \Phi(w(k), c(k)). \tag{3.2.20}$$

Equation (3.2.17) is related to equation (3.2.9) by the Riccati-type substitution

$$w(k) = \frac{c(k)\Psi(\Delta x(k))}{\Psi(x(k))}. \tag{3.2.21}$$

We recall the definition $\Psi(x) = |x|^{\alpha-1}\operatorname{sgn} x$ and let Ψ^{-1} be the inverse of Ψ, that is, $\Psi^{-1}(x) = |x|^{\beta-1}\operatorname{sgn} x$, where the constants α and β are mutually conjugate, that is,

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 \quad \text{with } \alpha > 1,\ \beta > 1. \tag{3.2.22}$$

Next, we will describe some properties of the function

$$\Phi(x, y) = \Phi(x, y, \alpha) = x\left(1 - \frac{y}{\Psi\left[\Psi^{-1}(x) + \Psi^{-1}(y)\right]}\right), \tag{3.2.23}$$

appearing in equation (3.2.17), which we will need in the following sections. Note that the function Φ is the "half-linear extension" of the function $x^2/(x+y)$ appearing in the Riccati difference equation associated to the linear difference equation (presented in Sections 1.3 and 2.3.3), and hence one can expect a similar behavior of these functions in a certain sense.

Lemma 3.2.6. *The function $\Phi(x, y, \alpha)$ has the following properties.*

(I_1) $\Phi(x, y, \alpha)$ *is continuously differentiable on*

$$\mathcal{D} = \left\{(x, y, \alpha) \in \mathbb{R} \times \mathbb{R} \times [1, \infty) : x \neq -y\right\}. \tag{3.2.24}$$

(I_2) *Let $y > 0$. Then $x\Phi_x(x, y, \alpha) \geq 0$ for $x + y > 0$, where $\Phi_x(x, y, \alpha) = 0$ if and only if $x = 0$.*

(I_3) *Let $x + y > 0$. Then $\Phi_y(x, y, \alpha) \leq 0$, where the equality holds if and only if $x = 0$.*

(I_4) *$\Phi(x, y, \alpha) \geq 0$ for $x + y > 0$, where the equality holds if and only if $x = 0$.*

(I_5) *Suppose that the sequence $(x(k), y(k))$, $k \in \mathbb{N}$, is such that $x(k)+y(k) > 0$ and there exists a constant $M > 0$ such that $y(k) \leq M$ for $k \in \mathbb{N}$. Then $\Phi(x(k), y(k), \alpha) \to 0$ as $k \to \infty$ implies $x(k) \to 0$ as $k \to \infty$. Moreover, $\liminf_{k\to\infty} y(k) \geq 0$.*

(I_6) *Let $\tilde{\Phi}(x, y, \alpha) = x - \Phi(x, y, \alpha)$. Then $\tilde{\Phi}(x, y, \alpha) = \tilde{\Phi}(y, x, \alpha)$ on $\mathcal{D}$ and $\tilde{\Phi}_x(x, y, \alpha) \geq 0$ for $x+y > 0$, where the equality holds if and only if $y = 0$. If $y \leq 1$, then $\tilde{\Phi}(x, y, \alpha) < 1$ for all $x + y > 0$.*

(I_7) *Let $x, y > 0$. Then $\Phi_\alpha(x, y, \alpha) \geq 0$.*

(I_8) *Suppose that* $\operatorname{sgn} y = \operatorname{sgn}(x + y)$ *and $y \neq 0$. Then,*

$$\Phi(x, y, \alpha) = \frac{(\alpha - 1)|x|^{\beta}|\xi|^{\alpha-2}}{\Psi\left[\Psi^{-1}(x) + \Psi^{-1}(y)\right]}, \tag{3.2.25}$$

where ξ is between $\Psi^{-1}(y)$ and $\Psi^{-1}(x) + \Psi^{-1}(y)$.

Proof. (I_1) is obvious.

Now we prove (I_2). One can easily compute that

$$\Phi_x(x,y,\alpha)=\frac{\left[\Psi^{-1}(x)+\Psi^{-1}(y)\right]^{\alpha}-|y|^{\alpha/(\alpha-1)}}{\left[\Psi^{-1}(x)+\Psi^{-1}(y)\right]^{\alpha}} \tag{3.2.26}$$

on $\mathcal{D}$. Suppose that $y>0$ and $x\geq 0$. Then clearly $\Phi_x(x,y,\alpha)\geq 0$. If we suppose $y>0$, $x\leq 0$, and $x+y>0$, then we have

$$\begin{aligned}\left[-|x|^{1/(\alpha-1)}+|y|^{1/(\alpha-1)}\right]^{\alpha}&\leq\alpha\left[|y|^{1/(\alpha-1)}-|x|^{1/(\alpha-1)}\right]^{\alpha-1}\left(-|x|^{1/(\alpha-1)}\right)\\&\leq 0,\end{aligned} \tag{3.2.27}$$

and hence, in this case $\Phi_x(x,y,\alpha)\leq 0$.

Clearly,

$$\Phi_y(x,y,\alpha)=\frac{-|x|^{\alpha/(\alpha-1)}}{\left[\Psi^{-1}(x)+\Psi^{-1}(y)\right]^{\alpha}}\leq 0 \tag{3.2.28}$$

for $x+y>0$, and hence (I_3) holds.

The statement (I_4), for the case $y>0$, follows from (I_2) and from the fact that $\Phi(0,y,\alpha)=0$. One can observe that the function $\Phi(x,y,\alpha)$ with arbitrary fixed $y<0$ and $x+y>0$ is increasing with respect to the first variable for $x>2^{\alpha-1}|y|$, decreasing for $|y|<x<2^{\alpha-1}|y|$, and

$$\Phi\left(2^{\alpha-1}|y|,y,\alpha\right)=2^{\alpha}|y|>0. \tag{3.2.29}$$

The statement now follows from the continuity of Φ.

Now we prove (I_5). Denote

$$\begin{aligned}\Gamma_{xy}^{++}&=\{k\in\mathbb{N}:x(k)>0\text{ and }y(k)>0\},\\\Gamma_{xy}^{-+}&=\{k\in\mathbb{N}:x(k)<0\text{ and }y(k)>0\},\\\Gamma_{y}^{-}&=\{k\in\mathbb{N}:y(k)<0\}.\end{aligned} \tag{3.2.30}$$

Assume for the sake of contradiction that $x(k)\not\to 0$ as $k\to\infty$. Then there exists $x^+>0$ (or $x^-<0$) such that $x(k_\ell)>x^+$ for $\{k_\ell\}\subset\Gamma_{xy}^{++}$ (or $x(k_\ell)\leq x^-$ for $\{k_\ell\}\subset I_{xy}^{-+}$). But, since Φ is monotone and bounded above with respect to y, we have

$$\Phi(x(k_\ell), y(k_\ell), \alpha) \geq \Phi(x(k_\ell), M, \alpha) \geq \Phi(x^+, M, \alpha) \tag{3.2.31}$$

or

$$\Phi(x(k_\ell), y(k_\ell), \alpha) \geq \Phi(x(k_\ell), M, \alpha) \geq \Phi(x^-, M, \alpha). \tag{3.2.32}$$

From this we see that $x^+ = 0$ (or $x^- = 0$) since $\Phi(x(k_\ell), y(k_\ell), \alpha) \to \infty$ and $\Phi(x^+, M, \alpha) = 0$ if and only if $x^+ = 0$ (or $\Phi(x^-, M, \alpha) = 0$ if and only if $x^- = 0$). Hence, such positive x^+ and negative x^- do not exist. Concerning the behavior of $x(k_\ell)$ in the case $\{k_\ell\} \subset \Gamma_y^-$, we note only that the condition $x(k)+y(k) > 0$ implies $x(k) > 0$, and this is essentially the same situation as above with $x^+ > 0$. The fact that $\liminf_{k\to\infty} y(k) \geq 0$ follows from the inequality $x(k) > -y(k) > 0$.

Now we address (I_6). For

$$\tilde{\Phi}(x, y, \alpha) = \frac{xy}{\Psi[\Psi^{-1}(x) + \Psi^{-1}(y)]}, \tag{3.2.33}$$

we clearly have

$$\tilde{\Phi}_x(x, y, \alpha) = \frac{|y|^{\alpha/(\alpha-1)}}{[\Psi^{-1}(x) + \Psi^{-1}(y)]^\alpha} > 0 \tag{3.2.34}$$

for $x + y > 0$ and $y \neq 0$. To prove the inequality $\tilde{\Phi}(x, y, \alpha) < 1$, note that $\tilde{\Phi}(x, y, \alpha) \leq \tilde{\Phi}(x, 1, \alpha) < 1$ for $x, y > 0$, and for other x, y such that $x + y > 0$, the desired inequality clearly holds.

If $x, y > 0$, then the function Φ can be rewritten as

$$\Phi(x, y, \alpha) = x\left[1 - \left(1 + \left(\frac{x}{y}\right)^{1/(\alpha-1)}\right)^{1-\alpha}\right]. \tag{3.2.35}$$

Now it is easy to see that

$$\begin{aligned}\Phi_\alpha(x, y, \alpha) &= \frac{x\left[\left(1+(x/y)^{1/(\alpha-1)}\right)\ln\left(1+(x/y)^{1/(\alpha-1)}\right) - (x/y)^{1/(\alpha-1)}\ln(x/y)^{1/(\alpha-1)}\right]}{\left(1 + (x/y)^{1/(\alpha-1)}\right)^\alpha} \\ &\geq 0,\end{aligned} \tag{3.2.36}$$

and hence (I_7) holds.

Finally, to show the statement (I_8), we simply apply the Lagrange mean value theorem. □

3.3. Reid's roundabout theorem and Sturmian theory

In this section we will formulate the discrete half-linear extension of the so-called *Picone identity*. This identity plays a crucial rôle in the proof of the "disconjugacy characterization theorem" or the so-called "Reid roundabout theorem" (see Theorem 2.6.1), which is the main result of this section. Finally, we employ the disconjugacy characterization theorem to obtain Sturm-type comparison and separation theorems.

3.3.1. The disconjugacy characterization theorem

Consider the second-order difference operators of the form

$$L_1 x(k) := \Delta\big(c_1(k)\Psi(\Delta x(k))\big) + q_1(k)\Psi(x(k+1)), \tag{3.3.1}$$

$$L_2 y(k) := \Delta\big(c_2(k)\Psi(\Delta y(k))\big) + q_2(k)\Psi(y(k+1)), \tag{3.3.2}$$

where $k \in [m,n]$ and $m,n \in \mathbb{Z}$ with $m \le n$, $q_i(k)$ for $i \in \{1,2\}$ are real-valued sequences defined on $[m,n]$, and the sequences $c_i(k)$ for $i \in \{1,2\}$ are real valued and defined on $[m,n+1]$ with $c_i(k) \neq 0$ for $i \in \{1,2\}$ on $[m,n+1]$.

Lemma 3.3.1 (Picone's identity). *Let x, y be defined on $[m,n+2]$ and let $y(k) \neq 0$ for $k \in [m,n+1]$. Then the equality*

$$\begin{aligned}
\Delta&\left\{\frac{x(k)}{\Psi(y(k))}\big[\Psi(y(k))c_1(k)\Psi(\Delta x(k)) - \Psi(x(k))c_2(k)\Psi(\Delta y(k))\big]\right\} \\
&= \big(q_2(k) - q_1(k)\big)\,|x_{k+1}|^{\alpha} + \big(c_1(k) - c_2(k)\big)\,|\Delta x(k)|^{\alpha} \\
&\quad + \frac{x(k+1)}{\Psi(y(k+1))}\big[L_1x(k)\Psi(y(k+1)) - L_2y(k)\Psi(x(k+1))\big] \\
&\quad + \left\{c_2(k)\,|\Delta x_k|^{\alpha} - \frac{c_2(k)\Psi(\Delta y(k))}{\Psi(y(k+1))}\,|x(k+1)|^{\alpha} + \frac{c_2(k)\Psi(\Delta y(k))}{\Psi(y(k))}\,|x(k)|^{\alpha}\right\}
\end{aligned} \tag{3.3.3}$$

holds for $k \in [m,n]$.

Proof. For $k \in [m,n]$ we have

$$\begin{aligned}
\Delta&\big[x(k)c_1(k)\Psi(\Delta x(k))\big] - x(k+1)L_1x(k) \\
&= x(k+1)\Delta\big(c_1(k)\Psi(\Delta x(k))\big) + \Delta x(k)c_1(k)\Psi(\Delta x(k)) \\
&\quad - x(k+1)\Delta\big(c_1(k)\Psi(\Delta x(k))\big) - x(k+1)q_1(k)\Psi(x(k+1)) \\
&= -q_1(k)\,|x(k+1)|^{\alpha} + c_1(k)\,|\Delta x(k)|^{\alpha}.
\end{aligned} \tag{3.3.4}$$

Also, for $k \in [m, n]$ we have

$$
\begin{aligned}
& L_2 y(k)\Psi(x(k+1))\frac{x(k+1)}{\Psi(y(k+1))} - \Delta\left[\frac{x(k)}{\Psi(y(k))}\Psi(x(k))c_2(k)\Psi(\Delta y(k))\right] \\
&\quad - c_2(k)|\Delta x(k)|^\alpha + \frac{c_2(k)\Psi(\Delta y(k))}{\Psi(y(k+1))}x(k+1)\Psi(x(k+1)) \\
&\quad - \frac{c_2(k)\Psi(\Delta y(k))}{\Psi(y(k))}x(k)\Psi(x(k)) \\
&= \frac{\Delta(c_2(k)\Psi(\Delta y(k)))}{\Psi(y(k+1))}x(k+1)\Psi(x(k+1)) + c_2(k)x(k+1)\Psi(x(k+1)) \\
&\quad - \left(\frac{\Delta x(k)\Psi(y(k)) - x(k)\Delta\Psi(y(k))}{\Psi(y(k))\Psi(y(k+1))}\right)\Psi(x(k))c_2(k)\Psi(\Delta y(k)) \\
&\quad - \frac{x(k+1)}{\Psi(y(k+1))}\left[\Psi(x(k+1))\Delta(c_2(k)\Psi(\Delta y(k))) + \Delta\Psi(x(k))c_2(k)\Psi(\Delta y(k))\right] \\
&\quad - c_2(k)|\Delta x(k)|^\alpha + \frac{c_2(k)\Psi(\Delta y(k))}{\Psi(y(k+1))}x(k+1)\Psi(x(k+1)) \\
&\quad - \frac{c_2(k)\Psi(\Delta y(k))}{\Psi(y(k))}x(k)\Psi(x(k)) \\
&= q_2(k)|x(k+1)|^\alpha - c_2(k)|\Delta x(k)|^\alpha \\
&\quad + \frac{1}{\Psi(y(k))\Psi(y(k+1))}\big[-c_2(k)\Delta x(k)\Psi(x(k))\Psi(y(k))\Psi(\Delta y(k)) \\
&\qquad\qquad + c_2(k)x(k)\Psi(\Delta y(k))\Delta\Psi(y(k)) \\
&\qquad\qquad - c_2(k)x(k+1)\Delta\Psi(x(k))\Psi(y(k))\Psi(\Delta y(k)) \\
&\qquad\qquad + c_2(k)x(k+1)\Psi(x(k+1))\Psi(y(k))\Psi(\Delta y(k)) \\
&\qquad\qquad - c_2(k)x(k)\Psi(x(k))\Psi(y(k+1))\Psi(\Delta y(k))\big] \\
&= q_2(k)|x(k+1)|^\alpha - c_2(k)|\Delta x(k)|^\alpha \\
&\quad + \frac{1}{\Psi(y(k))\Psi(y(k+1))}\big[c_2(k)x(k)\Psi(x(k))\Psi(y(k))\Psi(\Delta y(k)) \\
&\qquad\qquad - c_2(k)x(k)\Psi(x(k))\Psi(y(k))\Psi(\Delta y(k))\big] \\
&= q_2(k)|x(k+1)|^\alpha - c_2(k)|\Delta x(k)|^\alpha.
\end{aligned}
\tag{3.3.5}
$$

Combining these two equalities we obtain the desired result. □

The last term of (3.3.3) given in Lemma 3.3.1 can be rewritten as

$$H(x, y) - \frac{c_2(k)y(k)}{y(k+1)}, \tag{3.3.6}$$

where

$$H(x,y) = \frac{y(k+1)}{y(k)}\left|\Delta y(k)\right|^{\alpha} - \frac{y(k+1)\Psi(\Delta y(k))}{y(k)\Psi(y(k+1))}\left|x(k+1)\right|^{\alpha} + \frac{y(k+1)\Psi(\Delta y(k))}{y(k)\Psi(y(k))}\left|x(k)\right|^{\alpha}. \tag{3.3.7}$$

Using this fact we have the following lemma.

Lemma 3.3.2. *Let x, y be defined on $[m, n+1]$ and let $y(k) \neq 0$ on this interval. Then $H(x,y) \geq 0$ for $k \in [m,n]$, where equality holds if and only if $\Delta x = (x\Delta y)/y$.*

Proof. It is enough to verify the inequality

$$\frac{y(k+1)}{y(k)}\left|\Delta x(k)\right|^{\alpha} + \frac{y(k+1)\left|\Delta y(k)\right|^{\alpha-2}\Delta y(k)}{\left|y(k)\right|^{\alpha-2}y^2(k)}\left|x(k)\right|^{\alpha} \geq \frac{y(k+1)\left|\Delta y(k)\right|^{\alpha-2}\Delta y(k)}{y(k)\left|y(k+1)\right|^{\alpha-2}y(k+1)}\left|x(k+1)\right|^{\alpha} \tag{3.3.8}$$

for $k \in [m,n]$. Set $z(k) = y(k)/y(k+1)$. Then inequality (3.3.8) assumes the form

$$\frac{\left|\Delta x(k)\right|^{\alpha}}{z(k)} + \frac{\left|1-z(k)\right|^{\alpha-2}(1-z(k))}{\left|z(k)\right|^{\alpha}}\left|x(k)\right|^{\alpha} \geq \frac{1}{z(k)}\left|1-z(k)\right|^{\alpha-2}(1-z(k))\left|x(k+1)\right|^{\alpha}. \tag{3.3.9}$$

First, we prove the lemma under the condition $u_k = 1$ for $k \in [m,n]$. In this case, $H(x,y) = |\Delta x(k)|^{\alpha} \geq 0$, where the equality holds if and only if $\Delta x(k) = 0$, which holds if and only if $\Delta x(k) = x(k)\Delta y(k)/y(k)$.

In the remainder of this proof, when we write z we mean $z(k)$ (the same holds for the sequences s, u, v, and v_0). Now denote

$$\begin{aligned} J_0 &= \{k \in [m,n] : y(k) \neq y(k+1)\}, \\ J_1 &= \{k \in J_0 : x(k) = 0\}, \\ J_2 &= \{k \in J_0 : \Delta x(k) = 0\}. \end{aligned} \tag{3.3.10}$$

Hence we have four cases.

Case 1. $k \in J_0 \setminus (J_1 \cup J_2)$. Setting $s = \Delta x(k)$ and $u = x(k)$, we get $x(k+1) = s + u$. Inequality (3.3.8) becomes

$$\frac{|s|^\alpha}{z} + \frac{|1-z|^{\alpha-2}(1-z)}{|z|^\alpha}|u|^\alpha \geq \frac{1}{z}|1-z|^{\alpha-2}(1-z)|s+u|^\alpha, \tag{3.3.11}$$

and dividing by $|u|^\alpha$, the inequality (3.3.11) takes the form

$$\left|\frac{s}{u}\right|^\alpha \frac{1}{z} + \frac{1}{|z|^\alpha}|1-z|^{\alpha-2}(1-z) \geq \frac{1}{z}|1-z|^{\alpha-2}(1-z)\left|1+\frac{s}{u}\right|^\alpha. \tag{3.3.12}$$

Now denote

$$G(v;z) = \frac{|v|^\alpha}{z} - \frac{1}{z}|1-z|^{\alpha-2}(1-z)|1+v|^\alpha + \frac{1}{|z|^\alpha}|1-z|^{\alpha-2}(1-z), \tag{3.3.13}$$

where $v = s/u$. For $v = v_0 = (1-z)/z$, the following equality holds:

$$\begin{aligned} G(v_0;z) &= \frac{|1-z|^\alpha}{z|z|^\alpha} - \frac{1}{z|z|^\alpha}|1-z|^{\alpha-2}(1-z) + \frac{1}{|z|^\alpha}|1-z|^{\alpha-2}(1-z) \\ &= \frac{|1-z|^{\alpha-2}}{z|z|^\alpha}\left[(1-z)^2 - (1-z) + z - z^2\right] = 0. \end{aligned} \tag{3.3.14}$$

Differentiating G with respect to v, we obtain

$$G_v(v;z) = \alpha\frac{|v|^{\alpha-1}\operatorname{sgn} v}{z} - \frac{\alpha}{z}|1-z|^{\alpha-2}(1-z)|1+v|^{\alpha-1}\operatorname{sgn}(1+v), \tag{3.3.15}$$

and hence

$$G_v(v_0;z) = \alpha\frac{|1-z|^{\alpha-1}\operatorname{sgn}(1-z)}{|z|^{\alpha-1}z\operatorname{sgn} z} - \alpha\frac{|1-z|^{\alpha-2}(1-z)}{|z|^{\alpha-1}z\operatorname{sgn} z} = 0. \tag{3.3.16}$$

Further, we have

$$\begin{aligned} G_{vv}(v;z) &= \alpha(\alpha-1)\frac{1}{z}\left[\frac{|1-z|^{\alpha-2}}{|z|^{\alpha-2}} - \frac{|1-z|^{\alpha-2}(1-z)}{|z|^{\alpha-2}}\right] \\ &= \alpha(\alpha-1)\frac{|1-z|^{\alpha-2}}{z|z|^{\alpha-2}}[1-1+z] \\ &= \alpha(\alpha-1)\frac{|1-z|^{\alpha-2}}{|z|^{\alpha-2}} > 0. \end{aligned} \tag{3.3.17}$$

Since

$$\begin{aligned} G_v(v;z) = 0 &\Longleftrightarrow |v|^{\alpha-1}\operatorname{sgn} v = |1-z|^{\alpha-2}(1-z)|1+v|^{\alpha-1}\operatorname{sgn}(1+v) \\ &\Longleftrightarrow |v|^{\alpha-1}\operatorname{sgn} v = \big|(1-z)(1+v)\big|^{\alpha-1}\operatorname{sgn}\big[(1-z)(1+v)\big] \\ &\Longleftrightarrow v = 1+v-z-zv \\ &\Longleftrightarrow v = \frac{1-z}{z} \\ &\Longleftrightarrow v = v_0 \end{aligned} \tag{3.3.18}$$

holds, G_v just has a simple zero v_0. This occurs if and only if $\Delta x = (x\Delta y)/y$. In the opposite case, $G(v;z) > 0$.

Case 2. $k \in J_1 \setminus J_2$. Put $u = 0$ in (3.3.11) and suppose by contradiction that

$$\frac{\left|\Delta x(k)\right|^{\alpha}}{z} \le \frac{1}{z}|1-z|^{\alpha-2}(1-z)\left|\Delta x(k)\right|^{\alpha}. \tag{3.3.19}$$

Therefore,

$$\frac{1}{z} + |1-z|^{\alpha-2} \le \frac{|1-z|^{\alpha-2}}{z}. \tag{3.3.20}$$

Now, we distinguish the following three subcases.

Subcase a. If $z > 1$, then $1 + z|1-z|^{\alpha-2} \le |1-z|^{\alpha-2}$, or

$$0 < \frac{1}{|1-z|^{\alpha-1}} \le \operatorname{sgn}(1-z), \tag{3.3.21}$$

which is a contradiction.

Subcase b. If $0 < z < 1$, then the same computation as above holds, and hence we obtain a contradiction, since

$$\frac{1}{|1-z|^{\alpha-1}} \nleq 1, \quad \text{where } 0 < |1-z|^{\alpha-1} < 1. \tag{3.3.22}$$

Subcase c. If $z < 0$, then we have $1/|1-z|^{\alpha-1} \geq \operatorname{sgn}(1-z)$, which is again a contradiction, since

$$\frac{1}{|1-z|^{\alpha-1}} \ngeq 1, \quad \text{where } |1-z|^{\alpha-1} > 1. \tag{3.3.23}$$

Case 3. $k \in J_2 \setminus J_1$. Put $s = 0$ in (3.3.11) and suppose by contradiction that

$$\frac{|1-z|^{\alpha-2}(1-z)}{|z|^{\alpha}} \left| x(k) \right|^{\alpha} \leq \frac{1}{z} |1-z|^{\alpha-2}(1-z) \left| x(k) \right|^{\alpha}. \tag{3.3.24}$$

Consequently, $(1-z)/|z|^{\alpha} \leq (1-z)/z$. As in the proof of Case 2, we have the following subcases.

Subcase a. If $z > 1$, then $1/|z|^{\alpha} \geq 1/z$ so that $z \geq |z|^{\alpha}$.

Subcase b. If $0 < z < 1$, then $1/|z|^{\alpha} \leq 1/z$ so that $z \leq |z|^{\alpha}$.

Subcase c. If $z < 0$, then $1/|z|^{\alpha} \leq 1/z$ so that $z \geq |z|^{\alpha}$.

Obviously in every subcase we again arrive at a contradiction.

Case 4. $k \in J_1 \cap J_2$. Here, we see that $H(x, y) = 0$, since $x(k) = 0 = \Delta x(k)$. Note that this case occurs if and only if $\Delta x(k) = x(k)\Delta y(k)/y(k)$.

The proof is complete. □

Remark 3.3.3. Concerning the linear case, if we put $\alpha = 2$, then we get

$$H(x, y) = \left(\Delta x(k) - \frac{\Delta y(k)}{y(k)} x(k) \right)^2. \tag{3.3.25}$$

Consider equation (3.2.9) on the interval $[m, n]$ with $c(k) \neq 0$ on $[m, n+1]$. Now, we are ready to formulate the central statement of the oscillation theory of equation (3.2.9).

Theorem 3.3.4 (Reid's roundabout theorem). *All of the following statements are equivalent.*

(I) *Equation (3.2.9) is disconjugate on* $[m, n]$.

(II) *Equation (3.2.9) has a solution* x *without generalized zeros on* $[m, n+1]$.

(III) *The generalized Riccati difference equation (3.2.17) has a solution* w *on* $[m, n]$ *with* $c(k) + w(k) > 0$ *on* $[m, n]$.

(IV) *The functional* $\mathcal{F}$ *defined by equation (3.2.13) is positive definite on* $U(m, n)$.

Proof. First we show (I)$\Rightarrow$(II). Let y be a solution of (3.2.9) given by the initial conditions $y(m) = 0$ and $y(m+1) = 1$. It follows that $c(k)y(k)y(k+1) > 0$ for $k \in [m+1, n]$. Consider the solution $y_\varepsilon(k)$ satisfying the initial conditions $y_\varepsilon(m) = \varepsilon c(m)$ and $y_\varepsilon(m+1) = 1$. Then, $y_\varepsilon(k) \to y(k)$ as $\varepsilon \to 0$. If we choose $\varepsilon > 0$ sufficiently small, then $x(k) \equiv y_\varepsilon(k)$ satisfies $c(k)x(k)x(k+1) > 0$ for $k \in [m, n]$, that is, x has no generalized zero on $[m, n+1]$.

Now we prove (II)$\Rightarrow$(III). Assume that x is a solution of equation (3.2.9) satisfying $c(k)x(k)x(k+1) > 0$ on $[m, n]$. Using the Riccati-type substitution $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k))$, we have

$$\begin{aligned}\frac{c(k)x(k+1)}{x(k)} &= c(k)\left[\frac{x(k)+\Delta x(k)}{x(k)}\right] = c(k)\left[1+\frac{\Psi^{-1}(w(k))}{\Psi^{-1}(c(k))}\right]\\ &= \frac{c(k)}{\Psi^{-1}(c(k))}\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right].\end{aligned} \tag{3.3.26}$$

Since

$$\begin{aligned}\Psi\left(\frac{x(k)}{x(k+1)}\right) &= \Psi(c(k))\Psi\left(\frac{x(k)}{c(k)x(k+1)}\right)\\ &= \frac{c(k)}{\Psi\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]},\end{aligned} \tag{3.3.27}$$

we have

$$\begin{aligned}\Delta w(k) &= -q(k) - w(k)\left[1-\frac{\Psi(x(k))}{\Psi(x(k+1))}\right]\\ &= -q(k) - w(k)\left[1-\frac{c(k)}{\Psi\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]}\right].\end{aligned} \tag{3.3.28}$$

Now, (3.3.26) clearly implies $c(k) + w(k) > 0$, and hence (III) holds.

Now we address (III)$\Rightarrow$(IV). Assume that w is a solution of equation (3.2.17) with $c(k) + w(k) > 0$. Note that $x(k)$ given by $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k))$, that is,

$$\Delta x(k) = \Psi^{-1}\left(\frac{w(k)}{c(k)}\right)x(k), \tag{3.3.29}$$

is a solution of (3.2.9). From the Picone identity applied to $q_1(k) = q_2(k) = q(k)$, $c_1(k) = c_2(k) = c(k)$, $x(k) = \xi(k)$, and $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k))$, we obtain

$$\begin{aligned}&\Delta\left[\xi(k)c(k)\Psi(\Delta\xi(k))\right] - \Delta\left[|\xi(k)|^{\alpha}w(k)\right]\\ &\qquad = \xi(k+1)\Delta\left(c(k)\Psi(\Delta\xi(k))\right) + q(k)|\xi(k+1)|^{\alpha} + \tilde{G}(\xi, w),\end{aligned} \tag{3.3.30}$$

where

$$\tilde{G}(\xi,w)=c(k)\,|\,\Delta\xi(k)\,|^{\alpha}-\frac{c(k)w(k)}{\Psi\big[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\big]}\,|\,\xi(k+1)\,|^{\alpha}+w(k)\,|\,\xi(k)\,|^{\alpha}. \tag{3.3.31}$$

Hence

$$\tilde{G}(\xi,w)+\Delta\Big[w(k)\,|\,\xi(k)\,|^{\alpha}\Big]=c(k)\,|\,\Delta\xi(k)\,|^{\alpha}-q(k)\,|\,\xi(k+1)\,|^{\alpha}. \tag{3.3.32}$$

Summing both sides of this equation from m to n yields

$$\mathcal{F}(\xi)=w(n+1)\,|\,\xi(n+1)\,|^{\alpha}-w(m)\,|\,\xi(m)\,|^{\alpha}+\sum_{k=m}^{n}\tilde{G}(\xi,w). \tag{3.3.33}$$

Then $\mathcal{F}(\xi)\geq 0$, since $c(k)x(k+1)/x(k)>0$ by (3.3.9) and because of Lemma 3.3.2. In addition, if $\mathcal{F}(\xi)=0$, then again by Lemma 3.3.2, $\Delta\xi(k)=\xi(k)\Delta x(k)/x(k)$. Furthermore, we have $\xi(m)=0$ and therefore $\xi\equiv 0$. Consequently $\mathcal{F}(\xi)>0$ for all nontrivial admissible sequences.

Finally, we discuss (IV)⇒(I). Suppose to the contrary that equation (3.2.9) is not disconjugate on $[m,n]$. Then there exists a nontrivial solution x of equation (3.2.9) with $c(M)x(M)x(M+1)\leq 0$ for $x(M+1)\neq 0$ and $c(N)x(N)x(N+1)\leq 0$ for $x(N)\neq 0$, where $m+1\leq M+1<N\leq n$. Define

$$\xi(k)=\begin{cases}0 & \text{for } m\leq k\leq M,\\ x(k) & \text{for } M+1\leq k\leq N,\\ 0 & \text{for } N+1\leq k\leq n+1.\end{cases} \tag{3.3.34}$$

Then ξ is a nontrivial admissible sequence and hence $\mathcal{F}(\xi)>0$. Using summation by parts, we get

$$\begin{aligned}\mathcal{F}(\xi)&=\sum_{k=m}^{n}\Big[c(k)\,|\,\Delta\xi(k)\,|^{\alpha}-q(k)\,|\,\xi(k+1)\,|^{\alpha}\Big]\\ &=\big[\xi(k)c(k)\Psi(\Delta\xi(k))\big]\,\big|_{k=m}^{n+1}-\sum_{k=m}^{n}\xi(k+1)L(\xi(k))\\ &=-\sum_{k=M}^{N-1}\xi(k+1)L(\xi(k)),\end{aligned} \tag{3.3.35}$$

where

$$L(\xi(k))=\Delta\big(c(k)\Psi(\Delta\xi(k))\big)+q(k)\Psi(\xi(k)). \tag{3.3.36}$$

Now,

$$\begin{aligned}
\mathcal{F}(\xi) &= x(M+1)\big[-q(M)\Psi(x(M+1)) - \Delta\big(c(N-1)\Psi(\Delta\xi(N-1))\big)\big] \\
&\quad + x(N)\big[-q(N-1)\Psi(x(N)) - \Delta\big(c(N-1)\Psi(\Delta\xi(N-1))\big)\big] \\
&= x(M+1)\big[\Delta\big(c(M)\Psi(\Delta x(M))\big) - c(M+1)\Psi(\Delta\xi(M+1)) \\
&\qquad\qquad + c(M)\Psi(\Delta\xi(M))\big] \\
&\quad + x(N)\big[\Delta\big(c(N-1)\Psi(\Delta x(N-1))\big) - c(N)\Psi(\Delta\xi(N)) \\
&\qquad\qquad + c(N-1)\Psi(\Delta\xi(N-1))\big] \\
&= x(M+1)\big[c(M+1)\Psi(\Delta x(M+1)) - c(M)\Psi(\Delta x(M)) \\
&\quad - c(M+1)\Psi(\Delta x(M+1)) + c(M)\Psi(x(M+1))\big] \\
&\quad + x(N)\big[c(N)\Psi(\Delta x(N)) - c(N-1)\Psi(\Delta x(N-1)) \\
&\qquad\qquad + c(N)\Psi(x(N)) + (N-1)\Psi(\Delta x(N-1))\big] \\
&= G_1\big(x(M), x(M+1); c(M)\big) + G_2\big(x(N), x(N+1); c(N)\big),
\end{aligned} \tag{3.3.37}$$

where

$$\begin{aligned}
&G_1\big(x(M), x(M+1); c(M)\big) \\
&\qquad = x(M+1)c(M)\Psi(x(M+1)) - x(M+1)c(M)\Psi(\Delta y(M)), \\
&G_2\big(x(N), x(N+1); c(N)\big) \\
&\qquad = x(N)c(N)\Psi(\Delta x(N)) + x(N)c(N)\Psi(x(N)).
\end{aligned} \tag{3.3.38}$$

To show that $\mathcal{F}(\xi) \le 0$, it remains to verify that $G_1(x(M), x(M+1); c(M)) \le 0$ and $G_2(x(N), x(N+1); c(N)) \le 0$. We will examine the function G_2, that is, we will check the inequality

$$x(N)c(N)\Psi(\Delta x(N)) \le -x(N)c(N)\Psi(x(N)). \tag{3.3.39}$$

It holds if and only if

$$c(N)\Psi\left(\frac{\Delta x(N)}{x(N)}\right) \le -c(N). \tag{3.3.40}$$

Now, if $\Delta x(N) = 0$, then we get $G_2 = c(N)|x(N)|^{\alpha}$. Hence $c(N)$ must be negative, since we assume $c(N)x^2(N) \le 0$. Consequently, $G_2 < 0$. Further, let $\Delta x(N) \ne 0$. Setting $y = x(N+1)/x(N)$, we obtain

$$\tilde{G}_2\big(x; a(N)\big) = c(N)|x-1|^{\alpha-1}\operatorname{sgn}(x-1) + c(N). \tag{3.3.41}$$

Note that $G_2 < 0$ ($G_2 = 0$) if and only if $\tilde{G}_2 < 0$ ($\tilde{G}_2 = 0$). If $x(N+1) = 0$, then $x = 0$ and hence $\tilde{G}_2(0; a(N)) = 0$. Differentiating $\tilde{G}_2$ with respect to x, we obtain

$$\frac{\partial \tilde{G}_2}{\partial x} = (\alpha - 1)c(N)|x - 1|^{\alpha-2}. \tag{3.3.42}$$

Now, we distinguish the following two particular cases.

Case 1. $x > 0 \Leftrightarrow x(N)x(N+1) > 0 \Leftrightarrow c(N) < 0 \Leftrightarrow \partial\tilde{G}_2/\partial x < 0$.

Case 2. $x < 0 \Leftrightarrow x(N)x(N+1) < 0 \Leftrightarrow c(N) > 0 \Leftrightarrow \partial\tilde{G}_2/\partial x > 0$.

Therefore we have $G_2 < 0$. Similarly we can verify that $G_1 \le 0$ holds. Thus, we conclude that $\mathcal{F}(\xi) = G_1 + G_2 \le 0$, which is a contradiction. Hence (I) holds. □

3.3.2. Sturmian theory

Consider two equations $L_1x(k) = 0$ and $L_2y(k) = 0$, where the operators L_1 and L_2 are defined by (3.3.1) and (3.3.2), respectively. Denote

$$\mathcal{F}_{c_2,q_2}(\xi) = \sum_{k=m}^{n} \Big[c_2(k)\,|\,\Delta\xi(k)\,|^{\alpha} - q_2(k)\,|\,\xi(k+1)\,|^{\alpha}\Big]. \tag{3.3.43}$$

Then we have the following versions of Sturmian theorems.

Theorem 3.3.5 (Sturm's comparison theorem). *Suppose that $c_2(k) \ge c_1(k)$ and $q_1(k) \ge q_2(k)$ for $k \in [m,n]$. If the equation $L_1x(k) = 0$ is disconjugate on $[m,n]$, then $L_2y(k) = 0$ is also disconjugate on $[m,n]$.*

Proof. Suppose that $L_1x(k) = 0$ is disconjugate on $[m,n]$. Then Theorem 3.3.4 yields $\mathcal{F}(\xi) > 0$ for all admissible sequences ξ. For such an admissible ξ, we have $\mathcal{F}_{c_2,q_2}(\xi) \ge \mathcal{F}(\xi) > 0$. Hence $\mathcal{F}_{c_2,q_2}(\xi) > 0$ and thus $L_2y(k) = 0$ is disconjugate on $[m,n]$ by Theorem 3.3.4. □

Theorem 3.3.6 (Sturm's separation theorem). *Two nontrivial solutions x_1 and x_2 of $L_1x(k) = 0$, which are not proportional, cannot have a common zero. If x_1 satisfying $x_1(m) = 0$ has a generalized zero in $(m, n+1]$, then x_2 has a generalized zero in $(m, n+1]$. If x_1 has generalized zeros in $(m, m+1]$ and $(n, n+1]$, then x_2 has a generalized zero in $(m, n+1]$.*

Proof. It is sufficient to prove the part concerning the common zero of nonproportional solutions since the remaining part follows from Theorem 3.3.4. Suppose to the contrary that $x_1(\ell) = 0 = x_2(\ell)$ for some $\ell \in [m,n]$. Let $\tilde{x}$ be a solution of $L_1x(k) = 0$ such that $\tilde{x}(\ell) = 0$ and $\tilde{x}(\ell+1) = 1$. Then, $x_1 = A\tilde{x}$ and $x_2 = B\tilde{x}$, where A and B are suitable nonzero constants, are also nontrivial solutions of $L_1x(k) = 0$ satisfying $x_1(\ell) = 0$, $x_1(\ell+1) = A$ and $x_2(\ell) = 0$, $x_2(\ell+1) = B$, respectively. Hence $x_1 = Cx_2$, where $C = A/B$, which is a contradiction. □

3.4. Nonoscillation and conjugacy criteria

In this section, we will present nonoscillation theorems for the second-order half-linear difference equation (3.2.9). Some of these criteria are based on the Riccati technique which deals with the existence of a solution of the generalized Riccati difference inequality. The others are based on the so-called variational principle which is concerned with the equivalence (I)$\Leftrightarrow$(IV) from Theorem 3.3.4. Some criteria for equation (3.2.9) to be conjugate on $\mathbb{Z}$ are also given.

3.4.1. Nonoscillation theorems based on the Riccati technique

We consider the operator

$$Ly(k) = \Delta\big(c(k)\Psi(\Delta y(k))\big) + q(k)\Psi\big(y(k+1)\big), \tag{3.4.1}$$

where $c(k)$, $q(k)$, and $\psi(y)$ are defined as in equation (3.2.9).

First, we give the following two lemmas which are based on the Sturm-type separation theorem.

Lemma 3.4.1. *If there exists a sequence $y(k)$ such that*

$$c(k)y(k)y(k+1) > 0, \qquad y(k+1)Ly(k) \le 0 \tag{3.4.2}$$

for some $k \in [m,\infty)$, $m \in \mathbb{N}$, where the operator L is defined by (3.4.1), then equation (3.2.9) is nonoscillatory.

Proof. Suppose that a sequence $y(k)$ satisfies (3.4.2) on the discrete interval $[m,\infty)$. Then $\psi(k) = -y(k+1)Ly(k)$ is a nonnegative sequence on this discrete interval. Set $\tilde{c}(k) = c(k)$ and $\tilde{q}(k) = q(k) + [\psi(k)/|y(k+1)|^{\alpha}]$. Then $\tilde{q}(k) \ge q(k)$ and

$$\begin{aligned}\Delta\big(\tilde{c}(k)\Psi(\Delta y(k))\big) &+ \tilde{q}(k)\Psi\big(y(k+1)\big)\\ &= \Delta\big(c(k)\Psi(\Delta y(k))\big) + \left(q(k) + \frac{\psi(k)}{|y(k+1)|^{\alpha}}\right)\Psi\big(y(k+1)\big) = 0.\end{aligned} \tag{3.4.3}$$

Thus the equation

$$\Delta\big(\tilde{c}(k)\Psi(\Delta y(k))\big) + \tilde{q}(k)\Psi\big(y(k+1)\big) = 0 \tag{3.4.4}$$

is disconjugate on $[m,\infty)$. Therefore, equation (3.2.9) is disconjugate on $[m,\infty)$ by the Sturm comparison theorem (Theorem 3.3.5), and hence it is nonoscillatory. This completes the proof. □

Lemma 3.4.2. *Equation (3.2.9) is nonoscillatory if and only if there exists a sequence $w(k)$ with $c(k) + w(k) > 0$ for all large k such that*

$$\mathcal{R}[w(k)] \le 0, \tag{3.4.5}$$

where the operator $\mathcal{R}$ is defined by (3.2.17), satisfying $c(k)+w(k)>0$ in a neighborhood of infinity.

Proof. The "only if" part follows from Theorem 3.3.4. For the "if" part, let $w(k)$ satisfy $\mathcal{R}[w(k)] \le 0$ with $c(k)+w(k)>0$ on $[m,\infty)$, and let

$$y(k) = \prod_{j=m}^{k-1}\left[1+\Psi^{-1}\left(\frac{w(j)}{c(j)}\right)\right] \quad \text{for } k>m \tag{3.4.6}$$

be a solution of the first-order difference equation

$$\Delta y(k) = \Psi^{-1}\left(\frac{w(k)}{c(k)}\right)y(k) \quad \text{with } y(m)=1. \tag{3.4.7}$$

Thus, $y(k)\neq 0$ since

$$1+\Psi^{-1}\left(\frac{w(k)}{c(k)}\right) = \frac{1}{\Psi^{-1}(c(k))}\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right] \neq 0, \tag{3.4.8}$$

and

$$c(k)y(k)y(k+1) > 0, \tag{3.4.9}$$

since

$$\frac{\Psi(y(k))}{\Psi(y(k+1))} = \frac{1}{\Psi[1+\Delta y(k)/y(k)]} = \frac{c(k)}{\Psi[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))]}. \tag{3.4.10}$$

Further,

$$\begin{aligned}
&y(k+1)Ly(k)\\
&\quad= y(k+1)\left[\Delta\left(c(k)\Psi(\Delta y(k))\right)+q(k)\Psi(y(k+1))\right]\\
&\qquad-\frac{|y(k+1)|^{\alpha}c(k)\Psi(\Delta y(k))\Delta\Psi(y(k))}{\Psi(y(k))\Psi(y(k+1))}\\
&\qquad+\frac{|y(k+1)|^{\alpha}c(k)\Psi(\Delta y(k))\Delta\Psi(y(k))}{\Psi(y(k))\Psi(y(k+1))}\\
&\quad= y(k+1)\Psi(y(k+1))\frac{\Delta\left(c(k)\Psi(\Delta y(k))\right)\Psi(y(k))-c(k)\Psi(\Delta y(k))\Delta\Psi(y(k))}{\Psi(y(k))\Psi(y(k+1))}\\
&\qquad+|y(k+1)|^{\alpha}q(k)+|y(k+1)|^{\alpha}\frac{c(k)\Psi(\Delta y(k))}{\Psi(y(k))}\left[1-\frac{\Psi(y(k))}{\Psi(y(k+1))}\right]\\
&\quad= |y(k+1)|^{\alpha}\mathcal{R}[w(k)] \le 0
\end{aligned} \tag{3.4.11}$$

for $k\in[m,\infty)$. The conclusion now follows from Lemma 3.4.1. □

Now we assume that

$$c(k) > 0, \qquad \sum_{j=m}^{\infty} c^{1-\beta}(j) = \infty, \tag{3.4.12}$$

$$\sum^{\infty} q(k) = \lim_{k\to\infty} \sum^{k} q(j) \quad \text{is convergent,} \tag{3.4.13}$$

$$\lim_{k\to\infty} \frac{c^{1-\beta}(k)}{\sum^{k-1} c^{1-\beta}(j)} = 0. \tag{3.4.14}$$

We present the following nonoscillation criterion for equation (3.2.9).

Theorem 3.4.3. *Suppose that conditions (3.4.12), (3.4.13), and (3.4.14) hold. If*

$$\limsup_{k\to\infty} \left(\sum^{k-1} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=k}^{\infty} q(j)\right) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.4.15}$$

$$\liminf_{k\to\infty} \left(\sum^{k-1} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=k}^{\infty} q(j)\right) > -\left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.4.16}$$

then equation (3.2.9) is nonoscillatory.

Proof. By Lemma 3.4.2 it suffices to show that the generalized Riccati inequality (3.4.5) has a solution w with $c(k)+w(k) > 0$ in a neighborhood of infinity. Set

$$w(k) = a\left(\sum^{k-1} c^{1-\beta}(j)\right)^{1-\alpha} + \sum_{j=k}^{\infty} q(j), \tag{3.4.17}$$

where a is a suitable constant which will be specified later.

By the Lagrange mean value theorem, we obtain

$$\Delta\left(\sum^{k-1} c^{1-\beta}(j)\right) = (1-\alpha)c^{1-\beta}(k)\eta^{-\alpha}(k), \tag{3.4.18}$$

where

$$\sum^{k-1} c^{1-\beta}(j) \le \eta(k) \le \sum^{k} c^{1-\beta}(j). \tag{3.4.19}$$

Similarly,

$$\begin{aligned} w(k)&\left[\Psi\left(\Psi^{-1}(c(k)) + \Psi^{-1}(w(k))\right) - c(k)\right] \\ &= w(k)\left[\Psi\left(\Psi^{-1}(c(k)) + \Psi^{-1}(w(k))\right) - \Psi\left(\Psi^{-1}(c(k))\right)\right] \\ &= (\alpha-1)\left|\xi(k)\right|^{\alpha-2}\Psi^{-1}(w(k))w(k) \\ &= (\alpha-1)\left|\xi(k)\right|^{\alpha-2}\left|w(k)\right|^{\beta}, \end{aligned} \tag{3.4.20}$$

where $\xi(k)$ is between $\Psi^{-1}(c(k))$ and $\Psi^{-1}(c(k)) + \Psi^{-1}(w(k))$. Hence

$$c^{\beta-1}(k) - |w(k)|^{\beta-1} \le \xi(k) \le c^{\beta-1}(k) + |w(k)|^{\beta-1}. \tag{3.4.21}$$

Denote

$$A(k) = \left(\sum^{k-1} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=k}^{\infty} q(j)\right), \tag{3.4.22}$$

and let $a = [(\alpha-1)/\alpha]^{\alpha}$. Then

$$\begin{aligned}\frac{|w(k)|}{c(k)} &= \frac{1}{c(k)}\left|a\left(\sum^{k-1} c^{1-\beta}(j)\right)^{1-\alpha} + \sum_{j=k}^{\infty} q(j)\right| \\ &= \left[\frac{c^{1-\beta}(k)}{\sum^{k-1} c^{1-\beta}(j)}\right]^{\alpha-1} |a + A(k)| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty,\end{aligned} \tag{3.4.23}$$

and hence

$$c(k) + w(k) = c(k)\left[1 + \frac{w(k)}{c(k)}\right] > 0 \quad \text{for all large } k \in \mathbb{N}. \tag{3.4.24}$$

Now the assumptions (3.4.15) and (3.4.16) imply the existence of $\varepsilon_1 > 0$ such that

$$-\left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} + 2\varepsilon_1 < A(k) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} - 2\varepsilon_1 \tag{3.4.25}$$

for all sufficiently large k, say, $k > k_1$. Therefore,

$$|a + A(k)| + \varepsilon_1 < a^{1/\beta} - \varepsilon_1. \tag{3.4.26}$$

This clearly implies that there exists $\varepsilon > 0$ such that

$$|a + A(k)|(1+\varepsilon)^{1/\beta} < a^{1/\beta}(1-\varepsilon)^{1/\beta}, \tag{3.4.27}$$

and hence

$$|a + A(k)|^{\beta}(1+\varepsilon) < a(1-\varepsilon) \quad \text{for } k \ge k_1. \tag{3.4.28}$$

Now, for a given $\varepsilon > 0$, there exists $k_2 \in \mathbb{N}$ such that $c(k) > |w(k)|$ for $k \ge k_2$ and

$$\left[\frac{1}{\eta(k)}\sum^{k-1} c^{1-\beta}(j)\right]^{\alpha} \ge \left[\frac{1}{\sum^{k} c^{1-\beta}(j)}\sum^{k-1} c^{1-\beta}(j)\right]^{\alpha} > 1 - \varepsilon, \tag{3.4.29}$$

since (3.4.14) implies

$$\lim_{k\to\infty}\frac{\sum^{k} c^{1-\beta}(j)}{\sum^{k-1} c^{1-\beta}(j)} = \lim_{k\to\infty}\frac{c^{1-\beta}(k)+\sum^{k-1} c^{1-\beta}(j)}{\sum^{k-1} c^{1-\beta}(j)} = 1. \tag{3.4.30}$$

Next,

$$\begin{aligned}\frac{|\xi(k)|^{\alpha-2}c^{1-\beta}(k)}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}} &\le \frac{c^{\beta-1}(k)\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-2}}{c(k)\left[1+\Psi^{-1}(w(k)/c(k))\right]^{\alpha-1}}\\ &= \frac{\left[1+\Psi^{-1}(|w(k)|/c(k))\right]^{\alpha-2}}{\left[1+\Psi^{-1}(w(k)/c(k))\right]^{\alpha-1}}.\end{aligned} \tag{3.4.31}$$

Using the fact that $w(k)/c(k)\to 0$ as $k\to\infty$, we obtain

$$\frac{|\xi(k)|^{\alpha-2}c^{1-\beta}(k)}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}} < 1+\varepsilon. \tag{3.4.32}$$

Multiplying (3.4.28) by

$$(\alpha-1)c^{1-\beta}(k)\left(\sum^{k-1} c^{1-\beta}(j)\right)^{-\alpha}, \tag{3.4.33}$$

and using the above estimates, we obtain

$$\begin{aligned} 0 &> -(\alpha-1)ac^{1-\beta}(k)\left(\sum^{k-1} c^{1-\beta}(j)\right)^{-\alpha}[1-\varepsilon]\\ &\quad + (\alpha-1)|a+A(k)|^{\beta}c^{1-\beta}(k)\left(\sum^{k-1} c^{1-\beta}(j)\right)^{-\alpha}[1+\varepsilon]\\ &> (1-\alpha)ac^{1-\beta}(k)\eta^{-\alpha}(k)-q(k)+q(k)\\ &\quad + \frac{(\alpha-1)|a+A(k)|^{\beta}\left(\sum^{k-1} c^{1-\beta}(j)\right)^{-\alpha}|\xi(k)|^{\alpha-2}}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}}\\ &= \Delta w(k)+q(k)+\frac{(\alpha-1)|\xi(k)|^{\alpha-2}|w(k)|^{\beta}}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}}\\ &= \Delta w(k)+q(k)+w(k)\left[1-\frac{c(k)}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}}\right]\end{aligned} \tag{3.4.34}$$

for $k\ge\max\{k_1,k_2\}$. This completes the proof. □

Remark 3.4.4. (i) In equation (3.2.9) if $c(k) \equiv 1$, then conditions (3.4.12) and (3.4.14) are disregarded.

(ii) If condition (3.4.12) holds, then by the discrete L'Hôpital rule, condition (3.4.14) can be replaced by a stronger condition, namely,

$$\lim_{k\to\infty} \frac{c(k+1)}{c(k)} = 1 \quad \text{if this limit exists.} \tag{3.4.35}$$

Indeed, we have

$$\lim_{k\to\infty} \frac{c^{1-\beta}(k)}{\sum^{k-1} c^{1-\beta}(j)} = \lim_{k\to\infty} \frac{c^{1-\beta}(k+1) - c^{1-\beta}(k)}{c^{1-\beta}(k)} = 0. \tag{3.4.36}$$

However, there exist simple examples of the sequence $c(k)$, for example,

$$c(k) = \{1, 1, 2^{1-\alpha}, 2^{2(1-\alpha)}, 3^{1-\alpha}, 3^{2(1-\alpha)}, \dots, k^{1-\alpha}, k^{2(1-\alpha)}, \dots\}, \tag{3.4.37}$$

for which

$$\lim_{k\to\infty} \frac{c(k+1)}{c(k)} = \lim_{k\to\infty} k^{1-\alpha} \quad \text{does not exist,} \tag{3.4.38}$$

even though condition (3.4.14) holds.

The following nonoscillation criterion for equation (3.2.9) deals with the case when condition (3.4.12) is not satisfied, that is, when condition (3.4.12) is replaced by

$$c(k) > 0, \qquad \sum_{j=m}^{\infty} c^{1-\beta}(j) < \infty. \tag{3.4.39}$$

We present the following complementary case.

Theorem 3.4.5. *Suppose that condition (3.4.39) holds and*

$$\lim_{k\to\infty} \frac{c^{1-\beta}(k)}{\sum_{j=k}^{\infty} c^{1-\beta}(j)} = 0. \tag{3.4.40}$$

If

$$\limsup_{k\to\infty} \left(\sum_{j=k}^{\infty} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum^{k-1} q(j)\right) < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.4.41}$$

$$\liminf_{k\to\infty} \left(\sum_{j=k}^{\infty} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum^{k-1} q(j)\right) > -\left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.4.42}$$

then equation (3.2.9) is nonoscillatory.

Proof. The proof can be modelled according to that of Theorem 3.4.3. In fact, one can show that

$$w(k) = a\left(\sum_{j=k}^{\infty} c^{1-\beta}(j)\right)^{1-\alpha} + \sum_{j=k}^{\infty} q(j), \tag{3.4.43}$$

where $a = [(\alpha-1)/\alpha]^{\alpha}$ satisfies the inequality

$$\left| a + \left(\sum_{j=k}^{\infty} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum^{k-1} q(j)\right)\right|^{\beta} < a. \tag{3.4.44}$$

This fact implies that the generalized Riccati difference inequality (3.4.5) has a solution such that $c(k)+w(k)>0$ holds in a neighborhood of infinity. □

Remark 3.4.6. As in Remark 3.4.4(ii), condition (3.4.40) can be replaced by condition (3.4.35).

To obtain the subsequent results we introduce the following notation. Set

$$B(k) = \left(\sum^{k-1} c^{1-\beta}(j)\right)^{-1} \sum^{k-1}\left[\left(\sum^{j-1} c^{1-\beta}(i)\right)^{\alpha} q(j)\right]. \tag{3.4.45}$$

Let

$$\omega(\alpha) = \theta_{\min} + \left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.4.46}$$

where $\theta_{\min}$ is the least root of the equation

$$(\alpha-1)|v|^{\beta} + \alpha v + \left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} = 0. \tag{3.4.47}$$

Theorem 3.4.7. *Suppose that conditions (3.4.12), (3.4.13), and (3.4.14) hold. If*

$$\limsup_{k\to\infty} B(k) < \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.4.48}$$

$$\liminf_{k\to\infty} B(k) > \omega(\alpha), \tag{3.4.49}$$

then (3.2.9) is nonoscillatory.

Proof. We will show that the generalized Riccati difference equation (3.4.5) has a solution w with $c(k)+w(k)>0$ in a neighborhood of infinity. Set

$$w(k)=[C-B(k)]\left(\sum^{k-1}c^{1-\beta}(j)\right)^{1-\alpha}, \tag{3.4.50}$$

where C is a suitable constant which will be specified later.

As in the proof of Theorem 3.4.3, we obtain (3.4.18) and

$$w(k)[\Psi(\Psi^{-1}(c(k))+\Psi^{-1}(w(k)))-c(k)]=(\alpha-1)|\xi(k)|^{\alpha-2}|w(k)|^{\beta}. \tag{3.4.51}$$

Also, we see that

$$\Delta\left(\sum^{k-1}c^{1-\beta}(j)\right)=\alpha c^{1-\beta}(k)\mu^{\alpha-1}(k), \tag{3.4.52}$$

where $\mu(k)$ is between $\sum^{k-1}c^{1-\beta}(j)$ and $\sum^{k}c^{1-\beta}(j)$. Let

$$C=\left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}. \tag{3.4.53}$$

Then

$$\begin{aligned}\frac{|w(k)|}{c(k)}&=\frac{1}{c(k)}\left|\left(\sum^{k-1}c^{1-\beta}(j)\right)^{1-\alpha}[C-B(k)]\right|\\&=\left[\frac{c^{1-\beta}(k)}{\sum^{k-1}c^{1-\beta}(j)}\right]^{\alpha-1}|C-B(k)|\longrightarrow 0\quad\text{as } k\longrightarrow\infty\end{aligned} \tag{3.4.54}$$

according to (3.4.14), and hence

$$c(k)+w(k)=c(k)\left[1+\frac{w(k)}{c(k)}\right]>0\quad\text{for all large } k\in\mathbb{N}. \tag{3.4.55}$$

Now conditions (3.4.48) and (3.4.49) imply the existence of $\varepsilon_1>0$ such that

$$w(\alpha)+\varepsilon_1<B(k)<\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}-\varepsilon_1 \tag{3.4.56}$$

for all k sufficiently large, say, $k\geq k_1$. Thus,

$$\omega(\alpha)-C+\varepsilon_1=\theta_{\min}<B(k)-C<\theta_{\max}-\varepsilon_1, \tag{3.4.57}$$

where $\theta_{\max} = -[(\alpha-1)/\alpha]^{\alpha-1}$ is the greatest root of equation (3.4.47). Therefore, from (3.4.57) it follows that there exists $\varepsilon_2 > 0$ such that

$$(\alpha-1)\,|\,B(k)-C\,|^{\beta}+\alpha\big[B(k)-C\big]+C+\varepsilon_2<0 \quad \text{for } k\geq k_1, \tag{3.4.58}$$

and finally, this implies the existence of $\varepsilon > 0$ such that

$$\begin{aligned}(\alpha-1)\,|\,C-B(k)\,|^{\beta}(1+\varepsilon)-(\alpha-1)C(1-\varepsilon)&\\ +\alpha B(k)\big[1+\varepsilon\operatorname{sgn}B(k)\big]<0 \quad \text{for } k\geq k_1.&\end{aligned} \tag{3.4.59}$$

Multiplying (3.4.59) by $c^{1-\beta}(k)(\sum^{k-1}c^{1-\beta}(j))^{-\alpha}$, we obtain

$$\begin{aligned}0>(\alpha-1)c^{1-\beta}(k)\bigg(\sum^{k-1}c^{1-\beta}(j)\bigg)^{-\alpha}\,|\,C-B(k)\,|^{\beta}(1+\varepsilon)&\\ -(\alpha-1)c^{1-\beta}(k)C\bigg(\sum^{k-1}c^{1-\beta}(j)\bigg)^{-\alpha}(1-\varepsilon)&\\ +\alpha c^{1-\beta}(k)B(k)\bigg(\sum^{k-1}c^{1-\beta}(j)\bigg)^{-\alpha}\big[1+\varepsilon\operatorname{sgn}B(k)\big]&\end{aligned} \tag{3.4.60}$$

for $k \geq k_1$. We can choose ε such that we may add $q(k)-q(k)[1-\varepsilon\operatorname{sgn}q(k)]$ to the right-hand side of (3.4.60).

Now as in the proof of Theorem 3.4.3 for a given $\varepsilon > 0$ there exists $k_2 \in \mathbb{N}$ such that we have $c(k) > |w(k)|$,

$$\begin{aligned}&\bigg[\frac{1}{\eta(k)}\sum^{k-1}c^{1-\beta}(j)\bigg]^{\alpha}>1-\varepsilon,\\ &(\operatorname{sgn}q(k))\bigg[\frac{\sum^{k-1}c^{1-\beta}(j)}{\sum^{k}c^{1-\beta}(j)}\bigg]^{\alpha}>(\operatorname{sgn}q(k))\big[1-\varepsilon\operatorname{sgn}q(k)\big],\\ &(\operatorname{sgn}B(k))\big[1+\varepsilon\operatorname{sgn}B(k)\big]>(\operatorname{sgn}B(k))\bigg[\frac{\mu(k)}{\sum^{k}c^{1-\beta}(j)}\bigg]^{\alpha}\\ &\qquad=\frac{(\operatorname{sgn}B(k))\big[\sum^{k-1}c^{1-\beta}(j)\big]^{\alpha}\mu^{\alpha-1}(k)}{\big[\sum^{k}c^{1-\beta}(j)\big]^{\alpha}\big[\sum^{k-1}c^{1-\beta}(j)\big]^{\alpha-1}},\end{aligned} \tag{3.4.61}$$

and for $k \geq k_2$,

$$\frac{|\,\xi(k)\,|^{\alpha-2}c^{1-\beta}(k)}{\big[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\big]^{\alpha-1}}<\frac{\big[1+\Psi^{-1}(\,|\,w(k)\,|/c(k))\big]^{\alpha-2}}{\big[1+\Psi^{-1}(w(k)/c(k))\big]^{\alpha-1}}<1+\varepsilon. \tag{3.4.62}$$

Using the above estimates, we obtain from the preceding inequality

$$
\begin{aligned}
0 &> (\alpha-1)c^{1-\beta}(k)\left(\sum^{k-1} c^{1-\beta}(j)\right)^{-\alpha}\left|C-B(k)\right|^{\beta}(1+\varepsilon) \\
&\quad-(\alpha-1)c^{1-\beta}(k)C\left[\sum^{k-1} c^{1-\beta}(j)\right]^{-\alpha}\left[1+\varepsilon\operatorname{sgn} B(k)\right] \\
&\quad+q(k)-q(k)\left[1-\varepsilon\operatorname{sgn} q(k)\right] \\
&> -(\alpha-1)c^{1-\beta}(k)C\eta^{-\alpha}(k)-q(k)\frac{\left[\sum^{k-1} c^{1-\beta}(j)\right]^{2\alpha}}{\left[\sum^{k-1} c^{1-\beta}(j)\right]^{\alpha}\left[\sum^{k} c^{1-\beta}(j)\right]^{\alpha}} \\
&\quad+\frac{\alpha c^{1-\beta}(k)\mu^{\alpha-1}(k)B(k)}{\left[\sum^{k-1} c^{1-\beta}(j)\right]^{\alpha-1}\left[\sum^{k} c^{1-\beta}(j)\right]^{\alpha}}+q(k) \\
&\quad+\frac{(\alpha-1)\left[\sum^{k-1} c^{1-\beta}(j)\right]^{-\alpha}\left|C-B(k)\right|^{\beta}\left|\xi(k)\right|^{\alpha-2}}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}} \\
&= \Delta w(k)+q(k)+\frac{(\alpha-1)\left|\xi(k)\right|^{\alpha-2}\left|w(k)\right|^{\beta}}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}} \\
&= \Delta w(k)+q(k)+w(k)\left[1-\frac{c(k)}{\left[\Psi^{-1}(c(k))+\Psi^{-1}(w(k))\right]^{\alpha-1}}\right],
\end{aligned}
\tag{3.4.63}
$$

which means that inequality (3.4.5) has a solution w with $c(k)+w(k)>0$ in a neighborhood of infinity and hence equation (3.2.9) is nonoscillatory by Lemma 3.4.2. □

The following theorem is concerned with the nonoscillation of equation (3.2.9) when condition (3.4.12) is not satisfied.

Theorem 3.4.8. *Suppose that conditions (3.4.35) and (3.4.39) hold. If*

$$
\limsup_{k\to\infty}\left[\sum_{j=k}^{\infty} c^{1-\beta}(j)\right]^{-1}\sum_{j=k}^{\infty}\left[\left(\sum_{i=j}^{\infty} c^{1-\beta}(i)\right)^{\alpha} q(j)\right]<\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.4.64}
$$

$$
\liminf_{k\to\infty}\left[\sum_{j=k}^{\infty} c^{1-\beta}(j)\right]^{-1}\sum_{j=k}^{\infty}\left[\left(\sum_{i=j}^{k} c^{1-\beta}(i)\right)^{\alpha} q(j)\right]>\omega(\alpha), \tag{3.4.65}
$$

where $\omega(\alpha)$ is defined by (3.4.46), then equation (3.2.9) is nonoscillatory.

Proof. The proof can be modelled according to that of Theorem 3.4.7. In fact, one can show that under the conditions of the theorem the sequence

$$w(k) = -\left[\sum_{j=k}^{\infty} c^{1-\beta}(j)\right]^{1-\alpha}\left\{C - \left[\sum_{j=k}^{\infty} c^{1-\beta}(j)\right]^{-1}\sum_{j=k}^{\infty}\left[\left(\sum_{i=j}^{k} c^{1-\beta}(i)\right)^{\alpha} q(j)\right]\right\}, \tag{3.4.66}$$

where

$$C = \left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.4.67}$$

solves the inequality (3.4.5) with $c(k) + w(k) > 0$. □

In the case when condition (3.4.16) of Theorem 3.4.3 does not hold, we can employ the following criterion.

Let $\lambda \le 0$. Denote by $\rho_{\max}(\lambda)$ the greatest root of the equation

$$|y|^{1/\beta} + y + \lambda = 0. \tag{3.4.68}$$

Theorem 3.4.9. *Let conditions (3.4.12)–(3.4.14) hold. If*

$$\begin{gathered}\limsup_{k\to\infty} A(k) < \left[\rho_{\max}\left(\liminf_{k\to\infty} A(k)\right)\right]^{1/\beta} - \rho_{\max}\left(\liminf_{k\to\infty} A(k)\right),\\ -\infty < \liminf_{k\to\infty} A(k) \le -\left(\frac{2\alpha-1}{\alpha}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},\end{gathered} \tag{3.4.69}$$

where A is defined by (3.4.22), then equation (3.2.9) is nonoscillatory.

When condition (3.4.42) of Theorem 3.4.5 fails to hold, it is clear that in the same way as above one can show that there exists a similar complementary case to Theorem 3.4.5.

In the case when condition (3.4.49) of Theorem 3.4.7 fails to hold, we can use the following criterion.

Let $\lambda_1 > -1$ and $0 < \lambda_2 < [(\alpha-1)/\alpha]^{\alpha}$. Denote by $\sigma_{\max}(\lambda_1)$ the greatest root of the equation

$$(\alpha-1)|z|^{\beta} + \alpha z + \lambda_1 = 0, \tag{3.4.70}$$

and by $\delta(\lambda_2)$ the greatest root of the equation

$$(\alpha-1)|y|^{\beta} - (\alpha-1)y + \lambda_2 = 0. \tag{3.4.71}$$

Theorem 3.4.10. *Let conditions (3.4.12)–(3.4.14) hold. If*

$$
\begin{aligned}
\limsup_{k\to\infty} B(k) < \liminf_{k\to\infty} B(k) + \delta\Big(\liminf_{k\to\infty} B(k)\Big) \\
+ \sigma_{\max}\Big[\liminf_{k\to\infty} B(k) + \delta\Big(\liminf_{k\to\infty} B(k)\Big)\Big], \\
-\infty < \liminf_{k\to\infty} B(k) \le \omega(\alpha),
\end{aligned}
\tag{3.4.72}
$$

where $\omega(\alpha)$ is defined by (3.4.46), then equation (3.2.9) is nonoscillatory.

When condition (3.4.65) of Theorem 3.4.8 does not hold, then by the same way as above, one can show that there exists a similar complementary statement for Theorem 3.4.8.

3.4.2. Nonoscillation theorems based on the variational principle

The variational principle method is based on the equivalence (I)⇔(IV) from Theorem 3.3.4. More precisely, we will employ the following lemma.

Lemma 3.4.11. *Equation (3.2.9) is nonoscillatory if and only if there exists $m \in \mathbb{N}$ such that*

$$
\mathcal{F}(\xi; m, \infty) = \sum_{k=m}^{\infty} \Big[c(k)\,|\Delta\xi(k)|^{\alpha} - q(k)\,|\xi(k+1)|^{\alpha}\Big] > 0 \tag{3.4.73}
$$

for every nontrivial $\xi \in U(m)$, where

$$
U(m) = \Big\{\xi = \{\xi(k)\}_{k=1}^{\infty} : \xi(k) = 0,\ k \le m, \exists n > m : \xi(k) = 0,\ k \ge n\Big\}. \tag{3.4.74}
$$

Next, we will obtain nonoscillation criteria based on the variational method. For this, the crucial rôle is played by the half-linear discrete version of the Wirtinger-type inequality. In the proof of this inequality we need the following technical lemma.

Lemma 3.4.12. *Let*

$$
\mu = \begin{cases} \sup\limits_{t>s>0} \dfrac{1}{(t-s)}\left[\Psi^{-1}\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha(t-s)}\right) - s\right] & \text{if } \alpha \ge 2, \\[2ex] \sup\limits_{t>s>0} \dfrac{1}{(t-s)}\left[t - \Psi^{-1}\left(\dfrac{t^{\alpha}-s^{\alpha}}{\alpha(t-s)}\right)\right] & \text{if } \alpha \le 2. \end{cases} \tag{3.4.75}
$$

Then for given $B > A > 0$ and for $\xi = \lambda B + (1-\lambda)B$ given by the Lagrange mean value theorem, $B^{\alpha} - A^{\alpha} = \alpha\Psi(\xi)[B-A]$, $\max\{\lambda, 1-\lambda\} \le \mu$.

Proof. If $\alpha \geq 2$, then $\lambda \geq 1/2$, that is, $\max\{\lambda, 1-\lambda\} = \lambda$, and for $\alpha \leq 2$ we have $\lambda \leq 1/2$. The conclusion can be easily verified by a direct computation via the Lagrange mean value theorem applied to the function $f(t) = t^\alpha$, $t \geq 0$. □

Lemma 3.4.13 (Wirtinger's inequality). *Let $M(k)$ be a positive sequence such that $\Delta M(k)$ is of one sign for $k \geq N \in \mathbb{N}$. Then for every $x \in U(N)$,*

$$\sum_{k=N}^{\infty} |\Delta M(k)|\,|x(k+1)|^\alpha \leq \alpha^\alpha\big[\mu(1+\phi(N))\big]^{\alpha-1}\sum_{k=N}^{\infty}\frac{M^\alpha(k)}{|\Delta M(k)|^{\alpha-1}}|\Delta x(k)|^\alpha, \tag{3.4.76}$$

where

$$\phi(N) = \sup_{k\geq N}\frac{|\Delta M(k)|}{|\Delta M(k-1)|}, \tag{3.4.77}$$

μ is given in (3.4.75), and $U(N)$ is defined in (3.4.74).

Proof. Suppose that $\Delta M(k) > 0$ for $k \geq N$ (in case $\Delta M(k) < 0$ we would proceed in the same way). Using summation by parts, the Hölder inequality, the Lagrange mean value theorem, and the Jensen inequality for the convex function $f(\xi) = |\xi|^\alpha$, we obtain

$$\begin{aligned}
&\sum_{k=N}^{\infty}|\Delta M(k)|\,|x(k+1)|^\alpha \\
&\quad= \sum_{k=N}^{\infty} M(k)\Delta\Big(|x(k)|^\alpha\Big) \\
&\quad= \alpha\sum_{k=N}^{\infty} M(k)\,|\Psi(\xi(k))|\,|\Delta x(k)| \\
&\quad= \alpha\left[\sum_{k=N}^{\infty}\frac{M^\alpha(k)}{|\Delta M(k)|^{\alpha-1}}|\Delta x(k)|^\alpha\right]^{1/\alpha}\left[\sum_{k=N}^{\infty}|\Delta M(k)|\,|\Psi(\xi(k))|^\beta\right]^{1/\beta} \\
&\quad= \alpha\left[\sum_{k=N}^{\infty}\frac{M^\alpha(k)}{|\Delta M(k)|^{\alpha-1}}|\Delta x(k)|^\alpha\right]^{1/\alpha} \\
&\qquad\times\left[\sum_{k=N}^{\infty}|\Delta M(k)|\Big(\lambda_k|x(k)|^\alpha + (1-\lambda_k)|x(k+1)|^\alpha\Big)\right]^{1/\beta} \\
&\quad\leq \alpha\left[\sum_{k=N}^{\infty}\frac{M^\alpha(k)}{|\Delta M(k)|^{\alpha-1}}|\Delta x(k)|^\alpha\right]^{1/\alpha}\big(\max\{\lambda_k, 1-\lambda_k\}\big)^{1/\beta} \\
&\qquad\times\left[\sum_{k=N}^{\infty}\frac{|\Delta M(k)|}{|\Delta M(k-1)|}\Delta|M(k-1)|\,|x(k)|^\alpha + \sum_{k=N}^{\infty}|\Delta M(k)|\,|x(k+1)|^\alpha\right]^{1/\beta},
\end{aligned} \tag{3.4.78}$$

where $\xi(k) = \lambda_k x(k) + (1-\lambda_k)x(k+1)$ is a number between $x(k)$ and $x(k+1)$, that is, $\lambda_k \in [0,1]$. Now, by Lemma 3.4.12, $\max\{\lambda_k, 1-\lambda_k\} \leq \mu$, and since $x(k) = 0$ for $k \leq N$, we have

$$\begin{aligned}\sum_{k=N}^{\infty} \frac{|\Delta M(k)|}{|\Delta M(k-1)|} |\Delta M(k-1)| \, |x(k)|^{\alpha} \\ \leq \left(\sup_{k\geq N} \frac{|\Delta M(k)|}{|\Delta M(k-1)|}\right) \sum_{k=N}^{\infty} |\Delta M(k)| \, |x(k+1)|^{\alpha}.\end{aligned} \tag{3.4.79}$$

Consequently, we have

$$\begin{aligned}&\sum_{k=N}^{\infty} |\Delta M(k)| \, |x(k+1)|^{\alpha} \\ &\quad\leq \alpha\left[\mu\left(1+\sup_{k\geq N} \frac{|\Delta M(k)|}{|\Delta M(k-1)|}\right) \sum_{k=N}^{\infty} |\Delta M(k)| \, |x(k+1)|^{\alpha}\right]^{1/\beta} \\ &\qquad\times\left[\sum_{k=N}^{\infty} \frac{M^{\alpha}(k)|\Delta x(k)|^{\alpha}}{|\Delta M(k)|^{\alpha-1}}\right]^{1/\alpha},\end{aligned} \tag{3.4.80}$$

and hence

$$\sum_{k=N}^{\infty} |\Delta M(k)| \, |x(k+1)|^{\alpha} \leq \alpha^{\alpha}[\mu(1+\phi(N))]^{\alpha-1} \sum_{k=N}^{\infty} \frac{M^{\alpha}(k)}{|\Delta M(k)|^{\alpha-1}} |\Delta x(k)|^{\alpha}, \tag{3.4.81}$$

as required. This completes the proof. □

Theorem 3.4.14. *Suppose that condition (3.4.12) holds and*

$$\sum^{\infty} q^{+}(k) < \infty, \tag{3.4.82}$$

where $q^{+}(k) = \max\{0, q(k)\}$. *Assume*

$$\begin{aligned}\phi_1(N) &:= \left[\sup_{k\geq N} \frac{\sum^{k} c^{1-\beta}(j)}{\sum^{k-1} c^{1-\beta}(j)}\right]^{\alpha(\alpha-1)} < \infty, \\ \phi_2(N) &:= \left[\sup_{k\geq N} \frac{c(k)}{c(k-1)}\right]^{1-\beta} < \infty.\end{aligned} \tag{3.4.83}$$

Moreover, suppose that

$$0 < \limsup_{N\to\infty} [1+\phi_2(N)]^{\alpha-1}\phi_1(N) = \Phi < \infty. \tag{3.4.84}$$

If

$$\limsup_{k\to\infty}\left(\sum^{k-1} c^{1-\beta}(j)\right)^{\alpha-1}\sum_{j=k}^{\infty} q^{+}(j) < \left(\frac{1}{\alpha\mu^{\alpha-1}}\right)\left(\frac{1}{\Phi}\right)\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.4.85}$$

then equation (3.2.9) is nonoscillatory.

Proof. According to Lemma 3.4.11, it suffices to find $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty}\left[c(k)\,|\,\Delta x(k)\,|^{\alpha} - q(k)\,|\,x(k+1)\,|^{\alpha}\right] > 0 \tag{3.4.86}$$

for any nontrivial $x \in U(N)$. For this, let $M(k) = [\sum^{k-1} c^{1-\beta}(j)]^{1-\alpha}$. Then, using the Lagrange mean value theorem, we obtain

$$|\,\Delta M(k)\,| = (\alpha-1)\xi^{-\alpha}(k)\Delta\left(\sum^{k-1} c^{1-\beta}(j)\right) = (\alpha-1)\xi^{-\alpha}(k)c^{1-\beta}(k), \tag{3.4.87}$$

where

$$\sum^{k-1} c^{1-\beta}(k) < \xi(k) < \sum^{k} c^{1-\beta}(k). \tag{3.4.88}$$

Hence,

$$(\alpha-1)\left[\sum^{k} c^{1-\beta}(k)\right]^{-\alpha} c^{1-\beta}(k) \le |\,\Delta M(k)\,| \le (\alpha-1)\left[\sum^{k-1} c^{1-\beta}(k)\right]^{-\alpha} c^{1-\beta}(k). \tag{3.4.89}$$

Thus,

$$\begin{gathered}\frac{|\,\Delta M(k)\,|}{|\,\Delta M(k-1)\,|} \le \left[\frac{c(k)}{c(k-1)}\right]^{1-\beta},\\ \frac{M^{\alpha}(k)}{|\,\Delta M(k)\,|^{\alpha-1}} \le c(k)\left(\frac{1}{\alpha-1}\right)^{\alpha-1}\left[\frac{\sum^{k} c^{1-\beta}(j)}{\sum^{k-1} c^{1-\beta}(j)}\right]^{\alpha(\alpha-1)}.\end{gathered} \tag{3.4.90}$$

According to (3.4.85), there exists $\varepsilon > 0$ such that

$$\limsup_{k\to\infty}\left[\sum^{k-1} c^{1-\beta}(j)\right]^{\alpha-1}\sum_{j=k}^{\infty} q^{+}(j) < \frac{1}{\alpha\mu^{\alpha-1}}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\left(\frac{1}{\Phi+\varepsilon}\right), \tag{3.4.91}$$

and (3.4.84) implies the existence of $n_0 \in \mathbb{N}$ such that

$$\left[1+\phi_2(N)\right]^{\alpha-1}\phi_1(N) < \Phi+\varepsilon \quad \text{for } N \geq n_0. \tag{3.4.92}$$

Now, using summation by parts and applying the same idea as in the proof of Lemma 3.4.13, we have for any nontrivial $x \in U(N)$,

$$\begin{aligned}
&\sum_{k=N}^{\infty} q(k)\left|x(k+1)\right|^{\alpha} \\
&\quad\leq \sum_{k=N}^{\infty} q^{+}(k)\left|x(k+1)\right|^{\alpha} \\
&\quad= \sum_{k=N}^{\infty}\left(\sum_{j=N}^{\infty} q^{+}(j)\right)\Delta\left(\left|x(k)\right|^{\alpha}\right) \\
&\quad= \sum_{k=N}^{\infty}\frac{1}{M(k)}\left(\sum_{j=N}^{\infty} q^{+}(j)\right)M(k)\Delta\left(\left|x(k)\right|^{\alpha}\right) \\
&\quad< \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\frac{1}{\mu^{\alpha-1}[\Phi+\varepsilon]}\sum_{k=N}^{\infty} M(k)\left|\Psi(\xi(k))\right|\left|\Delta x(k)\right| \\
&\quad\leq \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\frac{1}{\mu^{\alpha-1}[\Phi+\varepsilon]}\left[\sum_{k=N}^{\infty}\frac{M^{\alpha}(k)}{\left|\Delta M(k)\right|^{\alpha-1}}\left|\Delta x(k)\right|^{\alpha}\right]^{1/\alpha} \\
&\qquad\times\left[\mu\left[1+\Phi_2(N)\right]\sum_{k=N}^{\infty}\left|\Delta M(k)\right|\left|x(k+1)\right|^{\alpha}\right]^{1/\beta} \\
&\quad\leq \left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\frac{1}{\mu^{\alpha-1}[\Phi+\varepsilon]}\left[\mu\left(1+\phi_2(N)\right)\right]^{1/\beta}\alpha^{\alpha/\beta}\left[\mu\left(1+\phi_2(N)\right)\right]^{(\alpha-1)/\beta} \\
&\qquad\times\left[\sum_{k=N}^{\infty}\frac{M(k)}{\left|\Delta M(k)\right|^{\alpha-1}}\left|\Delta x(k)\right|^{\alpha}\right]^{(1/\alpha)+(1/\beta)} \\
&\quad\leq \left[1+\phi_2(N)\right]^{\alpha-1}\phi_1(N)\frac{1}{[\Phi+\varepsilon]}\sum_{k=N}^{\infty} c(k)\left|\Delta x(k)\right|^{\alpha} \\
&\quad\leq \sum_{k=N}^{\infty} c(k)\left|\Delta x(k)\right|^{\alpha}.
\end{aligned} \tag{3.4.93}$$

Hence

$$\sum_{k=N}^{\infty}\left[c(k)\left|\Delta x(k)\right|^{\alpha} - q(k)\left|x(k+1)\right|^{\alpha}\right] > 0 \tag{3.4.94}$$

for every nontrivial $x \in U(N)$, as claimed. This completes the proof. □

The next result treats the case when $\sum^{\infty} c^{1-\beta}(k) < \infty$.

Theorem 3.4.15. *Suppose that condition (3.4.39) holds, and assume*

$$\phi_3(N) := \sup_{k \geq N} \left[\frac{\sum_{k+1}^{\infty} c^{1-\beta}(j)}{\sum_{k-1}^{\infty} c^{1-\beta}(j)} \right]^{\alpha} \left(\frac{c(k)}{c(k-1)} \right)^{1-\beta} < \infty, \qquad 0 < \limsup_{N \to \infty} \left[1 + \phi_3(N)\right]^{\alpha-1} = \Phi_1 < \infty. \tag{3.4.95}$$

If

$$\limsup_{k \to \infty} \left(\sum_{k}^{\infty} c^{1-\beta}(j) \right)^{\alpha-1} \left(\sum^{k-1} q^{+}(j) \right) < \frac{1}{\alpha \mu^{\alpha-1}} \left(\frac{1}{\Phi_1} \right) \left(\frac{\alpha-1}{\alpha} \right)^{\alpha-1}, \tag{3.4.96}$$

then equation (3.2.9) is nonoscillatory.

Proof. Let $M(k) = [\sum_{j=k}^{\infty} c^{1-\beta}(j)]^{-\alpha}$. Then, as in the proof of Theorem 3.4.14, one can easily obtain

$$\frac{|\Delta M(k)|}{|\Delta M(k-1)|} \leq \phi_3(N) = \left[\frac{\sum_{k-1}^{\infty} c^{1-\beta}(j)}{\sum_{k+1}^{\infty} c^{1-\beta}(j)} \right]^{\alpha} \left(\frac{c(k)}{c(k+1)} \right)^{1-\beta}, \qquad \frac{M^{\alpha}(k)}{|\Delta M(k)|^{\alpha-1}} \leq \left(\frac{1}{\alpha-1} \right)^{\alpha-1} c(k). \tag{3.4.97}$$

The rest of the proof is similar to the proof of Theorem 3.4.9 and hence we omit it here. □

3.4.3. Conjugacy criteria

In this subsection we give several sufficient conditions for equation (3.2.9) to possess a nontrivial solution having at least two generalized zeros on a given interval.

Theorem 3.4.16. *Assume that*

$$\text{there exists a constant } C > 0 \quad \text{such that} \quad c(k) \leq C \quad \forall k \in \mathbb{Z}. \tag{3.4.98}$$

Suppose that $q(k) \not\equiv 0$ and $\sum_{j=-\infty}^{\infty} q(j) = \lim_{k \to \infty} \sum_{j=-k}^{k} q(j)$ exists as a finite number. If

$$\sum_{j=-\infty}^{\infty} q(j) \geq 0, \tag{3.4.99}$$

then equation (3.2.9) is conjugate on $\mathbb{Z}$.

Proof. Suppose to the contrary that equation (3.2.9) is not conjugate on $\mathbb{Z}$. Then there exists a solution $x = \{x(k)\}$ of (3.2.9) such that $c(k)x(k)x(k+1) > 0$ for all $k \in \mathbb{Z}$. Hence, we can assume that there exists a solution w of equation (3.2.17) with $c(k) + w(k) > 0$ for $k \in \mathbb{Z}$, which is related to equation (3.2.9) by the substitution $w(k) = c(k)\Psi(\Delta x(k)/x(k))$. We claim that

$$\sum_{j=m}^{\infty} \Phi(w(j), c(j)) < \infty. \tag{3.4.100}$$

Otherwise,

$$\sum_{j=m}^{\infty} \Phi(w(j), c(j)) = \infty. \tag{3.4.101}$$

From equation (3.2.17), we have

$$w(k+1) = w(m) - \sum_{j=m}^{k} q(j) - \sum_{j=m}^{k} \Phi(w(j), c(j)) \quad \text{for } k \geq m. \tag{3.4.102}$$

From (3.4.101), (3.4.102), and the condition on q we obtain $\lim_{k\to\infty} w(k) = -\infty$. But this contradicts $w(k) > -c(k)$ for $k \geq m$. Thus, we must have (3.4.100). Moreover, a necessary condition for (3.4.100) to hold is $\Phi(w(k), c(k)) \to 0$ as $k \to \infty$, and this implies $w(k) \to 0$ as $k \to \infty$ by Lemma 3.2.6(I_5). Next, let $m, n \in \mathbb{N}_0$ with $m < n$. Summation of equation (3.2.17) from m to $n-1$ provides

$$w(n) = w(m) - \sum_{j=m}^{n-1} q(j) - \sum_{j=m}^{n-1} \Phi(w(j), c(j)), \tag{3.4.103}$$

and, similarly, by summation of equation (3.2.17) from $-n$ to $-m-1$, we get

$$w(-m) = w(-n) - \sum_{j=-n}^{-m-1} q(j) - \sum_{j=-n}^{-m-1} \Phi(w(j), c(j)). \tag{3.4.104}$$

Letting $n \to \infty$ and putting $m = 0$ in the above equalities, we obtain

$$\begin{aligned} -w(0) &= -\sum_{j=0}^{\infty} q(j) - \sum_{j=0}^{\infty} \Phi(w(j), c(j)), \\ w(0) &= -\sum_{j=-\infty}^{-1} q(j) - \sum_{j=-\infty}^{-1} \Phi(w(j), c(j)), \end{aligned} \tag{3.4.105}$$

since $w(k)$ and $w(-k)$ tend to zero as $k \to \infty$. Now, the addition of the last two equalities yields

$$\sum_{j=-\infty}^{\infty} q(j) = -\sum_{j=-\infty}^{\infty} \Phi(w(j), c(j)) < 0, \tag{3.4.106}$$

which contradicts condition (3.4.99). This completes the proof. □

The following result gives a condition which guarantees nonexistence of the solution of equation (3.2.9) without generalized zeros in a given finite discrete interval.

Theorem 3.4.17. *Let m, n be any integers such that $n \geq m+1$. If*

$$\sum_{j=m}^{n-1} q(j) \geq c(m) + c(n), \tag{3.4.107}$$

then equation (3.2.9) possesses no solution without generalized zeros in the interval $[m, n]$.

Proof. Suppose that there exists a solution x of (3.2.9) which has no generalized zeros on the interval $[m, n]$. Then the sequence $w(k) = c(k)\Phi(\Delta x(k))/\Phi(x(k))$ satisfies the equation (3.2.17) with $c(k)+w(k) > 0$ for $k \in [m, n-1]$. We will show that

$$\sum_{j=m}^{n-1} q(j) < c(m) + c(n). \tag{3.4.108}$$

For this, we proceed in the following two steps.

Step 1. Inequality (3.4.108) holds for $n = m+1$. From equation (3.2.17) we get

$$\begin{aligned} q(m) &= w(m) - w(m+1) - w(m) + \frac{w(m)c(m)}{\left[\Psi^{-1}(c(m)) + \Psi^{-1}(w(m))\right]^{\alpha-1}} \\ &< c(m+1) + \frac{w(m)c(m)}{\left[\Psi^{-1}(c(m)) + \Psi^{-1}(w(m))\right]^{\alpha-1}}, \end{aligned} \tag{3.4.109}$$

since $c(m+1) + w(m+1) > 0$. Now we rewrite the right-hand side of (3.4.109) as follows:

$$c(m+1) + c(m) + \frac{w(m)c(m) - c(m)\left[\Psi^{-1}(c(m)) + \Psi^{-1}(w(m))\right]^{\alpha-1}}{\left[\Psi^{-1}(c(m)) + \Psi^{-1}(w(m))\right]^{\alpha-1}}. \tag{3.4.110}$$

The numerator of the last fraction in (3.4.110) is negative if and only if

$$c^2(m)\Psi\left(\frac{\Delta x(m)}{x(m)}\right) < c^2(m)\Psi\left(\frac{x(m+1)}{x(m)}\right), \tag{3.4.111}$$

but this clearly holds. Hence $q(m) < c(m+1) + c(m)$ and (3.4.108) holds for $n = m+1$.

Step 2. Inequality (3.4.108) holds for $n \geq m+2$. Summation of equation (3.2.17) from $m+1$ to $n-1$ yields

$$\begin{aligned}\sum_{j=m+1}^{n-1} q(j) &= w(m+1) - w(n) - \sum_{j=m+1}^{n-1} \Phi(w(j), c(j)) \\ &< w(m+1) - w(m) < w(m+1) + c(n),\end{aligned} \tag{3.4.112}$$

since Lemma 3.2.6(I_4) holds. From equation (3.2.9) we have

$$c(m)\Psi\left(\frac{\Delta x(m)}{x(m+1)}\right) - q(m) = c(m+1)\frac{\Psi(\Delta x(m+1))}{\Psi(x(m+1))}. \tag{3.4.113}$$

Hence,

$$w(m+1) = c(m)\Psi\left(1 - \frac{x(m)}{x(m+1)}\right) - q(m) < c(m) - q(m), \tag{3.4.114}$$

since we have $c(m) > 0$ if and only if $x(m)x(m+1) > 0$. Using this result in (3.4.112), we get (3.4.108).

Therefore (3.4.108) holds in either case and the theorem is proved. □

The following corollary is an immediate consequence of Theorem 3.4.17.

Corollary 3.4.18. *Let $m_1, m_2 \in \mathbb{N}$ be such that $m_2 \geq m_1 + 2$. A sufficient condition for equation (3.2.9) to be conjugate on the interval $[m_1, m_2]$ is that either*

$$\sum_{j=m_1}^{m_2} q(j) \geq c(m_1) + c(m_2+1) \quad \text{or} \quad q(m_i) \geq c(m_i) + c(m_i+1), \quad i \in \{1,2\}. \tag{3.4.115}$$

The last criterion in this subsection gives a sufficient condition for conjugacy of equation (3.2.9) with $c(k) \equiv 1$, that is, of equation (3.2.11) on an interval $[n, \infty)$.

Theorem 3.4.19. *Suppose that $q(k) \geq 0$ for $k \in \mathbb{N}$. A sufficient condition for conjugacy of equation (3.2.11) on an interval $[n, \infty)$, $n \in \mathbb{N}$, is that there exist integers ℓ and m with $n < \ell < m$ such that*

$$\frac{1}{(\ell - n)^{\alpha-1}} < \sum_{j=\ell}^{m} q(j). \tag{3.4.116}$$

Proof. We will show that the solution x of equation (3.2.11) given by initial conditions $x(n) = 0$ and $x(n+1) = 1$ has a generalized zero in (n, ∞). Suppose not, then we can assume $x(k) > 0$ in (n, ∞) and $\Delta x(k) \geq 0$ in $[n, \infty)$, since if $\Delta x(k) < 0$ at some point in (n, ∞), then we would have a generalized zero in (n, ∞) by the condition $q(k) \geq 0$ (which implies $\Delta^2 x(k) \leq 0$). From equation (3.2.11) we obtain

$$\left(\Delta x(m+1)\right)^{\alpha-1} = \left(\Delta x(\ell)\right)^{\alpha-1} - \sum_{j=\ell}^{m} q(j)\left(x(j+1)\right)^{\alpha-1}. \tag{3.4.117}$$

Since $q(k) \geq 0$, using the discrete version of the mean value theorem, we have

$$\frac{x(\ell)}{\ell - n} = \frac{x(\ell) - x(n)}{\ell - n} \geq \Delta x(k) \geq \Delta x(\ell) \tag{3.4.118}$$

with some $k \in [n+1, \ell - 1]$. Thus

$$\left(x(\ell)\right)^{\alpha-1} \geq (\ell - n)^{\alpha-1}\left(\Delta x(\ell)\right)^{\alpha-1}. \tag{3.4.119}$$

Hence,

$$\begin{aligned}
&\left(\Delta x(\ell)\right)^{\alpha-1} - \sum_{j=\ell}^{m} q(j)\left(x(j+1)\right)^{\alpha-1} \\
&\quad \leq \left(\Delta x(\ell)\right)^{\alpha-1} - \sum_{j=\ell}^{m} q(j)\left(x(\ell)\right)^{\alpha-1} \\
&\quad \leq \left(\Delta x(\ell)\right)^{\alpha-1} - (\ell - n)^{\alpha-1}\left(\Delta x(\ell)\right)^{\alpha-1} \sum_{j=\ell}^{m} q(j) \\
&\quad = \left(\Delta x(\ell)\right)^{\alpha-1}\left[1 - (\ell - n)^{\alpha-1} \sum_{j=\ell}^{m} q(j)\right].
\end{aligned} \tag{3.4.120}$$

By condition (3.4.116), the factor in the bracket above is negative.

Next, we consider the following two cases.

(i) If $\Delta x(\ell) > 0$, then $\Delta x(m+1) < 0$, and hence there is a generalized zero in $(m+1, \infty)$.

(ii) If $\Delta x(\ell) = 0$, then $\Delta x(m) < 0$ since $\sum_{j=\ell}^{m} q(j)(x(j+1))^{\alpha-1} > 0$ by the assumption.

In either case, x has a generalized zero in $(m+1, \infty)$, and so equation (3.2.11) is conjugate on $[n, \infty)$. □

3.5. Oscillation criteria

In this section, we will present several oscillation results for equation (3.2.9). Some of these criteria are based on the Riccati technique and some others are derived by employing the variational principle.

3.5.1. Oscillation criteria based on the Riccati technique

First, we present the following result.

Theorem 3.5.1. *Assume that condition (3.4.98) holds and that equation (3.2.9) is nonoscillatory. Then the following statements are equivalent.*

(i) *It holds that*

$$\liminf_{k\to\infty} \sum_{j=1}^{k} q(j) > -\infty. \tag{3.5.1}$$

(ii) *For any solution x with $c(k)x(k)x(k+1) > 0$, $k \geq m$, for some $m \in \mathbb{N}$, the sequence $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k))$, $k \geq m$, satisfies*

$$\sum_{j=m}^{\infty} \Phi(w(j), c(j)) < \infty. \tag{3.5.2}$$

Moreover, this implies $\liminf_{k\to\infty} c(k) \geq 0$.

(iii) $\lim_{k\to\infty} \sum_{j=1}^{k} q(j)$ *exists (as a finite number).*

Proof. First we show (i)⇒(ii). In view of the fact that $\Phi(w(k), c(k)) \geq 0$, see Lemma 3.2.6(I_4), the sequence $\sum_{j=m}^{k} \Phi(w(j), c(j))$ is nondecreasing for $k \geq m$. Therefore, the limit of this sequence exists and is equal to either a finite (positive) number or to infinity. Suppose to the contrary there is a nonoscillatory solution of equation (3.2.9) such that

$$w(k) = \frac{c(k)\Psi(\Delta x(k))}{\Psi(x(k))} > -c(k), \tag{3.5.3}$$

$$\sum_{j=m}^{\infty} \Phi(w(j), c(j)) = \infty. \tag{3.5.4}$$

From equation (3.2.17) we have

$$w(k+1) = w(m) - \sum_{j=m}^{k} q(j) - \sum_{j=m}^{k} \Phi(w(j), c(j)) \quad \text{for } k \geq m. \tag{3.5.5}$$

Now from (3.5.1), (3.5.4), and (3.5.5) we obtain $\lim_{k\to\infty} w(k) = -\infty$. But this contradicts (3.5.3) since (3.4.98) holds. Therefore, we must have (3.5.2). Moreover, a necessary condition for (3.5.2) to hold is

$$\Phi(w(k), c(k)) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{3.5.6}$$

and thus $w(k) \to 0$ as $k \to \infty$ by Lemma 3.2.6(I_5). The condition $c(k) + w(k) > 0$ now implies $\liminf_{k\to\infty} c(k) \geq 0$.

Now we show (ii)⇒(iii). Let w be a sequence as in (ii). According to the above observation, we see that $w(k) \to 0$ as $k \to \infty$. Letting $k \to \infty$ in (3.5.5), we obtain the statement (iii).

The statement (iii)⇒(i) is obvious. □

The following result is a counterpart to Theorem 3.5.1.

Theorem 3.5.2. *Assume that condition (3.4.98) holds and that equation (3.2.9) is nonoscillatory. Then the following statements are equivalent.*

(i) $\liminf_{k\to\infty} \sum_{j=1}^{k} q(j) = -\infty$.

(ii) *There exists a solution x of equation (3.2.9) with $c(k)x(k)x(k+1) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$ such that (3.5.4) holds, where w satisfies (3.5.3).*

(iii) $\lim_{k\to\infty} \sum_{j=1}^{k} q(j) = -\infty$.

Proof. The statement (i)⇒(ii) follows from Theorem 3.5.1. For the statement (ii)⇒(iii), observe the following: from equation (3.2.17), using $c(k) + w(k) > 0$ for $k \geq m$ and condition (3.4.98), we get

$$\begin{aligned} \sum_{j=m}^{k} q(j) &= -w(k+1) + w(m) - \sum_{j=m}^{k} \Phi(w(j), c(j)) \\ &\leq C + w(m) - \sum_{j=m}^{k} \Phi(w(j), c(j)) \longrightarrow -\infty. \end{aligned} \tag{3.5.7}$$

Finally, the statement (iii)⇒(i) is trivial. □

The following theorem gives a necessary condition for nonoscillation of equation (3.2.9) in terms of the existence of a solution of the generalized Riccati difference equation in a "summation" form.

Theorem 3.5.3. *Let conditions (3.4.98) and (3.5.1) hold. If equation (3.2.9) is nonoscillatory, then there exists a sequence $w(k)$ such that $c(k) + w(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$, and*

$$w(k) = \sum_{j=k}^{\infty} q(j) + \sum_{j=m}^{\infty} \Phi(w(j), c(j)). \tag{3.5.8}$$

Proof. In view of the hypotheses of the theorem, Theorem 3.5.1(iii) holds. From equation (3.2.9), we get

$$w(k+1) = w(m) - \sum_{j=m}^{k} q(j) - \sum_{j=m}^{k} \Phi(w(j), c(j)) \quad \text{for } k \geq m, \tag{3.5.9}$$

and letting $k \to \infty$ in this equation, we obtain

$$0 = w(m) - \sum_{j=m}^{\infty} q(j) - \sum_{j=m}^{\infty} \Phi(w(j), c(j)), \tag{3.5.10}$$

by the proof of Theorem 3.5.1. Replacing m by k we obtain (3.5.8). □

It is clear that the necessary condition in Theorem 3.5.3 is also sufficient for nonoscillation of equation (3.2.9). One can easily verify this by applying the difference operator to both sides of (3.5.8) and taking into account that $c(k) + w(k) > 0$ for $k \geq m$. Then the statement follows from Theorem 3.3.4. However, the next theorem shows that such a type of condition guaranteeing nonoscillation can be somewhat weakened.

Theorem 3.5.4. *Let conditions (3.4.98) and (3.5.1) hold. Suppose that $c(k) > 0$ for all large k. If there exists a sequence $z(k)$ such that $c(k) + z(k) > 0$, $k \geq m$, for some $m \in \mathbb{N}$, and a constant M satisfying*

$$z(k) \geq M - \sum_{j=1}^{k-1} q(j) + \sum_{j=k}^{\infty} \Phi(z(j), c(j)) \geq 0 \tag{3.5.11}$$

or

$$z(k) \leq M - \sum_{j=1}^{k-1} q(j) + \sum_{j=k}^{\infty} \Phi(z(j), c(j)) \leq 0, \tag{3.5.12}$$

then equation (3.2.9) is oscillatory.

Proof. Suppose that conditions (3.4.98) and (3.5.1) hold and that there exists a constant M such that (3.5.11) or (3.5.12) holds. Let

$$w(k) = M - \sum_{j=1}^{k-1} q(j) + \sum_{j=k}^{\infty} \Phi(w(j), c(j)). \tag{3.5.13}$$

Then

$$\Delta w(k) = -q(k) - \Phi(w(k), c(k)). \tag{3.5.14}$$

Now $z(k) \geq w(k) \geq 0$ or $z(k) \leq w(k) \leq 0$, so $\Phi(z(k), c(k)) \geq \Phi(w(k), c(k))$ for $k \geq m$, by Lemma 3.2.6(I_2). Obviously, $c(k) + w(k) > 0$ and

$$\Delta w(k) + q(k) + \Phi\big(w(k), c(k)\big) \leq 0 \quad \text{for } k \geq m. \tag{3.5.15}$$

Now equation (3.2.9) is nonoscillatory by Lemma 3.4.2. □

The following corollary is an immediate consequence of Theorems 3.5.1 and 3.5.2.

Corollary 3.5.5. *Let condition (3.4.98) hold. A sufficient condition for equation (3.2.9) to be oscillatory is that either*

$$\lim_{k\to\infty} \sum_{j=1}^{k} q(j) = \infty \tag{3.5.16}$$

or

$$\liminf_{k\to\infty} \sum_{j=1}^{k} q(j) < \limsup_{k\to\infty} \sum_{j=1}^{k} q(j). \tag{3.5.17}$$

Note that the last criterion in Corollary 3.5.5 has no continuous analogue.

The following criterion is a consequence of Theorem 3.4.14, and it enables us to give an example showing that there exists an oscillatory equation (3.2.9) with $q(k)$ satisfying condition Theorem 3.5.2(iii).

Corollary 3.5.6. *If there exist two sequences of integers m_k and n_k with $n_k \geq m_k + 1$ such that $m_k \to \infty$ as $k \to \infty$ and*

$$\sum_{j=m_k}^{n_k-1} q(j) \geq c(m_k) + c(n_k), \tag{3.5.18}$$

then equation (3.2.9) is oscillatory.

Example 3.5.7. Let $m_k = 4k$ for $k \in \mathbb{N}$. Define $c(k) = 1$ and

$$q(m_k) = q(m_k + 1) = q(m_k + 2) = 1, \qquad q(m_k + 3) = -4 \quad \text{for } k \in \mathbb{N}. \tag{3.5.19}$$

Then

$$\sum_{j=m_k}^{m_k+2} q(j) = 3 > 2 = c(m_k) + c(n_k) \quad \forall k \in \mathbb{N}. \tag{3.5.20}$$

Equation (3.2.9) is oscillatory by Corollary 3.5.6. It is clear that $\sum_{j=1}^{\infty} q(j) = -\infty$.

Theorem 3.5.8. *Suppose that $c(k) \leq 1$ for $k \in \mathbb{N}$. If for all $m \in \mathbb{N}$ there exists $n \geq m$ such that*

$$\lim_{k\to\infty} \sum_{j=n}^{k} q(j) \geq 1, \tag{3.5.21}$$

then equation (3.2.9) is oscillatory.

Proof. Suppose to the contrary that equation (3.2.9) is nonoscillatory. Then there exist $m \in \mathbb{N}$ and a solution of equation (3.2.9) with $c(k)x(k)x(k+1) > 0$ for $k \geq m$. Therefore, we can consider equation (3.2.17) with $c(k) + w(k) > 0$, where $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k))$ for $k \geq m$. Pick $n \geq m$ such that (3.5.21) holds. Summing equation (3.2.17) from n to k, we obtain

$$w(k+1) = w(n) - \sum_{j=n}^{k} q(j) - \sum_{j=n}^{k} \Phi(w(j), c(j)) \quad \text{for } k \geq n. \tag{3.5.22}$$

Hence

$$w(k+1) = \tilde{\Phi}(w(n), c(n)) - \sum_{j=n}^{k} q(j) - \sum_{j=n+1}^{k} \Phi(w(j), c(j)), \tag{3.5.23}$$

where

$$\tilde{\Phi}(w(n), c(n)) = \frac{w(n)c(n)}{\left[\Psi^{-1}(c(n)) + \Psi^{-1}(w(n))\right]^{\alpha-1}}. \tag{3.5.24}$$

If $\sum_{j=n+1}^{\infty} \Phi(w(j), c(j)) = \infty$, then we get a contradiction from (3.5.23) since $w(k) > -c(k) \geq -1$ for $k \geq m$. Thus we can assume $\sum_{j=n+1}^{\infty} \Phi(w(j), c(j)) < \infty$. Then $\Phi(w(k), c(k)) \to 0$ and therefore $w(k) \to 0$ as $k \to \infty$, according to Lemma 3.2.6(I_5). In view of the fact that $\Phi(w(k), c(k)) \geq 0$ for $k \geq m$, we have

$$-w(k+1) \geq -\tilde{\Phi}(w(n), c(n)) + \sum_{j=n}^{k} q(j). \tag{3.5.25}$$

Hence, from (3.5.21) and Lemma 3.2.6(I_6), we have

$$0 \geq -\tilde{\Phi}(w(n), c(n)) + \limsup_{k\to\infty} \sum_{j=n}^{k} q(j) > 0, \tag{3.5.26}$$

which is a contradiction. □

The following results are oscillation criteria for equation (3.2.9) that are half-linear extensions of those proved in Section 1.11 for linear difference equations (1.1.2).

First, we give several auxiliary lemmas. We start with a result that claims that under certain conditions a positive solution of equation (3.2.9) has a positive difference.

In the sequel, we assume the following condition:

$$\sum_{j=k}^{\infty} q(j) \geq 0, \qquad \sum_{j=n_1}^{\infty} q(j) > 0 \tag{3.5.27}$$

for some $n_1 \geq n_0$ and all $k \geq n_0$. We also assume that

$$\text{condition (3.5.27) holds and} \quad q(k) \not\equiv 0 \quad \text{eventually.} \tag{3.5.28}$$

Lemma 3.5.9. *Assume that condition (3.5.28) holds. Further suppose*

$$c(k) > 0, \quad k \in \mathbb{N}, \qquad \sum_{j=1}^{\infty} c^{1-\beta}(j) = \infty \tag{3.5.29}$$

and let x be a nonoscillatory solution of equation (3.2.9) such that $x(k) > 0$ for all $k \geq m$. Then there exists $n \geq m$ such that $\Delta x(k) > 0$ for all $k \geq n$.

Proof. The proof is by contradiction. We will distinguish two cases.

Case 1. Suppose that $\Delta x(k) < 0$ for all large k, say $k \geq n \geq m$ (hence $\Delta\Psi(x(k)) < 0$). Without loss of generality, we may suppose that

$$\sum_{j=m}^{k} q(j) \geq 0 \tag{3.5.30}$$

holds for $k \geq n$ and $q(n) \geq 0$. Define

$$\phi(k) = \begin{cases} \displaystyle\sum_{j=n}^{k} q(j) & \text{for } k \geq n, \\ 0 & \text{for } k = n-1. \end{cases} \tag{3.5.31}$$

Then, for $k \geq n$ we have

$$\begin{aligned} \sum_{j=n}^{k} q(j)\Psi(x(j+1)) &= \sum_{j=n}^{k} (\Delta\phi(j-1))\Psi(x(j+1)) \\ &= \phi(k)\Psi(x(k+2)) - \sum_{j=n}^{k} \phi(j)\Delta\Psi(x(j+1)) \geq 0. \end{aligned} \tag{3.5.32}$$

Therefore,

$$c(k+1)\Psi(\Delta x(k+1)) - c(n)\Psi(\Delta x(n)) = \sum_{j=n}^{k} \Delta\big(c(j)\Psi(\Delta x(j))\big) \leq 0, \tag{3.5.33}$$

and so

$$\Psi(\Delta x(k+1)) \leq \frac{c(n)}{c(k+1)} \Psi(\Delta x(n)) \quad \text{for } k \geq n. \tag{3.5.34}$$

Hence

$$\begin{gathered}\Delta x(k+1) \leq c^{\beta-1}(n)(\Delta x(n)) c^{1-\beta}(k+1), \\ x(k+2) - x(n+1) = \sum_{j=n}^{k} \Delta x(j+1) \leq c^{\beta-1}(n)(\Delta x(n)) \sum_{j=n}^{k} c^{1-\beta}(j+1).\end{gathered} \tag{3.5.35}$$

Now we have $x(k) \to -\infty$ as $k \to \infty$, which is a contradiction.

Case 2. Assume that there exists a subsequence $k_\ell \to \infty$ such that $\Delta x(k_\ell) < 0$ and $\Delta x(k_\ell + 1) \geq 0$. The sequence $v(k) = -c(k)\Psi(\Delta x(k))/\Psi(x(k))$ is a solution of the generalized Riccati difference equation

$$\Delta v(k) = q(k) + \tilde{\Phi}(v(k), c(k)) \tag{3.5.36}$$

with $c(k) > v(k)$, where

$$\tilde{\Phi}(v(k), c(k)) = -v(k) \left[1 - \frac{c(k)}{\left[\Psi^{-1}(c(k)) - \Psi^{-1}(v(k))\right]^{\alpha-1}} \right] \geq 0. \tag{3.5.37}$$

Hence

$$0 \geq v(k_\ell + 1) - v(k_\ell) = \sum_{j=k_\ell}^{k_\ell+1} q(j) + \sum_{j=k_\ell}^{k_\ell+1} \tilde{\Phi}(v(j), c(j)). \tag{3.5.38}$$

Summing (3.5.38) as $k_{\ell+1} \to \infty$, we obtain $\sum_{j=k_\ell}^{\infty} q(j) < 0$, since $v(k) \not\equiv 0$, and this contradicts assumption (3.5.28).

In either case, we get a contradiction, and hence the proof is finished. □

In the next lemma a necessary condition for nonoscillation of equation (3.2.9) is given in the case when $\lim_{k\to\infty} \sum^{k} q(j)$ is convergent and the assumptions of Lemma 3.5.9 hold.

Lemma 3.5.10. *Let the assumptions of Lemma 3.5.9 hold and assume further that*

$$\sum_{j=1}^{\infty} q(j) = \lim_{k\to\infty} \sum_{j=1}^{k} q(j) \quad \textit{is convergent.} \tag{3.5.39}$$

Let x be a nonoscillatory solution of equation (3.2.9) such that $x(k) > 0$ for all $k \geq m$. Then there exists $n \geq m$ such that

$$w(k) \geq \sum_{j=k}^{\infty} q(j) + \sum_{j=k}^{\infty} \Phi(w(j), c(j)) \tag{3.5.40}$$

for $k \geq n$, where $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k)) > 0$, and the function Φ is defined by (3.2.19).

Proof. From Lemma 3.5.9 there exists an integer n such that $w(k) > 0$ for $k \geq n$ and $w(k)$ satisfies equation (3.2.17) for $k \geq n$. Summing equation (3.2.17) from n to $\ell > k \geq n$, we get

$$w(\ell+1) - w(k) + \sum_{j=k}^{\ell} q(j) + \sum_{j=k}^{\ell} \Phi(w(j), c(j)) = 0. \tag{3.5.41}$$

Therefore,

$$0 < w(\ell+1) \leq w(n) - \sum_{j=n}^{\ell} q(j) \quad \forall \ell > n, \tag{3.5.42}$$

and hence

$$w(k) \geq \sum_{j=k}^{\ell} q(j) + \sum_{j=k}^{\ell} \Phi(w(j), c(j)) \quad \text{for } \ell > k \geq n. \tag{3.5.43}$$

Letting $\ell \to \infty$, we obtain (3.5.40). □

The following lemma claims that under slightly stronger conditions than those of Lemma 3.5.10 we can estimate a positive sequence given by the Riccati-type substitution from above.

Lemma 3.5.11. *Let conditions (3.5.29), (3.5.39) hold and $q(k) \geq 0$ (and eventually nontrivial) for all $k \geq m$, and let x be a nonoscillatory solution of equation (3.2.9) such that $x(k) > 0$ for all $k \geq m$. Then $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k)) > 0$ for all $k \geq m$ satisfies $w(k) \to 0$ as $k \to \infty$. Further, the following inequality holds:*

$$w(k) \leq \left[\sum_{j=m}^{k-1} c^{1-\beta}(j)\right]^{1-\alpha} \quad \textit{for } k \geq m. \tag{3.5.44}$$

Proof. From Lemma 3.5.9 and equation (3.2.17), we have $w(k) > 0$ and for $k \geq m$,

$$\Delta w(k) + \Phi\big(w(k), c(k)\big) = w(k+1) - \frac{w(k)}{\big[c^{\beta-1}(k) + w^{\beta-1}(k)\big]^{\alpha-1}} \leq 0. \tag{3.5.45}$$

Hence

$$w^{\beta-1}(k+1)\big[c^{\beta-1}(k) + w^{\beta-1}(k)\big] - w^{\beta-1}(k)c^{\beta-1}(k) \leq 0, \tag{3.5.46}$$

and therefore

$$\begin{aligned}\Delta&\left[\sum_{j=1}^{k-1} c^{1-\beta}(j) - w^{1-\beta}(j)\right]\\ &\quad= \frac{\big[w(k)w(k+1)\big]^{\beta-1} + \big[c(k)w(k+1)\big]^{\beta-1} - \big[c(k)w(k)\big]^{\beta-1}}{\big[c(k)w(k)w(k+1)\big]^{\beta-1}} \leq 0\end{aligned} \tag{3.5.47}$$

for $k \geq m$. Summation of (3.5.47) from m to $k-1$ gives

$$\sum_{j=1}^{k-1} c^{1-\beta}(j) - w^{1-\beta}(k) \leq \sum_{j=1}^{m-1} c^{1-\beta}(j) - w^{1-\beta}(m) \leq \sum_{j=1}^{m-1} c^{1-\beta}(j) \tag{3.5.48}$$

for $k > m$ and so

$$w^{\beta-1}(k) \leq \left[\sum_{j=m}^{k-1} c^{1-\beta}(j)\right]^{-1}, \tag{3.5.49}$$

and $w(k) \to 0$ as $k \to \infty$ by condition (3.5.29). $\square$

We will also need the following lemma which is the well-known Young inequality.

Lemma 3.5.12 (Young's inequality). *If $\alpha > 1$ and $\beta > 1$ are mutually conjugate numbers, then*

$$|uv| \leq \frac{|u|^{\alpha}}{\alpha} + \frac{|v|^{\beta}}{\beta} \tag{3.5.50}$$

holds for all $u, v \in \mathbb{R}$.

To obtain the next oscillation results for equation (3.2.9), it will be convenient to employ the following notation. For a given $m \in \mathbb{N}$ we define the sequence $F(t;k)$ as follows:

$$F(0;k) = \sum_{j=k}^{\infty} q(j) \quad \text{for } k \geq m, \tag{3.5.51}$$

and for $k \geq m$ and $t \in \mathbb{N}$,

$$F(t;k) = \sum_{j=k}^{\infty} F(t-1;j)\left[1 - \frac{c(j)}{\left[\Psi^{-1}(t-1;j) + \Psi^{-1}(c(j))\right]^{\alpha-1}}\right] + F(0;k)$$
$$= \sum_{j=k}^{\infty} \Phi(F(t-1;j), c(j)) + F(0;k). \tag{3.5.52}$$

The following theorem shows that equation (3.2.9) is oscillatory provided the above series is not convergent for some $t \in \mathbb{N}$.

Theorem 3.5.13. *Let conditions (3.5.28), (3.5.29), and (3.5.39) hold. Then equation (3.2.9) is oscillatory if either one of the following hold.*

(I) *There exists $t \in \mathbb{N}$ such that $F(0;k), \ldots, F(t-1;k)$ satisfy*

$$\sum_{j=k}^{\infty} \Phi(F(t-1;j), c(j)) = \infty. \tag{3.5.53}$$

(II) *There exists an integer $n \geq m$ such that*

$$\limsup_{t\to\infty} F(t;n) = \infty. \tag{3.5.54}$$

Proof. Suppose, by contradiction, that equation (3.2.9) is nonoscillatory. Let (I) hold. If $t = 1$, then let x be a solution of equation (3.2.9) such that $x(k) > 0$ for $k \geq m$. Let $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k))$ for $k \geq m$. Then by Lemmas 3.5.9 and 3.5.12 we have

$$w(k) \geq \sum_{j=k}^{\infty} \Phi(w(j), c(j)) + \sum_{j=k}^{\infty} q(j) \geq F(0;k) \quad \text{for } k \geq n \geq m. \tag{3.5.55}$$

Hence

$$\sum_{j=k}^{\infty} \Phi(F(0;j), c(j)) \leq \sum_{j=k}^{\infty} \Phi(w(j), c(j)) < \infty \quad \text{for } k \geq n, \tag{3.5.56}$$

since the function $\Phi(u, v)$ is increasing with respect to the first variable for $u, v > 0$. But the last inequality contradicts condition (3.5.53) for $t = 1$. Similarly, if $t > 1$, then from inequality (3.5.40), we have $w(k) \geq F(i,k)$, $i = 0, 1, \ldots, t-1$ and $k \geq m$, and hence

$$\sum_{j=k}^{\infty} \Phi(F(t-1;j), c(j)) \leq \sum_{j=k}^{\infty} \Phi(w(j), c(j)) < \infty, \tag{3.5.57}$$

which again contradicts condition (3.5.53).

Suppose next that (II) holds. Obviously, in view of $w(k) \geq F(t;k)$ for $t \in \mathbb{N}_0$, we have from inequality (3.5.40) that $\limsup_{t\to\infty} F(t;n) \leq w(n) < \infty$, which is a contradiction. □

Theorem 3.5.14. *Let the assumptions of Lemma 3.5.11 hold and $F(t;k)$ be defined as above. Assume further that*

$$\limsup_{k\to\infty} \left[\sum_{j=m}^{k-1} c^{1-\beta}(j) \right]^{\alpha-1} F(t;k) > 1 \tag{3.5.58}$$

for some $t \in \mathbb{N}_0$. Then equation (3.2.9) is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (3.2.9). As in the proof of Theorem 3.5.13, we have $F(t;k) \leq w(k)$ for $k \geq m$ and $t \in \mathbb{N}_0$ and hence by Lemma 3.5.11,

$$F(t;k) \leq \left[\sum_{j=m}^{k-1} c^{1-\beta}(j) \right]^{1-\alpha} \quad \text{for } k \geq m,\ t \in \mathbb{N}_0. \tag{3.5.59}$$

But then

$$\limsup_{k\to\infty} \left[\sum_{j=m}^{k-1} c^{1-\beta}(j) \right]^{\alpha-1} F(t;k) \leq 1, \tag{3.5.60}$$

which contradicts condition (3.5.58). □

Employing the estimate from Lemma 3.5.11, we have the following criterion.

Theorem 3.5.15. *Let the assumptions of Lemma 3.5.11 hold and assume further that there exist two sequences of positive integers $\{m_\ell\}$ and $\{n_\ell\}$ with $m+1 \leq m_\ell < n_\ell$ and $m_\ell \to \infty$ as $\ell \to \infty$ such that*

$$\sum_{j=m_\ell}^{n_\ell} q(j) \geq c(n_\ell + 1) + \left[\sum_{j=m}^{m_\ell-1} c^{1-\beta}(j) \right]^{1-\alpha} \tag{3.5.61}$$

for all large k. Then equation (3.2.9) is oscillatory.

Proof. Let x be a nonoscillatory solution of (3.2.9), say, $x(k) > 0$ for $k \geq m$. As in the proof of Lemma 3.5.11, we suppose $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k)) > 0$ for $k \geq n \geq m$ and that $w(k)$ satisfies equation (3.2.17). Hence

$$\Delta w(k) + q(k) < 0 \quad \text{for } k \geq m, \tag{3.5.62}$$

which yields

$$w(n_\ell + 1) - w(m_\ell) + \sum_{j=m_\ell}^{n_\ell} q(j) < 0 \quad \text{for } m < m_\ell < n_\ell, \tag{3.5.63}$$

so that

$$\sum_{j=m_\ell}^{n_\ell} q(j) < w(m_\ell) - w(n_\ell + 1) < c(n_\ell + 1) + w(m_\ell). \tag{3.5.64}$$

Thus, by Lemma 3.5.11, we have

$$\sum_{j=m_\ell}^{n_\ell} q(j) < c(n_\ell + 1) + \left[\sum_{j=m}^{m_\ell - 1} c^{1-\beta}(j)\right]^{1-\alpha}, \tag{3.5.65}$$

which contradicts condition (3.5.61). □

Remark 3.5.16. Under the assumptions of Lemma 3.5.11, a necessary condition for the existence of a nonoscillatory solution of equation (3.2.9) is

$$q(k) < \left[\sum_{j=m}^{k-1} c^{1-\beta}(j)\right]^{1-\alpha} \quad \text{for all large } k \in \mathbb{N}. \tag{3.5.66}$$

Indeed, the inequality (3.5.62) implies

$$0 < \frac{w(k+1)}{w(k)} < 1 - \frac{q(k)}{w(k)} \quad \text{for } k \geq m. \tag{3.5.67}$$

Thus, $q(k) < w(k)$ for $k \geq m$, and so Lemma 3.5.11 implies (3.5.66).

As a further use of the estimate from Lemma 3.5.11, we give a criterion, where a "weighted sequence" appears in the assumptions.

Theorem 3.5.17. *In addition to the assumptions of Lemma 3.5.11 assume that there exists a sequence $\rho = \{\rho(k)\}$ with $\rho(k) > 0$ for $k \in \mathbb{N}$ such that*

$$\lim_{k\to\infty} \sum_{j=m}^{k} \rho(j)q(j) = \infty, \tag{3.5.68}$$

$$\lim_{k\to\infty}\left(\sum_{j=m}^{k}\rho(j)q(j)\right)^{-1}\left(\sum_{j=m}^{k}\frac{|\Delta\rho(j-1)|^{\alpha}}{\rho^{\alpha-1}(j)}c(j)\left[1+c^{1-\beta}(j)\left(\sum_{i=m}^{j-1}c^{1-\beta}(i)\right)^{-1}\right]^{\sigma}\right)=0, \tag{3.5.69}$$

where

$$\sigma = \begin{cases} \alpha - 1 & \text{for } 1 < \alpha \leq 2, \\ (\alpha - 1)^2 & \text{for } \alpha \geq 2. \end{cases} \tag{3.5.70}$$

Then equation (3.2.9) is oscillatory.

Proof. Let x be an eventually positive solution of equation (3.2.9). As in Lemma 3.5.11 we have equation (3.2.17) with $w(k) = c(k)\Psi(\Delta x(k))/\Psi(x(k)) > 0$ for $k \geq m$. Multiplying equation (3.2.17) by $\rho(k)$ and summing it from m to k we obtain

$$\sum_{j=m}^{k} \rho(j)q(j) = -\sum_{j=m}^{k} \rho(j)\Delta w(j) - \sum_{j=m}^{k} \rho(j)\Phi(w(j), c(j)). \tag{3.5.71}$$

Obviously,

$$\begin{aligned} -\sum_{j=m}^{k} \rho(j)\Delta w(j) &= \rho(m)w(m) - \rho(k+1)w(k+1) + \sum_{j=m}^{k} w(j+1)\Delta\rho(j) \\ &= \sum_{j=m}^{k} w(j)\Delta\rho(j-1) + w(m)\rho(m-1) - w(k+1)\rho(k) \\ &= \sum_{j=m}^{k} w(j)\Delta\rho(j-1) + w(m)\rho(m-1). \end{aligned} \tag{3.5.72}$$

Observe that from Lemma 3.5.11,

$$\begin{aligned} c^{\beta-1}(k) + w^{\beta-1}(k) &= c^{\beta-1}(k)\left[1 + c^{1-\beta}(k)w^{\beta-1}(k)\right] \\ &\leq c(k)\left[1 + \frac{c^{1-\beta}(k)}{\sum_{j=m}^{k-1} c^{1-\beta}(j)}\right], \end{aligned} \tag{3.5.73}$$

and by the Lagrange mean value theorem,

$$\Phi(w(k), c(k)) = \frac{(\alpha-1)w^{\beta}(k)\xi^{\alpha-2}(k)}{\left[c^{\beta-1}(k) + w^{\beta-1}(k)\right]^{\alpha-1}}, \tag{3.5.74}$$

where $c^{\beta-1}(k) \leq \xi(k) \leq c^{\beta-1}(k) + w^{\beta-1}(k)$.

Using (3.5.74) and the inequality (3.5.50), where we set

$$\begin{aligned} u &= w(k)\left(\Delta\rho(k-1)\right)\left[\beta\rho(k)\Phi(w(k), c(k))\right]^{1-\beta}, \\ v &= \left[\beta\rho(k)\Phi(w(k), c(k))\right]^{\beta-1}, \end{aligned} \tag{3.5.75}$$

we get

$$
\begin{aligned}
\sum_{j=m}^{k}\rho(j)q(j) &= w(m)\rho(m-1)+\sum_{j=m}^{k}w(j)\Delta\rho(j-1)-\sum_{j=m}^{k}\rho(j)\Phi(w(j),c(j)) \\
&\le w(m)\rho(m-1) \\
&\quad+\left(\frac{1}{\alpha\beta^{\alpha/\beta}(\alpha-1)^{\alpha/\beta}}\right)\sum_{j=m}^{k}\frac{|\Delta\rho(j-1)|^{\alpha}\left[c^{\beta-1}(j)+w^{\beta-1}(j)\right]^{(\alpha-1)^2}}{\rho^{\alpha-1}(j)\xi^{(\alpha-2)(\alpha-1)}(j)},
\end{aligned}
\tag{3.5.76}
$$

so by (3.5.73),

$$
\sum_{j=m}^{k}\rho(j)q(j)\le w(m)\rho(m-1)+\frac{1}{\alpha^{\alpha}}\sum_{j=m}^{k}\frac{|\Delta\rho(j-1)|^{\alpha}}{\rho^{\alpha-1}(j)}c(j)\left[1+\frac{c^{1-\beta}(j)}{\sum_{i=m}^{j-1}c^{1-\beta}(i)}\right]^{\sigma}.
\tag{3.5.77}
$$

Combining conditions (3.5.68) and (3.5.69) and the inequality (3.5.77), we now get the desired contradiction. □

Remark 3.5.18. If we assume that

$$
\liminf_{k\to\infty}c^{\beta-1}(k)\sum_{j=m}^{k-1}c^{1-\beta}(j)>0,
\tag{3.5.78}
$$

then condition (3.5.69) reduces to

$$
\lim_{k\to\infty}\left(\sum_{j=m}^{k}\rho(j)q(j)\right)^{-1}\sum_{j=m}^{k}\frac{|\Delta\rho(j-1)|^{\alpha}}{\rho^{\alpha-1}(j)}c(j)=0.
\tag{3.5.79}
$$

Indeed, if condition (3.5.78) holds, then

$$
\limsup_{k\to\infty}\frac{c^{1-\beta}(k)}{\sum_{j=m}^{k-1}c^{1-\beta}(j)}<\infty,
\tag{3.5.80}
$$

and in inequality (3.5.77) we have

$$
\left[1+\frac{c^{1-\beta}(k)}{\sum_{j=m}^{k-1}c^{1-\beta}(j)}\right]^{\sigma}\le[1+M]^{\sigma}
\tag{3.5.81}
$$

for each $k\ge m$ and a suitable constant M.

One can observe that condition (3.5.78) is not a severe restriction. For example, $c(k)=k^{\alpha-1}$, $c(k)=k^{-(\alpha-1)}$, $c(k)=e^{-k(\alpha-1)}$, and $c(k)\equiv 1$ all satisfy (3.5.78).

Choosing a suitable weighted sequence in Theorem 3.5.17, we get the following oscillation result.

Corollary 3.5.19. *Let the assumptions of Lemma 3.5.11 hold along with condition (3.5.78). Assume further that there exists $\mu > 0$ such that $\alpha - \mu > 1$ and*

$$\lim_{k\to\infty} \sum_{j=m}^{k} \left(\sum_{i=1}^{j} c^{1-\beta}(j) \right)^{\mu} q(j) = \infty. \tag{3.5.82}$$

Then equation (3.2.9) is oscillatory.

Proof. Define the sequence ρ in Theorem 3.5.17 by $\rho(k) = [\sum_{j=1}^{k} c^{1-\beta}(j)]^{\mu}$. We show that (3.5.79) holds. First, suppose that $0 < \mu < 1$. Then from the Lagrange mean value theorem

$$0 < \Delta\rho(k-1) \le \mu c^{1-\beta}(k) \left[\sum_{j=1}^{k} c^{1-\beta}(j) \right]^{\mu-1}. \tag{3.5.83}$$

From here, we have

$$\begin{aligned} \frac{[\Delta\rho(k-1)]^{\alpha}}{\rho^{\alpha-1}(k)} c(k) &\le \frac{\mu^{\alpha} c^{\alpha(1-\beta)}(k) \left[\sum_{j=1}^{k} c^{1-\beta}(j) \right]^{\alpha(\mu-1)}}{\left[\sum_{j=1}^{k} c^{1-\beta}(j) \right]^{(\alpha-1)\mu}} \\ &\le \mu^{\alpha} c^{1-\beta}(k) \left[\sum_{j=1}^{k-1} c^{1-\beta}(j) \right]^{\mu-\alpha}. \end{aligned} \tag{3.5.84}$$

Similarly, one can show that for $\mu \ge 1$ we get

$$\frac{[\Delta\rho(k-1)]^{\alpha}}{\rho^{\alpha-1}(k)} c(k) \le \mu^{\alpha-1} c^{1-\beta}(k) \left[\sum_{j=1}^{k} c^{1-\beta}(j) \right]^{\mu-\alpha}. \tag{3.5.85}$$

But from (3.5.78) there exists a positive constant M such that

$$\sum_{j=1}^{k} c^{1-\beta}(j) \le M \sum_{j=1}^{k-1} c^{1-\beta}(j) \quad \text{for } k \in \mathbb{N} \setminus \{1\}, \tag{3.5.86}$$

and hence (for μ satisfying the condition of the corollary, regardless whether or not $\mu < 1$ holds) we have

$$\sum_{j=m}^{k} \frac{[\Delta\rho(j-1)]^{\alpha}}{\rho^{\alpha-1}(j)} c(j) \le \mu^{\alpha} M \sum_{j=m}^{k} \left(\sum_{i=1}^{j} c^{1-\beta}(i) \right)^{\mu-\alpha} \left(\Delta \sum_{i=1}^{j-1} c^{1-\beta}(i) \right). \tag{3.5.87}$$

Setting $\xi(k) = \sum_{j=1}^{k} c^{1-\beta}(j)$ we get

$$\sum_{j=m}^{k} \frac{[\Delta\rho(j-1)]^{\alpha}}{\rho^{\alpha-1}(j)} c(j) \le \mu^{\alpha} M \sum_{j=m-1}^{k-1} \frac{\Delta\xi(j)}{\xi^{\alpha-\mu}(j+1)}. \tag{3.5.88}$$

Now we show that the sum on the right-hand side of (3.5.88) is convergent. Let

$$f(t) = \xi(k) + \big(\Delta\xi(k)\big)(t-k) \quad \text{for } k \le t \le k+1. \tag{3.5.89}$$

Then $f'(t) = \Delta\xi(k)$ and $f(t) \le \xi(k+1)$ for $k < t < k+1$, and hence we get

$$\frac{\Delta\xi(k)}{\xi^{\alpha-\mu}(k+1)} = \int_k^{k+1} \frac{\Delta\xi(k)}{\xi^{\alpha-\mu}(k+1)}\,dt \le \int_k^{k+1} \frac{f'(t)}{[f(t)]^{\alpha-\mu}}\,dt. \tag{3.5.90}$$

It follows that

$$\begin{aligned}\sum_{j=m-1}^{k-1} \frac{\Delta\xi(j)}{\xi^{\alpha-\mu}(j+1)} &\le \int_{m-1}^{k} \frac{f'(t)}{[f(t)]^{\alpha-\mu}}\,dt \\ &= \frac{1}{\alpha-\mu-1}\left[\left(\sum_{j=1}^{m-1} c^{1-\beta}(j)\right)^{1+\mu-\alpha} - \left(\sum_{j=1}^{k} c^{1-\beta}(j)\right)^{1+\mu-\alpha}\right].\end{aligned} \tag{3.5.91}$$

Therefore the condition (3.5.79) from Remark 3.5.18 holds, and so the statement follows from Theorem 3.5.17. □

3.5.2. Further oscillation criteria based on the Riccati technique

In the theory of oscillation of linear second-order differential equations of the form

$$x''(t) + q(t)x(t) = 0, \tag{3.5.92}$$

where $q \in C([1,\infty),\mathbb{R})$, two of the most important criteria are due to Hartman and Wintner. It was proved that if

$$\lim_{t\to\infty} C(t) = \infty \quad \text{(Wintner)} \tag{3.5.93}$$

or

$$-\infty < \liminf_{t\to\infty} C(t) < \limsup_{t\to\infty} C(t) \le \infty \quad \text{(Hartman)}, \tag{3.5.94}$$

where

$$C(t) = \frac{1}{t}\int_1^t \int_1^s q(u)\,du\,ds, \tag{3.5.95}$$

then equation (3.5.92) is oscillatory. Using the results from Section 1.11, we have observed that in the parallel statements to the above-mentioned, for second-order linear difference equations of the form

$$\Delta^2 x(k) + q(k)x(k) = 0, \tag{3.5.96}$$

where q is a sequence of real numbers, it is superfluous in a certain sense, to consider a "direct" discrete counterpart of the function $C(t)$, that is, the sequence

$$E(k) = \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{j} q(i), \tag{3.5.97}$$

since the assumption

$$\liminf_{k\to\infty}\sum_{j=1}^{k} q(j) < \limsup_{k\to\infty}\sum_{j=1}^{k} q(j) \tag{3.5.98}$$

guarantees oscillation of equation (3.5.96) (this criterion has no continuous analogue), and the existence of the finite limit

$$\lim_{k\to\infty}\sum_{j=1}^{k} q(j) = C, \quad \text{where } C \text{ is a constant} \tag{3.5.99}$$

implies the existence of $\lim_{k\to\infty} E(k)$ (hence $\liminf_{k\to\infty} E(k) < \limsup_{k\to\infty} E(k)$ implies (3.5.98)). Moreover, we can have $\lim_{k\to\infty}\sum_{j=1}^{k} q(j) = -\infty$ in criterion (3.5.98). These discrepancies between the discrete and the continuous cases are due to some specific properties of difference calculus.

In the half-linear case we have a similar situation. For example, Kandelaki et al. [169] investigated the oscillatory behavior of second-order half-linear differential equations of type (3.2.1) with $c(t) \equiv 1$ in the cases when

$$\lim_{t\to\infty} C_\alpha(t) \quad \text{exists or not,} \tag{3.5.100}$$

where

$$C_\alpha(t) = \frac{\alpha-1}{t^{\alpha-1}}\int_1^t s^{\alpha-2}\int_1^s q(u)\,du\,ds. \tag{3.5.101}$$

In particular,

$$\lim_{t\to\infty} C_\alpha(t) = \infty \quad \text{or} \quad -\infty < \liminf_{t\to\infty} C_\alpha(t) < \limsup_{t\to\infty} C_\alpha(t) \le \infty \tag{3.5.102}$$

are sufficient for the oscillation of equation (3.2.1) with $c(t) \equiv 1$.

From Corollary 3.5.5 it clearly follows that in the case of equation (3.2.11), similarly as for the linear equations, it suffices to deal with the limit of the sequence $\sum_{j=1}^{k} q(j)$ in this connection since (3.5.99) implies the existence of the discrete counterpart of (3.5.100), namely,

$$\lim_{k\to\infty}\frac{\alpha-1}{k^{\alpha-1}}\sum_{j=1}^{k} j^{\alpha-2}\sum_{i=1}^{j} q(i). \tag{3.5.103}$$

Hence, in this subsection we will always assume that (3.5.99) holds.

The following notation will be employed:

$$\begin{aligned} Q(k) &= (k+1)^{\alpha-1}\sum_{j=k+1}^{\infty} q(j), \qquad H(k) = \frac{1}{k+1}\sum_{j=1}^{k} j^{\alpha}q(j), \\ Q_* &= \liminf_{k\to\infty} Q(k), \qquad Q^* = \limsup_{k\to\infty} Q(k), \\ H_* &= \liminf_{k\to\infty} H(k), \qquad H^* = \limsup_{k\to\infty} H(k), \end{aligned} \tag{3.5.104}$$

and we will abbreviate $\Phi(w(k),1)$ by $\Phi(k)$.

Now we give oscillation criteria for equation (3.2.11), considered on $\mathbb{N}$, that are discretizations of the results by Kandelaki et al. [169] for equation (3.2.1) with $c(t) \equiv 1$.

Theorem 3.5.20. *Suppose that condition (3.5.99) holds. If*

$$\limsup_{k\to\infty} \frac{\sum_{j=1}^{k}(j+1)^{\alpha-2}\sum_{i=j+1}^{\infty} q(i)}{\sum_{j=1}^{k} 1/(j+1)} > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.5.105}$$

then equation (3.2.11) is oscillatory.

Proof. Suppose that equation (3.2.11) is nonoscillatory. Then according to Theorem 3.5.3, there exists a sequence $w(k)$ with $1+w(k)>0$ for $k \ge m$ for some $m \in \mathbb{N}$ satisfying

$$w(k+1) = \sum_{j=k+1}^{\infty} q(j) + \sum_{j=k+1}^{\infty} \Phi(j) = C - \sum_{j=1}^{k} q(j) + \sum_{j=k+1}^{\infty} \Phi(j) \tag{3.5.106}$$

for $k \ge m$, where C is defined by (3.5.99). Let ε_1 be given and let $\varepsilon > 0$ be such that

$$\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\alpha-1} \le 1+\varepsilon_1. \tag{3.5.107}$$

Obviously, since $w(k) \to 0$ as $k \to \infty$, there exists $m_1 = m_1(\varepsilon_1) \in \mathbb{N}$ such that

$$\begin{gathered} |w(k)| < 1, \\ \left[1 - |w(k)|^{1/(\alpha-1)}\right]^{\alpha-2} \ge 1-\varepsilon \quad \text{with } \alpha \ge 2, \\ \left[1 + |w(k)|^{1/(\alpha-1)}\right]^{\alpha-2} \ge 1-\varepsilon \quad \text{with } 1 < \alpha \le 2, \\ \left[1+\Psi(w(k))\right]^{\alpha-1} \le 1+\varepsilon \end{gathered} \tag{3.5.108}$$

for $k \ge m_1$. Now we can suppose that $m \ge m_1$.

First, assume that $\alpha \geq 2$. Multiplying both sides of (3.5.106) by $(k+1)^{\alpha-2}$ and applying the summation from m to k we get for $k \geq m$,

$$\begin{aligned} \sum_{j=m}^{k}(j+1)^{\alpha-2}w(j+1) = C\sum_{j=m}^{k}(j+1)^{\alpha-2} - \sum_{j=m}^{k}(j+1)^{\alpha-2}\sum_{i=1}^{j}q(i) \\ + \sum_{j=m}^{k}(j+1)^{\alpha-2}\sum_{i=j+1}^{\infty}\Phi(i). \end{aligned} \tag{3.5.109}$$

Set

$$a(k) = \frac{1}{k^{\alpha-1}}\sum_{j=1}^{k}(j+1)^{\alpha-2}\sum_{i=1}^{j}q(i) \tag{3.5.110}$$

(one can see that $\lim_{k\to\infty} a(k) = C/(\alpha-1)$). Clearly,

$$k^{\alpha-1}a(k) = \sum_{j=1}^{m-1}(j+1)^{\alpha-2}\sum_{i=1}^{j}q(i) + \sum_{j=m}^{k}(j+1)^{\alpha-2}\sum_{i=1}^{j}q(i), \tag{3.5.111}$$

and employing the Lagrange mean value theorem, we obtain

$$\begin{aligned} &\sum_{j=m}^{k}(j+1)^{\alpha-2}\sum_{i=j+1}^{\infty}\Phi(i) \\ &\quad\geq \sum_{j=m}^{k}\eta^{\alpha-2}(j)\sum_{i=j+1}^{\infty}\Phi(i) \\ &\quad= \frac{(k+1)^{\alpha-1}}{\alpha-1}\sum_{j=k+2}^{\infty}\Phi(j) - \frac{m^{\alpha-1}}{\alpha-1}\sum_{j=m+1}^{\infty}\Phi(j) + \sum_{j=m}^{k}\frac{(j+1)^{\alpha-1}}{\alpha-1}\Phi(j+1) \\ &\quad\geq \frac{(k+1)^{\alpha-1}}{\alpha-1}\sum_{j=k+2}^{\infty}\Phi(j) - \frac{m^{\alpha-1}}{\alpha-1}\sum_{j=m+1}^{\infty}\Phi(j), \end{aligned} \tag{3.5.112}$$

where $k \leq \eta(k) \leq k+1$. Now from (3.5.109), we have

$$\begin{aligned} &\sum_{j=m}^{k}\left[(j+1)^{\alpha-2}w(j+1) - \frac{(j+1)^{\alpha-1}}{\alpha-1}\Phi(j+1)\right] \\ &\qquad \geq C\sum_{j=1}^{k}(j+1)^{\alpha-2} - k^{\alpha-1}a(k) - K_1 \end{aligned} \tag{3.5.113}$$

for $k \geq m$, where

$$K_1 = -\sum_{j=1}^{m-1}(j+1)^{\alpha-2}\sum_{i=1}^{j}q(i) + \frac{m^{\alpha-1}}{\alpha-1}\sum_{j=m+1}^{\infty}\Phi(j) + C\sum_{j=1}^{m-1}(j+1)^{\alpha-2}. \qquad (3.5.114)$$

By Lemma 3.2.6(I_8), the summand on the left-hand side of (3.5.113) can be rewritten as

$$\frac{1}{j+1}\left[y - \frac{|y|^{\beta}\xi^{\alpha-2}(j+1)}{\left[1+\Psi^{-1}(w(j+1))\right]^{\alpha-2}}\right] \leq \frac{1}{j+1}\left[y - \frac{1-\varepsilon}{1+\varepsilon}|y|^{\beta}\right] = \frac{1}{j+1}f(y), \qquad (3.5.115)$$

where $y = (j+1)^{\alpha-1}w(j+1)$ and $f(y) = y - [(1-\varepsilon)/(1+\varepsilon)]|y|^{\beta}$. Now, one can easily compute

$$f(y) \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\alpha-1} \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\left[1+\varepsilon_1\right] \quad \text{for any } y. \qquad (3.5.116)$$

Using this fact in (3.5.113), we find

$$C\sum_{j=1}^{k}(j+1)^{\alpha-2} - k^{\alpha-1}a(k) \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\left[1+\varepsilon_1\right]\sum_{j=1}^{k}\frac{1}{j+1} + K_1 \qquad (3.5.117)$$

for $k \geq m$.

Assume next that $1 < \alpha < 2$ and denote

$$\bar{a}(k) = \frac{1}{k^{\alpha-1}}\sum_{j=1}^{k}j^{\alpha-2}\sum_{i=1}^{j}q(i). \qquad (3.5.118)$$

Multiplying both sides of equality (3.5.106) by $k^{\alpha-1}$, summing from m to k, and applying similar arguments as above yields

$$C\sum_{j=1}^{k}j^{\alpha-2} - k^{\alpha-1}\bar{a}(k) \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\left[1+\varepsilon_1\right]\sum_{j=1}^{k}\frac{1}{j+1} + K_2 \qquad (3.5.119)$$

for $k \geq m$, where

$$K_2 = -\sum_{j=1}^{m-1} j^{\alpha-2} \sum_{i=1}^{j} q(i) + \frac{m^{\alpha-1}}{\alpha-1} \sum_{j=m+1}^{\infty} \Phi(j) + C \sum_{j=1}^{m-1} j^{\alpha-2} + m[1+\varepsilon_1] \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}. \tag{3.5.120}$$

Now, we would like to rewrite the left-hand side of (3.5.119) into the same form as (3.5.117) because of the next consideration with the common value $\alpha > 1$. For this purpose, denote the left-hand side of (3.5.117) as $A(k)$ and that of (3.5.119) as $B(k)$. Obviously, there exist constants $M, N > 0$ (we still assume $1 < \alpha < 2$) such that

$$\begin{aligned} A(k) &= B(k) + C \sum_{j=1}^{k} \left[(j+1)^{\alpha-2} - j^{\alpha-2}\right] + \sum_{j=1}^{k} \left[j^{\alpha-2} - (j+1)^{\alpha-2}\right] \sum_{i=1}^{j} q(i) \\ &\leq B(k) + M \sum_{j=1}^{k} \left[j^{\alpha-2} - (j+1)^{\alpha-2}\right] \\ &\leq B(k) + M(2-\alpha) \sum_{j=1}^{k} j^{\alpha-3} \leq B(k) + N. \end{aligned} \tag{3.5.121}$$

Now, from (3.5.117), (3.5.119), and the last inequality, one can observe that there exists a constant $K > 0$ such that

$$\left(\sum_{j=1}^{k} \frac{1}{j+1}\right)^{-1} A(k) \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} [1+\varepsilon_1] + K\left(\sum_{j=1}^{k} \frac{1}{j+1}\right)^{-1} \tag{3.5.122}$$

for $\alpha > 1$ and sufficiently large k. But we have

$$\begin{aligned} A(k) &= \left(\sum_{j=1}^{\infty} q(j)\right) \sum_{j=1}^{k} (j+1)^{\alpha-2} - \sum_{j=1}^{k} (j+1)^{\alpha-2} \sum_{i=1}^{j} q(i) \\ &= \sum_{j=1}^{k} (j+1)^{\alpha-2} \sum_{i=j+1}^{\infty} q(i). \end{aligned} \tag{3.5.123}$$

Hence (3.5.122) contradicts (3.5.105). □

Corollary 3.5.21. *Suppose that*

$$\liminf_{k\to\infty}\left(\sum_{j=1}^{k}(j+1)^{\alpha-1}\right)\sum_{j=k+1}^{\infty}q(j)>-\infty, \tag{3.5.124}$$

$$\limsup_{k\to\infty}\frac{\sum_{j=1}^{k}q(j)\sum_{i=1}^{j-1}(i+1)^{\alpha-2}}{\sum_{j=1}^{k}1/(j+1)}>\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}. \tag{3.5.125}$$

Then equation (3.2.11) is oscillatory.

Proof. Define

$$\phi(k)=\begin{cases}\left(\sum_{j=1}^{k}(j+1)^{\alpha-2}\right)\sum_{j=1}^{k}q(j) & \text{for } k\in\mathbb{N},\\ 0 & \text{for } k=0.\end{cases} \tag{3.5.126}$$

Then the left-hand side of (3.5.122) can be rewritten using

$$\begin{aligned}A(k)&=C\sum_{j=1}^{k}(j+1)^{\alpha-2}-\sum_{j=1}^{k}(j+1)^{\alpha-2}\sum_{i=1}^{j}q(i)\\&=C\sum_{j=1}^{k}(j+1)^{\alpha-2}-\phi(k)+\phi(0)+\sum_{j=1}^{k}\left(q(j)\sum_{i=1}^{j-1}(i+1)^{\alpha-2}\right)\\&=\left(\sum_{j=1}^{k}(j+1)^{\alpha-2}\right)\sum_{j=k+1}^{\infty}q(j)+\sum_{j=1}^{k}\left(q(j)\sum_{i=1}^{j-1}(i+1)^{\alpha-2}\right).\end{aligned} \tag{3.5.127}$$

Now, employing conditions (3.5.124) and (3.5.125), we see that the assumption of Theorem 3.5.20 is fulfilled. □

The next result claims that under the condition (3.5.28) the left-hand side of (3.5.105) can be replaced by a somewhat simpler expression.

Corollary 3.5.22. *Suppose that condition (3.5.28) holds. If*

$$\limsup_{k\to\infty}\frac{\sum_{j=1}^{k}j^{\alpha-1}q(j)}{\sum_{j=1}^{k}1/(j+1)}>\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.5.128}$$

then equation (3.2.11) is oscillatory.

Proof. Let $m \in \mathbb{N}$ be such that $\sum_{j=k}^{\infty} q(j) \geq 0$ for $k \geq m$. Denote

$$Z(m) = \sum_{j=1}^{m-1} (j+1)^{\alpha-2} \sum_{i=j+1}^{\infty} q(i). \tag{3.5.129}$$

Using the Lagrange mean value theorem, the sequence $A(k)$ on the left-hand side of (3.5.122) can be rewritten in the following way for $\alpha \geq 2$:

$$\begin{aligned} A(k) &= Z(m) + \sum_{j=m}^{k} (j+1)^{\alpha-2} \sum_{i=j+1}^{\infty} q(i) \\ &\geq Z(m) + \sum_{j=m}^{k} \xi^{\alpha-2}(j) \sum_{i=j+1}^{\infty} q(i) \\ &= Z(m) + \sum_{j=m}^{k} \frac{\Delta j^{\alpha-1}}{\alpha-1} \sum_{i=j+1}^{\infty} q(i) \\ &\geq Z(m) + \sum_{j=m}^{k} \frac{j^{\alpha-1}}{\alpha-1} q(j) - \frac{m^{\alpha-1}}{\alpha-1} \sum_{j=m}^{\infty} q(j) \end{aligned} \tag{3.5.130}$$

for $k \geq m$, where $k \leq \xi(k) \leq k+1$. Similarly, as above, the left-hand side of (3.5.119) can be rewritten for $1 < \alpha < 2$ in the following way:

$$\begin{aligned} B(k) &= \sum_{j=1}^{k} j^{\alpha-2} \sum_{i=j+1}^{\infty} q(i) \\ &\geq \sum_{j=1}^{m-1} j^{\alpha-2} \sum_{i=j+1}^{\infty} q(i) + \sum_{j=m}^{k} \frac{j^{\alpha-1}}{\alpha-1} q(j) - \frac{m^{\alpha-1}}{\alpha-1} \sum_{j=m}^{\infty} q(j) \end{aligned} \tag{3.5.131}$$

for $k \geq m$. But both of these inequalities contradict condition (3.5.128), and hence equation (3.2.11) is oscillatory. □

Remark 3.5.23. One can show that $\sum_{j=1}^{k} 1/(1+j)$ in (3.5.105), (3.5.125), and (3.5.128) can be replaced by $\ln k$.

Corollary 3.5.24. *A sufficient condition for equation (3.2.11) to be oscillatory is that either*

$$Q_* > \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} \tag{3.5.132}$$

or that condition (3.5.28) holds and

$$H_* > \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}. \tag{3.5.133}$$

Proof. Suppose that condition (3.5.132) holds. Using summation by parts, we rewrite the sequence $A(k)$ on the left-hand side of (3.5.122) in the following way:

$$\begin{aligned} A(k) &= \sum_{j=1}^{k}(j+1)^{\alpha-2}\sum_{i=j+1}^{\infty}q(i) \\ &= \sum_{j=1}^{k}\frac{Q(j)}{j+1} = \frac{1}{k+1}\sum_{j=1}^{k}Q(j) + \sum_{j=1}^{k}\frac{1}{j(j+1)}\sum_{i=1}^{j-1}Q(i). \end{aligned} \tag{3.5.134}$$

From here, we immediately get the statement by Theorem 3.5.20. Note that the other possibility to prove this part of the statement is to apply the discrete L'Hôpital rule to the condition (3.5.105).

Suppose that condition (3.5.133) holds. Applying the difference operator to $H(k)$, we have

$$\Delta H(k) = \Delta\left(\frac{1}{k+1}\right)\sum_{j=1}^{k+1} j^{\alpha}q(j) + \frac{1}{k+1}(k+1)^{\alpha}q(k+1). \tag{3.5.135}$$

Summing (3.5.135) from 1 to k yields

$$\sum_{j=1}^{k}(j+1)^{\alpha-1}q(j+1) = \sum_{j=1}^{k}\frac{H(j+1)}{j+1} + H(k+1) - H(1), \tag{3.5.136}$$

and the statement now follows from Corollary 3.5.22 since condition (3.5.28) holds. ☐

Corollary 3.5.25. *Suppose that condition (3.5.28) holds. If*

$$\liminf_{k\to\infty}\left[Q(k) + H(k)\right] > \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.5.137}$$

then equation (3.2.11) is oscillatory.

Proof. Let $m \in \mathbb{N}$ be such that $\sum_{j=k}^{\infty} q(j) \geq 0$ for $k \geq m$. From (3.5.134) we have

$$A(k) = \frac{1}{k+1}\sum_{j=1}^{k}Q(j) + \sum_{j=1}^{k}\frac{1}{j(j+1)}\sum_{i=1}^{j-1}Q(i), \tag{3.5.138}$$

where $A(k)$ is the same as in (3.5.122). Further, using the Lagrange mean value theorem, we obtain

$$\begin{aligned}\frac{\alpha}{k+1}\sum_{j=1}^{k}Q(j) &= \frac{\alpha}{k+1}\sum_{j=1}^{m-1}Q(j)+\frac{\alpha}{k+1}\sum_{j=m}^{k}(j+1)^{\alpha-1}\sum_{i=j+1}^{\infty}q(i)\\ &\geq \frac{\alpha}{k+1}\sum_{j=1}^{m-1}Q(j)+\frac{\alpha}{k+1}\sum_{j=m}^{k}\xi^{\alpha-1}(j)\sum_{i=j+1}^{\infty}q(i)\\ &= Q(k)+H(k)+\frac{\alpha}{k+1}\sum_{j=1}^{m-1}Q(j)-\frac{m^{\alpha}}{k+1}\sum_{j=m}^{\infty}q(j).\end{aligned}\tag{3.5.139}$$

Now it is easy to see that the hypothesis of Theorem 3.5.20 is fulfilled. □

Next we present the following criterion, which completes in a certain sense the previous statement.

Theorem 3.5.26. *Suppose that condition (3.5.28) holds. If*

$$\limsup_{k\to\infty}\left[Q(k)+H(k)\right]>1,\tag{3.5.140}$$

then equation (3.2.11) is oscillatory.

Proof. Suppose that $x = \{x(k)\}$ is a nonoscillatory solution of equation (3.2.11), say, $x(k) > 0$ for $k \geq m_1$ for some $m_1 \in \mathbb{N}$. By Lemma 3.5.9 there exists $m_2 \geq m_1$ such that $\Delta x(k) > 0$ for $k \geq m_2$. We can employ the substitution $w(k) = \Psi(\Delta x(k)/x(k)) > 0$ for $k \geq m_2$, and then w satisfies the equation

$$\Delta w(k)+q(k)+\Phi(k)=0.\tag{3.5.141}$$

Let $\varepsilon > 0$ be given and $\varepsilon_1 > 0$ be such that

$$\alpha\left(1-\varepsilon_1\right)^{-1/(\beta-1)}-(\alpha-1)\left(1-\varepsilon_1\right)^{1/(\beta-1)}\leq 1+\varepsilon.\tag{3.5.142}$$

Let $m_3 = m_3(\varepsilon) \in \mathbb{N}$ be such that

$$\frac{1}{\left[1+|w(k)|^{1/(\alpha-1)}\right]^{\alpha-1}}\geq 1-\varepsilon_1\quad\text{for } k\geq m_3.\tag{3.5.143}$$

Set $m = \max\{m_2, m_3\}$. Multiplying both sides of (3.5.141) by k^{α} and summing from m to k, we obtain

$$\sum_{j=m}^{k}j^{\alpha}\Delta w(j)=-\sum_{j=m}^{k}j^{\alpha}q(j)-\sum_{j=m}^{k}j^{\alpha}\Phi(j).\tag{3.5.144}$$

Using summation by parts, we have

$$\sum_{j=m}^{k} j^{\alpha}\Delta w(j) = k^{\alpha}w(k+1) - (m-1)^{\alpha}w(m) - \sum_{j=m}^{k} \left(\Delta(j-1)^{\alpha}\right)w(j). \tag{3.5.145}$$

Now, by the Lagrange mean value theorem, we obtain for $k \geq m$,

$$\begin{aligned} k^{\alpha}w(k+1) &- (m-1)^{\alpha}w(m) + \sum_{j=m}^{k} j^{\alpha}q(j) \\ &= \sum_{j=m}^{k} \left[\left(\Delta(j-1)^{\alpha}\right)w(j) - j^{\alpha}\Phi(j)\right] \\ &= \sum_{j=m}^{k} \left[\alpha\eta^{\alpha-1}(j)w(j) - j^{\alpha}\Phi(j)\right] \\ &\leq \sum_{j=m}^{k} \left[\alpha j^{\alpha-1}w(j) - j^{\alpha}\frac{(\alpha-1)w^{\beta}(j)\xi^{\alpha-2}(j)}{\left[1+\Psi^{-1}(w(j))\right]^{\alpha-1}}\right] \\ &\leq \sum_{j=m}^{k} \left[\alpha j^{\alpha-1}w(j) - (\alpha-1)\left(j^{\alpha-1}w(j)\right)^{\beta}\left(1-\varepsilon_1\right)\right], \end{aligned} \tag{3.5.146}$$

where $1 \leq \xi(k) \leq 1 + w^{\beta-1}(k)$ and $k-1 \leq \eta(k) \leq k$. The summand of the last line of (3.5.146) can be rewritten as $f(x) = \alpha x - (\alpha-1)(1-\varepsilon_1)x^{\beta}$, where $x = j^{\alpha-1}w(j) > 0$. Now it is easy to see that

$$f(x) \leq \alpha\left(1-\varepsilon_1\right)^{-1/(\beta-1)} - (\alpha-1)\left(1-\varepsilon_1\right)^{1/(\beta-1)} \leq 1+\varepsilon. \tag{3.5.147}$$

Hence, from (3.5.146) we obtain

$$\begin{aligned} k^{\alpha}w(k+1) &+ \sum_{j=1}^{k} j^{\alpha}q(j) \\ &\leq (1+\varepsilon)(k+1-m) + (m-1)^{\alpha}w(m) + \sum_{j=1}^{m-1} j^{\alpha}q(j). \end{aligned} \tag{3.5.148}$$

Dividing (3.5.148) by $(k+1)$ and using (3.5.106), we get

$$\frac{k^{\alpha}}{k+1}\sum_{j=k+1}^{\infty} q(j) + H(k) \leq 1 + \varepsilon + \frac{1}{k+1}\left[-(1+\varepsilon)m + (m-1)^{\alpha}w(m) + \sum_{j=1}^{m-1} j^{\alpha}q(j)\right]. \tag{3.5.149}$$

Thus we have $(1-\varepsilon)Q(k)+H(k) \leq 1+2\varepsilon$ for all large k, which contradicts condition (3.5.140) since $\varepsilon > 0$ is arbitrary. □

Let

$$0 < \lambda < \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.5.150}$$

and denote by $\gamma_{\min}(\lambda)$ and $\gamma_{\max}(\lambda)$ the smallest and the greatest root, respectively, of the equation

$$|x|^{\beta} - x + \lambda = 0. \tag{3.5.151}$$

To prove the next oscillation criteria we need the following two lemmas.

Lemma 3.5.27. *Suppose that equation (3.2.11) is nonoscillatory,*

$$0 < Q_* \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \tag{3.5.152}$$

and x is a solution of equation (3.2.11) such that $x(k)x(k+1) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Then

$$\liminf_{k\to\infty} \frac{k^{\alpha-1}\Psi(\Delta x(k))}{\Psi(x(k))} \geq \gamma_{\min}(Q_*). \tag{3.5.153}$$

Proof. Let equation (3.2.11) be nonoscillatory. Then there exists $m \in \mathbb{N}$ such that $x(k)x(k+1) > 0$ for $k \geq m$. Set $A = \liminf_{k\to\infty} k^{\alpha-1}w(k)$, where as before $w(k) = \Psi(\Delta x(k)/x(k))$. If $A = \infty$, then there is nothing to prove. So, let $A < \infty$. If $Q_* = 0$, then (3.5.153) is trivial in view of (3.5.106). Thus, suppose $Q_* > 0$. For arbitrary $\varepsilon \in (0, Q_*)$ there exists $k[\varepsilon] > m$ such that

$$Q(k) > Q_* - \varepsilon \quad \text{for } k \geq k[\varepsilon]. \tag{3.5.154}$$

From (3.5.106) and (3.5.154) we have $k^{\alpha-1}w(k) \geq Q_* - \varepsilon$ for $k \geq k[\varepsilon]$. Hence $A \geq Q_*$ for $k \geq k[\varepsilon]$. Now we choose $k_1[\varepsilon] > k[\varepsilon]$ such that $k^{\alpha-1}w(k) \geq A - \varepsilon$ and

$$\frac{\xi^{\alpha-1}(k)}{\left[1 + w^{1/(1+\alpha)}(k)\right]^{\alpha-1}} \geq 1 - \varepsilon \quad \text{for } k \geq k_1[\varepsilon], \tag{3.5.155}$$

where $1 \le \xi(k) \le 1 + w^{1/(1+\alpha)}(k)$. Taking into account inequalities (3.5.154) and (3.5.155) from (3.5.106), we get

$$\begin{aligned} k^{\alpha-1} &\ge Q_* - \varepsilon + k^{\alpha-1}\sum_{j=k}^{\infty}\Phi(j) \\ &= Q_* - \varepsilon + \sum_{j=k}^{\infty}\frac{(\alpha-1)w^{\beta}(j)\xi^{\alpha-2}(j)}{\left[1+\Psi^{-1}\big(w(j)\big)\right]^{\alpha-1}} \\ &\ge Q_* - \varepsilon + (1-\varepsilon)k^{\alpha-1}(\alpha-1)\sum_{j=k}^{\infty}w^{\beta}(j) \\ &\ge Q_* - \varepsilon + (1-\varepsilon)(\alpha-1)k^{\alpha-1}\sum_{j=k}^{\infty}j^{-\alpha}(A-\varepsilon)^{\beta} \end{aligned} \tag{3.5.156}$$

for $k \ge k_1[\varepsilon]$. It is easy to see that $-\Delta(k^{1-\alpha}) \le (\alpha-1)k^{-\alpha}$ for $\alpha > 1$. Hence we get

$$k^{\alpha-1}w(k) \ge Q_* - \varepsilon + (1+\varepsilon)(A-\varepsilon)^{\beta} \quad \text{for } k \ge k_1[\varepsilon]. \tag{3.5.157}$$

Therefore $A \ge Q_* - \varepsilon + (1-\varepsilon)(A-\varepsilon)^{\beta}$. Thus, we have $A^{\beta} - A + Q_* \le 0$ since $\varepsilon > 0$ is arbitrary, and consequently $A \ge \gamma_{\min}(Q_*)$. □

Lemma 3.5.28. *Assume that (3.5.28) holds and (3.2.11) is nonoscillatory. If*

$$0 \le H_* \le \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.5.158}$$

and x is a solution of equation (3.2.11) such that $x(k)x(k+1) > 0$ for $k \ge m$ for some $m \in \mathbb{N}$, then

$$\limsup_{k\to\infty}\frac{k^{\alpha-1}\Psi(\Delta x(k))}{\Psi(x(k))} \le \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right). \tag{3.5.159}$$

Proof. Let equation (3.2.11) be nonoscillatory. Then there exists $m \in \mathbb{N}$ such that $x(k)x(k+1) > 0$ for $k \ge m$. Set $B = \limsup_{k\to\infty} k^{\alpha-1}w(k)$, where as before $w(k) = \Psi(\Delta x(k)/x(k))$. If $B = -\infty$, then there is nothing to prove. The inequality (3.5.159) is also valid for $B \le 0$, since $\gamma_{\max}[H_*/(\alpha-1)] \ge 0$, and hence we can assume that $B > 0$. From (3.5.148) we have

$$\frac{k^{\alpha}}{k+1}w(k) \le 1 + \varepsilon - H(k) + K\big(k, k[\varepsilon]\big) \quad \text{for } k \ge k[\varepsilon], \tag{3.5.160}$$

where

$$K\big(k, k[\varepsilon]\big) = \frac{1}{k+1}\left[\big(k[\varepsilon]-1\big)^{\alpha}w\big(k[\varepsilon]\big) - (1+\varepsilon)k[\varepsilon] + \sum_{j=1}^{k[\varepsilon]-1} j^{\alpha}q(j)\right] \tag{3.5.161}$$

for a given $\varepsilon > 0$ and sufficiently large $k[\varepsilon]$. Hence $B \le 1 - H_*$. Thus, the estimate (3.5.159) holds for $H_* = 0$ and hence we proceed with $H_* > 0$. The inequality (3.5.146) for $k \ge k[\varepsilon]$ can be written as

$$\frac{k^\alpha}{k+1} w(k+1) \le -H(k) + K\big(k, k[\varepsilon]\big) + \frac{(1+\varepsilon)k[\varepsilon]}{k+1} + \frac{1}{k+1} \sum_{j=k[\varepsilon]}^{k} \Big[\alpha j^{\alpha-1} w(j) - (\alpha-1)(1-\varepsilon)\big(j^{\alpha-1} w(j)\big)^\beta\Big]. \tag{3.5.162}$$

We can suppose that $k[\varepsilon]$ is so large that

$$k^{\alpha-1} w(k) \le B + \varepsilon, \quad H(k) \ge H_* - \varepsilon \quad \text{for } k \ge k[\varepsilon]. \tag{3.5.163}$$

Hence we get

$$\frac{k}{k+1} w(k+1) \le -H(k) + K\big(k, k[\varepsilon]\big) + \frac{(1+\varepsilon)k[\varepsilon]}{k+1} + \frac{k - k[\varepsilon] + 1}{k+1} \big[\alpha(B+\varepsilon) - (\alpha-1)(1-\varepsilon)(B+\varepsilon)^\beta\big]. \tag{3.5.164}$$

The behavior of the function $\alpha x - (\alpha-1)(1-\varepsilon)|x|^\beta$ is similar to that of the function $\alpha x - (\alpha-1)|x|^\beta$ for small $\varepsilon > 0$. In particular, it is increasing for $x \in (-\infty, 1)$ and we have $B \in (0,1)$. Further,

$$B \le -H_* + \varepsilon + \alpha(B+\varepsilon) - (\alpha-1)(\varepsilon-1)(B+\varepsilon)^\beta. \tag{3.5.165}$$

Therefore, $(\alpha-1)B^\beta - (\alpha-1)B + H_* \le 0$, since $\varepsilon > 0$ is arbitrary. Consequently $B \le \gamma_{\max}[H_*/(\alpha-1)]$. ☐

Now we present the following results.

Theorem 3.5.29. *A sufficient condition for equation (3.2.11) to be oscillatory is that either*

$$0 \le Q_* \le \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \qquad H^* > 1 - \gamma_{\min}(Q_*) \tag{3.5.166}$$

or that condition (3.5.28) holds and

$$0 \le H_* \le \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \qquad Q^* > \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right). \tag{3.5.167}$$

Proof. Suppose that equation (3.2.11) is nonoscillatory and that $x = \{x(k)\}$ is a solution of (3.2.11). There exists $m \in \mathbb{N}$ such that $x(k)x(k+1) > 0$ for $k \geq m$. By Lemma 3.5.27 (in the case that (3.5.166) holds) and by Lemma 3.5.28 (in the case that (3.5.167) holds), for arbitrary $\varepsilon > 0$, there exists $k[\varepsilon] \in \mathbb{N}$ such that for $k \geq k[\varepsilon]$,

$$\frac{k^\alpha}{k+1}w(k) > \gamma_{\min}(Q_*) - \varepsilon, \qquad k^{\alpha-1}w(k) < \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon, \tag{3.5.168}$$

where $w(k) = \Psi(\Delta x(k)/x(k))$. Suppose that $k[\varepsilon]$ is so large that (3.5.148) with $k[\varepsilon]$ instead of m holds and $K(k, k[\varepsilon]) < \varepsilon$ for $k \geq k[\varepsilon]$, where $K(k, k[\varepsilon])$ is defined by (3.5.161).

Now, if (3.5.166) holds, then from (3.5.148) we get for $k \geq k[\varepsilon]$,

$$\begin{aligned} H(k) &\leq -\frac{k^\alpha}{k+1}w(k+1) + 1 + \varepsilon + K(k, k[\varepsilon]) \\ &\leq -\gamma_{\min}(Q_*) + 1 + 2\varepsilon + K(k, k[\varepsilon]) \\ &\leq -\gamma_{\min}(Q_*) + 1 + 3\varepsilon, \end{aligned} \tag{3.5.169}$$

which contradicts (3.5.166).

If (3.5.167) holds, then from (3.5.106) we have

$$k^{\alpha-1}w(k) = Q(k-1) + k^{\alpha-1}\sum_{j=k}^{\infty}\Phi(j). \tag{3.5.170}$$

Thus $Q(k) \leq \gamma_{\max}[H_*/(\alpha-1)] + \varepsilon$, which contradicts (3.5.167). This completes the proof. □

Theorem 3.5.30. *Suppose*

$$0 \leq Q_* \leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}, \qquad 0 \leq H_* \leq \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}. \tag{3.5.171}$$

Then a sufficient condition for equation (3.2.11) to be oscillatory is that either

$$H^* > H_* + \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) - \gamma_{\min}(Q_*) \tag{3.5.172}$$

or

$$Q^* > Q_* + \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) - \gamma_{\min}(Q_*). \tag{3.5.173}$$

PROOF. Let $x = \{x(k)\}$ be a nonoscillatory solution of equation (3.2.11). There exists $m \in \mathbb{N}$ such that $x(k)x(k+1) > 0$ for $k \geq m$. Note that the condition $Q_* \geq 0$ implies that $\sum_{j=m}^{k} q(j) \geq 0$ for all $k \geq m$. Denote

$$M = \min\left\{1 - \gamma_{\max}\left(\frac{H_*}{\alpha - 1}\right), \gamma_{\min}(Q_*)\right\}. \tag{3.5.174}$$

By Lemmas 3.5.27 and 3.5.28 for arbitrary $\varepsilon \in (1, M)$ there exists $k[\varepsilon] \geq m$ such that

$$\frac{k^\alpha}{\alpha+1} w(k) > \gamma_{\min}(Q_*) - \varepsilon, \qquad k^{\alpha-1} w(k) < \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon, \tag{3.5.175}$$

respectively, for $k \geq k[\varepsilon]$. Let $k[\varepsilon]$ be so large that (3.5.146) (with ε instead of ε_1 and $k[\varepsilon]$ instead of m) and (3.5.155) hold for $k \geq k[\varepsilon]$.

Assume that (3.5.172) is satisfied. We will proceed with $H_* > 0$, since for $H_* = 0$ the condition (3.5.172) is equivalent to condition (3.5.166). Using the first inequality in (3.5.175) and the fact that $\gamma_{\max}[H_*/(\alpha - 1)] + \varepsilon < 1$ from (3.5.146), we obtain

$$\begin{aligned} H(k) \leq{} & -\gamma_{\min}(Q_*) + \varepsilon + K(k, k[\varepsilon]) + \frac{(1+\varepsilon)k[\varepsilon]}{k+1} + \alpha\left[\gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon\right] \\ & - (\alpha - 1)\left[\gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon\right]^\beta (1 - \varepsilon), \end{aligned} \tag{3.5.176}$$

where $K(k, k[\varepsilon])$ is defined by (3.5.161). Hence

$$\begin{aligned} H^* &\leq -\gamma_{\min}(Q_*) + \alpha\gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) - (\alpha-1)\left[\gamma_{\max}\left(\frac{H_*}{\alpha-1}\right)\right]^\beta \\ &\quad + (\alpha - 1)\left\{\gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) - \left[\gamma_{\max}\left(\frac{H_*}{\alpha-1}\right)\right]^\beta\right\} \\ &= -\gamma_{\min}(Q_*) + H_* + \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right), \end{aligned} \tag{3.5.177}$$

which contradicts (3.5.172).

Suppose that condition (3.5.173) holds. Using the inequalities (3.5.175), the properties of the function Φ and the same estimates as those in the proof of Lemma 3.5.27, we have for $k \geq k[\varepsilon]$ from (3.5.106) that

$$\begin{aligned}
Q(k-1) &\leq \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon - k^{\alpha-1}\sum_{j=k}^{\infty}\Phi\big(w(j),1\big) \\
&\leq \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon - k^{\alpha-1}\sum_{j=k}^{\infty}\Phi\left(\frac{\gamma_{\min}(Q_*)}{j^{\alpha-1}} - 1\right) \\
&\leq \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon \\
&\quad - (1-\varepsilon)k^{\alpha-1}(\alpha-1)\big[\gamma_{\min}(Q_*) - \varepsilon\big]^{\beta}\sum_{j=k}^{\infty}\Delta\left(-\frac{1}{(\alpha-1)j^{\alpha-1}}\right) \\
&= \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) + \varepsilon - (1-\varepsilon)\big[\gamma_{\min}(Q_*) - \varepsilon\big]^{\beta}.
\end{aligned} \tag{3.5.178}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$Q^* \leq \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) - \big[\gamma_{\min}(Q_*)\big]^{\beta} = \gamma_{\max}\left(\frac{H_*}{\alpha-1}\right) - \gamma_{\min}(Q_*) + Q_*, \tag{3.5.179}$$

which contradicts (3.5.173). This completes the proof. □

The following examples show that for any pair of numbers (r,s) with $r \leq s$, there exists a sequence $\{q(k)\}$ such that (3.5.99) holds and $Q_* = r$, $Q^* = s$ (or also $H_* = r$, $H^* = s$). Therefore Theorems 3.5.29 and 3.5.30 are meaningful.

Example 3.5.31. Let $r,s \in \mathbb{R}$ be arbitrary with $r \leq s$. Set $a = (r+s)/2$, $b = (s-r)/2$, and

$$q(k) = \Delta h(k), \quad \text{where } h(k) = -\frac{a}{k^{\alpha-1}} + \frac{(-1)^k b}{k^{\alpha-1}}. \tag{3.5.180}$$

Then $\sum_{j=1}^{k-1} q(j) = h(k)$ and $\lim_{k\to\infty}\sum_{j=1}^{k} q(j) = 0$. Hence

$$Q(k) = (k+1)^{\alpha-1}\big(-h(k+1)\big) = a - b(-1)^{k+1}, \tag{3.5.181}$$

and thus $Q_* = a - b = r$ and $Q^* = a + b = s$.

Example 3.5.32. Let r, s, a, and b be as in Example 3.5.31 and $q(k) = \Delta h(k)$, where

$$h(k) = -\frac{a}{(\alpha-1)k^{\alpha-1}} + \frac{(-1)^k b}{k^{\alpha-1}} + \frac{[a/(\alpha-1)]+b}{k^{\alpha}}. \tag{3.5.182}$$

Then $\sum_{j=1}^{k-1} q(j) = h(k)$ and $\lim_{k\to\infty}\sum_{j=1}^{k} q(j) = 0$. Hence

$$\begin{aligned}
H(k) &= \frac{1}{k+1}\left[j^{\alpha}h(j)\right]\Big|_{j=1}^{j=k+1} - \frac{1}{k+1}\sum_{j=1}^{k}(\Delta j^{\alpha})h(j+1) \\
&= (k+1)^{\alpha-1}h(k+1) - \frac{\alpha}{k+1}\sum_{j=1}^{k}\xi^{\alpha-1}(j)h(j+1) \\
&= -\frac{a}{\alpha-1} + (-1)^{k+1}b + \frac{(a/(\alpha-1))+b}{k+1} + \frac{\alpha a}{(\alpha-1)(k+1)}\sum_{j=1}^{k}\left(\frac{\xi(j)}{j+1}\right)^{\alpha-1} \\
&\quad - \frac{\alpha b}{k+1}\sum_{j=1}^{k}(-1)^j\left(\frac{\xi(j)}{j+1}\right)^{\alpha-1} - \frac{\alpha\left[(a/(\alpha-1))+b\right]}{k+1}\sum_{j=1}^{k}\frac{\xi^{\alpha-1}(j)}{(j+1)^{\alpha}},
\end{aligned} \tag{3.5.183}$$

where $j \le \xi(j) \le j+1$. Now it is easy to see that $H_* = a-b = r$ and $H^* = a+b = s$.

3.5.3. Oscillation criteria based on the variational principle

In this subsection, we will consider equation (3.2.9) on $[m,\infty)$ with $c(k) > 0$ for $k \ge m$. We will present half-linear discrete versions of two well-known oscillation results. To prove them, we employ the variational principle.

First, we give the following auxiliary lemma, which is the second mean value theorem (MVT) of summation calculus.

Lemma 3.5.33 (second MVT). *Let $n \in \mathbb{N}$ and the sequence $a(k)$ be monotonic for $k \in [K+n-1, L+n-1]$. Then for any sequence $b(n)$ there exist $n_1, n_2 \in [K, L-1]$ such that*

$$\begin{aligned}
\sum_{j=K}^{L-1} a(n+j)b(j) &\le a(K+n-1)\sum_{i=K}^{n_1-1} b(i) + a(L+n-1)\sum_{i=n_1}^{L-1} b(i), \\
\sum_{j=K}^{L-1} a(n+j)b(j) &\ge a(K+n-1)\sum_{i=K}^{n_2-1} b(i) + a(L+n-1)\sum_{i=n_2}^{L-1} b(i).
\end{aligned} \tag{3.5.184}$$

Proof. Suppose that $a(k)$ is nondecreasing, that is, $\Delta a(k) \ge 0$ (if $\Delta a(k) \le 0$, we proceed in the same way), and denote $B(j) = \sum_{i=k}^{j-1} b(i)$. Let $n_1, n_2 \in [K, L-1]$ be such that $B(n_1) \le B(j) \le B(n_2)$ for $j = K, \dots, L-1$. Using summation by parts, we get

$$\begin{aligned}\sum_{j=K}^{L-1} \Delta a(j+n-1)B(j) &= a(L+n-1)B(L) - a(K+n-1)B(K) - \sum_{j=K}^{L-1} a(j+n)b(j)\\ &= a(L+n-1)\sum_{i=K}^{L-1} b(i) - \sum_{j=K}^{L-1} a(j+n)b(j),\end{aligned} \tag{3.5.185}$$

and hence

$$\begin{aligned}B(n_1)\sum_{j=K}^{L-1}\Delta a(j+n-1) &= \sum_{i=K}^{n_1-1} b(i)\big[a(L+n-1)-a(K+n-1)\big]\\ &\le \sum_{j=K}^{L-1}\Delta a(j+n-1)B(j)\\ &= a(L+n-1)\sum_{i=K}^{L-1} b(i) - \sum_{j=K}^{L-1} a(j+n)b(j)\\ &\le B(n_2)\sum_{j=K}^{L-1}\Delta a(j+n-1)\\ &= \big[a(L+n-1)-a(K+n-1)\big]\sum_{i=K}^{n_2-1} b(i),\end{aligned} \tag{3.5.186}$$

which implies the required statement. □

Theorem 3.5.34 (Leighton-Wintner's oscillation criterion). *Suppose that condition (3.5.29) holds and*

$$\sum_{j=m}^{\infty} q(j) = \infty. \tag{3.5.187}$$

Then equation (3.2.9) is oscillatory.

Remark 3.5.35. From Lemma 3.4.11 it is clear that to prove an oscillation result for equation (3.2.9) it suffices to find for any $m \in \mathbb{N}$ a (nontrivial) admissible sequence $\xi \in U(m)$ for which $\mathcal{F}(\xi) \le 0$.

Proof of Theorem 3.5.34. According to Remark 3.5.35 it is sufficient to find for any $K \ge m$ a sequence x satisfying $x(k) = 0$ for $k \le K$ and $k \ge N+1$, where $N > K$ (then x is admissible) such that

$$\mathcal{F}(x;K,N) = \sum_{k=K}^{N}\Big[c(k)\,|\,\Delta x(k)\,|^{\alpha} - q(k)\,|\,x(k+1)\,|^{\alpha}\Big] \le 0. \tag{3.5.188}$$

Let $K < L < M < N$. Define the sequence $x(k)$ by

$$x(k) = \begin{cases} 0 & \text{for } k = m,\dots,K, \\ \left(\sum_{j=K}^{k-1} c^{1-\beta}(j)\right)\left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{-1} & \text{for } k = K+1,\dots,L+1, \\ 1 & \text{for } k = L+1,\dots,M, \\ \left(\sum_{j=k}^{N} c^{1-\beta}(j)\right)\left(\sum_{j=M}^{N} c^{1-\beta}(j)\right)^{-1} & \text{for } k = M,\dots,N, \\ 0 & \text{for } k \geq N+1. \end{cases} \tag{3.5.189}$$

Using summation by parts, we obtain

$$\begin{aligned} \mathcal{F}(x;K,N) &= \sum_{k=K}^{N}\left[c(k)|\Delta x(k)|^{\alpha} - q(k)|x(k+1)|^{\alpha}\right] \\ &= \sum_{k=K}^{L-1} c(k)|\Delta x(k)|^{\alpha} + c(L)|\Delta x(L)|^{\alpha} + \sum_{k=L+1}^{M-1} c(k)|\Delta x(k)|^{\alpha} \\ &\quad + \sum_{k=M}^{N} c(k)|\Delta x(k)|^{\alpha} - \sum_{k=K}^{N} q(k)|x(k+1)|^{\alpha} \\ &= \left[x(k)c(k)\Psi(\Delta x(k))\right]\Big|_{k=K}^{L} - \sum_{j=K}^{L-1} x(k+1)\Delta\big(c(k)\Psi(\Delta x(k))\big) \\ &\quad + c(L)\left(c^{1-\beta}(L)\right)^{\alpha}\left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{-\alpha} + \left[x(k)c(k)\Psi(\Delta x(k))\right]\Big|_{k=M}^{N+1} \\ &\quad - \sum_{k=M}^{N} x(k+1)\Delta\big(c(k)\Psi(\Delta x(k))\big) - \sum_{k=K}^{N} q(k)|x(k+1)|^{\alpha} \\ &= x(L)c(L)\Psi(\Delta x(L)) + c^{1-\beta}(L)\left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{-\alpha} \\ &\quad + x(M)c(M)\Psi(\Delta x(M)) - \sum_{k=K}^{N} q(k)|x(k+1)|^{\alpha} \\ &= \left(\sum_{k=K}^{L} c^{1-\beta}(k)\right)^{1-\alpha} + \left(\sum_{k=M}^{N} c^{1-\beta}(k)\right)^{1-\alpha} - \sum_{k=K}^{L} q(k)|x(k+1)|^{\alpha} \\ &\quad - \sum_{k=L+1}^{M-1} q(k) - \sum_{k=M}^{N} q(k)|x(k+1)|^{\alpha}. \end{aligned} \tag{3.5.190}$$

Further, the sequence x is strictly monotonic on $[K, L+1]$ and $[M, N+1]$ since

$$\begin{aligned}\Delta x(k) &= c^{1-\beta}(k)\left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{-1} > 0 \quad \text{for } k \in [K, L],\\ \Delta x(k) &= -c^{1-\beta}(k)\left(\sum_{j=M}^{N} c^{1-\beta}(j)\right)^{-1} < 0 \quad \text{for } k \in [M, N].\end{aligned} \tag{3.5.191}$$

Hence, by Lemma 3.5.33 there exists $N_1 \in [K, L]$ such that

$$\sum_{k=K}^{L} q(k)\left|x(k+1)\right|^{\alpha} \geq x(K)\sum_{k=K}^{N_1-1} q(k) + x(L+1)\sum_{k=N_1}^{L} q(k) = \sum_{k=N_1}^{L} q(k), \tag{3.5.192}$$

and similarly there exists $N_2 \in [M, N]$ for which

$$\sum_{k=M}^{N} q(k)\left|x(k+1)\right|^{\alpha} \geq x(M)\sum_{k=M}^{N_2-1} q(k) + x(N+1)\sum_{k=N_2}^{N} q(k) = \sum_{k=M}^{N_2-1} q(k). \tag{3.5.193}$$

Using these estimates, we have

$$\mathcal{F}(x; K, N) \leq \left(\sum_{k=K}^{L} c^{1-\beta}(k)\right)^{1-\alpha} + \left(\sum_{k=M}^{N} c^{1-\beta}(k)\right)^{1-\alpha} - \sum_{k=N_1}^{N_2-1} q(k). \tag{3.5.194}$$

Now denote $A = (\sum_{k=K}^{L} c^{1-\beta}(k))^{1-\alpha}$ and let $\varepsilon > 0$ be arbitrary. According to (3.5.187), the integer M can be chosen in such a way that $\sum_{j=N_1}^{k} q(j) \geq A + \varepsilon$ whenever $k > M$. Since condition (3.5.29) holds, $\sum_{k=M}^{N} c^{1-\beta}(k) \leq \varepsilon$ if N is sufficiently large. Summarizing the above estimates, if M and N are sufficiently large, then we have

$$\mathcal{F}(x; K, N) \leq A + \varepsilon - (A + \varepsilon) = 0, \tag{3.5.195}$$

which completes the proof. □

In the case when

$$\sum_{j=m}^{\infty} q(j) \quad \text{is convergent}, \tag{3.5.196}$$

we can use the following criterion.

Theorem 3.5.36 (Hinton-Lewis oscillation criterion). *Suppose that condition (3.5.29) holds and*

$$\lim_{k\to\infty}\left(\sum_{j=m}^{k} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=k}^{\infty} q(j)\right) > 1. \tag{3.5.197}$$

Then equation (3.2.9) is oscillatory.

Proof. Let the sequence x be the same as in the proof of Theorem 3.5.34. Hence we have

$$\begin{aligned}
\mathcal{F}(x;K,N) &\le \left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{1-\alpha} + \left(\sum_{j=M}^{N} c^{1-\beta}(j)\right)^{1-\alpha} - \sum_{j=N_1}^{N_2-1} q(j)\\
&= \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{1-\alpha}\Bigg[\left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{1-\alpha}\\
&\qquad - \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=N_1}^{N_2-1} q(j)\right)\\
&\qquad + \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=M}^{N} c^{1-\beta}(j)\right)^{1-\alpha}\Bigg],
\end{aligned} \tag{3.5.198}$$

where $m \le K < L < M < N$, $N_1 \in [K,L]$, and $N_2 \in [M,N]$.

Now let $\varepsilon > 0$ be such that the limit in (3.5.197) is greater than or equal to $1+4\varepsilon$. According to condition (3.5.197), K may be chosen in such a way that

$$\left(\sum_{j=m}^{K} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=k}^{\infty} q(j)\right) \ge 1+4\varepsilon \tag{3.5.199}$$

for $k > K$. Obviously there exists $L > K$ such that

$$\left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{1-\alpha} \le 1+\varepsilon. \tag{3.5.200}$$

In view of the fact that (3.5.199) holds, there exists $M > L$ such that

$$\left(\sum_{j=m}^{k} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=k}^{\ell} q(j)\right) \ge 1+2\varepsilon \tag{3.5.201}$$

for $\ell \geq M$. Finally, since $\sum_{j=m}^{\infty} c(j) = \infty$ holds, we have

$$\left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=M}^{N} c^{1-\beta}(j)\right)^{1-\alpha} \leq \varepsilon \tag{3.5.202}$$

if $N > M$ is sufficiently large.

Using these estimates and the fact that $\sum_{j=m}^{k} c^{1-\beta}(j)$ is positive and increasing with respect to $k \geq m$ and $\sum_{j=N_1}^{N_2} q(j)$ is positive if N_1 and N_2 are sufficiently large, we get

$$\begin{aligned}
\mathcal{F}(k) \leq \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{1-\alpha} \Bigg[& \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=K}^{L} c^{1-\beta}(j)\right)^{1-\alpha} \\
& - \left(\sum_{j=m}^{N_1} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=N_1}^{N_2+1} q(j)\right) \\
& + \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=M}^{N} c^{1-\beta}(j)\right)^{1-\alpha} \Bigg] \\
\leq \left(\sum_{j=m}^{L} c^{1-\beta}(j)\right)^{1-\alpha} & [1 + \varepsilon - 1 - 2\varepsilon + \varepsilon] \\
= 0, &
\end{aligned} \tag{3.5.203}$$

which yields the desired result. □

Remark 3.5.37. (i) We note that Theorem 1.8.35 of Leighton and Wintner and Theorem 3.5.34 with $\alpha = 2$ are similar.

(ii) In the particular case of Theorem 3.5.14 when $t = 0$, that is, under the assumptions of Lemma 3.5.11, the condition

$$\limsup_{k\to\infty} \left(\sum_{j=m}^{k-1} c^{1-\beta}(j)\right)^{\alpha-1} \left(\sum_{j=k}^{\infty} q(j)\right) > 1 \tag{3.5.204}$$

guarantees oscillation of equation (3.2.9). Note that using the variational technique, this criterion is similar to Theorem 3.5.36 without the condition $q(k) \geq 0$, however, with lim inf instead of lim sup in (3.5.204).

In the case when

$$c(k) > 0, \qquad \sum^{\infty} c^{1-\beta}(j) < \infty, \tag{3.5.205}$$

we will prove the following result by employing the so-called *reciprocity principle*. Here we present the idea of this reciprocity principle. We suppose that $c(k) > 0$

and $q(k) > 0$ for $k \geq m$. If we denote $y(k) = c(k)\Psi(x(k))$, where x is a solution of equation (3.2.9), then $y(k)$ satisfies the reciprocal equation

$$\Delta\big(q^{1-\beta}(k)\Psi_\beta(\Delta y(k))\big) + c^{1-\beta}(k+1)\Psi_\beta\big(y(k+1)\big) = 0, \tag{3.5.206}$$

where $\Psi_\beta(x) = |x|^{\beta-2}x$ and β is the conjugate number of α, that is, $(1/\alpha)+(1/\beta)=1$. Conversely, if $x(k) = q^{1-\beta}(k-1)\Psi_\beta(\Delta y(k-1))$, where $y(k)$ is a solution of equation (3.5.206), then $x(k)$ solves the original equation (3.2.9). Since the discrete version of Rolle's mean value theorem holds, we have the following equivalence: equation (3.2.9) is oscillatory (nonoscillatory) if and only if equation (3.5.206) is oscillatory (nonoscillatory).

Indeed, if $x(k)$ is an oscillatory solution of equation (3.2.9), then its difference and hence also $y(k) = c(k)\Psi(\Delta x(k))$ oscillate. Conversely, if $y(k)$ oscillates, then $x(k) = q^{1-\beta}(k)\Psi_\beta(\Delta y(k-1))$ oscillates as well.

Theorem 3.5.38. *Suppose $q(k) > 0$ for $k \geq m$ and condition (3.5.205) holds. If*

$$\lim_{k\to\infty}\left(\sum_{j=k+1}^{\infty} c^{1-\beta}(j)\right)^{\alpha-1}\left(\sum_{j=m}^{k} q(j)\right) > 1, \tag{3.5.207}$$

then equation (3.2.9) is oscillatory.

Proof. We will use the reciprocity principle. From condition (3.5.207),

$$\sum_{j=m}^{\infty} q^{(1-\beta)(1-\alpha)}(j) = \sum_{j=m}^{\infty} q(j) = \infty. \tag{3.5.208}$$

Therefore, by Theorem 3.5.36, equation (3.2.9) is oscillatory if

$$\lim_{k\to\infty}\left(\sum_{j=m}^{k} q^{(1-\beta)(1-\alpha)}(j)\right)^{\beta-1}\left(\sum_{j=k}^{\infty} c^{1-\beta}(j+1)\right) > 1, \tag{3.5.209}$$

which is equivalent to (3.5.207). Thus equation (3.2.9) is also oscillatory. □

3.6. Comparison theorems

In this section we will discuss some comparison theorems for equation (3.2.9) of types other than the classical Sturm type, and also comparison theorems for generalized Riccati difference equations. For this we need the following notation and results from algebra.

3.6.1. Some results from algebra

The space ℓ^∞ is the set of all real sequences defined on $\mathbb{N}$, where any individual sequence is bounded with respect to the usual supremum norm. It is well known that under the supremum norm ℓ^∞ is a Banach space. A subset S of a Banach space B is *relatively compact* if every sequence in S has a subsequence converging to an element of B. An ε-net for S is a set of elements of B such that each $x \in S$ is within a distance ε of some number of the net. A finite ε-net is an ε-net consisting of a finite number of elements.

Theorem 3.6.1. *A subset S of a Banach space B is relatively compact if and only if for each $\varepsilon > 0$ it has a finite ε-net.*

Definition 3.6.2. A set S of sequences in ℓ^∞ is *uniformly Cauchy* (or equi-Cauchy) if for every $\varepsilon > 0$ there exists an integer N such that $|x(i) - x(j)| < \varepsilon$ whenever $i, j > N$ for any $x = \{x(k)\}$ in S.

The following result can be considered as a discrete analogue of the Arzelà-Ascoli theorem.

Theorem 3.6.3 (Arzelá-Ascoli). *A bounded uniformly Cauchy subset S of ℓ^∞ is relatively compact.*

Theorem 3.6.4 (Schauder's fixed point theorem). *Let K be a closed convex subset of B. Suppose T is a mapping such that $T(K)$ is a subset of K. If T is continuous and $T(K)$ is relatively compact, then T has a fixed point.*

Next, we can formulate a fixed point theorem that is applicable for our setting in difference equations.

Theorem 3.6.5. *Let K be a closed, bounded, and convex subset of ℓ^∞. Suppose T is a continuous map such that $T(K)$ is contained in K, and suppose further that $T(K)$ is uniformly Cauchy. Then T has a fixed point in K.*

3.6.2. Some comparison theorems

Along with equation (3.2.9) consider the equation

$$\Delta\big(c_1(k)\Psi_{\bar{\alpha}}(\Delta y(k))\big) + q_1(k)\Psi_{\bar{\alpha}}\big(y(k+1)\big) = 0, \tag{3.6.1}$$

where $c_1(k) > 0$ and $\Psi_{\bar{\alpha}}(x) = |x|^{\bar{\alpha}-1}\operatorname{sgn} x$ with $\bar{\alpha} > 1$.

Now we present the following comparison result.

Theorem 3.6.6. *Assume that the sequences $\{q(k)\}$ and $\{q_1(k)\}$ satisfy condition (3.5.28). Let the series $\sum_{j=m}^{\infty} q(j)$ and $\sum_{j=m}^{\infty} q_1(k)$ be convergent and*

$$\sum_{j=k}^{\infty} q(j) \le \sum_{j=k}^{\infty} q_1(j) \quad \text{for all large } k \in \mathbb{N}. \tag{3.6.2}$$

Further, suppose that $0 < c_1(k) \le c(k)$, $\sum_{j=1}^{\infty} c_1^{1-\beta}(k) = \infty$, and $1 < \alpha \le \bar{\alpha}$. If equation (3.6.1) is nonoscillatory, then so is equation (3.2.9).

Proof. By Lemma 3.5.10, nonoscillation of equation (3.6.1) implies the existence of $m_1 \in \mathbb{N}$ such that

$$z(k) \geq \sum_{j=k}^{\infty} q_1(j) + \sum_{j=k}^{\infty} \Phi\big(z(j), c_1(j), \bar{\alpha}\big) =: Z(k) \tag{3.6.3}$$

for $k \geq m_1$ (clearly with $z(k) + c_1(k) > 0$). Let $m_2 \in \mathbb{N}$ be such that (3.6.2) holds and $\sum_{j=k}^{\infty} q(j) \geq 0$ for $k \geq m_2$. Set $m = \max\{m_1, m_2\}$ and define the set Ω and the mapping T by

$$\begin{gathered} \Omega = \big\{w \in \ell^{\infty} : 0 \leq w(k) \leq Z(k) \text{ for } k \geq m\big\}, \\ (Tw)(k) = \sum_{j=k}^{\infty} q(j) + \sum_{j=k}^{\infty} \Phi\big(w(j), c(j), \alpha\big) \quad \text{for } k \geq m,\ w \in \Omega. \end{gathered} \tag{3.6.4}$$

We will show that T has a fixed point in Ω. We must verify that

(i) Ω is a bounded, closed, and convex subset of ℓ^{∞},
(ii) T maps Ω into itself,
(iii) $T\Omega$ is relatively compact,
(iv) T is continuous.

Concerning (i), Ω is clearly bounded and convex. Let $y(t) = \{y(k,t)\}$, $t \in \mathbb{N}$, be any sequence in Ω such that $y(t)$ tends to y as $t \to \infty$ (in the supnorm). From our assumptions, for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $\sup_{k \geq m} |y(k,t) - y(k)| < \varepsilon$ for all $t \geq n$. Thus, for any fixed k, we must have $\lim_{t\to\infty} y(k,t) = y(k)$. Since $0 \leq y(k,t) \leq z(k)$ for all t, we have $0 \leq y(k) \leq z(k)$. Since $k \geq m$ is arbitrary, $y \in \Omega$.

Now we show (ii). Suppose that $w \in \Omega$ and define $y(k) = (Tw)(k)$ for $k \geq m$. Obviously, $y(k) \geq 0$ for $k \geq m$. We must show that $y(k) \leq Z(k)$ for $k \geq m$. We have

$$\begin{aligned} y(k) &= \sum_{j=k}^{\infty} q(j) + \sum_{j=k}^{\infty} \Phi\big(w(j), c(j), \alpha\big) \\ &\leq \sum_{j=k}^{\infty} q_1(j) + \sum_{j=k}^{\infty} \Phi\big(w(j), c_1(j), \alpha\big) \\ &\leq \sum_{j=k}^{\infty} q_1(j) + \sum_{j=k}^{\infty} \Phi\big(w(j), c_1(j), \bar{\alpha}\big) \\ &\leq \sum_{j=k}^{\infty} q_1(j) + \sum_{j=k}^{\infty} \Phi\big(z(j), c_1(j), \bar{\alpha}\big) \\ &= Z(k), \end{aligned} \tag{3.6.5}$$

by the assumptions of the theorem and by Lemma 3.2.6. Hence $T\Omega \subset \Omega$.

Now we address (iii). According to Theorem 3.6.3 it suffices to show that $T\Omega$ is uniformly Cauchy. Let $\varepsilon > 0$ be given. We show that there exists $N \in \mathbb{N}$ such that for any $k, \ell > N$, $|(Ty)(k) - (Ty)(\ell)| < \varepsilon$ for any $y \in \Omega$. Without loss of generality, suppose $k < \ell$. Then we have

$$\begin{aligned}
\left|(Ty)(k) - (Ty)(\ell)\right| &= \left| \sum_{j=k}^{\ell-1} q(j) + \sum_{j=k}^{\ell-1} \Phi(y(j), c(j), \alpha) \right| \\
&= \sum_{j=k}^{\ell-1} q(j) + \sum_{j=k}^{\ell-1} \Phi(y(j), c(j), \alpha)
\end{aligned} \tag{3.6.6}$$

for large k by Lemma 1.11.1. Taking into account the properties of $\Phi(y(k), c(k), \alpha)$ and $q(k)$, for any $\varepsilon > 0$ one can find $N \in \mathbb{N}$ such that

$$\sum_{j=k}^{\ell-1} q(j) < \frac{\varepsilon}{2}, \quad \sum_{j=k}^{\ell-1} \Phi(y(j), c(j), \alpha) < \frac{\varepsilon}{2} \quad \text{for } \ell > k > N. \tag{3.6.7}$$

From this and the above, $|(Ty)(k) - (Ty)(\ell)| < \varepsilon$. Hence $T\Omega$ is relatively compact.

Finally we prove (iv). Let $y(t) = \{y(k,t)\}$, $k \geq m$, be a sequence in Ω converging to y. We must show that $Ty(t)$ converges to Ty. Clearly, $Ty(t) \in T\Omega \subset \Omega$ for any t and also $Ty \in T\Omega \subset \Omega$. For any $\varepsilon > 0$ one can choose $M \geq m$ such that $(Ty(t))(k) < \varepsilon/2$ and $(Ty)(k) < \varepsilon/2$ for $k \geq M$ and for each $t \in \mathbb{N}$. Define

$$(T_1 y)(k, \ell) = \begin{cases} \displaystyle\sum_{j=k}^{\ell} q(j) + \sum_{j=k}^{\ell} \Phi(y(j), c(j), \alpha) & \text{for } \ell \geq k \geq m, \\ 0 & \text{for } \ell < k. \end{cases} \tag{3.6.8}$$

The mapping T_1 is obviously continuous. Therefore, for given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|(T_1 y(t))(k, \ell) - (T_1 y)(k, \ell)| < \varepsilon/2$ for $t \geq N$ and $k \geq m$. Now having chosen such M and N as above, the following estimate holds for any $k \geq m$:

$$\begin{aligned}
&\left|(Ty(t))(k) - (Ty)(k)\right| \\
&\quad = \left|(T_1 y(t))(k, M) + (Ty(t))(M+1) - (T_1 y)(k, M) - (Ty)(M+1)\right| \\
&\quad \leq \left|(T_1 y(t))(k, M) - (T_1 y)(k, M)\right| + \left|(Ty(t))(M+1) - (Ty)(M+1)\right| \\
&\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{3.6.9}$$

Since $k \geq m$ is arbitrary, we have that $Ty(t)$ converges to Ty.

Therefore, it follows from the Schauder fixed point theorem, Theorem 3.6.4, that there exists an element $w \in \Omega$ such that $w = Tw$. In view of the definition of T this (positive) sequence w satisfies the equation

$$w(k) = \sum_{j=k}^{\infty} q(j) + \sum_{j=k}^{\infty} \Phi(w(j), c(j), \alpha) \quad \text{for } k \geq m, \tag{3.6.10}$$

and hence also equation (3.2.17). Consequently, the sequence x given by

$$\begin{aligned} x(m) &= m_0 \neq 0, \\ x(k+1) &= \left(1 + \left(\frac{w(k)}{c(k)}\right)^{\beta-1}\right)x(k) \quad \text{for } k \geq m \end{aligned} \tag{3.6.11}$$

is a nonoscillatory solution of

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi\big(x(k+1)\big) = 0, \tag{3.6.12}$$

and hence this equation is nonoscillatory. □

Remark 3.6.7. A closer examination of the proof of Theorem 3.6.6 shows that the necessary condition for nonoscillation of equation (3.2.9) in Lemma 3.5.10 is also sufficient.

In the following result we compare nonoscillation of equation (3.2.9) with equations of the form

$$\Delta\big(c_1(k)\Psi(\Delta y(k))\big) + \lambda q(k)\Psi\big(y(k+1)\big) = 0, \tag{3.6.13}$$

where $c_1(k) > 0$ and $\lambda \in \mathbb{R}$.

Theorem 3.6.8. *Suppose that* $0 < c(k) \leq c_1(k)$ *for* $k \geq m \in \mathbb{N}$ *and let* $\lambda \in [0,1]$. *If equation (3.2.9) is nonoscillatory, then so is equation (3.6.13).*

Proof. Suppose that equation (3.2.9) is nonoscillatory. Let x be its solution such that $x(k) > 0$, $k \geq m$ for some $m \in \mathbb{N}$. Set $y(k) = x^\nu(k)$, where $\nu = \Psi^{-1}(\lambda)$. Then $\Delta y(k) \leq \nu x^{\nu-1}(k)\Delta x(k)$ and $\Delta y(k) \geq \nu x^{\nu-1}(k+1)\Delta x(k)$ by the Lagrange mean value theorem, since $\Delta x(k) \geq 0$ if and only if $x^{\nu-1}(k+1) \leq x^{\nu-1}(k)$ and $\Delta x(k) \leq 0$ if and only if $x^{\nu-1}(k+1) \geq x^{\nu-1}(k)$. Further,

$$\begin{aligned} &\Delta\big(c(k)\Psi(\Delta y(k))\big) \\ &\quad = c(k+1)\Psi\big(\Delta y(k+1)\big) - c(k)\Psi\big(\Delta y(k)\big) \\ &\quad \leq c(k+1)\Psi\big(\nu x^{\nu-1}(k+1)\Delta x(k+1)\big) - c(k)\Psi\big(\nu x^{\nu-1}(k+1)\Delta x(k)\big) \\ &\quad = \Psi\big(\nu x^{\nu-1}(k+1)\big)\Delta\big(c(k)\Psi(\Delta x(k))\big) \\ &\quad = -\Psi\big(\nu x^{\nu-1}(k+1)\big)q(k)\Psi\big(x(k+1)\big) \\ &\quad = -\Psi(\nu)q(k)\Psi\big(x^\nu(k+1)\big) \\ &\quad = -\lambda q(k)\Psi\big(y(k+1)\big). \end{aligned} \tag{3.6.14}$$

This implies that equation (3.6.13) is nonoscillatory by Lemma 3.4.1 and Theorem 3.3.5. □

If equation (3.6.13) is replaced by the more general equation

$$\Delta\big(c_1(k)\Psi(\Delta y(k))\big) + p(k)q_1(k)\Psi\big(y(k+1)\big) = 0, \tag{3.6.15}$$

where $\{p(k)\}$ is a sequence of positive real numbers, then we can present the following result.

Theorem 3.6.9. *Assume that* $0 < c(k) \le c_1(k)$, $q_1(k) \le q(k)$, $0 < p(k) \le 1$, *and* $\Delta p(k) \le 0$ *for* $k \ge m$. *Further, suppose that* $\sum^\infty c^{1-\beta}(k) = \infty$ *and (3.5.28) holds. Then nonoscillation of equation (3.2.9) implies nonoscillation of equation (3.6.15).*

Remark 3.6.10. The "oscillatory counterpart" of Theorems 3.6.8 and 3.6.9 are immediate. The formulations are left to the reader.

3.6.3. Comparison theorem for generalized Riccati difference equations

Along with equation (3.2.17), consider the generalized Riccati difference equation

$$\Delta v(k) + q_1(k) + \Phi\big(v(k), c_1(k)\big) = 0. \tag{3.6.16}$$

Theorem 3.6.11. *Suppose that* $c_1(k) \ge c(k)$ *and* $q_1(k) \le q(k)$ *for* $k \in [m,n]$. *Let* $w(k)$ *and* $v(k)$ *be solutions of equations (3.2.9) and (3.6.16), respectively, defined on* $[m,n]$. *If* $c(k) + w(k) > 0$ *on* $[m,n]$ *and* $v(m) \ge w(m)$, *then* $v(k) \ge w(k)$ *and* $c_1(k) + v(k) > 0$ *for* $k \in [m,n]$.

Proof. Let $w(k)$ and $v(k)$ be solutions of equations (3.2.17) and (3.6.16), respectively, such that $v(m) \ge w(m)$ (then $c_1(m) + v(m) > 0$) and $c(k) + w(k) > 0$ for $k \in [m,n]$. One can rewrite these equations as $w(k+1) = -q(k) + \tilde{\Phi}(w(k), c(k))$ and $v(k+1) = -q_1(k) + \tilde{\Phi}(v(k), c_1(k))$, respectively, where the function $\tilde{\Phi}$ is defined in Lemma 3.2.6. According to this lemma, $\tilde{\Phi}(x,y) \le \tilde{\Phi}(X,Y)$ for $x \le X$ and $y \le Y$ with $x + y > 0$. Hence

$$\begin{aligned} w(m+1) &= -q(m) + \tilde{\Phi}\big(w(m), c(m)\big) \le -q_1(m) + \tilde{\Phi}\big(v(m), c_1(m)\big) \\ &= v(m+1), \end{aligned} \tag{3.6.17}$$

and $c_1(m+1) + v(m+1) > 0$. Continuing this process step by step, we find that $v(k) \ge w(k)$ and $c_1(k) + v(k) > 0$ for $k \in [m,n]$. This completes the proof. □

3.7. More on oscillation, nonoscillation, and positive solutions

Consider the half-linear difference equation

$$\Delta\Psi^*\big(\Delta x(k-1)\big) + q(k)\Psi^*\big(x(k)\big) = 0, \tag{3.7.1}$$

where $\Psi^*(x) = x^{\alpha-1}$, $\alpha > 1$ is a real number, and $q(k) \ge 0$ for $k \in \mathbb{N}$.

In this section we will present some sufficient and/or necessary conditions under which equation (3.7.1) has a positive nondecreasing solution. However, if $\alpha-1$ is a quotient of two positive odd integers, it is clear that the initial value problem (3.7.1) with given real initial values $x(0)$ and $x(1)$ has exactly one solution. We organize this section as follows. In Section 3.7.1, we give some preparatory results. A lemma which bridges discrete and continuous functions is also given. In Sections 3.7.2 and 3.7.3 sufficient and/or necessary conditions for equation (3.7.1) to have a positive nondecreasing solution are derived. In the last subsection, we discuss some related oscillation and nonoscillation criteria under which $\alpha - 1$ is a quotient of two positive odd integers.

3.7.1. Preparatory lemmas

In order to obtain the main results, we need the following two lemmas.

Lemma 3.7.1 (Hardy's inequality). *If $\alpha > 1$, $a(k) \geq 0$, and $A(k) = \sum_{j=1}^{k} a(j)$ for $k \in \mathbb{N}$, then*

$$\sum_{k=1}^{n}\left(\frac{A(k)}{k}\right)^{\alpha} \leq \left(\frac{\alpha}{\alpha-1}\right)^{\alpha}\sum_{k=1}^{n} a^{\alpha}(k), \tag{3.7.2}$$

where the equality is possible only when $a(k) = 0$ for all $k \in \{1,2,\dots,n\}$.

With respect to two given real numbers $\sigma \geq -1$ and $\eta \leq -1$, a real vector $y = (y(0), y(1),\dots, y(n-1))$ is said to be admissible if it is nontrivial and satisfies $y(0)+\sigma y(1) = 0$, $y(k) > 0$ for $k \in \{1,2,\dots,n\}$, $\Delta y(k) \geq 0$ for $k \in \{1,2,\dots,n-1\}$, and $y(n+1) + \eta y(n) = 0$.

Lemma 3.7.2. *If $u = (u(0), u(1),\dots, u(n+1))$ is an admissible solution of*

$$\Delta\Psi^{*}\left(\Delta u(k-1)\right) + q(k)\Psi^{*}\left(u(k)\right) = 0 \quad \textit{for } k \in \mathbb{N}, \tag{3.7.3}$$

then

$$(1+\sigma)^{\alpha-1}u^{\alpha}(1) + \sum_{k=1}^{n-1}\left(\Delta u(k)\right)^{\alpha} - (-1-\eta)^{\alpha-1}u^{\alpha}(n) = \sum_{k=1}^{n} q(k)u^{\alpha}(k), \tag{3.7.4}$$

and for any admissible vector $v = (v(0), v(1),\dots, v(n+1))$,

$$(1+\sigma)^{\alpha-1}v^{\alpha}(1) + \sum_{k=1}^{n-1}\left(\Delta v(k)\right)^{\alpha} - (-1-\eta)^{\alpha-1}v^{\alpha}(n) = \sum_{k=1}^{n} q(k)v^{\alpha}(k). \tag{3.7.5}$$

As a consequence, we have the following theorem.

Theorem 3.7.3. *Equation (3.7.1) has a positive nondecreasing solution if and only if there is a real number $-1 < \sigma < 0$ such that*

$$\sum_{k=m-1}^{N} q(k)y^{\alpha}(k) < (1+\sigma)^{\alpha-1}y^{\alpha}(m+1) + \sum_{k=m+1}^{N-1}\big(\Delta y(k)\big)^{\alpha} \tag{3.7.6}$$

for any positive nondecreasing vector $(y(m+1), y(m+2),\dots,y(N))$, where $m \in \mathbb{N}$.

Proof. Let $m \in \mathbb{N}$. If equation (3.7.1) has a positive nondecreasing solution x, then $x(k) > 0$ and $\Delta x(k) \geq 0$ for $k \geq m$. Thus there exists a real number $\sigma \in (-1,0)$ such that $\sigma > -(x(m)/x(m+1))$, and hence it follows from (3.7.5) that (3.7.6) holds.

Conversely, suppose that there exists a real number $\sigma \in (-1,0)$ such that (3.7.6) holds for any positive nondecreasing vector $(y(m+1), y(m+2),\dots,y(N))$. This implies that $q(1) < (1+\sigma)^{\alpha-1}$. Let $\{x(k)\}$ be a solution of equation (3.7.1) determined by the conditions $x(0) = -\sigma$ and $x(1) = 1$. We assert that $\Delta x(k) > 0$ for $k > m$. Suppose to the contrary that N is the first positive integer such that $(\Delta x(N))^{\alpha-1} \leq 0$ and $x(k) > 0$ for $1 \leq k \leq N$. Define a vector $(y(1), y(2),\dots,y(N))$ by $y(k) = x(k)$ for $1 \leq k \leq N$. Then, according to (3.7.4)

$$\begin{aligned}
&(1+\sigma)^{\alpha-1}y^{\alpha}(1) + \sum_{k=1}^{N-1}\big(\Delta y(k)\big)^{\alpha} - \sum_{k=1}^{N} q(k)y^{\alpha}(k)\\
&\quad= (1+\sigma)^{\alpha-1}x^{\alpha}(1) + \sum_{k=1}^{N-1}\big(\Delta x(k)\big)^{\alpha} - \sum_{k=1}^{N} q(k)x^{\alpha}(k)\\
&\quad= \left(-1+\frac{x(N+1)}{x(N)}\right)^{\alpha-1}x^{\alpha}(N)\\
&\quad= x(N)\big(\Delta x(N)\big)^{\alpha-1} \leq 0,
\end{aligned} \tag{3.7.7}$$

which contradicts our assumption. □

Corollary 3.7.4. *Suppose*

$$\sum_{k=m+1}^{N} q(k)y^{\alpha}(k) < y^{\alpha}(m+1) + \sum_{k=m+1}^{N-1}\big(\Delta y(k)\big)^{\alpha}, \tag{3.7.8}$$

where $(y(m+1), y(m+2),\dots,y(N))$ is a positive decreasing vector. Here $m \in \mathbb{N}$. Then equation (3.7.1) has a positive nondecreasing solution for $k \in \mathbb{N}$.

Lemma 3.7.5. *Let $\alpha > 1$. Suppose $\phi(k) = ak+b$ with real constants a, b and $k \in \mathbb{N}$. Then*

(i) $\Delta\phi^{\gamma+1}(k) \geq a(\gamma+1)\phi^{\gamma}(k)$ *if $\gamma \geq 0$, $a \geq 0$, and $\phi(k) \geq 0$,*
(ii) $\Delta\phi^{\gamma+1}(k) > a(\gamma+1)\phi^{\gamma}(k+1)$ *if $-1 < \gamma < 0$, $a > 0$, and $\phi(k) \geq 0$,*

(iii) $\{\Delta\phi^{\gamma/\alpha}(k)\}^\alpha \le (a^{\alpha-1}/(\gamma-\alpha+1))(\gamma/\alpha)^\alpha \Delta\phi^{\gamma-\alpha+1}(k-1)$ *if* $0 \le \gamma \le \alpha$, $\gamma \ne \alpha - 1$, *and* $\phi(k-1) > 0$,

(iv) $\{\Delta\phi^{\gamma/\alpha}(k)\}^\alpha < (a^{\alpha-1}/(\gamma-\alpha+1))(\gamma/\alpha)^\alpha \Delta\phi^{\gamma-\alpha+1}(k+1)$ *if* $\gamma > \alpha$, $\phi(k) \ge 0$, *and* $a \ge 0$,

(v) $a\phi^\gamma(k) \ge (1/(\gamma+1))\Delta\phi^{\gamma+1}(k-1)$ *if* $\gamma \ge 0$, $\phi(k-1) \ge 0$, *and* $a \ge 0$,

(vi) $a\phi^\gamma(k) > (1/(\gamma+1))\Delta\phi^{\gamma+1}(k)$ *if* $\gamma < 0$, $\gamma \ne -1$, *and* $a > 0$.

Proof. Suppose $\gamma \ge 0$, $a \ge 0$, and $\phi(k) \ge 0$. Then, by the Lagrange mean value theorem, there exists $\xi(k) \in (k, k+1)$ such that

$$\Delta\phi^{\gamma+1}(k) = \phi^{\gamma+1}(k+1) - \phi^{\gamma+1}(k) = a(\gamma+1)\phi^\gamma(\xi(k)) \ge a(\gamma+1)\phi^\gamma(k), \tag{3.7.9}$$

which proves the validity of (i). Case (ii) is similarly proved.

Suppose $0 \le \gamma \le \alpha$, $\gamma \ne \alpha - 1$, $a \ge 0$, and $\phi(k-1) > 0$. Then there exists $\mu(k) \in (k, k+1)$ such that

$$\begin{aligned}
\{\Delta\phi^{\gamma/\alpha}(k)\}^\alpha &= \{\phi^{\gamma/\alpha}(k+1) - \phi^{\gamma/\alpha}(k)\}^\alpha \\
&= a^{\alpha-1}\left(\frac{\gamma}{\alpha}\right)^\alpha \{a\phi^{\gamma-\alpha}(\mu(k))\} \\
&\le a^{\alpha-1}\left(\frac{\gamma}{\alpha}\right)^\alpha \{a\phi^{\gamma-\alpha}(k)\} \\
&\le a^{\alpha-1}\left(\frac{\gamma}{\alpha}\right)^\alpha \int_{\phi(k-1)}^{\phi(k)} z^{\gamma-\alpha}\,dz \\
&= \frac{a^{\alpha-1}}{\gamma-\alpha+1}\left(\frac{\gamma}{\alpha}\right)^\alpha \Delta\phi^{\gamma-\alpha+1}(k-1),
\end{aligned} \tag{3.7.10}$$

which proves the validity of (iii). Case (iv) is similarly proved.

If $\gamma \ge 0$, $a \ge 0$, and $\phi(k-1) \ge 0$, then

$$\begin{aligned}
a\phi^\gamma(k) = a(ak+b)^\gamma &\ge \int_{\phi(k-1)}^{\phi(k)} z^\gamma\,dz \\
&= \frac{1}{\gamma+1}\left[\phi^{\gamma+1}(k) - \phi^{\gamma+1}(k-1)\right] \\
&= \frac{1}{\gamma+1}\Delta\phi^{\gamma+1}(k-1),
\end{aligned} \tag{3.7.11}$$

which proves the validity of (v). Case (vi) is similarly proved. □

3.7.2. Sufficient conditions

We will derive sufficient conditions for equation (3.7.1) to have a positive nondecreasing solution. To do so, let $x(0) = 0$ and $x(1) = 1$. If $q(1) \le [(\alpha-1)/\alpha]^\alpha < 1$, then there exists $x(2) \ge x(1) > 0$ satisfying equation (3.7.1) for $n = 1$.

From (3.7.1), we have

$$x(n)\big(\Delta x(n)\big)^{\alpha-1} = \big(\Delta x(n-1)\big)^{\alpha} + x(n-1)\big(\Delta x(n-1)\big)^{\alpha-1} - q(n)x^{\alpha}(n). \tag{3.7.12}$$

Summing (3.7.12) from $n = 1$ to $n = k$, we find

$$x(k)\big(\Delta x(k)\big)^{\alpha-1} = \sum_{n=1}^{k}\big(\Delta x(n-1)\big)^{\alpha} + x(0)\big(\Delta x(0)\big)^{\alpha-1} - \sum_{n=1}^{k} q(n)x^{\alpha}(n). \tag{3.7.13}$$

If (3.7.1) has a solution $\{x(n)\}_{n=0}^{k}$ for $1 \le n \le k-1$ satisfying $x(0) = 0$, $x(1) = 1$, and $\Delta x(n) \ge 0$ for $0 \le n \le k-1$, then

$$\begin{aligned} x(k)\big(\Delta x(k)\big)^{\alpha-1} &= \sum_{n=1}^{k}\big(\Delta x(n-1)\big)^{\alpha} - \sum_{n=1}^{k} q(n)\left[\sum_{j=0}^{n-1}\Delta x(j)\right]^{\alpha} \\ &= \sum_{n=1}^{k}\big(\Delta x(n-1)\big)^{\alpha} - \sum_{n=1}^{k} q(n)n^{\alpha}\phi^{\alpha}(n), \end{aligned} \tag{3.7.14}$$

where $\phi(n) = (1/n)\sum_{j=0}^{n-1}\Delta x(j)$. If

$$\max\left\{n^{\alpha}q(n) : 1 \le n \le k\right\} \le \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.7.15}$$

then by Lemma 3.7.1, we have

$$\begin{aligned} x(k)\big(\Delta x(k)\big)^{\alpha-1} &\ge \sum_{n=1}^{k}\big(\Delta x(n-1)\big)^{\alpha} - \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}\sum_{n=1}^{k}\phi^{\alpha}(n) \\ &> \sum_{n=1}^{k}\big(\Delta x(n-1)\big)^{\alpha} - \left(\frac{\alpha-1}{\alpha}\right)^{\alpha}\left(\frac{\alpha}{\alpha-1}\right)^{\alpha}\sum_{n=1}^{k}\big(\Delta x(n-1)\big)^{\alpha}. \end{aligned} \tag{3.7.16}$$

Hence there exists $x(k+1) \ge x(k) > 0$ satisfying equation (3.7.1) for $n = k$. Therefore we obtain the following result.

Theorem 3.7.6. *If*

$$q(k) \le \left(\frac{\alpha-1}{\alpha k}\right)^{\alpha} \quad \textit{for } k \ge 1, \tag{3.7.17}$$

then equation (3.7.1) has a positive nondecreasing solution for $k \ge 1$.

Theorem 3.7.7. *Suppose*

$$k^{\alpha-1}\sum_{j=k+1}^{\infty} q(j) \le \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}} = \psi(\alpha) \quad \text{for } k \ge 1. \tag{3.7.18}$$

Then equation (3.7.1) has a positive nondecreasing solution for $k \ge 1$.

Proof. Let $y = (y(m+1), y(m+2), \dots, y(N))$ be a positive nondecreasing vector and $y(m) = 0$, where $m \in \mathbb{N}_0$. Let $r(k) = -\sum_{j=k}^{\infty} q(j)$ for $k \ge m+1$. Then, by Hölder's inequality and Lemma 3.7.1, we obtain

$$\begin{aligned}
\sum_{k=m+1}^{N} q(k)y^{\alpha}(k) &= \sum_{k=m+1}^{N} y^{\alpha}(k)\Delta r(k) \\
&= r(N+1)y^{\alpha}(N) - \sum_{k=m}^{N-1} r(k+1)\Delta y^{\alpha}(k) \\
&\le \alpha\psi(\alpha)\sum_{k=m}^{N-1}\left(\frac{y(k+1)}{k}\right)^{\alpha-1}\Delta y(k) \\
&\le \alpha\psi(\alpha)\left[\sum_{k=m}^{N-1}(\Delta y(k))^{\alpha}\right]^{1/\alpha}\left[\sum_{k=m}^{N-1}\left(\frac{\theta(k)}{k-m+1}\right)^{\alpha}\right]^{1/\beta} \\
&< \alpha\psi(\alpha)\left[\sum_{k=m}^{N}(\Delta y(k))^{\alpha}\right]^{1/\alpha}\left\{\left(\frac{\alpha}{\alpha-1}\right)^{\alpha}\sum_{k=m}^{N-1}(\Delta y(k))^{\alpha}\right\}^{1/\beta} \\
&\le y^{\alpha}(m+1) + \sum_{k=m+1}^{N-1}(\Delta y(k))^{\alpha},
\end{aligned} \tag{3.7.19}$$

where $(1/\alpha) + (1/\beta) = 1$ and $\theta(k) = \sum_{j=m}^{k}\Delta y(j) = y(k+1)$. Now, by Corollary 3.7.4, equation (3.7.1) has a positive nondecreasing solution for $k \ge 1$. □

Corollary 3.7.8. *Suppose $\alpha \ge 2$ and*

$$\sum_{j=k}^{\infty} q(j) \le 2^{(1-\alpha)(k-1)}e^{2-2\alpha} \quad \text{for } k \ge 1. \tag{3.7.20}$$

Then equation (3.7.1) has a positive nondecreasing solution.

Proof. Let $a(k+1) = q(k)$ for $k \ge 1$. Then

$$\sum_{j=k+1}^{\infty} a(j) \le 2^{(1-\alpha)(k-1)}e^{2-2\alpha} \quad \text{for } k \ge 1. \tag{3.7.21}$$

Since $\alpha \geq 2$,

$$e^{2\alpha-2} - (\alpha - 1)\left(\frac{\alpha}{\alpha - 1}\right)^{\alpha} > \alpha e^{\alpha-1} - \alpha e \geq 0. \tag{3.7.22}$$

Thus

$$e^{2-2\alpha}2^{(1-\alpha)(k-1)} < \frac{k^{1-\alpha}}{\alpha - 1}\left(\frac{\alpha - 1}{\alpha}\right)^{\alpha} \quad \text{for } k \geq 1. \tag{3.7.23}$$

Then by Theorem 3.7.7, equation (3.7.1) has a positive nondecreasing solution. The proof is complete. □

3.7.3. Necessary conditions

First we will illustrate how Theorem 3.7.3 can be applied to yield a condition necessary for equation (3.7.1) to have a positive nondecreasing solution in a very simple case. If equation (3.7.1) has a positive nondecreasing solution, then (3.7.8) holds for any positive nondecreasing vector $(y(m+1), y(m+2), \dots, y(N))$.

Let $n \in \mathbb{N}$ be such that $m < n < N$ and let

$$y(k) = \begin{cases} \dfrac{k - m}{n - m} & \text{if } m + 1 \leq k \leq n, \\ 1 & \text{if } k \geq n. \end{cases} \tag{3.7.24}$$

Then

$$\begin{aligned} & y^{\alpha}(m+1) + \sum_{k=m+1}^{N-1} (\Delta y(k))^{\alpha} \\ & \quad = y^{\alpha}(m+1) + \sum_{k=m+1}^{n-1} (\Delta y(k))^{\alpha} + \sum_{k=n}^{N-1} (\Delta y(k))^{\alpha} = \left(\frac{1}{n - m}\right)^{\alpha-1}. \end{aligned} \tag{3.7.25}$$

By (3.7.8), we then have

$$\sum_{k=n+1}^{N} q(k) = \sum_{k=n+1}^{N} q(k)y^{\alpha}(k) \leq \sum_{k=m+1}^{N} q(k)y^{\alpha}(k) < \left(\frac{1}{n - m}\right)^{\alpha-1}. \tag{3.7.26}$$

Since N can be taken arbitrarily large, this last inequality thus implies

$$(n - m)^{\alpha-1} \sum_{k=n+1}^{\infty} q(k) \leq 1. \tag{3.7.27}$$

Let

$$q^* = \limsup_{n\to\infty} \left\{ n^{\alpha-1} \sum_{k=n+1}^{\infty} q(k) \right\}. \tag{3.7.28}$$

We see that $q^* \leq 1$. This means that we have proved the following result.

Theorem 3.7.9. *Let $m \in \mathbb{N}$. Suppose equation (3.7.1) has a positive nondecreasing solution. Then $q^* \leq 1$ and inequality (3.7.27) holds for any $n > m$, where q^* is as in (3.7.28).*

More generally, we have the following theorem.

Theorem 3.7.10. *Suppose equation (3.7.1) has a positive nondecreasing solution. Assume $0 < \gamma < \alpha - 1 < \delta$ and let λ be a real number. Then for any $m > \lambda$,*

$$\begin{aligned}\limsup_{n\to\infty}&\left[(n-\lambda)^{\alpha-1-\delta}\sum_{k=m+1}^{n}(k-\lambda)^{\delta}q(k)+(n-\lambda)^{\alpha-1-\gamma}\sum_{k=n+1}^{\infty}(k-\lambda)^{\gamma}q(k)\right]\\&\leq\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}+\frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}.\end{aligned}\tag{3.7.29}$$

Proof. Since equation (3.7.1) has a positive nondecreasing solution, by Theorem 3.7.3, (3.7.8) holds for any positive nondecreasing sequence $\{y(k)\}_{m+1}^{N}$. Let $n \in \mathbb{N}$ be such that $m < n < N$ and let

$$y(k)=\begin{cases}\left(\dfrac{k-\lambda}{n-\lambda}\right)^{\delta/\alpha} & \text{if } m<k\leq n,\\[2ex] \left(\dfrac{k-\lambda}{n-\lambda}\right)^{\gamma/\alpha} & \text{if } n\leq k\leq N.\end{cases}\tag{3.7.30}$$

Since

$$\Delta y(n)=\left(\frac{n+1-\lambda}{n-\lambda}\right)^{\gamma/\alpha}-1<\left(\frac{n+1-\lambda}{n-\lambda}\right)^{\delta/\alpha}-1,\tag{3.7.31}$$

it follows that

$$\begin{aligned}y^{\alpha}(m+1)&+\sum_{k=m+1}^{N-1}(\Delta y(k))^{\alpha}\\&<y^{\alpha}(m+1)+\sum_{k=m+1}^{n}\left[\left(\frac{k+1-\lambda}{n-\lambda}\right)^{\delta/\alpha}-\left(\frac{k-\lambda}{n-\lambda}\right)^{\delta/\alpha}\right]^{\alpha}\\&\quad+\sum_{k=n+1}^{N-1}\left[\left(\frac{k+1-\lambda}{n-\lambda}\right)^{\gamma/\alpha}-\left(\frac{k-\lambda}{n-\lambda}\right)^{\gamma/\alpha}\right]^{\alpha}.\end{aligned}\tag{3.7.32}$$

We distinguish the following two cases.

Case 1. If $\alpha - 1 < \delta \leq \alpha$, then by Lemma 3.7.5(iii) and the above calculation, we have

$$\begin{aligned}
&y^{\alpha}(m+1) + \sum_{k=m+1}^{N-1} (\Delta y(k))^{\alpha} \\
&\quad \leq y^{\alpha}(m+1) + \frac{n-\lambda}{\delta+\alpha+1}\left(\frac{\delta}{\alpha(n-\lambda)}\right)^{\alpha} \sum_{k=m+1}^{n} \Delta\left(\frac{k-1-\lambda}{n-\lambda}\right)^{\delta-\alpha+1} \\
&\qquad + \frac{n-\lambda}{\gamma-\alpha+1}\left(\frac{\gamma}{\alpha(n-\lambda)}\right)^{\alpha} \sum_{k=n+1}^{N-1} \Delta\left(\frac{k-1-\lambda}{n-\lambda}\right)^{\gamma-\alpha+1} \\
&\quad = y^{\alpha}(m+1) + \frac{n-\lambda}{\delta-\alpha+1}\left(\frac{\delta}{\alpha(n-\lambda)}\right)^{\alpha}\left[1 - \left(\frac{m-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}\right] \\
&\qquad + \frac{n-\lambda}{\gamma-\alpha+1}\left(\frac{\gamma}{\alpha(n-\lambda)}\right)^{\alpha}\left[-1 + \left(\frac{N-1-\lambda}{n-\lambda}\right)^{\gamma-\alpha+1}\right] \\
&\quad < \left(\frac{m+1-\lambda}{n-\lambda}\right)^{\delta} + (n-\lambda)^{1-\alpha}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha} + \frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right].
\end{aligned} \tag{3.7.33}$$

Since

$$\sum_{k=m+1}^{N} q(k)y^{\alpha}(k) = (n-\lambda)^{-\delta}\sum_{k=m+1}^{n}(k-\lambda)^{\delta}q(k) + (n-\lambda)^{-\gamma}\sum_{k=n+1}^{N}(k-\lambda)^{\gamma}q(k), \tag{3.7.34}$$

it follows from (3.7.8) and (3.7.34) that

$$\begin{aligned}
&(n-\lambda)^{-\delta}\sum_{k=m+1}^{n}(k-\lambda)^{\delta}q(k) - (n-\lambda)^{-\gamma}\sum_{k=n+1}^{N}(k-\lambda)^{\gamma}q(k) \\
&\quad < \left(\frac{m+1-\lambda}{n-\lambda}\right)^{\delta} + (n-\lambda)^{1-\alpha}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha} + \frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right].
\end{aligned} \tag{3.7.35}$$

Multiplying (3.7.35) by $(n-\lambda)^{\alpha-1}$ and letting $N \to \infty$, we obtain

$$\begin{aligned}
&(n-\lambda)^{\alpha-1-\delta}\sum_{k=m+1}^{n}(k-\lambda)^{\delta}q(k) + (n-\lambda)^{\alpha-1-\gamma}\sum_{k=n+1}^{N}(k-\lambda)^{\gamma}q(k) \\
&\quad \leq (m+1-\lambda)^{\alpha-1}\left(\frac{m+1-\lambda}{n-\lambda}\right)^{\delta-\alpha+1} + \frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha} + \frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha},
\end{aligned} \tag{3.7.36}$$

which implies (3.7.29) as desired.

Case 2. If $\delta > \alpha$, then by the calculation preceding Case 1 and Lemma 3.7.5(iii) and (iv), we have

$$\begin{aligned}
&y^{\alpha}(m+1)+\sum_{k=m+1}^{N-1}(\Delta y(k))^{\alpha}\\
&\quad< y^{\alpha}(m+1)+\frac{(n-\lambda)^{1-\alpha}}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}\sum_{k=m+1}^{n}\left[\left(\frac{k+2-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}-\left(\frac{k+1-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}\right]\\
&\qquad+\frac{(n-\lambda)^{1-\alpha}}{\gamma-\alpha+1}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\sum_{k=n+1}^{N-1}\left[\left(\frac{k-\lambda}{n-\lambda}\right)^{\gamma-\alpha+1}-\left(\frac{k-1-\lambda}{n-\lambda}\right)^{\gamma-\alpha+1}\right]\\
&\quad<\left(\frac{m+1-\lambda}{n-\lambda}\right)^{\delta}\\
&\qquad+(n-\lambda)^{1-\alpha}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}\left(\frac{n+2-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}+\frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right].
\end{aligned}\tag{3.7.37}$$

From this, (3.7.8), and (3.7.34), we obtain

$$\begin{aligned}
&(n-\lambda)^{-\delta}\sum_{k=m+1}^{n}(k-\lambda)^{\delta}q(k)+(n-\lambda)^{-\gamma}\sum_{k=n+1}^{N}(k-\lambda)^{\gamma}q(k)\\
&\quad<\left(\frac{m+1-\lambda}{n-\lambda}\right)^{\delta}\\
&\qquad+(n-\lambda)^{1-\alpha}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}\left(\frac{n+2-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}+\frac{1}{\alpha-1+\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right].
\end{aligned}\tag{3.7.38}$$

Multiplying (3.7.38) by $(n-\lambda)^{\alpha-1}$ and then letting $N\to\infty$, we find

$$\begin{aligned}
&(n-\lambda)^{\alpha-1-\delta}\sum_{k=m+1}^{n}(k-\lambda)^{\delta}q(k)+(n-\lambda)^{\alpha-1-\gamma}\sum_{k=n+1}^{N}(k-\lambda)^{\gamma}q(k)\\
&\quad\le(m+1-\lambda)^{\alpha-1}\left(\frac{m+1-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}+\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}\left(\frac{n+2-\lambda}{n-\lambda}\right)^{\delta-\alpha+1}\\
&\qquad+\frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha},
\end{aligned}\tag{3.7.39}$$

which implies (3.7.29) as required.

The proof is complete. □

Remark 3.7.11. (i) If we choose $\lambda = 1$ in Theorem 3.7.10, then (3.7.29) is reduced to

$$\begin{aligned} &\limsup_{n\to\infty}\left[n^{\alpha-1-\delta}\sum_{k=m+1}^{n}k^{\delta}q(k)+n^{\alpha-1-\gamma}\sum_{k=n+1}^{\infty}k^{\gamma}q(k)\right] \\ &\quad\le \frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}+\frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}. \end{aligned} \tag{3.7.40}$$

(ii) Since both terms on the left-hand side of (3.7.40) are nonnegative, we may let $\gamma = 0$ in (3.7.40) to obtain

$$\limsup_{n\to\infty}\left\{n^{\alpha-1-\delta}\sum_{k=m+1}^{n}k^{\delta}q(k)\right\}\le\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}. \tag{3.7.41}$$

(iii) If we let $\delta = \alpha$ in (3.7.40), then we find

$$\limsup_{n\to\infty}\left\{n^{\alpha-1-\gamma}\sum_{k=n+1}^{\infty}k^{\gamma}q(k)\right\}\le 1+\frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}. \tag{3.7.42}$$

(iv) If we let $\gamma = 0$ in (3.7.42), then we obtain $q^* \le 1$, where q^* is as in (3.7.28).

For the next result, we will need a sequence $\{s(k)\}_{k=1}^{\infty}$ defined by

$$s(k) = k^{\alpha-1-\gamma}\sum_{j=k+1}^{\infty}j^{\gamma}q(j), \tag{3.7.43}$$

where $0 \le \gamma < \alpha - 1$. Note that if equation (3.7.1) has a positive nondecreasing solution, then in view of (3.7.42), the sequence $\{s(k)\}$ is bounded.

Theorem 3.7.12. *Suppose $0 \le \gamma < \alpha - 1 < \delta$ and equation (3.7.1) has a positive nondecreasing solution. Then for any $m \in \mathbb{N}$,*

$$\begin{aligned} &\limsup_{n\to\infty}\left\{n^{\alpha-1-\delta}\sum_{k=m+1}^{n-1}k^{\delta-\alpha}s(k)\right\} \\ &\quad\le \frac{1}{\delta-\gamma}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha}+\frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right]. \end{aligned} \tag{3.7.44}$$

Proof. Note first that if we let

$$T(k) = k^{\gamma-\alpha+1}s(k) = \sum_{j=k+1}^{\infty}j^{\gamma}q(j)\quad\text{for } k \ge m, \tag{3.7.45}$$

then in view of (3.7.42), $T(k)$ is bounded and $\Delta T(k) = -(k+1)^{\gamma}q(k+1) \leq 0$, $k \geq m$ and $q(k) = -k^{-\gamma}\Delta T(k-1)$, $k \geq m+1$. Consider first the case $\delta-\gamma-1 \geq 0$. Summing the right-hand side of

$$\sum_{k=m+1}^{n} k^{\delta}q(k) = -\sum_{k=m+1}^{n} k^{\delta-\gamma}\Delta T(k-1) \tag{3.7.46}$$

by parts and using Lemma 3.7.5(i), we obtain

$$\begin{aligned}\sum_{k=m+1}^{n} k^{\delta}q(k) &= -n^{\delta-\gamma}T(n) + m^{\delta-\gamma}T(m) + \sum_{k=m}^{n-1} T(k)\Delta k^{\delta-\gamma}\\ &> -n^{\delta-\gamma}T(n) + \sum_{k=m+1}^{n-1} T(k)\Delta k^{\delta-\gamma}\\ &\geq -n^{\delta-\gamma}T(n) + \sum_{k=m+1}^{n-1} (\delta-\gamma)k^{\delta-\gamma-1}T(k)\\ &= -n^{\delta-\alpha+1}s(n) + (\delta-\gamma)\sum_{k=m+1}^{n-1} k^{\delta-\alpha}s(k).\end{aligned} \tag{3.7.47}$$

Next, we consider the case that $\delta-\gamma-1<0$. Since

$$\sum_{k=m+1}^{n} k^{\delta}q(k) = -\sum_{k=m+1}^{n} k^{\delta-\gamma}\Delta T(k-1) > -\sum_{k=m+1}^{n} (k-1)^{\delta-\gamma}\Delta T(k-1), \tag{3.7.48}$$

summing the last term in (3.7.48) by parts and using Lemma 3.7.5(ii), we obtain

$$\begin{aligned}\sum_{k=m+1}^{n} k^{\delta}q(k) &\geq -n^{\delta-\gamma}T(n) + m^{\delta-\gamma}T(m) + \sum_{k=m+1}^{n} T(k)\Delta(k-1)^{\delta-\gamma}\\ &> -n^{\delta-\gamma}T(n) + \sum_{k=m+1}^{n-1} (\delta-\gamma)k^{\delta-\gamma-1}T(k)\\ &= -n^{\delta-\alpha+1}s(n) + (\delta-\gamma)\sum_{k=m+1}^{n-1} k^{\delta-\alpha}s(k).\end{aligned} \tag{3.7.49}$$

In either case, it follows that

$$\begin{aligned}&(\delta-\gamma)n^{\alpha-1-\delta}\sum_{k=m+1}^{n-1} k^{\delta-\alpha}s(k)\\ &\quad\leq n^{\alpha-1-\delta}\sum_{k=m+1}^{n} k^{\delta}q(k) + n^{\alpha-1-\gamma}\sum_{k=n+1}^{\infty} k^{\gamma}q(k).\end{aligned} \tag{3.7.50}$$

The assertion now follows from Theorem 3.7.10. $\square$

Theorem 3.7.13. *If (3.7.1) has a positive nondecreasing solution and* $0 \le \gamma < \alpha - 1$, *then*

$$\liminf_{n\to\infty} n^{\alpha-1-\gamma} \sum_{k=n+1}^{\infty} k^{\gamma} q(k) \le \frac{1}{\alpha - 1 - \gamma}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}. \tag{3.7.51}$$

Proof. If $\alpha - 1 < \delta < \alpha$, then by Theorem 3.7.12, (3.7.44) holds for any $m \in \mathbb{N}$. Let ℓ be an integer such that $m + 1 < \ell < n - 1$. Then

$$\begin{aligned}
&\limsup_{n\to\infty}\left[n^{\alpha-1-\delta}\sum_{k=m+1}^{\ell-1} k^{\delta-\alpha}s(k) + n^{\alpha-1-\delta}\sum_{k=\ell}^{n-1} k^{\delta-\alpha}s(k)\right]\\
&\qquad\le \frac{1}{\delta-\gamma}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha} + \frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right].
\end{aligned} \tag{3.7.52}$$

Let $S(\ell) = \inf_{k\ge\ell} s(k)$ and use Lemma 3.7.5(vi) to derive

$$\begin{aligned}
n^{\alpha-1-\delta}\sum_{k=\ell}^{n-1} k^{\delta-\alpha}s(k) &\ge S(\ell)\sum_{k=\ell}^{n-1}\frac{1}{n}\left(\frac{k}{n}\right)^{\delta-\alpha}\\
&> S(\ell)\sum_{k=\ell}^{n-1}\frac{1}{\delta-\alpha+1}\Delta\left(\frac{k}{n}\right)^{\delta-\alpha+1}\\
&= \frac{S(\ell)}{\delta-\alpha+1}\left[1-\left(\frac{\ell}{n}\right)^{\delta-\alpha+1}\right].
\end{aligned} \tag{3.7.53}$$

Hence

$$\begin{aligned}
&\frac{1}{\delta-\gamma}\left[\frac{1}{\delta-\alpha+1}\left(\frac{\delta}{\alpha}\right)^{\alpha} + \frac{1}{\alpha-1-\gamma}\left(\frac{\gamma}{\alpha}\right)^{\alpha}\right]\\
&\qquad\ge \limsup_{n\to\infty} n^{\alpha-1-\delta}\sum_{k=\ell}^{n-1} k^{\delta-\alpha}s(k) \ge \frac{S(\ell)}{\delta-\alpha+1}.
\end{aligned} \tag{3.7.54}$$

Multiplying (3.7.54) by $\delta - \alpha + 1$ and letting $\delta \to \alpha - 1$, we have

$$S(\ell) \le \frac{1}{\alpha-1-\gamma}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha}, \tag{3.7.55}$$

which implies (3.7.51) as desired. □

As an immediate consequence, we may let $\gamma = 0$ in Theorem 3.7.13 to obtain the following result.

Theorem 3.7.14. *If equation (3.7.1) has a positive nondecreasing solution, then*

$$Q = \liminf_{k\to\infty} k^{\alpha-1} \sum_{n=k+1}^{\infty} q(n) \le \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}. \tag{3.7.56}$$

The number $(\alpha-1)^{\alpha-1}/\alpha^{\alpha}$ *is the best possible.*

To see that (3.7.56) is sharp, let

$$q(k) = \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \frac{1}{(k+1)^{\alpha}} \quad \text{for } k \in \mathbb{N}. \tag{3.7.57}$$

Then

$$\liminf_{k\to\infty} k^{\alpha-1} \sum_{n=k+1}^{\infty} \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \left(\frac{1}{n+1}\right)^{\alpha} = \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}, \tag{3.7.58}$$

and by Theorem 3.7.6, the equation

$$\Delta\Psi^*(x(k-1)) + \left(\frac{\alpha-1}{\alpha}\right)^{\alpha} \left(\frac{1}{k+1}\right)^{\alpha} \Psi^*(x(k)) = 0 \quad \text{for } k \in \mathbb{N} \tag{3.7.59}$$

has a positive solution.

Remark 3.7.15. For equation (3.2.11) or equation (3.7.1), if $\alpha - 1$ is a quotient of two positive odd integers and $q(k) \ge 0$ with $q(k) \not\equiv 0$ eventually, it is easy to see that if $\{x(k)\}$ is a solution of either equation such that $x(k) > 0$ for $k \ge m$ for some $m \in \mathbb{N}$, then $\Delta x(k) > 0$ for $k \ge m$, that is, $\{x(k)\}$ is eventually monotonic.

Based on Remark 3.7.15, the sufficient and necessary conditions in Sections 3.7.2 and 3.7.3 can be reduced to oscillation criteria for equation (3.7.1) with α and $q(k)$ as in Remark 3.7.15. As a consequence, we obtain the following results.

Theorem 3.7.16. *If equation (3.7.1), where* α *and* $q(k)$ *are as in Remark 3.7.15, is oscillatory, then*

$$\limsup_{k\to\infty} k^{\alpha-1} \sum_{j=k+1}^{\infty} q(j) \ge \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}. \tag{3.7.60}$$

Theorem 3.7.17. *If equation (3.7.1), where* α *and* $q(k)$ *are as in Remark 3.7.15, is nonoscillatory, then*

$$\begin{aligned} &\limsup_{k\to\infty} k^{\alpha-1} \sum_{j=k+1}^{\infty} q(j) \le 1, \\ &\liminf_{k\to\infty} k^{\alpha-1} \sum_{j=k+1}^{\infty} q(j) \le \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}. \end{aligned} \tag{3.7.61}$$

3.8. Strong oscillation

In the following we will employ Theorems 3.7.16 and 3.7.17 to classify the solutions of equation (3.7.1) and hence solutions of equations (3.2.9) and (3.2.11).

3.8.1. Strong oscillation, nonoscillation, and conditional oscillation

The class of equations (3.2.9) can be divided according to the following definition.

Definition 3.8.1. (i) Equation (3.2.9) is said to be *strongly oscillatory* if the equation

$$\Delta(c(k)\Psi(\Delta x(k))) + \lambda q(k)\Psi(x(k+1)) = 0 \tag{3.8.1}$$

is oscillatory for all $\lambda > 0$.

(ii) Equation (3.2.9) is said to be *strongly nonoscillatory* if equation (3.8.1) is nonoscillatory for all $\lambda > 0$.

(iii) Equation (3.2.9) is said to be *conditionally oscillatory* if equation (3.8.1) is oscillatory for some $\lambda > 0$ and nonoscillatory for some other $\lambda > 0$.

By Theorem 3.3.5, it follows that in the case of Definition 3.8.1(iii) there must exist a positive number $\gamma(q)$ such that equation (3.2.9) is oscillatory for $\lambda > \gamma(q)$ and nonoscillatory for $\lambda < \gamma(q)$. This number $\gamma(q)$ is called the *oscillation constant* of the sequence $\{q(k)\}$ (with respect to the sequence $\{c(k)\}$).

Now, we present strong oscillation and nonoscillation criteria. Note that here we consider only the case when $\sum^{\infty} q(j)$ is convergent, since if $\sum^{\infty} q(j) = \infty$ (and $\sum^{\infty} c^{1-\beta}(j) = \infty$ with $c(k) > 0$), then equation (3.2.9) is oscillatory by Theorem 3.5.34 and obviously also strongly oscillatory.

Theorem 3.8.2. *Assume that condition (3.5.29) holds and $\sum^{\infty} q(j)$ is convergent. Then the following statements hold.*

(I_1) *Suppose in addition that condition (3.4.14) holds and $\sum^{\infty} q(j) \geq 0$. If equation (3.2.9) is strongly oscillatory, then*

$$\limsup_{k\to\infty} \left(\sum_{j=m}^{k-1} c^{1-\beta}(j)\right)^{\alpha-1} \sum_{j=k}^{\infty} q(j) = \infty. \tag{3.8.2}$$

(I_2) *Suppose in addition that $q(k) \geq 0$. If (3.8.2) holds, then equation (3.2.9) is strongly oscillatory.*

(I_3) *Suppose in addition that $q(k) \geq 0$. If (3.2.9) is strongly nonoscillatory, then*

$$\lim_{k\to\infty} \left(\sum_{j=m}^{k-1} c^{1-\beta}(j)\right)^{\alpha-1} \sum_{j=k}^{\infty} q(j) = 0. \tag{3.8.3}$$

(I_4) *Suppose in addition that (3.4.14) holds. If (3.8.3) is satisfied, then equation (3.2.9) is strongly nonoscillatory.*

The following theorem provides information about the oscillation constant of conditionally oscillatory equations (3.2.11). We will use the symbol Q_* and Q^* introduced in (3.5.104) at the beginning of Section 3.5.2.

Theorem 3.8.3. *Suppose that* $0 < Q_* \le Q^* < \infty$ *and* $q(k) \ge 0$. *Then the oscillation constant* $\gamma(q)$ *of the equation (3.2.11) satisfies*

$$\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\frac{1}{Q^*} \le \gamma(q) \le \min\left\{\frac{1}{Q^*}, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\frac{1}{Q_*}\right\}. \tag{3.8.4}$$

In particular, if $Q_* = Q^*$, *then*

$$\gamma(q) = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\frac{1}{Q^*}. \tag{3.8.5}$$

Finally, we state and prove the following two oscillation and nonoscillation criteria for equation (3.2.11).

Theorem 3.8.4. *Let* $\{q(k)\}$ *and* $\{q_1(k)\}$, $k \in \mathbb{N}$, *be two sequences such that condition (3.5.28) holds and such that* $\sum_{j=k}^{\infty} q_1(j)$ *is positive for all large k. Further, let* $\gamma(q_1) \in (0, \infty)$ *be the oscillation constant of* $q_1(k)$. *If*

$$\Gamma_* := \liminf_{k\to\infty} \frac{\sum_{j=k}^{\infty} q(j)}{\sum_{j=k}^{\infty} q_1(j)} > \gamma(q_1), \tag{3.8.6}$$

then equation (3.2.11) is oscillatory.

Proof. The difference equation

$$\Delta(\Psi(\Delta x(k))) + \lambda q_1(k)\Psi(x(k+1)) = 0 \tag{3.8.7}$$

is oscillatory if $\lambda > \gamma(q_1)$. Thus by Theorem 3.7.16, we have

$$\limsup_{k\to\infty} k^{\alpha-1} \sum_{j=k+1}^{\infty} q_1(j) \ge \frac{(\alpha-1)^{\alpha-1}}{\lambda\alpha^\alpha}, \tag{3.8.8}$$

so that for any $1/(4\gamma(q_1)) > \varepsilon > 0$, there exists a positive integer $m_1 \ge m \in \mathbb{N}$ such that

$$k^{\alpha-1} \sum_{j=k+1}^{\infty} q_1(j) > \frac{(\alpha-1)^{\alpha-1}}{\lambda\alpha^\alpha} - \varepsilon \quad \text{for } k \ge m_1. \tag{3.8.9}$$

As $\lambda \to \gamma(q_1)$, we have

$$k^{\alpha-1} \sum_{j=k+1}^{\infty} q_1(j) \ge \frac{(\alpha-1)^{\alpha-1}}{\gamma(q_1)\alpha^\alpha} - \varepsilon \quad \text{for } k \ge m_1, \tag{3.8.10}$$

so that

$$\Gamma_* \leq \liminf_{k\to\infty} \frac{k^{\alpha-1}\sum_{j=k+1}^{\infty} q(j)}{(\alpha-1)^{\alpha-1}/\gamma(q_1)\alpha^{\alpha}-\varepsilon} = \frac{1}{(\alpha-1)^{\alpha-1}/\gamma(q_1)-\varepsilon}\liminf_{k\to\infty} k^{\alpha-1}\sum_{j=k+1}^{\infty} q(j). \tag{3.8.11}$$

This implies

$$\liminf_{k\to\infty} k^{\alpha-1}\sum_{j=k+1}^{\infty} q(j) > \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}. \tag{3.8.12}$$

Hence, by Theorem 3.7.17, (3.2.11) is oscillatory. □

Theorem 3.8.5. *Let $\{q(k)\}$ and $\{q_1(k)\}$, $k \in \mathbb{N}$, be two sequences such that condition (3.5.28) holds and $\sum_{j=k}^{\infty} q_1(j) > 0$ for all large k. Further, let $\gamma(q) \in (0,\infty)$ and $\gamma(q_1) \in (0,\infty)$ be the oscillation constants of $\{q(k)\}$ and $\{q_1(k)\}$, respectively. Then*

$$\Gamma_* \leq \frac{\gamma(q_1)}{\gamma(q)}, \tag{3.8.13}$$

and if

$$\Gamma^* := \limsup_{k\to\infty} \frac{\sum_{j=k}^{\infty} q(j)}{\sum_{j=k}^{\infty} q_1(j)} < \gamma(q_1), \tag{3.8.14}$$

then equation (3.2.11) is nonoscillatory.

Proof. Note that by the definition of $\gamma(q)$ the difference equation

$$\Delta(\Psi(\Delta x(k))) + [\gamma(q)-\varepsilon]q(k)\Psi(x(k+1)) = 0 \tag{3.8.15}$$

is nonoscillatory for any $\varepsilon > 0$. Thus by Theorem 3.8.4,

$$[\gamma(q)-\varepsilon]\Gamma^* = \liminf_{k\to\infty} \frac{[\gamma(q)-\varepsilon]\sum_{j=k}^{\infty} q(j)}{\sum_{j=k}^{\infty} q_1(j)} \leq \gamma(q_1). \tag{3.8.16}$$

Since ε is arbitrary, $\Gamma_* \leq \gamma(q_1)/\gamma(q)$ is clear. Next, since

$$\frac{1}{\Gamma^*} = \liminf_{k\to\infty} \frac{\sum_{j=k}^{\infty} q_1(j)}{\sum_{j=k}^{\infty} q(j)}, \tag{3.8.17}$$

by means of the first part of the theorem, we have $1/\Gamma^* \leq \gamma(q)/\gamma(q_1)$, or equivalently, $\gamma(q_1) \leq \Gamma^*\gamma(q)$. If equation (3.2.11) is oscillatory, then $\gamma(q) \leq 1$ by the definition of $\gamma(q)$, thus $\gamma(q_1) \leq \Gamma^*$ as required. □

For the strongly (non)oscillatory criteria of equation (3.2.11), we have the following result.

Theorem 3.8.6. *Equation (3.2.11) is strongly oscillatory if and only if*

$$\limsup_{k\to\infty} k^{\alpha-1}\sum_{j=k}^{\infty} q(j) = \infty. \tag{3.8.18}$$

Equation (3.2.11) is strongly nonoscillatory if and only if

$$\limsup_{k\to\infty} k^{\alpha-1}\sum_{j=k}^{\infty} q(j) = 0. \tag{3.8.19}$$

3.8.2. Generalized Euler difference equation

In this subsection we will investigate some oscillatory properties of the discrete generalized Euler equation

$$\Delta\big(\Psi(\Delta x(k))\big) + \frac{\gamma}{(k+1)^{\alpha}}\Psi\big(x(k+1)\big) = 0, \tag{3.8.20}$$

where $\gamma \in \mathbb{R}$.

Theorem 3.8.7. *To show that equation (3.8.20) is (non)oscillatory, the following four cases are distinguished.*

- (I_1) *If $\gamma > [(\alpha-1)/\alpha]^{\alpha}$, then equation (3.8.20) is oscillatory.*
- (I_2) *If $0 \le \gamma < [(\alpha-1)/\alpha]^{\alpha}$, then equation (3.8.20) is nonoscillatory.*
- (I_3) *If $\gamma < 0$, then equation (3.8.20) is nonoscillatory.*
- (I_4) *If $\gamma = [(\alpha-1)/\alpha]^{\alpha}$, then equation (3.8.20) is nonoscillatory.*

Proof. The statement (I_1) follows by Corollary 3.5.24 since

$$\begin{aligned} k^{\alpha-1}\sum_{j=k}^{\infty}\frac{\gamma}{(j+1)^{\alpha}} &\ge k^{\alpha-1}\gamma\int_{k+1}^{\infty}\frac{1}{x^{\alpha}}dx \\ &= \frac{\gamma k^{\alpha-1}}{(\alpha-1)(k+1)^{\alpha-1}} \\ &\ge \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}+\varepsilon \end{aligned} \tag{3.8.21}$$

for sufficiently small $\varepsilon > 0$.

The statement (I_2) follows by Theorem 3.6.1 since

$$\begin{aligned} k^{\alpha-1}\sum_{j=k}^{\infty}\frac{\gamma}{(j+1)^{\alpha}} &\leq k^{\alpha-1}\gamma\int_k^{\infty}\frac{1}{x^{\alpha}}dx \\ &= \frac{\gamma k^{\alpha-1}}{(\alpha-1)k^{\alpha-1}} \\ &\leq \frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}-\varepsilon \end{aligned} \tag{3.8.22}$$

for sufficiently small $\varepsilon > 0$.

The statement (I_3) follows by the Sturm-type comparison theorem (Theorem 3.3.5) and the result in case (I_2).

Finally we show (I_4). Indeed, according to Lemma 3.4.11 it is sufficient to find $m \in \mathbb{N}$ such that $\mathcal{F}(\xi; m, \infty) > 0$ for any nontrivial $\xi \in U(m)$. Put $m = 1$ and $\xi(k+1) = \sum_{j=1}^{k}\eta(j)$ for $k \in \mathbb{N}$, where η is such that ξ is admissible. Clearly $\Delta\xi(k) = \eta(k)$. Now, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} \mathcal{F}(\xi; 1, \infty) &= \sum_{k=1}^{n}\left[|\eta(k)|^{\alpha} - \frac{\gamma}{(k+1)^{\alpha}}\left|\sum_{j=1}^{k}\eta(j)\right|^{\alpha}\right] \\ &\geq \sum_{k=1}^{n}\left[|\eta(k)|^{\alpha} - \frac{\gamma}{k^{\alpha}}\left(\sum_{j=1}^{k}|\eta(j)|\right)^{\alpha}\right] \\ &> 0 \end{aligned} \tag{3.8.23}$$

by Lemma 3.7.1, and hence equation (3.8.20) is nonoscillatory. □

Now we conclude that equation (3.8.20) is oscillatory if $\gamma > [(\alpha-1)/\alpha]^{\alpha}$ and nonoscillatory otherwise. Thus, if we consider equation (3.8.20) as an equation of the form (3.8.1) with $c(k) \equiv 1$, more precisely, if $\lambda = \gamma$ and $q(k) = (k+1)^{-\alpha}$, then it is easy to see that the oscillation constant of such a sequence $\{q(k)\}$ is equal to $[(\alpha-1)/\alpha]^{\alpha}$.

3.9. Half-linear difference equations with damping term

Consider the second-order half-linear damped difference equation of the form

$$\Delta\left(c(k)\Psi(\Delta x(k))\right) + p(k)\Psi(\Delta x(k)) + q(k)\Psi(x(k+1)) = 0, \tag{3.9.1}$$

where $\{c(k)\}$, $\{p(k)\}$, and $\{q(k)\}$ are sequences of nonnegative real numbers with $c(k) > 0$ and $q(k) \not\equiv 0$ for $k \geq m$ for some $m \in \mathbb{N}$, and Ψ is defined as in equation (3.2.9).

We will study the oscillatory property of equation (3.9.1) via comparison with the oscillatory behavior of equation (3.2.9).

First, we prove the following two lemmas.

Lemma 3.9.1. *Assume that for some $m \in \mathbb{N}$,*

$$c(k) > p(k) \quad \text{for } k \geq m, \tag{3.9.2}$$

$$P(k) = \sum_{n=k}^{\infty} \left(\frac{1}{c(n)} \prod_{j=m}^{n-1} \left(1 - \frac{p(j)}{c(j)}\right)\right)^{\beta-1}, \qquad P(m) = \infty. \tag{3.9.3}$$

If $\{x(k)\}$ is a nonoscillatory solution of (3.9.1), then there is an integer $k_0 \geq m$ such that $x(k)\Delta x(k) > 0$ for all $k \geq k_0$.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (3.9.1). Without loss of generality we assume that $x(k) > 0$ for $k \geq m$; the proof for the case when $x(k)$ is eventually negative is similar and will be omitted.

We consider the following two cases for the behavior of $\{\Delta x(k)\}$.

Case 1. $\{\Delta x(k)\}$ is oscillatory. First, suppose that there exists an integer $k_1 \geq m$ such that $\Delta x(k_1) < 0$. From equation (3.9.1), we have

$$\begin{aligned} &\Delta\big(c(k_1)\Psi(\Delta x(k_1))\big)\Delta x(k_1) \\ &\quad = -p(k_1)\Psi(\Delta x(k_1))\Delta x(k_1) - q(k_1)\Psi(x(k_1+1))\Delta x(k_1), \end{aligned} \tag{3.9.4}$$

and so

$$\big[c(k_1+1)\Psi(\Delta x(k_1+1)) - c(k_1)\Psi(\Delta x(k_1))\big]\Delta x(k_1) > -p(k_1)\Psi(\Delta x(k_1))\Delta x(k_1). \tag{3.9.5}$$

Hence

$$\begin{aligned} &c(k_1+1)\Psi(\Delta x(k_1+1))\Delta x(k_1) \\ &\quad > \big(c(k_1) - p(k_1)\big)\Psi(\Delta x(k_1))\Delta x(k_1) > 0. \end{aligned} \tag{3.9.6}$$

Thus, we have $\Delta x(k_1+1) < 0$, and so by induction, we obtain $\Delta x(k) < 0$ for all $k \geq k_1$.

Next, suppose $\Delta x(k_1) = 0$. Then (3.9.1) implies $\Delta x(k_1+1) < 0$, and we are back to the above considered case. Thus, in either situation, we have $\Delta x(k) < 0$ for all $k \geq k_1 + 1$. This contradicts the assumption that $\{\Delta x(k)\}$ oscillates, so $\{\Delta x(k)\}$ is eventually of fixed sign.

Case 2. Let $\Delta x(k) < 0$ for $k \geq k_0 \geq n_0$. Define $u(k) = -c(k)\Psi(\Delta x(k))$. Then, from equation (3.9.1) we get

$$\Delta u(k) + \frac{p(k)}{c(k)}u(k) \geq 0. \tag{3.9.7}$$

Now (3.9.7) implies

$$u(k) \geq u(k_0) \prod_{j=k_0}^{k-1} \left(1 - \frac{p(j)}{c(j)}\right), \tag{3.9.8}$$

so

$$\Delta x(k) \leq -\Psi^{-1}(u(k_0))\Psi^{-1}\left(\frac{1}{c(k)} \prod_{j=k_0}^{k-1} \left(1 - \frac{p(j)}{c(j)}\right)\right). \tag{3.9.9}$$

Summing both sides of (3.9.9) from k_0 to $k-1$, we obtain

$$x(k) - x(k_0) \leq \Psi^{-1}(c(k_0))\Delta x(k_0) \sum_{i=k_0}^{k-1} \left(\frac{1}{c(i)} \prod_{j=k_0}^{i-1} \left(1 - \frac{p(j)}{c(j)}\right)\right)^{\beta-1} \tag{3.9.10}$$

for $k \geq k_0$. Now condition (3.9.3) implies that $\{x(k)\}$ is eventually negative, which is a contradiction.

This completes the proof. □

The following lemma extends Theorem 1.20.1 to the half-linear difference equation (3.2.9) and the inequality

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi\big(x(k+1)\big) \leq 0. \tag{3.9.11}$$

We assume that $c(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$ and

$$\sum^{\infty} c^{1-\beta}(j) = \infty. \tag{3.9.12}$$

Lemma 3.9.2. *Assume that condition (3.9.12) holds. If inequality (3.9.11) has an eventually nonnegative solution, then equation (3.2.9) also has an eventually nonnegative solution.*

Proof. Let $\{x(k)\}$ be an eventually nonnegative solution of inequality (3.9.11). By Lemma 3.5.9, we have $\Delta x(k) \geq 0$ eventually. Let

$$u(k) = c(k)\Psi(\Delta x(k)). \tag{3.9.13}$$

Then

$$\Delta x(k) = \Psi^{-1}\left(\frac{u(k)}{c(k)}\right) \tag{3.9.14}$$

for $k \geq m$ for some $m \in \mathbb{N}$. Summing both sides of (3.9.14) from m to k, we have

$$x(k+1) = x(m) + \sum_{j=m}^{k} \Psi^{-1}\left(\frac{u(j)}{c(j)}\right). \tag{3.9.15}$$

From inequality (3.9.11), we have

$$\Delta u(k) \leq -q(k)\Psi(x(k+1)), \tag{3.9.16}$$

and hence we see that

$$u(k) \geq \sum_{j=k}^{\infty} q(j)\Psi\left(x(m) + \sum_{i=m}^{j} \Psi^{-1}\left(\frac{u(i)}{c(i)}\right)\right). \tag{3.9.17}$$

Now we define a sequence of successive approximations $\{y(k,\ell)\}$ as follows:

$$\begin{aligned} y(k,0) &= u(k), \\ y(k,\ell+1) &= \sum_{j=k}^{\infty} q(j)\Psi\left(x(m) + \sum_{i=m}^{j} \Psi^{-1}\left(\frac{y(i,\ell)}{c(i)}\right)\right) \quad \text{for } \ell \in \mathbb{N}_0 \end{aligned} \tag{3.9.18}$$

for $k \geq m$. Obviously, we can prove that $0 \leq y(k,\ell) \leq u(k)$ and $y(k,\ell+1) \leq y(k,\ell)$ for $\ell \in \mathbb{N}_0$ and $k \geq m$. Then the sequence $\{y(k,\ell)\}$ is nonnegative and nonincreasing in ℓ for each k. This means that we may define $y(k) = \lim_{\ell\to\infty} y(k,\ell) \geq 0$. Since $0 \leq y(k) \leq y(k,\ell) \leq u(k)$ for all $\ell \in \mathbb{N}_0$ and $k \geq m$ and since

$$\sum_{i=m}^{j} \Psi^{-1}\left(\frac{y(i,\ell)}{c(i)}\right) \leq \sum_{i=m}^{j} \Psi\left(\frac{u(i)}{c(i)}\right), \tag{3.9.19}$$

the convergence of the series in the definition of $y(k,\ell)$ is uniform with respect to ℓ.

Taking the limit on both sides in the definition of $y(k,\ell)$, we have

$$y(k) = \sum_{j=k}^{\infty} q(j)\Psi\left(x(m) + \sum_{i=m}^{j} \Psi^{-1}\left(\frac{y(i)}{c(i)}\right)\right). \tag{3.9.20}$$

Therefore

$$\Delta y(k) = -q(k)\Psi(v(k+1)), \tag{3.9.21}$$

where

$$v(k+1) = x(m) + \sum_{i=m}^{k} \Psi^{-1}\left(\frac{y(i)}{c(i)}\right) \geq 0. \tag{3.9.22}$$

Now $\Delta v(k) = \Psi^{-1}(y(k)/c(k))$, or $y(k) = c(k)\Psi(\Delta v(k))$ and

$$\Delta\big(c(k)\Psi(\Delta v(k))\big) = \Delta y(k) = -q(k)\Psi(v(k+1)). \tag{3.9.23}$$

We showed that equation (3.2.9) has a nonnegative solution $\{v(k)\}$. This completes the proof. □

Now we can prove the following interesting criterion.

Theorem 3.9.3. *Let conditions (3.9.2), (3.9.3), and (3.9.12) hold. If equation (3.2.9) is oscillatory, then equation (3.9.1) is oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (3.9.1). By Lemma 3.9.1 we see that $\Delta x(k) > 0$ eventually. Thus, from equation (3.9.1) we get

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi\big(x(k+1)\big) \le 0 \quad \text{eventually.} \tag{3.9.24}$$

By applying Lemma 3.9.2, we see that equation (3.2.9) has an eventually positive solution, which is a contradiction. This completes the proof. □

In the case when condition (3.9.3) fails to hold, that is,

$$P(m) < \infty \quad \text{for } m \in \mathbb{N} \tag{3.9.25}$$

holds, we give the following lemma.

Lemma 3.9.4. *Let conditions (3.9.2) and (3.9.25) hold, and suppose*

$$\liminf_{k\to\infty} q(k)\Psi\big(P(k+1)\big) > 1. \tag{3.9.26}$$

If $\{x(k)\}$ is a nonoscillatory solution of equation (3.9.1), then there exists an integer $k_0 \ge m \in \mathbb{N}$ such that $x(k)\Delta x(k) > 0$ for $k \ge k_0$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (3.9.1), say, $x(k) > 0$ for $k \ge n_0$. From the proof of Lemma 3.9.1, we have that $\{\Delta x(k)\}$ is eventually of one sign.

Let $\Delta x(k) < 0$ for all $k \ge k_0$ for some $k_0 \ge n_0$. From equation (3.9.1), we have

$$\Delta u(k) + \frac{p(k)}{c(k)}u(k) \ge 0, \tag{3.9.27}$$

where $u(k) = -c(k)\Psi(\Delta x(k))$. As in the proof of Lemma 3.9.1, we obtain

$$x(k) - x(k_0) \le \Psi^{-1}\big(c(k_0)\big)\Delta x(k_0)\sum_{i=k_0}^{k-1}\left(\frac{1}{c(i)}\prod_{j=k_0}^{i-1}\left(1 - \frac{p(j)}{c(j)}\right)\right)^{\beta-1} \tag{3.9.28}$$

for $k \ge k_0$. Hence

$$x(k_0) \ge -\Psi^{-1}\big(c(k_0)\Psi(\Delta x(k_0))\big)\sum_{i=k_0}^{k}\left(\frac{1}{c(i)}\prod_{j=k_0}^{i-1}\left(1 - \frac{p(j)}{c(j)}\right)\right)^{\beta-1} \tag{3.9.29}$$

for $k \ge k_0$. Letting $k \to \infty$, we find

$$\Psi\big(x(k_0)\big) \ge -c(k_0)\Psi\big(\Delta x(k_0)\big)\Psi\big(P(k_0)\big). \tag{3.9.30}$$

From equation (3.9.1) we get

$$c(k+1)\Psi\big(\Delta x(k+1)\big) - \big(c(k) - p(k)\big)\Psi\big(\Delta x(k)\big) + q(k)\Psi\big(x(k+1)\big) = 0, \tag{3.9.31}$$

and so by (3.9.2) we have

$$c(k+1)\Psi\big(\Delta x(k+1)\big) + q(k)\Psi\big(x(k+1)\big) < 0 \quad \text{for } k \geq k_0. \tag{3.9.32}$$

From (3.9.30), inequality (3.9.32) implies

$$c(k+1)\Psi\big(\Delta x(k+1)\big) - c(k+1)q(k)\Psi\big(P(k+1)\big)\Psi\big(\Delta x(k+1)\big) < 0 \quad \text{for } k \geq k_0. \tag{3.9.33}$$

Thus $q(k)\Psi(P(k+1)) \leq 1$ for all $k \geq k_0$, which contradicts (3.9.26). This completes the proof. □

Theorem 3.9.5. *Let conditions (3.9.2), (3.9.12), (3.9.25), and (3.9.26) hold. If equation (3.2.9) is oscillatory, then equation (3.9.1) is oscillatory.*

In equation (3.9.1), if $\alpha = 2$, then equation (3.9.1) is reduced to the linear damped difference equation

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k)x(k+1) = 0. \tag{3.9.34}$$

The following two corollaries supplement our treatment of (3.9.34) from Chapter 1.

Corollary 3.9.6. *Assume that condition (3.9.2) holds and that either*

$$P^*(m) = \sum_{k=m}^{\infty} \frac{1}{c(k)} \prod_{j=m}^{k-1} \left(1 - \frac{p(j)}{c(j)}\right) = \infty \tag{3.9.35}$$

or

$$P^*(m) < \infty, \qquad \liminf_{k\to\infty} q(k)P^*(k+1) > 1. \tag{3.9.36}$$

If $\{x(k)\}$ is a nonoscillatory solution of equation (3.9.34), then $x(k)\Delta x(k) > 0$ eventually.

Corollary 3.9.7. *Let conditions (3.9.2),*

$$\sum^{\infty} \frac{1}{c(j)} = \infty, \tag{3.9.37}$$

and either condition (3.9.35) or (3.9.36) hold. If the equation

$$\Delta\big(c(k)\Delta x(k)\big) + q(k)x(k+1) = 0 \tag{3.9.38}$$

is oscillatory, then equation (3.9.34) is also oscillatory.

In equation (3.9.1), if $p(k)$ is nonpositive for $k \geq n_0 \in \mathbb{N}$, then we see that conditions (3.9.2) and (3.9.3) are automatically satisfied. Hence, if $\{x(k)\}$ is a nonoscillatory solution of equation (3.9.1), then $x(k)\Delta x(k) > 0$ eventually.

In the following result we consider the equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + p(k)\Psi(\Delta x(k)) + \big(q(k) + q^*(k)\big)\Psi\big(x(k+1)\big) = 0, \tag{3.9.39}$$

where $\{c(k)\}$, $\{q(k)\}$, and Ψ are as in equation (3.2.9), $\{p(k)\}$ is a sequence of nonpositive real numbers, and $\{q^*(k)\}$ is a sequence of nonnegative real numbers.

We will assume that

$$\sum^{\infty} p(j) > -\infty. \tag{3.9.40}$$

For each of the positive constants λ and μ, there exists an integer $N(\lambda, \mu)$ such that

$$\frac{\lambda}{c(k)}\,|p(k)| \leq \mu q^*(k) \quad \text{for every } k \geq N(\lambda, \mu). \tag{3.9.41}$$

Now we prove the following oscillation criterion for equation (3.9.39).

Theorem 3.9.8. *Let conditions (3.9.12), (3.9.40), and (3.9.41) hold. If equation (3.2.9) is oscillatory, then equation (3.9.39) is also oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (3.9.39), say, $x(k) > 0$ for $k \geq m \in \mathbb{N}$. It is easy to check that $\Delta x(k) > 0$ for $k \geq n_1 \geq m$. There exist a constant $a > 0$ and an integer $n_2 \geq n_1$ such that

$$x(k) \geq a, \qquad \Psi\big(x(k+1)\big) \geq \Psi(a) = \mu > 0. \tag{3.9.42}$$

Next, from equation (3.9.39), we have

$$\sum_{j=n_1}^{\infty} \frac{\Delta\big(c(j)\Psi(\Delta x(j))\big)}{\Psi\big(\Delta x(j)\big)} \leq -\sum_{j=n_1}^{\infty} p(j) < \infty. \tag{3.9.43}$$

Thus it is necessary that

$$\lim_{j\to\infty}\left[c(j+1)\frac{\Psi\big(\Delta x(j+1)\big)}{\Psi\big(\Delta x(j)\big)} - c(j)\right] = 0, \tag{3.9.44}$$

and hence we conclude that $\{c(k)\Psi(\Delta x(k))\}$ is a positive nonincreasing sequence. There exist a constant $\lambda > 0$ and an integer $k_1 \geq n_1$ with $0 < \Psi(\Delta x(k)) \leq \lambda/c(k)$ for $k \geq k_1$ and therefore

$$\frac{\lambda}{c(k)}p(k) \leq p(k)\Psi\big(\Delta x(k)\big) \quad \text{for } k \geq k_1. \tag{3.9.45}$$

Choose $N(\lambda,\mu) \geq \max\{k_1, n_2\}$. Hence, from equation (3.9.39) we have eventually

$$\begin{aligned}
&\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi\big(x(k+1)\big)\\
&\quad \leq \Delta\big(c(k)\Psi(\Delta x(k))\big) + \frac{\lambda}{c(k)}p(k) + \mu q^*(k) + q(k)\Psi\big(x(k+1)\big)\\
&\quad \leq \Delta\big(c(k)\Psi(\Delta x(k))\big) + p(k)\Psi(\Delta x(k)) + \big(q(k)+q^*(k)\big)\Psi\big(x(k+1)\big)\\
&\quad = 0.
\end{aligned} \tag{3.9.46}$$

By Lemma 3.9.2, equation (3.2.9) has a positive solution, which is a contradiction. This completes the proof. □

Example 3.9.9. Consider the half-linear damped difference equation

$$\Delta\left(\frac{1}{k}\Psi(\Delta x(k))\right) - e^{-k}\Psi(\Delta x(k)) + 2\Psi\big(x(k+1)\big) = 0 \quad \text{for } k \in \mathbb{N}, \tag{3.9.47}$$

where Ψ is as in equation (3.2.9). The oscillation of equation (3.9.47) follows from Theorem 3.9.8. For, we observe that $q(k) = q^*(k) = 1$ and so the equation

$$\Delta\left(\frac{1}{k}\Psi(\Delta x(k))\right) + \Psi\big(x(k+1)\big) = 0 \tag{3.9.48}$$

is oscillatory (see Theorem 3.5.34). Moreover, $\lambda|p(k)|/c(k) \leq \mu q^*(k)$, that is, $\lambda k e^{-k} \leq \mu$ is satisfied eventually for any of the positive constants λ and μ.

3.10. Half-linear difference equations with forcing term

In this section we will discuss the oscillatory behavior of the forced equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)\Psi\big(x(k+1)\big) = e(k), \tag{3.10.1}$$

where $c(k)$, $q(k)$, and Ψ are as in equation (3.2.9) and $\{e(k)\}$ is a sequence of real numbers, $c(k) > 0$ and $q(k) \geq 0$ for $k \geq m$ for some $m \in \mathbb{N}$.

Now we present the following result.

Theorem 3.10.1. *If for every $N \geq m$ for some $m \in \mathbb{N}$,*

$$\liminf_{k\to\infty} \sum_{j=N}^{k-1} e(j) = -\infty, \qquad \limsup_{k\to\infty} \sum_{j=N}^{k-1} e(j) = \infty, \tag{3.10.2}$$

$$\begin{aligned}
\liminf_{k\to\infty} \sum_{j=N}^{k-1} c^{1-\beta}(j)\Psi^{-1}\left(\sum_{i=N}^{j-1} e(i)\right) &= -\infty,\\
\limsup_{k\to\infty} \sum_{j=N}^{k-1} c^{1-\beta}(j)\Psi^{-1}\left(\sum_{i=N}^{j-1} e(i)\right) &= \infty,
\end{aligned} \tag{3.10.3}$$

then equation (3.10.1) is oscillatory.

Proof. Assume the contrary. Then without loss of generality, we can assume that there is a nonoscillatory solution $\{x(k)\}$ of equation (3.10.1), say, $x(k) > 0$ for $k \geq n_1 \geq m \in \mathbb{N}$. From (3.10.1), we have

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) \leq e(k) \quad \text{for } k \geq n_1. \tag{3.10.4}$$

Thus it follows from (3.10.4) that

$$c(k)\Psi(\Delta x(k)) \leq c(n_1)\Psi(\Delta x(n_1)) + \sum_{i=n_1}^{k-1} e(i). \tag{3.10.5}$$

By (3.10.3), there exists a sufficiently large integer $N \geq n_1$ such that $\Psi(\Delta x(N)) < 0$ and $\Psi(\Delta x(k)) < 0$ for $k \geq N$. Replacing n_1 by N in (3.10.5), we get

$$\begin{aligned} \Delta x(k) &\leq c^{1-\beta}(k)\Psi^{-1}\left(\sum_{i=N}^{k-1} e(i)\right), \\ x(k) &\leq x(N) + \sum_{j=N}^{k-1} c^{1-\beta}(j)\Psi^{-1}\left(\sum_{i=N}^{j-1} e(i)\right). \end{aligned} \tag{3.10.6}$$

Therefore, $\liminf_{k\to\infty} x(k) = -\infty$, which contradicts the fact that $x(k) > 0$ eventually. □

Theorem 3.10.2. *If for every constant $\gamma > 0$ and all large $N \geq m$ for some $m \in \mathbb{N}$,*

$$\liminf_{k\to\infty} \sum_{j=N}^{k-1} c^{1-\beta}(j)\Psi^{-1}\left(\sum_{i=N}^{j-1} e(i) + \gamma\right) = -\infty, \tag{3.10.7}$$

$$\limsup_{k\to\infty} \sum_{j=N}^{k-1} c^{1-\beta}(j)\Psi^{-1}\left(\sum_{i=N}^{j-1} e(i) - \gamma\right) = \infty, \tag{3.10.8}$$

then equation (3.10.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (3.10.1), say, $x(k) > 0$ for $k \geq m$. As in the proof of Theorem 3.10.1, we obtain (3.10.4) and (3.10.5). Now there exist a positive constant $\gamma_1 > 0$ and an integer $N \geq m$ such that

$$c(k)\Psi(\Delta x(k)) \leq \gamma_1 + \sum_{i=1}^{k-1} e(i) \quad \text{for } k \geq N+1, \tag{3.10.9}$$

so

$$\Delta x(j) \leq c^{1-\beta}(j)\Psi^{-1}\left(\gamma_1 + \sum_{i=N}^{j-1} e(i)\right) \quad \text{for } j \geq N+1. \tag{3.10.10}$$

Summing both sides of (3.10.10) from N to $k-1$ and taking lim inf as $k \to \infty$, we obtain

$$\liminf_{k\to\infty} x(k) \le x(N) + \liminf_{k\to\infty} \sum_{j=N}^{k-1} c^{1-\beta}(j)\Psi^{-1}\left(\gamma_1 + \sum_{i=N}^{j-1} e(i)\right) = -\infty, \qquad (3.10.11)$$

which contradicts the fact that $x(k) > 0$ for $k \ge n_0$. □

In equation (3.10.1) if $q(k) \le 0$ for $k \ge m \in \mathbb{N}$, we present the following criterion.

Theorem 3.10.3. *If $q(k) \le 0$ for $k \ge m \in \mathbb{N}$, for every $\gamma > 0$ and all large $N \ge n_0$ and conditions (3.10.7) and (3.10.8) are satisfied, then every bounded solution of equation (3.10.1) is oscillatory.*

Proof. Let $\{x(k)\}$ be a bounded nonoscillatory solution of (3.10.1), say, $x(k) > 0$ for $k \ge n_0 \in \mathbb{N}$. From (3.10.1), we have $\Delta(c(k)\Psi(\Delta x(k))) \ge e(k)$. As in the proof of Theorem 3.10.2, there exist a positive constant $\gamma_2 > 0$ and $N \ge n_0$ such that

$$\Delta x(j) \ge c^{1-\beta}(j)\Psi^{-1}\left(\sum_{i=N}^{j-1} e(i) - \gamma_2\right) \quad \text{for } j \ge N+1. \qquad (3.10.12)$$

Summing both sides of (3.10.12), taking the upper limit as $k \to \infty$, and employing condition (3.10.8), we obtain a contradiction to the boundedness of $\{x(k)\}$. □

The following examples illustrate the methods presented above.

Example 3.10.4. Consider the forced half-linear difference equation

$$\Delta\left(k(\Delta x(k))^{\alpha}\right) + q(k)x^{\alpha}(k+1) = (-1)^k k \quad \text{for } k \ge m \in \mathbb{N}, \qquad (3.10.13)$$

where $\alpha \ge 1$ is the ratio of two positive odd integers and $\{q(k)\}$ is a sequence of nonnegative real numbers.

It is easy to check that all the hypotheses of Theorems 3.10.1 and 3.10.2 are satisfied, and hence all solutions of equation (3.10.13) are oscillatory.

Example 3.10.5. Consider the forced half-linear difference equation

$$\Delta\left(\frac{1}{k}(\Delta x(k))^{\alpha}\right) - q(k)x^{\alpha}(k+1) = (-1)^k k \quad \text{for } k \ge m \in \mathbb{N}, \qquad (3.10.14)$$

where α and $q(k)$ are as in equation (3.10.13). All conditions of Theorem 3.10.3 are satisfied, and hence all bounded solutions of equation (3.10.14) are oscillatory.

Remark 3.10.6. When $\alpha - 1$ is a ratio of two positive odd integers, then conditions (3.10.3), (3.10.7), and (3.10.8) take the form

$$\liminf_{k\to\infty}\sum_{j=N}^{k-1}\left(\frac{1}{c(j)}\sum_{i=N}^{j-1}e(i)\right)^{\beta-1}, \qquad \limsup_{k\to\infty}\sum_{j=N}^{k-1}\left(\frac{1}{c(j)}\sum_{i=N}^{j-1}e(i)\right)^{\beta-1},$$
$$\liminf_{k\to\infty}\sum_{j=N}^{k-1}\left(\frac{1}{c(j)}\left[\sum_{i=N}^{j-1}e(i)+\gamma\right]\right)^{\beta-1}, \tag{3.10.15}$$
$$\limsup_{k\to\infty}\sum_{j=N}^{k-1}\left(\frac{1}{c(j)}\left[\sum_{i=N}^{j-1}e(i)-\gamma\right]\right)^{\beta-1},$$

respectively.

3.11. Notes and general discussions

(1) For many of the results from this chapter, see [241].

(2) The results of Sections 3.2 and 3.3 are taken from Řehák [247]. Note that the Picone identity, that is, Lemma 3.3.1 was used in [110] in order to investigate oscillatory properties of the forced second-order half-linear difference equation (3.10.1). We also refer to [168] for the half-linear continuous version of Picone's identity and some of its applications.

We observe that the Picone identity could also be used to show that absence of generalized zeros of the solution of equation (3.2.9) implies positive definiteness of the functional $\mathcal{F}$ (the implication Theorem 3.3.4(II)$\Rightarrow$(IV)). We also included the generalized Riccati equation in Theorem 3.3.4 since such an equivalence is important for applications of the presented theory.

(3) Theorems 3.4.3 and 3.4.5 are due to Došlý and Řehák [112] while Theorems 3.4.7–3.4.10 are taken from Řehák [242]. Theorems 3.4.14 and 3.4.15 are also taken from Došlý and Řehák [112]. The conjugacy criteria, that is, Theorems 3.4.16–3.4.19 are due to Řehák [246]. We note that the proofs of Theorems 3.4.9 and 3.4.10 are omitted since these proofs are similar to those of the continuous case, see [169]. The results of this section when specialized to linear difference equations, that is, equation (3.2.9) with $\alpha = 2$, are supplementing those from Chapter 1. The details are left to the reader.

(4) Theorems 3.5.1–3.5.8 are due to Řehák [246]. The rest of the results of Section 3.5.1 is taken from Řehák [244]. We note that Lemma 3.5.12 is given in [153]. These results are extensions of those presented in Section 1.11. The results of Section 3.5.2 are taken from Řehák [246]. These results are the discretization of those in [169]. The results of Section 3.5.3 are due to Řehák [247] while Lemma 3.5.33 is taken from Došlý [107]. We note that some of these results when specialized to linear

difference equations, that is, equation (3.2.9) with $\alpha = 2$, are new and are to be added to the results of Chapter 1.

(5) The results of Section 3.6 are taken from Řehák [248]. The preliminary results from analysis are extracted from Cheng and Patula [93].

(6) The results of Section 3.7 are due to Li and Yeh [197]. These results are extensions of the results obtained by Cheng et al. [94]. We note that Lemma 3.7.1 is taken from [153] and Lemma 3.7.2 is due to Cheng and Lu [92]. We also note that Cheng and Lu [92] showed that Hardy's inequality, that is, Lemma 3.7.1, can be viewed as a necessary condition for the existence of positive nondecreasing solutions of equation (3.7.1).

(7) The results of Section 3.8 are extracted from Řehák [248]. The so-called strong oscillation and further related concepts are introduced in [207] for the linear second order equation $x''(t) + q(t)x(t) = 0$. The proofs of Theorems 3.8.2 and 3.8.3 are essentially the same as in the half-linear continuous case (see [188]), and hence we omitted the details. We note that Theorems 3.8.4 and 3.8.5 are extensions of [94, Theorems 4.2 and 4.3].

(8) The results of Sections 3.9 and 3.10 for damped and forced equations, respectively, are discrete analogues of some of the results in [20, Chapter 3].

(9) Regarding interesting and intensive study of oscillation theory of half-linear differential equations, we refer the reader to [20], in particular Chapter 3 and parts of Chapters 4 and 5 in there.

(10) Oscillation theory for half-linear difference equations will be discussed again in the following chapters.

(11) Throughout this chapter we observe that the techniques of proofs that are needed in the discrete case are often different from the continuous case and mostly are complicated. This is due to the absence of the "chain rule" for computing the difference of the "composite" sequence (like $\Delta f(x(k))$, see [267]) and some other specific properties of difference (and summation) calculus. Also, we note that there exist certain limitations in the use of the linear approach to investigate half-linear equations. These limitations are due to the absence of transformation theory similar to that for linear equations or the impossibility of the extension of the so-called Casoratian to the half-linear discrete case. Because of these difficulties, a great number of open problems arises. Therefore, the reader must search for the concepts and results available in the linear case and re-establish them for the half-linear case.

4 Oscillation theory for nonlinear difference equations I

This chapter presents oscillation and asymptotic behavior of all solutions of second-order nonlinear difference equations. Section 4.1 deals with some oscillation criteria applicable to nonlinear difference equations with alternating coefficients. In Section 4.2 we will provide necessary and sufficient conditions for oscillation of superlinear difference equations. Oscillation criteria for sublinear difference equations are given in Section 4.3. Section 4.4 is concerned with oscillation characterization for nonlinear difference equations. In Section 4.5, we will present some new criteria for oscillation of damped difference equations. In Section 4.6, we will reduce the problem of asymptotically equivalent solutions of a second-order nonlinear difference equation to the boundedness of solutions of some difference equations of first order. Oscillation criteria for nonlinear difference equations via Liapunov's second method will be given in Section 4.7.

4.1. Oscillation criteria

In this chapter we will present oscillation criteria for second-order nonlinear difference equations of the form

$$\Delta^2 x(k-1) + q(k) f\big(x(k)\big) = 0, \tag{4.1.1}$$

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k) f\big(x(k)\big) = 0, \tag{4.1.2}$$

and the more general damped equation

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k) f\big(x(k+1)\big) = 0, \tag{4.1.3}$$

where

(i) $\{c(k)\}$ is a sequence of positive real numbers,
(ii) $\{p(k)\}$, $\{q(k)\}$ are sequences of real numbers,
(iii) $f \in C(\mathbb{R}, \mathbb{R})$ satisfies

$$x f(x) > 0 \quad \forall x \neq 0. \tag{4.1.4}$$

We say that f is *superlinear* if

$$\int_{\varepsilon}^{\infty} \frac{du}{f(u)} < \infty, \quad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty \quad \text{for } \varepsilon > 0, \tag{4.1.5}$$

and f is called *sublinear* if

$$\int_{0}^{\varepsilon} \frac{du}{f(u)} < \infty, \quad \int_{0}^{-\varepsilon} \frac{du}{f(u)} < \infty \quad \text{for } \varepsilon > 0. \tag{4.1.6}$$

The special case when $f(x) = |x|^{\gamma} \operatorname{sgn} x$ with $x \in \mathbb{R}$ and $\gamma > 0$ is of particular interest. In fact, the difference equations

$$\Delta^2 x(k-1) + q(k)\left|x(k)\right|^{\gamma} \operatorname{sgn} x(k) = 0, \tag{4.1.7}$$

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)\left|x(k)\right|^{\gamma} \operatorname{sgn} x(k) = 0, \tag{4.1.8}$$

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k)\left|x(k+1)\right|^{\gamma} \operatorname{sgn} x(k+1) = 0 \tag{4.1.9}$$

are prototypes of (4.1.1), (4.1.2), and (4.1.3), respectively and will be discussed extensively.

We will assume that

$$\sum^{\infty} \frac{1}{c(j)} = \infty, \tag{4.1.10}$$

$$f(u) - f(v) = F(u,v)(u-v) \quad \text{for } u, v \neq 0, \tag{4.1.11}$$

where F is a nonnegative function.

We begin with the following oscillation criterion for equation (4.1.2).

Theorem 4.1.1. *If conditions (4.1.4), (4.1.10), and (4.1.11) hold and*

$$\sum^{\infty} q(j) = \infty, \tag{4.1.12}$$

then equation (4.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. We will consider only this case since the proof for the case $x(k) < 0$ for $k \geq m$ for some $m \in \mathbb{N}$ is similar. Now

$$\Delta\left(\frac{c(k-1)\Delta x(k-1)}{f(x(k))}\right) = -q(k) - \frac{c(k)\left(\Delta x(k)\right)^2 F\left(x(k+1), x(k)\right)}{f(x(k)) f(x(k+1))}, \tag{4.1.13}$$

which in view of (4.1.11) provides

$$\Delta\left(\frac{c(k-1)\Delta x(k-1)}{f(x(k))}\right) \leq -q(k) \tag{4.1.14}$$

for $k \geq m$. Summing inequality (4.1.14) from $m+1$ to k gives

$$\frac{c(k)\Delta x(k)}{f(x(k+1))} \leq \frac{c(m)\Delta x(m)}{f(x(m+1))} - \sum_{j=m+1}^{k} q(j). \tag{4.1.15}$$

This implies that there exists $m_1 \geq m$ such that $\Delta x(k) < 0$ for $k \geq m_1$. Condition (4.1.12) also implies that there exists an integer $m_2 \geq m_1$ such that

$$\sum_{j=m_2+1}^{k} q(j) \geq 0 \quad \text{for } k \geq m_2 + 1. \tag{4.1.16}$$

Summing equation (4.1.2) from $m_2 + 1$ to k provides

$$\begin{aligned}
c(k)\Delta x(k) &= c(m_2)\Delta x(m_2) - \sum_{j=m_2+1}^{k} q(j) f(x(j+1)) \\
&= c(m_2)\Delta x(m_2) - f(x(k+1)) \sum_{j=m_2+1}^{k} q(j) + \sum_{i=m_2+1}^{k} \Delta f(i)\left[\sum_{j=m_2+1}^{i} q(j)\right] \\
&= c(m_2)\Delta x(m_2) - f(x(k+1)) \sum_{j=m_2+1}^{k} q(j) \\
&\quad + \sum_{i=m_2+1}^{k} F\left(x(i+1), x(i)\right)\Delta x(i)\left[\sum_{j=m_2+1}^{i} q(j)\right] \\
&\leq c(m_2)\Delta x(m_2)
\end{aligned} \tag{4.1.17}$$

for $k \geq m_2 + 1$. Thus

$$\Delta x(k) \leq c(m_2)\Delta x(m_2)\frac{1}{c(k)} \quad \text{for } k \geq m_2 + 1. \tag{4.1.18}$$

Summing (4.1.18) from $m_2 + 1$ to k provides

$$x(k+1) \le x(m_2+1) + c(m_2)\Delta x(m_2) \sum_{j=m_2+1}^{k} \frac{1}{c(j)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{4.1.19}$$

which contradicts the fact that $\{x(k)\}$ is eventually positive. □

For illustration we consider the following example.

Example 4.1.2. The equation

$$\Delta\big(k\Delta x(k-1)\big) + 2(2k+1)x(k) = 0 \tag{4.1.20}$$

has an oscillatory solution $x(k) = \{(-1)^k\}$. All conditions of Theorem 4.1.1 are satisfied.

Next we present the following theorem.

Theorem 4.1.3. *If conditions (4.1.4), (4.1.10), and (4.1.11) hold,*

$$\liminf_{k\to\infty} \sum_{j=m}^{k} q(j) \ge 0 \quad \text{for all large } m \in \mathbb{N}, \tag{4.1.21}$$

$$\limsup_{k\to\infty} \sum_{j=m}^{k} q(j) = \infty \quad \text{for all large } m \in \mathbb{N}, \tag{4.1.22}$$

then equation (4.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.2), say, $x(k) > 0$ for $k \ge m$ for some $m \in \mathbb{N}$. We consider the following three cases.

Case 1. If $\{\Delta x(k)\}$ is oscillatory, then there exists a sequence $k_n \to \infty$ such that $\Delta x(k_n) < 0$. We choose ℓ so large that (4.1.21) holds. Then summing (4.1.14) from $k_\ell + 1$ to k followed by taking lim sup as $k \to \infty$ provides

$$\limsup_{k\to\infty} \frac{c(k)\Delta x(k)}{f(x(k+1))} \le \frac{c(k_\ell)\Delta x(k_\ell)}{f(x(k_\ell+1))} + \limsup_{k\to\infty}\left[-\sum_{j=k_\ell+1}^{k} q(j)\right] < 0. \tag{4.1.23}$$

It follows from (4.1.23) that $\lim_{k\to\infty} \Delta x(k) < 0$. This contradicts the assumption that $\{\Delta x(k)\}$ oscillates.

Case 2. If $\Delta x(k) \ge 0$ for $k \ge m_1 \ge m$, then summation of (4.1.14) from $m_1 + 1$ to k yields

$$\frac{c(k)\Delta x(k)}{f(x(k+1))} \le \frac{c(m_1)\Delta x(m_1)}{f(x(m_1+1))} - \sum_{j=m_1+1}^{k} q(j), \tag{4.1.24}$$

and by condition (4.1.22) we have

$$\liminf_{k\to\infty} \frac{c(k)\Delta x(k)}{f(x(k+1))} = -\infty, \tag{4.1.25}$$

which is a contradiction.

Case 3. Suppose $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. We note that condition (4.1.21) implies the existence of an integer $m_2 \geq m_1$ such that (4.1.16) holds. The rest of the proof is similar to that of Theorem 4.1.1.

The proof is complete. □

To prove the next result, we need the following lemma.

Lemma 4.1.4. *Assume that condition (4.1.11) holds and the function F satisfies*

$$F(u,v) \geq \lambda > 0 \quad \forall u, v \neq 0, \text{ where } \lambda \text{ is a constant.} \tag{4.1.26}$$

Moreover, assume that there exist a subsequence $k_n \to \infty$ as $n \to \infty$ and $m \in \mathbb{N}$ with

$$\sum_{j=N}^{k_n} q(j) - \frac{c(k_n)}{\lambda} \geq 0 \quad \text{for } k_n \geq N,\ n \in \mathbb{N} \tag{4.1.27}$$

for every $N \geq m$. If $\{x(k)\}$ is a nonoscillatory solution of equation (4.1.2), then $x(k)\Delta x(k) > 0$ eventually.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define

$$w(k) = \frac{c(k-1)\Delta x(k-1)}{f(x(k-1))} \quad \text{for } k \geq m. \tag{4.1.28}$$

Then for $k \geq m$,

$$\begin{aligned} \Delta w(k) &= \frac{\Delta\big(c(k-1)\Delta x(k-1)\big)}{f(x(k))} - \frac{c(k-1)\Delta x(k-1)\Delta f(x(k-1))}{f(x(k-1))f(x(k))} \\ &= -q(k) - \frac{c(k-1)\Delta x(k-1)\Delta f(x(k-1))}{f(x(k-1))f(x(k))}. \end{aligned} \tag{4.1.29}$$

Using conditions (4.1.11) and (4.1.26), we see that $\Delta x(k-1)\Delta f(x(k-1)) \geq 0$ for $k \geq m$, and hence equation (4.1.29) yields $\Delta w(k) \leq -q(k)$ for $k \geq m$. Summing both sides of this inequality from $N \geq m$ to k, we find

$$w(k+1) - w(N) \leq -\sum_{j=N}^{k} q(j). \tag{4.1.30}$$

Note now that conditions (4.1.11) and (4.1.26) for all $k \geq N$ lead to

$$\begin{aligned} w(k) &= c(k-1)\frac{\Delta x(k-1)}{f(x(k-1))} \\ &= c(k-1)\left[\frac{1}{F(x(k),x(k-1))}\frac{f(x(k))}{f(x(k-1))} - \frac{1}{F(x(k),x(k-1))}\right] \\ &\geq c(k-1)\left[\frac{1}{F(x(k),x(k-1))}\frac{f(x(k))}{f(x(k-1))} - \frac{1}{\lambda}\right] \\ &\geq -\frac{c(k-1)}{\lambda}. \end{aligned} \tag{4.1.31}$$

Then (4.1.30) implies

$$w(N) \geq \sum_{j=N}^{k} q(j) - \frac{c(k)}{\lambda} \quad \forall k \geq N. \tag{4.1.32}$$

Let $\{k_n\}$ be such that (4.1.27) holds. Then

$$w(N) \geq \sum_{j=N}^{k_n} q(j) - \frac{c(k_n)}{\lambda} \geq 0 \quad \forall k_n \geq N. \tag{4.1.33}$$

Since $N \geq m$ is any arbitrary integer, it follows that $w(k) > 0$ for all $k \geq m$, and (4.1.28) implies $\Delta x(k-1) > 0$ for all $k \geq m$ as desired. This completes the proof. □

Theorem 4.1.5. *Assume that conditions (4.1.11), (4.1.22), and (4.1.26) are satisfied. Then equation (4.1.2) is oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (4.1.2), say, $x(k) > 0$ for $k \geq m \in \mathbb{N}$. By Lemma 4.1.4 we conclude that $\Delta x(k) > 0$ for $k \geq m$. Next, we proceed as in the proof of Lemma 4.1.4 to obtain (4.1.30), which in view of the fact that $w(k) > 0$ for $k \geq m$ implies that $-w(N) \leq -\sum_{j=N}^{k} q(j)$ for $N \geq m$ and hence

$$w(N) \geq \limsup_{k\to\infty} \sum_{j=N}^{k} q(j) = \infty, \tag{4.1.34}$$

which is impossible. This completes the proof. □

If the coefficients of equation (4.1.2) satisfy the condition

$$\limsup_{k\to\infty} \sum_{j=N}^{k} q(j) - \frac{c(k)}{\lambda} = \infty, \tag{4.1.35}$$

then both conditions (4.1.22) and (4.1.27) are satisfied. In this case we have the following immediate results.

Corollary 4.1.6. *If conditions (4.1.11), (4.1.26), and (4.1.35) hold, then equation (4.1.2) is oscillatory.*

Corollary 4.1.7. *If conditions (4.1.11) and (4.1.26) hold and*

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=N}^{k} \left[\sum_{j=N}^{i} q(j) - \frac{c(i)}{\lambda} \right] = \infty, \tag{4.1.36}$$

then equation (4.1.2) is oscillatory.

Proof. The proof follows from Corollary 4.1.6 as Lemma 1.13.3 yields

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=N}^{k} \left[\sum_{j=N}^{i} q(j) - \frac{c(i)}{\lambda} \right] \le \limsup_{k\to\infty} \sum_{j=N}^{k+1} q(j) - \frac{c(k+1)}{\lambda}, \tag{4.1.37}$$

that is, (4.1.36) implies (4.1.35), and hence equation (4.1.2) is oscillatory in view of Corollary 4.1.6. □

The following example illustrates the methods presented above.

Example 4.1.8. Consider the difference equation

$$\Delta^2 x(k-1) + \left(1 + \gamma k \sin\left(\frac{\pi(k-1)}{2}\right)\right)\left(x(k) + x^3(k)\right) = 0 \quad \text{for } k \in \mathbb{N}, \tag{4.1.38}$$

where γ is any real number. We will employ Corollary 4.1.7 to show that equation (4.1.38) is oscillatory. Clearly, condition (4.1.26) is satisfied with $\lambda = 1$. Furthermore,

$$\begin{aligned} \sum_{j=N}^{k} q(j) = (k+1)\left[1 - \frac{\gamma}{\sqrt{2}} \cos\left(\frac{\pi(2k-1)}{4}\right)\right] + \frac{\gamma}{2} \sin\left(\frac{\pi k}{2}\right) \\ - \frac{\gamma}{2} \sin\left(\frac{\pi(N-1)}{2}\right) + N\left[-1 + \frac{\gamma}{\sqrt{2}} \cos\left(\frac{\pi(2N-3)}{4}\right)\right], \end{aligned} \tag{4.1.39}$$

which leads to $\limsup_{k\to\infty} \sum_{j=N}^{k} q(j) = \infty$. Then (4.1.36) is satisfied, and by Corollary 4.1.7 we conclude that equation (4.1.38) is oscillatory. One such solution is $\{x(k)\}$, where $x(2k) = 0$ and $x(2k+1) = (-1)^k$ for $k \in \mathbb{N}_0$.

Here we will present two lemmas which are interesting in their own right and which will be employed in the proofs of the upcoming results.

We let $\mathbb{N}_m^\alpha = \{m, m+1, \dots, \alpha\}$, where $\alpha, m \in \mathbb{N}$ are such that $m < \alpha$ or $\alpha = \infty$. In the last case $\mathbb{N}_m^\alpha$ is denoted by $\mathbb{N}_m$.

Lemma 4.1.9. *Let $K(k, j, x)$ be defined on $\mathbb{N}_m \times \mathbb{N}_m \times \mathbb{R}^+$ such that for fixed k and j, K is a nondecreasing function of x. Let $\{p(k)\}$ be a given sequence and let $\{x(k)\}$ and $\{y(k)\}$ be defined on $\mathbb{N}_m$ satisfying for all $k \in \mathbb{N}_m$,*

$$
\begin{aligned}
x(k) &\geq p(k) + \sum_{j=m}^{k-1} K(k, j, x(k)), \\
y(k) &= p(k) + \sum_{j=m}^{k-1} K(k, j, y(k)),
\end{aligned}
\tag{4.1.40}
$$

respectively. Then $y(k) \leq x(k)$ for all $k \in \mathbb{N}_m$.

Proof. When $k = m$, the result is obvious. Suppose there exists an integer $t \in \mathbb{N}_m$ such that $y(t+1) > x(t+1)$ and $y(j) < x(j)$ for all $j \leq t$. Then,

$$
y(t+1) - x(t+1) \leq \sum_{j=m}^{k-1} \left[K(t+1, j, y(j)) - K(t+1, j, x(j))\right] \leq 0, \tag{4.1.41}
$$

which is a contradiction. □

Lemma 4.1.10. *Suppose that conditions (4.1.4) and (4.1.11) hold. Let $\{x(k)\}$ be an eventually positive (eventually negative) solution of equation (4.1.2) for $k \in \mathbb{N}_m^\alpha$ for some $m_1 \in \mathbb{N}$ such that $m \leq m_1 < \alpha < \infty$. Suppose there exist $N \in \mathbb{N}_{m_1}^\alpha$ and a positive constant b such that*

$$
\begin{aligned}
b \leq &-\frac{c(m_1)\Delta x(m_1)}{f(x(m_1))} + \sum_{j=m_1+1}^{k} q(j) \\
&+ \sum_{j=m_1+1}^{N} \frac{c(j-1)\left(\Delta x(j-1)\right)^2 F(x(j), x(j-1))}{f(x(j-1)) f(x(j))}
\end{aligned}
\tag{4.1.42}
$$

for all $k \in \mathbb{N}_{m_1}^\alpha$. Then $c(k)\Delta x(k) \leq -bf(x(N))$ $(c(k)\Delta x(k) \geq -bf(x(N)))$ for all $k \in \mathbb{N}_N^\alpha$.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $w(k)$ as in (4.1.28) and obtain (4.1.29). Summing both sides of equation (4.1.29) from $m_1 + 1$ to k provides

$$
\begin{aligned}
-\frac{c(k)\Delta x(k)}{f(x(k))} = &-\frac{c(m_1)\Delta x(m_1)}{f(x(m_1))} + \sum_{j=m_1+1}^{k} q(j) \\
&+ \sum_{j=m_1+1}^{k} \frac{c(j-1)\left(\Delta x(j-1)\right)^2 F(x(j), x(j-1))}{f(x(j-1)) f(x(j))}
\end{aligned}
\tag{4.1.43}
$$

for all $k \in \mathbb{N}_{m_1}^{\alpha}$. Thus from (4.1.42) we see that

$$-\frac{c(k)\Delta x(k)}{f(x(k))} \geq b + \sum_{j=N+1}^{k} \frac{c(j-1)\big(\Delta x(j-1)\big)^2 F\big(x(j), x(j-1)\big)}{f\big(x(j-1)\big) f\big(x(j)\big)} \tag{4.1.44}$$

for all $k \in \mathbb{N}_{m_1+1}^{\alpha}$. Since the sum in (4.1.44) is nonnegative, we have $c(k)\Delta x(k) \leq 0$ for all $k \in \mathbb{N}_N^{\alpha}$. Let $u(k) = -c(k)\Delta x(k)$. Then (4.1.44) becomes

$$u(k) \geq bf\big(x(k)\big) + \sum_{j=N+1}^{k} \frac{f\big(x(k)\big)F\big(x(j), x(j-1)\big)\big(-\Delta x(j-1)\big)}{f\big(x(j-1)\big)f\big(x(j)\big)} u(j-1). \tag{4.1.45}$$

Define

$$K(k, j, z) = \frac{f\big(x(k)\big)F\big(x(j), x(j-1)\big)\big(-\Delta x(j-1)\big)}{f\big(x(j-1)\big)f\big(x(j)\big)} z \quad \text{for } k, j \in \mathbb{N}_N^{\alpha},\ z \in \mathbb{R}^{+}. \tag{4.1.46}$$

Notice that for each fixed k and j, the function $K(k, j, z)$ is nondecreasing in z. Hence Lemma 4.1.9 applies with $p(k) = bf\big(x(k)\big)$ to obtain

$$v(k) = bf\big(x(k)\big) + \sum_{j=N+1}^{k} K\big(k, j, v(j)\big) \tag{4.1.47}$$

provided $v(j) \in \mathbb{R}^{+}$ for each $j \in \mathbb{N}_N^{\alpha}$. Multiplying the last equation by $1/f\big(x(k)\big)$ and then applying the operator Δ, we obtain $\Delta v(k) = 0$. Thus

$$v(k) = v(N) = bf\big(x(N)\big) \quad \forall k \in \mathbb{N}_N^{\alpha}. \tag{4.1.48}$$

By Lemma 4.1.9, $c(k)\Delta x(k) \leq -bf\big(x(N)\big)$ for all $k \in \mathbb{N}_N^{\alpha}$.

The proof for the case when $\{x(k)\}$ is eventually negative follows from a similar argument by taking $u(k) = c(k)\Delta x(k)$ and $p(k) = -bf\big(x(k)\big)$. □

We will assume that

$$\sum_{j=m\in\mathbb{N}}^{\infty} q(j) \quad \text{converges.} \tag{4.1.49}$$

Now we present the following result.

Theorem 4.1.11. *Suppose conditions (4.1.4), (4.1.10), (4.1.11), (4.1.26), and (4.1.49) hold. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.1.2) such that* $\liminf_{k\to\infty} |x(k)| > 0$. *Then*

$$\sum_{j=k+1}^{\infty} \frac{w^2(j)}{c(j-1)} \left(\frac{f\big(x(j-1)\big)F\big(x(j), x(j-1)\big)}{f\big(x(j)\big)} \right) \quad \textit{converges.} \tag{4.1.50}$$

For all sufficiently large k,

$$w(k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty, \tag{4.1.51}$$

$$w(k+1) = \sum_{j=k+1}^{\infty} q(j) + \sum_{j=k+1}^{\infty} \frac{w^2(j)}{c(j-1)} \left(\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \right), \tag{4.1.52}$$

where $w(k)$ is defined by (4.1.28).

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. It follows from Lemma 4.1.10 that

$$\Delta x(k) < 0, \quad c(k)\Delta x(k) \leq -bf(x(m_1)) \quad \text{for } k \in \mathbb{N}_{m_1}. \tag{4.1.53}$$

Summing both sides of the last inequality in (4.1.53) from m_1 to k, we have

$$x(k+1) \leq x(m_1) - bf(x(m_1)) \sum_{j=m_1}^{k} \frac{1}{c(j)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{4.1.54}$$

which contradicts the fact that $x(k) > 0$ for $k \in \mathbb{N}_m$. As in the proof of Lemma 4.1.10, we have

$$\begin{aligned} w(n+1) &+ \sum_{j=k+1}^{n} \frac{w^2(j)}{c(j-1)} \left(\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \right) \\ &= w(k+1) - \sum_{j=k+1}^{n} q(j). \end{aligned} \tag{4.1.55}$$

From (4.1.49), (4.1.50), and (4.1.55) we see that $a = \lim_{n\to\infty} w(n)$ exists, so that from (4.1.55) we have

$$w(k+1) = a + \sum_{j=k+1}^{\infty} q(j) + \sum_{j=k+1}^{\infty} \frac{w^2(j)}{c(j-1)} \left(\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \right) \tag{4.1.56}$$

for $k \in \mathbb{N}_{m_1}$. To prove that (4.1.51) and (4.1.52) hold, it suffices to show that $a = 0$. If $a < 0$, then (4.1.49) and (4.1.50) imply that there exists an integer $m_2 \geq m_1$ such that

$$\begin{gathered} \left| \sum_{j=m_2+1}^{k} q(j) \right| \leq -\frac{a}{4}, \\ \sum_{j=m_2+1}^{k} \frac{w^2(j)}{c(j-1)} \left(\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \right) \leq -\frac{a}{4} \end{gathered} \tag{4.1.57}$$

for all $k \in \mathbb{N}_{m_2}$. From (4.1.56) we see that (4.1.42) holds on $\mathbb{N}_{m_2}$. But then, by the argument given above, Lemma 4.1.10 and its proof lead to a contradiction of the fact that $x(k) > 0$ eventually. If $a > 0$, then it follows from (4.1.49), (4.1.50), and (4.1.56) that $\lim_{k\to\infty} w(k) = a$, so there exists an integer $m_3 > m_2$ such that $w(k) > a/2$ for all $k \in \mathbb{N}_{m_3}$. Next, we use (4.1.11) and (4.1.26) to obtain

$$\frac{w(k)}{c(k-1)}\frac{F(x(k),x(k-1))f(x(k-1))}{f(x(k))} \geq \frac{\lambda a}{2c(k-1)+\lambda a}. \tag{4.1.58}$$

Thus

$$\begin{aligned}\sum_{j=m_3+1}^{\infty}&\frac{w^2(j)}{c(j-1)}\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))}\\ &\geq \lim_{k\to\infty}\sum_{j=m_3+1}^{k}\frac{\lambda a^2}{2(2c(j-1)+\lambda a)} = \infty,\end{aligned} \tag{4.1.59}$$

which contradicts (4.1.50). This completes the proof that $a = 0$ for the case when $x(k) > 0$ eventually. The proof for $a = 0$ in the case when $x(k) < 0$ eventually is similar. □

Theorem 4.1.12. *Assume that conditions (4.1.11) and (4.1.26) hold and that there exists a sequence $\{\phi(k)\}$ such that*

$$\liminf_{k\to\infty}\sum_{j=N}^{k} q(j) \geq \phi(N) \quad \textit{for all sufficiently large } N \in \mathbb{N}, \tag{4.1.60}$$

$$\sum_{j=N}^{\infty}\frac{(\phi^+(j))^2}{1+\lambda\phi^+(j)} = \infty, \tag{4.1.61}$$

where $\phi^+(k) = \max\{\phi(k), 0\}$. Then equation (4.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.1), say, $x(k) > 0$ for $k \in \mathbb{N}_m$ for some $m \in \mathbb{N}$. Since condition (4.1.26) implies that $f(x)$ is strictly increasing for $x \neq 0$, we have $f(x(k)) > 0$ for all $k \in \mathbb{N}_m$, and hence by Theorem 4.1.11 with $c(k) \equiv 1$, we find

$$w(k) \geq \phi(k) \quad \text{for } k \in \mathbb{N}_m. \tag{4.1.62}$$

Next, let a subsequence $\{i_n\}_{n=1}^{\infty}$ be defined by $\{i_n\}_{n=1}^{\infty} = \{i \in \mathbb{N}_{m+1} : \phi(i) \geq 0\}$ and $i_n \to \infty$ as $n \to \infty$.

From (4.1.62) we see that

$$w(i_n) \geq \phi(i_n) \quad \text{for } i_n \in \mathbb{N}_{m+1}. \tag{4.1.63}$$

Also, (4.1.31) with $c(k) \equiv 1$ yields

$$\frac{f(x(i_n-1))F(x(i_n),x(i_n-1))}{f(x(i_n))} \geq \frac{\lambda}{1+\lambda w(i_k)} \quad \text{for } i_k \in \mathbb{N}_{m+1}. \tag{4.1.64}$$

From (4.1.64), (4.1.63), and the fact that the function $g(y) = y^2/(1+\lambda y)$ is increasing on $\mathbb{R}^+$, we have

$$\begin{aligned}
&\sum_{i=k}^{\infty} w^2(i)\frac{f(x(i-1))F(x(i_n),x(i_n-1))}{f(x(i))} \\
&\quad \geq \sum_{n=1}^{\infty} w^2(i_n)\frac{f(x(i_n-1))F(x(i_n),x(i_n-1))}{f(x(i_n))} \\
&\quad \geq \lambda\sum_{n=1}^{\infty}\frac{w^2(i_n)}{1+\lambda w(i_n)} = \sum_{n=1}^{\infty} g(w(i_n)) \\
&\quad \geq \lambda\sum_{n=1}^{\infty} g(\phi(i_n)) = \lambda\sum_{i=k}^{\infty}\frac{(\phi^+(i))^2}{1+\lambda\phi^+(i)}.
\end{aligned} \tag{4.1.65}$$

In view of (4.1.50), we see that

$$\lambda\sum_{i=n}^{\infty}\frac{(\phi^+(i))^2}{1+\lambda\phi^+(i)} < \infty \quad \text{for } n \in \mathbb{N}_{m+1}, \tag{4.1.66}$$

which contradicts condition (4.1.61). This completes the proof. □

We note that if either

$$\sum^{\infty}(\phi^+(i))^2 \quad \text{or} \quad \sum^{\infty}\frac{(\phi^+(i))^2}{1+\lambda\phi^+(i)} \tag{4.1.67}$$

is convergent, then $\lim_{k\to\infty}\phi^+(k) = 0$. Hence there exists a real number $\eta > 0$ such that $\phi^+(k) \leq \eta$ for all $k \in \mathbb{N}$ and

$$(\phi^+(k))^2 \geq \frac{(\phi^+(k))^2}{1+\lambda\phi^+(k)} \geq \frac{1}{1+\lambda\eta}(\phi^+(k))^2 \quad \text{for } k \in \mathbb{N}, \tag{4.1.68}$$

which implies that the convergence of any of the above two series yields the convergence of the other one.

The following result is immediate.

Corollary 4.1.13. *If conditions (4.1.11), (4.1.26), and (4.1.60) hold and*

$$\sum_{j=N}^{\infty}(\phi^+(j))^2 = \infty \quad \textit{for all sufficiently large } N \in \mathbb{N}, \tag{4.1.69}$$

then equation (4.1.1) is oscillatory.

Example 4.1.14. Consider equation (4.1.1) with $\{q(k)\}$ defined by

$$q(k) = (-1)^k \frac{2k^2 + 8k + 7}{(k+1)(k+2)} - \Delta \frac{1}{\sqrt{k+1}}. \tag{4.1.70}$$

Now,

$$\sum_{j=N}^{k} q(j) = (-1)^k + \frac{(-1)^k}{k+2} - \frac{1}{\sqrt{k+2}} + (-1)^N + \frac{(-1)^N}{N+1} + \frac{1}{\sqrt{N+1}}, \tag{4.1.71}$$

and hence

$$\liminf_{k\to\infty} \sum_{j=N}^{k} q(j) = -1 + (-1)^N + \frac{(-1)^N}{N+1} + \frac{1}{\sqrt{N+1}} \quad \text{for } N \in \mathbb{N}. \tag{4.1.72}$$

Accordingly, there exists a sequence $\{\phi(k)\}$ defined by

$$\phi(k) = -1 + (-1)^k + \frac{(-1)^k}{k+1} + \frac{1}{\sqrt{k+1}} \quad \text{for } k \in \mathbb{N}, \tag{4.1.73}$$

and satisfying condition (4.1.60). Furthermore,

$$\phi^+(k) = \begin{cases} \dfrac{1}{k+1} + \dfrac{1}{\sqrt{k+1}} & \text{if } k \in \mathbb{N} \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.74}$$

Therefore

$$\sum_{j=1}^{\infty} \left(\phi^+(j)\right)^2 \geq \sum_{j=1}^{\infty} \frac{1}{2j+1} = \infty. \tag{4.1.75}$$

In view of Corollary 4.1.13, equation (4.1.1) with $q(k)$ defined by (4.1.70) is oscillatory.

The following result is concerned with the oscillation of equation (4.1.1) when condition (4.1.61) fails to hold, that is,

$$\sum_{j=N}^{\infty} \frac{\left(\phi^+(j)\right)^2}{1 + \lambda\phi^+(j)} < \infty. \tag{4.1.76}$$

In this case we define the sequence $\{h_\ell(k)\}$ as follows:

$$\begin{gathered} h_0(k) = \phi(k), \qquad h_1(k) = \sum_{j=k}^{\infty} \frac{\left(h_0^+(j)\right)^2}{1 + \lambda h_0^+(j)}, \\ h_{i+1}(k) = \sum_{j=k}^{\infty} \frac{\left[\left(h_0(j) + \lambda h_i(j)\right)^+\right]^2}{1 + \lambda\left(h_0(j) + \lambda h_i(j)\right)^+} \quad \text{for } i \in \mathbb{N}. \end{gathered} \tag{4.1.77}$$

We will need the condition

$$\begin{aligned}&\text{there exists } L \in \mathbb{N} \text{ such that } h_\ell(k) \text{ exists} \quad \text{for } \ell \in \{1,2,\dots,L-1\}\\&\text{and } h_L(k) \text{ does not exist}, \quad k \in \mathbb{N}.\end{aligned} \tag{4.1.78}$$

Theorem 4.1.15. *If conditions (4.1.11), (4.1.26), (4.1.61), and (4.1.78) hold, then equation (4.1.1) is oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (4.1.1), say, $x(k) > 0$ for $k \in \mathbb{N}_m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 4.1.12, we obtain (4.1.52) and (4.1.62), from which we get

$$\sum_{j=k}^{\infty} w^2(j)\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \geq \lambda h_1(k). \tag{4.1.79}$$

By using (4.1.79) in (4.1.52), we obtain $w(k) \geq h_0(k) + \lambda h_1(k)$. Similarly, one can derive

$$\sum_{j=k}^{\infty} w^2(j)\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \geq \lambda h_2(k). \tag{4.1.80}$$

Repeating this process, we get $w(k) > h_0(k) + \lambda h_{L-1}(k)$ and hence

$$\sum_{j=k}^{\infty} w^2(j)\frac{f(x(j-1))F(x(j),x(j-1))}{f(x(j))} \geq \lambda h_L(k), \tag{4.1.81}$$

which contradicts (4.1.50). This completes the proof. □

Remark 4.1.16. In Theorem 4.1.11, the assumptions (4.1.11) and (4.1.26) can be replaced by

$$f'(x) \geq 0 \quad \text{for } x \neq 0, \qquad \lim_{|x|\to\infty} |f(x)| = \infty. \tag{4.1.82}$$

In this case the requirement on the nonoscillatory solution $\{x(k)\}$ of equation (4.1.2) to have $\liminf_{k\to\infty} |x(k)| > 0$ can be disregarded.

4.2. Superlinear oscillation

In this section we are concerned with the oscillation of equations (4.1.1)–(4.1.3) when the function f satisfies condition (4.1.5), or equations (4.1.7)–(4.1.9) with $\gamma > 1$.

We first prove the following result.

Theorem 4.2.1. *Suppose that conditions (4.1.4), (4.1.5), (4.1.10), (4.1.11), and (4.1.21) hold and in addition*

$$\sum^{\infty} q(j) < \infty, \tag{4.2.1}$$

$$\lim_{k\to\infty} \sum_{i=m\in\mathbb{N}}^{k} \left[\frac{1}{c(i)} \sum_{j=i+1}^{\infty} q(j)\right] = \infty. \tag{4.2.2}$$

Then equation (4.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Summing equation (4.1.13) from $m_1 + 1$ for any integer $m_1 \geq m$ to k, we obtain

$$\frac{c(k)\Delta x(k)}{f(x(k+1))} \leq \frac{c(m_1)\Delta x(m_1)}{f(x(m_1+1))} - \sum_{j=m_1+1}^{k} q(j). \tag{4.2.3}$$

Now we distinguish the following three cases.

Case 1. If $\Delta x(k) > 0$ for all $k \geq m_1$, then by condition (4.2.1) we find

$$0 \leq \frac{c(m_1)\Delta x(m_1)}{f(x(m_1+1))} - \sum_{j=m_1+1}^{\infty} q(j), \tag{4.2.4}$$

and therefore for $k \geq m_1$

$$\sum_{j=k+1}^{\infty} q(j) \leq \frac{c(k)\Delta x(k)}{f(x(k+1))}, \tag{4.2.5}$$

so

$$\frac{1}{c(k)} \sum_{j=k+1}^{\infty} q(j) \leq \frac{\Delta x(k)}{f(x(k+1))}. \tag{4.2.6}$$

Summing (4.2.6) from m_1 to k, we get

$$\sum_{i=m_1}^{k} \left[\frac{1}{c(i)} \sum_{j=i+1}^{\infty} q(j)\right] \leq \sum_{i=m_1}^{k} \frac{\Delta x(i)}{f(x(i+1))} \leq \int_{x(m_1)}^{x(k)} \frac{du}{f(u)}. \tag{4.2.7}$$

By (4.2.2), the left-hand side of (4.2.7) tends to ∞ as $k \to \infty$, while the right-hand side of (4.2.7) is finite by (4.2.6).

Case 2. If $\{\Delta x(k)\}$ changes signs, (i.e., is oscillatory), then there exists a sequence $k_n \to \infty$ as $n \to \infty$ such that $\Delta x(k_n) < 0$. Choose N so large that (4.1.21) holds. Then summing (4.1.14) from $k_n + 1$ to k followed by taking lim sup provides

$$\limsup_{k\to\infty} \frac{c(k)\Delta x(k)}{f(x(k+1))} \le \frac{c(k_n)\Delta x(k_n)}{f(x(k_n+1))} + \limsup_{k\to\infty}\left[-\sum_{j=k_n+1}^{k} q(j)\right] < 0. \tag{4.2.8}$$

It follows that $\lim_{k\to\infty} \Delta x(k) < 0$. This contradicts the assumption that $\{\Delta x(k)\}$ oscillates.

Case 3. Suppose that $\Delta x(k) < 0$ for $k \ge m_1 \ge m \in \mathbb{N}$. We note that condition (4.1.21) implies the existence of an integer $m_2 \ge m_1$ such that (4.1.16) holds. The rest of the proof is similar to that of Theorem 4.1.1.

This completes the proof. □

Corollary 4.2.2. *Let the hypotheses of Theorem 4.2.1 be satisfied except for condition (4.1.5). Then all bounded solutions of equation (4.1.2) are oscillatory.*

Proof. The condition (4.1.5) is used only in Case 1 of the proof of Theorem 4.2.1. Suppose $\{x(k)\}$ is a bounded nonoscillatory solution of equation (4.1.2). In Case 1 we have $x(k) > 0$ and $\Delta x(k) > 0$ for $k \ge m_1 \ge m \in \mathbb{N}$. Hence, in view of (4.1.11), we have $f(x(k)) \ge f(x(m_1))$ for $k \ge m_1$. It follows from (4.2.7) that

$$\begin{aligned}\sum_{i=m_1}^{k}\left[\frac{1}{c(i)}\sum_{j=i+1}^{\infty} q(j)\right] &\le \sum_{i=m_1}^{k} \frac{\Delta x(i)}{f(x(i+1))} \le \frac{1}{f(x(m_1))}\sum_{i=m_1}^{k} \Delta x(i) \\ &= \frac{1}{f(x(m_1))}\left[x(k+1) - x(m_1)\right] < \infty.\end{aligned} \tag{4.2.9}$$

This contradicts condition (4.2.2) and completes the proof. □

Remark 4.2.3. In the proof of Lemma 4.1.10 and Theorem 4.1.11, if we define $w(k)$ by $w(k) = c(k-1)\Delta x(k-1)/f(x(k))$ for $k \ge m \in \mathbb{N}$, then it follows from equation (4.1.2) that

$$\Delta\left(\frac{c(k-1)\Delta x(k-1)}{f(x(k))}\right) = -q(k) - \frac{c(k)\Delta x(k)\Delta f(x(k))}{f(x(k))f(x(k+1))}. \tag{4.2.10}$$

Summing (4.2.10) from m to k we have

$$-\frac{c(m-1)\Delta x(m-1)}{f(x(m))} + \sum_{j=m}^{k} \frac{c(j)\Delta x(j)\Delta f(x(j))}{f(x(j))f(x(j+1))} + \sum_{j=m}^{k} q(j) = -\frac{c(k)\Delta x(k)}{f(x(k+1))}. \tag{4.2.11}$$

Next, if the assumptions of Theorem 4.1.11 hold, then equation (4.1.52) takes the form

$$\frac{c(k)\Delta x(k)}{f(x(k+1))} = \sum_{j=k+1}^{\infty} q(j) + \sum_{j=k+1}^{\infty} \frac{c(j)\Delta x(j)\Delta f(x(j))}{f(x(j))f(x(j+1))}. \tag{4.2.12}$$

Now we are ready to give an alternative proof of Theorem 4.2.1 without the assumption (4.1.5).

Alternative proof of Theorem 4.2.1. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since f is nondecreasing, the second term in (4.2.12) is nonnegative. Hence

$$\frac{c(k)\Delta x(k)}{f(x(k+1))} \geq \sum_{j=k+1}^{\infty} q(j), \tag{4.2.13}$$

so

$$\frac{\Delta x(k)}{f(x(k+1))} \geq \frac{1}{c(k)} \sum_{j=k+1}^{\infty} q(j). \tag{4.2.14}$$

Summing inequality (4.2.14) from m to k we get

$$\sum_{i=m}^{k} \frac{\Delta x(i)}{f(x(i+1))} \geq \sum_{i=m}^{k} \frac{1}{c(i)} \sum_{j=i+1}^{\infty} q(j). \tag{4.2.15}$$

We define $u(t) = x(i) + (t-i)\Delta x(i)$ for $i \leq t \leq i+1$. If we have $\Delta x(i) \geq 0$, then $x(i) \leq u(t) \leq x(i+1)$ and

$$\frac{\Delta x(i)}{f(x(i+1))} \leq \frac{u'(t)}{f(u(t))} \leq \frac{\Delta x(i)}{f(x(i))}. \tag{4.2.16}$$

If we have $\Delta x(i) < 0$, then $x(i+1) \leq u(t) \leq x(i)$ and (4.2.16) also holds. From (4.2.15) and (4.2.16), we obtain

$$\int_{u(m)}^{\infty} \frac{dy}{f(y)} \geq \int_{m}^{k+1} \frac{du(t)}{f(u(t))} \geq \sum_{i=m}^{k} \frac{1}{c(i)} \sum_{j=i+1}^{\infty} q(j). \tag{4.2.17}$$

Let $G(z) = \int_z^{\infty} dy/f(y)$. Then (4.2.17) implies

$$G(u(m)) \geq \sum_{i=m}^{k} \frac{1}{c(i)} \sum_{j=i+1}^{\infty} q(j), \tag{4.2.18}$$

which contradicts condition (4.2.2).

Similarly, one can prove that equation (4.1.2) does not possess eventually negative solutions. This completes the proof. □

The following example illustrates the methods presented above.

Example 4.2.4. Consider the difference equation

$$\Delta\left(\frac{1}{k^2}\Delta x(k-1)\right)+\frac{2(2k^2+2k+1)}{k^2(k+1)^2}x^{\gamma}(k)=0 \quad \text{for } k\in\mathbb{N}, \tag{4.2.19}$$

where $\gamma>0$ is the ratio of two odd integers. Clearly,

$$\sum^{\infty}q(j)=\sum^{\infty}\frac{2(2j^2+2j+1)}{j^2(j+1)^2}=2\sum^{\infty}\left[\frac{1}{j^2}+\frac{1}{(j+1)^2}\right]<\infty. \tag{4.2.20}$$

To see that condition (4.2.2) is satisfied, we note that

$$\begin{aligned}
\sum_{i=m\in\mathbb{N}}^{\infty}\left[\frac{1}{c(i)}\sum_{j=i+1}^{\infty}q(j)\right] &= 2\sum_{i=m}^{\infty}\left[i^2\sum_{j=i+1}^{\infty}\frac{2j^2+2j+1}{j^2(j+1)^2}\right]\\
&= 2\sum_{i=m}^{\infty}\left[i^2\left(\sum_{j=i+1}^{\infty}\frac{1}{j^2}+\sum_{j=i+1}^{\infty}\frac{1}{(j+1)^2}\right)\right]\\
&\geq 2\sum_{i=m}^{\infty}\left[i^2\sum_{j=i+1}^{2i}\frac{1}{j^2}\right]\\
&\geq 2\sum_{i=m}^{\infty}\left[i^2\sum_{j=i+1}^{2i}\frac{1}{(2j)^2}\right]\\
&= 2\sum_{i=m}^{\infty}\frac{i}{4}=\infty.
\end{aligned} \tag{4.2.21}$$

All conditions of Corollary 4.2.2 are satisfied, and hence all bounded solutions of equation (4.2.19) are oscillatory. One such solution is $x(k)=(-1)^k$. We also note that equation (4.2.19) is oscillatory by Theorem 4.2.1.

Theorem 4.2.5. *If conditions (4.1.4), (4.1.5), (4.1.11), and (4.1.21) hold and*

$$\limsup_{k\to\infty}\sum_{j=m\in\mathbb{N}}^{k}jq(j)=\infty, \tag{4.2.22}$$

then equation (4.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since (4.1.21) holds, we see from the proof of Theorem 4.1.3 that $\{\Delta x(k)\}$ is eventually of one sign. From equation (4.1.1) we get that for $k \geq m$,

$$\begin{aligned}\Delta\left(\frac{k\Delta x(k-1)}{f(x(k))}\right) &= -kq(k) + \frac{\Delta x(k)}{f(x(k+1))} - k\frac{F(x(k+1),x(k))(\Delta x(k))^2}{f(x(k))f(x(k+1))}\\ &\leq -kq(k) + \frac{\Delta x(k)}{f(x(k+1))}.\end{aligned} \tag{4.2.23}$$

Now we consider the following two cases.

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Summing (4.2.23) from $m_1 + 1$ to k gives

$$\begin{aligned}\sum_{j=m_1+1}^{k} jq(j) &\leq \frac{(m_1+1)\Delta x(m_1)}{f(x(m_1+1))} - \frac{(k+1)\Delta x(k)}{f(x(k+1))} + \sum_{j=m_1+1}^{k}\frac{\Delta x(j)}{f(x(j+1))}\\ &\leq \frac{(m_1+1)\Delta x(m_1)}{f(x(m_1+1))} + \sum_{j=m_1+1}^{k}\frac{\Delta x(j)}{f(x(j+1))}\\ &\leq \frac{(m_1+1)\Delta x(m_1)}{f(x(m_1+1))} + \int_{x(m_1+1)}^{x(k+1)}\frac{dy}{f(y)} < \infty,\end{aligned} \tag{4.2.24}$$

which is a contradiction to condition (4.2.22).

Case 2. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. Condition (4.2.22) implies that there exists an integer $m_2 \geq m_1$ such that

$$\sum_{j=m_2+1}^{k} jq(j) \geq 0 \quad \forall k \geq m_2 + 1. \tag{4.2.25}$$

Multiplying equation (4.1.1) by k and using summation by parts, we obtain

$$\begin{aligned}(k+1)\Delta x(k) &= (m_2+1)\Delta x(m_2) + \sum_{j=m_2+1}^{k}\Delta x(j) - \sum_{j=m_2+1}^{k} jq(j)f(x(j))\\ &\leq (m_2+1)\Delta x(m_2) - f(x(k+1))\sum_{j=m_2+1}^{k} jq(j)\\ &\quad + \sum_{i=m_2+1}^{k}\Delta f(x(i))\left(\sum_{j=m_2+1}^{i} jq(j)\right)\\ &\leq (m_2+1)\Delta x(m_2)\end{aligned} \tag{4.2.26}$$

for all $k \geq m_2$. Thus

$$x(k+1) \leq x(m_2) + (m_2+1)\Delta x(m_2) \sum_{j=m_2}^{k} \frac{1}{1+j} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{4.2.27}$$

which is again a contradiction.

A similar proof works if $\{x(k)\}$ is eventually negative. □

Example 4.2.6. The difference equation

$$\Delta^2 x(k-1) + 2x^{\gamma}(k) = 0, \tag{4.2.28}$$

where $\gamma > 0$ is an odd integer, has an oscillatory solution $x(k) = (-1)^k$. All conditions of Theorem 4.2.5 are satisfied.

Corollary 4.2.7. *If the hypotheses of Theorem 4.2.5 are satisfied except for condition (4.1.5), then all bounded solutions of equation (4.1.1) are oscillatory.*

Theorem 4.2.8. *Suppose that conditions (4.1.4), (4.1.5), (4.1.10), (4.1.11), and (4.1.21) hold. If*

$$\sum^{\infty} q(j)C(j,m) = \infty \quad \textit{for } m \in \mathbb{N}, \tag{4.2.29}$$

where

$$C(k,m) = \sum_{j=m}^{k} \frac{1}{c(j-1)}, \quad k \geq m \textit{ for some } m \in \mathbb{N}, \tag{4.2.30}$$

then equation (4.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since condition (4.1.21) holds, from the proof of Theorem 4.1.3, we see that $\{\Delta x(k)\}$ does not oscillate. Define

$$y(k) = \frac{c(k-1)\Delta x(k-1)}{f(x(k))} C(k,m) \quad \text{for } k \geq m. \tag{4.2.31}$$

Then

$$\begin{aligned} \Delta y(k) &= -C(k,m)q(k) + \frac{\Delta x(k)}{f(x(k+1))} \\ &\quad - \frac{c(k)F(x(k+1),x(k))C(k,m)(\Delta x(k))^2}{f(x(k))f(x(k+1))} \\ &\leq -C(k,m)q(k) + \frac{\Delta x(k)}{f(x(k+1))}. \end{aligned} \tag{4.2.32}$$

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Summing the above inequality from $m_1 + 1$ to k yields

$$\begin{aligned} y(k+1) &\leq y(m_1+1) - \sum_{j=m_1+1}^{k} q(j)C(j,m) + \sum_{j=m_1+1}^{k} \frac{\Delta x(j)}{f(x(j+1))} \\ &\leq y(m_1+1) - \sum_{j=m_1+1}^{k} q(j)C(j,m) + \int_{x(m_1+1)}^{x(k+1)} \frac{du}{f(u)} \\ &\longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{4.2.33}$$

which contradicts the fact that $y(k) \geq 0$ eventually.

Case 2. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. In view of condition (4.2.29), there exists an integer $m_2 \geq m_1$ such that

$$\sum_{j=m_2+1}^{k} q(j)C(j,m_1) \geq 0 \quad \text{for } k \geq m_2 + 1. \tag{4.2.34}$$

Now, multiplying equation (4.1.2) by $C(k-1,m_1)$, we obtain

$$C(k-1,m_1)\Delta\big(c(k-1)\Delta x(k-1)\big) = -C(k-1,m_1)q(k)f(x(k)), \tag{4.2.35}$$

which after summing by parts from $m_2 + 1$ to $k > m_2$ and using (4.2.34) gives

$$\begin{aligned} &C(k,m_1)c(k)\Delta x(k) \\ &\quad = C(m_2,m_1)c(m_2)\Delta x(m_2) + \sum_{j=m_2+1}^{k} c(j)\Delta x(j)\left(\frac{1}{c(j-1)}\right) \\ &\qquad - \sum_{j=m_2+1}^{k} C(j-1,m_1)q(j)f(x(j)) \\ &\quad \leq C(m_2,m_1)c(m_2)\Delta x(m_2) - f(x(k+1)) \sum_{j=m_2+1}^{k} C(j-1,m_1)q(j) \\ &\qquad + \sum_{i=m_2+1}^{k} \Delta f(x(i))\left[\sum_{j=m_2+1}^{i} C(j-1,m_1)q(j)\right] \\ &\quad \leq C(m_2,m_1)c(m_2)\Delta x(m_2) < 0. \end{aligned} \tag{4.2.36}$$

Thus we get

$$\Delta x(k) \leq \frac{C(m_2,m_1)}{C(k,m_1)}\frac{c(m_2)}{c(k)}\Delta x(m_2). \tag{4.2.37}$$

Summing (4.2.37) from m_2+1 to k gives

$$x(k+1) \leq x(m_2+1) + [C(m_2,m_1)c(m_2)\Delta x(m_2)] \sum_{j=m_2+1}^{k} \frac{\Delta C(j,m_1)}{C(j,m_1)}. \tag{4.2.38}$$

Now, since

$$\frac{\Delta C(j,m_1)}{C(j,m_1)} = \frac{\int_j^{j+1} C'(t,m_1)\,dt}{C(j,m_1)} \geq \int_j^{j+1} \frac{C'(t,m_1)}{C(t,m_1)}\,dt = \ln \frac{C(j+1,m_1)}{C(j,m_1)}, \tag{4.2.39}$$

it follows from (4.2.38) and (4.1.10) that

$$\begin{aligned} x(k+1) &\leq x(m_2+1) + [C(m_2,m_1)c(m_2)\Delta x(m_2)] \ln\left(\frac{C(k+1,m_1)}{C(m_2+1,m_1)}\right) \\ &\longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{4.2.40}$$

But this contradicts the assumption that $\{x(k)\}$ is eventually positive.

This completes the proof. □

Corollary 4.2.9. *If the hypotheses of Theorem 4.2.8 are satisfied except for condition (4.1.5), then every bounded solution of equation (4.1.2) oscillates.*

Example 4.2.10. Consider the difference equation

$$\Delta^2 x(k-1) + [1+2(-1)^{k+1}]|x(k)|^{\gamma} \operatorname{sgn} x(k) = 0 \quad \text{for } k \geq m \in \mathbb{N}, \tag{4.2.41}$$

where $\gamma > 1$. Clearly,

$$\sum_{j=m}^{k} jq(j) = \frac{k(k+1)}{2} + (-1)^k\left(\frac{2k+1}{2}\right) - \frac{m(m-1)}{2} - (-1)^{m-1}\left(\frac{2m-1}{2}\right), \tag{4.2.42}$$

and hence $\sum_{j=m}^{\infty} jq(j) = \infty$. All conditions of Theorem 4.2.8 are satisfied, and hence equation (4.2.41) is oscillatory. One such oscillatory solution is $\{x(k)\}$, where $x(2k) = 0$ and $x(2k+1) = (-1)^k(2/3)^{1/(\gamma-1)}$ for $k \in \mathbb{N}_0$.

4.3. Sublinear oscillation

This section deals with the oscillation of equations (4.1.1)–(4.1.3) when the function f satisfies condition (4.1.6), or equations (4.1.7)–(4.1.9) with $0 < \gamma < 1$.

We first present the following result.

Theorem 4.3.1. *Suppose (4.1.4), (4.1.6), (4.1.11), and (4.1.21) hold. If for every constant M,*

$$\sum^{\infty}\left[\frac{M}{c(i)}-\frac{1}{c(i)}\sum_{j=m\in\mathbb{N}}^{i}q(j)\right]=-\infty, \tag{4.3.1}$$

then equation (4.1.2) is oscillatory.

Proof. Suppose that $\{x(k)\}$ is a nonoscillatory solution of equation (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since condition (4.1.21) holds, we see from the proof of Theorem 4.1.3 that $\{\Delta x(k)\}$ is eventually of one sign. Define $w(k)$ by (4.1.28) and obtain (4.1.29). Then

$$\Delta\left(\frac{c(k-1)\Delta x(k-1)}{f(x(k-1))}\right) \leq -q(k) \quad \text{for } k \geq m. \tag{4.3.2}$$

We consider the following two cases.

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Summing (4.3.2) from $m_1 + 1$ to k gives

$$\frac{c(k)\Delta x(k)}{f(x(k))} \leq \frac{c(m_1)\Delta x(m_1)}{f(x(m_1))}-\sum_{j=m_1+1}^{k}q(j), \tag{4.3.3}$$

so

$$\frac{\Delta x(k)}{f(x(k))} \leq \frac{L}{c(k)}-\frac{1}{c(k)}\sum_{j=m_1+1}^{k}q(j), \tag{4.3.4}$$

where $L = (c(m_1)\Delta x(m_1))/f(x(m_1))$. Again we sum (4.3.4) from $m_1 + 1$ to k to obtain

$$\sum_{i=m_1+1}^{k}\frac{\Delta x(i)}{f(x(i))} \leq \sum_{i=m_1+1}^{k}\left[\frac{L}{c(i)}-\frac{1}{c(i)}\sum_{j=m_1+1}^{i}q(j)\right] \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{4.3.5}$$

which contradicts the fact that the left-hand side is nonnegative.

Case 2. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. Then from (4.3.4) we find

$$\begin{aligned}\sum_{i=m_1+1}^{k}\left[\frac{1}{c(i)}\sum_{j=m_1+1}^{i}q(j)-\frac{L}{c(i)}\right] &\leq \sum_{i=m_1+1}^{k}\frac{\Delta x(i)}{f(x(i))}\\ &\leq \int_{x(k+1)}^{x(m_1+1)}\frac{du}{f(u)}\\ &\leq \int_{0}^{x(m_1+1)}\frac{du}{f(u)} < \infty,\end{aligned} \tag{4.3.6}$$

which contradicts condition (4.3.1).

This completes the proof. □

Corollary 4.3.2. *Suppose (4.1.4), (4.1.11), (4.1.21), and (4.3.1) hold. Then all bounded solutions of equation (4.1.2) are oscillatory.*

Proof. We note that condition (4.1.6) is used only in Case 2 of the proof of Theorem 4.3.1. Let $\{x(k)\}$ be a bounded nonoscillatory solution of equation (4.1.2). In Case 2 of the proof of Theorem 4.3.1, we have $x(k) > 0$ and $\Delta x(k) < 0$ for $k \geq m_1$. Hence, $x(k)$ decreases to a constant $\ell > 0$ as $k \to \infty$ and we have $f(x(k)) \geq f(\ell) > 0$ for $k \geq m_1$. It follows from (4.3.6) that

$$\begin{aligned}\sum_{i=m_1+1}^{k}\left[\frac{1}{c(i)}\sum_{j=m_1+1}^{i}q(j)-\frac{L}{c(i)}\right] &\leq \frac{1}{f(\ell)}\sum_{i=m_1+1}^{k}\Delta x(i)\\ &= \frac{1}{f(\ell)}\left[x(k+1)-x(m_1+1)\right]\\ &< \infty\end{aligned} \tag{4.3.7}$$

as $k \to \infty$, which contradicts (4.3.1). This completes the proof. □

Theorem 4.3.3. *If there is a positive and nondecreasing sequence $\{\xi(k)\}$ such that for all large $m \in \mathbb{N}$,*

$$\lim_{k\to\infty}\frac{\xi(k+1)}{\psi(k,m+1)} < \infty, \tag{4.3.8}$$

$$\lim_{k\to\infty}\frac{1}{\psi(k,m+1)}\sum_{i=m+1}^{k}\sum_{j=m+1}^{i}\rho(i)q(j) = \infty, \tag{4.3.9}$$

and conditions (4.1.6) and (4.1.11) hold, where

$$\rho(k) = \frac{\xi(k+1)}{c(k)}, \qquad \psi(k,m+1) = \sum_{j=m+1}^{k}\rho(j), \tag{4.3.10}$$

then equation (4.1.2) is oscillatory.

Proof. Suppose that $\{x(k)\}$ is a nonoscillatory solution of equation (4.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 4.3.1, we obtain (4.3.2). Summing (4.3.2) from $m+1$ to k, we have

$$\frac{\Delta x(k)}{f(x(k))} + \frac{1}{c(k)} \sum_{j=m+1}^{k} q(j) \leq \frac{1}{c(k)} \frac{c(m)\Delta x(m)}{f(x(m))}. \tag{4.3.11}$$

Multiplying (4.3.11) by $\xi(k+1)$ and summing from $m+1$ to k, we obtain

$$\sum_{i=m+1}^{k} \frac{\xi(i+1)\Delta x(i)}{f(x(i))} + \sum_{i=m+1}^{k} \sum_{j=m+1}^{i} \rho(i)q(j) \leq \frac{c(m)\Delta x(m)}{f(x(m))}\psi(k, m+1). \tag{4.3.12}$$

Let $r(t) = x(k) + (t-k)\Delta x(k)$ for $m \leq k \leq t \leq k+1$. Then $r'(t) = \Delta x(k)$ for $k < t < k+1$ and

$$\frac{\Delta x(k)}{f(x(k))} \geq \frac{r'(t)}{f(r(t))} \geq \frac{\Delta x(k)}{f(x(k+1))} \quad \text{for } m \leq k < t < k+1. \tag{4.3.13}$$

Using (4.3.13) in (4.3.12) we find

$$\sum_{i=m+1}^{k} \xi(i+1) \int_{x(i)}^{x(i+1)} \frac{du}{f(u)} + \sum_{i=m+1}^{k} \sum_{j=m+1}^{i} \rho(i)q(j) \leq \frac{c(m)\Delta x(m)}{f(x(m))}\psi(k, m+1). \tag{4.3.14}$$

Now let $G(k) = \int_0^{x(k)} du/f(u)$ for $k \geq m$. Since (4.1.6) holds, $G(k) > 0$ for $k \geq m$. Furthermore,

$$\begin{aligned}
&\sum_{i=m+1}^{k} \xi(i+1) \int_{x(i)}^{x(i+1)} \frac{du}{f(u)} \\
&\quad = \sum_{i=m+1}^{k} \xi(i+1)\Delta G(i) \qquad\qquad (4.3.15)\\
&\quad = \xi(k+1)G(k+1) - \sum_{i=m+1}^{k-1} G(i+1)\Delta\xi(i+1) - \xi(m+1)G(m+1).
\end{aligned}$$

There are two cases to consider. Suppose first that there is a sequence $k_n \to \infty$ and

$$\xi(k_n+1)G(k_n+1) - \sum_{i=m+1}^{k_n-1} G(i+1)\Delta\xi(i+1) \geq 0. \tag{4.3.16}$$

Then from (4.3.15), we have

$$\sum_{i=m+1}^{k_n} \sum_{j=m+1}^{i} \rho(i)q(j) \leq \frac{c(m)\Delta x(m)}{f(x(m))}\psi(k_n, m+1) + \xi(m+1)G(m+1), \tag{4.3.17}$$

so

$$\frac{1}{\psi(k_n, m+1)} \sum_{i=m+1}^{k_n} \sum_{j=m+1}^{i} \rho(i)q(j) \leq \frac{c(m)\Delta x(m)}{f(x(m))} + \frac{\xi(m+1)G(m+1)}{\psi(k_n, m+1)}. \tag{4.3.18}$$

The right-hand side of (4.3.18) is bounded in view of (4.3.8), while the left-hand side diverges to ∞ in view of (4.3.9). This is a contradiction.

Next, suppose there is an integer $m_1 \geq m+1$ such that for every $k \geq m_1$ we have

$$\xi(k+1)G(k+1) - \sum_{i=m+1}^{k-1} G(i+1)\Delta\xi(i+1) < 0. \tag{4.3.19}$$

Set

$$M(n) = \sum_{i=m+1}^{n-1} G(i+1)\Delta\xi(i+1) \quad \text{for } n \geq m_1 \geq m+1. \tag{4.3.20}$$

We claim that

$$M(k) \leq \frac{M(m_1)}{\xi(m_1+1)}\xi(k+1) \quad \text{for } k \geq m_1+1. \tag{4.3.21}$$

Indeed, this follows from

$$\begin{aligned} \Delta\left[\frac{M(k)}{\xi(k+1)}\right] &= \frac{M(k+1)}{\xi(k+2)} - \frac{M(k)}{\xi(k+1)} \\ &= \frac{\xi(k+1)\Delta M(k) - M(k)\Delta\xi(k+1)}{\xi(k+1)\xi(k+2)} \\ &= \frac{\xi(k+1)G(k+1)\Delta\xi(k+1) - M(k)\Delta\xi(k+1)}{\xi(k+1)\xi(k+2)}, \end{aligned} \tag{4.3.22}$$

since in view of (4.3.19) we have

$$\Delta\left[\frac{M(k)}{\xi(k+1)}\right] < 0 \quad \text{for } k \geq m_1+1. \tag{4.3.23}$$

Finally, from (4.3.12), (4.3.15), (4.3.19), and (4.3.21) we arrive at

$$\begin{aligned} &\xi(k+1)G(k+1) + \sum_{i=m+1}^{k} \sum_{j=m+1}^{i} \rho(i)q(j) \\ &\quad \leq \frac{M(m_1)}{\xi(m_1+1)}\xi(k+1) + \xi(m+1)G(m+1) + \frac{c(m)\Delta x(m)}{f(x(m))}\psi(k, m+1), \end{aligned} \tag{4.3.24}$$

so

$$\frac{\xi(k+1)G(k+1)}{\psi(k,m+1)}+\frac{1}{\psi(k,m+1)}\sum_{i=m+1}^{k}\sum_{j=m+1}^{i}\rho(i)q(j)$$
$$\leq \frac{M(m_1)}{\xi(m_1+1)}\frac{\xi(k+1)}{\psi(k,m+1)}+G(m+1)\frac{\xi(m+1)}{\psi(k,m+1)}+\frac{c(m)\Delta x(m)}{f(x(m))}. \tag{4.3.25}$$

In view of (4.3.8), the right-hand side of (4.3.25) is bounded, while the left-hand side of (4.3.25) diverges to ∞. This contradiction completes the proof. □

The following result is immediate.

Corollary 4.3.4. *If*

$$\lim_{k\to\infty}\frac{1}{k}\sum_{i=m}^{k}\sum_{j=m}^{i}q(j)=\infty \quad \text{for all large } m\in\mathbb{N}, \tag{4.3.26}$$

then equation (4.1.7) with $\gamma\in(0,1)$ is oscillatory.

Next, we give the following result.

Theorem 4.3.5. *Suppose $f'(x)\geq 0$ for $x\neq 0$ and that (4.1.4) and (4.1.6) hold. If*

$$\limsup_{k\to\infty}\frac{1}{\sum_{i=m}^{k}(1/c(i))}\left[\sum_{i=m}^{k}\frac{1}{c(i)}\left(\sum_{j=m}^{i}q(j)\right)\right]=\infty \tag{4.3.27}$$

for all large $m\in\mathbb{N}$, then equation (4.1.2) is oscillatory.

Proof. Suppose that $\{x(k)\}$ is a nonoscillatory solution of equation (4.1.2), say, $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. Define $w(k)$ by (4.1.28) and obtain (4.1.29). Then we obtain (4.3.2). Summing (4.3.2) from $m+1$ to k we have

$$\frac{\Delta x(k)}{f(x(k))}-\frac{1}{c(k)}w(m)\leq -\frac{1}{c(k)}\sum_{j=m+1}^{k}q(j). \tag{4.3.28}$$

Again summing (4.3.28) from $m+1$ to k, we obtain

$$\sum_{i=m+1}^{k}\frac{\Delta x(i)}{f(x(i))}-w(m)\sum_{i=m+1}^{k}\frac{1}{c(i)}\leq -\sum_{i=m+1}^{k}\frac{1}{c(i)}\sum_{j=m+1}^{i}q(j). \tag{4.3.29}$$

Proceeding as in the proof of Theorem 4.3.3 we get (4.3.13). From (4.3.13) and (4.3.29) we conclude

$$\sum_{i=m+1}^{k}\int_{x(i)}^{x(i+1)}\frac{du}{f(u)}-w(m)\sum_{i=m+1}^{k}\frac{1}{c(i)}\leq -\sum_{i=m+1}^{k}\frac{1}{c(i)}\sum_{j=m+1}^{i}q(j), \tag{4.3.30}$$

so

$$\begin{aligned}
&-\frac{1}{\sum_{i=m+1}^{k}(1/c(i))}\left[\sum_{i=m+1}^{k}\int_{x(i)}^{x(i+1)}\frac{du}{f(u)}\right]+w(m)\\
&\qquad\geq\frac{1}{\sum_{i=m+1}^{k}(1/c(i))}\left[\sum_{i=m+1}^{k}\frac{1}{c(i)}\sum_{j=m+1}^{i}q(j)\right].
\end{aligned}\tag{4.3.31}$$

Taking the upper limit of both sides of (4.3.31) as $k\to\infty$, we arrive at the desired contradiction. This completes the proof. □

The following corollaries are immediate.

Corollary 4.3.6. *Suppose $f'(x)\geq 0$ for $x\neq 0$ and that (4.1.4) and (4.1.6) hold. If*

$$\limsup_{k\to\infty}\frac{1}{k}\sum_{i=m}^{k}\sum_{j=m}^{i}q(j)=\infty\quad\text{for all large } m\in\mathbb{N},\tag{4.3.32}$$

then equation (4.1.1) is oscillatory.

Corollary 4.3.7. *If $\gamma\in(0,1)$ and condition (4.3.32) holds, then equation (4.1.7) is oscillatory.*

Remark 4.3.8. Corollary 4.3.7 improves Corollary 4.3.4.

Theorem 4.3.9. *Suppose (4.1.4), (4.1.6), (4.1.10), (4.1.11), and (4.1.26) hold. Suppose further that $-f(-xy)\geq f(xy)\geq f(x)f(y)$ for $x,y\neq 0$. If*

$$Q(k)=\sum_{j=k}^{\infty}q(j)\geq 0\quad\text{for } k\in\mathbb{N},\tag{4.3.33}$$

$$\sum_{j=m\in\mathbb{N}}^{\infty}\frac{Q^2(j+1)}{c(j)}f\big(C(j-1,m)\big)=\infty,\tag{4.3.34}$$

where

$$C(k-1,m)=\sum_{j=m}^{k-1}\frac{1}{c(j)},\tag{4.3.35}$$

then equation (4.1.2) is oscillatory.

Proof. Assume that equation (4.1.2) has a nonoscillatory solution $\{x(k)\}$, and we may assume that $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. Since (4.3.33) holds, Theorem 4.1.11 implies that $\Delta x(k)\geq 0$ for $k\geq m$. From (4.1.52) we have

$$\Delta x(k)\geq\frac{Q(k+1)f(x(k))}{c(k)}\quad\text{for } k\geq m.\tag{4.3.36}$$

Summing equation (4.1.2) from $k+1$ to s, we obtain

$$c(s)\Delta x(s) - c(k)\Delta x(k) + \sum_{j=k+1}^{s} q(j)f\big(x(j)\big) = 0, \tag{4.3.37}$$

or for $s > k+1 \geq m+1$,

$$\begin{aligned} c(k)\Delta x(k) = {} & c(s)\Delta x(s) - Q(s+1)f\big(x(s+1)\big) + Q(k+1)f\big(x(k+1)\big) \\ & + \sum_{j=k+1}^{s} Q(j+1)\Delta f\big(x(j)\big). \end{aligned} \tag{4.3.38}$$

Note that

$$\sum_{j=k+1}^{\infty} Q(j+1)F\big(x(j+1),x(j)\big)\Delta x(j) < \infty, \tag{4.3.39}$$

since otherwise (4.3.38) would imply that $c(s)\Delta x(s) - Q(s+1)f(x(s+1)) \to -\infty$ as $s \to \infty$, which contradicts (4.3.36). Therefore, letting $s \to \infty$ in (4.3.38), we find

$$c(k)\Delta x(k) = a + Q(k+1)f\big(x(k+1)\big) + \sum_{j=k+1}^{\infty} Q(j+1)F\big(x(j+1),x(j)\big)\Delta x(j) \tag{4.3.40}$$

for $k \geq m$, where a denotes the finite limit

$$a = \lim_{s\to\infty}\big[c(s)\Delta x(s) - Q(s+1)f\big(x(s+1)\big)\big] \geq 0. \tag{4.3.41}$$

Define

$$\begin{aligned} G_1(k) &= \sum_{j=k}^{\infty} Q(j+1)F\big(x(j+1),x(j)\big)\Delta x(j) \quad \text{for } k \geq m, \\ G_2(k) &= \sum_{j=k}^{\infty} \frac{Q^2(j+1)}{c(j)}F\big(x(j+1),x(j)\big)f\big(x(j)\big) \quad \text{for } k \geq m. \end{aligned} \tag{4.3.42}$$

We see that $G_1(k) \geq G_2(k)$ for $k \geq m$, and hence $G_2(k)$ is well defined for $k \geq m$ and is convergent, that is, $G_2(k) < \infty$ for $k \geq m$. Thus, it follows from (4.3.40) that $c(k)\Delta x(k) \geq G_1(k+1) \geq G_2(k+1)$ for $k \geq m$, or

$$\Delta x(k) \geq \frac{G_2(k+1)}{c(k)} \quad \text{for } k \geq m. \tag{4.3.43}$$

Summing (4.3.43) from m to $k-1$, we obtain

$$x(k) \geq \sum_{j=m}^{k-1} \frac{G_2(j+1)}{c(j)} \geq G_2(k)C(k-1,m) \quad \text{for } k \geq m+1, \tag{4.3.44}$$

so that for $k \geq m+1$,

$$\frac{Q^2(k+1)f(x(k))F(x(k+1),x(k))}{c(k)f(G_2(k))} \geq \frac{\lambda}{c(k)}Q^2(k+1)f(C(k-1,m)). \tag{4.3.45}$$

Now

$$\Delta G_2(k) = -\frac{Q^2(k+1)f(x(k))}{c(k)}F(x(k+1),x(k)) \tag{4.3.46}$$

implies

$$-\frac{\Delta G_2(k)}{f(G_2(k))} \geq \frac{\lambda}{c(k)}Q^2(k+1)f(C(k-1,m)) \quad \text{for } k \geq m+1. \tag{4.3.47}$$

Summing (4.3.47) from m to $k-1$ and letting $k \to \infty$ and using the fact that

$$\int_{G_2(k)}^{G_2(m)} \frac{du}{f(u)} \geq \sum_{j=m}^{k-1} \frac{-\Delta G_2(j)}{f(G_2(j))}, \tag{4.3.48}$$

we obtain

$$\lambda \sum_{j=m}^{\infty} \frac{Q^2(j+1)}{c(j)} f(C(j-1,m)) \leq \int_0^{G_2(m)} \frac{du}{f(u)} < \infty, \tag{4.3.49}$$

which contradicts (4.3.34). This completes the proof. □

The following corollaries are immediate.

Corollary 4.3.10. *Let the hypotheses of Theorem 4.3.9 hold except for condition (4.1.10). If*

$$\sum_{j=m\in\mathbb{N}}^{\infty} Q^2(j+1)f(j) = \infty, \tag{4.3.50}$$

then equation (4.1.1) is oscillatory.

Corollary 4.3.11. *If condition (4.3.33) holds and*

$$\sum_{j=m\in\mathbb{N}}^{\infty} j^{\gamma}Q^2(j) = \infty, \tag{4.3.51}$$

then equation (4.1.7) with $\gamma \in (0,1)$ is oscillatory.

Theorem 4.3.12. *Suppose (4.1.4) and (4.1.6) hold, $f'(x) \geq 0$ for $x \neq 0$, and*

$$f'(x)F(x) \geq \frac{1}{a} > 0 \quad \text{for } x \neq 0, \tag{4.3.52}$$

where a is a positive constant and $F(x) = \int_0^x du/f(u)$. If

$$Q(k) = \sum_{j=k}^{\infty} q(j) \quad \text{exists} \tag{4.3.53}$$

and

$$\lim_{k\to\infty} \sum_{j=m}^{k} \left[(j+1)^{\lambda} - j^{\lambda}\right]\left(\sum_{i=j+1}^{\infty} q(i)\right) = \infty, \tag{4.3.54}$$

where $\lambda = 1/(a+1) < 1$, then equation (4.1.1) is oscillatory.

Proof. We claim that condition (4.1.82) in Remark 4.1.16 holds under the assumptions of the theorem. In fact, from the hypotheses we see that

$$\frac{f'(x)}{f(x)} \geq \frac{1}{a}\frac{F'(x)}{F(x)} \quad \text{for } x > 0. \tag{4.3.55}$$

Integrating (4.3.55), we obtain

$$\left(\frac{f(x)}{f(x_0)}\right)^{a} \geq \frac{F(x)}{F(x_0)} \quad \text{for } x \geq x_0 > 0. \tag{4.3.56}$$

Hence

$$af'(x)(f(x))^{a} \geq \frac{(f(x))^{a}}{F(x)} \geq \frac{(f(x_0))^{a}}{F(x_0)}. \tag{4.3.57}$$

Integrating (4.3.57) from x_0 to x, we obtain

$$\left(\frac{a}{a+1}\right)\left[(f(x))^{a+1} - (f(x_0))^{a+1}\right] \geq \frac{(f(x_0))^{a}}{F(x_0)}(x - x_0). \tag{4.3.58}$$

It follows that $\lim_{x\to\infty} f(x) = \infty$. Similarly, we can prove that $\lim_{x\to-\infty} f(x) = -\infty$.

Now let $\{x(k)\}$ be a nonoscillatory solution of equation (4.1.1), say, $x(k) > 0$ for all $k \geq m$ for some $m \in \mathbb{N}$. By Theorem 4.1.11, equation (4.1.52) holds. Define $r(t) = x(i) + (t-i)\Delta x(i)$ for $i \leq t \leq i+1$ and

$$\phi(t) = t^{\lambda-1}\int_0^{r(t)} \frac{dy}{f(y)} \quad \text{for } i \leq t \leq i+1. \tag{4.3.59}$$

We claim that

$$\lim_{k\to\infty}\frac{\phi(k)}{k^{\lambda}}=\lim_{k\to\infty}\frac{1}{k}\int_0^{r(k)}\frac{dy}{f(y)}=0. \tag{4.3.60}$$

In fact, as in the proof of Theorem 4.3.3, (4.2.12) holds. Hence

$$\int_i^{i+1}\frac{\Delta x(i)}{f(x(i+1))}dt\le\int_i^{i+1}\frac{r'(t)}{f(r(t))}dt\le\int_i^{i+1}\frac{\Delta x(i)}{f(x(i))}dt, \tag{4.3.61}$$

$$\frac{1}{k}\sum_{i=m_1\ge m}^{i-1}\frac{\Delta x(i)}{f(x(i+1))}\le\frac{1}{k}\int_{r(m_1)}^{r(k)}\frac{dy}{f(y)}\le\frac{1}{k}\sum_{i=m_1}^{k-1}\frac{\Delta x(i)}{f(x(i))}. \tag{4.3.62}$$

From (4.1.52) we see that

$$\frac{1}{k}\sum_{i=m_1}^{k-1}\frac{\Delta x(i)}{f(x(i+1))}\longrightarrow 0\quad\text{as } k\longrightarrow\infty. \tag{4.3.63}$$

Since

$$\sum_{i=k}^{\infty}\frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))}=\sum_{i=k}^{\infty}\left[\frac{\Delta x(i)}{f(x(i))}-\frac{\Delta x(i)}{f(x(i+1))}\right]<\infty, \tag{4.3.64}$$

it follows that

$$\lim_{k\to\infty}\frac{\Delta x(k)}{f(x(k))}=\lim_{k\to\infty}\frac{\Delta x(k)}{f(x(k+1))}=0. \tag{4.3.65}$$

Hence

$$\frac{1}{k}\sum_{i=m_1}^{k-1}\frac{\Delta x(i)}{f(x(i))}\longrightarrow 0\quad\text{as } k\longrightarrow\infty. \tag{4.3.66}$$

Combining (4.3.62), (4.3.63), and (4.3.66), we obtain (4.3.60).

Next we claim that

$$\int_t^{\infty}\frac{\phi(s)}{s^{\lambda+1}}ds<\infty\quad\text{for } t\ge m_1. \tag{4.3.67}$$

In fact, from (4.3.59) we get

$$\frac{r'(t)}{f(r(t))}=t^{1-\lambda}\phi'(t)+(1-\lambda)t^{-\lambda}\phi(t)\quad\text{for } i\le t\le i+1. \tag{4.3.68}$$

Now

$$\frac{f'(r(t))(r'(t))^2}{f^2(r(t))} \geq \frac{1}{a}\left[t^{1-\lambda}\frac{(\phi'(t))^2}{\phi(t)} + 2(1-\lambda)t^{-\lambda}\phi'(t) + (1-\lambda)^2 t^{-\lambda-1}\phi(t)\right]$$
$$= \frac{t^{1-\lambda}(\phi'(t))^2}{a\phi(t)} + 2\lambda t^{-\lambda}\phi'(t) + \lambda(1-\lambda)t^{-1-\lambda}\phi(t). \tag{4.3.69}$$

On the other hand, we see that

$$\begin{aligned}\int_{m_1}^{\infty}\frac{f'(r(s))(r'(s))^2}{f^2(r(s))}ds &= \sum_{i=m_1}^{\infty}\int_i^{i+1}\frac{f'(r(s))(r'(s))^2}{f^2(r(s))}ds \\ &= \sum_{i=m_1}^{\infty}\int_i^{i+1}\frac{f'(r(s))r'(s)}{f^2(r(s))}ds\,\Delta x(i) \\ &= \sum_{i=m_1}^{\infty}\left[\frac{1}{f(x(i))} - \frac{1}{f(x(i+1))}\right]\Delta x(i) \\ &= \sum_{i=m_1}^{\infty}\frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))},\end{aligned} \tag{4.3.70}$$

that is,

$$\int_{m_1}^{\infty}\frac{f'(r(s))(r'(s))^2}{f^2(r(s))}ds = \sum_{i=m_1}^{\infty}\frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))}. \tag{4.3.71}$$

Thus

$$\int_{m_1}^{\infty}\frac{f'(r(s))(r'(s))^2}{f^2(r(s))}ds < \infty. \tag{4.3.72}$$

Combining (4.3.69) and (4.3.72), we find

$$\int_{m_1}^{\infty} s^{-\lambda}\phi'(s)ds < \infty. \tag{4.3.73}$$

In view of (4.3.73) and (4.3.60), we obtain

$$\int_{m_1}^{\infty}\frac{\phi(s)}{s^{\lambda+1}}ds < \infty, \tag{4.3.74}$$

and hence (4.3.67) holds.

We now define

$$w(t) = \phi(t) - (1+\lambda)t^{\lambda}\int_t^{\infty}\frac{\phi(s)}{s^{\lambda+1}}ds \quad \text{for } k \leq t \leq k+1. \tag{4.3.75}$$

Then from (4.3.59), (4.2.12), (4.1.52), (4.3.69), and (4.3.71), we have

$$\begin{aligned}
w'(t) &= \phi'(t) - (1+\lambda)\lambda t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds + (1+\lambda)t^{\lambda}\frac{\phi(t)}{t^{\lambda+1}} \\
&= t^{\lambda-1}\frac{r'(t)}{f(r(t))} + (\lambda-1)t^{\lambda-2}\int_0^{r(t)}\frac{dy}{f(y)} - (1+\lambda)\lambda t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&\quad + (1+\lambda)\frac{\phi(t)}{t} \\
&= t^{\lambda-1}\frac{r'(t)}{f(r(t))} + 2\lambda\frac{\phi(t)}{t} - \lambda(1+\lambda)t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&\ge t^{\lambda-1}\frac{\Delta x(k)}{f(x(k+1))} + 2\lambda\frac{\phi(t)}{t} - \lambda(1+\lambda)t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&= t^{\lambda-1}\left[\sum_{i=k+1}^\infty q(i) + \sum_{i=k+1}^\infty \frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))}\right] + 2\lambda\frac{\phi(t)}{t} \\
&\quad - \lambda(1+\lambda)t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&= t^{\lambda-1}\sum_{i=k+1}^\infty q(i) - t^{\lambda-1}\frac{\Delta f(x(k))\Delta x(k)}{f(x(k))f(x(k+1))} + t^{\lambda-1}\int_t^\infty \frac{f'(r(s))(r'(s))^2}{f^2(r(s))}ds \\
&\quad + 2\lambda\frac{\phi(t)}{t} - \lambda(\lambda+1)t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&\ge t^{\lambda-1}\sum_{i=k+1}^\infty q(i) - t^{\lambda-1}\frac{\Delta f(x(k))\Delta x(k)}{f(x(k))f(x(k+1))} \\
&\quad + t^{\lambda-1}\int_t^\infty \left[\frac{s^{1-\lambda}(\phi'(s))^2}{a\phi(s)} + 2\lambda s^{-\lambda}\phi'(s) + \lambda(1-\lambda)s^{-\lambda-1}\phi(s)\right]ds + 2\lambda\frac{\phi(t)}{t} \\
&\quad - \lambda(\lambda+1)t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&= t^{\lambda-1}\sum_{i=k+1}^\infty q(i) - t^{\lambda-1}\frac{\Delta f(x(k))\Delta x(k)}{f(x(k))f(x(k+1))} + t^{\lambda-1}\int_t^\infty \frac{s^{1-\lambda}(\phi'(s))^2}{a\phi(s)}ds \\
&\quad + 2\lambda t^{\lambda-1}\left[-\frac{\phi(t)}{t^{\lambda}} + \lambda\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds\right] + t^{\lambda-1}\lambda(1-\lambda)\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&\quad + 2\lambda\frac{\phi(t)}{t} - \lambda(1+\lambda)t^{\lambda-1}\int_t^\infty \frac{\phi(s)}{s^{\lambda+1}}ds \\
&= t^{\lambda-1}\sum_{i=k+1}^\infty q(i) - t^{\lambda-1}\frac{\Delta f(x(k))\Delta x(k)}{f(x(k))f(x(k+1))} + t^{\lambda-1}\int_t^\infty \frac{s^{1-\lambda}(\phi'(s))^2}{a\phi(s)}ds \\
&\ge t^{\lambda-1}\sum_{i=k+1}^\infty q(i) - t^{\lambda-1}\frac{\Delta f(x(k))\Delta x(k)}{f(x(k))f(x(k+1))}.
\end{aligned} \tag{4.3.76}$$

We note that $(1+k)^{\lambda} - k^{\lambda}$ is decreasing and bounded for all large k. In view of (4.1.50),

$$\begin{aligned}\sum_{i=m}^{k}\int_{i}^{i+1} s^{\lambda-1}\frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))}ds &= \frac{1}{\lambda}\sum_{i=m}^{k}\left[(i+1)^{\lambda}-i^{\lambda}\right]\frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))}\\ &\le M\sum_{i=m}^{k}\frac{\Delta f(x(i))\Delta x(i)}{f(x(i))f(x(i+1))}<\infty\end{aligned} \tag{4.3.77}$$

as $k \to \infty$, where M is a positive constant.

On the other hand, we see that

$$\int_{k}^{k+1} s^{\lambda-1}\left(\sum_{i=k+1}^{\infty} q(i)\right)ds = \frac{1}{\lambda}\left[(k+1)^{\lambda}-k^{\lambda}\right]\sum_{i=k+1}^{\infty} q(i). \tag{4.3.78}$$

Due to condition (4.3.54), we know that

$$\begin{aligned}\sum_{j=m}^{k}\int_{j}^{j+1} s^{k-1}\left(\sum_{i=j+1}^{\infty} q(i)\right)ds &= \frac{1}{\lambda}\sum_{j=m}^{k}\left[(j+1)^{\lambda}-j^{\lambda}\right]\left(\sum_{i=j+1}^{\infty} q(i)\right)\\ &\longrightarrow \infty \quad \text{as } k \longrightarrow \infty.\end{aligned} \tag{4.3.79}$$

Combining (4.3.76), (4.3.77), and (4.3.79), we obtain

$$w(k) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.3.80}$$

It follows that $\phi(t) \to \infty$ as $t \to \infty$. Let $k_n = \sup\{k : \phi(k) \le n\}$. Clearly $k_n \to \infty$ as $n \to \infty$. For $k \ge k_n$, $\phi(k) \ge n \ge \phi(k_n)$. Hence

$$\begin{aligned} w(k_n) &= \phi(k_n) - (1+\lambda)k_n^{\lambda}\int_{k_n}^{\infty}\frac{\phi(s)}{s^{\lambda+1}}ds\\ &\le \phi(k_n) - (1+\lambda)k_n^{\lambda}\phi(k_n)\int_{k_n}^{\infty}\frac{ds}{s^{\lambda+1}}\\ &= \phi(k_n)\left[1-(1+\lambda)k_n^{\lambda}\frac{k_n^{-\lambda}}{\lambda}\right]\\ &= -\frac{1}{\lambda}\phi(k_n) < 0,\end{aligned} \tag{4.3.81}$$

which contradicts (4.3.80). Similarly, one can prove that equation (4.1.1) has no eventually negative solution. This completes the proof. □

4.4. Oscillation characterizations

In this section we are interested in characterizing the oscillation of more general equations of type (4.1.2), namely,

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + f\big(k, x(k)\big) = 0, \tag{4.4.1}$$

where $k \in \mathbb{N}(m) = \{m, m+1, \ldots\}$, $m \in \mathbb{N}$ is fixed, $\{c(k)\}$ is a sequence of positive real numbers, $f : \mathbb{N}(m) \times \mathbb{R} \to \mathbb{R}$ is for any $k \in \mathbb{N}(m)$ continuous as a function of $x \in \mathbb{R}$, and $xf(k,x) > 0$ for $x \neq 0$ and $k \in \mathbb{N}(m)$, that is, in obtaining necessary and sufficient conditions for equation (4.4.1) to be oscillatory.

We classify equations of the form of (4.4.1) according to the nonlinearity of $f(k,x)$ with respect to x.

Definition 4.4.1. Equation (4.4.1) is called *superlinear* if for each k,

$$\frac{|f(k,x)|}{|x|} \leq \frac{|f(k,y)|}{|y|} \quad \text{for } |x| < |y|,\ xy > 0, \tag{4.4.2}$$

and *strongly superlinear* if there exists a number $\alpha > 1$ such that for each k,

$$\frac{|f(k,x)|}{|x|^\alpha} \leq \frac{|f(k,y)|}{|y|^\alpha} \quad \text{for } |x| < |y|,\ xy > 0. \tag{4.4.3}$$

Definition 4.4.2. Equation (4.4.1) is called *sublinear* if for each k,

$$\frac{|f(k,x)|}{|x|} \geq \frac{|f(k,y)|}{|y|} \quad \text{for } |x| < |y|,\ xy > 0, \tag{4.4.4}$$

and *strongly sublinear* if there exists a number $\beta \in (0,1)$ such that for each k,

$$\frac{|f(k,x)|}{|x|^\beta} \geq \frac{|f(k,y)|}{|y|^\beta} \quad \text{for } |x| < |y|,\ xy > 0. \tag{4.4.5}$$

We will use the sequence $\{C(k,m)\}$ defined by $C(k,m) = \sum_{j=m}^{k-1} 1/c(j)$ for $k \in \mathbb{N}(m)$, and throughout this section we assume that

$$C(k,m) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{4.4.6}$$

The following lemma gives information on the behavior of nonoscillatory solutions of equation (4.4.1).

Lemma 4.4.3. *If $\{x(k)\}$ is a nonoscillatory solution of equation (4.4.1), then there exist positive constants a_1, a_2 and $m_1 \in \mathbb{N}(m)$ such that*

$$x(k)\Delta x(k) > 0 \quad \forall k \in \mathbb{N}(m_1), \tag{4.4.7}$$
$$a_1 \leq \big|x(k)\big| \leq a_2 C(k,m) \quad \forall k \in \mathbb{N}(m_1). \tag{4.4.8}$$

Proof. We may suppose that $\{x(k)\}$ is an eventually positive solution of equation (4.4.1), since a similar argument holds if $\{x(k)\}$ is eventually negative. There exists an integer $m_1 \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$ such that $x(k) > 0$ for all $k \in \mathbb{N}(m_1)$. It follows from (4.4.1) that $\Delta(c(k)\Delta x(k)) \le 0$, that is, the sequence $\{c(k)\Delta x(k)\}$ is nonincreasing for $k \in \mathbb{N}(m_1)$. If $c(m_2)\Delta x(m_2) = a < 0$ for some $m_2 \in \mathbb{N}(m_1)$, then $c(k)\Delta x(k) \le a$ for all $k \in \mathbb{N}(m_2)$, and hence $x(k) \le x(m_2) + aC(k, m_2) \to -\infty$ as $k \to \infty$, which is a contradiction. Therefore we must have $c(k)\Delta x(k) > 0$ for all $k \in \mathbb{N}(m_1)$, that is, (4.4.7) holds. Further, since $\{c(k)\Delta x(k)\}$ is nonincreasing for $k \in \mathbb{N}(m_1)$, we may write $\Delta x(k) \le c(m_1)\Delta x(m_1)/c(k)$, which implies

$$x(k) \le x(m_1) + c(m_1)\Delta x(m_1)C(k, m_1) \quad \forall k \in \mathbb{N}(m_1), \tag{4.4.9}$$

and since $\{x(k)\}$ is increasing, (4.4.8) follows. □

According to Lemma 4.4.3, among all nonoscillatory solutions of equation (4.4.1), those which are asymptotic to nonzero constants as $k \to \infty$, may be regarded as the "minimal" solutions, and those which are asymptotic to sequences of the form $\{aC(k, m)\}$, $a \neq 0$, as $k \to \infty$, may be regarded as the "maximal" solutions.

Next we will give necessary and sufficient conditions for the existence of these two special types of nonoscillatory solutions.

Theorem 4.4.4. *Let equation (4.4.1) be either superlinear or sublinear. A necessary and sufficient condition for equation (4.4.1) to have a nonoscillatory solution* $\{x(k)\}$ *such that* $\lim_{k\to\infty} x(k) = \text{constant} \neq 0$ *is that*

$$\sum^{\infty} C(k, m)|f(k, a)| < \infty, \tag{4.4.10}$$

for some constant $a \neq 0$.

Proof. First we show necessity. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.4.1) which is asymptotic to a nonzero constant as $k \to \infty$. Without loss of generality, we may suppose that $x(k) > 0$ for $k \ge m_1 \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$. By Lemma 4.4.3, $\Delta x(k) > 0$ for all $k \in \mathbb{N}(m_1)$, and so there are positive constants a_1 and a_2 such that

$$a_1 \le x(k) \le a_2 \quad \forall k \in \mathbb{N}(m_1). \tag{4.4.11}$$

Using the identity

$$\Delta(C(k, m)c(k)\Delta x(k)) = C(k+1, m)\Delta(c(k)\Delta x(k)) + \Delta x(k), \tag{4.4.12}$$

from equation (4.4.1), we have

$$\begin{aligned} C(k, m)c(k)\Delta x(k) + x(m_1) + \sum_{j=m_1+1}^{k} C(j, m)f(j, x(j)) \\ = C(m_1, m)c(m_1)\Delta x(m_1) + x(k) \end{aligned} \tag{4.4.13}$$

for all $k \in \mathbb{N}(m_1)$. By the boundedness of $\{x(k)\}$ and positivity of $\{\Delta x(k)\}$, (4.4.13) implies

$$\sum_{j=m_1+1}^{\infty} C(j,m) f(j,x(j)) < \infty. \tag{4.4.14}$$

In turn this implies that

$$\sum^{\infty} C(j,m) f(j,a_1) < \infty \quad \text{if equation (4.4.1) is superlinear,} \tag{4.4.15}$$

$$\sum^{\infty} C(j,m) f(j,a_2) < \infty \quad \text{if equation (4.4.1) is sublinear.} \tag{4.4.16}$$

Now we show sufficiency. Assume that (4.4.10) holds with $a > 0$ (a similar argument can be applied if $a < 0$). Choose $m_1 \in \mathbb{N}(m)$ so large that

$$\sum_{j=m_1}^{\infty} C(j,m) f(j,a) \le \frac{b}{2}, \tag{4.4.17}$$

where

$$b = \begin{cases} \dfrac{a}{2} & \text{if equation (4.4.1) is superlinear,} \\ a & \text{if equation (4.4.1) is sublinear,} \end{cases} \tag{4.4.18}$$

and consider the equation

$$x(k) = b + \sum_{j=m_1}^{k-1} C(j,m) f(j,x(j)) + C(k,m) \sum_{j=k}^{\infty} f(j,x(j)). \tag{4.4.19}$$

It is easy to verify that a solution of (4.4.19) is also a solution of equation (4.4.1). Now we will show that (4.4.19) has a bounded nonoscillatory solution. Consider the Banach space B of all bounded functions $x : \mathbb{N}(m_1) \to \mathbb{R}$, $m_1 \ge m \in \mathbb{N}$, with norm $\|x\| = \sup_{k \ge m_1} |x(k)|$ and let

$$S = \{x \in B : b \le x(k) \le 2b,\ k \in \mathbb{N}(m_1)\}. \tag{4.4.20}$$

Clearly, S is a bounded, convex, and closed subset of B. For $k \in \mathbb{N}(m_1)$, we define an operator $T : S \to S$ by

$$Tx(k) = b + \sum_{j=m_1}^{k-1} C(j,m) f(j,x(j)) + C(k,m) \sum_{j=k}^{\infty} f(j,x(j)). \tag{4.4.21}$$

Next, we will show that T satisfies the hypotheses of Schauder's fixed point theorem, that is, Theorem 3.6.4.

First we show that T maps S into itself. In fact, if $x \in S$, then $Tx(k) \geq b$ for all $k \in \mathbb{N}(m_1)$ and by (4.4.17),

$$\begin{aligned} Tx(k) &\leq b + \sum_{j=m_1}^{\infty} C(j,m) f(j,x(j)) \\ &\leq b + 2\sum_{j=m_1}^{\infty} C(j,m) f(j,a) \leq 2b \end{aligned} \tag{4.4.22}$$

for all $k \in \mathbb{N}(m_1)$. Therefore, $TS \subset S$.

Next we show that T is continuous. Let $\varepsilon > 0$ and choose $m_2 \in \mathbb{N}(m_1)$ so large that $\sum_{j=m_2}^{\infty} C(j,m) f(j,a) < \varepsilon$. Let $x = \{x(k)\}$ and for each i, let $x^i = \{x^i(k)\}_{i=1}^{\infty}$ be a sequence in S such that $\lim_{i\to\infty} \|x^i - x\| = 0$. Since S is closed, $x \in S$. Then using (4.4.21), by superlinearity or sublinearity, we get

$$\begin{aligned} |Tx^i(k) - Tx(k)| &\leq \sum_{j=m_1}^{m_2-1} C(j,m) |f(j,x^i(j)) - f(j,x(j))| \\ &\quad + C(m_2,m) \sum_{j=m_2}^{\infty} |f(j,x^i(j)) - f(j,x(j))| \\ &\leq \sum_{j=m_1}^{m_2-1} C(j,m) |f(j,x^i(j)) - f(j,x(j))| \\ &\quad + 4\sum_{j=m_2}^{\infty} C(j,m) f(j,a), \end{aligned} \tag{4.4.23}$$

from which, by the continuity of f, it follows that

$$\limsup_{i\to\infty,\, k\geq m_1} |Tx^i(k) - Tx(k)| = 0, \tag{4.4.24}$$

so T is continuous.

Finally we show that $T(S)$ is relatively compact. It suffices to show that $T(S)$ is uniformly Cauchy. Let $x \in S$ and $k > n \in \mathbb{N}(m_1)$. Then we have

$$\begin{aligned} Tx(k) - Tx(n) &= \sum_{j=n}^{k-1} C(j,m) f(j,x(j)) + C(k,m) \sum_{j=k}^{\infty} f(j,x(j)) \\ &\quad - C(n,m) \sum_{j=n}^{\infty} f(j,x(j)), \end{aligned} \tag{4.4.25}$$

which implies

$$|Tx(k) - Tx(n)| \leq 3\sum_{j=n}^{\infty} C(j,m) f(j,x(j)) \leq 6\sum_{j=n}^{\infty} C(j,m) f(j,a). \tag{4.4.26}$$

By (4.4.10) the last sum tends to zero as $n \to \infty$, so for given $\varepsilon > 0$, there exists $m_2 \in \mathbb{N}(m_1)$ such that for all $x \in S$, $|Tx(k) - Tx(n)| < \varepsilon$ for all $k, m \in \mathbb{N}(m_2)$.

By Theorem 3.6.4 we conclude that there exists $x \in S$ such that $x = Tx$, that is, x is a solution of equation (4.4.19), and since

$$\Delta x(k) = \frac{1}{c(k)} \sum_{j=k+1}^{\infty} f(j, x(j)) > 0 \quad \forall k \in \mathbb{N}(m_1), \tag{4.4.27}$$

the solution $\{x(k)\}$ tends monotonically to a positive constant in $[b, 2b]$ as $k \to \infty$. This completes the proof. □

The following corollary is immediate.

Corollary 4.4.5. *Equation (4.1.8) has a bounded nonoscillatory solution if and only if*

$$\sum^{\infty} C(j, m) q(j) < \infty \quad \forall m \in \mathbb{N}. \tag{4.4.28}$$

A characterization of the oscillation situation for the strongly superlinear equation (4.4.1) is given in the following result.

Theorem 4.4.6. *Let equation (4.4.1) be strongly superlinear. A necessary and sufficient condition for equation (4.4.1) to be oscillatory is that*

$$\sum^{\infty} C(j, m) |f(j, a)| = \infty \tag{4.4.29}$$

for all constants $a \neq 0$ and $m \in \mathbb{N}$.

Proof. The necessity of the given condition follows from the sufficiency part of Theorem 4.4.4. To prove the sufficiency part suppose there exists a nonoscillatory solution $\{x(k)\}$ of equation (4.4.1). Without loss of generality we may suppose that $x(k) > 0$ for all $k \geq m_1 \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$. Then by Lemma 4.4.3 there exists a constant $a_1 > 0$ such that $x(k) \geq a_1$ and $\Delta x(k) > 0$ for all $k \in \mathbb{N}(m_1)$. Let $\alpha > 1$ be the strong superlinearity constant of equation (4.4.1). Set $y(k) = c(k)\Delta x(k)$ for $k \in \mathbb{N}(m_1)$. Using (4.4.1), the strong superlinearity, and the fact that $\{x(k)\}$ is increasing, we obtain

$$\begin{aligned} &\Delta\big(C(k, m) y(k) x^{-\alpha}(k+1)\big) \\ &\quad \leq y(k) x^{-\alpha}(k+1) \Delta C(k, m) + C(k+1, m) \Delta y(k) x^{-\alpha}(k+1) \\ &\quad \leq \Delta x(k) x^{-\alpha}(k+1) - a_1^{-\alpha} C(k+1, m) f(k+1, a_1) \end{aligned} \tag{4.4.30}$$

for $k \in \mathbb{N}(m_1)$. From this, taking into consideration the fact that

$$\frac{\Delta x(k)}{x^{\alpha}(k+1)} \leq \int_{x(k)}^{x(k+1)} \frac{du}{u^{\alpha}} \quad \text{for } k \in \mathbb{N}(m_1), \tag{4.4.31}$$

we get

$$a_1^{-\alpha} \sum_{j=m_1+1}^{k} C(j,m)f(j,a_1) \le \int_{x(m_1)}^{x(k)} \frac{du}{u^\alpha} + C(m_1,m)y(m_1)x^{-\alpha}(m_1+1), \tag{4.4.32}$$

which contradicts (4.4.29). This completes the proof. □

Corollary 4.4.7. *Equation (4.1.8) with $\gamma > 1$ is oscillatory if and only if*

$$\sum^{\infty} C(j,m)q(j) = \infty \quad \text{for } m \in \mathbb{N}. \tag{4.4.33}$$

Theorem 4.4.8. *Let equation (4.4.1) be either superlinear or sublinear. A necessary and sufficient condition for equation (4.4.1) to have a nonoscillatory solution $\{x(k)\}$ with the property $\lim_{k\to\infty} x(k)/C(k,m) = \text{constant} \ne 0$, $k \in \mathbb{N}(m)$ for $m \in \mathbb{N}$, is that*

$$\sum^{\infty} |f(j, aC(j,m))| < \infty \tag{4.4.34}$$

for some constant $a \ne 0$.

Proof. First we show necessity. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.4.1) such that $\lim_{k\to\infty} x(k)/C(k,m) = b \ne 0$, where b is a constant. We may suppose $b > 0$. Hence there are positive constants a_1, a_2 and $m_1 \in \mathbb{N}(m)$, $m \in \mathbb{N}$, such that

$$a_1C(k,m) \le x(k) \le a_2C(k,m) \quad \forall k \in \mathbb{N}(m_1). \tag{4.4.35}$$

Summing equation (4.4.1) from $m_1 + 1$ to k, we obtain

$$c(m_1)\Delta x(m_1) - c(k)\Delta x(k) = \sum_{j=m_1+1}^{k} f(j, x(j)), \tag{4.4.36}$$

from which in view of Lemma 4.4.3,

$$\sum^{\infty} f(j, x(j)) < \infty. \tag{4.4.37}$$

From (4.4.35) and (4.4.37) it follows that

$$\begin{aligned} &\sum^{\infty} f(j, a_1C(j,m)) < \infty \quad \text{if equation (4.4.1) is superlinear,} \\ &\sum^{\infty} f(j, a_2C(j,m)) < \infty \quad \text{if equation (4.4.1) is sublinear.} \end{aligned} \tag{4.4.38}$$

Now we show sufficiency. Suppose (4.4.34) holds with $a > 0$. Set

$$b = \begin{cases} \dfrac{a}{2} & \text{if equation (4.4.1) is superlinear,} \\ a & \text{if equation (4.4.1) is sublinear.} \end{cases} \tag{4.4.39}$$

Choose $m_1 \in \mathbb{N}(m)$ so large that

$$\sum_{j=m_1}^{\infty} f(j, aC(j,m)) \leq \frac{b}{2}. \tag{4.4.40}$$

The required solution of equation (4.4.1) is obtained as a solution of the equation

$$x(k) = bC(k,m) + \sum_{j=m_1}^{k-1} C(j,m) f(j, x(j)) + C(k,m) \sum_{j=k}^{\infty} f(j, x(j)). \tag{4.4.41}$$

To prove that (4.4.41) has a solution, with the help of Schauder's fixed point theorem, we introduce the linear space B_C of all functions $x : \mathbb{N}(m_1) \to \mathbb{R}$ such that

$$\|x\|_C = \sup_{k \geq m_1} \frac{|x(k)|}{C^2(k,m)} < \infty. \tag{4.4.42}$$

It is obvious that B_C is a Banach space with norm $\|\cdot\|_C$. Define

$$S = \{x \in B_C : bC(k,m) \leq x(k) \leq 2bC(k,m),\ k \in \mathbb{N}(m_1)\}. \tag{4.4.43}$$

Clearly S is a bounded, convex, and closed subset of B_C.

For $k \in \mathbb{N}(m_1)$, we define the operator $T : S \to S$ by the formula

$$Tx(k) = bC(k,m) + \sum_{j=m_1}^{k-1} C(j,m) f(j, x(j)) + C(k,m) \sum_{j=k}^{\infty} f(j, x(j)). \tag{4.4.44}$$

As in the proof of the sufficiency part of Theorem 4.4.4 it can be shown that T is a continuous operator which maps S into a compact subset of S. Therefore, by Theorem 3.6.4, T has a fixed point $x \in S$ which provides a solution $x = \{x(k)\}$ of equation (4.4.1). This solution has the required asymptotic property, since by L'Hôpital's rule and (4.4.41) we have

$$\lim_{k \to \infty} \frac{x(k)}{C(k,m)} = \lim_{k \to \infty} \frac{\Delta x(k)}{\Delta C(k,m)} = b. \tag{4.4.45}$$

This sketches the proof of the sufficiency part and completes the proof of the theorem. □

Corollary 4.4.9. *Equation (4.1.8) has a nonoscillatory solution* $\{x(k)\}$ *such that* $\lim_{k\to\infty} x(k)/C(k,m) = \text{constant} \neq 0$, $m \in \mathbb{N}$, *if and only if*

$$\sum^{\infty} C^{\gamma}(j,m)q(j) < \infty. \tag{4.4.46}$$

Next, we pass to the case of strongly sublinear equations (4.4.1).

Theorem 4.4.10. *Let equation (4.4.1) be strongly sublinear. A necessary and sufficient condition for equation (4.4.1) to be oscillatory is that*

$$\sum^{\infty} \left| f(j, aC(j,m)) \right| = \infty \tag{4.4.47}$$

for all constants $a \neq 0$ *and* $m \in \mathbb{N}$.

Proof. The necessity part is an immediate consequence of Theorem 4.4.8. It remains to prove the sufficiency part. Let $\{x(k)\}$ be a nonoscillatory solution of (4.4.1). We may suppose that $x(k) > 0$ for $k \geq m_1 \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$. Then the sequence $\{c(k)\Delta x(k)\}$ is decreasing for $k \in \mathbb{N}(m_1)$ and by Lemma 4.4.3, $\Delta x(k) > 0$ for all $k \in \mathbb{N}(m_1)$. We observe that

$$x(k) > c(k-1)\Delta x(k-1)C(k,m_1) \quad \text{for } k \in \mathbb{N}(m_1). \tag{4.4.48}$$

By Lemma 4.4.3 and (4.4.48) there exist positive constants a_1 and a_2 such that

$$a_1 c(k-1)\Delta x(k-1)C(k,m) \leq x(k) \leq a_2 C(k,m) \quad \forall k \in \mathbb{N}(m_1). \tag{4.4.49}$$

Let $\beta \in (0,1)$ be the strong sublinearity constant of equation (4.4.1). The first inequality of (4.4.49) implies that

$$\left(c(k-1)\Delta x(k-1)\right)^{-\beta} \geq x^{-\beta}(k)\left[a_1 C(k,m)\right]^{\beta} \quad \forall k \in \mathbb{N}(m_1). \tag{4.4.50}$$

Using the mean value theorem, equation (4.4.1), (4.4.50), and the second inequality of (4.4.49) together with the strong sublinearity, we see that

$$\begin{aligned}
&\Delta\left(c(k-1)\Delta x(k-1)\right)^{1-\beta} \\
&\quad \leq (1-\beta)\left(c(k-1)\Delta x(k-1)\right)^{-\beta}\Delta\left(c(k-1)\Delta x(k-1)\right) \\
&\quad \leq (\beta-1)x^{-\beta}(k)\left[a_1 C(k,m)\right]^{\beta} f(k,x(k)) \\
&\quad \leq (\beta-1)\left(\frac{a_1}{a_2}\right)^{\beta} f(k, a_2 C(k,m))
\end{aligned} \tag{4.4.51}$$

for all $k \in \mathbb{N}(m_1)$. Summation of the above inequalities shows that

$$\sum^{\infty} f(k, a_2 C(k,m)) < \infty, \tag{4.4.52}$$

which contradicts condition (4.4.47). This completes the proof. $\square$

Corollary 4.4.11. *Equation (4.1.8) with $\gamma \in (0,1)$ is oscillatory if and only if*

$$\sum^{\infty} C^{\gamma}(j,m)q(j) = \infty \quad \text{for } m \in \mathbb{N}. \tag{4.4.53}$$

The following two corollaries are the discrete analogues of the continuous versions of the well-known results of Atkinson and Belohorec.

Corollary 4.4.12. *Let conditions (4.1.4) and (4.1.5) hold and suppose $f'(x) \geq 0$ for $x \neq 0$. Equation (4.1.1) is oscillatory if and only if*

$$\sum^{\infty} jq(j) = \infty. \tag{4.4.54}$$

Corollary 4.4.13. *Equation (4.1.7) with $\gamma \in (0,1)$ is oscillatory if and only if*

$$\sum^{\infty} j^{\gamma}q(j) = \infty. \tag{4.4.55}$$

Next, we will consider equation (4.1.8) when the sequence $\{c(k)\}$ satisfies

$$C(k) = \sum_{j=k}^{\infty} \frac{1}{c(j)}, \qquad C(m) < \infty \quad \text{for } m \in \mathbb{N}, \tag{4.4.56}$$

and obtain some necessary and sufficient conditions for the oscillation of equation (4.1.8) with either $\gamma > 1$ or $0 < \gamma < 1$.

We will need the following results from the analysis.

Definition 4.4.14. Let (X,d) be a metric space and let $T : X \to X$. We say that T is a *contraction mapping* on X if there exists a number $r \in [0,1)$ such that $d(Tx,Ty) \leq rd(x,y)$ for every $x, y \in X$.

Theorem 4.4.15 (Banach's contraction principle). *Let (X,d) be a complete metric space and let T be a contraction mapping on X. Then T has exactly one fixed point on X, that is, there exists exactly one $\bar{x} \in X$ such that $T\bar{x} = \bar{x}$.*

Theorem 4.4.16. *Assume that condition (4.4.56) holds, $\gamma > 1$, and $q(k) \geq 0$ eventually. Then equation (4.1.8) is oscillatory if and only if*

$$\sum_{j=m+1}^{\infty} C^{\gamma}(j)q(j) = \infty \quad \text{for } m \in \mathbb{N}. \tag{4.4.57}$$

Proof. Suppose that there exists a nonoscillatory solution $\{x(k)\}$ of equation (4.1.8). Without loss of generality assume that $x(k) > 0$ for $k \in \mathbb{N}(m)$. Hence

$$\Delta\big(c(k-1)\Delta x(k-1)\big) \leq 0 \quad \text{for } k \in \mathbb{N}(m), \tag{4.4.58}$$

and thus $c(k)\Delta x(k) \le c(m)\Delta x(m)$ for $k \in \mathbb{N}(m)$, or

$$\Delta x(k) \le \frac{c(m)\Delta x(m)}{c(k)} \quad \text{for } k \in \mathbb{N}(m). \tag{4.4.59}$$

Summing (4.4.59) from m to k we obtain

$$x(k+1) - x(m) \le c(m)\Delta x(m) \sum_{j=m}^{k} \frac{1}{c(j)} \quad \text{for } k \in \mathbb{N}(m). \tag{4.4.60}$$

So $x(k)$ is bounded from above. From (4.4.60) we have

$$x(m) \ge -\big(c(m)\Delta x(m)\big) \sum_{j=m}^{k} \frac{1}{c(j)} \quad \text{for } k \in \mathbb{N}(m). \tag{4.4.61}$$

Letting $k \to \infty$ we find

$$x(m) \ge -\big(c(m)\Delta x(m)\big)C(m), \tag{4.4.62}$$

where m is an arbitrarily large number.

From (4.4.58) there are two possible cases of $\{\Delta x(k)\}$.

Case 1. Suppose $\Delta x(k) \ge 0$ for $k \in \mathbb{N}(m)$. Summing equation (4.1.8) from $m+1$ to k we have

$$c(k)\Delta x(k) - c(m)\Delta x(m) + \sum_{j=m+1}^{k} q(j)x^{\gamma}(j) = 0, \tag{4.4.63}$$

so

$$\sum_{j=m+1}^{k} q(j)x^{\gamma}(j) \le c(m)\Delta x(m). \tag{4.4.64}$$

Letting $k \to \infty$ in (4.4.64) we obtain

$$\sum_{j=m+1}^{\infty} q(j)x^{\gamma}(j) < \infty. \tag{4.4.65}$$

Since $x(k)$ is nondecreasing, there exists a constant $a > 0$ such that $x(k) \ge a > 0$ for $k \in \mathbb{N}(m)$. Thus there exists an integer $m_1 \ge m$ such that $x(k) \ge C(k)$ for $k \in \mathbb{N}(m_1)$. Combining (4.4.65) and the above inequality, we have

$$\sum_{j=m_1+1}^{\infty} C^{\gamma}(j)q(j) < \infty, \tag{4.4.66}$$

which contradicts (4.4.57).

Case 2. Suppose $\Delta x(k) < 0$ for $k \in \mathbb{N}(m)$. By the Lagrange mean value theorem we see

$$\begin{aligned}\Delta\Big(\big[c(k-1)\Delta x(k-1)\big]^{1-\gamma}\Big) &= (1-\gamma)\xi^{-\gamma}\Delta\big(c(k-1)\Delta x(k-1)\big)\\ &= (1-\gamma)\big(-q(k)x^{\gamma}(k)\big)\xi^{-\gamma},\end{aligned} \tag{4.4.67}$$

where $c(k)\Delta x(k) < \xi < c(k-1)\Delta x(k-1)$. By (4.4.62) we see that

$$x(k) \geq -c(k)\Delta x(k)C(k) \quad \text{for } k \in \mathbb{N}(m). \tag{4.4.68}$$

Now equation (4.4.67) implies that

$$\begin{aligned}\Delta\Big(\big[c(k-1)\Delta x(k-1)\big]^{1-\gamma}\Big) &\leq (1-\gamma)\big(-q(k)x^{\gamma}(k)\big)\big(c(k)\Delta x(k)\big)^{-\gamma}\\ &\leq (1-\gamma)q(k)\big(c(k)\Delta x(k)C(k)\big)^{\gamma}\big(c(k)\Delta x(k)\big)^{-\gamma}.\end{aligned} \tag{4.4.69}$$

Hence

$$\Delta\Big(\big[c(k-1)\Delta x(k-1)\big]^{1-\gamma}\Big) \leq -(\gamma-1)q(k)C^{\gamma}(k) \quad \text{for } k \in \mathbb{N}(m). \tag{4.4.70}$$

Summing (4.4.70) from $m+1$ to k we have

$$\big(c(k)\Delta x(k)\big)^{1-\gamma} - \big(c(m)\Delta x(m)\big)^{1-\gamma} \leq -(\gamma-1)\sum_{j=m+1}^{k} q(j)C^{\gamma}(j), \tag{4.4.71}$$

and hence

$$-\big(c(k)\Delta x(k)\big)^{1-\gamma} \geq (\gamma-1)\sum_{j=m+1}^{k} q(j)C^{\gamma}(j) - \big(c(m)\Delta x(m)\big)^{1-\gamma}. \tag{4.4.72}$$

Letting $k \to \infty$ in (4.4.72) we find

$$\sum_{j=m+1}^{\infty} C^{\gamma}(j)q(j) < \infty, \tag{4.4.73}$$

which contradicts (4.4.57).

This completes the proof of the sufficiency part of the theorem.

To prove the necessity of condition (4.4.57), we will show that if (4.4.73) holds, then equation (4.1.8) has a nonoscillatory solution.

Consider the Banach space $B = \ell_{\infty}^{m}$ of all sequences $x = \{x(k)\}$, $k \in \mathbb{N}(m)$ with the norm

$$\|x\| = \sup_{k\in\mathbb{N}(m)} \frac{|x(k)|}{C(k)}. \tag{4.4.74}$$

Next, we define a closed, convex, and bounded subset S of B by

$$S = \left\{ x \in \ell_\infty^m : \frac{C(k)}{2} \le x(k) \le C(k),\ k \in \mathbb{N}(m) \right\}. \tag{4.4.75}$$

Define an operator $T : S \to B$ such that

$$\begin{aligned} Tx(m) &= C(m), \\ Tx(k) &= \frac{C(k)}{2} + C(k) \sum_{j=m}^{k-1} q(j)x^\gamma(j) + \sum_{j=k}^{\infty} C(j)q(j)x^\gamma(j) \quad \text{for } k \in \mathbb{N}(m). \end{aligned} \tag{4.4.76}$$

We will show that $TS \subset S$. It is obvious that $Tx(k) \ge (1/2)C(k)$ for $k \in \mathbb{N}(m)$. Choose m so large that $\sum_{j=m}^{\infty} C^\gamma(j)q(j) < 1/(8\gamma)$. Hence, for $k \in \mathbb{N}(m)$,

$$Tx(k) \le C(k)\left[\frac{1}{2} + \sum_{j=m}^{k-1} C^\gamma(j)q(j) + \sum_{j=k}^{\infty} C^\gamma(j)q(j)\right] < C(k). \tag{4.4.77}$$

Consequently, $Tx(k) \le C(k)$ for $k \in \mathbb{N}(m)$.

Now, we will show that the operator T is a contraction mapping on S. For $x, y \in S$, we have

$$\begin{aligned} &\frac{1}{C(k)} \left| Tx(k) - Ty(k) \right| \\ &\quad \le \sum_{j=m}^{k-1} q(j) \left| x^\gamma(j) - y^\gamma(j) \right| + \frac{1}{C(k)} \sum_{j=k}^{\infty} C(j)q(j) \left| x^\gamma(j) - y^\gamma(j) \right| \\ &\quad \le \sum_{j=m}^{k-1} C^\gamma(j)q(j) \left| \left(\frac{x(j)}{C(j)}\right)^\gamma - \left(\frac{y(j)}{C(j)}\right)^\gamma \right| \\ &\qquad + \sum_{j=k}^{\infty} C^\gamma(j)q(j) \left| \left(\frac{x(j)}{C(j)}\right)^\gamma - \left(\frac{y(j)}{C(j)}\right)^\gamma \right| \\ &\quad \le 2\gamma \sum_{j=m}^{k-1} C^\gamma(j)q(j) \left| \frac{x(j)}{C(j)} - \frac{y(j)}{C(j)} \right| + 2\gamma \sum_{j=k}^{\infty} C^\gamma(j)q(j) \left| \frac{x(j)}{C(j)} - \frac{y(j)}{C(j)} \right| \\ &\quad \le \frac{1}{2} \|x - y\|, \end{aligned} \tag{4.4.78}$$

and hence $\|Tx - Ty\| \le (1/2)\|x - y\|$, that is, T is a contraction mapping on S. Therefore, by Theorem 4.4.15, T has a unique fixed point $x \in S$. It is easy to check that $x = \{x(k)\}$ is a nonoscillatory solution of equation (4.1.8). This completes the proof. $\square$

Next we consider the sublinear case, that is, $0 < \gamma < 1$.

Theorem 4.4.17. *Assume that condition (4.4.56) holds,* $0 < \gamma < 1$, *and* $q(k) \geq 0$ *eventually. Then equation (4.1.8) is oscillatory if and only if*

$$\sum_{j=m+1}^{\infty} C(j)q(j) = \infty \quad \textit{for } m \in \mathbb{N}. \tag{4.4.79}$$

Proof. First we show sufficiency. Assume that x with $x(k) > 0$ for $k \in \mathbb{N}(m)$ is a solution of (4.1.8). Therefore, $\Delta(c(k-1)\Delta x(k-1)) \leq 0$ for $k \in \mathbb{N}(m)$. Next, we distinguish the following two possible cases.

Case 1. If $\Delta x(k) \geq 0$ for $k \in \mathbb{N}(m)$, we have (4.4.65) and (4.4.66). For all large k we have $C(k) \leq 1$ and $C^{\gamma}(k) \geq C(k)$. From (4.4.66) we get $\sum_{j=m_1+1}^{\infty} C(j)q(j) < \infty$, which contradicts condition (4.4.79).

Case 2. If $\Delta x(k) < 0$ for $k \in \mathbb{N}(m)$, then by summing equation (4.1.8) from $m+1$ to k one can easily find

$$-\Delta x(k) \geq \frac{1}{c(k)} \sum_{j=m+1}^{k} q(j)x^{\gamma}(j) \quad \text{for } k \in \mathbb{N}(m). \tag{4.4.80}$$

We consider the difference $\Delta(x^{2\varepsilon}(k))$ where $\varepsilon > 0$ such that $2\varepsilon < 1-\gamma$. By the Lagrange mean value theorem

$$\begin{aligned} -\Delta\big(x^{2\varepsilon}(k)\big) &= -2\varepsilon\xi^{2\varepsilon-1}\Delta x(k) \\ &\geq 2\varepsilon\xi^{2\varepsilon-1}\left(\frac{1}{c(k)}\right)\sum_{j=m+1}^{k} q(j)x^{\gamma}(j) \\ &\geq 2\varepsilon x^{2\varepsilon-1}(k)\left(\frac{1}{c(k)}\right)\sum_{j=m+1}^{k} q(j)x^{\gamma}(j), \end{aligned} \tag{4.4.81}$$

where $x(k+1) < \xi < x(k)$ and $x(k)$ is decreasing. Thus,

$$-\Delta\big(x^{2\varepsilon}(k)\big) \geq \frac{2\varepsilon}{c(k)} \sum_{j=m+1}^{k} q(j)x^{\gamma+2\varepsilon-1}(j). \tag{4.4.82}$$

Since $0 < x(k) \leq b_1$ for $k \in \mathbb{N}(m)$, where $b_1 > 0$ is a constant, there exists a positive number b such that

$$-\Delta\big(x^{2\varepsilon}(k)\big) \geq \frac{b}{c(k)} \sum_{j=m+1}^{k} q(j). \tag{4.4.83}$$

Summing (4.4.83) from $m+1$ to k we have

$$x^{2\varepsilon}(m+1) - x^{2\varepsilon}(k+1) \geq b \sum_{i=m+1}^{k} \frac{1}{c(i)} \sum_{j=m+1}^{i} q(j). \tag{4.4.84}$$

By rearranging the double sum in (4.4.84) we obtain

$$x^{2\varepsilon}(m+1) - x^{2\varepsilon}(k+1) \geq b \sum_{j=m+1}^{k} q(j) \sum_{i=j}^{k} \frac{1}{c(i)} \tag{4.4.85}$$

and so, letting $k \to \infty$, we have

$$\sum_{j=m+1}^{\infty} C(j)q(j) < \infty, \tag{4.4.86}$$

which is a contradiction to condition (4.4.79).

Now we show necessity. To this end, we consider the Banach space $B = \ell_\infty^m$ of all real sequences $x = \{x(k)\}$, $k \in \mathbb{N}(m)$, with the norm

$$\|x\| = \sup_{k \in \mathbb{N}(m)} C(k)\,|x(k)|. \tag{4.4.87}$$

Define a closed, convex, and bounded subset S of B by

$$S = \left\{ x(k) \in B : \frac{1}{2} \leq x(k) \leq 1,\ k \in \mathbb{N}(m) \right\}. \tag{4.4.88}$$

Define an operator $T : S \to B$ such that

$$\begin{gathered} Tx(m) = 1, \\ Tx(k) = \frac{1}{2} + C(k) \sum_{j=m}^{k-1} q(j)x^{\gamma}(j) + \sum_{j=k}^{\infty} C(j)q(j)x^{\gamma}(j) \quad \text{for } k \in \mathbb{N}(m). \end{gathered} \tag{4.4.89}$$

We choose m so large that $C(k) \leq 1$ for $k \in \mathbb{N}(m)$ and $\sum_{j=m+1}^{\infty} C(j)q(j) < 1/4$, since (4.4.86) holds.

It is obvious that $1/2 \leq Tx(k) \leq 1$ for $k \in \mathbb{N}(m)$, that is, $TS \subset S$. It is easy to prove that T is a contraction on S under assumption (4.4.86). Therefore, T has a unique fixed point $x \in S$, and x is a nonoscillatory solution of equation (4.1.8). $\square$

Remark 4.4.18. We note that Theorems 4.4.16 and 4.4.17 can also be obtained for the more general equation (4.4.1). In this case, conditions (4.4.57) and (4.4.79) can be replaced by

$$\sum_{j=m+1}^{\infty} f\big(j, aC(j)\big) = \infty \quad \text{for some constant } a \neq 0,\ m \in \mathbb{N}, \tag{4.4.90}$$

and equation (4.4.1) is strongly superlinear, and

$$\sum_{j=m+1}^{\infty} C(j) f(j, a) = \infty \quad \text{for some constant } a \neq 0,\ m \in \mathbb{N}, \tag{4.4.91}$$

and equation (4.4.1) is sublinear, respectively. The details are left to the reader.

4.5. Oscillation of damped nonlinear difference equations

In this section we will present some results for the oscillation of damped nonlinear difference equations of the form (4.1.3) and (4.1.9).

In Section 3.9 we gave some results on the oscillation of half-linear damped difference equations. Here and by applying the same technique, we can extend Corollaries 3.9.6 and 3.9.7 to equation (4.1.3).

Lemma 4.5.1. *Let condition (4.1.4) hold, $f'(x) \geq 0$ for $x \neq 0$,*

$$c(k) > p(k), \tag{4.5.1}$$

$$\sum_{k=m}^{\infty} \frac{1}{c(k)} \prod_{j=m}^{k-1} \left(1 - \frac{p(j)}{c(j)}\right) = \infty \quad \textit{for } m \in \mathbb{N}. \tag{4.5.2}$$

If $\{x(k)\}$ is a nonoscillatory solution of equation (4.1.3), then $x(k)\Delta x(k) > 0$ eventually.

Theorem 4.5.2. *In addition to the assumptions of Lemma 4.5.1 assume that condition (4.1.10) holds and the equation*

$$\Delta\big(c(k)\Delta x(k)\big) + q(k) f\big(x(k+1)\big) = 0 \tag{4.5.3}$$

is oscillatory. Then equation (4.1.3) is oscillatory.

Next, we prove the following result.

Theorem 4.5.3. *Suppose that conditions (4.1.4) and (4.1.11) hold. Moreover assume that there exists a positive sequence $\{\rho(k)\}$, $k \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$, such that*

$$\sum^{\infty} \frac{\eta^2(j)}{c(j)\rho(j)} < \infty, \tag{4.5.4}$$

where $\eta(k) = c(k)\Delta\rho(k) - p(k)\rho(k+1)$ *for* $k \in \mathbb{N}(m)$,

$$\sum^{\infty} \rho(j+1)q(j) = \infty, \tag{4.5.5}$$

$$\sum^{\infty} \frac{1}{c(j)\rho(j)} = \infty. \tag{4.5.6}$$

Then equation (4.1.3) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.1.3), say, $x(k) > 0$ for $k \geq m \in \mathbb{N}$. Now we consider the identity

$$\begin{aligned} \Delta\left(\frac{\rho(k)c(k)\Delta x(k)}{f(x(k))}\right) &= \rho(k+1)\frac{\Delta(c(k)\Delta x(k))}{f(x(k+1))} + \frac{c(k)\Delta x(k)\Delta\rho(k)}{f(x(k+1))} \\ &\quad - \frac{\rho(k)c(k)\Delta x(k)\Delta f(x(k))}{f(x(k))f(x(k+1))}, \end{aligned} \tag{4.5.7}$$

from which, by using (4.1.3), we conclude

$$\begin{aligned} \Delta\left(\frac{\rho(k)c(k)\Delta x(k)}{f(x(k))}\right) &= -\rho(k+1)q(k) + \eta(k)\frac{\Delta x(k)}{f(x(k+1))} \\ &\quad - \frac{\rho(k)c(k)F(x(k+1),x(k))(\Delta x(k))^2}{f(x(k))f(x(k+1))}. \end{aligned} \tag{4.5.8}$$

Summing (4.5.8) on both sides from m to $k-1$, we obtain

$$\begin{aligned} &\frac{\rho(k)c(k)\Delta x(k)}{f(x(k))} + \sum_{j=m}^{k-1} \frac{\rho(j)c(j)F(x(j+1),x(j))(\Delta x(j))^2}{f(x(j))f(x(j+1))} \\ &\quad - \sum_{j=m}^{k-1} \eta(j)\frac{\Delta x(j)}{f(x(j+1))} + \sum_{j=m}^{k-1} \rho(j+1)q(j) = \frac{\rho(m)c(m)\Delta x(m)}{f(x(m))}. \end{aligned} \tag{4.5.9}$$

Using Schwarz's inequality we find

$$\left(\sum_{j=m}^{k-1} \frac{\eta(j)}{\sqrt{\rho(j)c(j)}}\sqrt{\rho(j)c(j)}\frac{\Delta x(j)}{f(x(j+1))}\right)^2 \leq L^2 \sum_{j=m}^{k-1} \frac{\rho(j)c(j)(\Delta x(j))^2}{f^2(x(j+1))} \tag{4.5.10}$$

with

$$L^2 = \sum_{j=m}^{\infty} \frac{\eta^2(j)}{\rho(j)c(j)}, \quad \text{where } L \text{ is a positive constant.} \tag{4.5.11}$$

Using (4.5.10) in (4.5.9) we have

$$\begin{aligned} &\frac{\rho(k)c(k)\Delta x(k)}{f(x(k))} - L\left(\sum_{j=m}^{k-1} \frac{\rho(j)c(j)(\Delta x(j))^2}{f^2(x(j+1))}\right)^{1/2} + \sum_{j=m}^{k-1} \rho(j+1)q(j) \\ &\quad + \sum_{j=m}^{k-1} \frac{\rho(j)c(j)F(x(j+1),x(j))(\Delta x(j))^2}{f(x(j))f(x(j+1))} \le \frac{\rho(m)c(m)\Delta x(m)}{f(x(m))}. \end{aligned} \tag{4.5.12}$$

Note that

$$\sum_{j=m}^{k-1} \frac{\rho(j)c(j)F(x(j+1),x(j))(\Delta x(j))^2}{f(x(j))f(x(j+1))} - L\left(\sum_{j=m}^{k-1} \frac{\rho(j)c(j)(\Delta x(j))^2}{f^2(x(j+1))}\right)^{1/2} \tag{4.5.13}$$

remains bounded from below as $k \to \infty$. Thus, by (4.5.5), we see from (4.5.12) that

$$\frac{\rho(k)c(k)\Delta x(k)}{f(x(k))} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \tag{4.5.14}$$

Hence there exists an integer $m_1 \in \mathbb{N}(m)$ such that

$$\Delta x(k) < 0 \quad \text{for } k \in \mathbb{N}(m_1). \tag{4.5.15}$$

It is easy to check that there exist an integer $m_2 \in \mathbb{N}(m_1)$ and a positive constant b such that for $k \in \mathbb{N}(m_2)$,

$$\frac{\rho(k)c(k)\Delta x(k)}{f(x(k))} + \sum_{j=m_2}^{k-1} \frac{\rho(j)c(j)F(x(j+1),x(j))(\Delta x(j))^2}{f(x(j))f(x(j+1))} \le -b. \tag{4.5.16}$$

Hence

$$\xi(k) \ge bf(x(k)) + \sum_{j=m_2}^{k-1} \frac{f(x(k))F(x(j+1),x(j))(-\Delta x(j))}{f(x(j))f(x(j+1))}\xi(j), \tag{4.5.17}$$

where $\xi(k) = -\rho(k)c(k)\Delta x(k)$. Define

$$K(k,j,y) = \frac{f(x(k))F(x(j+1),x(j))(-\Delta x(j))}{f(x(j))f(x(j+1))}y \tag{4.5.18}$$

for $k \in \mathbb{N}(m_2)$ and $y \in \mathbb{R}^+$. Note that $K(k,j,y)$ is nondecreasing in y for each fixed k and j. Applying Lemma 4.1.9, we obtain $\xi(k) \ge z(k)$ for $k \in \mathbb{N}(m_2)$, where

$$z(k) = bf(x(k)) + \sum_{j=m_2}^{k-1} \frac{f(x(k))F(x(j+1),x(j))(-\Delta x(j))}{f(x(j))f(x(j+1))}z(j) \tag{4.5.19}$$

provided that $z(j) \in \mathbb{R}^+$ for $j \in \mathbb{N}(m_2)$. Dividing (4.5.19) by $f(x(k))$ and then applying the difference operator Δ, it is easy to verify that $\Delta z(k) \equiv 0$. Therefore $\xi(k) \geq z(k) = bf(x(m_2))$ for $k \in \mathbb{N}(m_2)$. Thus

$$\Delta x(k) \leq -\frac{bf(x(m_2))}{\rho(k)c(k)} \quad \text{for } k \in \mathbb{N}(m_2). \tag{4.5.20}$$

Summing (4.5.20) from m_2 to $k-1$, we have

$$x(k) \leq x(m_2) - bf(x(m_2)) \sum_{j=m_2}^{k-1} \frac{1}{\rho(j)c(j)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{4.5.21}$$

which is a contradiction to the fact that $x(k) > 0$ eventually. This completes the proof. □

Remark 4.5.4. (i) Theorem 4.5.3 is obtained without explicit sign assumptions on the sequences $\{p(k)\}$ and $\{q(k)\}$. We also note that Theorem 4.5.3 can be applied to superlinear, linear, as well as sublinear equations.

(ii) One can easily derive many interesting oscillation criteria for the equations (4.1.3) and (4.1.9) by letting $\rho(k) = k^\alpha$ with $\alpha \geq 0$ or $\rho(k) = \sum_{j=m}^{k-1} 1/c(j)$ for $k \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$. The formulation of such results are left to the reader.

Next, we present the following illustrative examples.

Example 4.5.5. Consider the damped difference equation

$$\Delta\left(\frac{1}{k}\Delta x(k)\right) + \frac{(-1)^k}{(k+1)^2}\Delta x(k) + \left[\frac{2}{k(k+1)} + \frac{2(-1)^{\alpha k}}{(k+1)^2}\right]x^\alpha(k+1) = 0 \tag{4.5.22}$$

for $k \geq m \in \mathbb{N}$, where α is the ratio of two positive odd integers. Here,

$$c(k) = \frac{1}{k}, \qquad p(k) = \frac{(-1)^k}{(k+1)^2}, \qquad q(k) = \frac{2}{k(k+1)} + \frac{2(-1)^{\alpha k}}{(k+1)^2}, \qquad f(x) = x^\alpha. \tag{4.5.23}$$

Applying Theorem 4.5.3 with $\rho(k) = k$, we see that all the hypotheses of this theorem are satisfied for equation (4.5.22). Hence we conclude that equation (4.5.22) is oscillatory. One such oscillatory solution of (4.5.22) is $x(k) = (-1)^k$.

Example 4.5.6. Consider the damped difference equation

$$\Delta^2 x(k) + \frac{1}{k+1}\Delta x(k) + \frac{a+b(-1)^k}{(k+1)^2}x^\alpha(k+1) = 0 \quad \text{for } k \geq m \in \mathbb{N}, \tag{4.5.24}$$

where a, b are real constants and α is the ratio of two positive odd integers. It is easy to check that all the assumptions of Theorem 4.5.3 with $\rho(k) = k$ are satisfied, and hence we conclude that equation (4.5.24) is oscillatory.

Example 4.5.7. Consider the damped difference equation

$$\Delta\big(\sqrt{k}\Delta x(k)\big) + \frac{(-1)^k}{2(k+1)}\Delta x(k) + \frac{(-1)^k}{\sqrt{k+1}[2-(-1)^k]}x^{\alpha}(k+1) = 0, \quad k \ge m \in \mathbb{N}, \tag{4.5.25}$$

where α is the ratio of two positive odd integers. Here we can apply Theorem 4.5.3 with $\rho(k) = \sqrt{k}$ and conclude that equation (4.5.25) is oscillatory.

Next, we will consider the more general damped equation

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k)f\big(x(k+1)\big)g\big(\Delta x(k)\big) = 0 \quad \text{for } k \in \mathbb{N}, \tag{4.5.26}$$

where $\{c(k)\}$, $\{p(k)\}$, $\{q(k)\}$, and f are defined as in (4.1.3) and $g \in C(\mathbb{R},\mathbb{R})$. We will assume that

$$p(k) \ge 0, \quad q(k) \ge 0 \quad \text{eventually}, \tag{4.5.27}$$

$$xf(x) > 0, \quad g(x) > 0 \quad \text{for } x \ne 0, \tag{4.5.28}$$

$$f'(x) \ge 0 \quad \text{for } x \ne 0, \qquad -f(-xy) \ge f(xy) \ge f(x)f(y) \quad \text{for } xy \ne 0, \tag{4.5.29}$$

$$g'(x) \ge 0 \quad \text{for } x < 0, \qquad g'(x) \le 0 \quad \text{for } x > 0, \tag{4.5.30}$$

$$g(-xy) \ge g(xy) \ge g(x)g(y) \quad \text{for } xy \ne 0. \tag{4.5.31}$$

As in Lemma 4.5.1, we can obtain the following result.

Lemma 4.5.8. *Let conditions (4.5.1), (4.5.2), (4.5.27), and (4.5.28) hold. If x is a nonoscillatory solution of equation (4.5.26), then $x(k)\Delta x(k) > 0$ eventually.*

Next, we present the following lemma.

Lemma 4.5.9. *If $\{x(k)\}$ is eventually positive such that $\Delta(c(k)\Delta x(k)) \le 0$ and $\Delta x(k) > 0$ eventually and*

$$C(k,m) = \sum_{j=m}^{k-1}\frac{1}{c(j)} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \qquad C(m,m) = 0, \tag{4.5.32}$$

then there exists a number $\theta \in (0,1)$ such that

$$x(k+1) \ge x(k) \ge \theta C(k,m)c(k)\Delta x(k) \quad \text{eventually, } m \in \mathbb{N}. \tag{4.5.33}$$

Proof. Clearly,

$$\begin{aligned} x(k) - x(m) &= \sum_{j=m}^{k-1} \Delta x(j) = \sum_{j=m}^{k-1} \frac{1}{c(j)}\big(c(j)\Delta x(j)\big) = \sum_{j=m}^{k-1} \Delta C(j,m)\big(c(j)\Delta x(j)\big) \\ &= C(k,m)c(k)\Delta x(k) - C(m,m)c(m)\Delta x(m) \\ &\quad - \sum_{j=m}^{k-1} C(j+1,m)\Delta\big(c(j)\Delta x(j)\big) \\ &\geq C(k,m)c(k)\Delta x(k). \end{aligned} \tag{4.5.34}$$

Now the conclusion follows. □

Next, we prove the following result.

Theorem 4.5.10. *In addition to the assumptions of Lemma 4.5.8, assume that (4.5.29), (4.5.31), and (4.5.32) hold, $(fg)'(x) \geq 0$ for $x \neq 0$, and*

$$\int_{+0} \frac{du}{f(u)g(u)} < \infty, \qquad \int_{-0} \frac{du}{f(u)g(u)} < \infty. \tag{4.5.35}$$

If

$$\sum^{\infty} q(j) f\big(C(j,m)\big) g\left(\frac{1}{c(j)}\right) = \infty \quad \text{for } j \geq m \in \mathbb{N}, \tag{4.5.36}$$

then equation (4.5.26) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.5.26), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. By Lemma 4.5.8 there exists an integer $m_1 \geq m$ such that $\Delta x(k) > 0$ for all $k \in \mathbb{N}(m_1)$, and by Lemma 4.5.9, there exist an integer $m_2 \in \mathbb{N}(m_1)$ and a constant $\theta \in (0,1)$ such that (4.5.33) holds for all $k \in \mathbb{N}(m_2)$. From equation (4.5.26) it follows that for $k \in \mathbb{N}(m_2)$,

$$\Delta\big(c(k)\Delta x(k)\big) + q(k) f\big(\theta C(k,m)c(k)\Delta x(k)\big) g\left(\frac{c(k)\Delta x(k)}{c(k)}\right) \leq 0. \tag{4.5.37}$$

Using condition (4.5.29) and (4.5.31) in (4.5.37), we find

$$\Delta y(k) + f(\theta) q(k) f\big(C(k,m)\big) g\left(\frac{1}{c(k)}\right) f\big(y(k)\big) g\big(y(k)\big) \leq 0 \quad \text{for } k \in \mathbb{N}(m_2), \tag{4.5.38}$$

where $y(k) = c(k)\Delta x(k)$ for $k \in \mathbb{N}(m_2)$. Observe that for $y(k) \geq u \geq y(k+1)$, we have $f(u)g(u) \leq f\big(y(k)\big)g\big(y(k)\big)$ and therefore,

$$\frac{\Delta y(k)}{f\big(y(k)\big)g\big(y(k)\big)} \leq \frac{\Delta y(k)}{f(u)g(u)}. \tag{4.5.39}$$

Using (4.5.39) in (4.5.38) and summing from m_2 to k, we obtain

$$\begin{aligned} f(\theta)\sum_{j=m_2}^{k} q(j)f(C(j,m))g\left(\frac{1}{c(j)}\right) &\leq \sum_{j=m_2}^{k-1}\int_{y(j+1)}^{y(j)}\frac{du}{f(u)g(u)} \\ &\leq \int_{y(k)}^{y(m_2)}\frac{du}{f(u)g(u)} \\ &< \infty \end{aligned} \tag{4.5.40}$$

as $k \to \infty$, which contradicts (4.5.36). This completes the proof. □

Next, we present the following useful comparison result.

Theorem 4.5.11. *In addition to the hypotheses of Lemma 4.5.8, suppose that (4.5.30)–(4.5.32) hold and $(fg)'(x) \geq 0$ for $x \neq 0$. If for every constant $\delta > 0$ the equation*

$$\Delta(c(k)\Delta x(k)) + g(\delta)g\left(\frac{1}{C(k,m)c(k)}\right)q(k)f(x(k+1))g(x(k+1)) = 0 \tag{4.5.41}$$

is oscillatory, then equation (4.5.26) is also oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.5.26), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 4.5.10 we obtain $\Delta x(k) > 0$ and

$$\Delta x(k) \leq \frac{1}{\theta C(k,m)c(k)}x(k+1) \quad \forall k \in \mathbb{N}(m_2). \tag{4.5.42}$$

Using condition (4.5.30) we have for $k \in \mathbb{N}(m_2)$,

$$\Delta(c(k)\Delta x(k)) + g\left(\frac{1}{\theta}\right)g\left(\frac{1}{C(k,m)c(k)}\right)q(k)f(x(k+1))g(x(k+1)) \leq 0. \tag{4.5.43}$$

Arguing as in the proof of Lemma 3.9.2 we conclude that equation (4.5.41) has an eventually positive solution, which is a contradiction. □

We note that Theorem 4.5.10 can be applied to damped equations of the form

$$\Delta(c(k)\Delta x(k)) + p(k)\Delta x(k) + q(k)|x(k+1)|^{\alpha}|\Delta x(k)|^{\beta}\operatorname{sgn} x(k+1) = 0, \tag{4.5.44}$$

which is a special case of equation (4.4.37), where $\alpha > 0$ and $\beta \geq 0$ are real constants.

For the oscillation of equation (4.5.44) we give the following result.

Corollary 4.5.12. *Let (4.5.1), (4.5.2), (4.5.27), and (4.5.32) hold. If* $0 < \alpha + \beta < 1$ *and*

$$\sum^{\infty} q(j)c^{\alpha}(j,m)c^{-\beta}(j) = \infty \quad \textit{for } m \in \mathbb{N}, \tag{4.5.45}$$

then equation (4.5.44) is oscillatory.

We also see that Theorem 4.5.11 can be applied to equations of the form

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k)\frac{|x(k+1)|^{\alpha}}{1+|\Delta x(k)|^{\beta}}\operatorname{sgn} x(k+1) = 0, \tag{4.5.46}$$

which is a special case of equation (4.5.26), where $\alpha > 0$ and $\beta \geq 0$ are real constants. Here, we let $f(x) = |x|^{\alpha}\operatorname{sgn} x$ and $g(y) = 1/[1+|y|^{\beta}]$. Clearly, $g'(y) \geq 0$ when $y < 0$, $g'(y) \leq 0$ when $y > 0$, and

$$\begin{aligned} g(-xy) = g(xy) &= \frac{1}{1+|xy|^{\beta}} \geq \frac{1}{(1+|x|^{\beta})(1+|y|^{\beta})} \\ &= g(x)g(y) \quad \text{for } xy \neq 0. \end{aligned} \tag{4.5.47}$$

Thus, we may state the following immediate result.

Corollary 4.5.13. *Let conditions (4.5.1), (4.5.2), (4.5.27), and (4.5.32) hold. If the function* $x^{\alpha}\operatorname{sgn} x/[1+|x|^{\beta}]$ *is nondecreasing for* $x \neq 0$ *and if there exists a constant* $\theta \in (0,1)$ *such that the equation*

$$\Delta\big(c(k)\Delta x(k)\big) + \left(\frac{1}{1+\theta^{\beta}}\right)\left(\frac{1}{1+c(k)C(k,m)}\right)\frac{|x(k+1)|^{\alpha}}{1+|x(k+1)|^{\beta}}\operatorname{sgn} x(k+1) = 0, \tag{4.5.48}$$

where $m \in \mathbb{N}$, *is oscillatory, then equation (4.5.46) is oscillatory as well.*

In Theorem 4.5.10 if condition (4.5.35) is violated, we present the following result.

Theorem 4.5.14. *Suppose that the hypotheses of Lemma 4.5.8 hold and assume that (4.5.29), (4.5.31), and (4.5.32) are satisfied. If*

$$\frac{f(x)g(x)}{x} \geq \ell > 0, \quad \textit{where } \ell \textit{ is a constant}, \tag{4.5.49}$$

and there exists a constant $\theta \in (0,1)$ *such that*

$$\limsup_{k\to\infty}\left[q(k)f\big(C(k,m)\big)g\left(\frac{1}{c(k)}\right)\right] > \frac{1}{\ell f(\theta)} \quad \textit{for } m \in \mathbb{N}, \tag{4.5.50}$$

then equation (4.5.26) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (4.5.26), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 4.5.10 we obtain

$$\Delta y(k) + f(\theta)q(k)f(C(k,m))g\left(\frac{1}{c(k)}\right)f(y(k))g(y(k)) \leq 0 \tag{4.5.51}$$

for $k \in \mathbb{N}(m_2)$. Using condition (4.5.49) in (4.5.51), we obtain

$$y(k) \geq y(k) - y(k+1) \geq \ell f(\theta)q(k)f(C(k,m))g\left(\frac{1}{c(k)}\right)y(k), \tag{4.5.52}$$

so

$$1 \geq \ell f(\theta)q(k)f(C(k,m))g\left(\frac{1}{c(k)}\right). \tag{4.5.53}$$

Taking lim sup on both sides of (4.5.53), we arrive at the desired contradiction. The proof is complete. □

Next we assume that

$$f(x)g(x) \geq |x|^{\alpha} \operatorname{sgn} x \quad \text{for } x \neq 0, \tag{4.5.54}$$

where $\alpha > 0$ is a constant and obtain the following result.

Theorem 4.5.15. *Let the assumptions of Theorem 4.5.14 hold except for (4.5.49) and (4.5.50); instead, assume that (4.5.54) holds. If $0 < \alpha \leq 1$ and there exists a constant $\theta \in (0,1)$ such that the equation*

$$\Delta y(k) + f(\theta)q(k)f(C(k,m))g\left(\frac{1}{c(k)}\right)|x(k+1)|^{\alpha} \operatorname{sgn} x(k+1) = 0 \quad \textit{for } m \in \mathbb{N} \tag{4.5.55}$$

is oscillatory, then equation (4.5.26) is also oscillatory.

Proof. The proof follows from the inequality (4.5.39) and a simple comparison result. The details are left to the reader. □

Remark 4.5.16. In equation (4.5.26), if $c(k) \equiv 1$ for $k \in \mathbb{N}$, then inequality (4.5.33) reduces to

$$x(k+1) \geq x(k) \geq \frac{k}{2}\Delta x(k) \quad \text{for all large } k \in \mathbb{N}. \tag{4.5.56}$$

In this case, condition (4.5.31) in Theorems 4.5.10, 4.5.14, and 4.5.15 is disregarded. Clearly, condition (4.5.36) becomes $\sum^{\infty} q(j)f(j) = \infty$, equation (4.5.41) takes the form

$$\Delta^2 x(k) + g(\delta)g\left(\frac{2}{k}\right)q(k)f(x(k+1))g(x(k+1)) = 0 \quad \text{for } k \geq m \in \mathbb{N}, \tag{4.5.57}$$

condition (4.5.50) becomes

$$\limsup_{k\to\infty}\left[f(k)q(k)\right] \geq \frac{1}{\ell f(1/2)g(1)}, \tag{4.5.58}$$

and equation (4.5.55) takes the form

$$\Delta y(k) + f\left(\frac{1}{2}\right)g(1)q(k)f(k)\left|x(k+1)\right|^{\alpha}\operatorname{sgn} x(k+1) = 0. \tag{4.5.59}$$

Formulations of the above results for equation (4.5.26) with $c(k) \equiv 1$ can now be easily obtained, and the details are left to the reader.

4.6. Asymptotic behavior for nonlinear difference equations

We will consider the asymptotic behavior of solutions of the nonlinear difference equation

$$\Delta^2 x(k) + f\left(k, x(k), x(k+1)\right) = b(k), \tag{4.6.1}$$

where $\{b(k)\} \subset \mathbb{R}$ and $f : \mathbb{N}_m \times \mathbb{R}^2 \to \mathbb{R}$ with $\mathbb{N}_m = \{m, m+1, \dots\}$ for some $m \in \mathbb{N}$.

It is known that in the case when the function f is "small" in some sense, the solutions of equation (4.6.1) are asymptotically equivalent to the solutions of the equation

$$\Delta^2 z(k) = b(k) \tag{4.6.2}$$

as $k \to \infty$.

In this section, we will consider the above problem using a comparison method. In fact, we reduce the problem of asymptotically equivalent solutions of equation (4.6.1) to the boundedness of solutions of some difference equations of first order.

We will assume that there exists a function $F : \mathbb{N}_m \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ which is nondecreasing with respect to the last two arguments such that

$$\left|f\left(k, a_1, a_2\right)\right| \leq F\left(k, |a_1|, |a_2|\right) \quad \text{for } k \in \mathbb{N}, \tag{4.6.3}$$

where a_1 and a_2 are real numbers.

Now we prove the following lemma.

Lemma 4.6.1. *Let $\{E(k)\}$ be a sequence of positive real numbers such that*

$$\Delta E(k) > 0, \quad \Delta^2 E(k) \geq 0 \quad \text{for } k \in \mathbb{N}_m, \tag{4.6.4}$$

and let x be a solution of equation (4.6.1) satisfying the initial conditions

$$e_0 = x(m), \qquad e_1 = \Delta x(m). \tag{4.6.5}$$

Then

$$|x(k+i)| \leq [e+g(k)]E(k+i) \quad \textit{for } i \in \{0,1\},\ k \in \mathbb{N}_m, \tag{4.6.6}$$

where

$$e = \frac{e_0}{E(m)} + \frac{e_1}{\Delta E(m)}, \qquad g(k) = \max\left\{\frac{\Delta x(j)}{\Delta E(j)} : m \leq j \leq k\right\}. \tag{4.6.7}$$

Proof. Let $\{x(k)\}$ be a solution of equation (4.6.1) satisfying the initial conditions (4.6.5). From (4.6.7) and (4.6.4), it follows that

$$|\Delta x(k)| \leq g(k)\Delta E(k) \leq \left[g(k) + \frac{e_1}{\Delta E(m)}\right]\Delta E(k) \quad \text{for } k \in \mathbb{N}_m. \tag{4.6.8}$$

In view of the nondecreasing character of g and E and the inequality (4.6.8), we find

$$\begin{aligned}
|x(k)| &= \left|x(m) + \sum_{j=m}^{k-1} \Delta x(j)\right| \\
&\leq e_0 + \sum_{j=m}^{k-1}\left[g(j) + \frac{e_1}{\Delta E(m)}\right]\Delta E(j) \\
&\leq e_0 + \left[g(k-1) + \frac{e_1}{\Delta E(m)}\right](E(k) - E(m)) \\
&\leq \left[g(k) + \frac{e_1}{\Delta E(m)} + \frac{e_0}{E(m)}\right]E(k) \\
&= [g(k)+e]E(k).
\end{aligned} \tag{4.6.9}$$

Using the equality $x(k+1) = x(k) + \Delta x(k)$, one can easily find

$$|x(k+1)| \leq [g(k)+e]E(k+1), \tag{4.6.10}$$

which completes the proof. □

To obtain the main results of this section, we will consider the following two cases:

(I) $\sum^\infty b(j)$ is divergent,
(II) $\sum^\infty b(j)$ is convergent.

Theorem 4.6.2. *Let $\sum^\infty b(j)$ be divergent and let the sequence $\{E(k)\}$ be as in Lemma 4.6.1. In addition suppose that*

$$\lim_{k\to\infty} \Delta E(k) = \infty, \tag{4.6.11}$$

$$\lim_{k\to\infty} \frac{\sum_{j=m}^{k-1} b(j)}{\Delta E(k)} = \gamma = \text{constant}. \tag{4.6.12}$$

If every solution of the first-order difference equation

$$\Delta y(k) = \frac{1}{\Delta E(k)} F(k, E(k)|y(k)|, E(k+1)|y(k)|) \tag{4.6.13}$$

is bounded, then every solution $\{x(k)\}$ of equation (4.6.1) has the property

$$\lim_{k\to\infty} \frac{\Delta^i x(k)}{\Delta^i E(k)} = \gamma \quad \textit{for } i \in \{0,1\}. \tag{4.6.14}$$

Proof. Summing both sides of equation (4.6.1) from m to $k-1$ and next dividing by $\Delta E(k)$, we obtain

$$\frac{\Delta x(k)}{\Delta E(k)} = \frac{\Delta x(m)}{\Delta E(k)} + \frac{1}{\Delta E(k)} \sum_{j=m}^{k-1} b(j) - \frac{1}{\Delta E(k)} \sum_{j=m}^{k-1} f(j, x(j), x(j+1)). \tag{4.6.15}$$

From (4.6.11) and (4.6.12) it follows that the first term on the right-hand side of (4.6.15) tends to zero and the second term tends to γ as $k \to \infty$. We will show that

$$\lim_{k\to\infty} \frac{1}{\Delta E(k)} \sum_{j=m}^{k-1} f(j, x(j), x(j+1)) = 0. \tag{4.6.16}$$

Using conditions (4.6.3), (4.6.4), and (4.6.11) in (4.6.15), we get

$$\begin{aligned} \frac{|\Delta x(k)|}{\Delta E(k)} &\le \frac{|\Delta x(m)|}{\Delta E(k)} + \frac{|\sum_{j=m}^{k-1} b(j)|}{\Delta E(k)} + \frac{1}{\Delta E(k)} \sum_{j=m}^{k-1} |f(j, x(j), x(j+1))| \\ &\le H_1 + \sum_{j=m}^{k-1} \frac{1}{\Delta E(j)} F(j, |x(j)|, |x(j+1)|), \end{aligned} \tag{4.6.17}$$

where

$$H_1 = \sup_{k\ge m} \frac{1}{\Delta E(k)} \left[|\Delta x(m)| + \left| \sum_{j=m}^{k-1} b(j) \right| \right]. \tag{4.6.18}$$

Applying Lemma 4.6.1 for $k \in \mathbb{N}_m$, we have

$$|x(k+i)| \le [e + g(k)] E(k+i) \quad \text{for } i \in \{0,1\}, \tag{4.6.19}$$

where e and g are as in (4.6.7). Let $h(k) = e + g(k)$. Then

$$|x(k+i)| \le h(k) E(k+i) \quad \text{for } i \in \{0,1\},\ k \in \mathbb{N}_m. \tag{4.6.20}$$

Now from (4.6.17), we derive

$$h(k) = e + g(k) \le H_2 + \sum_{j=m}^{k-1} \frac{1}{\Delta E(j)} F\big(j, E(j)h(j), E(j+1)h(j)\big), \tag{4.6.21}$$

where $H_2 = e + H_1$.

Let $\{y(k)\}$ be a solution of equation (4.6.13) satisfying the initial condition

$$y(m) = 2H_2. \tag{4.6.22}$$

Then from (4.6.21) it follows that

$$h(k) < y(k) = 2H_2 + \sum_{j=m}^{k-1} \frac{1}{\Delta E(j)} F\big(j, E(j)\,|\,y(j)\,|, E(j+1)\,|\,y(j+1)\,|\big). \tag{4.6.23}$$

Since all solutions of (4.6.13) are bounded, we have

$$h(k) \le H_3 = \sup\big\{\,|\,y(k)\,| : m \le k < \infty\big\}, \tag{4.6.24}$$

and for any $H \in \mathbb{R}^+$,

$$\sum_{j=m}^{\infty} \frac{1}{\Delta E(j)} F\big(j, HE(j), HE(j+1)\big) < \infty. \tag{4.6.25}$$

Therefore, (4.6.4), (4.6.8), and (4.6.25) give

$$\lim_{k\to\infty} \frac{1}{\Delta E(k)} \sum_{j=m}^{k-1} F\big(j, H_3E(j), H_3E(j+1)\big) = 0. \tag{4.6.26}$$

From (4.6.2), (4.6.20), and (4.6.24), we find for $j \in \mathbb{N}_m$,

$$\begin{aligned} \big|\,f\big(j, x(j), x(j+1)\big)\,\big| &\le F\big(j, |\,x(j)\,|, |\,x(j+1)\,|\big) \\ &\le F\big(j, h(j)E(j), h(j)E(j+1)\big) \\ &\le F\big(j, H_3E(j), H_3E(j+1)\big). \end{aligned} \tag{4.6.27}$$

Thus (4.6.27) and (4.6.26) give (4.6.16), which completes the proof. □

Theorem 4.6.3. *Let $\sum^{\infty} b(j)$ be convergent. If every solution of the first-order difference equation*

$$\Delta y(k) = F\big(k, (k+2)\,|\,y(k)\,|, (k+3)\,|\,y(k)\,|\big) \tag{4.6.28}$$

is bounded, then every solution x of equation (4.6.1) has the property

$$\lim_{k\to\infty} \frac{x(k)}{k} = \delta, \quad \lim_{k\to\infty} \Delta x(k) = \delta, \quad \text{where } \delta \text{ is a constant.} \tag{4.6.29}$$

Proof. Summing both sides of equation (4.6.1) from m to $k-1$, we obtain

$$\Delta x(k) = \Delta x(m) + \sum_{j=m}^{k-1} b(j) - \sum_{j=m}^{k-1} f\big(j, x(j), x(j+1)\big). \tag{4.6.30}$$

First, we note that

$$\lim_{k\to\infty} \left[\Delta x(m) + \sum_{j=m}^{k-1} b(j)\right] = \text{constant}. \tag{4.6.31}$$

Next, we will show that

$$\lim_{k\to\infty} \sum_{j=m}^{k-1} f\big(j, x(j), x(j+1)\big) = \text{constant}. \tag{4.6.32}$$

From condition (4.6.3) and (4.6.31), we obtain

$$\begin{aligned} |\Delta x(k)| &\le |\Delta x(m)| + \left|\sum_{j=m}^{k-1} b(j)\right| + \sum_{j=m}^{k-1} \left|f\big(j, x(j), x(j+1)\big)\right| \\ &\le M_1 + \sum_{j=m}^{k-1} F\big(j, |x(j)|, |x(j+1)|\big), \end{aligned} \tag{4.6.33}$$

where

$$M_1 = \sup_{k\ge m} \left[|\Delta x(m)| + \left|\sum_{j=m}^{k-1} b(j)\right| \right]. \tag{4.6.34}$$

In Lemma 4.6.1, let $E(k) = k+2$. Then,

$$|x(k+i)| \le [e+g(k)](k+2+i) \quad \text{for } i \in \{0,1\},\ k \in \mathbb{N}_m, \tag{4.6.35}$$

where $g(k) = \max\{|\Delta x(j)| : m \le j \le k\}$ and $e = [e_0/(m+2)] + e_1$. If we set $h(k) = e + g(k)$, then

$$|x(k+i)| \le (k+2+i)h(k) \quad \text{for } i \in \{0,1\},\ k \in \mathbb{N}_m. \tag{4.6.36}$$

Hence from (4.6.33), we obtain

$$h(k) = e + g(k) \le M_2 + \sum_{j=m}^{k-1} F\big(j, (j+2)h(j), (j+3)h(j)\big), \tag{4.6.37}$$

where $M_2 = e + M_1$.

Let $\{y(k)\}$ be a solution of equation (4.6.28) satisfying the initial condition $y(m) = 2M_2$. Then from (4.6.37) it follows that

$$h(k) < y(k) = 2M_2 + \sum_{j=m}^{k-1} F\big(j,(j+2)\,|\,y(j)\,|,(j+3)\,|\,y(j)\,|\,\big). \tag{4.6.38}$$

Since all solutions of equation (4.6.28) are bounded, we have

$$h(k) \le M_3 = \sup\big\{\,|\,y(k)\,| : m \le k < \infty\big\}, \tag{4.6.39}$$

and for any $M \in \mathbb{R}^+$,

$$\sum_{j=m}^{\infty} F\big(j,(j+2)M,(j+3)M\big) < \infty. \tag{4.6.40}$$

Combining (4.6.3) with (4.6.36) and (4.6.39) yields for $j \ge m$,

$$\begin{aligned}
\big|\,f(j,x(j),x(j+1))\,\big| &\le F\big(j,\,|\,x(j)\,|,\,|\,x(j+1)\,|\,\big)\\
&\le F\big(j,(j+2)h(j),(j+3)h(j)\big)\\
&\le F\big(j,(j+2)M_3,(j+3)M_3\big).
\end{aligned} \tag{4.6.41}$$

From the above inequality and (4.6.40) it follows that (4.6.32) holds. Therefore, the right-hand side of equality (4.6.30) has a finite limit. Thus, we conclude that

$$\lim_{k\to\infty} \Delta x(k) = \delta = \text{constant}, \tag{4.6.42}$$

which completes the proof of the theorem. □

The comparison theorems presented above give us the possibility to determine asymptotic behavior of solutions of difference equations of second order based on the boundedness of solutions of corresponding first-order difference equations. For many classes of first-order difference equations we can impose conditions which will guarantee the boundedness of solutions. Therefore, based on Theorems 4.6.2 and 4.6.3, we can give explicitly the conditions implying asymptotic properties of solutions presented in (4.6.14) or (4.6.29). For example, consider nonlinear difference equations of the form

$$\Delta^2 x(k) + q_1(k)\,|\,x(k)\,|^{\gamma_1}\operatorname{sgn} x(k) + q_2(k)\,|\,x(k+1)\,|^{\gamma_2}\operatorname{sgn} x(k+1) = 0, \tag{4.6.43}$$

where $\gamma_i \in [0,1]$ are constants and $\{q_i(k)\}$ are sequences of real numbers for $i \in \{1,2\}$.

Now, we present the following asymptotic results for equation (4.6.43).

Corollary 4.6.4. *Let $\{E(k)\}$ be a sequence of positive constants which satisfy conditions (4.6.4), (4.6.11), and (4.6.12). If*

$$\sum_{j=m}^{\infty} \frac{1}{\Delta E(j)}\left[|q_1(j)|E^{\gamma_1}(j) + |q_2(j)|E^{\gamma_2}(j+1)\right] < \infty, \tag{4.6.44}$$

then every solution $\{x(k)\}$ of equation (4.6.43) has the property

$$\lim_{k\to\infty} \frac{\Delta^i x(k)}{\Delta^i E(k)} = \gamma = \text{constant} \quad \textit{for } i \in \{0,1\}. \tag{4.6.45}$$

Proof. By Theorem 4.6.2 it suffices to show that under condition (4.6.44) every solution of difference equations of the form

$$\Delta y(k) = \frac{1}{\Delta E(k)}\left[|q_1(k)|E^{\gamma_1}(k)|y(k)|^{\gamma_1} + |q_2(k)|E^{\gamma_2}(k+1)|y(k)|^{\gamma_2}\right] \tag{4.6.46}$$

is bounded. Summing both sides of equation (4.6.46) from m to $k-1$, we get

$$\begin{aligned} |y(k)| &\le |y(m)| \\ &\quad + \sum_{j=m}^{k-1} \frac{1}{\Delta E(j)}\left[|q_1(j)|E^{\gamma_1}(j)|y(j)|^{\gamma_1} + |q_2(j)|E^{\gamma_2}(j+1)|y(j)|^{\gamma_2}\right] \\ &= s(k). \end{aligned} \tag{4.6.47}$$

Therefore

$$\begin{aligned} \Delta s(k) &= \frac{1}{\Delta E(k)}\left[|q_1(k)|E^{\gamma_1}(k)|y(k)|^{\gamma_1} + |q_2(k)|E^{\gamma_2}(k+1)|y(k)|^{\gamma_2}\right] \\ &\le \frac{1}{\Delta E(k)}\left[|q_1(k)|E^{\gamma_1}(k)s^{\gamma_1}(k) + |q_2(k)|E^{\gamma_2}(k+1)s^{\gamma_2}(k)\right] \\ &\le s^{\gamma^*}(k)\frac{1}{\Delta E(k)}\left[|q_1(k)|E^{\gamma_1}(k) + |q_2(k)|E^{\gamma_2}(k+1)\right], \end{aligned} \tag{4.6.48}$$

where $\gamma^* = \max\{\gamma_1, \gamma_2\}$. Since

$$\begin{aligned} \int_{s(m)}^{s(k)} u^{-\gamma^*}\,du &\le \sum_{j=m}^{k-1} \frac{\Delta s(j)}{s^{\gamma^*}(j)} \\ &\le \sum_{j=m}^{k-1} \frac{1}{\Delta E(j)}\left[|q_1(j)|E^{\gamma_1}(j) + |q_2(j)|E^{\gamma_2}(j+1)\right], \end{aligned} \tag{4.6.49}$$

we obtain for $\gamma^* = 1$,

$$s(k) \leq s(m) \exp\left(\sum_{j=m}^{k-1} \frac{1}{\Delta E(j)}\left[\left|q_1(j)\right| E^{\gamma_1}(j) + \left|q_2(j)\right| E^{\gamma_2}(j+1)\right]\right), \tag{4.6.50}$$

and for $0 \leq \gamma^* < 1$,

$$s(k) \leq \left[s^{1-\gamma^*}(m) + (1-\gamma^*)\sum_{j=m}^{k-1} \frac{1}{\Delta E(j)}\left(\left|q_1(j)\right| E^{\gamma_1}(j) + \left|q_2(j)\right| E^{\gamma_2}(j+1)\right)\right]^{1/(1-\gamma^*)}. \tag{4.6.51}$$

From the above inequalities and (4.6.44) it follows that there exists a positive constant M such that $|y(k)| \leq s(k) \leq M$ for $k \geq m$, which completes the proof. □

Corollary 4.6.5. *Suppose that $\sum_{j=m}^{\infty} b(j)$ is convergent. If*

$$\sum_{j=m}^{\infty} \left[(j+2)^{\gamma_1}\left|q_1(j)\right| + (j+3)^{\gamma_2}\left|q_2(j)\right|\right] < \infty, \tag{4.6.52}$$

then every solution of equation (4.6.43) has the property

$$\lim_{k\to\infty} \frac{x(k)}{k} = \delta, \qquad \lim_{k\to\infty} \Delta x(k) = \delta = \text{constant}. \tag{4.6.53}$$

Proof. The proof can be modelled according to that of Corollary 4.6.4 and hence is omitted. □

Next, we will give some explicit conditions which will guarantee the asymptotic properties (4.6.14) or (4.6.29) of solutions of equation (4.6.1). This can be done by imposing an extra condition on the function F, namely, F is homogeneous in the last two arguments, that is,

$$F(k, \lambda a_1, \lambda a_2) = \lambda^{\beta} F(k, a_1, a_2) \quad \text{for some } \beta \in (0, 1]. \tag{4.6.54}$$

Proceeding in a manner analogous to the proofs of Corollaries 4.6.4 and 4.6.5 we can prove the following theorems.

Theorem 4.6.6. *Let conditions (4.6.3), (4.6.4), (4.6.11), (4.6.12), and (4.6.54) hold. Moreover, let*

$$\sum_{j=m}^{\infty} \frac{1}{\Delta E(j)} F\left(j, E(j), E(j+1)\right) < \infty. \tag{4.6.55}$$

Then every solution x of equation (4.6.1) has property (4.6.14).

Theorem 4.6.7. *Suppose that $\sum_{j=m}^{\infty} b(j)$ is convergent and conditions (4.6.3) and (4.6.54) hold. If*

$$\sum_{j=m}^{\infty} F(j,(j+2),(j+3)) < \infty, \tag{4.6.56}$$

then every solution x of equation (4.6.1) has property (4.6.29).

4.7. Oscillation criteria via Liapunov's second method

In this section we will employ Liapunov's second method to investigate the oscillatory behavior of solutions of the second-order nonlinear difference equation

$$\Delta(c(k)\Delta x(k)) + f(k, x(k+1), \Delta x(k)) = 0, \tag{4.7.1}$$

where $k \in \mathbb{N}(m) = \{m, m+1, \dots\}$ for some $m \in \mathbb{N}$, $\{c(k)\}$ is a sequence of positive real numbers, and f is a real-valued function defined on $\mathbb{N}(m) \times \mathbb{R}^2$.

In system form, equation (4.7.1) can be written as

$$\begin{aligned} \Delta x(k) &= \frac{y(k)}{c(k)}, \\ \Delta y(k) &= -f\left(k, x(k+1), \frac{y(k)}{c(k)}\right). \end{aligned} \tag{4.7.2}$$

Definition 4.7.1. The function $v(k,x,y)$ is called a *Liapunov function* for the system (4.7.2) if $v(k,x,y)$ is defined and continuous on its domain of definition and is locally Lipschitzian in (x,y). Further, define $\Delta v_{(4.7.2)}(k,x,y)$ by

$$\begin{aligned} \Delta v_{(4.7.2)}(k, x(k), y(k)) = v\left(k+1, x(k) + \frac{y(k)}{c(k)}, y(k) - f\left(k, x(k+1), \frac{y(k)}{c(k)}\right)\right) \\ - v(k, x(k), y(k)). \end{aligned} \tag{4.7.3}$$

Now we present the following oscillation criterion for equation (4.7.1).

Theorem 4.7.2. *Suppose there exist two functions $V : \mathbb{N}(m_1) \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $W : \mathbb{N}(m_1) \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$, $m_1 \in \mathbb{N}(m)$ for some large $m \in \mathbb{N}$, such that*

(i) *$V(k,x,y) \to \infty$ uniformly for $x > 0$ and $|y| < \infty$ as $k \to \infty$ and $W(k,x,y) \to \infty$ uniformly for $x < 0$ and $|y| < \infty$ as $k \to \infty$,*
(ii) *$\Delta V_{(4.7.2)}(k,x,y) \le 0$ for all sufficiently large k, where $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) > 0$ for all large k,*
(iii) *$\Delta W_{(4.7.2)}(k,x,y) \le 0$ for all sufficiently large k, where $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) < 0$ for all large k.*

Then equation (4.7.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.7.1), say, $x(k) > 0$ for $k \in N(m)$ for some $m \in \mathbb{N}$. By condition (i), there exists $m_1 \in \mathbb{N}(m)$ such that

$$V(m_1, x(m_1), y(m_1)) < V(k, x(k), y(k)) \quad \text{for } k \in \mathbb{N}(m_1). \tag{4.7.4}$$

Next, from condition (ii) it follows that

$$V(m_1, x(m_1), y(m_1)) \geq V(k, x(k), y(k)) \quad \text{for } k \in \mathbb{N}(m_1), \tag{4.7.5}$$

which contradicts (4.7.4). For the case $x(k) < 0$ for all large k, we consider the function $W(k, x(k), y(k))$ and arrive at the same contradiction. □

We need the following lemmas.

Lemma 4.7.3. *Suppose for $x(k) > 0$, $|y(k)| < \infty$, and $k \in \mathbb{N}(m_1)$, where $m_1 \in \mathbb{N}(m)$ is sufficiently large, there exists a function $v(k, x, y) : \mathbb{N}(m_1) \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that*

- (i_1) *$yv(k, x, y) > 0$ for $y \neq 0$, $k \in \mathbb{N}(m_1)$, $x > 0$ and $v(k, x, y) = 0$ for $y = 0$,*
- (i_2) *$\Delta v_{(4.7.2)}(k, x, y) \leq -\beta(k)$, where $\{\beta(k)\}$, $k \in \mathbb{N}(m_1)$ is a sequence of real numbers such that*

$$\liminf_{k\to\infty} \sum_{j=m_1}^{k-1} \beta(j) \geq 0 \quad \textit{for all large } m_1 \in \mathbb{N}. \tag{4.7.6}$$

Further, assume that for all sufficiently large $m_1^* \in \mathbb{N}(m)$ there exist $m_2 \in \mathbb{N}(m_1^*)$ and a function $w(k, x, y) : \mathbb{N}(m_2) \times \mathbb{R}^+ \times \mathbb{R}^- \to \mathbb{R}$ such that for $k \in \mathbb{N}(m_2)$,

- (i_3) *$y \leq w(k, x, y)$ and $w(m_2, x(m_2), y(m_2)) \leq b(y(m_2))$, where $b : \mathbb{R} \to \mathbb{R}$ satisfies $b(0) = 0$ and $b(y) < 0$ for $y \neq 0$,*
- (i_4) *$\Delta w(k, x, y) \leq -\rho(k) w(k, x, y)$, where $\{\rho(k)\}$ is a sequence of nonnegative real numbers such that*

$$\sum_{k=m_1^*}^{\infty} \frac{1}{c(k)} \prod_{j=m_1^*}^{k-1} (1 - \rho(j)) = \infty. \tag{4.7.7}$$

Then, if $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) > 0$ for all large k, then $y(k) \geq 0$ for all large k.

Proof. Assume that there is a sequence of integers $\{k_s\}$, $s \in \mathbb{N}_0$ such that $k_s \to \infty$ as $s \to \infty$ and $y(k_s) < 0$. Further, assume that $k_s \geq m_1 \in \mathbb{N}(m)$ is so large that

$$x(k) > 0, \quad \liminf_{k\to\infty} \sum_{j=k_s}^{k-1} \beta(j) \geq 0 \quad \text{for } k \in \mathbb{N}(k_s). \tag{4.7.8}$$

By (i_2), for the function $v(k,x(k),y(k))$, $k \geq k_s$, we have

$$v(k,x(k),y(k)) \leq v(k_s,x(k_s),y(k_s)) - \sum_{j=k_s}^{k-1} \beta(j) \quad \text{for } k \in \mathbb{N}(k_s). \tag{4.7.9}$$

Thus, in view of $v(k_s,x(k_s),y(k_s)) < 0$, (4.7.8) implies that there is $m_2 \in \mathbb{N}(k_s)$ such that for $k \in \mathbb{N}(m_2)$,

$$\sum_{j=k_s}^{k-1} \beta(j) \geq \frac{1}{2} v(k_s,x(k_s),y(k_s)). \tag{4.7.10}$$

From (4.7.9) it follows that

$$v(k,x(k),y(k)) \leq \frac{1}{2} v(k_s,x(k_s),y(k_s)) < 0 \quad \text{for } k \in \mathbb{N}(m_2). \tag{4.7.11}$$

Thus $y(k) < 0$ for $k \in \mathbb{N}(m_2)$.

For $m_2 \in \mathbb{N}(k_s)$ there is an integer $m_3 \in \mathbb{N}(m_2)$ such that for $k \in \mathbb{N}(m_3)$,

$$\Delta w_{(4.7.2)}(k,x(k),y(k)) \leq -\rho(k) w(k,x(k),y(k)), \tag{4.7.12}$$

that is,

$$w(k,x(k),y(k)) \leq (1-\rho(k-1)) w(k-1,x(k-1),y(k-1)). \tag{4.7.13}$$

By (i_3) it follows that

$$\begin{aligned} y(k) &\leq w(k,x(k),y(k)) \\ &\leq w(m_3,x(m_3),y(m_3)) \prod_{j=m_3}^{k-1} (1-\rho(j)) \\ &\leq b(y(m_3)) \prod_{j=m_3}^{k-1} (1-\rho(j)), \end{aligned} \tag{4.7.14}$$

that is,

$$c(k)\Delta x(k) = y(k) \leq b(y(m_3)) \prod_{j=m_3}^{k-1} (1-\rho(j)). \tag{4.7.15}$$

Summing (4.7.15) from m_3 to $k-1$, we obtain

$$x(k) \leq x(m_3) + b(y(m_3)) \sum_{i=m_3}^{k-1} \frac{1}{c(i)} \prod_{j=m_3}^{i-1} (1-\rho(j)) \quad \text{for } k \in \mathbb{N}(m_3), \tag{4.7.16}$$

which in view of condition (4.7.7), implies that $x(k) \to -\infty$ as $k \to \infty$. This contradiction proves that $y(k) \geq 0$ eventually. This completes the proof. □

Lemma 4.7.4. *Suppose for $x(k) < 0$, $|y(k)| < \infty$, and $k \in \mathbb{N}(m_1)$, where $m_1 \in \mathbb{N}(m)$ is sufficiently large, there exists a function $v(k,x,y) : \mathbb{N}(m_1) \times \mathbb{R}^- \times \mathbb{R} \to \mathbb{R}$ such that*

(ii_1) *$yv(k,x,y) < 0$ for $y \neq 0$, $k \in \mathbb{N}(m_1)$, $x < 0$ and $v(k,x,y) = 0$ for $y = 0$,*
(ii_2) *$\Delta v_{(4.7.2)}(k,x,y) \leq -\beta(k)$, where $\{\beta(k)\}$, $k \in \mathbb{N}(m_1)$, is a sequence of real numbers such that condition (4.7.6) holds.*

Further, assume that for all sufficiently large $m_1^* \in \mathbb{N}(m)$, there exist $m_2 \in \mathbb{N}(m_1^*)$ and a function $w(k,x,y) : \mathbb{N}(m_2) \times \mathbb{R}^- \times \mathbb{R}^+ \to \mathbb{R}$ such that

(ii_3) *$-y(k) \leq w(k,x,y)$ and $w(m_2,x(m_2),y(m_2)) \leq b(y(m_2))$, where the function $b : \mathbb{R} \to \mathbb{R}$ satisfies $b(0) = 0$ and $b(y) < 0$ for $y \neq 0$,*
(ii_4) *$\Delta w_{(4.7.2)}(k,x,y) \leq -\rho(k)w(k,x,y)$, where $\{\rho(k)\}$ is a sequence of nonnegative real numbers such that condition (4.7.7) holds.*

Then, if $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) < 0$ eventually, then $y(k) \leq 0$ eventually.

Proof. The proof is similar to that of Lemma 4.7.3. □

Lemma 4.7.5. *In addition to the assumptions of Lemma 4.7.3 assume that there exists a function $u(k,x,y) : \mathbb{N}(m_1) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, where $m_1 \in \mathbb{N}(m)$ is such that for $k \in \mathbb{N}(m_1)$,*

(iii_1) *$u(k,x,y) \to \infty$ uniformly for k, x as $y \to \infty$,*
(iii_2) *$\Delta u_{(4.7.2)}(k,x,y) \leq 0$ for all sufficiently large k.*

Then, if $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) > 0$ eventually, then $y(k)$ is eventually bounded.

Proof. By Lemma 4.7.3 there exists a large $m_1 \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$ such that $x(k) > 0$ and $y(k) \geq 0$ for $k \in \mathbb{N}(m_1)$. From condition (iii_2), it follows that $u(k,x(k),y(k)) \leq u(m_1,x(m_1),y(m_1))$ for $k \in \mathbb{N}(m_1)$. Now, if there exists a sequence $\{k_s\}$, $s \in \mathbb{N}_0$, which satisfies $k_s \to \infty$ as $s \to \infty$ such that $y(k_s) \to \infty$ as $s \to \infty$, then we have

$$u(k_s,x(k_s),y(k_s)) \leq u(m_1,x(m_1),y(m_1)), \tag{4.7.17}$$

which contradicts condition (iii_1). Thus, $\{y(k)\}$ is eventually bounded. □

Lemma 4.7.6. *In addition to the assumptions of Lemma 4.7.4 assume that there exists a function $u(k,x,y) : \mathbb{N}(m_1) \times \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}$ such that for $k \in \mathbb{N}(m_1)$,*

(iv_1) *$u(k,x,y) \to \infty$ uniformly for k, x as $y(k) \to -\infty$,*
(iv_2) *$\Delta u_{(4.7.2)}(k,x,y) \leq 0$ for all sufficiently large k.*

Then, if $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) < 0$ eventually, then $y(k)$ is eventually bounded.

Proof. The proof is similar to that of Lemma 4.7.5. □

Now, we are ready to prove the following result.

Theorem 4.7.7. *In addition to the assumptions of Lemmas 4.7.5 and 4.7.6 assume that for each $\delta > 0$ and $n > 0$, there exist $\ell(\delta, n) \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$ and functions $V(k,x,y)$ and $W(k,x,y)$ such that V is defined on*

$$\Omega_1 := \{k \in \mathbb{N}(\ell(\delta,n)) : x > \delta,\ 0 \le y \le n\}, \tag{4.7.18}$$

and W is defined on

$$\Omega_2 = \{k \in \mathbb{N}(\ell(\delta,n)) : x < -\delta,\ -n \le y \le 0\}, \tag{4.7.19}$$

and

(i) *$V(k,x,y)$ and $W(k,x,y)$ tend to infinity, uniformly for x and y as $k \to \infty$,*
(ii) *$\Delta V_{(4.7.2)}(k,x,y) \le 0$ for $(k,x,y) \in \Omega_1$,*
(iii) *$\Delta W_{(4.7.2)}(k,x,y) \le 0$ for $(k,x,y) \in \Omega_2$.*

Then equation (4.7.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (4.7.1), say, $x(k) > 0$ for $k \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$. By Lemma 4.7.3 there exists an integer $m_1 \in \mathbb{N}(m)$ such that $x(k) > 0$ and $y(k) \ge 0$ for $k \in \mathbb{N}(m_1)$. By Lemma 4.7.5 there is a constant $n > 0$ such that $0 \le y(k) \le n$ for $k \in \mathbb{N}(m_1)$. Since $\Delta x(k) = y(k)/c(k) \ge 0$ for all $k \in \mathbb{N}(m_1)$, we have $x(k) \ge x(m_1)$ for all $k \in \mathbb{N}(m_1)$. Consider the Liapunov function $V(k,x,y)$ defined for $k \in \mathbb{N}(\ell(\delta,n))$, $x > \delta$, $0 \le y \le m$, where $\delta = x(m_1)/2$ and we assume that $\ell \ge m_1$. Then by the same argument as in the proof of Theorem 4.7.2 we arrive at a contradiction. When $x(k) < 0$ eventually, we use Lemma 4.7.6 and the function $W(k,x,y)$ to obtain a similar contradiction. This completes the proof. □

Next, we will apply the obtained results to equation (4.7.1).

Theorem 4.7.8. *Assume that the following conditions are satisfied.*

(α_1) $\sum^{\infty} 1/c(k) = \infty$.
(α_2) *For $k \in \mathbb{N}$ and $x \ge 0$ there exist $\{q_1(k)\}$ and $f_1 \in C(\mathbb{R},\mathbb{R})$ such that*
  (i) *$f_1(u) - f_1(v) = g_1(u,v)(u-v)$, where g_1 is nonnegative,*
  (ii) *$\lim_{k\to\infty} \sum_{j=m}^{k-1} q_1(j) \ge 0$ for all large $m \in \mathbb{N}$,*
  (iii) *$xf_1(x) > 0$ for $x \ne 0$ and for all k, $x \ge 0$, $|y| < \infty$,*

$$f(k,x,y) \ge q_1(k) f_1(x). \tag{4.7.20}$$

(α_3) *For $k \in \mathbb{N}$ and $x \le 0$ there exist $\{q_2(k)\}$ and $f_2 \in C(\mathbb{R},\mathbb{R})$ such that*
  (iv) *$f_2(u) - f_2(v) = g_2(u,v)(u-v)$, where g_2 is nonnegative,*
  (v) *$\lim_{k\to\infty} \sum_{j=m}^{k-1} q_2(j) \ge 0$ for all large $m \in \mathbb{N}$,*
  (vi) *$xf_2(x) > 0$ for $x \ne 0$ and for all k, $x \le 0$, $|y| < \infty$,*

$$f(k,x,y) \le q_2(k) f_2(x). \tag{4.7.21}$$

Then, if $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) > 0$ eventually, then $y(k) \geq 0$ eventually.

Proof. To show this, we assume that (α_1), (ii), (4.7.20), (v), (4.7.21) hold for $k \in \mathbb{N}(m_1)$ for $m_1 \geq m \in \mathbb{N}$. For $k \in \mathbb{N}(m_1)$, $x > 0$, $|y| < \infty$ we define the function v by $v(k, x, y) = y/f_1(y)$. It is easy to see that the function $v(k, x, y)$ satisfies the hypotheses of Lemma 4.7.3 with $\beta(k) = q_1(k)$ such that $m_2 \in \mathbb{N}(m_1)$ and $\sum_{j=m_2}^{k-1} q_1(j) \geq 0$ for all $k \in \mathbb{N}(m_2)$. Next, $w(k, x, y) = y + f_1(x) \sum_{j=m_2}^{k-1} q_1(j)$ defined for all $k \in \mathbb{N}(m_2)$, $x > 0$ and $y < 0$ satisfies the hypotheses of Lemma 4.7.3 with $\rho(k) = 0$. In this case, the conclusion follows from Lemma 4.7.3. □

Similarly, for the function $v(k, x, y) = y/f_2(x)$, $k \in \mathbb{N}(m_1)$, $m_1 \geq m \in \mathbb{N}$, $x < 0$, $|y| < \infty$ and $w(k, x, y) = -y - f_2(x) \sum_{j=m_2}^{k-1} q_2(j)$, $k \in \mathbb{N}(m_2)$, $x < 0$, $y > 0$ from Lemma 4.7.4 it follows that if $(x(k), y(k))$ is a solution of the system (4.7.2) such that $x(k) < 0$ eventually, then $y(k) \leq 0$ eventually.

Now we can present alternative proofs for the following known oscillation criteria for equation (4.7.1).

Corollary 4.7.9. *If in addition to the conditions Theorem 4.7.8(α_1)–(α_3),*

$$\sum_{j=0}^{\infty} q_i(j) = \infty \quad \textit{for } i \in \{1, 2\}, \tag{4.7.22}$$

then equation (4.7.1) is oscillatory.

Proof. For $k \geq m_1 \in \mathbb{N}(m)$ for some $m \in \mathbb{N}$, $x > 0$, and $|y| < \infty$, we let

$$V(k, x, y) = \begin{cases} \dfrac{y}{f_1(x)} + \displaystyle\sum_{j=0}^{k-1} q_1(j) & \text{if } y \geq 0, \\ \displaystyle\sum_{j=0}^{k-1} q_1(j) & \text{if } y < 0. \end{cases} \tag{4.7.23}$$

It is easy to see that $V(k, x, y)$ satisfies the assumptions of Theorem 4.7.2. Similarly, the function

$$W(k, x, y) = \begin{cases} \displaystyle\sum_{j=0}^{k-1} q_2(j) & \text{if } y > 0, \\ \dfrac{y}{f_2(x)} + \displaystyle\sum_{j=0}^{k-1} q_2(j) & \text{if } y \leq 0 \end{cases} \tag{4.7.24}$$

satisfies the conditions of Theorem 4.7.2. Hence the conclusion follows. □

Corollary 4.7.10. *If in addition to the conditions Theorem 4.7.8*(α_1)–(α_3) *there exist a constant* $g > 0$ *and positive sequences* $\{\rho_i(k)\}$ *for* $i \in \{1,2\}$ *with* $g_i(u,v) \geq g$ *for* $i \in \{1,2\}$ *and*

$$\sum_{j=0}^{k-1} \rho_i(j)\left[q_i(j) - \frac{1}{4g}c(j)\left(\frac{\Delta\rho_i(j)}{\rho_i(j)}\right)^2\right] \longrightarrow \infty \quad \text{as } k \longrightarrow \infty \text{ for } i \in \{1,2\}, \tag{4.7.25}$$

then equation (4.7.1) is oscillatory.

Proof. It suffices to show that the functions

$$\begin{aligned} V(k,x,y) &= \frac{y}{f_1(x)}\rho_1(k) + \sum_{j=0}^{k-1}\rho_1(j)\left[q_1(j) - \frac{c(j)}{4g}\left(\frac{\Delta\rho_1(j)}{\rho_1(j)}\right)^2\right], \quad x > 0,\ y \geq 0, \\ W(k,x,y) &= \frac{y}{f_2(x)}\rho_2(k) + \sum_{j=0}^{k-1}\rho_2(j)\left[q_2(j) - \frac{c(j)}{4g}\left(\frac{\Delta\rho_2(j)}{\rho_2(j)}\right)^2\right], \quad x < 0,\ y \leq 0 \end{aligned} \tag{4.7.26}$$

satisfy the conclusions of Theorem 4.7.2. Here we omit the details. □

Finally, as applications of Theorem 4.7.7 we will provide two oscillation criteria for the more general difference equation

$$\Delta^2 x(k) + p(k,x(k),\Delta x(k))\Delta x(k) + f(k,x(k+1),\Delta x(k)) = 0, \tag{4.7.27}$$

which in system form can be written as

$$\begin{aligned} \Delta x(k) &= y(k), \\ \Delta y(k) &= -p(k,x(k),y(k))y(k) - f(k,x(k+1),y(k)). \end{aligned} \tag{4.7.28}$$

For equation (4.7.27) we will assume that

(γ_1) $f : \mathbb{N}\times\mathbb{R}^2 \to \mathbb{R}$ and $xf(k,x,y) > 0$ for $x \neq 0$,

(γ_2) $p : \mathbb{N}\times\mathbb{R}^2 \to \mathbb{R}$ and there exist two sequences $\{\xi_i(k)\}$, $i \in \{1,2\}$, of real numbers such that

  (i) $-\xi_1(k) \leq p(k,x,y) \leq \xi_2(k)$ for $k \in \mathbb{N}$, $|x| < \infty$, $|y| < \infty$,

  (ii) $0 \leq \xi_1(k) < 1$, $k \in \mathbb{N}$,

(γ_3) for any $\delta > 0$ and $n > 0$ there exist an integer $\ell(\delta,n)$ and a nonnegative sequence $\{\sigma(k)\}$ defined for $k \in \mathbb{N}(\ell(\delta,n))$ such that

  (i) $\sum_{j=\ell(\delta,n)}^{k-1}\sigma(j) \to \infty$ as $k \to \infty$,

  (ii) $|f(k,x,y)| \geq \sigma(k)$ for $|x| \geq \delta$, $|y| \leq n$, and $xy \geq 0$,

(γ_4) $\sum_{j=0}^{\infty}\xi_1(j) < \infty$ and $\lim_{k\to\infty}\sum_{i=0}^{k-1}[\prod_{j=0}^{i-1}(1-\xi_2(j))] = \infty$.

Corollary 4.7.11. *If* (γ_1)–(γ_4) *hold, then (4.7.27) is oscillatory.*

Corollary 4.7.12. *Let conditions* (γ_3) *and* (γ_4) *of Corollary 4.7.11 be replaced by*

(γ_3') *for any* $\delta > 0$ *there exist an integer* $\ell(\delta)$ *and a nonnegative sequence* $\{\sigma(k)\}$ *such that*
 (i) $\sum_{j=0}^{k-1}(1-\xi_1(j))\sum_{i=\ell(\delta)}^{j-1}\sigma(i) \to \infty$ *as* $k \to \infty$,
 (ii) $|f(k,x,y)| \geq \sigma(k)$, $|x| \geq \delta$, $xy \geq 0$,

(γ_4') $\lim_{k\to\infty}\sum_{i=0}^{k-1}[\prod_{j=0}^{i-1}(1-\xi_2(j))] = \infty$.

Then the conclusion of Corollary 4.7.11 holds.

Proof of Corollaries 4.7.11 and 4.7.12. To prove Corollary 4.7.11 it suffices to show the following. The functions

$$
\begin{aligned}
v(k,x,y) &= \begin{cases} y\prod_{j=0}^{k-1}[1-\xi_1(j)] & \text{if } k \in \mathbb{N},\ x>0,\ y \geq 0, \\ y\prod_{j=0}^{k-1}[1+\xi_2(j)] & \text{if } k \in \mathbb{N},\ x>0,\ y<0, \end{cases} \\
w(k,x,y) &= y \quad \text{if } k \in \mathbb{N}(m_1),\ m_1 \in \mathbb{N}(m),\ x>0,\ y<0, \\
u(k,x,y) &= y^2\left[\sum_{j=0}^{k-1}(1-\xi_1(j))\right]^2 \quad \text{if } k \in \mathbb{N},\ x>0,\ y>n>0
\end{aligned}
\tag{4.7.29}
$$

(with large n) satisfy the conditions of Lemma 4.7.5.

Also the functions

$$
\begin{aligned}
v(k,x,y) &= \begin{cases} -y\prod_{j=0}^{k-1}[1+\xi_2(j)] & \text{if } k \in \mathbb{N},\ x<0,\ y \geq 0, \\ -y\prod_{j=0}^{k-1}[1-\xi_1(j)] & \text{if } k \in \mathbb{N},\ x<0,\ y<0, \end{cases} \\
w(k,x,y) &= -y \quad \text{if } k \in \mathbb{N}(m_1),\ m_1 \in \mathbb{N}(m),\ x<0,\ y>0, \\
u(k,x,y) &= y^2\left[\sum_{j=0}^{k-1}(1-\xi_1(j))\right]^2 \quad \text{if } k \in \mathbb{N},\ x<0,\ y<0
\end{aligned}
\tag{4.7.30}
$$

satisfy the conditions of Lemma 4.7.6.

Next for each $\delta > 0$ and $n > 0$ the functions

$$V(k,x,y) = y\prod_{j=0}^{k-1}(1-\xi_1(j)) + \left[\prod_{j=0}^{\infty}(1-\xi_1(j))\right]\sum_{j=\ell(\delta,n)}^{k-1}\sigma(j) \tag{4.7.31}$$

for $k \in \mathbb{N}(\ell(\delta,n))$, $x > \delta$, and $0 \le y \le n$; and

$$W(k,x,y) = -y\prod_{j=0}^{k-1}(1-\xi_1(j)) + \left[\prod_{j=\ell(\delta,n)}^{k-1}(1-\xi_1(j))\right]\sum_{j=\ell(\delta,n)}^{k-1}\sigma(j) \tag{4.7.32}$$

for $k \in \mathbb{N}(\ell(\delta,n))$, $x < -\delta$, and $y \le 0$ satisfy the hypotheses of Theorem 4.7.7, and hence we conclude that equation (4.7.27) is oscillatory.

To prove Corollary 4.7.12 we will consider the functions v, w, and u given above and replace the functions V and W by the following two functions: for each $\delta > 0$,

$$V(k,x,y) = y\prod_{j=0}^{k-1}(1-\xi_1(j)) + \left[\prod_{j=0}^{k-1}(1-\xi_1(j))\right]\sum_{j=\ell(\delta)}^{k-1}\sigma(j) \tag{4.7.33}$$

for $k \in \mathbb{N}(\ell(\delta))$, $x > \delta$, and $y \ge 0$; and

$$W(k,x,y) = -y\prod_{j=0}^{k-1}(1-\xi_1(j)) + \left[\prod_{j=0}^{k-1}(1-\xi_1(j))\right]\sum_{j=\ell(\delta)}^{k-1}\sigma(j) \tag{4.7.34}$$

for $k \in \mathbb{N}(\ell(\delta))$, $x < -\delta$, and $y \le 0$. Both functions V and W satisfy all conditions of Theorem 4.7.7, and hence we conclude that (4.7.27) is oscillatory. □

4.8. Notes and general discussions

(1) Theorems 4.1.1 and 4.1.3 are taken from Thandapani [264] and are special cases of those of Wong and Agarwal [283]. Lemma 4.1.4 and Theorem 4.1.5 are due to Grace and El-Morshedy [143]. Lemmas 4.1.9, 4.1.10, and Theorem 4.1.11 are extensions of the results obtained by Thandapani et al. [267]. Theorems 4.1.12 and 4.1.15 are extensions of results due to Grace and El-Morshedy [143].

(2) The superlinear oscillation criteria of Section 4.2 are taken from Thandapani [264] and also extracted from Wong and Agarwal [283].

(3) Theorem 4.3.1 is due to Thandapani [264], Theorem 4.3.3 is taken from Li and Cheng [196], and Theorem 4.3.5 is due to Grace and El-Morshedy [143]. Theorem 4.3.9 is taken from Zhang and Chen [290], and Theorem 4.3.12 is extracted from Li [198].

(4) Theorems 4.4.4–4.4.10 are due to Szmanda [263] while Theorems 4.4.16 and 4.4.17 are taken from Zhang [289].

(5) The results of Sections 4.1–4.3 are extendable to perturbed difference equations of the form

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + Q\big(k, x(k)\big) = P\big(k, x(k), \Delta x(k)\big), \tag{4.8.1}$$

where $\{c(k)\}$ is a sequence of positive real numbers, $Q : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, and $P : \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}$. In this case we will assume that there exist real sequences $\{p(k)\}$, $\{q(k)\}$ and $f : \mathbb{R} \to \mathbb{R}$ such that $xf(x) > 0$ for $x \neq 0$ and

$$\frac{Q(k,u)}{f(u)} \geq q(k), \qquad \frac{P(k,u,v)}{f(u)} \leq p(k) \quad \text{for } u, v \neq 0. \tag{4.8.2}$$

Now, if $\{x(k)\}$ is a nonoscillatory solution of equation (4.8.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$, then equation (4.8.1) is reduced to inequality

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + \big[q(k) - p(k)\big] f\big(x(k)\big) \leq 0 \quad \text{eventually}, \tag{4.8.3}$$

which behaves as equation (4.1.2) when $x(k) > 0$ eventually.

(6) Theorem 4.5.2 is extracted from Theorem 3.9.3, and Theorem 4.5.3 is new. Theorems 4.5.10–4.5.15 are related to those of Thandapani and Pandian [269].

(7) The results of Section 4.6 are extracted from Gleska and Werbowski [129].

(8) The results of Section 4.7 are taken from Agarwal and Wong [25, Section 11] and He [158].

5 Oscillation theory for nonlinear difference equations II

In this chapter we will investigate oscillation and asymptotic behavior of solutions of certain second-order nonlinear difference equations. In Section 5.1 we will provide sufficient conditions for the oscillation of nonlinear equations. Superlinear and sublinear oscillation criteria are also included. In Section 5.2 we will establish existence criteria of eventually positive and eventually negative monotone solutions of nonlinear difference equations. In Section 5.3 we will develop full characterization of limit behavior of all positive decreasing solutions as well as all positive increasing solutions in terms of the coefficients of the equations under consideration.

5.1. Oscillation criteria

In this chapter we are concerned with oscillation criteria for second-order nonlinear difference equations of the form

$$\Delta\Psi\big(\Delta x(k-1)\big) + q(k)f\big(x(k)\big) = 0, \tag{5.1.1}$$

$$\Delta\big(c(k-1)\Psi\big(\Delta x(k-1)\big)\big) + q(k)f\big(x(k)\big) = 0, \tag{5.1.2}$$

and the more general equation

$$\Delta\big(c(k)\Psi\big(\Delta x(k)\big)\big) + p(k)\Psi\big(\Delta x(k)\big) + q(k)f\big(x(k+1)\big) = 0, \tag{5.1.3}$$

where

(i) $\{c(k)\}$ is a sequence of positive real numbers,
(ii) $\{p(k)\}$ and $\{q(k)\}$ are sequences of real numbers,
(iii) $f \in C(\mathbb{R},\mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$,
(iv) $\Psi \in C(\mathbb{R},\mathbb{R})$ is one of the following:
  (I_1) $\Psi(x) = |x|^\alpha \operatorname{sgn} x$ with $\alpha \geq 1$, or $\Psi(x) = x^\alpha$ with α is the ratio of two positive odd integers,
  (I_2) $\Psi(x) = x^\alpha$ with $0 < \alpha = m/n$, where $m \in \mathbb{N}$ is even and $n \in \mathbb{N}$ is odd,
  (I_3) $x\Psi(x) > 0$ and $\Psi'(x) > 0$ for $x \neq 0$.

We will assume that

$$\sum^{\infty} \Psi^{-1}\left(\frac{1}{c(j)}\right) = \infty, \tag{5.1.4}$$

$$f(u) - f(v) = F(u,v)(u - v) \quad \text{for } u, v \neq 0, \text{ where } F \text{ is nonnegative.} \tag{5.1.5}$$

5.1.1. Oscillation criteria—(I_1)

We will study the oscillation of equations (5.1.1)–(5.1.3) when Ψ is as in (I_1).

Theorem 5.1.1. *Suppose (5.1.4) and (5.1.5) hold. If*

$$\sum^{\infty} q(j) = \infty, \tag{5.1.6}$$

then equation (5.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. The proof of the case $x(k) < 0$ for $k \geq m$ is similar. From equation (5.1.2) for $k \geq m$, we have

$$\Delta\left(\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k))}\right) = -q(k) - \frac{c(k)F(x(k+1),x(k))\Psi(\Delta x(k))\Delta x(k)}{f(x(k+1))f(x(k))}, \tag{5.1.7}$$

which in view of (5.1.6) and the fact that $x\Psi(x) \geq 0$, gives

$$\Delta\left(\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k))}\right) \leq -q(k) \quad \text{for } k \geq m. \tag{5.1.8}$$

Summing (5.1.8) from $m+1$ to k gives

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \leq \frac{c(m)\Psi(\Delta x(m))}{f(x(m+1))} - \sum_{j=m+1}^{k} q(j), \tag{5.1.9}$$

which implies that there exists an integer $m_1 \geq m$ such that $\Delta x(k) < 0$ for $k \geq m_1$. Condition (5.1.6) also implies that there exists an integer $m_2 \geq m_1$ such that

$$\sum_{j=m_2+1}^{k} q(j) \geq 0. \tag{5.1.10}$$

Summing equation (5.1.2) from m_2+1 to k and then using summation by parts, we get

$$\begin{aligned} c(k)\Psi(\Delta x(k)) &= c(m_2)\Psi(\Delta x(m_2)) - \sum_{j=m_2+1}^{k} q(j)f(x(j)) \\ &= c(m_2)\Psi(\Delta x(m_2)) - f(x(k+1))\sum_{j=m_2+1}^{k} q(j) \\ &\quad + \sum_{i=m_2+1}^{k} F(x(i+1),x(i))\Delta x(i)\left[\sum_{j=m_2+1}^{i} q(j)\right] \\ &\le c(m_2)\Psi(\Delta x(m_2)) \end{aligned} \tag{5.1.11}$$

for $k \ge m_2+1$. Since $\Delta x(k) < 0$ for $k \ge m_1$, it follows that

$$\Delta x(k) \le -\left|\Psi^{-1}(\Delta x(m_2))\right|\Psi^{-1}(c(m_2))\Psi^{-1}\left(\frac{1}{c(k)}\right) \quad \text{for } k \ge m_2+1. \tag{5.1.12}$$

Summing (5.1.12) from m_2+1 to k provides

$$\begin{aligned} x(k+1) &\le x(m_2+1) - \left|\Psi^{-1}(\Delta x(m_2))\right|\Psi^{-1}(c(m_2))\sum_{j=m_2+1}^{k}\Psi^{-1}\left(\frac{1}{c(j)}\right) \\ &\longrightarrow -\infty \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{5.1.13}$$

This contradicts the fact that $x(k) > 0$ eventually. □

Example 5.1.2. The nonlinear difference equation

$$\Delta\left(k\left|\Delta x(k-1)\right|^{\alpha-1}\Delta x(k-1)\right) + 2^{\alpha}(2k+1)y(k) = 0 \tag{5.1.14}$$

has an oscillatory solution $x(k) = (-1)^k$. All conditions of Theorem 5.1.1 are satisfied, and hence equation (5.1.14) is oscillatory.

Next, we present the following result.

Theorem 5.1.3. *Let conditions (5.1.4) and (5.1.5) hold. If*

$$\liminf_{k\to\infty} \sum_{j=m\in\mathbb{N}}^{k} q(j) \ge 0, \tag{5.1.15}$$

$$\limsup_{k\to\infty} \sum_{j=m\in\mathbb{N}}^{k} q(j) = \infty, \tag{5.1.16}$$

then equation (5.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. We distinguish the following three possible cases.

Case 1. Suppose that $\{\Delta x(k)\}$ is oscillatory. So there exists a sequence $\{k_n\} \to \infty$ as $n \to \infty$ such that $\Delta x(k_n) < 0$. We choose n so large that (5.1.15) holds. Then, summing (5.1.8) from $k_n + 1$ to k followed by taking lim sup as $k \to \infty$ and using (5.1.15), we have

$$\limsup_{k\to\infty} \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = \frac{c(k_n)\Psi(\Delta x(k_n))}{f(x(k_n+1))} - \liminf_{k\to\infty} \sum_{j=k_n+1}^{k} q(j) < 0. \tag{5.1.17}$$

It follows from (5.1.17) that $\lim_{k\to\infty} \Delta x(k) < 0$. This contradicts the assumption that $\{\Delta x(k)\}$ oscillates.

Case 2. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. We sum (5.1.8) from $m_1 + 1$ to k to get

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \leq \frac{c(m_1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} - \sum_{j=m_1+1}^{k} q(j), \tag{5.1.18}$$

and by condition (5.1.16) we obtain

$$\liminf_{k\to\infty} \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = -\infty, \tag{5.1.19}$$

which is a contradiction.

Case 3. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. We note that condition (5.1.15) implies the existence of an integer $m_2 \geq m_1$ such that (5.1.10) holds. The rest of the proof is similar to that of Theorem 5.1.1 and hence is omitted.

The proof is complete. □

The following lemma extends Lemma 4.1.10.

Lemma 5.1.4. *Suppose that $\{x(k)\}$ is a positive (negative) solution of equation (5.1.2) for $k \in \mathbb{N}_m^\gamma = \{m, m+1, \dots, \gamma\}$ for some $m \in \mathbb{N}$, and assume that there exist an integer $m_1 \in \mathbb{N}_m^\gamma$ and a constant $b > 0$ such that*

$$-\frac{c(m)\Psi(\Delta x(m))}{f(x(m+1))} + \sum_{j=m+1}^{k} q(j) + \sum_{j=m+1}^{m_1} \frac{c(j)\Psi(\Delta x(j))F(x(j+1), x(j))\Delta x(j)}{f(x(j))f(x(j+1))} \geq b. \tag{5.1.20}$$

Then

$$c(k)\Psi(\Delta x(k)) \leq (\geq) - bf(x(m_1+1)), \quad k \in \mathbb{N}_{m_1}^\gamma. \tag{5.1.21}$$

Proof. From equation (5.1.2) we have

$$\Delta\left(\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k))}\right) = -q(k) - \frac{c(k)\Psi(\Delta x(k))F(x(k+1),x(k))\Delta x(k)}{f(x(k))f(x(k+1))}. \tag{5.1.22}$$

Summing (5.1.22) from $m+1$ to $k \in \mathbb{N}_{m_1}^{\gamma}$ and using (5.1.20), we find

$$\begin{aligned}
-\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} &= -\frac{c(m)\Psi(\Delta x(m))}{f(x(m+1))} + \sum_{j=m+1}^{k} q(j) \\
&\quad + \sum_{j=m+1}^{k} \frac{c(j)\Psi(\Delta x(j))F(x(j+1),x(j))}{f(x(j))f(x(j+1))} \\
&\geq b + \sum_{j=m_1+1}^{k} \frac{c(j)\Psi(\Delta x(j))\Delta x(j)F(x(j+1),x(j))}{f(x(j))f(x(j+1))} > 0.
\end{aligned} \tag{5.1.23}$$

Now we consider the following two cases.

Case 1. Suppose that $\{x(k)\}$ is eventually positive. Then (5.1.23) implies that $-c(k)\Psi(\Delta x(k)) > 0$, or equivalently $\Delta x(k) < 0$, $k \in \mathbb{N}_{m_1}^{\gamma}$. Let $u(k) = -c(k)\Psi(\Delta x(k))$. Then (5.1.23) becomes

$$u(k) \geq bf(x(k+1)) + \sum_{j=m_1+1}^{k} \frac{f(x(k+1))[-\Delta x(j)]F(x(j+1),x(j))}{f(x(j))f(x(j+1))} u(j). \tag{5.1.24}$$

Define

$$K(k,j,z) = \frac{f(x(k+1))[-\Delta x(j)]F(x(j+1),x(j))}{f(x(j))f(x(j+1))} z \tag{5.1.25}$$

for $k, j \in \mathbb{N}_{m_1}^{\gamma}$ and $z \in \mathbb{R}^+$. Since $\Delta x(k) < 0$ for $k \in \mathbb{N}_{m_1}^{\gamma}$, we observe that for fixed k and j, the function $K(k,j,\cdot)$ is nondecreasing. With $p(k) = bf(x(k+1))$ we apply Lemma 4.1.9 to get

$$u(k) \geq v(k) \quad \text{for } k \in \mathbb{N}_{m_1}^{\gamma}, \tag{5.1.26}$$

where $v(k)$ satisfies

$$v(k) = bf(x(k+1)) + \sum_{j=m_1+1}^{k} \frac{f(x(k+1))[-\Delta x(j)]F(x(j+1),x(j))}{f(x(j))f(x(j+1))}v(j) \tag{5.1.27}$$

provided $v(k) \in \mathbb{R}^+$ for all $k \in \mathbb{N}_{m_1}^{\gamma}$. From (5.1.27) we find

$$\begin{aligned}\Delta\left[\frac{v(k)}{f(x(k+1))}\right] &= \Delta\left[b + \sum_{j=m_1+1}^{k} \frac{[-\Delta x(j)]F(x(j+1),x(j))}{f(x(j))f(x(j+1))}v(j)\right] \\ &= \frac{[-\Delta x(k+1)]F(x(k+2),x(k+1))}{f(x(k+1))f(x(k+2))}v(k+1).\end{aligned} \tag{5.1.28}$$

On the other hand,

$$\Delta\left[\frac{v(k)}{f(x(k+1))}\right] = \frac{\Delta v(k)}{f(x(k+1))} - \frac{\Delta x(k+1)F(x(k+2),x(k+1))}{f(x(k+1))f(x(k+2))}v(k+1). \tag{5.1.29}$$

Equating (5.1.28) and (5.1.29), we get $\Delta v(k) = 0$ and $v(k) = v(m_1) = bf(x(m_1+1))$ for $k \in \mathbb{N}_{m_1}^{\gamma}$. The inequality (5.1.21) is now immediate from (5.1.26).

Case 2. Suppose that $\{x(k)\}$ is eventually negative. Then inequality (5.1.23) gives $c(k)\Psi(\Delta x(k)) > 0$, or equivalently $\Delta x(k) > 0$ for $k \in \mathbb{N}_{m_1}^{\gamma}$. Let $u(k) = c(k)\Psi(\Delta x(k))$. It follows from (5.1.23) that

$$u(k) \geq -bf(x(k+1)) + \sum_{j=m_1+1}^{k} \frac{[-f(x(k+1))]\Delta x(j)F(x(j+1),x(j))}{f(x(j))f(x(j+1))}u(j). \tag{5.1.30}$$

With $K(k,j,z)$ defined as in (5.1.25), we note that for fixed k and j, the function $K(k,j,\cdot)$ is nondecreasing. Applying Lemma 4.1.9 with $p(k) = -bf(y(k))$, we get (5.1.26), where $v(k)$ satisfies

$$v(k) = -bf(x(k+1)) + \sum_{j=m_1+1}^{k} \frac{[-f(x(k+1))]\Delta x(j)F(x(j+1),x(j))}{f(x(j))f(x(j+1))}v(j). \tag{5.1.31}$$

As in Case 1, $\Delta v(k) = 0$, and hence $v(k) = v(m_1) = -bf(x(m_1+1))$ for $k \in \mathbb{N}_{m_1}^{\gamma}$. Then inequality (5.1.26) immediately reduces to (5.1.21).

The proof is complete. $\square$

Next, we present the following result.

Theorem 5.1.5. *Suppose that (5.1.4) and (5.1.5) hold, and assume*

$$\lim_{|x|\to\infty} |f(x)| = \infty, \tag{5.1.32}$$

$$-\infty < \lim_{k\to\infty} \sum_{j=m+1}^{k} q(j) < \infty \quad \text{for } m \in \mathbb{N}. \tag{5.1.33}$$

If $\{x(k)\}$ is a nonoscillatory solution of equation (5.1.2), then

$$\sum_{j=m+1}^{k} \frac{c(j)\Psi(\Delta x(j))F(x(j+1),x(j))}{f(x(j))f(x(j+1))}\Delta x(j) < \infty, \tag{5.1.34}$$

$$\lim_{k\to\infty} \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = 0, \tag{5.1.35}$$

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = \sum_{j=k+1}^{\infty} q(j) + \sum_{j=k+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))F(x(j+1),x(j))}{f(x(j))f(x(j+1))}\Delta x(j) \tag{5.1.36}$$

for all sufficiently large k.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for all $k \geq m$ for some $m \in \mathbb{N}$. Suppose that (5.1.34) does not hold. In view of condition (5.1.33) there exists $m_1 \geq m$ such that (5.1.20) holds. Therefore, by Lemma 5.1.4, $c(k)\Psi(\Delta x(k)) \leq -bf(x(m_1+1))$ for $k \in \mathbb{N}_{m_1}$, where $b > 0$ is a constant. Thus,

$$\Delta x(k) \leq -\Psi^{-1}(bf(x(m_1+1)))\Psi^{-1}\left(\frac{1}{c(k)}\right). \tag{5.1.37}$$

In view of (5.1.4), relation (5.1.37) implies that $\{x(k)\}$ is eventually negative, which is a contradiction. Hence (5.1.34) is proved.

Next, to prove (5.1.35) and (5.1.36), we sum (5.1.22) from $m+1$ to k to obtain

$$\begin{aligned}\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = \frac{c(m)\Psi(\Delta x(m))}{f(x(m+1))} &- \sum_{j=m+1}^{k} q(j) \\ &- \sum_{j=m+1}^{k} \frac{c(j)\Psi(\Delta x(j))F(x(j+1),x(j))}{f(x(j))f(x(j+1))}\Delta x(j).\end{aligned} \tag{5.1.38}$$

In view of (5.1.33) and (5.1.34), it follows from (5.1.38) that

$$\sigma = \lim_{k\to\infty} \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \tag{5.1.39}$$

exists. Letting $k \to \infty$ in (5.1.38) and replacing m by k provides

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = \sigma + \sum_{j=k+1}^{\infty} q(j) + \sum_{j=k+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))F(x(j+1),x(j))}{f(x(j))f(x(j+1))}\Delta x(j). \tag{5.1.40}$$

We claim that $\sigma = 0$.

Case 1. If $\sigma < 0$, then we choose m_2 so large that

$$\left|\sum_{j=m_2+1}^{k} q(j)\right| \le -\frac{\sigma}{4} \quad \text{for } k \in \mathbb{N}_{m_2}, \tag{5.1.41}$$

$$\sum_{j=m_2+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))F(x(j+1),x(j))}{f(x(j))f(x(j+1))}\Delta x(j) < -\frac{\sigma}{4}. \tag{5.1.42}$$

If we take $m = m_1 = m_2$ in Lemma 5.1.4, then all the assumptions of Lemma 5.1.4 hold and so

$$\Delta x(k) \le -\Psi^{-1}(bf(x(m_2+1)))\Psi^{-1}\left(\frac{1}{c(k)}\right) \quad \text{for } k \in \mathbb{N}_{m_2}, \tag{5.1.43}$$

which in view of (5.1.4) contradicts the positivity of $\{x(k)\}$.

Case 2. If $\sigma > 0$ from (5.1.40), then we have

$$\lim_{k\to\infty} \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} = \sigma > 0, \tag{5.1.44}$$

which implies that $\Delta x(k) > 0$ eventually. Hence there exists $m_1 \ge m \in \mathbb{N}$ such that

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \ge \frac{\sigma}{2} \quad \text{for } k \in \mathbb{N}_{m_1}. \tag{5.1.45}$$

Therefore

$$\sum_{j=m_1+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))\Delta f(x(j))}{f(x(j))f(x(j+1))} \ge \frac{\sigma}{2}\sum_{j=m_1+1}^{\infty} \frac{\Delta f(x(j))}{f(x(j))}. \tag{5.1.46}$$

Define $r(t) = f(x(j)) + (t-j)\Delta f(x(j))$ for $j \le t \le j+1$. It is easy to check that $r'(t) = \Delta f(x(j))$ and $f(x(j)) \le r(t) \le f(x(j+1))$ for $j \le t \le j+1$. Hence,

$$\frac{\Delta f(x(j))}{f(x(j))} = \int_j^{j+1} \frac{\Delta f(x(j))}{f(x(j))}dt = \int_j^{j+1} \frac{r'(t)}{f(x(j))}dt \ge \int_j^{j+1} \frac{r'(t)}{r(t)}dt. \tag{5.1.47}$$

Therefore we obtain

$$\begin{aligned}
\infty &> \sum_{j=m_1+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))\Delta f(x(j))}{f(x(j))f(x(j+1))} \\
&\geq \frac{\sigma}{2}\sum_{j=m_1+1}^{\infty} \frac{\Delta f(x(j))}{f(x(j))} \\
&\geq \frac{\sigma}{2}\sum_{j=m_1+1}^{\infty} \int_j^{j+1} \frac{r'(t)}{r(t)}dt \\
&= \frac{\sigma}{2}\lim_{k\to\infty} \ln\left(\frac{r(k)}{r(m_1+1)}\right).
\end{aligned} \tag{5.1.48}$$

Hence $\ln r(t) < \infty$, which implies that $f(x(k)) < \infty$ as $k \to \infty$. From (5.1.32), $\{x(k)\}$ is bounded.

On the other hand from above and the monotonicity of f, we have

$$c(k)\Psi(\Delta x(k)) \geq \frac{\sigma}{2} f(x(k+1)) \geq \frac{\sigma}{2} f(x(m_1+1)), \tag{5.1.49}$$

and so

$$\Delta x(k) \geq \Psi^{-1}\left(\frac{\sigma}{2} f(x(m_1+1))\right)\Psi^{-1}\left(\frac{1}{c(k)}\right) \quad \text{for } k \in \mathbb{N}_{m_1}. \tag{5.1.50}$$

By (5.1.4) it follows that $\lim_{k\to\infty} x(k) = \infty$, which contradicts the boundedness of $\{x(k)\}$.

This completes the proof. □

Next, we obtain a sufficient condition for the oscillation of equation (5.1.2) subject to the condition

$$\frac{F(u,v)\Psi^{-1}(u)}{f(u)} \geq \lambda > 0, \quad \text{where } \lambda \text{ is a constant and } u, v \neq 0. \tag{5.1.51}$$

We note that if (5.1.33) holds, then $h_0(k) = \sum_{j=k+1}^{\infty} q(j)$, $k \in \mathbb{N}_m$, is finite. Assume that $h_0(k) \geq 0$ for all sufficiently large k. Define for $\ell \in \mathbb{N}$ the series

$$\begin{aligned}
h_1(k) &= \sum_{j=k+1}^{\infty} h_0(j)\Psi^{-1}(h_0(j))\Psi^{-1}\left(\frac{1}{c(j)}\right), \\
h_{\ell+1}(k) &= \sum_{j=k+1}^{\infty} [h_0(j)+\lambda h_\ell(j)]\Psi^{-1}[h_0(j)+\lambda h_\ell(j)]\Psi^{-1}\left(\frac{1}{c(j)}\right).
\end{aligned} \tag{5.1.52}$$

We introduce the following condition.

Condition (H). For every $\lambda > 0$ there exists a positive constant L such that $h_\ell(k)$ is finite for $\ell \in \{1, 2, \dots, L-1\}$ and $h_L(k)$ is infinite (or does not exist).

Theorem 5.1.6. *Suppose conditions (5.1.4), (5.1.5), (5.1.32), (5.1.33), (5.1.51), and (H) hold. Then equation (5.1.2) is oscillatory.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Hence, by Theorem 5.1.5, $\{x(k)\}$ satisfies (5.1.34) and (5.1.36), that is, for $k \in \mathbb{N}_m$, we have

$$\begin{aligned}\frac{c(k)\Delta x(k)}{f(x(k+1))} &= h_0(k) + \sum_{j=k+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))\Delta x(j) F(x(j+1), x(j))}{f(x(j)) f(x(j+1))} \\ &\geq h_0(k) \geq 0.\end{aligned} \tag{5.1.53}$$

Now

$$\begin{aligned}\Delta x(k) &\geq \Psi^{-1}\left(\frac{1}{c(k)}\right)\Psi^{-1}(h_0(k))\Psi^{-1}(f(x(k+1))) \\ &\geq \Psi^{-1}\left(\frac{1}{c(k)}\right)\Psi^{-1}(h_0(k))\Psi^{-1}(f(x(k))).\end{aligned} \tag{5.1.54}$$

From this and (5.1.51), we obtain

$$\begin{aligned}&\sum_{j=k+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))\Delta x(j) F(x(j+1), x(j))}{f(x(j)) f(x(j+1))} \\ &\quad \geq \lambda \sum_{j=k+1}^{\infty} h_0(j)\Psi^{-1}(h_0(j))\Psi^{-1}\left(\frac{1}{c(j)}\right) = \lambda h_1(k)\end{aligned} \tag{5.1.55}$$

for $k \in \mathbb{N}_m$. If $L = 1$ in condition (H), then the right-hand side of (5.1.55) is infinite. This is a contradiction to (5.1.34).

Next, it follows from (5.1.53) and (5.1.55) that

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \geq h_0(k) + \lambda h_1(k) \quad \text{for } k \in \mathbb{N}_m, \tag{5.1.56}$$

and as before we obtain for $k \in \mathbb{N}_m$,

$$\begin{aligned}&\sum_{j=k+1}^{\infty} \frac{c(j)\Psi(\Delta x(j))\Delta x(j) F(x(j+1), x(j))}{f(x(j)) f(x(j+1))} \\ &\quad \geq \lambda \sum_{j=k+1}^{\infty} [h_0(j) + \lambda h_1(j)]\Psi^{-1}[h_0(j) + \lambda h_1(j)]\Psi^{-1}\left(\frac{1}{c(j)}\right) = \lambda h_2(k).\end{aligned} \tag{5.1.57}$$

If $L = 2$ in condition (H), then once again we get a contradiction to (5.1.34). A similar argument yields a contradiction for any integer $L > 2$. This completes the proof. □

Remark 5.1.7. It is easy to see that the above results are also applicable to equation (5.1.2) when Ψ satisfies (I_3) and the following condition: for sufficiently small u and every $v > 0$,

$$\Psi(u)\Psi(v) \leq \Psi(uv) \leq \Psi(u)[-\Psi(-v)]. \tag{5.1.58}$$

5.1.2. Superlinear oscillation

An equation of the form of (5.1.1), (5.1.2), or (5.1.3) is called *superlinear* if

$$\int^{+\infty} \frac{du}{\Psi^{-1}(f(u))} < \infty, \qquad \int^{-\infty} \frac{du}{\Psi^{-1}(f(u))} < \infty. \tag{5.1.59}$$

Now we present the following result.

Theorem 5.1.8. *Suppose conditions (5.1.4), (5.1.5), (5.1.15) and (5.1.59) hold. If*

$$\sum^{\infty} q(j) < \infty, \tag{5.1.60}$$

$$\lim_{k\to\infty} \sum_{j=m\in\mathbb{N}}^{k} \Psi^{-1}\left[\frac{1}{c(j)} \sum_{i=j+1}^{\infty} q(i)\right] = \infty, \tag{5.1.61}$$

then equation (5.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. We distinguish the following three cases.

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. We sum (5.1.8) from $m_1 + 1$ to k to get

$$0 \leq \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \leq \frac{c(m_1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} - \sum_{j=m_1+1}^{k} q(j). \tag{5.1.62}$$

In view of (5.1.26) it follows from (5.1.62) that

$$0 \leq \frac{c(m_1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} - \sum_{j=m_1+1}^{\infty} q(j), \tag{5.1.63}$$

and therefore for $k \geq m_1$,

$$\sum_{j=k+1}^{\infty} q(j) \leq \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))}, \tag{5.1.64}$$

so

$$\Psi^{-1}\left[\frac{1}{c(k)} \sum_{j=k+1}^{\infty} q(j)\right] \leq \frac{\Delta x(k)}{\Psi^{-1}(f(x(k+1)))}. \tag{5.1.65}$$

Summing (5.1.65) from m_1 to k, we get

$$\begin{aligned}\sum_{j=m_1}^{k} \Psi^{-1}\left[\frac{1}{c(j)}\sum_{i=j+1}^{\infty} q(i)\right] &\le \sum_{j=m_1}^{k} \frac{\Delta x(j)}{\Psi^{-1}(f(x(j+1)))} \\ &\le \int_{x(m_1)}^{x(k+1)} \frac{du}{\Psi^{-1}(f(u))}.\end{aligned} \tag{5.1.66}$$

By (5.1.61), the left-hand side of (5.1.66) tends to ∞ as $k \to \infty$. However, the right-hand side of (5.1.66) is finite by (5.1.59).

Case 2. Suppose that $\{\Delta x(k)\}$ is oscillatory. So there exists a sequence $\{k_n\} \to \infty$ such that $\Delta x(k_n) < 0$. We choose n so large that (5.1.15) holds. Then, summing (5.1.8) from $k_n + 1$ to k followed by taking lim sup as $k \to \infty$ provides

$$\limsup_{k\to\infty} \frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \le \frac{c(k_n)\Psi(\Delta x(k_n))}{f(x(k_n+1))} + \limsup_{k\to\infty}\left[-\sum_{j=k_n+1}^{\infty} q(j)\right] < 0. \tag{5.1.67}$$

It follows from the above inequality that $\lim_{k\to\infty} \Delta x(k) < 0$. This contradicts the assumption that $\{\Delta x(k)\}$ oscillates.

Case 3. Suppose that $\Delta x(k) < 0$ for $k \ge m_1 \ge m$. We note that condition (5.1.3) implies the existence of an integer $m_2 \ge m_1$ such that (5.1.10) holds. The rest of the proof is similar to that of Theorem 5.1.1 and hence is omitted.

The proof is complete. □

Corollary 5.1.9. *If the hypotheses of Theorem 5.1.8 are satisfied except for condition (5.1.59), then all bounded solutions of equation (5.1.2) are oscillatory.*

Proof. The condition (5.1.59) is used only in Case 1 of the proof of Theorem 5.1.8. Suppose $\{x(k)\}$ is a bounded nonoscillatory solution of equation (5.1.2). In Case 1 we have $x(k) > 0$ and $\Delta x(k) \ge 0$ for $k \ge m_1$. Hence, in view of (5.1.5), we have $f(x(k+1)) \ge f(x(m_1))$ for $k \ge m_1$. It follows from (5.1.66) that

$$\begin{aligned}\sum_{j=m_1}^{k} \Psi^{-1}\left[\frac{1}{c(j)}\sum_{i=j+1}^{\infty} q(i)\right] &\le \sum_{j=m_1}^{k} \frac{\Delta x(j)}{\Psi^{-1}(f(x(j+1)))} \\ &\le \frac{1}{\Psi^{-1}(f(x(m_1)))}\sum_{j=m_1}^{k} \Delta x(j) \\ &= \frac{1}{\Psi^{-1}(f(x(m_1)))}\left[x(k+1) - x(m_1)\right] \\ &< \infty\end{aligned} \tag{5.1.68}$$

as $k \to \infty$, which contradicts (5.1.61). □

Example 5.1.10. Consider the nonlinear difference equation

$$\Delta\left(\frac{1}{k^2}\left|\Delta x(k-1)\right|^{\alpha-1}\Delta x(k-1)\right)+2^\alpha\frac{2k^2+2k+1}{k^2(k+1)^2}x(k)=0 \quad \text{for } k\geq m\in\mathbb{N}, \tag{5.1.69}$$

where α is a positive constant. Now

$$\sum^\infty q(j)=2^\alpha\sum^\infty\frac{2j^2+2j+1}{j^2(j+1)^2}=2^\alpha\sum^\infty\left[\frac{1}{j^2}+\frac{1}{(j+1)^2}\right]<\infty, \tag{5.1.70}$$

and so (5.1.60) holds. To see that (5.1.61) is satisfied, we note that

$$\begin{aligned}\sum_{j=m}^\infty\left[\frac{1}{c(j)}\sum_{i=j+1}^\infty q(i)\right]^{1/\alpha} &= 2\sum_{j=m}^\infty\left[j^2\sum_{i=j+1}^\infty\frac{2i^2+2i+1}{i^2(i+1)^2}\right]^{1/\alpha}\\ &= 2\sum_{j=m}^\infty\left[j^2\left(\sum_{i=j+1}^\infty\frac{1}{i^2}+\sum_{i=j+1}^\infty\frac{1}{(i+1)^2}\right)\right]^{1/\alpha}\\ &\geq 2\sum_{j=m}^\infty\left[j^2\sum_{i=j+1}^{2j}\frac{1}{i^2}\right]^{1/\alpha}\\ &\geq 2\sum_{j=m}^\infty\left[j^2\sum_{i=j+1}^{2j}\frac{1}{(2i)^2}\right]^{1/\alpha}\\ &= 2\sum_{j=m}^\infty\left(\frac{j}{4}\right)^{1/\alpha}\\ &= \infty.\end{aligned} \tag{5.1.71}$$

Hence the conclusion of Corollary 5.1.9 follows, and thus all bounded solutions of equation (5.1.69) are oscillatory. One such solution is given by $(-1)^k$.

Remark 5.1.11. In equation (5.1.69) if we let $0<\alpha<1$, then $f(x(k))=x(k)$ also satisfies condition (5.1.59). Hence it follows from Theorem 5.1.8 that all solutions of equation (5.1.69) are oscillatory when $0<\alpha<1$.

Theorem 5.1.12. *Suppose $\alpha\geq 1$, conditions (5.1.5), (5.1.15), and (5.1.59) hold. If*

$$\limsup_{k\to\infty}\sum_{j=m}^k jq(j)=\infty \quad \textit{for all large } m\in\mathbb{N}, \tag{5.1.72}$$

then equation (5.1.1) is oscillatory.

Proof. Assume that $\{x(k)\}$ is a nonoscillatory solution of equation (5.1.1), say, $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. Since condition (5.1.15) holds, from the proof of Theorem 5.1.8 we see that $\{\Delta x(k)\}$ does not oscillate. Now we consider the identity

$$\begin{aligned}\Delta\left[\frac{k\Psi(\Delta x(k-1))}{f(x(k))}\right] &= -kq(k) + \frac{\Psi(\Delta x(k))}{f(x(k+1))} \\ &\quad - k\frac{F(x(k+1),x(k))\Psi(\Delta x(k))}{f(x(k))f(x(k+1))}\Delta x(k),\end{aligned} \tag{5.1.73}$$

which gives rise to

$$\Delta\left[\frac{k\Psi(\Delta x(k-1))}{f(x(k))}\right] \le -kq(k) + \frac{\Psi(\Delta x(k))}{f(x(k+1))} \quad \text{for } k \ge m. \tag{5.1.74}$$

Next, we consider the following two cases.

Case 1. Suppose that $\Delta x(k) \ge 0$ for $k \ge m_1 \ge m$. Summing (5.1.74) from $m_1 + 1$ to k gives

$$\begin{aligned}\sum_{j=m_1+1}^{k} jq(j) &\le \frac{(m_1+1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} - \frac{(k+1)\Psi(\Delta x(k))}{f(x(k+1))} + \sum_{j=m_1+1}^{k}\frac{\Psi(\Delta x(j))}{f(x(j+1))} \\ &\le \frac{(m_1+1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} + \sum_{j=m_1+1}^{k}\frac{\Psi(\Delta x(j))}{f(x(j+1))} \\ &\le \frac{(m_1+1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} + \Psi\left[\sum_{j=m_1+1}^{k}\frac{\Delta x(j)}{\Psi^{-1}(f(x(j+1)))}\right] \\ &\le \frac{(m_1+1)\Psi(\Delta x(m_1))}{f(x(m_1+1))} + \Psi\left[\int_{x(m_1+1)}^{x(k+1)}\frac{du}{\Psi^{-1}(f(u))}\right].\end{aligned} \tag{5.1.75}$$

By condition (5.1.72), the left-hand side of (5.1.75) tends to ∞ as $k \to \infty$, whereas the right-hand side is finite by condition (5.1.59).

Case 2. Suppose that $\Delta x(k) < 0$ for $k \ge m_1 \ge m$. Condition (5.1.72) implies the existence of an integer $m_2 \ge m_1$ such that

$$\sum_{j=m_2+1}^{k} jq(j) \ge 0 \quad \text{for } k \ge m_2 + 1. \tag{5.1.76}$$

Multiplying equation (5.1.1) by k and summing by parts from m_2+1 to k, we have

$$
\begin{aligned}
&(k+1)\Psi(\Delta x(k))\\
&\quad = (m_2+1)\Psi(\Delta x(m_2)) + \sum_{j=m_2+1}^{k} \Psi(\Delta x(j)) - \sum_{j=m_2+1}^{k} jq(j)f(x(j))\\
&\quad = (m_2+1)\Psi(\Delta x(m_2)) + \sum_{j=m_2+1}^{k} \Psi(\Delta x(j)) - f(x(k+1)) \sum_{j=m_2+1}^{k} jq(j)\\
&\qquad + \sum_{j=m_2+1}^{k} \Delta f(x(j)) \sum_{i=m_2+1}^{j} iq(i)\\
&\quad = (m_2+1)\Psi(\Delta x(m_2)) + \sum_{j=m_2+1}^{k} \Psi(\Delta x(j)) - f(x(k+1)) \sum_{j=m_2+1}^{k} jq(j)\\
&\qquad + \sum_{j=m_2+1}^{k} F(x(j+1),x(j))\Delta x(j)\left[\sum_{i=m_2+1}^{j} iq(i)\right]\\
&\quad \le (m_2+1)\Psi(\Delta x(m_2)).
\end{aligned}
\tag{5.1.77}
$$

It follows that

$$
\Delta x(k) \le -\Psi^{-1}(m_2+1)\,|\,\Delta x(m_2)\,|\,\Psi^{-1}\left(\frac{1}{k+1}\right) \quad \text{for } k \ge m_2+1. \tag{5.1.78}
$$

Once again, we sum (5.1.78) from m_2+1 to k to get

$$
x(k+1) \le x(m_2+1) - \Psi^{-1}(m_2+1)\,|\,\Delta x(m_2)\,|\sum_{j=m_2+1}^{k} \Psi^{-1}\left(\frac{1}{j+1}\right) \longrightarrow -\infty \tag{5.1.79}
$$

as $k \to \infty$, and this contradicts the assumption that $\{x(k)\}$ is eventually positive.

This completes the proof. □

Example 5.1.13. The difference equation

$$
\Delta\left(\,|\,\Delta x(k-1)\,|^{\alpha-1}\Delta x(k-1)\right) + 2^{\alpha+1}x^3(k) = 0 \quad \text{for } k \in \mathbb{N}, \tag{5.1.80}
$$

where $1 \le \alpha < 3$, has an oscillatory solution $x(k) = (-1)^k$. All conditions of Theorem 5.1.12 are satisfied, and hence equation (5.1.80) is oscillatory.

Corollary 5.1.14. *Suppose $\alpha \ge 1$, conditions (5.1.5), (5.1.15), and (5.1.72) hold. Then all bounded solutions of equation (5.1.1) are oscillatory.*

Proof. The proof is similar to the proof of Corollary 5.1.9 and hence is omitted. □

Example 5.1.15. The difference equation

$$\Delta\Big(\left|\Delta x(k-1)\right|^{\alpha-1}\Delta x(k-1)\Big)+2^{\alpha+1}\left|x(k)\right|^{\gamma}\operatorname{sgn}x(k)=0\quad\text{for } k\in\mathbb{N},\tag{5.1.81}$$

where $\alpha \geq 1$ and $\gamma > 0$, has an oscillatory solution $x(k) = (-1)^k$. We note that the hypotheses of Corollary 5.1.14 are satisfied, and hence we conclude that all bounded solutions of equation (5.1.81) are oscillatory.

Next, we will present the following result when

$$\sum^{\infty}\frac{1}{c(j-1)}<\infty.\tag{5.1.82}$$

Theorem 5.1.16. *Suppose $\alpha > 1$, conditions (5.1.4), (5.1.5), (5.1.15), (5.1.59), and (5.1.82) hold. If*

$$\sum^{\infty}q(j)C(j,m)=\infty,\tag{5.1.83}$$

where

$$C(k,m)=\sum_{j=m}^{k}\frac{1}{c(j-1)},\tag{5.1.84}$$

then equation (5.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since condition (5.1.15) holds, from the proof of Theorem 5.1.8 we see that $\{\Delta x(k)\}$ does not oscillate. Next, we set

$$y(k)=\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k))}C(k,m).\tag{5.1.85}$$

Then for $k \geq m$,

$$\begin{aligned}\Delta y(k)&=\frac{\Delta\big(c(k-1)\Psi(\Delta x(k-1))\big)}{f(x(k))}C(k,m)+\frac{\Psi(\Delta x(k))}{f(x(k+1))}\\&\quad-\frac{c(k)F(x(k+1),x(k))\Delta x(k)\Psi(\Delta x(k))}{f(x(k))f(x(k+1))}C(k,m)\\&\leq -q(k)C(k,m)+\frac{\Psi(\Delta x(k))}{f(x(k+1))}.\end{aligned}\tag{5.1.86}$$

Next, we consider the following two cases.

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Summing (5.1.86) from $m_1 + 1$ to k yields

$$\begin{aligned} y(k+1) &\leq y(m_1+1) - \sum_{j=m_1+1}^{k} q(j)C(j,m) + \sum_{j=m_1+1}^{k} \frac{\Psi(\Delta x(j))}{f(x(j+1))} \\ &\leq y(m_1+1) - \sum_{j=m_1+1}^{k} q(j)C(j,m) + \Psi\left[\sum_{j=m_1+1}^{k} \frac{\Delta x(j)}{\Psi^{-1}(f(x(j+1)))}\right] \\ &\leq y(m_1+1) - \sum_{j=m_1+1}^{k} q(j)C(j,m) + \Psi\left[\int_{x(m_1+1)}^{x(k+1)} \frac{du}{\Psi^{-1}(f(u))}\right]. \end{aligned} \tag{5.1.87}$$

By conditions (5.1.59) and (5.1.83), the right-hand side of (5.1.87) tends to $-\infty$ as $k \to \infty$. However, the left-hand side is nonnegative.

Case 2. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. In view of condition (5.1.83) there exists an integer $m_2 \geq m_1$ such that

$$\sum_{j=m_2+1}^{k} q(j)C(j,m_1) \geq 0 \quad \text{for } k \geq m_2 + 1. \tag{5.1.88}$$

Now, multiplying equation (5.1.2) by $C(k,m_1)$, summing by parts from m_2+1 to k, and using (5.1.88) provides for $k \geq m_2 + 1$,

$$\begin{aligned} &C(k+1,m_1)c(k)\Psi(\Delta x(k)) - C(m_2+1,m_1)c(m_2)\Psi(\Delta x(m_2)) \\ &\quad = \sum_{j=m_2+1}^{k} \Psi(\Delta x(j)) - \sum_{j=m_2+1}^{k} C(j,m_1)q(j)f(x(j)) \\ &\quad = \sum_{j=m_2+1}^{k} \Psi(\Delta x(j)) - f(x(k+1)) \sum_{j=m_2+1}^{k} C(j,m_1)q(j) \\ &\qquad + \sum_{j=m_2+1}^{k} \Delta f(x(j))\left[\sum_{i=m_2+1}^{j} C(i,m_1)q(i)\right] \\ &\quad \leq 0, \end{aligned} \tag{5.1.89}$$

so, for $k \geq m_2 + 1$,

$$\Delta x(k) \leq -\Psi^{-1}[C(m_2+1,m_1)c(m_2)]\,|\,\Delta x(m_2)\,|\,\Psi^{-1}\left[\frac{1}{C(k+1,m_1)c(k)}\right]. \tag{5.1.90}$$

Summing (5.1.90) from m_2+1 to k gives

$$x(k+1) \leq x(m_2+1) - \Psi^{-1}\left[C(m_2+1,m_1)c(m_2)\right]\left|\Delta x(m_2)\right|\Psi^{-1}\left(\frac{1}{\theta}\right)\sum_{j=m_2+1}^{k}\Psi^{-1}\left(\frac{1}{c(j)}\right), \tag{5.1.91}$$

where $0<\theta:=C(\infty,m_1)<\infty$ (by (5.1.82)). By condition (5.1.4), the right-hand side of (5.1.91) tends to $-\infty$ as $k\to\infty$. But this contradicts the assumption that $\{x(k)\}$ is eventually negative.

The proof is complete. □

The following corollary is immediate.

Corollary 5.1.17. *If the hypotheses of Theorem 5.1.16 hold except for condition (5.1.59), then all bounded solutions of equation (5.1.2) are oscillatory.*

Next, we will give the following oscillation result for equation (5.1.2) when

$$C(k,m)\longrightarrow\infty \quad \text{as } k\longrightarrow\infty, \qquad \frac{c(k)}{c(k-1)}\leq 1 \quad \text{for } k\geq m\in\mathbb{N}, \tag{5.1.92}$$

where $C(k,m)$ is given by (5.1.84).

Theorem 5.1.18. *Suppose $\alpha\geq 1$, conditions (5.1.5), (5.1.15), (5.1.59), and (5.1.92) hold. If*

$$\sum^{\infty}C(j-1,m)q(j)=\infty, \tag{5.1.93}$$

then equation (5.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. As in Theorem 5.1.8, we see that $\{\Delta x(k)\}$ is eventually of one sign. Set

$$y(k)=\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k))}C(k-1,m) \quad \text{for } k\geq m+1. \tag{5.1.94}$$

It follows that

$$\begin{aligned}\Delta y(k) &= C(k-1,m)\frac{\Delta\big(c(k-1)\Psi(\Delta x(k-1))\big)}{f(x(k))}+\frac{c(k)}{c(k-1)}\frac{\Psi(\Delta x(k))}{f(x(k+1))}\\ &\quad -\frac{c(k)F(x(k+1),x(k))C(k-1,m)\Psi(\Delta x(k))\Delta x(k)}{f(x(k))f(x(k+1))}\\ &\leq -C(k-1,m)q(k)+\frac{c(k)}{c(k-1)}\frac{\Psi(\Delta x(k))}{f(x(k+1))}.\end{aligned} \tag{5.1.95}$$

Now we consider the following two cases.

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Since condition (5.1.92) holds, (5.1.95) provides

$$\Delta y(k) \leq -C(k-1,m)q(k) + \frac{\Psi(\Delta x(k))}{f(x(k+1))} \quad \text{for } k \geq m_1. \tag{5.1.96}$$

Thus, as in Theorem 5.1.16 we obtain

$$\begin{aligned} y(k+1) &\leq y(m_1+1) - \sum_{j=m_1+1}^{k} C(j-1,m)q(j) + \Psi\left[\int_{x(m_1+1)}^{x(k+1)} \frac{du}{\Psi^{-1}(f(u))}\right] \\ &\longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{5.1.97}$$

which is a contradiction to the fact that $y(k) \geq 0$ eventually.

Case 2. Suppose $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. In view of condition (5.1.93) there exists an integer $m_2 \geq m_1$ such that

$$\sum_{j=m_2+1}^{k} C(j-1,m_1)q(j) \geq 0 \quad \text{for } k \geq m_2+1. \tag{5.1.98}$$

Multiplying equation (5.1.2) by $C(k-1,m_1)$, summing by parts from m_2+1 to k, and using (5.1.98) provides for $k \geq m_2+1$,

$$\begin{aligned} &C(k,m_1)c(k)\Psi(\Delta x(k)) - C(m_2,m_1)c(m_2)\Psi(\Delta x(m_2)) \\ &\quad = \sum_{j=m_2+1}^{k} \frac{c(j)}{c(j-1)}\Psi(\Delta x(j)) - \sum_{j=m_2+1}^{k} C(j-1,m_1)q(j)f(x(j)) \\ &\quad = \sum_{j=m_2+1}^{k} \frac{c(j)}{c(j-1)}\Psi(\Delta x(j)) - f(x(k+1)) \sum_{j=m_2+1}^{k} C(j-1,m_1)q(j) \\ &\qquad + \sum_{j=m_2+1}^{k} F(x(j+1),x(j))\Delta x(j)\left[\sum_{i=m_2+1}^{j} C(i-1,m_1)q(i)\right] \\ &\quad \leq 0, \end{aligned} \tag{5.1.99}$$

so, for $k \geq m_2+1$,

$$\Delta x(k) \leq -\Psi^{-1}[C(m_2,m_1)c(m_2)]\,|\,\Delta x(m_2)\,|\,\Psi^{-1}\left[\frac{1}{C(k,m_1)c(k)}\right]. \tag{5.1.100}$$

Again, we sum (5.1.100) from m_2+1 to k to get

$$\begin{aligned}
&x(k+1)\\
&\quad\le x(m_2+1)-\Psi^{-1}\big[C(m_2,m_1)c(m_2)\big]\,|\Delta x(m_2)|\sum_{j=m_2+1}^{k}\Psi^{-1}\left[\frac{\Delta C(j,m_1)}{C(j,m_1)}\right]\\
&\quad\le x(m_2+1)-\Psi^{-1}\big[C(m_2,m_1)c(m_2)\big]\,|\Delta x(m_2)|\,\Psi^{-1}\left[\sum_{j=m_2+1}^{k}\frac{\Delta C(j,m_1)}{C(j,m_1)}\right].
\end{aligned}\tag{5.1.101}$$

From this, since

$$\frac{\Delta C(j,m_1)}{C(j,m_1)}=\frac{\int_j^{j+1}C'(t,m_1)\,dt}{C(j,m_1)}\ge\int_j^{j+1}\frac{C'(t,m_1)}{C(t,m_1)}dt=\ln\frac{C(j+1,m_1)}{C(j,m_1)},\tag{5.1.102}$$

it follows that

$$\begin{aligned}
x(k+1)&\le x(m_2+1)\\
&\qquad-\Psi^{-1}\big[C(m_2,m_1)c(m_2)\big]\,|\Delta x(m_2)|\,\Psi^{-1}\left[\ln\left(\frac{C(k+1,m_1)}{C(m_2+1,m_1)}\right)\right]\\
&\qquad\longrightarrow-\infty\quad\text{as }k\longrightarrow\infty.
\end{aligned}\tag{5.1.103}$$

But this contradicts the fact that $x(k)>0$ eventually.

This completes the proof. □

Corollary 5.1.19. *If the conditions of Theorem 5.1.18 are satisfied except for condition (5.1.59), then all bounded solutions of equation (5.1.2) are oscillatory.*

We will employ Theorem 5.1.5 to obtain the following sufficient condition for the oscillation of equation (5.1.2).

Theorem 5.1.20. *Assume that conditions (5.1.4), (5.1.5), and (5.1.59) hold. If $\sum_{j=m}^{\infty}q(j)$ exists and*

$$\lim_{k\to\infty}\sum_{j=m}^{k}\Psi^{-1}\left(\frac{1}{c(j)}\right)\Psi^{-1}\left(\sum_{i=j+1}^{\infty}q(i)\right)=\infty\quad\text{for } m\in\mathbb{N},\tag{5.1.104}$$

then equation (5.1.2) is oscillatory.

Proof. Suppose the contrary. Without loss of generality, we assume that $\{x(k)\}$ is an eventually positive solution of equation (5.1.2). We claim that under condition (5.1.59), condition (5.1.32) is true. If not, then there exists a constant $a > 0$ such that $f(x) \le a$ for $x > 0$. By (5.1.59), we have

$$\int^{\infty} \frac{du}{\Psi^{-1}(u)} \le \int^{\infty} \frac{du}{\Psi^{-1}(f(u))} < \infty, \tag{5.1.105}$$

which is a contradiction. It is easy to see that identity (5.1.36) holds and

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k+1))} \ge \sum_{j=k+1}^{\infty} q(j) \quad \text{for } k \ge m \in \mathbb{N}, \tag{5.1.106}$$

so

$$\frac{\Delta x(k)}{\Psi^{-1}(f(x(k+1)))} \ge \Psi^{-1}\left(\frac{1}{c(k)}\right)\Psi^{-1}\left(\sum_{j=k+1}^{\infty} q(j)\right). \tag{5.1.107}$$

Summing (5.1.107) from m to k, we get

$$\sum_{j=m}^{k} \frac{\Delta x(j)}{\Psi^{-1}(f(x(j+1)))} \ge \sum_{j=m}^{k} \Psi^{-1}\left(\frac{1}{c(j)}\right)\Psi^{-1}\left(\sum_{i=j+1}^{\infty} q(i)\right). \tag{5.1.108}$$

We define $r(t) = x(j) + (t-j)\Delta x(j)$ for $j \le t \le j+1$. If $\Delta x(j) \ge 0$, then $x(j) \le r(t) \le x(j+1)$ and

$$\frac{\Delta x(j)}{\Psi^{-1}(f(x(j+1)))} \le \frac{r'(t)}{\Psi^{-1}(f(r(t)))} \le \frac{\Delta x(j)}{\Psi^{-1}(f(x(j)))}. \tag{5.1.109}$$

If $\Delta x(j) < 0$, then $x(j+1) \le r(t) \le x(j)$ and (5.1.109) also holds. From (5.1.107) and (5.1.109), we obtain

$$\begin{aligned} \sum_{j=m}^{k} \Psi^{-1}\left(\frac{1}{c(j)}\right)\Psi^{-1}\left(\sum_{i=j+1}^{\infty} q(i)\right) &\le \int_{m}^{k+1} \frac{dr(t)}{\Psi^{-1}(f(r(t)))} \\ &\le \int_{r(m)}^{\infty} \frac{du}{\Psi^{-1}(f(u))} < \infty. \end{aligned} \tag{5.1.110}$$

This implies by letting $G(x) = \int_x^{\infty} du/\Psi^{-1}(f(u))$ that

$$\sum_{j=m}^{k} \Psi^{-1}\left(\frac{1}{c(j)}\right)\Psi^{-1}\left(\sum_{i=j+1}^{\infty} q(i)\right) \le G(r(m)), \tag{5.1.111}$$

which contradicts condition (5.1.104). This completes the proof. $\square$

5.1.3. Sublinear oscillation

An equation of the form (5.1.1), (5.1.2), or (5.1.3) is called *sublinear* if

$$\int^{+0} \frac{du}{\Psi^{-1}(f(u))} < \infty, \qquad \int^{-0} \frac{du}{\Psi^{-1}(f(u))} < \infty. \tag{5.1.112}$$

Now we present the following results.

Theorem 5.1.21. *Suppose $\alpha \geq 1$, conditions (5.1.5), (5.1.15), and (5.1.112) hold. If for every constant M,*

$$\sum^{\infty} \left[\frac{M}{c(j)} - \frac{1}{c(j)} \sum_{i=m}^{j} q(i) \right] = -\infty \quad \text{for } m \in \mathbb{N}, \tag{5.1.113}$$

then equation (5.1.2) is oscillatory.

Proof. Suppose that $\{x(k)\}$ is a nonoscillatory solution of equation (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 5.1.3, $\{\Delta x(k)\}$ is eventually of one sign. Next, we consider the identity

$$\begin{aligned} \Delta &\left[\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k-1))} \right] \\ &\quad = -q(k) - \frac{c(k-1)F(x(k), x(k-1))\Delta x(k-1)\Psi(\Delta x(k-1))}{f(x(k-1))f(x(k))}, \end{aligned} \tag{5.1.114}$$

which implies

$$\Delta \left[\frac{c(k-1)\Psi(\Delta x(k-1))}{f(x(k-1))} \right] \leq -q(k) \quad \text{for } k \geq m. \tag{5.1.115}$$

Now we distinguish the following two cases.

Case 1. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Summing (5.1.115) from $m_1 + 1$ to k gives

$$\frac{c(k)\Psi(\Delta x(k))}{f(x(k))} \leq \frac{c(m_1)\Psi(\Delta x(m_1))}{f(x(m_1))} - \sum_{j=m_1+1}^{k} q(j), \tag{5.1.116}$$

so

$$\frac{\Psi(\Delta x(k))}{f(x(k))} \leq \frac{M}{c(k)} - \frac{1}{c(k)} \sum_{j=m_1+1}^{k} q(j), \tag{5.1.117}$$

where $M = c(m_1)\Psi(\Delta x(m_1))/f(x(m_1))$. Again, we sum (5.1.117) from $m_1 + 1$ to

k to get

$$\sum_{j=m_1+1}^{k} \frac{\Psi(\Delta x(j))}{f(x(j))} \leq \sum_{j=m_1+1}^{k} \left[\frac{M}{c(j)} - \frac{1}{c(j)} \sum_{i=m_1+1}^{j} q(i) \right]. \tag{5.1.118}$$

By condition (5.1.113), the right-hand side of (5.1.118) tends to $-\infty$ as $k \to \infty$, whereas the left-hand side is nonnegative.

Case 2. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. Then, from (5.1.118) we find

$$\begin{aligned} -\sum_{j=m_1+1}^{k} \left[\frac{M}{c(j)} - \frac{1}{c(j)} \sum_{i=m_1+1}^{j} q(i) \right] &\leq \sum_{j=m_1+1}^{k} \frac{\Psi(|\Delta x(j)|)}{f(x(j))} \\ &\leq \Psi\left[\sum_{j=m_1+1}^{k} \frac{|\Delta x(j)|}{\Psi^{-1}(f(x(j)))} \right] \\ &\leq \Psi\left[\int_{x(k+1)}^{x(m_1+1)} \frac{du}{\Psi^{-1}(f(u))} \right] \\ &\leq \Psi\left[\int_{0}^{x(m_1+1)} \frac{du}{\Psi^{-1}(f(u))} \right]. \end{aligned} \tag{5.1.119}$$

By (5.1.113), the left-hand side of (5.1.119) tends to ∞ as $k \to \infty$, whereas the right-hand side is finite by condition (5.1.112).

This completes the proof. □

Corollary 5.1.22. *Suppose that the hypotheses of Theorem 5.1.21 are satisfied except for condition (5.1.112). Then every bounded solution of equation (5.1.2) is oscillatory.*

Proof. The condition (5.1.112) is used only in Case 2 of the proof of Theorem 5.1.21. Let $\{x(k)\}$ be a bounded nonoscillatory solution of equation (5.1.2). In Case 2 of the proof of Theorem 5.1.21 we have $x(k) > 0$ and $\Delta x(k) < 0$ for $k \geq m_1$. Hence $x(k) \downarrow a > 0$ as $k \to \infty$ and $f(x(k)) \geq f(a) > 0$ for $k \geq m_1$. It follows from (5.1.118) that

$$\begin{aligned} -\sum_{j=m_1+1}^{k} \left[\frac{M}{c(j)} - \frac{1}{c(j)} \sum_{i=m_1+1}^{j} q(i) \right] &\leq \Psi\left[\sum_{j=m_1+1}^{k} \frac{|\Delta x(j)|}{\Psi^{-1}(f(x(j)))} \right] \\ &\leq \Psi\left[\frac{1}{\Psi^{-1}(f(a))} \sum_{j=m_1+1}^{k} |\Delta x(j)| \right] \\ &= \Psi\left[\frac{1}{\Psi^{-1}(f(a))} (x(m_1+1) - x(k+1)) \right] \\ &< \infty, \end{aligned} \tag{5.1.120}$$

which contradicts condition (5.1.113). This completes the proof. □

Finally, we present the following result which extends Theorem 4.3.9 to equation (5.1.2).

Theorem 5.1.23. *Suppose that conditions (5.1.4), (5.1.5), and (5.1.32) hold. Also let $Q(k) = \sum_{j=k}^{\infty} q(j) \geq 0$ for $k \geq m$ for some $m \in \mathbb{N}$ and $F(u,v) \geq \lambda > 0$ for $u, v \neq 0$. If*

$$\int^{+0} \frac{du}{\Psi^{-1}(f(\Psi^{-1}(u)))} < \infty, \qquad \int^{-0} \frac{du}{\Psi^{-1}(f(\Psi^{-1}(u)))} < \infty, \tag{5.1.121}$$
$$-f(-uv) \geq f(uv) \geq f(u)f(v) \quad \text{for } u, v \neq 0,$$

$$\sum_{j=m}^{\infty} Q(j+1)\Psi^{-1}(Q(j+1))\Psi^{-1}(f(C(j)))\Psi^{-1}\left(\frac{1}{c(j)}\right) = \infty, \tag{5.1.122}$$

where

$$C(k) = \sum_{j=m}^{k-1} \Psi^{-1}\left(\frac{1}{c(j)}\right) \quad \text{for } m \in \mathbb{N}, \tag{5.1.123}$$

then equation (5.1.2) is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since $Q(k) \geq 0$ for $k \geq m$, Theorem 5.1.5 implies that $\Delta x(k) \geq 0$ for $k \geq m$. From (5.1.36)

$$\Psi(\Delta x(k)) \geq \frac{Q(k+1)}{c(k)} f(x(k+1)) \quad \text{for } k \geq m. \tag{5.1.124}$$

Summing equation (5.1.2) from $k \geq m$ to s, we obtain

$$c(s)\Psi(\Delta x(s)) - c(k)\Psi(\Delta x(k)) + \sum_{j=k+1}^{s} q(j) f(x(j)) = 0, \tag{5.1.125}$$

so for $s \geq k \geq m$,

$$\begin{aligned} c(k)\Psi(\Delta x(k)) &= c(s)\Psi(\Delta x(s)) - Q(s+1) f(x(s+1)) \\ &\quad + Q(k+1) f(x(k+1)) + \sum_{j=k+1}^{s} Q(j+1)\Delta f(x(j)). \end{aligned} \tag{5.1.126}$$

Note that

$$\sum_{j=k+1}^{\infty} Q(j+1) F(x(j+1), x(j)) \Delta x(j) < \infty, \tag{5.1.127}$$

since otherwise, (5.1.126) would imply $c(s)\Psi(\Delta x(s)) - Q(s+1)f(x(s+1)) \to -\infty$ as $s \to \infty$, which contradicts (5.1.124). Therefore, letting $s \to \infty$ in (5.1.126), we find for $k \geq m$,

$$\begin{aligned} c(k)\Psi(\Delta x(k)) &= a + Q(k+1)f(x(k+1)) \\ &\quad + \sum_{j=k+1}^{\infty} Q(j+1)F(x(j+1),x(j))\Delta x(j), \end{aligned} \tag{5.1.128}$$

where a denotes the finite limit

$$a = \lim_{s\to\infty}\left[c(s)\Psi(\Delta x(s)) - Q(s+1)f(x(s+1))\right] \geq 0. \tag{5.1.129}$$

Define

$$G_1(k) = \sum_{j=k}^{\infty} Q(j+1)F(x(j+1),x(j))\Delta x(j) \quad \text{for } k \geq m, \tag{5.1.130}$$

$$\begin{aligned} G_2(k) = \sum_{j=k}^{\infty} & Q(j+1)\Psi^{-1}(Q(j+1))\Psi^{-1}(f(x(j+1))) \\ & \times \Psi^{-1}\left(\frac{1}{c(j)}\right)F(x(j+1),x(j)) \quad \text{for } k \geq m. \end{aligned} \tag{5.1.131}$$

From (5.1.124) and (5.1.130) we see that $G_1(k) \geq G_2(k)$, and hence $G_2(k)$ is well defined for $k \geq m$ and is convergent, that is, $G_2(k) < \infty$ for $k \geq m$. Thus it follows from (5.1.128) that

$$c(k)\Psi(\Delta x(k)) \geq G_1(k+1) \geq G_2(k+1) \quad \text{for } k \geq m, \tag{5.1.132}$$

so

$$\Delta x(k) \geq \Psi^{-1}(G_2(k+1))\Psi^{-1}\left(\frac{1}{c(k)}\right) \quad \text{for } k \geq m. \tag{5.1.133}$$

Summing (5.1.133) from m to $k-1$, we obtain for $k \geq m$,

$$\begin{aligned} x(k) &\geq \sum_{j=m}^{k-1} \Psi^{-1}(G_2(j+1))\Psi^{-1}\left(\frac{1}{c(j)}\right) \\ &\geq \Psi^{-1}(G_2(k+1)) \sum_{j=m}^{k-1} \Psi^{-1}\left(\frac{1}{c(j)}\right) \\ &= \Psi^{-1}(G_2(k))C(k). \end{aligned} \tag{5.1.134}$$

Using this, (5.1.5), (5.1.121), and the fact that $\Delta x(k) \ge 0$ for $k \ge m$, we have for $k \ge m$,

$$\begin{aligned}\frac{Q(k+1)\Psi^{-1}(Q(k+1))\Psi^{-1}(f(x(k+1)))\Psi^{-1}(1/c(k))F(x(k+1),x(k))}{\Psi^{-1}(f(\Psi^{-1}(G_2(k))))}\\ \ge \lambda Q(k+1)\Psi^{-1}(Q(k+1))\Psi^{-1}(f(C(k)))\Psi^{-1}\left(\frac{1}{c(k)}\right).\end{aligned} \tag{5.1.135}$$

Since

$$\Delta G_2(k) = -Q(k+1)\Psi^{-1}(Q(k+1))\Psi^{-1}(f(x(k+1)))\Psi^{-1}\left(\frac{1}{c(k)}\right)F(x(k+1),x(k)), \tag{5.1.136}$$

inequality (5.1.135) for $k \ge m$ takes the form

$$\frac{-\Delta G_2(k)}{\Psi^{-1}(f(\Psi^{-1}(G_2(k))))} \ge \lambda Q(k+1)\Psi^{-1}(Q(k+1))\Psi^{-1}(f(C(k)))\Psi^{-1}\left(\frac{1}{c(k)}\right). \tag{5.1.137}$$

Summing (5.1.137) from m to $k-1$, letting $k \to \infty$, and using the fact that

$$\int_{G_2(k)}^{G_2(m)} \frac{dt}{\Psi^{-1}(f(\Psi^{-1}(t)))} \ge \sum_{j=m}^{k-1} \frac{-\Delta G_2(j)}{\Psi^{-1}(f(\Psi^{-1}(G_2(j))))}, \tag{5.1.138}$$

we obtain

$$\begin{aligned}\lambda \sum_{j=m}^{\infty} Q(j+1)\Psi^{-1}(Q(j+1))\Psi^{-1}(f(C(j)))\Psi^{-1}\left(\frac{1}{c(j)}\right)\\ \le \int_0^{G_2(m)} \frac{dt}{\Psi^{-1}(f(\Psi^{-1}(t)))} < \infty,\end{aligned} \tag{5.1.139}$$

which contradicts condition (5.1.122). This completes the proof. □

Remark 5.1.24. We also note that the results presented above for superlinear and sublinear oscillation can be extended to equations of type (5.1.2) when Ψ satisfies (I_3) and either condition (5.1.58) or for any appropriate sequence $\{y(k)\}$, Ψ satisfies $\sum^k \Psi(y(j)) \le \Psi(\sum^k y(j))$.

For the special case of equation (5.1.2)

$$\Delta\Big(c(k-1)\,|\Delta x(k-1)|^{\alpha-1}\Delta x(k-1)\Big) + q(k)\,|x(k)|^{\beta}x(k) = 0, \tag{5.1.140}$$

where α and β are positive constants, we extract the following immediate result from Theorem 5.1.23.

Corollary 5.1.25. *Suppose* $0 < \beta/\alpha^2 < 1$ *and* $Q(k) = \sum_{j=k}^{\infty} q(j) \geq 0$. *If*

$$\sum_{j=m}^{\infty} Q^{1+(1/\alpha)}(j+1)\left(\frac{C^{\beta}(j)}{c(j)}\right)^{1/\alpha} = \infty, \tag{5.1.141}$$

where $C(k) = \sum_{j=m}^{k-1} c^{-1/\alpha}(j) \to \infty$ *as* $k \to \infty$, *then equation (5.1.140) is oscillatory.*

5.1.4. Oscillation criteria—(I_2)

We will study the oscillatory and asymptotic behavior of equation (5.1.2) when Ψ satisfies (I_2).

Equation (5.1.2) is said to have "Property A" ("Property B") if every solution (every bounded solution) of equation (5.1.2) has any of the following properties:

(i) $\{x(k)\}$ is oscillatory,

(ii) $\{x(k)\}$ is weakly oscillatory, that is, $\{x(k)\}$ is nonoscillatory, while $\{\Delta x(k)\}$ is oscillatory,

(iii) $x(k)$ converges monotonically to zero as $k \to \infty$.

Assume that the function f satisfies the condition

$$\begin{aligned} &-f(-x) = f(x) \quad \forall x \in \mathbb{R} \quad (\text{i.e., } f \text{ is odd}) \\ &\text{and condition (5.1.5) holds.} \end{aligned} \tag{5.1.142}$$

Let $\{x(k)\}$ be a nonoscillatory solution of equation (5.1.2). If $x(k) > 0$ eventually, then it satisfies (5.1.2), and when $x(k) < 0$ eventually, we set $-x(k) = y(k)$. In this case $y(k)$ satisfies the equation

$$\Delta\Big(c(k-1)\big(\Delta y(k-1)\big)^{\alpha}\Big) - q(k)f\big(y(k)\big) = 0. \tag{5.1.143}$$

So, in the study of equation (5.1.2), we will consider both the equations (5.1.2) when $x(k) > 0$ eventually and (5.1.143) when $x(k) < 0$ eventually. Next, consider the condition

$$\begin{aligned} &f(-x) = f(x), \quad f(0) = 0 \quad (\text{i.e., } f \text{ is even}) \\ &\text{and condition (5.1.5) holds for } u, v \geq 0. \end{aligned} \tag{5.1.144}$$

In this case equation (5.1.2) is satisfied for both solutions $x(k) > 0$ and $x(k) < 0$ eventually.

Now we present the following result.

Theorem 5.1.26. *Let conditions (5.1.6), (5.1.15), and (5.1.142) hold. If*

$$\sum^{\infty}\left[\frac{1}{c(j)}\sum_{i=m}^{j} q(i)\right]^{1/\alpha} = \infty, \tag{5.1.145}$$

then equation (5.1.2) has Property B.

Proof. Let $\{x(k)\}$ be a bounded nonoscillatory solution of equation (5.1.2). We consider the following two cases.

Case 1. Suppose that $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. We have for $k \geq m$ the identity

$$\Delta\left[\frac{c(k-1)(\Delta x(k-1))^{\alpha}}{f(x(k))}\right] = -q(k) - \frac{c(k)F(x(k+1),x(k))(\Delta x(k))^{\alpha+1}}{f(x(k))f(x(k+1))}. \tag{5.1.146}$$

Now we distinguish the following two subcases.

Subcase a. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. Then, from (5.1.146) we obtain

$$\Delta\left[\frac{c(k-1)(\Delta x(k-1))^{\alpha}}{f(x(k))}\right] \leq -q(k) \quad \text{for } k \geq m_1. \tag{5.1.147}$$

Summing (5.1.147) from $m_1 + 1$ to k, we get

$$\frac{c(k)(\Delta x(k))^{\alpha}}{f(x(k+1))} \leq \frac{c(m_1)(\Delta x(m_1))^{\alpha}}{f(x(m_1+1))} - \sum_{j=m_1+1}^{k} q(j) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{5.1.148}$$

which contradicts the fact that the left-hand side of the above inequality is non-negative.

Subcase b. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. We proceed as in the proof of Theorem 5.1.1, and by using (5.1.5) and (5.1.10) we have

$$c(k)(\Delta x(k))^{\alpha} \leq c(m_2)(\Delta x(m_2))^{\alpha} - f(x(k+1)) \sum_{j=m_2+1}^{k} q(j). \tag{5.1.149}$$

Since $\{x(k)\}$ is eventually positive and $\{\Delta x(k)\}$ is eventually negative, $x(k) \to a$ as $k \to \infty$, where a is a positive constant. There exists $m_2 \geq m_1$ such that $x(k+1) \geq a$ for $k \geq m_2$. Consequently, it follows that $f(x(k+1)) \geq f(a) > 0$ for $k \geq m_2$. Hence from (5.1.149) we find

$$\begin{aligned} c(k)(\Delta x(k))^{\alpha} &\leq c(m_2)(\Delta x(m_2))^{\alpha} \\ &\quad - f(a) \sum_{j=m_2+1}^{k} q(j) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{5.1.150}$$

which is a contradiction to the fact that the left-hand side of the above inequality is positive.

Case 2. Suppose that $x(k) < 0$ for $k \geq m$. Set $-x(k) = y(k)$ in equation (5.1.2) and obtain (5.1.143). From equation (5.1.143) for $k \geq m$, we get

$$\Delta\left[\frac{c(k-1)(\Delta y(k-1))^{\alpha}}{f(y(k))}\right] = q(k) - \frac{c(k)F(y(k+1),y(k))(\Delta y(k))^{\alpha+1}}{f(y(k))f(y(k+1))}. \tag{5.1.151}$$

Next, we consider the following two subcases.

Subcase a. Suppose that $\Delta y(k) < 0$ for $k \geq m_1 \geq m$. Then from (5.1.151) we have

$$\Delta\left[\frac{c(k-1)(\Delta y(k-1))^{\alpha}}{f(y(k))}\right] \geq q(k) \quad \text{for } k \geq m_1. \tag{5.1.152}$$

Summing (5.1.152) from $m_1 + 1$ to k, we find

$$\frac{c(k)(\Delta y(k))^{\alpha}}{f(y(k+1))} \geq \frac{c(m_1)(\Delta y(m_1))^{\alpha}}{f(y(m_1+1))} + \sum_{j=m_1+1}^{k} q(j) \geq \sum_{j=m_1+1}^{k} q(j). \tag{5.1.153}$$

Since $\Delta y(k) < 0$ and $y(k) > 0$ for $k \geq m_1$, $y(k) \to a > 0$ as $k \to \infty$ and $y(k+1) \geq a$ for $k \geq m_1$. Consequently, $f(y(k+1)) \geq f(a) > 0$ for $k \geq m_1$. Hence, from (5.1.153) we obtain

$$c(k)(\Delta y(k))^{\alpha} \geq f(y(k+1)) \sum_{j=m_1+1}^{k} q(j) \geq f(a) \sum_{j=m_1+1}^{k} q(j). \tag{5.1.154}$$

Therefore we have

$$-\Delta y(k) \geq f^{1/\alpha}(a)\left[\frac{1}{c(k)} \sum_{j=m_1+1}^{k} q(j)\right]^{1/\alpha}. \tag{5.1.155}$$

Summing (5.1.155) from $m_1 + 1$ to k, we find

$$y(m_1+1) \geq y(k+1) + f^{1/\alpha}(a) \sum_{j=m_1+1}^{k} \left[\frac{1}{c(j)} \sum_{i=m_1+1}^{j} q(i)\right]^{1/\alpha} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \tag{5.1.156}$$

which contradicts the fact that the left-hand side of inequality (5.1.156) is bounded.

Subcase b. Suppose that $\Delta y(k) \geq 0$ for $k \geq m_1 \geq m$. Now, summing equation (5.1.143) from $m_1 + 1$ to k we get for $k \geq m_1$,

$$\begin{aligned} c(k)(\Delta y(k))^{\alpha} &= c(m_1)(\Delta y(m_1))^{\alpha} + \sum_{j=m_1+1}^{k} q(j) f(y(j)) \\ &\geq f(y(m_1+1)) \sum_{j=m_1+1}^{k} q(j). \end{aligned} \tag{5.1.157}$$

Thus

$$\Delta y(k) \geq f^{1/\alpha}(y(m_1+1)) \left[\frac{1}{c(k)} \sum_{j=m_1+1}^{k} q(j) \right]^{1/\alpha} \quad \text{for } k \geq m_1. \tag{5.1.158}$$

Summing (5.1.158) from $m_1 + 1$ to k, we have

$$\begin{aligned} y(k) &\geq y(m_1+1) + f^{1/\alpha}(y(m_1+1)) \sum_{j=m_1+1}^{k} \left[\frac{1}{c(j)} \sum_{i=m_1+1}^{j} q(i) \right]^{1/\alpha} \\ &\longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{5.1.159}$$

which is a contradiction to the fact that $y(k)$ is bounded.

This completes the proof. □

Example 5.1.27. Consider the difference equation

$$\Delta\left(k^{5/3}(k-1)^{2/3}(\Delta x(k-1))^{2/3}\right) + kx(k) = 0 \quad \text{for } k \geq 2. \tag{5.1.160}$$

Here, $q(k) = k$ and $f(x) = x$. Clearly, $\sum^{\infty} q(j) = \sum^{\infty} j = \infty$ and

$$\begin{aligned} \sum^{\infty} \left[\frac{1}{c(j)} \sum_{i=m}^{j} q(i) \right]^{1/\alpha} &= \sum^{\infty} \left[\frac{1}{(j+1)^{5/3} j^{2/3}} \sum_{i=m}^{j} i \right]^{3/2} \\ &= \sum^{\infty} \frac{(j+m)^{3/2}(j-m+1)^{3/2}}{(j+1)^{5/2} j} \\ &\geq \sum^{\infty} \frac{(j-m+1)^{3/2}}{(j+1)^2} \\ &= \infty. \end{aligned} \tag{5.1.161}$$

All conditions of Theorem 5.1.26 are satisfied, and hence equation (5.1.160) has Property B. In fact, equation (5.1.160) has a solution $x(k) = -1/k \to 0$ monotonically as $k \to \infty$.

Example 5.1.28. Consider the difference equation

$$\Delta\Big(k\big(\Delta x(k-1)\big)^{2/3}\Big) + q(k)x(k) = 0 \quad \text{for } k \geq 1, \tag{5.1.162}$$

where

$$q(k) = \begin{cases} 1 & \text{when } k \text{ is odd,} \\ \dfrac{1}{2} & \text{when } k \text{ is even.} \end{cases} \tag{5.1.163}$$

Clearly, all conditions of Theorem 5.1.26 are satisfied, and hence equation (5.1.162) has Property B. Equation (5.1.162) has a nonoscillatory solution

$$x(k) = \begin{cases} -1 & \text{if } k \text{ is odd,} \\ -2 & \text{if } k \text{ is even.} \end{cases} \tag{5.1.164}$$

We note that $\{\Delta x(k)\}$ is given by

$$\Delta x(k) = \begin{cases} -1 & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even,} \end{cases} \tag{5.1.165}$$

and is oscillatory.

Theorem 5.1.29. *In Theorem 5.1.26 the condition (5.1.6) can be replaced by the condition (5.1.60) and*

$$\sum^{\infty}\left[\frac{1}{c(j)}\sum_{i=j+1}^{\infty} q(i)\right]^{1/\alpha} = \infty. \tag{5.1.166}$$

Proof. The condition (5.1.6) is used only in the proof of Case 1 in Theorem 5.1.26. We will consider this case, and the following two subcases.

Subcase a. Suppose that $\Delta x(k) \geq 0$ for $k \geq m_1 \geq m$. The proof of this subcase is similar to that of Corollary 5.1.9, and hence we omit the details.

Subcase b. Suppose that $\Delta x(k) < 0$ for $k \geq m_1 \geq m$. As in the proof of Subcase b from Case 1 in Theorem 5.1.26, we obtain

$$c(k)\big(\Delta x(k)\big)^{\alpha} \leq c(m_2)\big(\Delta x(m_2)\big)^{\alpha} - f(a)\sum_{j=m_2+1}^{k} q(j). \tag{5.1.167}$$

Therefore we obtain

$$c(m_2)\big(\Delta x(m_2)\big)^{\alpha} \geq f(a)\sum_{j=m_2+1}^{k} q(j), \tag{5.1.168}$$

and hence for $k \geq m_2 + 1$,

$$c(k)\big(\Delta x(k)\big)^{\alpha} \geq f(a) \sum_{j=k+1}^{\infty} q(j), \tag{5.1.169}$$

so

$$\Delta x(k) \leq -\big(f(a)\big)^{1/\alpha} \left[\frac{1}{c(k)} \sum_{j=k+1}^{\infty} q(j) \right]^{1/\alpha}. \tag{5.1.170}$$

Summing (5.1.170) from $m_2 + 1$ to k we have

$$x(k+1) \leq x(m_2+1) - \big(f(a)\big)^{1/\alpha} \sum_{j=m_2+1}^{k} \left[\frac{1}{c(j)} \sum_{i=j+1}^{\infty} q(i) \right]^{1/\alpha} \longrightarrow -\infty \tag{5.1.171}$$

as $k \to \infty$, which contradicts the fact that $x(k) > 0$ eventually.

This completes the proof. □

Next, we say that equation (5.1.2) has "Property C" if every solution $\{x(k)\}$ of equation (5.1.2) has any of the properties (i)–(iii) of Property A, or

(iv) $|x(k)|$ diverges monotonically as $k \to \infty$.

From the proof of Theorem 5.1.26, the following result is immediate.

Theorem 5.1.30. *Let the hypotheses of Theorem 5.1.26 hold. Then equation (5.1.2) has Property C.*

Example 5.1.31. Consider the difference equation

$$\Delta\Big(k\big(\Delta x(k-1)\big)^{\alpha}\Big) + \frac{1}{k}x(k) = 0 \quad \text{for } k \geq 2. \tag{5.1.172}$$

Here $q(k) = 1/k$ for $k \geq 2$ and $f(x) = x$. We note that all conditions of Theorem 5.1.30 are satisfied, and hence equation (5.1.172) has Property C. One such solution of equation (5.1.172) is given by $x(k) = -k$ which satisfies (iv) in Property C.

Also, from the proof of Theorems 5.1.8 and 5.1.29 the following result is immediate.

Theorem 5.1.32. *In addition to the assumptions of Theorem 5.1.29 assume that*

$$\int^{+\infty} \frac{du}{f^{1/\alpha}(u)} < \infty, \qquad \int^{-\infty} \frac{du}{f^{1/\alpha}(u)} < \infty. \tag{5.1.173}$$

Then equation (5.1.2) has Property C.

Example 5.1.33. Consider the difference equation

$$\Delta\Big(k\big(\Delta x(k-1)\big)^2\Big) + q(k)x^3(k) = 0 \quad \text{for } k \in \mathbb{N}, \tag{5.1.174}$$

where

$$q(k) = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ \dfrac{1}{8} & \text{if } k \text{ is even}. \end{cases} \tag{5.1.175}$$

Equation (5.1.174) has a nonoscillatory solution

$$x(k) = \begin{cases} -1 & \text{if } k \text{ is odd}, \\ -2 & \text{if } k \text{ is even}, \end{cases} \tag{5.1.176}$$

Note that $\{\Delta x(k)\}$ is oscillatory. It is easy to check that all conditions of Theorem 5.1.32 are satisfied, and hence equation (5.1.174) has Property C.

Finally, we present the following result for equation (5.1.2) when condition (5.1.144) is satisfied.

Theorem 5.1.34. *Suppose that conditions (5.1.6) and (5.1.144) are satisfied. Then equation (5.1.2) has Property A.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (5.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$ (the proof for the case when $x(k) < 0$ for $k \geq m$ is similar). From (5.1.144) we obtain $f(x) > 0$ for $x \neq 0$. The rest of the proof is similar to that of Theorem 5.1.1 and hence is omitted. □

Example 5.1.35. The difference equation

$$\Delta\left(\frac{k^2}{k-1}\big(\Delta x(k-1)\big)^2\right) + \frac{k}{k-1}x^2(k) = 0 \quad \text{for } k \geq 2 \tag{5.1.177}$$

has nonoscillatory solutions $x_1(k) = -1/k$ and $x_2(k) = 1/k$. Both tend to zero monotonically as $k \to \infty$, while the difference equation

$$\Delta\left(\frac{k^2(k-1)}{(2k-1)^2}\big(\Delta x(k-1)\big)^2\right) + \frac{k}{k-1}x^2(k) = 0 \quad \text{for } k \geq 2 \tag{5.1.178}$$

has an oscillatory solution $x(k) = (-1)^k/k$. We note that all conditions of Theorem 5.1.34 are satisfied, and hence we conclude that equations (5.1.177) and (5.1.178) have Property A.

Remark 5.1.36. (i) Condition (5.1.142) can be replaced by condition (5.1.5) and $xf(x) > 0$ for $x \neq 0$. The results involving this condition remain valid.

(ii) Condition (5.1.144) can be replaced by

$$\begin{gathered} xf(x) \neq 0 \quad \text{for } x \neq 0, \qquad f(0) \geq 0, \\ f(u) - f(v) = \begin{cases} F(u,v)(u-v) & \text{for } u, v \geq 0, \\ -F(u,v)(u-v) & \text{for } u, v \leq 0, \end{cases} \end{gathered} \tag{5.1.179}$$

where $F(u,v)$ is a nonnegative function, and the obtained results which involve condition (5.1.144) remain valid.

(iii) The function Ψ which satisfies (I_2) can also be replaced by a function that satisfies a condition of type (5.1.179).

(iv) Other results similar to those obtained when the function Ψ satisfies (I_1) can also be obtained when the function Ψ satisfies (I_2). The details are left to the reader.

5.1.5. Oscillation criteria for damped equations

In Section 3.9 we have established some oscillation criteria for a special case of equation (5.1.3), namely, (3.9.1). Here we will extend (without proofs) some of these results to equation (5.1.3), where the function Ψ satisfies either (I_1) or (I_3), $p(k) \geq 0$, and $q(k) > 0$ for $k \geq m$. In fact, Lemmas 3.9.1 and 3.9.2 and Theorem 3.9.3 will take, respectively, the following forms.

Lemma 5.1.37. *Assume that $c(k) > p(k)$ for $k \geq m \in \mathbb{N}$ and*

$$\sum_{n=m}^{\infty} \Psi^{-1}\left[\frac{1}{c(n)} \prod_{j=m}^{n-1}\left(1 - \frac{p(j)}{c(j)}\right)\right] = \infty. \tag{5.1.180}$$

If $\{x(k)\}$ is a nonoscillatory solution of equation (5.1.3), then $x(k)\Delta x(k) > 0$ eventually.

Lemma 5.1.38. *Let conditions (5.1.4) and (5.1.5) hold. If the inequality*

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)f\big(x(k+1)\big) \leq 0 \tag{5.1.181}$$

has an eventually positive solution, then so does the difference equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)f\big(x(k+1)\big) = 0. \tag{5.1.182}$$

Theorem 5.1.39. *Let $c(k) > p(k)$ for $k \geq m \in \mathbb{N}$ and conditions (5.1.5) and (5.1.180) hold. If equation (5.1.182) is oscillatory, then so is equation (5.1.3).*

Remark 5.1.40. We note that the damping term in equation (5.1.3) preserves the oscillatory character of the undamped equation (5.1.182). Now, according to Theorem 5.1.39 one can derive many oscillation criteria for equation (5.1.3) by applying the above obtained results to equation (5.1.182). The formulation of such results are left to the reader.

For illustration, we consider the following examples.

Example 5.1.41. Consider the damped difference equation

$$\Delta\Big(k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)\Big)+\frac{1}{2}k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)+2^{\alpha-1}(3k+2)\,|\,x(k+1)\,|^{\beta-1}x(k+1) = 0 \quad \text{for } k\in\mathbb{N}, \tag{5.1.183}$$

where $\alpha \geq 1$ and $\beta > 0$. Here $c(k) = k$ and $p(k) = k/2$. It is easy to see that the associated undamped equation

$$\Delta\Big(k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)\Big) + 2^{\alpha-1}(3k+2)\,|\,x(k+1)\,|^{\beta-1}x(k+1) = 0 \quad \text{for } k\in\mathbb{N} \tag{5.1.184}$$

is oscillatory by Theorem 5.1.1. All conditions of Theorem 5.1.39 are satisfied, and hence equation (5.1.183) is oscillatory. One such solution is given by $x(k) = (-1)^k$.

Example 5.1.42. The damped difference equation

$$\Delta\Big(k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)\Big)-\frac{1}{2}k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)+\frac{k-2}{2(k+1)}x(k+1) = 0 \quad \text{for } k>2, \tag{5.1.185}$$

where $\alpha \geq 1$, has a nonoscillatory solution $x(k) = k$. Only the condition on the sign of $\{p(k)\}$ is violated.

Example 5.1.43. The damped difference equation

$$\Delta\Big(k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)\Big)+\frac{1}{2}k\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)-\frac{(k+2)}{2(k+1)}x(k+1) = 0 \quad \text{for } k\in\mathbb{N} \tag{5.1.186}$$

with $\alpha \geq 1$ has a nonoscillatory solution $x(k) = k$. Only the condition on the sign of $\{q(k)\}$ is violated.

5.2. Monotone solutions of nonlinear difference equations

In this section we will consider the nonlinear difference equations

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)f\big(x[g(k+1)]\big) = 0, \tag{5.2.1}$$

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + p(k)\Psi(\Delta x(k)) + q(k)f\big(x[g(k+1)]\big) = 0, \tag{5.2.2}$$

where

(i) $\{c(k)\}$ and $\{q(k)\}$ are eventually positive real sequences,
(ii) $\{p(k)\}$ is a sequence of real numbers,
(iii) $\{g(k)\}$ is an increasing sequence of integers such that $\lim_{k\to\infty} g(k) = \infty$,
(iv) $f \in C(\mathbb{R},\mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$,
(v) $\Psi \in C(\mathbb{R},\mathbb{R})$ and Ψ satisfies either (I_1) or (I_2) or (I_3).

Here, we will establish an existence criterion of an eventually positive monotone solution of the equation (5.2.2). We will also offer a criterion for the existence of monotone solutions of equation (5.2.1) provided that the function Ψ satisfies (I_2).

We will need the following result.

Theorem 5.2.1 (Knaster-Tarski fixed point theorem). *Let X be a partially ordered Banach space with ordering $\leq$. Let S be a subset of X with the following properties: the infimum of S belongs to X and every nonempty subset of S has a supremum which belongs to S. Let $T : S \to S$ be an increasing mapping, that is, $x \leq y$ implies $Tx \leq Ty$. Then T has a fixed point in S.*

5.2.1. Existence of positive monotone solutions of (5.2.2)

In this subsection we suppose that Ψ satisfies (I_1). In addition, we will assume that

(vi) $\{\rho(k)\}$ is a positive solution of the difference equation

$$c(k)\Delta\rho(k) - p(k)\rho(k+1) = 0. \tag{5.2.3}$$

Now we present the following existence result.

Theorem 5.2.2. *Suppose condition (5.1.5) holds,*

$$c(k) - p(k) > 0 \quad \textit{for } k \geq m \in \mathbb{N}, \tag{5.2.4}$$

$$\sum_{n=m}^{\infty} \Psi^{-1}\left[\frac{1}{c(n)\rho(n)} \sum_{j=m}^{n-1} q(j)\rho(j+1)\right] < \infty, \tag{5.2.5}$$

$$\sum_{j=m}^{\infty} q(j)\rho(j+1) = \infty. \tag{5.2.6}$$

Then equation (5.2.2) has a positive solution which is monotonically decreasing.

Proof. Let a be an arbitrary, but fixed, positive number. Condition (5.2.5) implies that there exists $k \geq m$ such that

$$\sum_{n=k}^{\infty} \Psi^{-1}\left[\frac{1}{c(n)\rho(n)} \sum_{j=N}^{n-1} q(j)\rho(j+1)\right] \leq \frac{a}{2\Psi^{-1}(f(a))} \quad \text{for } k \geq N. \tag{5.2.7}$$

Let B_N be the Banach space of all real sequences $x = \{x(k)\}$, $k \geq n$, with supnorm $\|x\| = \sup_{k \geq N} |x(k)|$. Define a partial ordering on the Banach space B_N as follows: for given $x, y \in B_N$, $x \leq y$ means that $x(k) \leq y(k)$ for $k \geq N$. Let

$$S = \left\{x \in B_N : \frac{a}{2} \leq x(k) \leq a,\ k \geq N\right\}. \tag{5.2.8}$$

We also define $T : S \to B_N$ by

$$Tx(k) = \frac{a}{2} + \sum_{n=k}^{\infty} \Psi^{-1}\left[\frac{1}{c(n)\rho(n)} \sum_{j=N}^{n-1} q(j)\rho(j+1) f(x[g(j+1)])\right] \quad \text{for } k \geq N. \tag{5.2.9}$$

For $x \in S$, we have $Tx(k) \geq a/2$, and by (5.2.7) we find

$$\begin{aligned} Tx(k) &\leq \frac{a}{2} + \sum_{n=k}^{\infty} \Psi^{-1}\left[\frac{1}{c(n)\rho(n)} \sum_{j=N}^{n-1} q(j)\rho(j+1) f(a)\right] \\ &\leq \frac{a}{2} + \Psi^{-1}(f(a)) \frac{a}{2\Psi^{-1}(f(a))} \\ &= a. \end{aligned} \tag{5.2.10}$$

Thus $TS \subseteq S$. Further, for all $x, y \in S$ with $x \geq y$, we find $Tx \geq Ty$ and therefore T is an increasing mapping. Hence, by Theorem 5.2.1 there exists $x \in S$ such that $Tx = x$, that is,

$$x(k) = \frac{a}{2} + \sum_{n=k}^{\infty} \Psi^{-1}\left[\frac{1}{c(n)\rho(n)} \sum_{j=N}^{n-1} q(j)\rho(j+1) f(x[g(j+1)])\right]. \tag{5.2.11}$$

Clearly, the sequence $\{x(k)\}$ given by (5.2.11) is a positive solution of equation (5.2.2).

Now we will show that the solution $\{x(k)\}$ given by (5.2.11) is nonincreasing, that is, $\Delta x(k) \leq 0$ for $k \geq N$. For this, we consider the following two cases.

Case 1. Assume that $\{\Delta x(k)\}$ is oscillatory.

(a_1) Suppose there exists $N_1 \geq N$ such that $\Delta x(N_1) < 0$. Let $k = N_1$ in equation (5.2.2) and then multiply the resulting equation by $\Delta x(N_1)$ to obtain

$$\begin{aligned}
&\Delta\big(c(N_1)\Psi(\Delta x(N_1))\big)\Delta x(N_1) \\
&\quad = -p(N_1)\Psi(\Delta x(N_1)) - q(N_1) f(x[g(N_1+1)])\Delta x(N_1) \\
&\quad > -p(N_1)\Psi(\Delta x(N_1))\Delta x(N_1).
\end{aligned} \tag{5.2.12}$$

Using (5.2.4) we get

$$\begin{aligned}
&c(N_1+1)\Psi(\Delta x(N_1+1))\Delta x(N_1) \\
&\qquad > [c(N_1) - p(N_1)]\Psi(\Delta x(N_1))\Delta x(N_1) > 0,
\end{aligned} \tag{5.2.13}$$

which implies that $\Delta x(N_1+1) < 0$. By induction, we obtain $\Delta x(k) < 0$ for all $k \geq N_1$, contradicting the assumption that $\{\Delta x(k)\}$ oscillates.

(a_2) Suppose there exists N_1 such that $\Delta x(N_1) = 0$. Then letting $k = N_1$ in equation (5.2.2) leads to

$$c(N_1+1)\Psi(\Delta x(N_1+1)) = -q(N_1) f(x[g(N_1+1)]) < 0, \tag{5.2.14}$$

which implies that $\Delta x(N_1+1) < 0$, that is, Case 1(a_1). We have seen that this contradicts the assumption that $\{\Delta x(k)\}$ is oscillatory.

Case 2. Assume that $\Delta x(k) > 0$ for $k \geq N \geq m$. Choose $N_1 \geq N$ so $\Delta x[g(k)] > 0$ for $k \geq N_1$. Put $u(k) = \rho(k)v(k)/f(x[g(k)])$ for $k \geq N_1$, where $v(k) = c(k)\Psi(\Delta x(k))$. Clearly, $u(k) > 0$ for $k \geq N_1$. By a direct computation, we find for $k \geq N_1$,

$$\begin{aligned}
\Delta u(k) &= -q(k)\rho(k+1) - \frac{v(k)\rho(k)F(x[g(k+1)], x[g(k)])\Delta x[g(k)]}{f(x[g(k)]) f(x[g(k+1)])} \\
&\leq -q(k)\rho(k+1),
\end{aligned} \tag{5.2.15}$$

which on summing from N_1 to k provides

$$\sum_{j=N_1}^{k} q(j)\rho(j+1) \leq u(N_1) - u(k+1) \leq u(N_1) < \infty. \tag{5.2.16}$$

Letting $k \to \infty$ in (5.2.16) we get a contradiction to (5.2.6).

This completes the proof. $\square$

5.2.2. Existence of monotone solutions of (5.2.1)

Throughout this subsection we will assume that the function Ψ satisfies (I_2).

Now we present the following results.

Theorem 5.2.3. *Suppose that conditions (5.1.5) and (5.1.6) hold,*

$$\sum^{\infty}\left[\frac{1}{c(n)}\sum_{j=m}^{n-1}q(j)\right]^{1/\alpha}<\infty, \tag{5.2.17}$$

$$g(k)\geq k \quad \text{for } k\geq m \text{ for all large } m\in\mathbb{N}. \tag{5.2.18}$$

Then equation (5.2.1) has an eventually negative solution which is monotonically increasing.

Proof. Condition (5.2.17) implies that there exists $N\geq m$ so that

$$\sum_{n=k}^{\infty}\left[\frac{1}{c(n)}\sum_{j=N}^{n-1}q(j)\right]^{1/\alpha}\leq\frac{a}{\left[-f(-2a)\right]^{1/\alpha}}\quad\text{for } k\geq N, \tag{5.2.19}$$

where a is an arbitrary, but fixed, positive real number. Let B_N be the Banach space of all bounded sequences $x=\{x(k)\}$, $k\geq N$, with supnorm $\|x\|=\sup_{k\geq N}|x(k)|$. Define a partial ordering on B_N as follows: for given $x,y\in B_N$, $x\leq y$ means that $x(k)\leq y(k)$ for $k\geq N$. Let

$$S=\left\{x\in B_N:-2a\leq x(k)\leq -a,\ k\geq N\right\}. \tag{5.2.20}$$

We also define $T:S\to B_N$ by

$$Tx(k)=-a-\sum_{n=k}^{\infty}\left[-\frac{1}{c(n)}\sum_{j=N}^{n-1}q(j)f\left(x[g(j+1)]\right)\right]^{1/\alpha}. \tag{5.2.21}$$

For all $x\in S$, we have $Tx(k)\leq -a$, and using (5.1.5) and (5.2.19) we get

$$\begin{aligned}
Tx(k)&\geq -a-\sum_{n=k}^{\infty}\left[-\frac{1}{c(n)}\sum_{j=N}^{n-1}q(j)f(-2a)\right]^{1/\alpha}\\
&=-a-\left[-f(-2a)\right]^{1/\alpha}\sum_{n=k}^{\infty}\left[\frac{1}{c(n)}\sum_{j=N}^{n-1}q(j)\right]^{1/\alpha}\\
&\geq -a-\left[-f(-2a)\right]^{1/\alpha}\frac{a}{\left[-f(-2a)\right]^{1/\alpha}}\\
&=-2a.
\end{aligned} \tag{5.2.22}$$

Thus $TS\subseteq S$. Further, for $x,y\in S$ with $x\geq y$, in view of (5.1.5), we find $Tx\geq Ty$ and therefore T is an increasing mapping. Hence by Theorem 5.2.1, there exists

$x \in S$ such that $Tx = x$, that is,

$$x(k) = -a - \sum_{n=k}^{\infty} \left[-\frac{1}{c(n)} \sum_{j=N}^{n-1} q(j) f\big(x[g(j+1)]\big) \right]^{1/\alpha} \quad \text{for } k \geq N. \tag{5.2.23}$$

Clearly, the sequence $\{x(k)\}$ given by (5.2.23) is a negative solution of equation (5.2.1) which is monotonically increasing, since

$$\Delta x(k) = \left[-\frac{1}{c(k)} \sum_{j=N}^{k-1} q(j) f\big(x[g(j+1)]\big) \right]^{1/\alpha} \geq 0. \tag{5.2.24}$$

This completes the proof. □

5.3. Bounded, unbounded, and monotone properties

In this section we will study positive increasing and decreasing solutions of second order nonlinear difference equations of the form

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) = q(k) f\big(x(k+1)\big), \tag{5.3.1}$$

where

(i) $\{c(k)\}$ and $\{q(k)\}$ are positive real sequences for $k \in \mathbb{N}$,
(ii) $f \in C(\mathbb{R}, \mathbb{R})$ with $x f(x) > 0$ for $x \neq 0$,
(iii) $\Psi \in C(\mathbb{R}, \mathbb{R})$ satisfies (I_1) given in Section 5.1.

5.3.1. Positive decreasing solutions

The qualitative behavior of solutions $\{x(k)\}$ of (5.3.1) satisfying $x(k)\Delta x(k) < 0$ for $k \in \mathbb{N}$ is investigated by giving necessary and sufficient conditions and hence fully characterizing the existence of monotone and zero-converging solutions. These criteria involve only the asymptotic behavior of the sequences $\{c(k)\}$ and $\{q(k)\}$.

It is easy to show that every solution of equation (5.3.1) is eventually monotone. As in Section 1.16, the set of solutions of equation (5.3.1) is divided into two classes with respect to their monotonicity properties. Here, we recall these results in a slightly more general form together with simple proofs which differ from those quoted in Section 1.16.

Lemma 5.3.1. *Denote the set of nontrivial solutions of equation (5.3.1) by S. Then any $x \in S$ is eventually monotone and belongs to one of the following two classes:*

$$\begin{aligned} M^{+} &= \{x \in S : \exists m \in \mathbb{N} \text{ such that } x(k)\Delta x(k) > 0 \text{ for } k \geq m\}, \\ M^{-} &= \{x \in S : x(k)\Delta x(k) < 0 \text{ for } k \in \mathbb{N}\}. \end{aligned} \tag{5.3.2}$$

Proof. Let $\{x(k)\}$ be a nontrivial solution of equation (5.3.1) and consider the sequence $\{F(k)\}$ given by $F(k) = c(k)x(k)\Psi(\Delta x(k))$. Then

$$\begin{aligned}\Delta F(k) &= \Delta\big(c(k)\Psi(\Delta x(k))\big)x(k+1) + c(k)\Psi(\Delta x(k))\Delta x(k)\\ &= q(k)f(x(k+1))x(k+1) + c(k)\Psi(\Delta x(k))\Delta x(k)\\ &\geq 0.\end{aligned} \tag{5.3.3}$$

Thus $\{F(k)\}$ is a nondecreasing sequence. Since $\{x(k)\}$ is not eventually constant, there are two possibilities to consider.

(1) There exists $m \in \mathbb{N}$ such that $F(k) > 0$ for $k \geq m$.

(2) $F(k) < 0$ for $m \geq 1$.

In the first case it follows that $x(k)\Delta x(k) > 0$ for $k \geq m$, and so $\{x(k)\}$ is eventually monotonic. In the second case, we see that $x(k)\Delta x(k) < 0$ for $k \geq 1$. Without loss of generality assume that $x(1) > 0$ and $\Delta x(1) < 0$. We claim that $\{x(k)\}$ is a positive decreasing solution of (5.3.1). Now, if $x(2) < 0$, then we obtain from equation (5.3.1)

$$\Delta x(2) = \Psi^{-1}\left(\frac{c(1)}{c(2)}\Psi(\Delta x(1)) + \frac{q(1)}{c(2)}f(x(2))\right) < 0, \tag{5.3.4}$$

which is a contradiction so that the assertion follows. □

In this subsection we will study the class M^-. As in Section 1.16, in view of Lemma 5.3.1, the class M^- can be divided into the following two subclasses:

$$\begin{aligned}M_B^- &= \Big\{x \in S : x(k)\Delta x(k) < 0 \text{ for } m \in \mathbb{N},\ \lim_{k\to\infty} x(k) = \ell \neq 0\Big\},\\ M_0^- &= \Big\{x \in S : x(k)\Delta x(k) < 0 \text{ for } m \in \mathbb{N},\ \lim_{k\to\infty} x(k) = 0\Big\}.\end{aligned} \tag{5.3.5}$$

We will show that a crucial rôle in our consideration is played by

$$\begin{gathered}Y_1 = \lim_{k\to\infty}\sum_{n=1}^{k}\Psi^{-1}\left(\sum_{j=1}^{n}\frac{q(j)}{c(n)}\right), \qquad Y_2 = \lim_{k\to\infty}\sum_{n=1}^{k}\Psi^{-1}\left(\sum_{j=n}^{k}\frac{q(j)}{c(n)}\right),\\ Y_3 = \lim_{k\to\infty}\sum_{n=1}^{k}\Psi^{-1}\left(\frac{1}{c(n)}\right), \qquad Y_4 = \lim_{k\to\infty}\sum_{n=1}^{k}q(n).\end{gathered} \tag{5.3.6}$$

Concerning the asymptotic behavior of solutions in the class M^{-1}, the following cases are significant and will be examined in the sequel:

(i_1) $Y_2 = \infty$,

(i_2) $Y_1 = \infty$ and $Y_2 < \infty$,

(i_3) $Y_1 < \infty$ and $Y_2 < \infty$.

Next, we present some relationships between the convergence or divergence of Y_i, $i \in \{1, 2, 3, 4\}$.

Lemma 5.3.2. *The following hold.*

(a_1) *If $Y_1 < \infty$, then $Y_3 < \infty$.*
(a_2) *If $Y_2 < \infty$, then $Y_4 < \infty$.*
(a_3) *If $Y_2 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.*
(a_4) *If $Y_1 = \infty$, then $Y_3 = \infty$ or $Y_4 = \infty$.*
(a_5) *$Y_1 < \infty$ and $Y_2 < \infty$ if and only if $Y_3 < \infty$ and $Y_4 < \infty$.*

Proof. To show (a_1), let $m_1 \in (0, k)$. Since

$$\begin{aligned}\sum_{n=1}^{k} \Psi^{-1}\left(\sum_{j=1}^{n} \frac{q(j)}{c(n)}\right) > \sum_{n=1}^{m_1} \Psi^{-1}\left(\sum_{j=1}^{n} \frac{q(j)}{c(n)}\right)& \\ + \left[\Psi^{-1}\left(\sum_{j=1}^{m_1} q(j)\right)\right]\left[\sum_{n=m_1}^{k} \Psi^{-1}\left(\frac{1}{c(n)}\right)\right]&,\end{aligned} \tag{5.3.7}$$

the assertion follows, and (a_2) follows as in (a_1). Next, (a_3) and (a_4) follow from the inequalities

$$\begin{aligned}\sum_{n=1}^{k} \Psi^{-1}\left(\sum_{j=n}^{k} \frac{q(j)}{c(n)}\right) &\le \Psi^{-1}\left(\sum_{j=1}^{k} q(j)\right) \sum_{n=1}^{k} \Psi^{-1}\left(\frac{1}{c(n)}\right), \\ \sum_{n=1}^{k} \Psi^{-1}\left(\sum_{j=1}^{n} \frac{q(j)}{c(n)}\right) &\le \Psi^{-1}\left(\sum_{j=1}^{k} q(j)\right) \sum_{n=1}^{k} \Psi^{-1}\left(\frac{1}{c(n)}\right).\end{aligned} \tag{5.3.8}$$

Finally, (a_5) immediately follows from (a_1)–(a_4). □

5.3.1.1. The discrete and continuous case

Some basic problems arising in the asymptotic theory of differential equations are the continuability to infinity of solutions and the uniqueness of solutions to the initial conditions. In contrast to the continuous case, for difference equation (5.3.1) every solution is continuable to infinity and there is only one solution satisfying $x(m) = x^0$ and $\Delta x(m) = x^1$ for some $m \in \mathbb{N}$. Thus, solutions of equation (5.3.1) continuously depend on the initial conditions, since the uniqueness of solutions by itself implies the continuous dependence of solutions on initial conditions.

Concerning the asymptotic behavior of solutions, the following differences are of particular interest. Differential equations can possess solutions that either become identically zero for all large t or are noncontinuable to infinity. In the study of qualitative behavior of solutions of

$$\left(c(t)x'(t)\right)' = q(t)f\left(x(t)\right), \tag{5.3.9}$$

$$\left(c(t)\Psi\left(x'(t)\right)\right)' = q(t)f\left(x(t)\right), \tag{5.3.10}$$

where $c, q \in C([t_0, \infty), \mathbb{R}^+)$ and $\Psi, f \in C(\mathbb{R}, \mathbb{R})$ such that Ψ satisfies (I_1) and $xf(x) > 0$ for $x \neq 0$, such facts play a crucial rôle, and concepts of singular solutions of the first or second kind are used (see, e.g., [174]). It is easy to show that every solution of (5.3.1) is continuable to infinity, that is, no singular solutions of second kind exist for (5.3.1). Analogously, if for some $m \in \mathbb{N}$ we have $x(m) > 0$ ($x(m) < 0$), $x(m+1) = 0$, then $x(m+2) < 0$ ($x(m+2) > 0$), that is, no singular solutions of first kind exist for equation (5.3.1) as well.

Other discrepancies consist in these facts. For equation (5.3.10), the class M^+ (M^-) can be empty; with regard to the class M^+ see, for example, [174, Corollary 17.4], concerning the class M^- see, for example, [174, Corollary 17.3]. This is not true for equation (5.3.1), since every solution of (5.3.1) is continuable to infinity. In view of Lemma 5.3.1, any solution $\{x(k)\}$ satisfying the initial condition $x(1)\Delta x(1) > 0$ belongs to M^+, and so in the discrete case M^+ is nonempty. For instance, the differential equation $x''(t) = x^2(t)\operatorname{sgn} x(t)$ does not have solutions in the class M^+ in view of the quoted results in [174], whereas the corresponding difference equation $\Delta^2 x(k) = x^2(k+1)\operatorname{sgn} x(k+1)$ has positive increasing solutions.

Concerning the class M^-, for equations (5.3.9) and (5.3.10) such a class can be empty, as the equation

$$x''(t) = \sqrt{|x(t)|} \tag{5.3.11}$$

shows (see, e.g., the quoted results in [174]). This fact has no discrete analogy, as the following consequence of continuous dependence on initial value shows.

Theorem 5.3.3. *Equation (5.3.1) has at least one solution in the class M^-.*

Proof. For a positive fixed real number a, let $x = \{x(k)\}$ be a solution of equation (5.3.1) such that $x(1) = a$ and $\Delta x(1) = b$. Consider the set S given by

$$S = \{b < 0 : \exists m \in \mathbb{N} \text{ such that } x(m)x(m+1) \leq 0\}. \tag{5.3.12}$$

Clearly, $S \neq \emptyset$, since any real negative b_1 such that $b_1 < -a$ belongs to S. Thus

$$\bar{b} = \sup S \quad \text{exists.} \tag{5.3.13}$$

Obviously, $\bar{b} \leq 0$. We will show that the solution $\bar{x} = \{\bar{x}(k)\}$ such that $\bar{x}(1) = a$ and $\Delta\bar{x}(1) = \bar{b}$ belongs to M^-.

Assume $\bar{x} \in M^+$. There are two possible cases to consider.

(i_1) There exists an integer $m_1 > 1$ such that $\bar{x}(k) < 0$ for every $k \geq m_1$ and $\Delta\bar{x}(k) < 0$.

(i_2) There exists an integer $m_1 \geq 1$ such that $\Delta\bar{x}(k) > 0$ for every $k > m_1$ and $\bar{x}(k) > 0$.

In case (i_1), take $d > \bar{b}$ such that $d - \bar{b}$ is sufficiently small. The continuous dependence on initial conditions implies that the solution y satisfying the initial conditions $y(1) = a$ and $\Delta y(1) = d$ satisfies $y(m_1)y(m_1+1) \le 0$ (i.e., the solution y has a generalized zero at m_1). But this is a contradiction to (5.3.13). In case (i_2), take $d < \bar{b}$ such that $|d - \bar{b}|$ is sufficiently small. Lemma 4.4.3 and the continuous dependence on initial conditions imply that the solution y satisfying the initial conditions $y(1) = a$ and $\Delta y(1) = d$ is positive for every $k \in \mathbb{N}$, which again contradicts (5.3.13). $\square$

From Theorem 5.3.3, the difference equation

$$\Delta^2 x(k) = \sqrt{|x(k+1)|} \tag{5.3.14}$$

has a positive decreasing solution, whereas, as already claimed, the corresponding differential equation (5.3.11) does not have positive decreasing solutions.

5.3.1.2. Sufficient conditions

The following results deal with the existence of solutions of equation (5.3.1) in the classes M_0^- and M_B^-. We start by recalling the following result that may be considered the discrete analog of the Lebesgue dominated convergence theorem.

Lemma 5.3.4 (Lebesgue dominated convergence theorem). *Let $\{a(i,j)\}$ be a double real sequence with $a(i,j) \ge 0$ for $i, j \in \mathbb{N}$. Assume that the series $\sum_{j=1}^{\infty} a(i,j)$ is totally convergent, that is, there exists a sequence $\{b(j)\}$ such that $b(j) \ge a(i,j)$ with $\sum_{j=1}^{\infty} b(j) < \infty$, and let $\lim_{i\to\infty} a(i,j) = A(j)$ for every $j \in \mathbb{N}$. Then the series $\sum_{j=1}^{\infty} A(j)$ converges and*

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} a(i,j) = \sum_{j=1}^{\infty} A(j). \tag{5.3.15}$$

Theorem 5.3.5. *If $Y_2 < \infty$, then for equation (5.3.1), $M_B^- \ne \emptyset$.*

Proof. Let $M_f = \max_{v\in[1/2,1]} f(v)$ and choose $m \in \mathbb{N}$ such that

$$\Psi^{-1}(M_f)\left[\sum_{n=m}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)} \sum_{j=n}^{\infty} q(j)\right)\right] \le \frac{1}{2}. \tag{5.3.16}$$

Denote by $\ell^\infty(m)$ the Banach space of all bounded sequences defined for all integers $k \ge m$ and endowed with the topology of the supnorm. Set

$$S = \left\{x = \{x(k)\} \in \ell^\infty(m) : \frac{1}{2} \le x(k) \le 1\right\}. \tag{5.3.17}$$

Clearly, S is a bounded, closed, and convex subset of $\ell^\infty(m)$. Now consider the operator $T : S \to \ell^\infty(m)$ defined by

$$Tx(k) = u(k) = \frac{1}{2} + \sum_{n=k}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)} \sum_{j=n}^{\infty} q(j) f(x(j+1))\right). \tag{5.3.18}$$

We will show that T satisfies the hypotheses of Theorem 3.6.5.

(1) T maps S into itself. In fact, if $x \in S$, then from (5.3.16) and (5.3.18), we have

$$\frac{1}{2} \leq u(k) \leq \frac{1}{2} + \Psi^{-1}(M_f)\left[\sum_{n=k}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)} \sum_{j=n}^{\infty} q(j)\right)\right] \leq 1. \tag{5.3.19}$$

Therefore, $T(S) \subseteq S$.

(2) T is continuous in S. Let $\{x^{(i)}\} \subset S$ converge to $x \in \ell^\infty(m)$. Since S is closed, $x \in S$. Let $x^{(i)} = \{x^{(i)}(k)\}$, $x = \{x(k)\}$, $T(x^{(i)}) = \{u^{(i)}(k)\}$, and $Tx = \{u(k)\}$. Then for every $k \geq m$,

$$\begin{aligned} \|T(x^{(i)}) - T(x)\| &= \sup_{k \geq m} \left| u^{(i)}(k) - u(k) \right| \\ &= \sup_{k \geq m} \left| \sum_{n=k}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)} \sum_{j=n}^{\infty} q(j) f(x^{(i)}(j+1))\right) \right. \\ &\qquad \left. - \sum_{n=k}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)} \sum_{j=n}^{\infty} q(j) f(x(j+1))\right) \right| \\ &\leq \sum_{n=n_0}^{\infty} a(i,n), \end{aligned} \tag{5.3.20}$$

where

$$\begin{aligned} a(i,n) = \Psi^{-1}\left(\frac{1}{c(n)}\right) &\left| \Psi^{-1}\left(\sum_{j=n}^{\infty} q(j) f(x^{(i)}(j+1))\right) \right. \\ &\left. - \Psi^{-1}\left(\sum_{j=n}^{\infty} q(j) f(x(j+1))\right) \right|. \end{aligned} \tag{5.3.21}$$

The series $\sum_{j=n}^{\infty} q(j) f(x^{(i)}(j+1))$ is totally convergent, since, for every $j \geq m$, $q(j) f(x^{(i)}(j+1)) \leq M_f q(j)$ and by Lemma 5.3.2(a_2), $Y_4 < \infty$. In view of the continuity of f we have

$$\lim_{i \to \infty} q(j) f(x^{(i)}(j+1)) = q(j) f(x(j+1)) \quad \text{for } j \geq m. \tag{5.3.22}$$

Thus, by Lemma 5.3.4, for every $n \geq m$,

$$\lim_{i \to \infty} \left| \Psi^{-1}\left(\sum_{j=n}^{\infty} q(j) f(x^{(i)}(j+1))\right) - \Psi^{-1}\left(\sum_{j=n}^{\infty} q(j) f(x(j+1))\right) \right| = 0. \tag{5.3.23}$$

Consequently,

$$\lim_{i \to \infty} a(i,n) = 0 \quad \text{for every } n \geq m. \tag{5.3.24}$$

Since $Y_2 < \infty$ and

$$\begin{aligned} a(i,n) &\le \Psi^{-1}\left(\sum_{j=n}^{\infty}\frac{q(j)}{c(n)}f\left(x^{(i)}(j+1)\right)\right)+\Psi^{-1}\left(\sum_{j=n}^{\infty}\frac{q(j)}{c(n)}f\left(x(j+1)\right)\right)\\ &\le 2\Psi^{-1}\left(M_f\right)\Psi^{-1}\left(\sum_{j=n}^{\infty}\frac{q(j)}{c(n)}\right), \end{aligned} \tag{5.3.25}$$

the series $\sum_{n=m}^{\infty} a(i,n)$ is totally convergent. Applying again Lemma 5.3.4, from (5.3.20) and (5.3.24) we obtain

$$\lim_{i\to\infty}\left\|T\left(x^{(i)}\right)-T(x)\right\| \le \lim_{i\to\infty}\sum_{n=m}^{\infty}a(i,n)=\sum_{n=m}^{\infty}\lim_{i\to\infty}a(i,m)=0, \tag{5.3.26}$$

that is, T is continuous in S.

(3) $T(S)$ is relatively compact. As in Section 3.6, it is sufficient to prove that $T(S)$ is uniformly Cauchy in the topology of $\ell^\infty(m)$. Let $x \in S$ and $m_2 > m_1 \ge m$. From (5.3.18), we see that

$$\begin{aligned} \left|u(m_1)-u(m_2)\right| &= \left|\sum_{n=m_1}^{\infty}\Psi^{-1}\left(\sum_{j=n}^{\infty}\frac{q(j)}{c(n)}f\left(x(j+1)\right)\right)\right.\\ &\qquad \left.-\sum_{n=m_2}^{\infty}\Psi^{-1}\left(\sum_{j=n}^{\infty}\frac{q(j)}{c(n)}f\left(x(j+1)\right)\right)\right|\\ &\le \Psi^{-1}\left(M_f\right)\sum_{n=m_1}^{m_2-1}\Psi^{-1}\left(\sum_{j=n}^{\infty}\frac{q(j)}{c(n)}\right). \end{aligned} \tag{5.3.27}$$

From the hypotheses, it is clear that for given $\varepsilon > 0$ there exists an integer $k_\varepsilon \ge m$ such that for all $m_2 > m_1 \ge k_\varepsilon$, $|Tx(m_1) - Tx(m_2)| < \varepsilon$. Thus, $T(S)$ is uniformly Cauchy and hence $T(S)$ is relatively compact.

Applying Theorem 3.6.5, there exists $x \in S$ such that $Tx = x$, that is,

$$x(k)=\frac{1}{2}+\sum_{n=k}^{\infty}\Psi^{-1}\left(\frac{1}{c(n)}\sum_{j=n}^{\infty}q(j)f\left(x(j+1)\right)\right). \tag{5.3.28}$$

It is easy to see that $\{x(k)\}$ is a solution of equation (5.3.1). Since

$$\Psi\left(\Delta x(k)\right)=-\frac{1}{c(k)}\sum_{j=k}^{\infty}q(j)f\left(x(j+1)\right)<0, \tag{5.3.29}$$

and $1/2 \le x(k) \le 1$, we see that $\{x(k)\}$ is an eventually positive decreasing solution of equation (5.3.1) with $\lim_{k\to\infty} x(k) = \ell \ne 0$. Hence $M_B^- \ne \emptyset$. This completes the proof. $\square$

Remark 5.3.6. (i) From the proof of Theorem 5.3.5, we note that no growth conditions are needed on the nonlinear function f.

(ii) Theorem 5.3.5 gives a sufficient condition so that equation (5.3.1) has at least one solution x in M_B^-. However, as we will see from the following example, $Y_2 < \infty$ is not enough to ensure that equation (5.3.1) has at least one solution $x \in M_0^-$.

Example 5.3.7. Consider the linear difference equation

$$\Delta^2 x(k) = \frac{2}{k(k+1)^2} x(k+1) \quad \text{for } k \in \mathbb{N}. \tag{5.3.30}$$

Clearly, $\{x(k)\}$ with $x(k) = 1 + (1/k)$ is a solution of equation (5.3.30) that belongs to M_B^-. By a standard computation, the sequence $\{w(k)\}$ with

$$w(k) = \frac{k+1}{k} \sum_{j=1}^{k-1} \frac{j}{j+2} \tag{5.3.31}$$

is a linearly independent solution of equation (5.3.30). Since

$$w(k) \geq \frac{k+1}{k} \frac{k-1}{3} = \frac{(k-1)(k+1)}{3k}, \tag{5.3.32}$$

we find $\{w(k)\} \in M_\infty^+$.

If equation (5.3.30) has a solution $\{y(k)\}$ in the class M_0^-, then because $\{x(k)\}$ and $\{y(k)\}$ are linearly independent, every solution of (5.3.30) would be bounded, which is a contradiction. Hence $M_0^- = \emptyset$.

Next we present the following result.

Theorem 5.3.8. *If $Y_1 < \infty$ and $Y_2 < \infty$, then $M_0^- \neq \emptyset$ for equation (5.3.1).*

Proof. In view of Lemma 5.3.2, $Y_3 < \infty$ and $Y_4 < \infty$. Now choose $m \geq 1$ such that

$$\max_{v \in [0, Y_3]} f(v) \sum_{j=m+1}^{\infty} q(j) < \frac{1}{2}. \tag{5.3.33}$$

Let S be a nonempty subset of $\ell^\infty(m)$ given by

$$S = \left\{ x = \{x(k)\} \in \ell^\infty(m) : 0 \leq x(k) \leq \sum_{n=m}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)}\right) \right\}. \tag{5.3.34}$$

The set S is bounded, closed, and convex. Let $T : S \to \ell^\infty(m)$ be defined by

$$Tx(k) = u(k) = \sum_{n=k}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)}\right) \Psi^{-1}\left[1 - \sum_{j=m}^{n} q(j-1) f(x(j))\right]. \tag{5.3.35}$$

Note that for every $x = \{x(k)\} \in S$, $0 \le x(k) \le Y_3$. In view of (5.3.33), we find

$$0 \le \sum_{j=m}^{n} q(j-1) f(x(j)) \le \max_{v \in [0,Y_3]} f(v) \sum_{j=m}^{\infty} q(j-1) < \frac{1}{2}. \tag{5.3.36}$$

Hence, taking into account that $\Psi^{-1}(1/2) \le 1$, we have for all $k \ge m$,

$$0 < u(k) \le \sum_{n=k}^{\infty} \Psi^{-1}\left(\frac{1}{c(n)}\right), \tag{5.3.37}$$

that is, $T(S) \subseteq S$. Now it suffices to prove that T is continuous and $T(S)$ is relatively compact. This may be accomplished by applying Lemma 5.3.4 and an argument similar to that given in the proof of Theorem 5.3.5(2) and (3). In fact, the continuity of T follows from the estimation

$$\|T(x^{(i)}) - T(x)\| = \sup_{k \ge m} |u^{(i)}(k) - u(k)| \le \sum_{n=m}^{\infty} \gamma(i,n), \tag{5.3.38}$$

where

$$\begin{aligned} \gamma(i,n) = \Psi^{-1}\left(\frac{1}{c(n)}\right) \Bigg| \Psi^{-1}\left[1 - \sum_{j=m}^{n} b(j-1) f(x^{(i)}(j))\right] \\ - \Psi^{-1}\left[1 - \sum_{j=m}^{n} b(j-1) f(x(j))\right] \Bigg|, \end{aligned} \tag{5.3.39}$$

and the relative compactness of $T(S)$ follows from

$$\begin{aligned} |u(m_1) - u(m_2)| &\le \sum_{n=m_1}^{m_2-1} \Psi^{-1}\left(\frac{1}{c(n)}\right) \Psi^{-1}\left[1 - \sum_{j=m}^{n} q(j-1) f(x(j))\right] \\ &\le \sum_{n=m_1}^{m_2-1} \Psi^{-1}\left(\frac{1}{c(n)}\right). \end{aligned} \tag{5.3.40}$$

Here, we omit the details. □

Now, from Theorems 5.3.5 and 5.3.8 we obtain the following.

Corollary 5.3.9. *If $Y_1 < \infty$ and $Y_2 < \infty$, then both classes M_0^- and M_B^- are nonempty for equation (5.3.1).*

Next, we will study the asymptotic behavior of all positive decreasing solutions of equation (5.3.1). If we assume

$$\limsup_{x \to \infty} \frac{f(x)}{\Psi(x)} < \infty, \tag{5.3.41}$$

then the conditions $Y_1 < \infty$ and $Y_2 < \infty$ in Corollary 5.3.9 also become necessary in order that both classes M_0^- and M_B^- are nonempty.

We will need the following lemma which can be proved by direct computation.

Lemma 5.3.10. *If $Y_3 = \infty$, then for any solution $\{x(k)\}$ of equation (5.3.1) in the class M^- it holds that* $\lim_{k\to\infty} c(k)\Psi(\Delta x(k)) = 0$.

Now we present the following result.

Theorem 5.3.11. *Assume that $Y_1 = \infty$, $Y_2 < \infty$, and condition (5.3.41) hold. Then every solution $\{x(k)\}$ of equation (5.3.1) in the class M^- tends to a nonzero limit as $k \to \infty$, that is, $M^- = M_B^- \neq \emptyset$ and $M_0^- = \emptyset$.*

Proof. Let $\{x(k)\}$ be a solution of equation (5.3.1) in the class M_0^-. Without loss of generality assume $0 < x(k) < 1$ and $\Delta x(k) < 0$ for $k \geq 1$. By Lemma 5.3.2, $Y_3 = \infty$ and thus, by Lemma 5.3.10, $\lim_{k\to\infty} c(k)\Psi(\Delta x(k)) = 0$. By (5.3.41) there exists a constant $M > 0$ such that $f(x(k+1)) \leq M\Psi(x(k+1))$ for $k \in \mathbb{N}$. Summing equation (5.3.1) from k to ∞, we obtain

$$\begin{aligned} -c(k)\Psi(\Delta x(k)) &= \sum_{n=k}^{\infty} q(n) f(x(n+1)) \\ &< M \sum_{n=k}^{\infty} q(n)\Psi(x(n+1)) \\ &< M\Psi(x(k+1)) \sum_{n=k}^{\infty} q(n). \end{aligned} \tag{5.3.42}$$

Thus

$$\frac{\Delta x(k)}{x(k)} > -\mu(k), \tag{5.3.43}$$

where

$$\mu(k) = \Psi^{-1}(M)\Psi^{-1}\left(\frac{1}{c(k)} \sum_{n=k}^{\infty} q(n)\right). \tag{5.3.44}$$

Since $Y_2 < \infty$, we have $\lim_{k\to\infty} \mu(k) = 0$. Thus $1 - \mu(k) \in (0, 1)$ for all $k \geq m$ for some $m \in \mathbb{N}$. From (5.3.43), we see that $x(k+1) > (1 - \mu(k))x(k)$, that is,

$$x(k) > \prod_{j=m}^{k-1} (1 - \mu(j))x(m). \tag{5.3.45}$$

Thus we conclude that

$$P_\mu = \prod_{j=m}^{\infty} (1 - \mu(j)) > 0 \tag{5.3.46}$$

and is convergent. From (5.3.45), as $k \to \infty$, we get

$$0 = \lim_{k\to\infty} x(k) \geq \prod_{j=m}^{\infty} (1 - \mu(j))x(m) = P_\mu x(m) > 0, \tag{5.3.47}$$

which is a contradiction. □

Example 5.3.12. The difference equation

$$\Delta^2 x(k) = \frac{6\sqrt{k^{(2)}}}{(k+1)^{(4)}}\sqrt{|x(k+1)|} \quad \text{for } k > 2, \tag{5.3.48}$$

where $k^{(n)} = k(k-1)\cdots(k-n+1)$, has a solution x with $x(k) = 1/(k-1)^{(2)}$ which belongs to the class M_0^-. Only condition (5.3.41) of Theorem 5.3.11 is violated.

From the above results, we have the following.

Theorem 5.3.13. (a_1) *Assume (i_1) holds. Then for (5.3.1), $M^- = M_0^- \neq \emptyset$ and $M_B^- = \emptyset$.*

(a_2) *Assume (i_2) holds. If condition (5.3.41) holds, then for equation (5.3.1), $M^- = M_B^- \neq \emptyset$ and $M_0^- = \emptyset$.*

(a_3) *Assume (i_3) holds. Then for equation (5.3.1), $M_B^- \neq \emptyset$ and $M_0^- \neq \emptyset$.*

Proof. We first show (a_1). In view of Theorem 5.3.3, it suffices to show that $M_B^- = \emptyset$. Assume that equation (5.3.1) has a solution $x = \{x(k)\}$ in the class M_B^-. Without loss of generality suppose $x(k) > 0$ and $\Delta x(k) < 0$ for $k \in \mathbb{N}$ and denote $\lim_{k\to\infty} x(k) = \ell_x$. Summing equation (5.3.1) from k to ∞, we obtain

$$-\lambda_x - c(k)\Psi(\Delta x(k)) = \sum_{j=k}^{\infty} q(j) f(x(j+1)), \tag{5.3.49}$$

where $-\lambda_x = \lim_{k\to\infty} c(k)\Psi(\Delta x(k))$. Since $x(j) > \ell_x > 0$ for all $j \in \mathbb{N}$ and $\lambda_x \geq 0$, (5.3.49) implies

$$-c(k)\Psi(\Delta x(k)) \geq b \sum_{j=k}^{\infty} q(j), \tag{5.3.50}$$

where $b = \min_{u\in[\ell_x, x(1)]} f(u)$. Hence

$$x(k+1) \leq x(1) - \Psi^{-1}(b) \sum_{n=1}^{k} \Psi^{-1}\left(\frac{1}{c(n)} \sum_{j=n}^{\infty} b(j)\right). \tag{5.3.51}$$

Now, as $k \to \infty$, we arrive at the desired conclusion.

The assertion (a_2) follows from Theorem 5.3.11, and (a_3) is the same as in Corollary 5.3.9. □

Remark 5.3.14. (i) The results of this subsection are presented in a form so that they can be easily extended to equation (5.3.1) with Ψ as in (I_3).

(ii) Some of the results presented above when specialized to second-order linear difference equations are not discussed in Section 1.16. Therefore they supplement those given in Section 1.16. The formulation of such results are left to the reader.

5.3.2. Positive increasing solutions

We will study positive increasing solutions of equation (5.3.1). A full characterization of limit behavior of all these solutions in terms of $\{c(k)\}$ and $\{q(k)\}$ is established.

As in Section 1.16.4, we denote by M_B^+ and M_∞^+ the subsets of M^+ consisting of bounded and unbounded solutions of (5.3.1), respectively. A solution $x \in M_\infty^+$ is said to be strongly increasing if $\lim_{k\to\infty} c(k)\Psi(\Delta x(k)) = \infty$ and regularly increasing otherwise. The subset of strongly increasing solutions will be denoted by $M_\infty^+(S)$, and the set of regular increasing solutions will be denoted by $M_\infty^+(R)$. Then

$$\begin{aligned} M_B^+ &= \Big\{\{x(k)\} \in M^+ : \lim_{k\to\infty} |x(k)| < \infty\Big\}, \\ M_\infty^+(R) &= \Big\{\{x(k)\} \in M^+ : \lim_{k\to\infty} |x(k)| = \infty,\ \lim_{k\to\infty} c(k)|\Psi(\Delta x(k))| < \infty\Big\}, \\ M_\infty^+(S) &= \Big\{\{x(k)\} \in M^+ : \lim_{k\to\infty} |x(k)| = \infty,\ \lim_{k\to\infty} c(k)|\Psi(\Delta x(k))| = \infty\Big\}, \\ M^+ &= M_B^+ \cup M_\infty^+ = M_B^+ \cup M_\infty^+(R) \cup M_\infty^+(S). \end{aligned} \tag{5.3.52}$$

In this subsection such subsets are fully characterized by means of the convergence or divergence of the two series

$$Z_1 = \lim_{k\to\infty} \sum_{n=2}^{k} \Psi^{-1}\left(\sum_{j=1}^{n-1} \frac{q(j)}{c(n)}\right), \qquad Z_2 = \lim_{k\to\infty} \sum_{n=2}^{k} q(n)\left(\sum_{j=1}^{n-1} \Psi^{-1}\left(\frac{1}{c(j+1)}\right)\right). \tag{5.3.53}$$

The following cases are significant and will be examined in the sequel:

(j_1) $Z_1 = \infty$ and $Z_2 = \infty$,
(j_2) $Z_1 = \infty$ and $Z_2 < \infty$,
(j_3) $Z_1 < \infty$.

5.3.2.1. Bounded solutions

Here we will consider the existence of bounded solutions of (5.3.1).

Proposition 5.3.15. *If equation (5.3.1) has solutions in the class M_B^+, then $Z_1 < \infty$.*

Proof. Let $\{x(k)\}$ be a solution of equation (5.3.1) in M_B^+, say, $x(k) > 0$ and $\Delta x(k) > 0$ for $k \ge m \ge 1$ and $\lim_{k\to\infty} x(k) = \ell_x < \infty$. From equation (5.3.1) it follows that

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) = q(k)f(x(k+1)) \geq q(k)L_f, \tag{5.3.54}$$

where $L_f = \min_{u\in[x(m),\ell_x]} f(u)$. Summing (5.3.54) from m to $k-1$, we get

$$c(k)\Psi(\Delta x(k)) \geq c(m)\Psi(\Delta x(m)) + L_f \sum_{j=m}^{k-1} q(j) \geq L_f \sum_{j=m}^{k-1} q(j). \tag{5.3.55}$$

Summing (5.3.55) from $m+1$ to k, we have

$$x(k+1) \geq x(m+1) + \sum_{n=m+1}^{k} \Psi^{-1}\left(\frac{L_f}{c(n)} \sum_{j=m}^{n-1} q(j)\right). \tag{5.3.56}$$

Since

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k-1} q(j)}{\sum_{j=m}^{k-1} q(j)} < \infty, \tag{5.3.57}$$

it follows that

$$\lim_{n\to\infty} \frac{\Psi^{-1}\big(\sum_{j=1}^{n-1} q(j)/c(n)\big)}{\Psi^{-1}\big(\sum_{j=m}^{n-1} q(j)/c(n)\big)} < \infty, \tag{5.3.58}$$

and so $Z_1 < \infty$. □

Theorem 5.3.16. *Assume $Z_1 < \infty$ and*

$$\lim_{|u|\to\infty} \frac{f(u)}{\Psi(u)} < \infty. \tag{5.3.59}$$

Then every solution of equation (5.3.1) in the class M^+ is bounded, that is, $M_\infty^+ = \emptyset$.

Proof. In view of condition (5.3.59), there exist two positive constants a and b such that

$$\frac{f(u)}{\Psi(u)} < a \quad \text{for } u > b. \tag{5.3.60}$$

Assume there exists an unbounded solution $\{x(k)\}$ of equation (5.3.1), and without loss of generality, suppose $x(k) > b$ and $\Delta x(k) > 0$ for $k \geq m \geq 1$. From equation (5.3.1), we obtain

$$\begin{aligned}\Delta\left[\frac{c(k)\Psi(\Delta x(k))}{\Psi(x(k))}\right] &= \frac{\Delta\big(c(k)\Psi(\Delta x(k))\big)}{\Psi(x(k+1))} + c(k)\Psi(\Delta x(k))\Delta\left(\frac{1}{\Psi(x(k))}\right)\\ &= \frac{q(k)f(x(k+1))}{\Psi(x(k+1))} - \frac{c(k)\Psi(\Delta x(k))\Delta\Psi(x(k))}{\Psi(x(k))\Psi(x(k+1))}.\end{aligned} \tag{5.3.61}$$

Summing (5.3.61) from m to $k-1$, we get

$$\frac{c(k)\Psi(\Delta x(k))}{\Psi(x(k))} \le H + \sum_{j=m}^{k-1} \frac{q(j) f(x(j+1))}{\Psi(x(j+1))} \le H + a \sum_{j=m}^{k-1} q(j), \tag{5.3.62}$$

where $H = c(m)\Psi(\Delta x(m))/\Psi(x(m))$. Now we consider the following two cases.

Case 1. If $Y_4 = \lim_{k\to\infty} \sum_{j=1}^{k} q(j) < \infty$, then there exists a positive constant H_1 such that

$$\frac{c(k)\Psi(\Delta x(k))}{\Psi(x(k))} \le H_1 \quad \text{or} \quad \frac{\Delta x(k)}{x(k)} \le \Psi^{-1}\left(\frac{H_1}{c(k)}\right) =: \gamma(k). \tag{5.3.63}$$

Since $Z_1 < \infty$ implies $Y_3 = \lim_{k\to\infty} \sum_{j=1}^{k} \Psi^{-1}(1/c(j)) < \infty$, we get $\lim_{k\to\infty} \gamma(k) = 0$. Thus

$$x(k+1) \le [1+\gamma(k)]x(k) \quad \text{or} \quad x(k+1) \le \prod_{j=m}^{k} [1+\gamma(j)]x(m). \tag{5.3.64}$$

Using the fact

$$Y_3 < \infty \Longleftrightarrow \sum_{j=m}^{\infty} \gamma(j) < \infty \Longleftrightarrow \prod_{j=m}^{\infty} [1+\gamma(j)] < \infty \tag{5.3.65}$$

as $k \to \infty$, we obtain a contradiction.

Case 2. If $Y_4 = \infty$, then choose $m_1 > m$ such that $H < a\sum_{j=m}^{m_1-1} q(j)$. From (5.3.62), we obtain for $k \ge m_1$

$$\frac{c(k)\Psi(\Delta x(k))}{\Psi(x(k))} \le 2a \sum_{j=m}^{k-1} q(j), \tag{5.3.66}$$

so

$$\frac{\Delta x(k)}{x(k)} \le \Psi^{-1}(2a)\Psi^{-1}\left(\sum_{j=m}^{k-1} q(j)\right) =: \mu(k). \tag{5.3.67}$$

Since $Z_1 < \infty$, we see that $\sum_{j=m}^{\infty} \mu(j) < \infty$. Proceeding as in the proof of Case 1, we arrive at the desired contradiction.

This completes the proof. $\square$

Example 5.3.17. The difference equation

$$\Delta\big((k-3)(k-4)\Delta x(k)\big) = \frac{20}{k^{(4)}} f(x(k+1)) \quad \text{for } k > 4 \tag{5.3.68}$$

has an unbounded solution $\{x(k)\}$ with $x(k) = (k-1)^{(4)}$. All conditions of Theorem 5.3.16 are satisfied except for condition (5.3.59).

If no conditions on the growth at infinity of the nonlinearity f are assumed, then the following result holds.

Theorem 5.3.18. *If $Z_1 < \infty$, then equation (5.3.1) has solutions in the class M_B^+.*

Proof. Let $M_f = \max_{v\in[1/2,1]} f(v)$ and choose $m \geq 2$ such that

$$\Psi^{-1}(M_f)\left[\sum_{n=m}^{\infty}\Psi^{-1}\left(\sum_{j=m-1}^{n-1}\frac{b(j)}{c(n)}\right)\right] \leq \frac{1}{2}. \tag{5.3.69}$$

Let $S \subseteq \ell^\infty(m)$ be defined as in the proof of Theorem 5.3.5 and consider the operator $T : S \to \ell^\infty(m)$ defined by

$$Tx(k) = y(k) = \frac{1}{2} + \sum_{n=m}^{k-1}\Psi^{-1}\left(\sum_{j=m-1}^{n-1}\frac{q(j)}{c(n)}f(x(j+1))\right). \tag{5.3.70}$$

We will show that the operator T satisfies the hypotheses of Theorem 3.6.5.

(1) T maps S into itself. In view of (5.3.69), for $k \geq m+1$ we find

$$\frac{1}{2} \leq y(k) \leq \frac{1}{2} + \Psi^{-1}(M_f)\left[\sum_{n=m}^{k-1}\Psi^{-1}\left(\sum_{j=m-1}^{n-1}\frac{q(j)}{c(n)}\right)\right] \leq 1. \tag{5.3.71}$$

(2) T is continuous in S. Let $\{x^{(i)}\}$ be a sequence in S converging to x in $\ell^\infty(m)$. Since S is closed, $x \in S$. Let $x^{(i)} = \{x^{(i)}(k)\}$, $x = \{x(k)\}$, $T(x^{(i)}) = \{y^{(i)}(k)\}$, and $Tx = \{y(k)\}$. Now, for every $k \geq m+1$,

$$\begin{aligned} |y^{(i)}(k) - y(k)| = \Bigg|&\sum_{n=m}^{k-1}\Psi^{-1}\left(\sum_{j=m-1}^{n-1}\frac{q(j)}{c(n)}f(x^{(i)}(j+1))\right) \\ &- \sum_{n=m}^{k-1}\Psi^{-1}\left(\sum_{j=m-1}^{n-1}\frac{q(j)}{c(n)}f(x(j+1))\right)\Bigg| \\ \leq& \sum_{n=m}^{k-1}a(i,n), \end{aligned} \tag{5.3.72}$$

where

$$\begin{aligned} a(i,n) = \Psi^{-1}\left(\frac{1}{c(n)}\right)\Bigg|&\Psi^{-1}\left(\sum_{j=m-1}^{n-1}q(j)f(x^{(i)}(j+1))\right) \\ &- \Psi^{-1}\left(\sum_{j=m-1}^{n-1}q(j)f(x(j+1))\right)\Bigg|. \end{aligned} \tag{5.3.73}$$

Hence

$$\begin{aligned} \|T(x^{(i)}) - T(x)\| &= \sup_{k\geq m} |y^{(i)}(k) - y(k)| \\ &\leq \max\left\{ |x^{(i)}(m) - x(m)|, \sup_{k\geq m+1} \sum_{n=m}^{k-1} a(i,n) \right\}. \end{aligned} \tag{5.3.74}$$

By the continuity of f we get $\lim_{i\to\infty} q(j)f(x^{(i)}(j+1)) = q(j)f(x(j+1))$ for every $j \geq m$. Thus,

$$\lim_{i\to\infty} \left| \Psi^{-1}\left(\sum_{j=m-1}^{n-1} q(j)f(x^{(i)}(j+1)) \right) - \Psi^{-1}\left(\sum_{j=m-1}^{n-1} q(j)f(x(j+1)) \right) \right| = 0. \tag{5.3.75}$$

Consequently,

$$\lim_{i\to\infty} a(i,n) = 0 \quad \text{for every } n \geq m+1. \tag{5.3.76}$$

Since $Z_1 < \infty$ and

$$\begin{aligned} a(i,n) &\leq \Psi^{-1}\left(\sum_{j=m-1}^{n-1} \frac{q(j)}{c(n)} f(x^{(i)}(j+1)) \right) + \Psi^{-1}\left(\sum_{j=m-1}^{n-1} \frac{q(j)}{c(n)} f(x(j+1)) \right) \\ &\leq 2\Psi^{-1}(M_f)\Psi^{-1}\left(\sum_{j=m-1}^{n-1} \frac{q(j)}{c(n)} \right), \end{aligned} \tag{5.3.77}$$

the series $\sum_{n=m}^{\infty} a(i,n)$ is totally convergent. Applying Lemma 5.3.4 and using (5.3.74) and (5.3.76), we obtain $\|T(x^{(i)}) - T(x)\| \to 0$ as $i \to \infty$, that is, T is continuous in S.

(3) $T(S)$ is relatively compact. As in the proof of Theorem 5.3.5(3), let $x \in S$ and $m_2 > m_1 \geq m+1$. From (5.3.70), we see that

$$\begin{aligned} |y(m_2) - y(m_1)| &= \sum_{n=m_1}^{m_2-1} \Psi^{-1}\left(\sum_{j=m-1}^{n-1} \frac{q(j)}{c(n)} f(x(j+1)) \right) \\ &\leq \Psi^{-1}(M_f) \sum_{n=m_1}^{m_2-1} \Psi^{-1}\left(\sum_{j=m-1}^{n-1} \frac{q(j)}{c(n)} \right), \end{aligned} \tag{5.3.78}$$

and as in the proof of Theorem 5.3.5(3), the Cauchy criterion gives the relatively compactness of $T(S)$.

Applying Theorem 3.6.5, there exists $x \in S$ such that $Tx = x$, that is, for $k \geq m+1$,

$$x(k) = \frac{1}{2} + \sum_{n=m}^{k-1} \Psi^{-1}\left(\sum_{j=m-1}^{n-1} \frac{q(j)}{c(n)} f\left(x(j+1)\right)\right). \tag{5.3.79}$$

It is easy to show that $\{x(k)\}$ for $k \geq m+1$ is a solution of equation (5.3.1) in the class M_B^+. This completes the proof. □

5.3.2.2. Unbounded solutions

Next, we will study the existence of unbounded solutions of a special case of equation (5.3.1), namely the equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) = q(k)\Phi(x(k+1)), \tag{5.3.80}$$

where $\Phi(x) = |x|^{\gamma-1}x$ and γ is a positive constant, by employing the so-called reciprocity principle which is stated as follows: if $x = \{x(k)\}$ is a solution of equation (5.3.80), then $\{y(k)\}$ with $y(k) = c(k)\Psi(\Delta x(k))$ is a solution of the reciprocal equation

$$\Delta\left(\frac{1}{\Phi^{-1}(q(k))}\Phi^{-1}(\Delta y(k))\right) = \frac{1}{\Psi^{-1}(c(k+1))}\Psi^{-1}(y(k+1)). \tag{5.3.81}$$

We observe that for solutions $\{x(k)\}$ of equation (5.3.80) and $\{y(k)\}$ of equation (5.3.81) with $y(k) = c(k)\Psi(\Delta x(k))$, $\{x(k)\} \in M^+$ if and only if $\{y(k)\} \in M^+$. We also note that the series Z_2 for equation (5.3.81) plays the same rôle as Z_1 for equation (5.3.80) and vice versa.

Now, by using the reciprocity principle, we obtain the following.

Corollary 5.3.19. *Assume $\alpha \geq \gamma$ and $Z_2 < \infty$. Then every solution $\{x(k)\}$ of equation (5.3.80) in M^+ satisfies* $\lim_{k\to\infty} c(k)\Psi(\Delta x(k)) < \infty$, *that is, $M_\infty^+(S) = \emptyset$.*

Proof. The assertion follows by applying Theorem 5.3.16 to equation (5.3.81). □

Corollary 5.3.20. (a_1) *If $Z_2 = \infty$, then the sequence $\{c(k)\Psi(\Delta x(k))\}$ in M^+ is unbounded.*

(a_2) *If $Z_1 = \infty$ and $Z_2 = \infty$, then every solution of equation (5.3.80) is strongly increasing, that is, $M^+ = M_\infty^+(S) \neq \emptyset$.*

(a_3) *If $Z_1 = \infty$ and $Z_2 < \infty$, then equation (5.3.80) has solutions in $M_\infty^+(R)$, that is, $M_\infty^+(R) \neq \emptyset$.*

Proof. The assertion (a_1) follows by applying Proposition 5.3.15 to equation (5.3.81), while (a_2) follows from Proposition 5.3.15 and (a_1). By Proposition 5.3.15, $M^+ = M_\infty^+$. Applying Theorem 5.3.18 to equation (5.3.81), we obtain the assertion (a_3). □

In order to extend the previous results to the more general equation (5.3.1), we need the following comparison criterion.

Consider the difference equations

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) = q_1(k) f_1\big(x(k+1)\big), \tag{5.3.82}$$
$$\Delta\big(c(k)\Psi(\Delta y(k))\big) = q_2(k) f_2\big(y(k+1)\big), \tag{5.3.83}$$

where $\{c(k)\}$ and $\{q_i(k)\}$ for $i \in \{1,2\}$ are positive sequences for $k \in \mathbb{N}$ and $f_i(u) \in C(\mathbb{R},\mathbb{R})$ with $u f_i(u) > 0$ for $u \neq 0$ and $i \in \{1,2\}$.

Theorem 5.3.21. *Suppose $q_2(k) \geq q_1(k)$ and there exists a positive constant a such that*

$$\big|f_2(u)\big| \geq \big|f_1(u)\big| \quad \text{for } |u| \geq a, \tag{5.3.84}$$

and f_1 or f_2 are nondecreasing for $|u| > a$.

Let $\{x(k)\}$ be a solution of (5.3.82) such that $x(1) > a$ and $x(1)\Delta x(1) > 0$. Then for any solution $\{y(k)\}$ of (5.3.83) in M^+ with $|y(1)| \geq |x(1)|$, $x(1)y(1) > 0$, and $|\Delta y(1)| \geq |\Delta x(1)|$, it holds that

$$\big|y(k)\big| \geq \big|x(k)\big|, \quad \big|\Psi(\Delta y(k))\big| \geq \big|\Psi(\Delta x(k))\big| \quad \text{for } k \in \mathbb{N}. \tag{5.3.85}$$

Proof. Without loss of generality, we consider solutions $\{x(k)\}$ starting with a positive value. In view of Lemma 5.3.1, the sequences $\{x(k)\}$ and $\{y(k)\}$ are increasing, and so $x(k) > a$ and $y(k) > a$. Define $d(k) = y(k) - x(k)$ for $k \in \mathbb{N}$. Clearly, $d(1) \geq 0$ and $\Delta d(1) \geq 0$, that is, $d(2) \geq d(1)$. We will show that the sequence $\{d(k)\}$ is nondecreasing. Assume there exists $m \geq 2$ such that

$$0 \leq d(i) \leq d(i+1) \quad \text{for } 1 \leq i \leq m-1, \qquad d(m) > d(m+1). \tag{5.3.86}$$

Let $\{G(k)\}$ be the sequence

$$G(k) = c(k)\big[\Psi(\Delta y(k)) - \Psi(\Delta x(k))\big]. \tag{5.3.87}$$

Then

$$\begin{aligned} \Delta G(k) &= q_2(k) f_2\big(y(k+1)\big) - q_1(k) f_1\big(x(k+1)\big) \\ &\geq q_1(k)\big[f_2\big(y(k+1)\big) - f_1\big(x(k+1)\big)\big]. \end{aligned} \tag{5.3.88}$$

We find

$$\begin{aligned} \Delta G(k) &\geq q_1(k)\big[f_2\big(y(k+1)\big) - f_2\big(x(k+1)\big)\big], \\ \Delta G(k) &\geq q_1(k)\big[f_1\big(y(k+1)\big) - f_1\big(x(k+1)\big)\big]. \end{aligned} \tag{5.3.89}$$

Taking into account $d(m) \geq 0$ and the monotonicity of f_1 or f_2, we get

$$\Delta G(m-1) \geq 0. \tag{5.3.90}$$

From (5.3.86) it follows that

$$\begin{aligned} \Delta y(m) - \Delta x(m) &= d(m+1) - d(m) < 0, \\ \Delta y(m-1) - \Delta x(m-1) &= d(m) - d(m-1) \geq 0. \end{aligned} \tag{5.3.91}$$

Hence $\Delta G(m-1) = G(m) - G(m-1) < 0$, which contradicts (5.3.90). Consequently, the sequence $\{d(k)\}$ is nondecreasing and therefore $d(k) \geq 0$. Since $\Delta y(k) - \Delta x(k) = d(k+1) - d(k) \geq 0$, the monotonicity of Ψ yields $\Psi(\Delta y(k)) \geq \Psi(\Delta x(k))$. This completes the proof. □

Now we present the following result.

Theorem 5.3.22. *Assume that condition (5.3.41) holds. If $Z_2 < \infty$, then equation (5.3.1) has no solution in the class $M_\infty^+(S)$, that is, $M_\infty^+(S) = \emptyset$.*

Proof. In view of condition (5.3.41), there exist two positive constants L and a such that $f(u) \leq L\Psi(u)$ for $u \geq a$. Let $\{x(k)\}$ be a solution of (5.3.1) in the class $M_\infty^+(S)$, and without loss of generality, assume $x(k) \geq a$ and $\Delta x(k) > 0$ for $k \in \mathbb{N}$. From Theorem 5.3.21 with $f_2(x) = L\Psi(x)$, $f_1(x) = f(x)$, $q_1(k) = q_2(k) = q(k)$, for any solution $\{y(k)\}$ of

$$\Delta\big(c(k)\Psi(\Delta y(k))\big) = q(k)\Psi\big(y(k+1)\big), \tag{5.3.92}$$

with $y(1) \geq x(1)$ and $\Delta y(1) \geq \Delta x(1)$, we find

$$\Psi\big(\Delta y(k)\big) - \Psi\big(\Delta x(k)\big) \geq 0 \quad \text{for } k \in \mathbb{N}. \tag{5.3.93}$$

In view of Corollary 5.3.19, equation (5.3.92) does not have solutions in the class $M_\infty^+(S)$, and so inequality (5.3.93) gives the desired contradiction as $k \to \infty$. □

Example 5.3.23. Consider the difference equation

$$\Delta^2 x(k) = \frac{2}{e^{k(k+1)} - 1} f\big(x(k+1)\big) \quad \text{for } k > 1, \tag{5.3.94}$$

where $f(x) = |e^x - 1| \operatorname{sgn} x$. The sequence $\{x(k)\}$ with $x(k) = k(k-1)$ is an unbounded solution of equation (5.3.94) and belongs to the class $M_\infty^+(S)$. In this case, for any $\alpha > 1$, we see that $Z_2 < \infty$ and condition (5.3.41) is violated.

Theorem 5.3.24. *Assume that*

$$\limsup_{|u|\to\infty} \frac{f(u)}{\Phi(u)} < \infty, \tag{5.3.95}$$

where $\Phi(u) = |u|^{\gamma-1}u$ with $\gamma > 1$. If $Z_1 = \infty$ and $Z_2 < \infty$, then equation (5.3.1) has solutions in the class $M_\infty^+(R)$, that is, $M_\infty^+(R) \neq \emptyset$.

Proof. In view of condition (5.3.95), there exist two positive constants a and L such that $f(u) \le L\Phi(u)$ for $u \ge a$. Taking into account Corollary 5.3.20(a_3), there exists a solution $\{y(k)\}$ of

$$\Delta\big(c(k)\Psi(\Delta y(k))\big) = q(k)L\Phi\big(y(k+1)\big) \tag{5.3.96}$$

in the class $M_\infty^+(R)$ and without loss of generality assume $y(k) \ge a$ and $\Delta y(k) > 0$ for $k > 1$. Let $\{x(k)\}$ be a solution of (5.3.1) with $x(1) = y(1)$ and $\Delta x(1) = \Delta y(1)$. From Theorem 5.3.21 with $f_2(u) = L\Phi(u)$, $f_1(u) = f_2(u)$, $q(k) = q_1(k) = q_2(k)$, we find $\Psi(\Delta y(k)) - \Psi(\Delta x(k)) \ge 0$ which implies $\lim_{k\to\infty} c(k)\Psi(\Delta x(k)) < \infty$. From Proposition 5.3.15, the solution $\{x(k)\}$ belongs to the class M_∞^+, and the proof is complete. □

If condition (5.3.95) is not satisfied, that is, the nonlinearity f is strongly increasing at infinity, then Theorem 5.3.24 may fail, as the following example shows.

Example 5.3.25. Consider the equation

$$\Delta^2 x(k) = e^{-k} f\big(x(k+1)\big) \quad \text{for } k \in \mathbb{N}, \tag{5.3.97}$$

where $f \in C(\mathbb{R},\mathbb{R})$ and $xf(x) > 0$ for $x \ne 0$ and $|f(x)| = e^{x^2}$ for $|x| \ge 1$. We claim that for equation (5.3.97) the class $M_\infty^+(R)$ is empty. Let $\{x(k)\}$ be a solution of equation (5.3.97) in the class M^+. Since $Z_1 = \infty$, $\{x(k)\}$ is unbounded, and so, without loss of generality, we can suppose $x(k) \ge 1$ for $k \in \mathbb{N}$ and $\Delta x(1) > 0$. Since $\Delta^2 x(k) > 0$, the sequence $\{\Delta x(k)\}$ is increasing and we have

$$x(k+1) > x(k) + \Delta x(1) > x(k-1) + 2\Delta x(1) > \cdots > k\Delta x(1). \tag{5.3.98}$$

Choose $m \in \mathbb{N}$ so large that $k\Delta x(1) \ge 1$ for $k \ge m$. From equation (5.3.97), for $k \ge m$ we obtain

$$\Delta^2 x(k) > \exp\Big(-k + k^2\big(\Delta x(1)\big)^2\Big). \tag{5.3.99}$$

Summing (5.3.99) from m to $k-1$, we have

$$\Delta x(k) > \Delta x(m) + \sum_{j=m}^{k-1} \exp\Big(-j + j^2\big(\Delta x(1)\big)^2\Big), \tag{5.3.100}$$

which implies the assertion. In this case $Z_1 = \infty$, and for any $\gamma > 1$, we have $Z_2 < \infty$ and $\lim_{u\to\infty} f(u)/\Phi(u) = \infty$, that is, condition (5.3.95) of Theorem 5.3.24 is violated.

Theorem 5.3.26. *Assume that*

$$\liminf_{|u|\to\infty} \frac{f(u)}{\Phi(u)} > 0, \tag{5.3.101}$$

where $\Phi(u) = |u|^{\gamma-1}u$ with $\gamma > 1$. If $Z_1 = \infty$ and $Z_2 = \infty$, then every solution of equation (5.3.1) in the class M^+ belongs to $M_\infty^+(S)$, that is, $M_\infty^+(R) = \emptyset = M_B^+$.

Proof. In view of Proposition 5.3.15 it suffices to show that $M_\infty^+(R) = \emptyset$. From condition (5.3.101) there exist two constants ℓ and a such that $f(u) \geq \ell\Phi(u)$ for $u \geq a$. Let $\{x(k)\}$ be a solution of equation (5.3.1) in the class $M_\infty^+(R)$, and without loss of generality, assume $x(k) \geq a$ and $\Delta x(k) > 0$ for $k \in \mathbb{N}$. Let $\{z(k)\}$ be a solution of

$$\Delta\big(c(k)\Psi(\Delta z(k))\big) = q(k)\ell\Phi\big(z(k+1)\big) \tag{5.3.102}$$

with $z(1) = x(1)$ and $\Delta z(1) = \Delta x(1)$. From Theorem 5.3.21 with $f_2(u) = f(u)$, $f_1(u) = \ell\Phi(u)$, $q_1(k) = q_2(k) = q(k)$, we have

$$\Psi\big(\Delta x(k)\big) - \Psi\big(\Delta z(k)\big) \geq 0. \tag{5.3.103}$$

In view of Corollary 5.3.20(a_2), $\{z(k)\}$ belongs to the class $M_\infty^+(S)$. Now the desired contradiction is obtained from (5.3.103) as $k \to \infty$. This contradiction completes the proof. □

Example 5.3.27. Consider the equation

$$\Delta\left(\frac{1}{k+1}\Delta x(k)\right) = \frac{2k}{k+2} f\big(x(k+1)\big) \quad \text{for } k > 1, \tag{5.3.104}$$

where $f \in C(\mathbb{R},\mathbb{R})$ with $xf(x) > 0$ for $x \neq 0$ and $|f(u)| = 1/|u|$ for $|u| \geq 1$. Equation (5.3.104) has an unbounded solution $\{x(k)\}$ with $x(k) = k(k-1)$ which belongs to the class $M_\infty^+(R)$. Here, $Z_1 = \infty$, $Z_2 = \infty$, and $\lim_{|u|\to\infty} f(u)/\Phi(u) = 0$ for any $\gamma > 1$, that is, condition (5.3.101) of Theorem 5.3.26 is violated.

The results of this subsection are summarized as follows.

Theorem 5.3.28. *For equation (5.3.1), the following hold.*

(1) *Assume (j_1) holds. Then $M_B^+ = \emptyset$. In addition, if condition (5.3.101) holds, then $M_\infty^+(\mathbb{R}) = \emptyset$ and $M_\infty^+(S) \neq \emptyset$, that is, every solution of equation (5.3.1) in M^+ is strongly monotone.*
(2) *Assume (j_2) holds. Then $M_B^+ = \emptyset$. In addition, if condition (5.3.95) holds, then $M_\infty^+(\mathbb{R}) \neq \emptyset$. If condition (5.3.41) holds, then $M_\infty^+(S) = \emptyset$. Consequently, if both conditions (5.3.41) and (5.3.95) hold, then every solution of equation (5.3.1) is regularly increasing.*
(3) *Assume (j_3) holds. Then $M_B^+ \neq \emptyset$ and $M_\infty^+(\mathbb{R}) = \emptyset$. In addition, if condition (5.3.41) holds, then $M_\infty^+(S) = \emptyset$, that is, every solution of equation (5.3.1) in M^+ is bounded.*

Proof. We first show (1). From Proposition 5.3.15, we obtain $M_B^+ = \emptyset$, and by Theorem 5.3.26, $M_\infty^+(\mathbb{R}) = \emptyset$. Since $M^+ \neq 0$, we must have $M_B^+(S) = \emptyset$.

Now we show (2). Again, from Proposition 5.3.15, it follows that $M_B^+ = \emptyset$. Also we see that $M_\infty^+(\mathbb{R}) \neq \emptyset$ by Theorem 5.3.24 and $M_\infty^+(S) = \emptyset$ by Theorem 5.3.22.

Finally we prove (3). By Theorem 5.3.18, it follows that $M_B^+ \neq \emptyset$. We will show that $M_\infty^+(\mathbb{R}) = \emptyset$. Assume that there exists a solution $\{x(k)\}$ of equation (5.3.1) in the class $M_\infty^+(\mathbb{R})$. Without loss of generality, assume that $x(k) > 0$ and $\Delta x(k) > 0$ for $k \geq m \in \mathbb{N}$. Since $c(k)\Psi(\Delta x(k))$ is bounded, there exists a positive constant b such that

$$\Delta x(k) \leq \Psi^{-1}\left(\frac{b}{c(k)}\right). \tag{5.3.105}$$

Summing (5.3.105) from m to $k-1$, we have

$$x(k) \leq x(m) + \sum_{j=m}^{k-1} \Psi^{-1}\left(\frac{b}{c(j)}\right). \tag{5.3.106}$$

Since $Z_1 < \infty$, we have $Y_3 < \infty$, and from (5.3.106) as $k \to \infty$, we arrive at a contradiction. Finally, the assertion $M_\infty^+(S) = \emptyset$ follows from Theorem 5.3.16. This completes the proof. □

Next, for equation (5.3.80) we obtain the following consequence.

Corollary 5.3.29. *For equation (5.3.80) the following hold.*

(1) *Assume (j_1) holds. Then every solution of equation (5.3.80) in M^+ is strongly increasing.*

(2) *Assume (j_2) and $\alpha \geq \gamma$ hold. Then every solution of equation (5.3.80) in M^+ is regularly increasing.*

(3) *Assume (j_3) holds. Then all solutions of equation (5.3.80) are bounded.*

Example 5.3.30. The difference equation

$$\Delta\big(|\Delta x(k)|\Delta x(k)\big) = \left(1 - \frac{1}{e}\right)^2\left(1 - \frac{1}{e^2}\right)e^{1-k}x(k+1) \quad \text{for } k \in \mathbb{N} \tag{5.3.107}$$

has a bounded solution $x(k) = e^{-k}$. Clearly, by Corollary 5.3.29(3), all solutions of equation (5.3.107) are bounded.

Remark 5.3.31. (i) The results of this section when specialized to equations of the form

$$\Delta\big(c(k)\Delta x(k)\big) = q(k)f\big(x(k+1)\big) \tag{5.3.108}$$

supplement the results of Chapter 4. The details are left to the reader.

(ii) It can be easily checked that most of the results of this section with minor modifications are also valid for equation (5.3.1) when Ψ is defined as in (I_3).

5.4. Notes and general discussions

(1) Theorems 5.1.1 and 5.1.3 are taken from Wong and Agarwal [283]. Lemma 5.1.4 and Theorems 5.1.5 and 5.1.6 are due to Wong and Agarwal [284] without any restrictions on the difference of solutions of the considered equations. For other related works we refer to Thandapani [265] and Li [198]. We note that Lemma 5.1.4 and Theorems 5.1.5 and 5.1.6 can also be extended to cover more general equations of the form

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)f\big(x(k+1)\big) = e(k), \tag{5.4.1}$$

where $\{c(k)\}$, $\{q(k)\}$, f, and Ψ are as in equation (5.1.2) and $\{e(k)\}$ is a sequence of real numbers. Under the additional condition

$$\sum_{j=m\geq 1}^{\infty} \big|e(j)\big| < \infty, \tag{5.4.2}$$

one can easily see that the conclusions of Theorems 5.1.5 and 5.1.6 are replaced by "every solution $\{x(k)\}$ of equation (5.4.1) is either oscillatory or satisfies $\liminf_{k\to\infty} |x(k)| = 0$." The details are left to the reader.

(2) Theorems 5.1.8–5.1.20 for superlinear equations and Theorem 5.1.21 for sublinear equations are due to Wong and Agarwal [283]. Theorem 5.1.23 is taken from Li [198]. Theorems 5.1.26–5.1.34 are new and they are corrections of some of the results due to Wong and Agarwal [282]. Lemmas 5.1.37 and 5.1.38 and Theorem 5.1.39 are related to results from Section 3.9.

(3) Theorem 5.2.2 is taken from Wong and Agarwal [283], and Theorem 5.2.3 is due to Peng et al. [220].

(4) The results of Section 5.3 are taken from Cecchi, Došlá, and Marini [80, 81]. Theorems 5.3.5 and 5.3.8 answer some open problems formulated for $\alpha = 1$ due to Thandapani et al. [268]. Some of the results of this section may be extended to equation (5.4.1), and when $\alpha = 1$ we refer the reader to Thandapani and Ravi [270] and Cheng and Zhang [95].

(5) It would be interesting to obtain oscillation and/or asymptotic behavior of equation (5.4.1) under an appropriate assumption on the sequence $\{e(k)\}$.

6 Oscillation theory for difference equations with deviating arguments

Difference equations with or without deviating arguments are in fact recurrence relations. The effect of such deviating arguments may not be as clear as that in the continuous counterpart of such difference equations. For example, one can easily see that both functional differential equations

$$x''(t) - 4x[t-\tau] = 0 \qquad \text{and} \qquad x''(t) - e^{\tau}(e-1)^2 x[t-\tau] = 0 \tag{6.0.1}$$

are oscillatory for appropriate $\tau > 0$ and both equations are nonoscillatory if $\tau = 0$. For the discrete counterpart of these equations, namely, the difference equations

$$\Delta^2 x(k) - 4x[k-\tau] = 0 \qquad \text{and} \qquad \Delta^2 x(k) - e^{\tau}(e-1)^2 x[t-\tau] = 0, \tag{6.0.2}$$

respectively, we see that the first equation is oscillatory if τ is an even, nonnegative integer and has an oscillatory solution $\{x(k)\}$ with $x(k) = (-1)^k$, while the other equation has a nonoscillatory solution $\{x(k)\}$ with $x(k) = e^k$.

It is a well-known fact that there is a striking similarity between the qualitative theories of functional differential equations with deviating arguments and difference equations with deviating arguments. Moreover, it turns out that it makes sense to study qualitative properties of difference equations with deviating arguments including very important oscillation theory. Therefore, the purpose of this chapter is to investigate the oscillatory behavior of certain difference equations with deviating arguments. In Section 6.1, we establish some oscillation criteria for certain linear and nonlinear difference equations with deviating arguments, and in Section 6.2 we extend some of the results obtained in Section 6.1 and present some other oscillation criteria for more general difference equations including half-linear difference equations with deviating arguments. Oscillation criteria for linear difference equations with deviating arguments via their associated characteristic equations are given in Section 6.3. In Section 6.4, we present many criteria for the oscillation and almost oscillation of linear and nonlinear damped difference equations with deviating arguments, while oscillation results for forced difference equations with deviating arguments are given in Section 6.5.

6.1. Oscillation criteria (I)

In this section we will be concerned with the oscillation of all solutions of second-order difference equations of the form

$$\Delta\big(c(k)\Delta x(k)\big) + q(k)f\big(x[g(k)]\big) = 0, \tag{6.1.1}$$

and the mixed-type equation

$$\Delta\big(c(k)\Delta x(k)\big) - q(k)f\big(x[g(k)]\big) - q_1(k)f_1\big(x[g_1(k)]\big) = 0, \tag{6.1.2}$$

where

(i) $\{c(k)\}$, $\{q(k)\}$, $\{q_1(k)\}$ are sequences of real numbers with $c(k) > 0$ and $q(k) \geq 0$ and $q_1(k) \geq 0$ eventually,
(ii) $f, f_1 \in C(\mathbb{R}, \mathbb{R})$ with $xf(x) > 0$, $xf_1(x) > 0$, $f'(x) \geq 0$, and $f_1'(x) \geq 0$ for $x \neq 0$,
(iii) $\{g(k)\}$ and $\{g_1(k)\}$ are sequences of nondecreasing nonnegative integers with $\lim_{k\to\infty} g(k) = \infty$ and $\lim_{k\to\infty} g_1(k) = \infty$.

6.1.1. Oscillation of equation (6.1.1)

First, we establish the following auxiliary results.

Lemma 6.1.1. *Suppose that*

$$\sum^{\infty} \frac{1}{c(j)} = \infty. \tag{6.1.3}$$

If $\{x(k)\}$ is a nonoscillatory solution of equation (6.1.1), then $x(k)\Delta x(k) > 0$ eventually.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). Then there exists an integer $m \in \mathbb{N}$ such that $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$. Since $\Delta(c(k)\Delta x(k)) \leq 0$ eventually, we see that the sequence $\{\Delta x(k)\}$ is eventually of one sign. Assume that there exists an integer $m_1 \geq m$ such that $\Delta x(m_1) \leq 0$. From equation (6.1.1), it follows that $\{c(k)\Delta x(k)\}$ is a nonincreasing sequence, and thus there exists an integer $m_2 \geq m_1$ such that for all $k \geq m_2 + 1$ we have

$$c(k)\Delta x(k) \leq c(m_2+1)\Delta x(m_2+1) < c(m_1)\Delta x(m_1) \leq 0, \tag{6.1.4}$$

so

$$\Delta x(k) \leq \frac{c(m_2+1)}{c(k)}\Delta x(m_2+1) < 0. \tag{6.1.5}$$

Summing both sides of (6.1.5) from m_2 to $k-1$, we obtain

$$x(k) \le x(m_2) + c(m_2+1)\Delta x(m_2+1) \sum_{j=m_2}^{k-1} \frac{1}{c(j)} \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{6.1.6}$$

which contradicts the fact that $x(k) > 0$ eventually. Thus $\Delta x(k) > 0$ eventually, and this completes the proof. □

Lemma 6.1.2. *If condition (6.1.3) holds and $\{x(k)\}$ is an eventually positive solution of equation (6.1.1), then*

$$x(k) \ge C(k-1, m)c(k)\Delta x(k) \quad \text{eventually, with } m \in \mathbb{N}, \tag{6.1.7}$$

where

$$C(k, \ell) = \sum_{j=\ell}^{k} \frac{1}{c(j)} \quad \text{for } k \ge \ell \ge m \in \mathbb{N}. \tag{6.1.8}$$

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). There exists $m \in \mathbb{N}$ such that $x(k) > 0$, $x[g(k)] > 0$, and $\Delta x(k) > 0$ for $k \ge m$. From equation (6.1.1), it follows that the sequence $\{c(k)\Delta x(k)\}$ is nonincreasing. Thus

$$\begin{aligned} x(k) \ge x(k) - x(m) &= \sum_{j=m}^{k-1} \Delta x(j) = \sum_{j=m}^{k-1} \frac{c(j)\Delta x(j)}{c(j)} \\ &\ge c(k-1)\Delta x(k-1) \sum_{j=m}^{k-1} \frac{1}{c(j)} \\ &\ge \left(c(k) \sum_{j=m}^{k-1} \frac{1}{c(j)} \right) \Delta x(k) = C(k-1, m)c(k)\Delta x(k), \end{aligned} \tag{6.1.9}$$

as required. □

We will assume that

$$0 \le g(k) \le k, \qquad \{c(k)\} \text{ is a nondecreasing sequence}, \tag{6.1.10}$$

$$f(u) - f(v) = F(u, v)(u - v) \quad \text{for } u, v \ne 0, \text{ where } F \text{ is nonnegative}. \tag{6.1.11}$$

Lemma 6.1.3. *If conditions (6.1.3), (6.1.10), and (6.1.11) hold and $\{x(k)\}$ is a nonoscillatory solution of equation (6.1.1), then $w(k) = c(k)\Delta x(k)/f(x[g(k)])$ is a positive solution of the Riccati-type inequality*

$$\Delta w(k) + q(k) + \left(\frac{F(x[g(k+1)], x[g(k)])}{c[g(k)]} \right) w(k)w(k+1) \le 0 \tag{6.1.12}$$

eventually.

Proof. From Lemma 6.1.1 we have $w(k) > 0$ eventually. Also,

$$\begin{aligned}\Delta w(k) &= \frac{c(k+1)\Delta x(k+1)}{f(x[g(k+1)])} - \frac{c(k)\Delta x(k)}{f(x[g(k)])}\\ &= -q(k) - \left(\frac{F(x[g(k+1)], x[g(k)])}{c[g(k)]}\right)\left(\frac{c[g(k)]\Delta x[g(k)]}{c(k)\Delta x(k)}\right) w(k)w(k+1).\end{aligned} \tag{6.1.13}$$

Since $g(k) \le k$, we see that

$$\frac{c[g(k)]\Delta x[g(k)]}{c(k)\Delta x(k)} \ge 1, \tag{6.1.14}$$

and so,

$$\Delta w(k) + q(k) + \frac{F(x[g(k+1)], x[g(k)])}{c[g(k)]} w(k)w(k+1) \le 0. \tag{6.1.15}$$

Hence w satisfies (6.1.12). □

Now, if we assume that

$$F(u,v) \ge \lambda, \quad \text{where } \lambda \text{ is a positive constant,} \tag{6.1.16}$$

then inequality (6.1.12) implies

$$\Delta w(k) + q(k) + \lambda \frac{w(k)w(k+1)}{c[g(k)]} \le 0 \tag{6.1.17}$$

eventually. From (6.1.17) one can easily see that $\{w(k)\}$ is a positive nonincreasing sequence and $w(k+1) \le w(k)$. Thus (6.1.17) implies

$$\Delta w(k) + q(k) + \lambda \frac{w^2(k+1)}{c[g(k)]} \le 0 \quad \text{eventually.} \tag{6.1.18}$$

Lemma 6.1.4. *Let conditions (6.1.3), (6.1.10), (6.1.11), and (6.1.16) hold and*

$$Q(k) = \sum_{j=k}^{\infty} q(j) \quad \textit{exists for every } k \ge m \textit{ for some } m \in \mathbb{N}. \tag{6.1.19}$$

If $\{x(k)\}$ is a nonoscillatory solution of (6.1.1), then $w(k) = c(k)\Delta x(k)/f(x[g(k)])$ satisfies the inequality

$$\begin{gathered} w(k) \ge Q(k) + \sum_{j=k}^{\infty} \frac{\lambda w^2(j+1)}{c[g(j)]},\\ \lim_{k\to\infty} w(k) = 0.\end{gathered} \tag{6.1.20}$$

Proof. Summing both sides of (6.1.18) from k to $s-1 \geq k \geq m \in \mathbb{N}$ and letting $s \to \infty$, we obtain

$$\lim_{s\to\infty} w(s) - w(k) + Q(k) + \sum_{j=k}^{\infty} \frac{\lambda w^2(j+1)}{c[g(j)]} \leq 0. \tag{6.1.21}$$

Since $\{w(k)\}$ is nonincreasing with $w(k) > 0$, we find that $\beta = \lim_{s\to\infty} w(s)$ exists and $\beta \geq 0$. If $\beta > 0$, then $w(k) \geq \beta/2$ for all sufficiently large k. From (6.1.21) one can easily see that

$$\sum_{j=k}^{\infty} \frac{w^2(j+1)}{c[g(j)]} < \infty \quad \forall k \geq m \text{ for some } m \in \mathbb{N}, \tag{6.1.22}$$

but

$$\infty > \sum_{j=m}^{\infty} \frac{w^2(j+1)}{c[g(j)]} \geq \frac{\beta^2}{4} \sum_{j=m}^{\infty} \frac{1}{c[g(j)]} \geq \frac{\beta^2}{4} \sum_{j=m}^{\infty} \frac{1}{c(j)}. \tag{6.1.23}$$

This contradiction implies $\beta = 0$. From (6.1.21), we have

$$w(k) \geq Q(k) + \sum_{j=k}^{\infty} \frac{\lambda w^2(j+1)}{c[g(j)]}. \tag{6.1.24}$$

This completes the proof. □

Note that $\sum^{\infty} 1/c[g(j)] = \infty$, $0 \leq g(k) \leq k$, does not imply $\sum^{\infty} 1/c(j) = \infty$. In this case, let $c(k) = k^2$ and $g(k) = \sqrt{k}$ for $k \in \mathbb{N}$. Then $\sum^{\infty} 1/c[g(j)] = \sum^{\infty} 1/j$ diverges while $\sum^{\infty} 1/c(j) = \sum^{\infty} 1/j^2$ converges.

In the case of an advanced equation (6.1.1), that is, $g(k) \geq k+1$ and f is not a monotonic function (i.e., condition (ii) is violated), Lemma 6.1.4 takes the following form.

Lemma 6.1.5. *Let $g(k) \geq k+1$ for $k \in \mathbb{N}$, condition (6.1.3) hold, and*

$$\frac{f(x)}{x} \geq \gamma > 0 \quad \text{for } x \neq 0. \tag{6.1.25}$$

If $\{x(k)\}$ is an eventually positive solution of (6.1.1), then $w(k) = c(k)\Delta x(k)/x(k)$ is *eventually positive and satisfies*

$$\begin{gathered}\lim_{k\to\infty} w(k) = 0,\\ \sum_{j=k}^{\infty} q(j) \prod_{i=j}^{g(j)-1}\left(1+\frac{w(i)}{c(i)}\right) < \infty,\\ \sum_{j=k}^{\infty} \frac{w(j)w(j+1)}{c(j)} < \infty,\\ w(k) \geq \gamma \sum_{j=k}^{\infty} q(j) \prod_{i=j}^{q(j)-1}\left(1+\frac{w(i)}{c(i)}\right) + \sum_{j=k}^{\infty} \frac{w(j)w(j+1)}{c(j)}.\end{gathered} \tag{6.1.26}$$

Proof. Let $\{x(k)\}$ be an eventually positive solution of (6.1.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. By Lemma 6.1.1, we see that $w(k) = c(k)\Delta x(k)/x(k) > 0$ for $k \geq m_1 \geq m$. Now

$$\frac{w(k)}{c(k)} = \frac{x(k+1)-x(k)}{x(k)}, \quad \text{so} \quad \frac{x(k+1)}{x(k)} = 1 + \frac{w(k)}{c(k)}. \tag{6.1.27}$$

Thus

$$\prod_{j=m}^{k-1}\left(\frac{x(j+1)}{x(j)}\right) = \prod_{j=m}^{k-1}\left(1+\frac{w(j)}{c(j)}\right), \tag{6.1.28}$$

so

$$x(k) = x(m)\prod_{j=m}^{k-1}\left(1+\frac{w(j)}{c(j)}\right). \tag{6.1.29}$$

Hence it follows that

$$\begin{aligned}\Delta w(k) &= \frac{c(k+1)\Delta x(k+1)}{x(k+1)} - \frac{c(k)\Delta x(k)}{x(k)}\\ &= -q(k)\frac{f(x[g(k)])}{x(k)} - \frac{w(k)w(k+1)}{c(k)}\\ &\leq -\gamma q(k)\frac{x[g(k)]}{x(k)} - \frac{w(k)w(k+1)}{c(k)}\\ &= -\gamma q(k)\prod_{j=k}^{g(k)-1}\left(1+\frac{w(j)}{c(j)}\right) - \frac{w(k)w(k+1)}{c(k)}.\end{aligned} \tag{6.1.30}$$

Therefore

$$\Delta w(k) + \gamma q(k)\prod_{j=k}^{g(k)-1}\left(1+\frac{w(j)}{c(j)}\right) + \frac{w(k)w(k+1)}{c(k)} \leq 0. \tag{6.1.31}$$

Summing (6.1.31) from $m_2 \geq m_1$ to $k-1$, we obtain

$$w(k) - w(m_2) + \sum_{j=m_2}^{k-1} \gamma q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{w(i)}{c(i)}\right) + \sum_{j=m_2}^{k-1} \frac{w(j)w(j+1)}{c(j)} \leq 0. \tag{6.1.32}$$

From (6.1.31), we have $\Delta w(k) \leq 0$. Thus $0 \leq \lim_{k\to\infty} w(k) \leq \beta < \infty$. From (6.1.32), we find

$$\begin{gathered} \sum_{j=k}^{\infty} q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{w(i)}{c(i)}\right) < \infty, \\ \sum_{j=k}^{\infty} \frac{w(j)w(j+1)}{c(j)} < \infty. \end{gathered} \tag{6.1.33}$$

If $\beta > 0$, then there exists $m_3 \geq m_2$ such that $w(k)w(k+1) \geq \beta^2/4$ for $k \geq m_3$. Therefore

$$\sum_{j=m_3}^{k-1} \frac{w(j)w(j+1)}{c(j)} \geq \frac{\beta^2}{4} \sum_{j=m_3}^{k-1} \frac{1}{c(j)} \longrightarrow \infty \quad \text{as } k \longrightarrow \infty, \tag{6.1.34}$$

which contradicts $\sum_{j=m_3}^{\infty} w(j)w(j+1)/c(j) < \infty$. Thus $\lim_{k\to\infty} w(k) = 0$ and

$$w(m_3) \geq \gamma \sum_{j=m_3}^{\infty} q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{w(i)}{c(i)}\right) + \sum_{j=m_3}^{\infty} \frac{w(j)w(j+1)}{c(j)}. \tag{6.1.35}$$

This completes the proof. □

We will also need the following two lemmas. Consider the delay difference equation and inequalities:

$$\Delta x(k) + q(k)x[k - d(k)] = 0, \tag{6.1.36}$$

$$\Delta x(k) + q(k)x[k - d(k)] \leq 0, \tag{6.1.37}$$

$$\Delta x(k) + q(k)x[k - d(k)] \geq 0, \tag{6.1.38}$$

where $\{q(k)\}$ is a sequence of nonnegative real numbers and $\{d(k)\}$ is a sequence of positive integers such that $\{k - d(k)\}$ is increasing and $\lim_{k\to\infty}(k - d(k)) = \infty$.

Lemma 6.1.6. *If*

$$\liminf_{k\to\infty}\left[\frac{1}{d(k)}\sum_{j=k-d(k)}^{k-1} q(j)\right] > \limsup_{k\to\infty}\frac{[d(k)]^{d(k)}}{[1+d(k)]^{1+d(k)}}, \tag{6.1.39}$$

then the following statements hold.

(i) *Inequality (6.1.37) has no eventually positive solution.*
(ii) *Inequality (6.1.38) has no eventually negative solution.*
(iii) *Every solution of equation (6.1.36) oscillates.*

In the case when $d(k) \equiv d \in \mathbb{N}$, condition (6.1.39) is reduced to

$$\liminf_{k\to\infty}\sum_{j=k-d}^{k-1} q(j) > \left(\frac{d}{1+d}\right)^{d+1}. \tag{6.1.40}$$

For the advanced difference equation and inequalities

$$\Delta x(k) - q(k)x[k+a(k)] = 0, \tag{6.1.41}$$

$$\Delta x(k) - q(k)x[k+a(k)] \geq 0, \tag{6.1.42}$$

$$\Delta x(k) - q(k)x[k+a(k)] \leq 0, \tag{6.1.43}$$

where $\{q(k)\}$ is a sequence of nonnegative real numbers, $\{a(k)\}$ is a sequence of positive integers, $a(k) > 1$ for all $k \in \mathbb{N}$, and $\{k+a(k)\}$ is increasing, we have the following lemma.

Lemma 6.1.7. *If*

$$\liminf_{k\to\infty}\left[\frac{1}{a(k)-1}\sum_{j=k+1}^{k+a(k)-1} q(j)\right] > \limsup_{k\to\infty}\frac{(a(k)-1)^{a(k)-1}}{(a(k))^{a(k)}}, \tag{6.1.44}$$

then the following statements hold.

(I) *Inequality (6.1.42) has no eventually positive solution.*
(II) *Inequality (6.1.43) has no eventually negative solution.*
(III) *Every solution of equation (6.1.41) oscillates.*

Also, for the case when $a(k) \equiv a \in \mathbb{N}$, condition (6.1.44) is reduced to

$$\liminf_{k\to\infty}\sum_{j=k+1}^{k+a-1} q(j) > \left(\frac{a-1}{a}\right)^{a}. \tag{6.1.45}$$

Finally, we will need the following useful lemma.

Lemma 6.1.8. *In Lemmas 6.1.6 and 6.1.7 let $d(k) \equiv d > 0$, $a(k) \equiv a > 0$ and*

$$\sum_{j=k-d}^{k-1} q(j) > 0, \quad \sum_{j=k+1}^{k+a-1} q(j) > 0 \quad \text{for all large } k \in \mathbb{N}. \tag{6.1.46}$$

Then the following statements hold.

(I_1) *If the delay difference inequality (6.1.37) has an eventually positive solution, then the delay difference equation (6.1.36) has an eventually positive solution.*

(I_2) *If the advanced difference inequality (6.1.42) has an eventually positive solution, then the advanced difference equation (6.1.41) has an eventually positive solution.*

To obtain the next result, define for $n \in \mathbb{N}_0$ and every constant $\lambda > 0$

$$\begin{aligned} h_0(k) &= Q(k) > 0, \\ h_1(k) &= \sum_{j=k}^{\infty} \frac{h_0^2(j+1)}{c[g(k)]}, \\ &\vdots \\ h_{n+1}(k) &= \sum_{j=k}^{\infty} \frac{\left[h_0(j+1) + \lambda h_n(j+1)\right]^2}{c[g(j)]}. \end{aligned} \tag{6.1.47}$$

If $h_\ell(k)$ for $\ell \in \{0, 1, \ldots, n\}$ exist, then $h_{n+1}(k) \geq h_n(k)$ and $\lim_{k\to\infty} h_n(k) = 0$.

Theorem 6.1.9. *Let the hypotheses of Lemma 6.1.4 hold. If equation (6.1.1) has a nonoscillatory solution, then h_n, $n \in \mathbb{N}_0$, in (6.1.47) are defined and*

$$\lim_{n\to\infty} h_n(k) = h(k) < \infty. \tag{6.1.48}$$

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.1.1), say, $x(k) \neq 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set $w(k) = c(k)\Delta x(k)/f(x[g(k)])$. Then, from Lemma 6.1.4 we have

$$w(k) \geq Q(k) + \sum_{j=k}^{\infty} \frac{\lambda w^2(j+1)}{c[g(j)]}, \tag{6.1.49}$$

and hence $w(k) \geq Q(k) = h_0(k)$ or $w^2(k+1) \geq h_0^2(k+1)$. Thus

$$h_1(k) \leq \sum_{j=k}^{\infty} \frac{w^2(j+1)}{c[g(j)]}, \qquad w(k) \geq h_0(k) + \lambda h_1(k). \tag{6.1.50}$$

Now, by induction we find $w(k) \geq h_0(k) + \lambda h_n(k)$. Thus

$$w(k) \geq h_{n+1}(k) = \sum_{j=k}^{\infty} \frac{\left[h_0(j+1) + h_n(j+1)\right]^2}{c[g(j)]}. \tag{6.1.51}$$

Thus the sequence $\{h_n(k)\}$ is bounded. Note that $\{h_n(k)\}$ is nondecreasing which implies that (6.1.47) is defined and (6.1.48) holds. This completes the proof. $\square$

From the proof of Theorem 6.1.9 we can easily obtain the following sufficient condition for equation (6.1.1) to be oscillatory.

Theorem 6.1.10. *Let the hypotheses of Lemma 6.1.4 hold. Then equation (6.1.1) is oscillatory if one of the following conditions is satisfied:*

(i_1) *h_n in (6.1.47) exist for $n \in \{1, 2, \ldots, N-1\}$ but h_N does not exist, where $N \in \mathbb{N}$,*

(i_2) *h_n in (6.1.47) exist, but for all sufficiently large $m \in \mathbb{N}$, there is $m^* \geq m$ such that $\lim_{n\to\infty} h_n(m^*) = \infty$.*

Example 6.1.11. Consider the difference equation

$$\Delta\left(\frac{1}{k^2}\Delta x(k)\right) + \left[\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right]x[k-1] = 0 \quad \text{for } k \in \mathbb{N}. \tag{6.1.52}$$

It is easy to see that for (6.1.52), the hypotheses of Lemma 6.1.4 are satisfied and $h_0(k) = Q(k) = 1/\sqrt{k} < \infty$. But

$$h_1(k) = \sum_{j=k}^{\infty} \frac{h_0^2(j+1)}{c[g(j)]} = \sum_{j=k}^{\infty} \frac{(j-1)^2}{j+1} = \infty. \tag{6.1.53}$$

Thus, by Theorem 6.1.10, equation (6.1.52) is oscillatory.

In the proof of Theorem 6.1.9, if $\{x(k)\}$ is a nonoscillatory solution of equation (6.1.1) and $w(k) = c(k)\Delta x(k)/f(x[g(k)])$, then we see that

$$w(k) \geq Q(k) + \sum_{j=k}^{\infty} \lambda \frac{w^2(j+1)}{c[g(j)]}. \tag{6.1.54}$$

Let $u(k) = \sum_{j=k}^{\infty} w^2(j+1)/c[g(j)]$, then $w(k) \geq Q(k) + u(k)$ and

$$\begin{aligned} -\Delta u(k) &= \frac{w^2(k+1)}{c[g(k)]} \\ &\geq \frac{1}{c[g(k)]}\left[Q(k+1) + u(k+1)\right]^2 \\ &\geq \frac{4Q(k+1)u(k+1)}{c[g(k)]}, \end{aligned} \tag{6.1.55}$$

that is,

$$u(k) - u(k+1) \geq \frac{4Q(k+1)u(k+1)}{c[g(k)]}, \tag{6.1.56}$$

so

$$\frac{u(k+1)}{u(k)} \leq \left(1 + \frac{4Q(k+1)}{c[g(k)]}\right)^{-1}, \tag{6.1.57}$$

and hence we see that

$$u(k) \leq u(m) \prod_{j=m}^{k-1} \left(1 + \frac{4Q(j+1)}{c[g(j)]}\right)^{-1}. \tag{6.1.58}$$

Note that

$$u(k) = \sum_{j=k}^{\infty} \frac{w^2(j+1)}{c[g(j)]} \geq \sum_{j=k}^{\infty} \frac{h_0^2(j+1)}{c[g(j)]} = h_1(k), \tag{6.1.59}$$

so $w(k) \geq h_0(k) + \lambda h_1(k)$. We then have

$$u(k) = \sum_{j=k}^{\infty} \frac{w^2(j+1)}{c[g(j)]} \geq \sum_{j=k}^{\infty} \frac{1}{c[g(j)]} \left(h_0(j+1) + \lambda h_1(j+1)\right)^2 = h_2(k), \tag{6.1.60}$$

and by induction we conclude that $u(k) \geq h_n(k)$ for all $n \in \mathbb{N}$. It then follows from the above that

$$h_n(k) \prod_{j=m}^{k-1} \left(1 + \frac{4Q(j+1)}{c[g(j)]}\right) \leq u(m) \quad \text{for some } m \in \mathbb{N}. \tag{6.1.61}$$

Thus

$$\lim_{n \to \infty} h_n(k) \prod_{j=m}^{k-1} \left(1 + \frac{4Q(j+1)}{c[g(j)]}\right) \leq u(m), \tag{6.1.62}$$

that is,

$$h(k) \prod_{j=m}^{k-1} \left(1 + \frac{4Q(j+1)}{c[g(j)]}\right) \leq u(m). \tag{6.1.63}$$

Therefore we conclude that $h_n(k)$ and $h(k)$ in Theorem 6.1.9 satisfy

$$\begin{aligned}
&\limsup_{k\to\infty} h_n(k)\prod_{j=m}^{k-1}\left(1+\frac{4Q(j+1)}{c[g(j)]}\right)<\infty,\\
&\limsup_{k\to\infty} h(k)\prod_{j=m}^{k-1}\left(1+\frac{4Q(j+1)}{c[g(j)]}\right)<\infty
\end{aligned} \tag{6.1.64}$$

for some $m\in\mathbb{N}$.

Based on the above computations, one can easily observe the following result.

Theorem 6.1.12. *Let the hypotheses of Lemma 6.1.4 hold. Then (6.1.1) is oscillatory if one of the following conditions is satisfied:*

(I_1) *there exists $N\in\mathbb{N}$ such that*

$$\limsup_{k\to\infty} h_N(k)\prod_{j=m}^{k-1}\left(1+\frac{4Q(j+1)}{c[g(j)]}\right)=\infty, \tag{6.1.65}$$

(I_2) $\limsup_{k\to\infty} h(k)\prod_{j=m}^{k-1}(1+4Q(j+1)/c[g(j)])=\infty$,

(I_3) $\lim_{k\to\infty}\sum_{j=m_1}^{k}\prod_{i=m}^{j-1}(1+4Q(i+1)/c[g(i)])<\infty$ *and there exists $N\in\mathbb{N}$ such that* $\lim_{k\to\infty}\sum_{j=m_1}^{k}h_N(j)=\infty$, *where $m_1\geq m$ for some $m\in\mathbb{N}$.*

To obtain similar results as those given above for (6.1.1) when $g(k)\geq k+1$, define a sequence $\{h_n(k)\}$ for $n\in\mathbb{N}_0$ as follows:

$$\begin{aligned}
h_0(k)&=\sum_{j=k}^{\infty}\gamma q(j),\\
h_1(k)&=\sum_{j=k}^{\infty}\gamma q(j)\prod_{i=j}^{g(j)-1}\left(1+\frac{h_0(i)}{c(i)}\right)+\sum_{j=k}^{\infty}\frac{h_0(j)h_0(j+1)}{c(j)},\\
&\ \ \vdots\\
h_{n+1}(k)&=\sum_{j=k}^{\infty}\gamma q(j)\prod_{i=j}^{g(j)-1}\left(1+\frac{h_n(i)}{c(i)}\right)+\sum_{j=k}^{\infty}\frac{h_n(j)h_n(j+1)}{c(j)},
\end{aligned} \tag{6.1.66}$$

where $\gamma>0$ is a constant.

We note that $h_n(k)$, $n\in\mathbb{N}_0$ are defined, that is, exist and are finite. We also see that $h_{n+1}(k)\geq h_n(k)$ and $\lim_{k\to\infty}h_n(k)=0$ for all $n\in\mathbb{N}$.

Now we prove the following result.

Theorem 6.1.13. *Let the hypotheses of Lemma 6.1.5 hold. If (6.1.1) has a nonoscillatory solution, then every $h_n(k)$ in (6.1.66) is defined and* $\lim_{n\to\infty}h_n(k)=h(k)<\infty$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.1.1), say, $x(k) > 0$ eventually. Let $w(k) = c(k)\Delta x(k)/x(k)$. By Lemma 6.1.5, we see that $\{w(k)\}$ is a decreasing sequence, $\lim_{k\to\infty} w(k) = 0$, and

$$w(k) \geq \gamma \sum_{j=k}^{\infty} q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{w(i)}{c(i)}\right) + \sum_{j=k}^{\infty} \frac{w(j)w(j+1)}{c(j)}. \tag{6.1.67}$$

Consequently, we have

$$\begin{aligned} w(k) &\geq \sum_{j=k}^{\infty} \gamma q(j) = h_0(k), \\ w(k) &\geq \sum_{j=k}^{\infty} \gamma q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{h_0(i)}{c(i)}\right) + \sum_{j=k}^{\infty} \frac{h_0(j)h_0(j+1)}{c(j)} = h_1(k), \end{aligned} \tag{6.1.68}$$

and by induction, we can easily obtain that

$$w(k) \geq \sum_{j=k}^{\infty} \gamma q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{h_n(i)}{c(i)}\right) + \sum_{j=k}^{\infty} \frac{h_n(j)h_n(j+1)}{c(j)} = h_{n+1}(k). \tag{6.1.69}$$

Hence, (6.1.66) is bounded, so $\lim_{n\to\infty} h_n(k) = h(k) < \infty$. □

Restating Theorem 6.1.13 as sufficient condition for equation (6.1.1) to be oscillatory, we have the following result.

Theorem 6.1.14. *Let the hypotheses of Lemma 6.1.5 hold. Equation (6.1.1) is oscillatory if one of the following conditions is satisfied.*

(i) *There exists $N \in \mathbb{N}$ such that $h_N(k)$ in (6.1.66) is not defined.*
(ii) *Every $h_n(k)$ in (6.1.66), $n \in \mathbb{N}_0$ is defined, but for any $m \in \mathbb{N}$ there exists $m_1 \geq m$ such that $\lim_{n\to\infty} h_n(m_1) = \infty$.*

Example 6.1.15. Consider the advanced difference equation

$$\Delta\left(\frac{1}{k^3}\Delta x(k)\right) + \frac{1}{k^2}x[g(k)] = 0 \quad \text{for } k \in \mathbb{N}, \tag{6.1.70}$$

where $g(k) \geq k+1$, $\{g(k)\}$ is a monotone increasing integer sequence. We let $c(k) = 1/k^3$ and $q(k) = 1/k^2$. Now

$$
\begin{aligned}
h_0(k) &= \sum_{j=k}^{\infty} q(j) = \sum_{j=k}^{\infty} \frac{1}{j^2} < \infty, \\
h_1(k) &= \sum_{j=k}^{\infty} q(j) \prod_{i=j}^{g(j)-1} \left(1 + \frac{h_0(i)}{c(i)}\right) + \sum_{j=k}^{\infty} \frac{h_0(j)h_0(j+1)}{c(j)} \\
&\geq \sum_{j=k}^{\infty} \frac{1}{j^2} \prod_{i=j}^{g(j)-1} \left(1 + i^3 \sum_{\tau=i}^{\infty} \frac{1}{\tau^2}\right) \\
&\geq \sum_{j=k}^{\infty} \frac{1}{j^2} \left(1 + \frac{j^3}{j^2}\right) \\
&\geq \sum_{j=k}^{\infty} \frac{1}{j^2}(1+j) \\
&\geq \sum_{j=k}^{\infty} \frac{1}{j},
\end{aligned}
\tag{6.1.71}
$$

that is, $h_1(k)$ is not defined, and hence equation (6.1.70) is oscillatory by Theorem 6.1.14.

We will assume that

$$
f(x) \operatorname{sgn} x \geq |x|^{\gamma} \quad \text{for } x \neq 0, \tag{6.1.72}
$$

where $\gamma > 0$ is a constant.

Theorem 6.1.16. *Let condition (6.1.72) hold with $\gamma = 1$, $g(k) = k - d(k)$, $\{d(k)\}$ is a sequence of positive integers such that $\{k - d(k)\}$ is increasing. If*

$$
\liminf_{k\to\infty} \left[\frac{1}{d(k)} \sum_{j=k-d(k)}^{k-1} C(j - d(j) - 1, m) q(j)\right] > \limsup_{k\to\infty} \frac{[d(k)]^{d(k)}}{[1 + d(k)]^{1+d(k)}}, \tag{6.1.73}
$$

where $m \in \mathbb{N}$ is large and C is as in (6.1.8), then equation (6.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). There exists an integer $m \in \mathbb{N}$ such that $x(k) > 0$, $x[g(k)] > 0$, and $\Delta x(k) > 0$ for $k \geq m$. By Lemma 6.1.2 there exists an integer $m_1 \geq m$ such that

$$
x[g(k)] \geq C(g(k) - 1, m_1) c[g(k)] \Delta x[g(k)] \quad \text{for } k \geq m_1. \tag{6.1.74}
$$

Using condition (6.1.72) with $\gamma = 1$ and (6.1.74) in equation (6.1.1) provides

$$
\Delta y(k) + q(k) C(g(k) - 1, m_1) y[g(k)] \leq 0, \tag{6.1.75}
$$

where $y(k) = c(k)\Delta x(k)$ for $k \geq m$. But in view of condition (6.1.73), it follows from Lemma 6.1.6(i) that the inequality (6.1.75) cannot have an eventually positive solution. This contradicts the fact that $y(k)$, $k \geq m$, is eventually positive and completes the proof. □

The following corollary is an immediate consequence of Theorem 6.1.16.

Corollary 6.1.17. *In Theorem 6.1.16, let* $d(k) \equiv d \in \mathbb{N}$. *If*

$$\liminf_{k\to\infty}\left[\sum_{j=k-d}^{k-1} C(j-d-1,m)q(j)\right] > \left(\frac{d}{1+d}\right)^{1+d}, \tag{6.1.76}$$

then equation (6.1.1) is oscillatory.

Theorem 6.1.18. *Let condition (6.1.72) hold with* $0 < \gamma < 1$. *If*

$$\sum^{\infty} C^{\gamma}(g(j)-1,m)q(j) = \infty, \quad \text{where } m \in \mathbb{N} \text{ is large}, \tag{6.1.77}$$

then equation (6.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). As in the proof of Theorem 6.1.16 we obtain (6.1.74) for $k \geq m_1$. Using (6.1.72) with $0 < \gamma < 1$ and (6.1.74) in equation (6.1.1), we find

$$\Delta y(k) + C^{\gamma}(g(k)-1,m_1)q(k)y^{\gamma}(k) \leq 0, \tag{6.1.78}$$

so

$$-\frac{\Delta y(k)}{y^{\gamma}(k)} \geq C^{\gamma}(g(k)-1,m_1)q(k) \quad \text{for } k \geq m_1. \tag{6.1.79}$$

Summing both sides of the inequality (6.1.79) from $m_1 + 1$ to k, we obtain

$$\begin{aligned}\sum_{j=m_1+1}^{k} C^{\gamma}(g(j)-1,m_1)q(j) &\leq \sum_{j=m_1+1}^{k} \frac{\Delta y(j)}{y^{\gamma}(j)} \\ &\leq \int_{y(k+1)}^{y(m_1+1)} u^{-\gamma}du \\ &< \int_{0}^{y(m_1+1)} u^{-\gamma}du \\ &< \infty,\end{aligned} \tag{6.1.80}$$

which contradicts condition (6.1.77). This completes the proof. □

Theorem 6.1.19. *Let conditions (6.1.3), (6.1.11), and (6.1.16) hold and suppose that* $0 \le g(k) \le k$. *If there exists a sequence* $\{\rho(k)\}$ *such that* $\rho(k) > 0$ *for* $k \ge m \in \mathbb{N}$ *and*

$$\limsup_{k\to\infty} \sum_{j=m}^{k} \left[\rho(j)q(j) - \frac{1}{4\lambda} c[g(j)] \frac{[\Delta\rho(j)]^2}{\rho(j)}\right] = \infty, \tag{6.1.81}$$

then equation (6.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). There exists an integer $m \in \mathbb{N}$ such that $x(k) > 0$, $x[g(k)] > 0$, and by Lemma 6.1.1, $\Delta x(k) > 0$ for $k \ge m$. Define

$$w(k) = \frac{\rho(k)c(k)\Delta x(k)}{f(x[g(k)])} \quad \text{for } k \ge m. \tag{6.1.82}$$

Then for $k \ge m$,

$$\begin{aligned}\Delta w(k) &= \frac{\rho(k)\Delta\big(c(k)\Delta x(k)\big)}{f(x[g(k)])} + \frac{\Delta\rho(k)c(k+1)\Delta x(k+1)}{f(x[g(k+1)])} \\ &\quad - \frac{\rho(k)c(k+1)\Delta x(k+1)\big[f(x[g(k+1)]) - f(x[g(k)])\big]}{f(x[g(k)])f(x[g(k+1)])} \\ &\le -\rho(k)q(k) + \frac{\Delta\rho(k)}{\rho(k+1)} w(k+1) - \lambda \frac{\rho(k)}{\rho(k+1)} \frac{\Delta x[g(k)]}{f(x[g(k)])} w(k+1).\end{aligned} \tag{6.1.83}$$

Since $\{c(k)\Delta x(k)\}$ is a nonincreasing sequence and $\{x(k)\}$ is an increasing sequence, we have

$$\frac{f(x[g(k+1)])}{f(x[g(k)])} \ge 1, \qquad c[g(k)]\Delta x[g(k)] \ge c(k+1)\Delta x(k+1) \tag{6.1.84}$$

for $k \ge m_1$ for some $m_1 \ge m$. Using (6.1.84) in (6.1.83), we get for $k \ge m_1$,

$$\begin{aligned}\Delta w(k) &\le -\rho(k)q(k) + \frac{\Delta\rho(k)}{\rho(k+1)} w(k+1) - \lambda \frac{\rho(k)}{\rho^2(k+1)c[g(k)]} w^2(k+1) \\ &= -\rho(k)q(k) + \frac{1}{4\lambda} \frac{(\Delta\rho(k))^2 c[g(k)]}{\rho(k)} \\ &\quad - \left[\sqrt{\frac{\lambda\rho(k)}{c[g(k)]}} \frac{w(k+1)}{\rho(k+1)} - \frac{\Delta\rho(k)\sqrt{c[g(k)]}}{2\sqrt{\lambda\rho(k)}}\right]^2 \\ &\le -\left[\rho(k)q(k) - \frac{1}{4\lambda} \frac{(\Delta\rho(k))^2}{\rho(k)} c[g(k)]\right].\end{aligned} \tag{6.1.85}$$

Summing both sides of (6.1.85) from m_1 to $k \geq m_1$, we have

$$w(k+1) \leq w(m_1) - \sum_{j=m_1}^{k} \left[\rho(j)q(j) - \frac{1}{4\lambda}\frac{(\Delta\rho(j))^2}{\rho(j)}c[g(j)]\right], \tag{6.1.86}$$

which yields

$$\sum_{j=m_1}^{k} \left[\rho(j)q(j) - \frac{1}{4\lambda}\frac{(\Delta\rho(j))^2}{\rho(j)}c[g(j)]\right] \leq w(m_1) < \infty, \tag{6.1.87}$$

and this contradicts (6.1.81). □

Remark 6.1.20. We note that conditions (6.1.11) and (6.1.16) can be replaced by condition (6.1.25). Also, the function f need not be monotonic. Then we assume that there exists a nondecreasing function $h \in C(\mathbb{R}, \mathbb{R})$ with $|f(x)| \geq |h(x)|$ and $xh(x) > 0$ for $x \neq 0$.

Corollary 6.1.21. *In Theorem 6.1.19, condition (6.1.81) can be replaced by*

$$\begin{gathered} \limsup_{k\to\infty} \sum_{j=m\in\mathbb{N}}^{k} \rho(j)q(j) = \infty, \\ \lim_{k\to\infty} \sum_{j=m}^{k} c[g(j)]\frac{(\Delta\rho(j))^2}{\rho(j)} < \infty. \end{gathered} \tag{6.1.88}$$

The following example illustrates the methods presented above.

Example 6.1.22. Consider the second-order delay difference equation

$$\Delta(c(k)\Delta x(k)) + \frac{\theta}{k^2}x[g(k)] = 0 \quad \text{for } k \in \mathbb{N}, \tag{6.1.89}$$

where $c(k)$ and $g(k) \leq k$ are as in equation (6.1.1) and $\theta > 0$ is a constant. We let $\rho(k) = k$. Then $\Delta\rho(k) = 1$. Now, we conclude the following.

(i) If $c(k) \equiv 1$, then equation (6.1.89) is oscillatory by Theorem 6.1.19 for any $\theta > 1/4$.

(ii) If $c(k) = 1/k^\alpha$, where α is any positive constant, then equation (6.1.89) is oscillatory by Corollary 6.1.21 for any $\theta > 0$.

Next, we will present some superlinear oscillation criteria for equation (6.1.1).

Theorem 6.1.23. *Let $g(k) = k - n$ with $n \in \mathbb{N}_0$, suppose conditions (6.1.3) and (6.1.11) hold, and*

$$\int^{\infty} \frac{du}{f(u)} < \infty, \qquad \int^{-\infty} \frac{du}{f(u)} < \infty. \tag{6.1.90}$$

If there exists a sequence $\{\rho(k)\}$ *such that*

$$\rho(k) > 0, \quad \Delta\rho(k) \geq 0, \quad \Delta\big(c[k-n]\Delta\rho(k)\big) \leq 0 \quad \textit{for } k \geq m \in \mathbb{N}, \tag{6.1.91}$$

$$\sum^{\infty} \rho(j)q(j) = \infty, \tag{6.1.92}$$

then equation (6.1.1) is oscillatory.

Proof. Assume that equation (6.1.1) has an eventually positive solution $\{x(k)\}$. There exists an integer $m \in \mathbb{N}$ such that $x(k) > 0$, $x[k-n] > 0$, and $\Delta x(k) > 0$ for $k \geq m+n$. Define $w(k)$ as in (6.1.82) with $g(k) = k-n$ and obtain

$$\Delta w(k) \leq -\rho(k)q(k) + \Delta\rho(k)\frac{c(k+1)\Delta x(k+1)}{f(x[k-n+1])} \quad \text{for } k \geq m. \tag{6.1.93}$$

In view of the monotonicity of $\{c(k)\Delta x(k)\}$ and $\{c(k-n)\Delta\rho(k)\}$, it follows that

$$\Delta w(k) \leq -\rho(k)q(k) + \big(c(m-n)\Delta\rho(m)\big)\frac{\Delta x[k-n]}{f(x[k+1-n])} \quad \text{for } k \geq m. \tag{6.1.94}$$

Now, for $x[k-n] \leq x \leq x[k+1-n]$, we have

$$\frac{1}{f(x[k+1-n])} \leq \frac{1}{f(x)}, \tag{6.1.95}$$

and so

$$\frac{\Delta x[k-n]}{f(x[k+1-n])} \leq \int_{x[k-n]}^{x[k+1-n]} \frac{dx}{f(x)}. \tag{6.1.96}$$

Using (6.1.96) in (6.1.94) and summing both sides from m to $k \geq m$, we obtain

$$\sum_{j=m}^{k} \rho(j)q(j) \leq w(m) - w(k+1) + \big(c(m-n)\Delta\rho(m)\big)\int_{x[m-n]}^{x[k+1-n]} \frac{dx}{f(x)} \tag{6.1.97}$$

for $k \geq m$. Since $w(k) \geq 0$ for $k \geq m$, it follows from (6.1.97) by condition (6.1.90) that $\sum^{\infty} \rho(j)q(j) < \infty$, which contradicts condition (6.1.92). $\square$

Example 6.1.24. Consider the superlinear difference equation

$$\Delta\left(\frac{1}{2k+1}\Delta x(k)\right) + \frac{2}{(k-n)^3}x^3[k-n] = 0, \tag{6.1.98}$$

where n is an odd positive integer, $k > n \geq 1$. Let $\rho(k) = k^2$. Then all conditions of Theorem 6.1.23 are satisfied, and hence equation (6.1.98) is oscillatory. One such solution of equation (6.1.98) is $\{x(k)\}$ with $x(k) = k(-1)^k$.

Theorem 6.1.25. *If conditions (6.1.91) and (6.1.92) of Theorem 6.1.23 are replaced by*

$$\sum_{j=m\in\mathbb{N}}^{\infty} \frac{1}{c(j)} \sum_{i=j+n+1}^{\infty} q(i) = \infty, \tag{6.1.99}$$

then the conclusion of Theorem 6.1.23 holds.

Proof. Let $\{x(k)\}$ be an eventually positive solution of (6.1.1). There exists $m \in \mathbb{N}$ such that $x(k) > 0$, $x[k-n] > 0$, and $\Delta x(k) > 0$ for $k \geq m+n$. Summing both sides of equation (6.1.1) with $g(k) = k - n$ from k to $N \geq k \geq m+n = m_1$, we get

$$c(N+1)\Delta x(N+1) - c(k)\Delta x(k) + \sum_{j=k}^{N} q(j) f\left(x[j-n]\right) = 0. \tag{6.1.100}$$

Letting $N \to \infty$, we obtain

$$\sum_{j=k}^{\infty} q(j) f\left(x[j-n]\right) \leq c(k)\Delta x(k), \tag{6.1.101}$$

and so

$$\sum_{j=k+n+1}^{\infty} q(j) f\left(x[j-n]\right) \leq c(k)\Delta x(k). \tag{6.1.102}$$

In view of the monotonicity of $\{x(k)\}$ and f, we find that

$$\frac{1}{c(k)} \sum_{j=k+n+1}^{\infty} q(j) \leq \frac{\Delta x(k)}{f\left(x[k+1]\right)} \leq \int_{x(k)}^{x(k+1)} \frac{du}{f(u)} \quad \text{for } k \geq m_1. \tag{6.1.103}$$

Summing both sides of (6.1.103) from m_1 to k, we obtain

$$\sum_{i=m_1}^{k} \frac{1}{c(i)} \sum_{j=i+n+1}^{\infty} q(j) \leq \int_{x(m_1)}^{x(k+1)} \frac{du}{f(u)} < \int_{x(m_1)}^{\infty} \frac{du}{f(u)} < \infty, \tag{6.1.104}$$

which contradicts condition (6.1.99). This completes the proof. □

For a more general sublinear oscillation criterion which extends Theorem 6.1.18, we present the following result.

Theorem 6.1.26. *Let $0 \le g(k) \le k$ and assume that conditions (6.1.3) and (6.1.11) hold,*

$$-f(-uv) \ge f(uv) \ge f(u)f(v) \quad \text{for } uv > 0, \tag{6.1.105}$$

$$\int^{+0} \frac{du}{f(u)} < \infty, \qquad \int^{-0} \frac{du}{f(u)} < \infty. \tag{6.1.106}$$

If

$$\sum_{j=m\in\mathbb{N}}^{\infty} q(j)f\big(C(g(j)-1,m)\big) = \infty, \tag{6.1.107}$$

where C is as in (6.1.8), then equation (6.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). As in the proof of Theorem 6.1.16, we obtain (6.1.74). From equation (6.1.1), by assumption, we obtain

$$\Delta y(k) + q(k)f\big(C(g(k)-1,m)\big)f\big(y(k)\big) \le 0, \tag{6.1.108}$$

where $y(k) = c(k)\Delta x(k)$. For $y(k+1) \le v \le y(k)$, we have $1/f(y(k)) \le 1/f(v)$ and so

$$q(k)f\big(C(g(k)-1,m)\big) \le \int_{y(k+1)}^{y(k)} \frac{dv}{f(v)} \quad \text{for } k \ge m_1 \ge m \in \mathbb{N}. \tag{6.1.109}$$

Summing both sides of (6.1.109) from m_1 to k, we obtain

$$\sum_{j=m_1}^{k} q(j)f\big(C(g(j)-1,m)\big) \le \int_{y(k+1)}^{y(m_1)} \frac{dv}{f(v)} < \int_0^{y(m_1)} \frac{dv}{f(v)} < \infty, \tag{6.1.110}$$

which contradicts condition (6.1.107). This completes the proof. □

In equation (6.1.1) if the function f satisfies $f \in C(\mathbb{R},\mathbb{R})$,

$$xf(x) > 0 \quad \text{for } x \ne 0, \qquad \liminf_{|u|\to\infty} \big|f(u)\big| > 0, \tag{6.1.111}$$

then the following result applies.

Theorem 6.1.27. *Let conditions (6.1.3) and (6.1.111) hold. If*

$$\sum^{\infty} q(j) = \infty, \tag{6.1.112}$$

then equation (6.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.1.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. By Lemma 6.1.1, there exists an integer $m_1 \geq m$ such that $x[g(k)] > 0$, $\Delta x(k) > 0$, and $\Delta(c(k)\Delta x(k)) \leq 0$ for $k \geq m_1$.

Define the constants α and β by $\alpha = \lim_{k\to\infty} c(k)\Delta x(k)$ and $\beta = \lim_{k\to\infty} x[g(k)]$. Then $\alpha \geq 0$ is finite and $\beta > 0$ is finite or infinite. We will consider the following two cases.

Case 1. $\beta > 0$ is finite. For this case, we have from the continuity of the function f that $\lim_{k\to\infty} f(x[g(k)]) = f(\beta) > 0$. Thus, we may choose a positive integer $m_2 \geq m$ such that

$$f(x[g(k)]) > \frac{1}{2} f(\beta) \quad \text{for } k \geq m_2. \tag{6.1.113}$$

Using (6.1.112) in equation (6.1.1), we obtain

$$\Delta(c(k)\Delta x(k)) + \frac{1}{2} f(\beta) q(k) \leq 0 \quad \text{for } k \geq m_2. \tag{6.1.114}$$

Summing both sides of (6.1.114) from m_2 to $k \geq m_2$, we have

$$c(k+1)\Delta x(k+1) - c(m_2)\Delta x(m_2) + \frac{1}{2} f(\beta) \sum_{j=m_2}^{k} q(j) \leq 0. \tag{6.1.115}$$

Using (6.1.112) and letting $k \to \infty$ in (6.1.115), we get $\lim_{k\to\infty} c(k)\Delta x(k) = -\infty$, which contradicts the fact that $\Delta x(k) > 0$ for $k \geq m_1$.

Case 2. $\beta = \infty$. For this case, we have from the condition (6.1.111) that $\liminf_{k\to\infty} f(x[g(k)]) > 0$, and so we may choose a small positive constant a and a positive integer $m_3 \geq m$ sufficiently large such that

$$f(x[g(k)]) \geq a \quad \text{for } k \geq m_3. \tag{6.1.116}$$

Using (6.1.116) in equation (6.1.1), we obtain

$$\Delta(c(k)\Delta x(k)) + a q(k) \leq 0 \quad \text{for } k \geq m_3. \tag{6.1.117}$$

The rest of the proof is similar to that of Case 1 and hence is omitted.

This completes the proof. $\square$

In equation (6.1.1), if $\{q(k)\}$ is not eventually of one sign, then we have the following result.

Theorem 6.1.28. *Let condition (6.1.3) hold,* $0 \le g(k) \le k$, $c(k) \le 1$, *and assume that the function* f *is as in equation (6.1.1) and there exists a constant* $M \ge 0$ *such that*

$$\limsup_{u \to \infty} \frac{u}{f(u)} = M. \tag{6.1.118}$$

If condition (6.1.112) holds, then the difference $\{\Delta x(k)\}$ *of every solution* $\{x(k)\}$ *of equation (6.1.1) oscillates.*

Proof. If not, then equation (6.1.1) has a solution $\{x(k)\}$ such that its difference $\{\Delta x(k)\}$ is nonoscillatory. There are two cases to consider.

Case 1. Assume that $\{\Delta x(k)\}$ is eventually negative. Then there exists $m \in \mathbb{N}$ such that

$$\Delta x(k) < 0 \quad \text{for } k \ge m, \tag{6.1.119}$$

and so $\{x(k)\}$ is strictly decreasing for $k \ge m$ which implies that $\{x(k)\}$ is also nonoscillatory. Then there is a positive integer $m_1 \ge m$ such that $x[g(k+1)]x[g(k)] > 0$ for $k \ge m_1$. Define $w(k)$ as in (6.1.82) with $\rho(k) = 1$ and obtain

$$\Delta w(k) \le -q(k) \quad \text{for } k \ge m_1. \tag{6.1.120}$$

Summing both sides of (6.1.120) from m_1 to k, we have

$$w(k+1) - w(m_1) < -\sum_{j=m_1}^{k} q(j) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{6.1.121}$$

which implies that

$$f(x[g(k)]) > 0, \quad \text{or} \quad x[g(k)] > 0 \quad \text{eventually.} \tag{6.1.122}$$

From (6.1.121), we can choose a positive integer $m_2 \ge m_1$ with $w(k) \le -(M+1)$ for $k \ge m_2$, that is,

$$\Delta x(k) + \frac{M+1}{c(k)} f(x[g(k)]) \le 0 \quad \text{for } k \ge m_2. \tag{6.1.123}$$

Set $\lim_{k\to\infty} x(k) = \beta$. Then $\beta \geq 0$. We claim that $\beta = 0$. If $\beta > 0$, then we have $\lim_{k\to\infty} f(x[g(k)]) = f(\beta) > 0$ by the continuity of f. Choose a positive integer $m_3 \geq m_2$ sufficiently large such that

$$f(x[g(k)]) > \frac{1}{2} f(\beta) \quad \text{for } k \geq m_3. \tag{6.1.124}$$

Using (6.1.124) in inequality (6.1.123), we have

$$\Delta x(k) + \frac{1}{2}\frac{M+1}{c(k)} f(\beta) \leq 0. \tag{6.1.125}$$

Summing both sides of (6.1.125) from m_3 to k, we obtain

$$x(k+1) - x(m_3) + \frac{1}{2}(M+1) f(\beta) \sum_{j=m_3}^{k} \frac{1}{c(j)} \leq 0, \tag{6.1.126}$$

which implies that $\lim_{k\to\infty} x(k+1) = -\infty$. This contradicts (6.1.122). Hence $\lim_{k\to\infty} x(k) = 0$. In view of (6.1.118) and (6.1.122), we have

$$\limsup_{k\to\infty} \frac{x[g(k)]}{f(x[g(k)])} \leq M. \tag{6.1.127}$$

From this, we choose a positive integer $m_4 \geq m_3$ such that

$$\frac{x[g(k)]}{f(x[g(k)])} < M+1 \quad \text{for } k \geq m_4, \tag{6.1.128}$$

that is, $x[g(k)] < (M+1) f(x[g(k)])$ for $k \geq m_4$. By inequality (6.1.123), we have $c(k)\Delta x(k) + x[g(k)] < 0$, which implies

$$0 < c(k)x(k+1) + \big(x[g(k)] - c(k)x(k)\big) < 0 \quad \text{for } k \geq m_4. \tag{6.1.129}$$

This is a contradiction.

Case 2. Assume that $\{\Delta x(k)\}$ is eventually positive. The proof is similar to that of Case 1 and hence is omitted.

This completes the proof. □

The following example illustrates the methods presented above.

Example 6.1.29. Consider the second-order difference equation

$$\Delta^2 x(k) + q(k)x(k-1) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{6.1.130}$$

where ($n \in \mathbb{N}_0$)

$$q(k) = \begin{cases} 1 - \dfrac{1}{n+1} & \text{for } k = 3n, \\ (n+1)^2(n+2) & \text{for } k = 3n+1, \\ -2\dfrac{n+1}{n+2} & \text{for } k = 3n+2. \end{cases} \tag{6.1.131}$$

Obviously $\{q(k)\}$ is not eventually of one sign and $\sum^{\infty} q(j) = \infty$. Since $f(x) = x$, conditions (6.1.111) and (6.1.118) are satisfied. Equation (6.1.130) has a nonoscillatory solution

$$x(-1) = 1, \qquad x(k) = \begin{cases} \dfrac{1}{n+1} & \text{for } k = 3n, \\ 1 & \text{for } k \neq 3n, \end{cases} \tag{6.1.132}$$

where its difference $\{\Delta x(k)\}$ is oscillatory.

We conclude that the condition $q(k) \geq 0$ eventually in Theorem 6.1.27 cannot be dropped. We also note that the hypotheses of Theorem 6.1.28 are satisfied for equation (6.1.130).

Next, we will consider a special case of equation (6.1.1) when $g(k) = k + n$ with $n \in \mathbb{N}$, namely the advanced equation

$$\Delta\big(c(k)\Delta x(k)\big) + q(k)x(k+n) = 0. \tag{6.1.133}$$

The relations between members of the class of equations of form (6.1.133) will be established via Riccati-type transformations. In the sequel, the convention that an empty product is equal to one will be adopted.

Lemma 6.1.30. *If condition (6.1.3) holds and $\{x(k)\}$ is an eventually positive solution of equation (6.1.133), then the sequence $\{\Delta x(k)\}$ is eventually positive. Furthermore, the sequence $\{w(k)\}$ defined by $w(k) = c(k)\Delta x(k)/x(k)$, $k \in \mathbb{N}$, converges to zero and satisfies*

$$\Delta w(k) = -q(k)\prod_{i=1}^{n-1}\left(1 + \frac{w(k+i)}{c(k+i)}\right) - \frac{w^2(k)}{c(k) + w(k)}. \tag{6.1.134}$$

Proof. From (6.1.133), $\Delta(c(k)\Delta x(k)) \le 0$ for $k \ge m$ for some $m \in \mathbb{N}$, and by Lemma 6.1.1, we conclude that the sequence $\{c(k)\Delta x(k)\}$ is eventually positive and nonincreasing so that it has a limit $\beta \ge 0$ as $k \to \infty$. If $\beta = 0$, then $\lim_{k\to\infty} w(k) = 0$ since $\{x(k)\}$ is eventually nondecreasing. If $\beta > 0$, then $\Delta x(k) \ge \beta/c(k)$ eventually. Thus, by condition (6.1.3), $\lim_{k\to\infty} x(k) = \infty$, which implies $\lim_{k\to\infty} w(k) = 0$.

Next, note that

$$\frac{x(k+1)}{x(k)} = 1 + \frac{w(k)}{c(k)} \quad \text{for } k \ge m \in \mathbb{N}. \tag{6.1.135}$$

Thus

$$\begin{aligned}\Delta w(k) &= \frac{1}{x(k+1)}\Delta\big(c(k)\Delta x(k)\big) + c(k)\Delta x(k)\Delta\left(\frac{1}{x(k)}\right)\\ &= -q(k)\left(\frac{x(k+n)}{x(k+1)}\right) - \frac{c(k)\big(\Delta x(k)\big)^2}{x(k)x(k+1)}\\ &= -q(k)\prod_{i=1}^{n-1}\left(1+\frac{w(k+i)}{c(k+i)}\right) - \frac{w^2(k)}{c(k)+w(k)}.\end{aligned} \tag{6.1.136}$$

This completes the proof. □

Note that if we sum (6.1.134) from k to infinity, then the sequence $\{w(k)\}$ defined in Lemma 6.1.30 satisfies

$$w(k) = \sum_{j=k}^{\infty} q(j)\prod_{i=1}^{n-1}\left(1+\frac{w(j+i)}{c(j+i)}\right) + \sum_{j=k}^{\infty}\frac{w^2(j)}{c(j)+w(j)}. \tag{6.1.137}$$

Lemma 6.1.31. *Suppose that condition (6.1.3) holds. Equation (6.1.133) has a nonoscillatory solution if and only if there exists an eventually positive sequence $\{w(k)\}$ which satisfies*

$$\Delta w(k) + \frac{w^2(k)}{c(k)+w(k)} \le -q(k)\prod_{i=1}^{n-1}\left(1+\frac{w(k+i)}{c(k+i)}\right) \tag{6.1.138}$$

eventually.

Proof. Necessity follows from Lemma 6.1.30. To prove sufficiency, assume that $\{w(k)\}$ is an eventually positive sequence which satisfies (6.1.138). Then $\{\Delta w(k)\}$ is eventually nonpositive so that $\{w(k)\}$ converges to some nonnegative constant β. If $\beta > 0$, then

$$\beta - w(k) + \sum_{j=k}^{\infty}\frac{w^2(j)}{c(j)+w(j)} < 0. \tag{6.1.139}$$

By condition (6.1.3), the sum in (6.1.139) is divergent, which is a contradiction. Thus $\{w(k)\}$ converges to zero, and summing (6.1.138) from k to infinity provides

$$w(k) \geq \sum_{j=k}^{\infty} q(j) \prod_{i=1}^{n-1} \left(1 + \frac{w(j+i)}{c(j+i)}\right) + \sum_{j=k}^{\infty} \frac{w^2(j)}{c(j) + w(j)} \tag{6.1.140}$$

eventually. This implies

$$\sum_{j=k}^{\infty} \frac{w^2(j)}{c(j) + w(j)} < \infty, \qquad \sum_{j=k}^{\infty} q(j) \prod_{i=1}^{n-1} \left(1 + \frac{w(j+i)}{c(j+i)}\right) < \infty. \tag{6.1.141}$$

We claim that equation (6.1.133) has an eventually positive solution. To this end, let $m \in \mathbb{N}$ be large enough so that $w(k) > 0$ and (6.1.140) holds for $k \geq m$. Consider the set of sequences

$$X = \{x = \{x(k)\} : 0 \leq x(k) \leq w(k),\ k \geq m\}. \tag{6.1.142}$$

Define a mapping T on X by

$$(Tx)(k) = \sum_{j=k}^{\infty} q(j) \prod_{i=1}^{n-1} \left(1 + \frac{x(j+i)}{c(j+i)}\right) + \sum_{j=k}^{\infty} \frac{x^2(j)}{c(j) + x(j)} \quad \text{for } k \geq m. \tag{6.1.143}$$

It is easy to verify that T maps X into itself and that when $\{x(k)\}$ and $\{y(k)\}$ are two sequences in X with $x(k) \leq y(k)$ for all $k \geq m$, then $(Tx)(k) \leq (Ty)(k)$. As a consequence, if we define a sequence $\{u^1, u^2, \dots\}$ by $u^1(k) = 0$ for $k = m, m+1, \dots$ and inductively for $j \in \mathbb{N}$, $u^{j+1}(k) = (Tu^j)(k)$ for $k = m, m+1, \dots$, then for $k \geq m$, $u^j(k) \leq u^{j+1}(k) \leq w(k)$ for $j \in \mathbb{N}$. Hence there is a positive sequence $\{v(k)\}$, $k \geq m$, such that $\lim_{j\to\infty} u^j(k) = v(k) \leq w(k)$ for $k \geq m$, so that $v(k) = (Tv)(k)$, or equivalently,

$$v(k) = \sum_{j=k}^{\infty} q(j) \prod_{i=1}^{n-1} \left(1 + \frac{v(j+i)}{c(j+i)}\right) + \sum_{j=k}^{\infty} \frac{v^2(j)}{c(j) + v(j)} \quad \text{for } k \geq m. \tag{6.1.144}$$

If we now define $x(m) = 1$ and

$$x(k+1) = x(k)\left(1 + \frac{v(k)}{c(k)}\right) \quad \text{for } k \geq m, \tag{6.1.145}$$

then we can easily verify that $\{x(k)\}$ is an eventually positive solution of equation (6.1.133) for $k \geq m$. This completes the proof. □

We are now ready to prove the following result.

Theorem 6.1.32. *If condition (6.1.3) holds and equation (6.1.133) has a nonoscillatory solution for some $n = N > 1$, then for any $1 \leq n < M$, the corresponding equation (6.1.133) has a nonoscillatory solution.*

Proof. If (6.1.133) has a nonoscillatory solution for some $n = N > 1$, then there exists an eventually positive sequence $\{w(k)\}$ which satisfies (6.1.138), where $n = N$. This implies that (6.1.138) holds also for $1 \leq n < N$. Now the proof follows from Lemmas 6.1.30 and 6.1.31. □

Example 6.1.33. Consider the difference equation

$$\Delta^2 x(k) + \frac{4k^2 + 8k + 2}{k(k+2)} x(k+1) = 0 \quad \text{for } k \in \mathbb{N}, \tag{6.1.146}$$

which is oscillatory since

$$\frac{4k^2 + 8k + 2}{k(k+2)} > \frac{1}{4k^2} \quad \text{eventually.} \tag{6.1.147}$$

By Theorem 6.1.32, one can observe that the advanced equation

$$\Delta^2 x(k) + \frac{4k^2 + 8k + 2}{k(k+2)} x(k+n) = 0 \quad \text{for } k \in \mathbb{N},\ n > 1 \tag{6.1.148}$$

is oscillatory. One such oscillatory solution of equation (6.1.148) is $\{x(k)\}$, where $x(k) = (-1)^{k-n+1}/(k-n+1)$, $k > n-1$. Thus we conclude that an oscillation criterion for equation (6.1.133) with $n = 1$ is also an oscillation criterion for equation (6.1.133) with $n > 1$, and for oscillation criteria for equation (6.1.133) with $n = 1$, we refer the reader to the oscillation results given in Chapter 1.

6.1.2. Comparison theorems

Here we will compare the oscillatory properties of equation (6.1.1) with those of the equation

$$\Delta\big(c_1(k)\Delta y(k)\big) + q_1(k) f\big(y[g_1(k)]\big) = 0, \tag{6.1.149}$$

where $\{c_1(k)\}$ and $\{q_1(k)\}$ are positive sequences of real numbers, $\{g_1(k)\}$ is a sequence of nonnegative integers, and the function f is as in equation (6.1.1). We will assume that $\sum^{\infty} 1/c_1(j) = \infty$.

Now we prove the following comparison results.

Theorem 6.1.34. *Suppose that $c_1(k) \geq c(k)$, $q(k) \geq q_1(k)$, and $g(k) \geq g_1(k)$ for $k \geq m$ for some $m \in \mathbb{N}$ and equation (6.1.1) has a nonoscillatory solution. Then equation (6.1.149) also has a nonoscillatory solution.*

Proof. Without loss of generality, let $\{x(k)\}$ be an eventually positive solution of equation (6.1.1). There exists an integer $m_1 \geq m$ such that $\Delta x(k) > 0$ for $k \geq m_1$ and $g_1(k) \geq m$ for $k \geq m_1$. From equation (6.1.1), it is easy to see that

$$\begin{aligned}
\Delta x(k) &\geq \frac{1}{c(k)} \sum_{j=k}^{\infty} q(j) f(x[g(j)]), \\
x(k) &\geq x(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c(i)} \sum_{j=i}^{\infty} q(j) f(x[g(j)]) \\
&\geq x(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c_1(i)} \sum_{j=i}^{\infty} q_1(j) f(x[g_1(j)]).
\end{aligned} \tag{6.1.150}$$

Now we will show that equation (6.1.149) has a solution $\{y(k)\}$ such that

$$x(m_1) \leq y(k) \leq x(k) \quad \forall k \geq m_1. \tag{6.1.151}$$

Clearly, this $y(k)$ is an eventually positive solution of equation (6.1.149). For this, we define the sequence $\{y_n(k)\}_{n=1}^{\infty}$ as follows: for $n \in \mathbb{N}$, $y_1(k) = x(k)$, $k \geq m$, and

$$y_{n+1}(k) = \begin{cases} y_n(k) & \text{for } m \leq k \leq m_1 - 1, \\ x(m_1) + \displaystyle\sum_{i=m_1}^{k-1} \frac{1}{c_1(i)} \sum_{j=i}^{\infty} q_1(j) f(y_n[g_1(j)]) & \text{for } k \geq m_1. \end{cases} \tag{6.1.152}$$

Then for $k \geq m_1$, (6.1.152) provides

$$y_2(k) = x(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c_1(i)} \sum_{j=i}^{\infty} q_1(j) f(x[g_1(j)]) \leq x(k) = y_1(k). \tag{6.1.153}$$

By induction, we find $x(m_1) \leq y_{n+1}(k) \leq y_n(k)$ for $k \geq m_1$. Hence for $k \geq m_1$, $\{y_n(k)\}$ converges monotonically to some $y(k)$ as $n \to \infty$, and clearly this $y(k)$ satisfies (6.1.151). Further, in view of (6.1.152) we have

$$y(k) = x(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c_1(i)} \sum_{j=i}^{\infty} q_1(j) f(y[g_1(j)]) \quad \text{for } k \geq m_1. \tag{6.1.154}$$

Now it is easy to check that $\{y(k)\}$ is a solution of equation (6.1.149). This completes the proof. $\square$

Theorem 6.1.35. *Suppose for $k \geq m$ for some $m \in \mathbb{N}$, $g(k) \geq g_1(k)$ and $g(k) - g_1(k)$ is bounded and condition (6.1.3) holds. Then equation (6.1.1) is oscillatory if and only if the equation*

$$\Delta(c(k)\Delta y(k)) + q(k)f(y[g_1(k)]) = 0 \tag{6.1.155}$$

is oscillatory.

Proof. Let $M \in \mathbb{N}$ be such that

$$g(k) - g_1(k) \leq M \quad \text{for } k \geq m. \tag{6.1.156}$$

Suppose that equation (6.1.1) has a nonoscillatory solution. Then, by Theorem 6.1.34, equation (6.1.155) also has a nonoscillatory solution. Next, suppose that equation (6.1.155) has a nonoscillatory solution. Without loss of generality, let $\{y(k)\}$ be an eventually positive solution of equation (6.1.155). There exists an integer $m_1 \geq m$ such that $\Delta y(k) > 0$ and $g(k) - M \geq m$ for $k \geq m_1$.

Define $u(k) = y(k - M)$. Then, since $y(k)$ is increasing for $k \geq m_1$ in view of (6.1.156), we have

$$u[g(k)] = y[g(k) - M] \leq y[g_1(k)] \quad \text{for } k \geq m_1. \tag{6.1.157}$$

As in the proof of Theorem 6.1.34, we obtain

$$y(k) \geq y(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c(i)} \sum_{j=i}^{\infty} q(j)f(y[g_1(j)]) \quad \text{for } k \geq m_1. \tag{6.1.158}$$

Inequalities (6.1.158) and (6.1.157) imply for $k \geq m_1 + M$ that

$$u(k) = y(k - M) \geq y(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c(i)} \sum_{j=i}^{\infty} q(j)f(u[g(j)]). \tag{6.1.159}$$

We will now show that equation (6.1.1) has a solution $\{x(k)\}$ such that

$$y(m_1) \leq x(k) \leq u(k) \quad \text{for } k \geq m_1 + M. \tag{6.1.160}$$

Clearly, this $x(k)$ is an eventually positive solution of equation (6.1.1). For this let the sequence $\{x_n(k)\}_{n=1}^{\infty}$ be defined as follows: for $n \in \mathbb{N}$, $x_1(k) = u(k)$, $k \geq m$, and

$$x_{n+1}(k) = \begin{cases} x_n(k) & \text{for } m \leq k \leq m_1 + M - 1, \\ y(m_1) + \displaystyle\sum_{i=m_1}^{k-1} \frac{1}{c(i)} \sum_{j=i}^{\infty} q(j)f(x_n[g(j)]) & \text{for } k \geq m_1 + M. \end{cases} \tag{6.1.161}$$

Then, in view of (6.1.159) it follows from (6.1.161) that $x_2(k) \le u(k) = x_1(k)$ for $k \ge m_1 + M$, and inductively we find $y(m_1) \le x_{n+1}(k) \le x_n(k)$ for $k \ge m_1 + M$. Hence, for $k \ge m_1 + M$, $\{x_n(k)\}$ converges monotonically to some $x(k)$ as $n \to \infty$. Obviously, $x(k)$ satisfies (6.1.160) and the relation

$$x(k) = y(m_1) + \sum_{i=m_1}^{k-1} \frac{1}{c(i)} \sum_{j=i}^{\infty} q(j) f\big(x[g(j)]\big) \quad \text{for } k \ge m_1 + M. \tag{6.1.162}$$

Now it can be verified that $x(k)$ is indeed a solution of the equation (6.1.1). This completes the proof. □

The following corollary is immediate.

Corollary 6.1.36. *Suppose that $|k - g(k)|$ is bounded for $k \ge m$ for some $m \in \mathbb{N}$ and condition (6.1.3) holds. Then equation (6.1.1) is oscillatory if and only if the equation*

$$\Delta\big(c(k)\Delta y(k)\big) + q(k) f\big(y(k)\big) = 0 \tag{6.1.163}$$

is oscillatory.

Example 6.1.37. The difference equation

$$\Delta^2 y(k) + \frac{2}{(k-1)(k+1)^2(k+2)} y(k) = 0 \quad \text{for } k \in \mathbb{N} \setminus \{1\} \tag{6.1.164}$$

is not oscillatory as it has a nonoscillatory solution given by $y(k) = k - (1/k)$. It follows from Theorem 6.1.34 and Corollary 6.1.36 that the equation

$$\Delta^2 x(k) + \frac{2}{(k-1)(k+1)^2(k+2)} y[g(k)] = 0 \quad \text{for } k \in \mathbb{N} \setminus \{1\} \tag{6.1.165}$$

is also not oscillatory for $\{g(k)\}$ as in (6.1.1) such that $g(k) \le k$ or $|k - g(k)|$ is bounded.

From Theorem 6.1.35 we see that a special case of equation (6.1.1), namely, the difference equation

$$\Delta\big(c(k)\Delta x(k)\big) + q(k) \left| x[k-\tau] \right|^{\gamma} \operatorname{sgn} x[k-\tau] = 0 \quad \text{with } \gamma > 0, \tag{6.1.166}$$

where τ is a positive constant, is oscillatory if and only if the difference equation

$$\Delta\big(c(k)\Delta y(k)\big) + q(k) \left| y(k) \right|^{\gamma} \operatorname{sgn} y(k) = 0 \quad \text{with } \gamma > 0 \tag{6.1.167}$$

is oscillatory.

6.1.3. Oscillation of equation (6.1.2)

We begin by considering difference inequalities of the form

$$\{\Delta(c(k)\Delta x(k)) - q(k)f(x[g(k)])\}\operatorname{sgn} x[g(k)] \geq 0, \tag{6.1.168}$$

where $c(k)$, $g(k)$, and $f(x)$ are as in equation (6.1.1).

Let $\{x(k)\}$ be a nonoscillatory solution of inequality (6.1.168). It is easy to see that $\{\Delta x(k)\}$ is eventually of one sign, so that either one of the following hold:

(I_1) $x(k)\Delta x(k) < 0$ eventually,
(I_2) $x(k)\Delta x(k) > 0$ eventually.

Clearly, $\{x(k)\}$ is bounded or unbounded according to whether (I_1) or (I_2) holds.

We denote the sets of all solutions, all oscillatory solutions, and all nonoscillatory solutions of (6.1.149) by $\mathcal{S}$, $\mathcal{O}$, and $\mathcal{N}$, respectively. It is clear that $\mathcal{S} = \mathcal{O} \cup \mathcal{N}$. Now, because of the assumptions on the coefficients of (6.1.149), $\mathcal{N}$ has a decomposition $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2$, where

$$\begin{aligned} \mathcal{N}_0 &= \{x : x \text{ is nonoscillatory, solves (6.1.168), and satisfies } (\mathrm{I}_1)\}, \\ \mathcal{N}_2 &= \{x : x \text{ is nonoscillatory, solves (6.1.168), and satisfies } (\mathrm{I}_2)\}. \end{aligned} \tag{6.1.169}$$

We will need the following two lemmas.

Lemma 6.1.38. *Let $\{x(k)\}$ be an eventually positive sequence with $\Delta x(k) > 0$ and $\Delta(c(k)\Delta x(k)) \geq 0$ eventually, where $\{c(k)\}$ satisfies (6.1.3). Then for $\mu - 1 \geq \tau \geq m$ for some $m \in \mathbb{N}$, it holds that*

$$x(\mu) \geq C(\mu - 1, \tau)c(\tau)\Delta x(\tau), \tag{6.1.170}$$

where C is as in (6.1.8).

Proof. For $\mu - 1 \geq \tau \geq m$,

$$x(\mu) - x(\tau) = \sum_{j=\tau}^{\mu-1} \Delta x(j) = \sum_{j=\tau}^{\mu-1} \frac{c(j)\Delta x(j)}{c(j)}. \tag{6.1.171}$$

Since $c(j)\Delta x(j) \geq c(\tau)\Delta x(\tau)$ for $j \geq \tau$, we find

$$x(\mu) \geq x(\mu) - x(\tau) \geq \left(\sum_{j=\tau}^{\mu-1} \frac{1}{c(j)}\right)(c(\tau)\Delta x(\tau)). \tag{6.1.172}$$

This completes the proof. $\square$

Lemma 6.1.39. *Let $\{x(k)\}$ be an eventually positive sequence with $\Delta x(k) < 0$ and $\Delta(c(k)\Delta x(k)) \geq 0$ eventually, where $\{c(k)\}$ satisfies (6.1.3). Then for $\tau \geq \mu \geq m$ with some $m \in \mathbb{N}$, it holds that*

$$x(\mu) \geq C(\tau,\mu)\big(- c(\tau)\Delta x(\tau)\big), \tag{6.1.173}$$

where C is as in (6.1.8).

Proof. For $\tau \geq \mu \geq m$,

$$x(\mu) - x(\tau+1) = \sum_{j=\mu}^{\tau} -\Delta x(j) = \sum_{j=\mu}^{\tau} \frac{-c(j)\Delta x(j)}{c(j)}. \tag{6.1.174}$$

Since $-c(j)\Delta x(j) \geq -c(\tau)\Delta x(\tau)$ for $j \leq \tau$, we have

$$x(\mu) \geq x(\mu) - x(\tau+1) \geq \left(\sum_{j=\mu}^{\tau} \frac{1}{c(j)}\right)\big(- c(\tau)\Delta x(\tau)\big). \tag{6.1.175}$$

This completes the proof. □

In the case when $\{g(k)\}$ is a retarded argument it may happen that inequality (6.1.168) with some restrictions on f admits no bounded nonoscillatory solutions, that is, $\mathcal{N}_0 = \emptyset$, as the following theorems show.

Theorem 6.1.40. *Suppose that $g(k) < k$ for $k \geq m$ for some $m \in \mathbb{N}$, condition (6.1.3) holds, and one of the following conditions holds:*

(i_1) *condition (6.1.25) and*

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k-1} q(\ell)C\big(g(k),g(\ell)\big) > \frac{1}{\gamma}, \tag{6.1.176}$$

(i_2) *condition (6.1.25) and*

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k} \frac{1}{c(\ell)} \sum_{j=\ell}^{k} q(j) > \frac{1}{\gamma}, \tag{6.1.177}$$

(i_3) *conditions (6.1.105), (6.1.106), and*

$$\sum^{\infty} q(j)f\big(C(j,g(j))\big) = \infty, \tag{6.1.178}$$

where C is as in (6.1.8).

Then $\mathcal{N}_0 = \emptyset$, that is, every bounded solution of (6.1.168) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of inequality (6.1.168) in $\mathcal{N}_0$, say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. There exists an integer $m_1 \geq m$ such that

$$\Delta x(k) < 0 \quad \text{for } k \geq m_1. \tag{6.1.179}$$

By applying Lemma 6.1.39 with $\tau = g(k)$ and $\mu = g(\ell)$, $k \geq \ell \geq m_2 > m_1$, where m_2 satisfies $\min_{k \geq m_2} g(k) \geq m_1$, inequality (6.1.173) becomes

$$x[g(\ell)] \geq C(g(k), g(\ell))\, y[g(k)] \quad \text{for } k \geq \ell \geq m_2, \tag{6.1.180}$$

where $y(k) = -c(k)\Delta x(k) > 0$ for $k \geq \ell$.

If (i_1) holds, then, using (6.1.25) and (6.1.180) in (6.1.168), we have

$$\Delta y(\ell) + \gamma q(\ell) C(g(k), g(\ell))\, y[g(k)] \leq 0 \quad \text{for } k \geq \ell \geq m_2. \tag{6.1.181}$$

Summing (6.1.181) from $g(k)$ to $k-1$, we get

$$y(k) - y[g(k)] + \gamma \sum_{\ell=g(k)}^{k-1} q(\ell) C(g(k), g(\ell))\, y[g(k)] \leq 0 \tag{6.1.182}$$

or

$$y[g(k)] \left[\gamma \sum_{\ell=g(k)}^{k-1} q(\ell) C(g(k), g(\ell)) - 1 \right] \leq 0, \tag{6.1.183}$$

which contradicts condition (6.1.176).

Next we assume that (i_2) holds. Summing inequality (6.1.168) from s to $k-1$ with $k \geq s+1 \geq m_1+1$, we obtain

$$c(k)\Delta x(k) - c(s)\Delta x(s) \geq \gamma \sum_{\ell=s}^{k-1} q(\ell) x[g(\ell)]. \tag{6.1.184}$$

In view of (6.1.179) it follows from (6.1.184) that

$$-\Delta x(s) \geq \frac{\gamma}{c(s)} \sum_{\ell=s}^{k-1} q(\ell) x[g(\ell)] \quad \text{for } k \geq s+1 \geq m_1+1. \tag{6.1.185}$$

Now we write

$$x(\ell) = x(k) + \sum_{s=\ell}^{k-1} [-\Delta x(s)] \quad \text{for } k \geq \ell + 1 \geq m_1 + 1, \tag{6.1.186}$$

which on using (6.1.185) yields

$$x(\ell) \geq \gamma \sum_{s=\ell}^{k-1} \frac{1}{c(s)} \sum_{j=s}^{k-1} q(j)x[g(j)] \quad \text{for } k \geq \ell + 1 \geq m_1 + 1. \tag{6.1.187}$$

Since $g(k-1) < k-1$, in the above inequality we may substitute $\ell = g(k-1)$ to obtain

$$x[g(k-1)] \geq \gamma x[g(k-1)] \left(\sum_{s=g(k-1)}^{k-1} \frac{1}{c(s)} \sum_{j=s}^{k-1} q(j) \right), \tag{6.1.188}$$

so

$$x[g(k-1)] \left[1 - \gamma \sum_{s=g(k-1)}^{k-1} \frac{1}{c(s)} \sum_{j=s}^{k-1} q(j) \right] \geq 0 \quad \text{for } k \geq m_2 + 1, \tag{6.1.189}$$

which contradicts condition (6.1.177).

Finally we assume that (i_3) holds. Applying Lemma 6.1.39 with $\tau = k$ and $\mu = g(k)$, inequality (6.1.173) becomes

$$x[g(k)] \geq C(k, g(k))y(k) \quad \text{for } k \geq m_2. \tag{6.1.190}$$

Using (6.1.105) and (6.1.190) in inequality (6.1.168), we obtain

$$\Delta y(k) + q(k) f(C(k, g(k))) f(y(k)) \leq 0 \quad \text{for } k \geq m_2. \tag{6.1.191}$$

The rest of the proof is similar to the proof of Theorem 6.1.26 and hence we omit it here. □

Theorem 6.1.41. *Let conditions (6.1.3) and (6.1.25) hold, $g(k) = k - d(k)$, $\{d(k)\}$ is a sequence of positive integers. If there exists a constant $\alpha \in (0,1)$ such that*

$$\liminf_{k\to\infty} \frac{1}{d_1(k)} \sum_{j=k-d_1(k)}^{k-1} q(j)C(j - d_1(j), j - d(j)) > \limsup_{k\to\infty} \frac{1}{\gamma} \frac{[d_1(k)]^{d_1(k)}}{[1 + d_1(k)]^{1+d_1(k)}}, \tag{6.1.192}$$

where $d_1(k) = \alpha d(k)$, then $\mathcal{N}_0 = \varnothing$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of inequality (6.1.168) in $\mathcal{N}_0$, say $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Then there exists an integer $m_1 \geq m$ such that (6.1.179) holds for $k \geq m_1$. Applying Lemma 6.1.39 with $\mu = k - d(k)$ and $\tau = k - d_1(k)$, $d_1(k) = \alpha d(k)$ and $0 < \alpha < 1$, we obtain

$$x[k - d(k)] \geq C(k - d_1(k), k - d(k))\, y[k - d_1(k)] \quad \text{for } k \geq m_2 \geq m_1. \tag{6.1.193}$$

Using (6.1.25) and (6.1.193) in inequality (6.1.168), we have

$$\Delta y(k) + \gamma q(k) C(k - d_1(k), k - d(k))\, y[k - d_1(k)] \leq 0 \quad \text{for } k \geq m_2. \tag{6.1.194}$$

The rest of the proof is similar to the proof of Theorem 6.1.16 and hence we omit it here. □

A duality to Theorem 6.1.40 holds in the case when $\{g(k)\}$ is an advanced argument.

Theorem 6.1.42. *Suppose that condition (6.1.3) holds and one of the following conditions holds:*

(ii$_1$) *$g(k) > k$, $k \geq m$ for some $m \in \mathbb{N}$, condition (6.1.25) holds, and*

$$\limsup_{k \to \infty} \sum_{\ell=k}^{g(k)-1} q(\ell) C(g(\ell) - 1, g(k)) > \frac{1}{\gamma}, \tag{6.1.195}$$

(ii$_2$) *$g(k) > k + 1$, $k \geq m$ for some $m \in \mathbb{N}$, condition (6.1.25) holds, and*

$$\limsup_{k \to \infty} \sum_{\ell=k}^{g(k)-1} \frac{1}{c(\ell)} \sum_{j=k}^{\ell-1} q(j) > \frac{1}{\gamma}, \tag{6.1.196}$$

(ii$_3$) *$g(k) > k + 1$, $k \geq m$ for some $m \in \mathbb{N}$, condition (6.1.90) holds, and*

$$\sum^{\infty} q(j) C(g(j) - 1, j + 1) = \infty, \tag{6.1.197}$$

where C is as in (6.1.8).

Then $\mathcal{N}_2 = \emptyset$, that is, every unbounded solution of (6.1.168) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of inequality (6.1.168) in $\mathcal{N}_2$, say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. There exists an integer $m_1 \geq m$ such that

$$\Delta x(k) > 0 \quad \text{for } k \geq m_1. \tag{6.1.198}$$

Applying Lemma 6.1.38 with $\mu = g(\ell)$ and $\tau = g(k)$, $\ell \geq k \geq m_2 \geq m_1$, inequality (6.1.170) takes the form

$$x[g(\ell)] \geq C\big(g(\ell) - 1, g(k)\big)y[g(k)] \quad \text{for } \ell \geq k \geq m_2, \tag{6.1.199}$$

where $y(k) = c(k)\Delta x(k) > 0$ for $k \geq m_1$.

If (ii$_1$) holds, then, using (6.1.25) and (6.1.199) in (6.1.168), we have

$$\Delta y(\ell) \geq \gamma q(\ell)C\big(g(\ell) - 1, g(k)\big)y[g(k)] \quad \text{for } \ell \geq k \geq m_2. \tag{6.1.200}$$

Summing (6.1.200) from k to $g(k) - 1$, we get

$$y[g(k)] \geq y[g(k)] - y(k) \geq \gamma y[g(k)] \sum_{\ell=k}^{g(k)-1} q(\ell)C\big(g(\ell) - 1, g(k)\big), \tag{6.1.201}$$

so

$$y[g(k)]\left\{1 - \gamma \sum_{\ell=k}^{g(k)-1} q(\ell)C\big(g(\ell) - 1, g(k)\big)\right\} \geq 0 \quad \text{for } k \geq m_2 + 1, \tag{6.1.202}$$

which contradicts condition (6.1.195).

Next we assume that (ii$_2$) holds. Summing (6.1.168) from k to $s - 1$ with $s \geq k + 1 \geq m_1 + 1$, we obtain

$$c(s)\Delta x(s) - c(k)\Delta x(k) \geq \gamma \sum_{\ell=k}^{s-1} q(\ell)x[g(\ell)]. \tag{6.1.203}$$

In view of (6.1.198) it is clear from (6.1.203) that

$$\Delta x(s) \geq \gamma \frac{1}{c(s)} \sum_{\ell=k}^{s-1} q(\ell)x[g(\ell)] \quad \text{for } s \geq k + 1 \geq m_1 + 1. \tag{6.1.204}$$

Summing (6.1.204) from k to $\ell - 1$ with $\ell \geq k + 1 \geq m_1 + 1$, we find

$$x(\ell) \geq \gamma \sum_{s=k}^{\ell-1} \frac{1}{c(s)} \sum_{j=k}^{s-1} q(j)x[g(j)]. \tag{6.1.205}$$

Since $g(k) \geq k+1$ in (6.1.205), we may set $\ell = g(k)$ to get

$$x[g(k)] \geq \gamma \sum_{s=k}^{g(k)-1} \frac{1}{c(s)} \sum_{j=k}^{s-1} q(j)x[g(j)] \geq \gamma x[g(k)] \sum_{s=k}^{g(k)-1} \frac{1}{c(s)} \sum_{j=k}^{s-1} q(j), \tag{6.1.206}$$

so

$$x[g(k)]\left\{1-\gamma \sum_{s=k}^{g(k)-1} \frac{1}{c(s)} \sum_{j=k}^{s-1} q(j)\right\} \geq 0 \quad \text{for } k \geq m_2+1 \geq m_1+1, \tag{6.1.207}$$

which contradicts condition (6.1.196).

Finally we assume that (ii$_3$) holds. Summing (6.1.168) from m_1 to $k-1$, we find

$$\Delta x(k) \geq \frac{1}{c(k)} \sum_{j=m_1}^{k-1} q(j) f(x[g(j)]) \quad \text{for } k \geq m_1. \tag{6.1.208}$$

Choose $m_2 > m_1$ and let $g^* = \max_{m_1 \leq k \leq m_2} g(k)$. Dividing (6.1.208) by $f(x(k+1))$ and summing from m_1+1 to g^*, we obtain

$$\begin{aligned} \sum_{k=m_1+1}^{g^*} \frac{\Delta x(k)}{f(x(k+1))} &\geq \sum_{k=m_1+1}^{g^*} \frac{1}{c(k)} \sum_{j=m_1}^{k-1} q(j) \frac{f(x[g(j)])}{f(x(k+1))} \\ &\geq \sum_{j=m_1}^{m_2} q(j)\left(\sum_{k=j+1}^{g(j)-1} \frac{1}{c(k)} \frac{f(x[g(j)])}{f(x(k+1))}\right). \end{aligned} \tag{6.1.209}$$

Since

$$\frac{f(x[g(j)])}{f(x(k+1))} \geq 1 \quad \text{for } j+1 \leq k \leq g(k)-1,\ m_1 \leq k \leq m_2, \tag{6.1.210}$$

we find that

$$\sum_{j=m_1}^{m_2} C(g(j)-1, j+1) q(j) \leq \sum_{k=m_1+1}^{g^*} \frac{\Delta x(k)}{f(x(k+1))} \leq \int_{x(m_1+1)}^{x(g^*+1)} \frac{du}{f(u)}. \tag{6.1.211}$$

Letting $m_2 \to \infty$ in the above inequality and using condition (6.1.90), we obtain a contradiction to condition (6.1.197). $\square$

Theorem 6.1.43. *Let condition (6.1.3) and (6.1.25) hold, $g(k) = k + a(k)$, $\{a(k)\}$ is a sequence of positive integers, $a(k) > 1$. If there exists a constant $\beta \in (0, 1)$ such that*

$$\begin{aligned} \liminf_{k\to\infty} \frac{1}{a_1(k) - 1} & \sum_{j=k+1}^{k+a(k)-1} q(j)C(j + a(j) - 1, j + a_1(j)) \\ & > \frac{1}{\gamma} \limsup_{k\to\infty} \frac{[a_1(k) - 1]^{a_1(k)-1}}{[a_1(k)]^{a_1(k)}}, \end{aligned} \tag{6.1.212}$$

where $a_1(k) = \beta a(k)$, then $\mathcal{N}_2 = \emptyset$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of inequality (6.1.168), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. There exists an integer $m_1 \geq m$ such that (6.1.198) holds for $k \geq m_1$. Next, we apply Lemma 6.1.38 with $\mu = k + a(k)$ and $\tau = k + a_1(k)$ and obtain

$$x[k + a(k)] \geq C(k + a(k) - 1, k + a_1(k))\, y[k + a_1(k)] \quad \text{for } k \geq m_2 \geq m_1. \tag{6.1.213}$$

Using (6.1.25) and (6.1.213) in inequality (6.1.168), we have

$$\Delta y(k) \geq \gamma q(k)C(k + a(k) - 1, k + a_1(k))\, y[k + a_1(k)] \quad \text{for } k \geq m_2. \tag{6.1.214}$$

But in view of condition (6.1.212), it follows from Lemma 6.1.7(I) that (6.1.214) has no eventually positive solution. This contradicts $y(k) = c(k)\Delta x(k) > 0$ for $k \geq m_1$. This completes the proof. □

Based on the above results, we are now ready to state some interesting oscillation criteria for the mixed type equation (6.1.2).

Theorem 6.1.44. *Let condition (6.1.3) hold. Equation (6.1.2) is oscillatory if one of the following holds:*

- (iii_1) *condition Theorem 6.1.42(ii_3) with f, g, q replaced by f_1, g_1, q_1, respectively, and either Theorem 6.1.40(i_1) or (i_2) or the hypotheses of Theorem 6.1.41,*
- (iii_2) *condition Theorem 6.1.42(ii_3) with f, g, q replaced by f_1, g_1, q_1, respectively, and condition Theorem 6.1.40(i_3),*
- (iii_3) *condition Theorem 6.1.40(i_3) and either Theorem 6.1.18(ii_1) or (ii_2), or the hypotheses of Theorem 6.1.43 with f, g, q replaced by f_1, g_1, q_1, respectively,*
- (iii_4) *condition Theorem 6.1.40(i_1) or (i_2), or the hypotheses of Theorem 6.1.41 and condition Theorem 6.1.42(ii_1) or (ii_2), or the hypotheses of Theorem 6.1.43 with f, g, q replaced by f_1, g_1, q_1, respectively.*

Example 6.1.45. Consider the mixed type of difference equation

$$\Delta^2 x(k) = q(k)x^{\alpha}[\tau(k)] + p(k)x^{\beta}[\sigma(k)], \tag{6.1.215}$$

where each α or β is a ratio of two positive odd integers, $\{p(k)\}$ and $\{q(k)\}$ are sequences of positive real numbers, and $\{\tau(k)\}$ and $\{\sigma(k)\}$ are increasing sequences of positive integers. Equation (6.1.215) is oscillatory if one of the following is satisfied:

(I_1) $\beta > 1$, $\sigma(k) > k+1$, $\sum^{\infty}[\sigma(j) - j - 1]p(j) = \infty$, and either one of
  (i) $0 < \alpha < 1$, $\tau(k) < k$, $\sum^{\infty}[j - \tau(j)]q(j) = \infty$,
  (ii) $\alpha = 1$, $\tau(k) < k$, $\limsup_{k\to\infty} \sum_{j=\tau(k)}^{k-1}[\tau(k) + 1 - \tau(j)]q(j) > 1$,

(I_2) $\beta = 1$, $\sigma(k) > k+1$, $\limsup_{k\to\infty} \sum_{j=k}^{g(k)-1}[\sum_{i=k}^{j-1} p(i)] > 1$, and either one of
  (i) $0 < \alpha < 1$, $\tau(k) < k$, $\sum^{\infty}[j - \tau(j)]q(j) = \infty$,
  (ii) $\alpha = 1$, $\tau(k) < k$, $\limsup_{k\to\infty} \sum_{j=\tau(k)}^{k-1}[\tau(k) + 1 - \tau(j)]q(j) > 1$,
  (iii) $\alpha = 1$, $\tau(k) < k$, $\limsup_{k\to\infty} \sum_{j=g(k)}^{k}[\sum_{i=j}^{k} q(i)] > 1$.

6.2. Oscillation criteria (II)

This section deals with the oscillation of certain second-order nonlinear difference equations of the form

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)f\big(x[g(k)]\big) = 0, \tag{6.2.1}$$

and the mixed-type equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) = q(k)f\big(x[g(k)]\big) + q_1(k)f_1\big(x[g_1(k)]\big), \tag{6.2.2}$$

where

(i) $\{c(k)\}$, $\{q(k)\}$, and $\{q_1(k)\}$ are sequences of real numbers such that $c(k) > 0$, $q(k) \ge 0$, and $q_1(k) \ge 0$ eventually,
(ii) $f, f_1 \in C(\mathbb{R},\mathbb{R})$, $xf(x) > 0$, $xf_1(x) > 0$, $f'(x) \ge 0$, and $f_1'(x) \ge 0$ for $x \ne 0$,
(iii) $\{g(k)\}$ and $\{g_1(k)\}$ are sequences of nondecreasing nonnegative integers with $\lim_{k\to\infty} g(k) = \infty$ and $\lim_{k\to\infty} g_1(k) = \infty$,
(iv) $\Psi \in C(\mathbb{R},\mathbb{R})$ satisfies either one of the following:
  (i_1) $\Psi(x) = |x|^{\alpha}\operatorname{sgn} x$, $\alpha \ge 1$,
  (i_2) $\Psi(x) = x^{\alpha}$, α is the ratio of two positive odd integers,
  (i_3) $x\Psi(x) > 0$ and $\Psi'(x) > 0$ for $x \ne 0$.

6.2.1. Oscillation of equation (6.2.1)

We will need the following two lemmas.

Lemma 6.2.1. *If X and Y are nonnegative, then*

$$X^{\gamma}+(\gamma-1)Y^{\gamma}-\gamma XY^{\gamma-1}\geq 0 \quad \textit{with } \gamma>1, \tag{6.2.3}$$

where the equality holds if and only if $X=Y$.

Lemma 6.2.2. *Consider the inequality*

$$\Delta x(k)+q(k)f(x[k-\tau])\leq 0 \quad \textit{eventually}, \tag{6.2.4}$$

where $\{q(k)\}$ is a sequence of nonnegative real numbers, $\tau\in\mathbb{N}$, $f\in C(\mathbb{R},\mathbb{R})$, $xf(x)>0$, and $f'(x)\geq 0$ for $x\neq 0$ and $\sum_{j=k+1}^{k+\tau}q(j)>0$ for all large k. If (6.2.4) has an eventually positive solution, then the equation

$$\Delta x(k)+q(k)f(x[k-\tau])=0 \tag{6.2.5}$$

also has an eventually positive solution.

In what follows we will assume that

$$\sum^{\infty}\Psi^{-1}\left(\frac{1}{c(j)}\right)=\infty, \tag{6.2.6}$$

$$0\leq g(k)\leq k \quad \text{for } k\in\mathbb{N}, \tag{6.2.7}$$

$$f(u)-f(v)=F(u,v)(u-v) \quad \text{for } u,v\neq 0, \text{ where } F \text{ is nonnegative}, \tag{6.2.8}$$

$$\frac{\Psi^{-1}(f(u))}{f(u)}F(u,v)\geq\lambda>0 \quad \text{for } u,v\neq 0, \text{ where } \lambda \text{ is a constant}. \tag{6.2.9}$$

Now we are ready to extend Theorem 6.1.19 to equation (6.2.1).

Theorem 6.2.3. *Let conditions (6.2.6)–(6.2.9) hold and Ψ satisfy either (i_1) or (i_2). If there exists a positive sequence $\{\rho(k)\}$ such that*

$$\limsup_{k\to\infty}\sum_{j=m\in\mathbb{N}}^{k}\left[\rho(j)q(j)-\frac{1}{\alpha+1}\left(\frac{\alpha}{\lambda(\alpha+1)}\right)^{\alpha}\frac{(\Delta\rho(j))^{\alpha+1}}{\rho^{\alpha}(j)}c[g(j)]\right]=\infty, \tag{6.2.10}$$

then equation (6.2.1) is oscillatory.

Proof. Assume for the sake of contradiction that equation (6.2.1) has a nonoscillatory solution $\{x(k)\}$ and that $\{x(k)\}$ is eventually positive. There exists an integer $m \in \mathbb{N}$ such that $x[g(k)] > 0$ for $k \geq m$. From equation (6.2.1), we find that the sequence $\{\Psi^{-1}(c(k))\Delta x(k)\}$ is decreasing for $k \geq m$ and so either $\Delta x(k) > 0$ or $\Delta x(k) < 0$ for $k \geq m_1$ for some $m_1 \geq m$. However, the latter case together with condition (6.2.6) violates the positivity of $x(k)$ and hence

$$\Delta x(k) > 0 \quad \text{for } k \geq m_1. \tag{6.2.11}$$

Define

$$w(k) = \rho(k)\frac{c(k)\Psi(\Delta x(k))}{f(x[g(k)])} \quad \text{for } k \geq m_1. \tag{6.2.12}$$

Then for $k \geq m_1$ we have

$$\begin{aligned}
\Delta w(k) &= \rho(k)\frac{\Delta\big(c(k)\Psi(\Delta x(k))\big)}{f(x[g(k)])} + (\Delta\rho(k))\frac{c(k+1)\Psi(\Delta x(k+1))}{f(x[g(k+1)])}\\
&\quad - \rho(k)c(k+1)\Psi(\Delta x(k+1))\frac{F(x[g(k+1)],x[g(k)])}{f(x[g(k+1)])f(x[g(k)])}\Delta x[g(k)]\\
&= -\rho(k)q(k) + \frac{\Delta\rho(k)}{\rho(k+1)}w(k+1)\\
&\quad - \frac{\rho(k)}{\rho(k+1)}\left(\frac{F(x[g(k+1)],x[g(k)])}{f(x[g(k)])}\right)\Delta x[g(k)]w(k+1).
\end{aligned} \tag{6.2.13}$$

Since $\{\Psi^{-1}(c(k))\Delta x(k)\}$ is decreasing and $\{x(k)\}$ is increasing, we have for $k \geq m_1$,

$$\frac{f(x[g(k+1)])}{f(x[g(k)])} \geq 1, \qquad \Psi^{-1}(c[g(k)])\Delta x[g(k)] \geq \Psi^{-1}(c(k+1))\Delta x(k+1). \tag{6.2.14}$$

Using (6.2.9) and (6.2.14) in (6.2.13), we have for $k \geq m_1$,

$$\Delta w(k) \leq -\rho(k)q(k) + \frac{\Delta\rho(k)}{\rho(k+1)}w(k+1) - \lambda\frac{\rho(k)w^{(1+\alpha)/\alpha}(k+1)}{\rho^{(1+\alpha)/\alpha}(k+1)c^{1/\alpha}[g(k)]}. \tag{6.2.15}$$

Set

$$\begin{aligned}
X &= (\lambda\rho(k))^{\alpha/(\alpha+1)}\left(\frac{c^{-1/(\alpha+1)}[g(k)]}{\rho(k+1)}\right)w(k+1) \quad \text{with } \gamma = \frac{\alpha+1}{\alpha} > 1,\\
Y &= \left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\left(\frac{\Delta\rho(k)}{\rho(k+1)}\right)^{\alpha}\left[\lambda^{-\alpha/(\alpha+1)}\rho^{-\alpha/(\alpha+1)}(k)\rho(k+1)c^{1/(1+\alpha)}[g(k)]\right]^{\alpha}
\end{aligned} \tag{6.2.16}$$

in Lemma 6.2.1 to conclude that

$$\begin{aligned}\frac{\Delta\rho(k)}{\rho(k+1)}w(k+1)-\lambda\frac{\rho(k)}{\rho^{(1+\alpha)/\alpha}(k)c^{1/\alpha}[g(k)]}w^{(\alpha+1)/\alpha}(k+1)&\\ \le\frac{1}{\lambda^{\alpha}}\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(k))^{\alpha+1}}{\rho^{\alpha}(k)}c[g(k)],&\end{aligned}\tag{6.2.17}$$

and therefore

$$\Delta w(k)\le-\rho(k)q(k)+\frac{1}{\lambda^{\alpha}}\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{(\Delta\rho(k))^{\alpha+1}}{\rho^{\alpha}(k)}c[g(k)]\quad\text{for } k\ge m_1.\tag{6.2.18}$$

Summing both sides of (6.2.18) from m_1 to $k \ge m_1$, we get

$$w(k+1)\le w(m_1)-\sum_{j=m_1}^{k}\left[\rho(j)q(j)-\frac{1}{\alpha+1}\left(\frac{\alpha}{\lambda(\alpha+1)}\right)^{\alpha}\frac{(\Delta\rho(j))^{\alpha+1}}{\rho^{\alpha}(j)}c[g(j)]\right].\tag{6.2.19}$$

Taking lim sup on both sides of (6.2.19) as $k \to \infty$ and using condition (6.2.10), we have a contradiction to the fact that $w(k+1) > 0$ for $k \ge m_1$. This completes the proof. □

Next, we will reduce the study of the oscillatory properties of solutions of equation (6.2.1) to those of equation (5.1.1) so that desirable generalizations of some oscillation criteria for equation (5.1.1) to equation (6.2.1) of the same type become immediate.

We will consider the difference inequality

$$\Delta\big(c(k)\Psi(\Delta x(k))\big)+F\big(k,x[g(k)]\big)\le 0,\tag{6.2.20}$$

and the difference equation

$$\Delta\big(c(k)\Psi(\Delta y(k))\big)+F\big(k,y[g(k)]\big)=0,\tag{6.2.21}$$

where $c(k)$, $g(k)$, and Ψ are as in equation (6.2.1), $F(k,x)$ is a continuous function on $\mathbb{N}(m)\times\mathbb{R}$, where $\mathbb{N}(m)=\{m,m+1,\dots\}$, $m\in\mathbb{N}$, $F(k,x)$ is nondecreasing in x, and $\operatorname{sgn}F(k,x)=\operatorname{sgn}x$. We also assume that $\{g(k)\}$ is an increasing sequence of positive integers with $\lim_{k\to\infty}g(k)=\infty$.

Lemma 6.2.4. *Let the function F and $\{g(k)\}$ be as above, the function Ψ satisfy (i_1) or (i_2), and condition (6.2.6) hold. If the inequality (6.2.20) has an eventually positive solution, then so does the equation (6.2.21).*

Proof. Let $\{x(k)\}$ be an eventually positive solution of inequality (6.2.20), and let $m_1 \in \mathbb{N}(m)$ be such that $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m_1$. As in the proof of Theorem 6.2.3, we see that $\Delta x(k) > 0$ for $k \geq m_2$ with $m_2 \in \mathbb{N}(m_1)$. Summing inequality (6.2.20) from k to $u \geq k \geq m_2$ and letting $u \to \infty$, we find

$$c(k)\Psi(\Delta x(k)) \geq \sum_{j=k}^{\infty} F(j, x[g(j)]) \tag{6.2.22}$$

or

$$\Delta x(k) \geq \Psi^{-1}\left[\frac{1}{c(k)} \sum_{j=k}^{\infty} F(j, x[g(j)])\right]. \tag{6.2.23}$$

Summing the above inequality from m_2 to $k-1$, we obtain for $k \geq m_2$,

$$x(k) \geq x(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{j=i}^{\infty} F(j, x[g(j)])\right]. \tag{6.2.24}$$

Let $N = \min\{m_2, \inf_{k \geq m_2} g(k)\}$ and consider the Banach space B_N of all real sequences $y = \{y(k)\}$, $k \geq N$, with the supnorm $\|y\| = \sup_{k \geq N} |y(k)|$. We define a set S by

$$S = \{y \in B_n : 0 \leq y(k) \leq x(k),\ k \geq N\}. \tag{6.2.25}$$

Clearly, S is a bounded, convex, and closed subset of B_N. Next, define an operator $T : S \to B_N$ by

$$(Ty)(k) = \begin{cases} x(m_2) + \displaystyle\sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{j=i}^{\infty} F(j, y[g(j)])\right] & \text{for } k \geq m_2, \\ x(k) & \text{for } N \leq k \leq m_2. \end{cases} \tag{6.2.26}$$

Now, T is continuous, and if $y \in S$, then $(Ty)(k) \geq x(m_2) \geq 0$ and

$$(Ty)(k) \leq x(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{j=i}^{\infty} F(j, x[g(j)])\right] \leq x(k). \tag{6.2.27}$$

Thus $TS \subset S$. Therefore, by the Schauder fixed point theorem, T has a fixed point $y \in S$. Moreover, $y(k) = (Ty)(k)$ satisfies

$$y(k) = x(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{j=i}^{\infty} F(j, y[g(j)])\right] \tag{6.2.28}$$

for $k \geq m_2$, from which it follows that $\{y(k)\}$ is an eventually positive solution of equation (6.2.21). □

Next we compare the oscillatory properties of equation (6.2.21) with those of equations of the form

$$\Delta\big(c(k)\Psi(\Delta y(k))\big) + H\big(k, y[\tau(k)]\big) = 0, \tag{6.2.29}$$

where $c(k)$ and Ψ are as in equation (6.2.21), $\{\tau(k)\}$ is an increasing sequence of positive integers with $\lim_{k\to\infty} \tau(k) = \infty$, and $H(k, y)$ is a continuous function on $\mathbb{N}(m) \times \mathbb{R}$ which is nondecreasing in y with $\operatorname{sgn} H(k, y) = \operatorname{sgn} y$.

Theorem 6.2.5. *Let condition (6.2.6) hold, the functions F and H be as given above, and the function Ψ satisfy (i_1) or (i_2). Assume that*

$$g(k) \ge \tau(k) \quad \text{for } k \in \mathbb{N}(m), \tag{6.2.30}$$

$$F(k, y) \operatorname{sgn} y \ge H(k, y) \operatorname{sgn} y \quad \text{for } k \in \mathbb{N}(m). \tag{6.2.31}$$

If equation (6.2.29) is oscillatory, then equation (6.2.21) is oscillatory.

Proof. Suppose to the contrary that equation (6.2.21) has an eventually positive solution $\{y(k)\}$. As in the proof of Lemma 6.2.4, $\{\Delta y(k)\}$ is eventually positive, so in view of (6.2.30) and (6.2.31), there exists $m_1 \in \mathbb{N}(m)$ such that for $k \ge m_1$, $y[g(k)] \ge y[\tau(k)]$ and $F(k, y[g(k)]) \ge H(k, y[\tau(k)])$. It follows that $\{y(k)\}$ satisfies the inequality

$$\Delta\big(c(k)\Psi(\Delta y(k))\big) + H\big(k, y[\tau(k)]\big) \le 0 \quad \text{for } k \ge m_1, \tag{6.2.32}$$

and so Lemma 6.2.4 implies that equation (6.1.25) has an eventually positive solution, which is a contradiction. □

Now we present the following result.

Theorem 6.2.6. *Let the hypotheses of Lemma 6.2.4 hold and $g(k) \le k$ for $k \in \mathbb{N}(m)$. If equation (6.2.21) has a nonoscillatory solution, then the equation*

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + F\big(g^{-1}(k), x(k)\big) = 0 \tag{6.2.33}$$

has a nonoscillatory solution, where g^{-1} denotes the inverse function of g.

Proof. Let $\{y(k)\}$ be a nonoscillatory solution of equation (6.2.21), say, $y(k) > 0$ for $k \ge m_1 \in \mathbb{N}(m)$. As in the proof of Lemma 6.2.4, we see that for $k \ge m_2$,

$$y(k) \ge y(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{j=i}^{\infty} F\big(j, y[g(j)]\big)\right]. \tag{6.2.34}$$

Let $v = g(j)$. Then

$$\sum_{j=i}^{\infty} F\big(j, y[g(j)]\big) \ge \sum_{v=g(i)}^{\infty} F\big(g^{-1}(v), y(v)\big). \tag{6.2.35}$$

Since $g(k) \leq k$,

$$\sum_{j=i}^{\infty} F(j, y[g(j)]) \geq \sum_{v=i}^{\infty} F(g^{-1}(v), y(v)). \tag{6.2.36}$$

Thus inequality (6.2.34) implies

$$y(k) \geq y(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{v=i}^{\infty} F(g^{-1}(v), y(v))\right] \quad \text{for } k \geq m_2. \tag{6.2.37}$$

Define the integer N, the Banach space B_N, and $S \subset B_N$ in the same way as in the proof of Lemma 6.2.4. Define an operator $T : S \to B_N$ by

$$(Tz)(k) = \begin{cases} y(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{v=i}^{\infty} F(g^{-1}(v), z(v))\right] & \text{for } k \geq m_2, \\ y(k) & \text{for } N \leq k \leq m_2. \end{cases} \tag{6.2.38}$$

It is not difficult to see that the hypotheses of Schauder's fixed point theorem are satisfied, and so there exists $z \in S$ such that $z = Tz$, that is,

$$z(k) = y(m_2) + \sum_{i=m_2}^{k-1} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{v=i}^{\infty} F(g^{-1}(v), z(v))\right] \quad \text{for } k \geq m_2. \tag{6.2.39}$$

From this expression it is easy to see that $z = \{z(k)\}$ is a positive solution of equation (6.2.33) for $k \geq m_2$. □

A duality to Theorem 6.2.6 holds in the case when $g(k)$ is an advanced argument. In fact, we have the following.

Theorem 6.2.7. *Let the hypotheses of Lemma 6.2.4 hold and $g(k) \geq k$ for $k \in \mathbb{N}(m)$. If equation (6.2.33) has a nonoscillatory solution, then equation (6.2.21) also has a nonoscillatory solution.*

Next, we present an alternative proof of Theorem 6.2.6. First, we prove the following lemma.

Lemma 6.2.8. *Let the function F and the sequence $\{g(k)\}$ be as in equation (6.2.21). If the inequality*

$$\Delta x(k) - F(k, x[g(k)]) \geq 0 \tag{6.2.40}$$

has an eventually positive solution, then so does the equation

$$\Delta y(k) - F(k, y[g(k)]) = 0. \tag{6.2.41}$$

PROOF. Let $\{x(k)\}$ be an eventually positive solution of equation (6.2.40), say, $x(k) > 0$ for $k \geq m_1 \in \mathbb{N}(m)$ and $g(k) \geq m$ for $k \geq m_1$. Then $x(k)$ satisfies the inequality

$$x(k) \geq x(m_1) + \sum_{j=m_1}^{k-1} F(j, x[g(j)]). \tag{6.2.42}$$

Define a sequence of successive approximations $\{y^i(k)\}$ as follows:

$$\begin{aligned} y^0(k) &= x(k), \\ y^{i+1}(k) &= x(m_1) + \sum_{j=m_1}^{k-1} F(j, y^i[g(j)]) \quad \text{for } i \in \mathbb{N}. \end{aligned} \tag{6.2.43}$$

Obviously we can prove that $0 \leq y^i(k) \leq x(k)$ and $y^{i+1}(k) \leq y^i(k)$ for $i \in \mathbb{N}_0$ and $k \geq m_1$. Thus, the sequence $\{y^i(k)\}$ is nonnegative and nonincreasing in i for each k. Hence, we may define $y(k) = \lim_{i\to\infty} y^i(k) \geq 0$. Since $0 \leq y(k) \leq y^i(k) \leq x(k)$ for all $i \geq 0$ and since $F(k, y^i[g(k)]) \leq F(k, x[g(k)])$, the convergence of the series in (6.2.43) is uniform with respect to i. Taking the limit on both sides of (6.2.43), we have

$$y(k) = x(m_1) + \sum_{j=m_1}^{k-1} F(k, y[g(k)]). \tag{6.2.44}$$

Hence $\{y(k)\}$ satisfies equation (6.2.41), and the proof is complete. □

ALTERNATIVE PROOF OF THEOREM 6.2.6. Let $\{y(k)\}$ be a nonoscillatory solution of equation (6.1.21), say, $y(k) > 0$ for $k \geq m_1 \in \mathbb{N}(m)$. As in the proof of Theorem 6.2.6, we obtain (6.2.36) and

$$\begin{aligned} \Delta y(k) &\geq \Psi^{-1}\left[\frac{1}{c(k)} \sum_{j=k}^{\infty} F(j, y[g(j)])\right] \\ &\geq \Psi^{-1}\left[\frac{1}{c(k)} \sum_{v=k}^{\infty} F(g^{-1}(v), y(v))\right]. \end{aligned} \tag{6.2.45}$$

By applying a slight extension of Lemma 6.2.8 to the inequality of the above type, we arrive at the desired conclusion. The details are easy and hence we omit them here. □

The following results are immediate consequences of those presented above. Also they extend some of the results obtained in Section 3.7 for equation (3.7.1) to a special case of equation (6.2.21), namely, the equation

$$\Delta(\Psi(\Delta x(k))) + q(k)\Psi(x[g(k)]) = 0, \tag{6.2.46}$$

where the function Ψ satisfies (i_2), $\{g(k)\}$ is as in equation (6.2.21), and $\{q(k)\}$ is a positive real sequence. Now, we have the following.

Theorem 6.2.9. *Suppose that*

$$\sum_{j=m\in\mathbb{N}}^{\infty} q(j) < \infty, \tag{6.2.47}$$

and $g(k) \geq k$ *for* $k \in \mathbb{N}(m)$ *with* $m \in \mathbb{N}$.

(a_1) *If either*

$$\limsup_{k\to\infty} k^{\alpha} \sum_{j=k+1}^{\infty} q(j) > 1 \tag{6.2.48}$$

or

$$\liminf_{k\to\infty} k^{\alpha} \sum_{j=k+1}^{\infty} q(j) > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{6.2.49}$$

then equation (6.2.46) is oscillatory.

(a_2) *If*

$$\limsup_{k\to\infty} g^{\alpha}(k) \sum_{j=k+1}^{\infty} q(j) < \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{6.2.50}$$

then equation (6.2.46) has a nonoscillatory solution.

Theorem 6.2.10. *Suppose that (6.2.47) holds and* $g(k) \leq k$ *for* $k \in \mathbb{N}(m)$.

(b_1) *If either*

$$\limsup_{k\to\infty} g^{\alpha}(k) \sum_{j=k+1}^{\infty} q(j) > 1 \tag{6.2.51}$$

or

$$\liminf_{k\to\infty} g^{\alpha}(k) \sum_{j=k+1}^{\infty} q(j) > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{6.2.52}$$

then equation (6.2.46) is oscillatory.

(b_2) *If*

$$\limsup_{k\to\infty} k^{\alpha} \sum_{j=k+1}^{\infty} q(j) < \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}, \tag{6.2.53}$$

then equation (6.2.46) has a nonoscillatory solution.

Remark 6.2.11. We note that some of the results of Chapters 3 and 5 may also be obtained for equation (6.2.1) via applying the comparison result Theorem 6.2.6. The formulation of such results are easy and left to the reader.

Next, we have the following comparison result. We will compare the oscillatory properties of solutions of equation (6.2.1) with those of a certain first-order difference equation.

We will use the abbreviation

$$C[k,\ell] = \sum_{j=\ell}^{k} \left[\Psi^{-1}(c(j))\right]^{-1} \quad \text{for } k \geq \ell \geq m, \text{ for some } m \in \mathbb{N}. \tag{6.2.54}$$

Theorem 6.2.12. *Let conditions (6.1.105) and (6.2.6) hold, $g(k) = k - \tau$, where $\tau \in \mathbb{N}$, and the function Ψ satisfy either (i_1) or (i_2). If the first-order delay difference equation*

$$\Delta y(k) + q(k) f(C[k-\tau-1,m])(f \circ \Psi^{-1})(y[k-\tau]) = 0 \quad \textit{for } k \geq m+\tau,\ m \in \mathbb{N} \tag{6.2.55}$$

is oscillatory, then equation (6.2.1) is also oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.2.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Proceeding as in the proof of Theorem 6.2.3, we obtain $\Delta x(k) > 0$ and that $\{\Psi^{-1}(c(k))\Delta x(k)\}$ is a nonincreasing sequence for $k \geq m_1$ for some $m_1 \geq m$. Now

$$x(k) - x(m_1) = \sum_{j=m_1}^{k-1} \Delta x(j) = \sum_{j=m_1}^{k-1} \left[\frac{1}{\Psi^{-1}(c(j))}\right]\left[\Psi^{-1}(c(j))\Delta x(j)\right]. \tag{6.2.56}$$

Using the fact that $\{\Psi^{-1}(c(j))\Delta x(j)\}$ is nonincreasing for $k \geq m_1 + 1$, we obtain

$$x(k) \geq C[k-1,m_1]\Psi^{-1}(c(k))\Delta x(k) \tag{6.2.57}$$

for $k \geq m_2 + 1$ for some $m_2 \geq m_1$.

Using condition (6.1.105) and (6.2.57) in equation (6.2.1), we have

$$\Delta y(k) + q(k) f(C[k-\tau-1,m_1]) f(\Psi^{-1}(y[k-\tau])) \leq 0 \tag{6.2.58}$$

for $k \geq m_2 + \tau$, where $y(k) = a(k)\Psi(\Delta x(k))$, $k \geq m_2 + 1$. But in view of Lemma 6.2.2, equation (6.2.55) has an eventually positive solution, which is a contradiction. This completes the proof. $\square$

The following corollary is immediate.

Corollary 6.2.13. *Let the hypotheses of Theorem 6.2.12 hold. If for all large $m \in \mathbb{N}$, $k \geq m + \tau$ either*

$$\liminf_{k\to\infty} \sum_{j=k-\tau}^{k-1} q(j) f\left(C[j-\tau-1, m]\right) \geq \frac{1}{\gamma}\left(\frac{\tau}{\tau+1}\right)^{\tau+1} \tag{6.2.59}$$

provided

$$\frac{f \circ \Psi^{-1}(x)}{x} \geq \gamma > 0 \quad \text{for } x \neq 0, \text{ where } \gamma \text{ is a constant}, \tag{6.2.60}$$

or

$$\sum_{j=m}^{\infty} q(j) f\left(C[j-\tau-1, m]\right) = \infty \tag{6.2.61}$$

provided

$$\int^{+0} \frac{du}{f \circ \Psi^{-1}(u)} < \infty, \qquad \int^{-0} \frac{du}{f \circ \Psi^{-1}(u)} < \infty, \tag{6.2.62}$$

then equation (6.2.1) is oscillatory.

Next, we present the following result.

Theorem 6.2.14. *Let condition (6.2.6) hold and the function Ψ satisfy either (i_1) or (i_2), $g(k) \geq k+1$ for $k \geq m$,*

$$\int^{\infty} \frac{du}{\Psi^{-1} \circ f(u)} < \infty, \qquad \int^{-\infty} \frac{du}{\Psi^{-1} \circ f(u)} < \infty, \tag{6.2.63}$$

$$Q(m) < \infty \quad \text{for } m \in \mathbb{N}, \text{ where } Q(k) = \sum_{j=k}^{\infty} q(j). \tag{6.2.64}$$

If either

$$\sum_{j=m}^{\infty} \Psi^{-1}\left[\frac{Q(j)}{c(j)}\right] = \infty \tag{6.2.65}$$

or conditions (6.2.8), (6.2.9), and

$$\limsup_{k\to\infty} \sum_{j=m}^{k} \left(\frac{1}{c(j)}\left[Q(j) + \lambda \sum_{i=j}^{\infty} c^{-1/\alpha}[g(i)] Q^{(1+\alpha)/\alpha}(i)\right]\right)^{1/\alpha} = \infty \tag{6.2.66}$$

hold, then equation (6.2.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.2.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define

$$w(k) = \frac{c(k)\Psi(\Delta x(k))}{f(x[g(k)])} \quad \text{for } k \geq m, \tag{6.2.67}$$

and proceeding as in the proof of Theorem 6.2.3, we obtain $\Delta w(k) \leq -q(k)$ for $k \geq m$. Summing both sides of this inequality from k to u and letting $u \to \infty$ in the resulting inequality, we obtain

$$\Psi(\Delta x(k)) \geq \frac{1}{c(k)} Q(k) f(x[g(k)]) \geq \frac{1}{c(k)} Q(k) f(x(k+1)) \tag{6.2.68}$$

for $k \geq m$, so

$$\frac{\Delta x(k)}{\Psi^{-1} \circ f(x(k+1))} \geq \Psi^{-1}\left(\frac{1}{c(k)} Q(k)\right) \quad \text{for } k \geq m. \tag{6.2.69}$$

Thus it follows that

$$\sum_{j=m+1}^{k} \Psi^{-1}\left(\frac{1}{c(j)} Q(j)\right) \leq \int_{x(m+1)}^{x(k+1)} \frac{du}{\Psi^{-1} \circ f(u)} < \infty. \tag{6.2.70}$$

This contradicts condition (6.2.65).

Once again as in the proof of Theorem 6.2.3, we obtain

$$\Delta w(k) \leq -q(k) - \frac{\lambda}{a^{1/\alpha}[g(k)]} w^{(1+\alpha)/\alpha}(k) \quad \text{for } k \geq m_1 \geq m. \tag{6.2.71}$$

Summing both sides of (6.2.71) from k to u, noting $w(k) \geq Q(k)$, and then letting $u \to \infty$, we have

$$\Psi(\Delta x(k)) \geq \frac{1}{c(k)}\left[Q(k) + \lambda \sum_{j=k}^{\infty} a^{-1/\alpha}[g(j)] Q^{(1+\alpha)/\alpha}(j)\right] f(x(k+1)) \tag{6.2.72}$$

for $k \geq m_1 \geq m$, or

$$\frac{\Delta x(k)}{\Psi^{-1} \circ f(x(k+1))} \geq \left(\frac{1}{c(k)}\left[Q(k) + \lambda \sum_{j=k}^{\infty} a^{-1/\alpha}[g(j)] Q^{(1+\alpha)/\alpha}(j)\right]\right)^{1/\alpha}. \tag{6.2.73}$$

Summing (6.2.73) from m_1+1 to k, we obtain

$$\int_{x(m_1+1)}^{x(k+1)} \frac{du}{\Psi^{-1}\circ f(u)} \geq \sum_{j=m_1+1}^{k} \left(\frac{1}{c(j)} \left[Q(j) + \lambda \sum_{i=j}^{\infty} a^{-1/\alpha}[g(i)]Q^{1+1/\alpha}(i) \right] \right)^{1/\alpha}. \tag{6.2.74}$$

Taking lim sup on both sides of (6.2.74) and using condition (6.2.66), we obtain a contradiction to condition (6.2.63). This completes the proof. □

6.2.2. Oscillation of equation (6.2.2)

First, we will consider the difference inequality

$$\{\Delta(c(k)\Psi(\Delta x(k))) - q(k)f(x[g(k)])\} \operatorname{sgn} x[g(k)] \geq 0, \tag{6.2.75}$$

where $c(k)$, $g(k)$, $f(x)$, and $\Psi(x)$ are as in equation (6.2.1).

Let $\{x(k)\}$ be a nonoscillatory solution of inequality (6.2.75). It is easy to see that $\{\Delta x(k)\}$ is eventually of constant sign, so that either

(I_1) $x(k)\Delta x(k) < 0$ eventually, or
(I_2) $x(k)\Delta x(k) > 0$ eventually.

Clearly, $\{x(k)\}$ is bounded or unbounded according to whether (I_1) or (I_2) holds.

When f is not a monotonic function, we will assume that

$$\frac{f(x)}{\Psi(x)} \geq \gamma > 0 \quad \text{for } x \neq 0, \text{ where } \gamma \text{ is a constant.} \tag{6.2.76}$$

Now we will prove the following result.

Theorem 6.2.15. *Suppose that $g(k) < k$ for $k \geq m$ for some $m \in \mathbb{N}$, condition (6.2.6) holds, Ψ satisfies (i_1) or (i_2), and one of the following holds:*

(1) *condition (6.2.76) and*

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k-1} q(\ell)\Psi(C[g(k), g(\ell)]) > \frac{1}{\gamma}, \tag{6.2.77}$$

(2) *condition (6.2.76) and*

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k} \Psi^{-1}\left[\frac{\gamma}{c(\ell)} \sum_{j=\ell}^{k} q(j) \right] > 1, \tag{6.2.78}$$

(3) *conditions (6.1.105), (6.2.62), and*

$$\sum^{\infty} q(j)f(C[j, g(j)]) = \infty, \tag{6.2.79}$$

where C is as in (6.2.54).

Then all bounded solutions of (6.2.75) are oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory bounded solution of (6.2.75), say, $x(k) > 0$, $k \geq m$, for some $m \in \mathbb{N}$. By (I_1),

$$\Delta x(k) < 0 \quad \text{for } k \geq m_1 \text{ for some } m_1 \geq m. \tag{6.2.80}$$

Now, for $\tau + 1 \geq \mu \geq m_1$, we get

$$x(\mu) - x(\tau+1) = -\sum_{j=\mu}^{\tau} \Delta x(j) = \sum_{j=\mu}^{\tau} \left[- \frac{\Psi^{-1}(c(j))\Delta x(j)}{\Psi^{-1}(c(j))} \right]. \tag{6.2.81}$$

Since $\{-\Psi^{-1}(c(k))\Delta x(k)\}$ is an increasing sequence, we can easily obtain

$$x(\mu) \geq C[\tau,\mu]\big(- \Psi^{-1}(c(\tau))\Delta x(\tau)\big). \tag{6.2.82}$$

Suppose that (1) holds. Let $\tau = g(k)$ and $\mu = g(\ell)$ in (6.2.82). Then we get

$$x[g(\ell)] \geq C[g(k),g(\ell)]\big(- \Psi^{-1}(c[g(k)])\Delta x[g(k)]\big) \quad \text{for } k \geq \ell \geq m_2, \tag{6.2.83}$$

where $m_2 > m_1$ satisfies

$$\min_{k \geq m_2} g(k) \geq m_1. \tag{6.2.84}$$

In view of (6.2.80) and (6.2.76), it follows from (6.2.83) that for $k \geq \ell \geq m_2$,

$$\begin{aligned} q(\ell) f(x[g(\ell)]) &\geq \gamma q(\ell)\Psi(x[g(\ell)]) \\ &\geq \gamma q(\ell)\Psi(C[g(k),g(\ell)])\Psi(y[g(k)]), \end{aligned} \tag{6.2.85}$$

where $y(k) = -\Psi^{-1}(c(k))\Delta x(k) > 0$ for $k \geq m_2$. Using the above inequality in (6.2.75) we have

$$\Delta\Psi(- y(\ell)) \geq \gamma q(\ell)\Psi(C[g(k),g(\ell)])\Psi(y[g(k)]) \quad \text{for } k \geq \ell \geq m_2. \tag{6.2.86}$$

Summing (6.2.86) from $g(k)$ to $k-1$ (noting that $g(k) < k$), we get

$$\Psi(y[g(k)]) \sum_{\ell=g(k)}^{k-1} \gamma q(\ell)\Psi(C[g(k),g(\ell)]) - [- \Psi(- y[g(k)])] \leq \Psi(- y(k)). \tag{6.2.87}$$

Since Ψ satisfies (i_1) or (i_2) one can easily see that

$$\Psi(y[g(k)])\left\{\gamma\sum_{\ell=g(k)}^{k-1}q(\ell)\Psi(C[g(k),g(\ell)])-1\right\}\leq 0\quad\text{for } k\geq m_2,\tag{6.2.88}$$

which is a contradiction to condition (6.2.77).

Next, suppose that (2) holds. We sum (6.2.75) from μ to $k-1$, to obtain

$$c(k)\Psi(\Delta x(k))-c(\mu)\Psi(\Delta x(\mu))\geq\sum_{\ell=\mu}^{k-1}q(\ell)f(x[g(\ell)])\quad\text{for } k\geq\mu+1\geq m_1+1.\tag{6.2.89}$$

In view of (6.2.80) it follows from (6.2.89) that

$$-\Delta x(\mu)\geq\Psi^{-1}\left[\frac{1}{c(\mu)}\sum_{\ell=\mu}^{k-1}q(\ell)f(x[g(\ell)])\right]\quad\text{for } k\geq\mu+1\geq m_1+1.\tag{6.2.90}$$

Now we write

$$x(\ell)=x(k)+\sum_{\mu=\ell}^{k-1}[-\Delta x(\mu)]\quad\text{for } k\geq\ell+1\geq m_1+1.\tag{6.2.91}$$

Using (6.2.76) and (6.2.91) in (6.2.90), we find

$$x(\ell)\geq\sum_{\mu=\ell}^{k-1}\Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=\mu}^{k-1}q(j)\Psi(x[g(j)])\right]\quad\text{for } k\geq\ell+1\geq m_1+1.\tag{6.2.92}$$

Since $g(k-1)<k-1$, in (6.2.92) we may set $\ell=g(k-1)$ to get

$$\begin{aligned}x[g(k-1)]&\geq\sum_{\mu=g(k-1)}^{k-1}\Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=\mu}^{k-1}q(j)\Psi(x[g(j)])\right]\\&\geq\sum_{\mu=g(k-1)}^{k-1}\Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=\mu}^{k-1}q(j)\Psi(x[g(k-1)])\right]\\&=x[g(k-1)]\sum_{\mu=g(k-1)}^{k-1}\Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=\mu}^{k-1}q(j)\right].\end{aligned}\tag{6.2.93}$$

This inequality is equivalent to

$$x[g(k-1)]\left\{1-\sum_{\mu=g(k-1)}^{k-1}\Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=\mu}^{k-1}q(j)\right]\right\}\geq 0\quad\text{for } k\geq m_2\geq m_1+1,\tag{6.2.94}$$

which is a contradiction to condition (6.2.78).

Finally suppose that (3) holds. In inequality (6.2.82), set $\tau = k$ and $\mu = g(k)$ to obtain

$$x[g(k)] \geq C(k, g(k))y(k) \quad \text{for } k \geq m_2, \tag{6.2.95}$$

where $y(k)$ is as defined above. Using (6.1.105) and (6.2.95) in (6.2.75), we can easily find

$$-\Delta\Psi(y(k)) \geq q(k)f(C[k, g(k)])f(y(k)) \quad \text{for } k \geq m_2, \tag{6.2.96}$$

or

$$-\Delta z(k) \geq q(k)f(C[k, g(k)])f(\Psi^{-1}(z(k))) \quad \text{for } k \geq m_2, \tag{6.2.97}$$

where $z(k) = \Psi(y(k))$. The rest of the proof is similar to that of Theorem 6.1.26 and hence is omitted. This completes the proof. □

The obtained results may be extended to inequalities of type (6.2.75) when the function Ψ satisfies (i_3) and the condition

$$-\Psi(-xy) \geq \Psi(xy) \geq \Psi(x)\Psi(y) \quad \text{for } x, y > 0. \tag{6.2.98}$$

Now, when $g(k)$ in inequality (6.2.75) is an advanced argument, we present the following result.

Theorem 6.2.16. *Suppose that condition (6.2.6) holds, the function Ψ satisfies (i_1) or (i_2) and one of the following holds:*

(1) *$g(k) > k$, $k \geq m$, for some $m \in \mathbb{N}$, conditions (6.2.76) and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g(k)-1} q(\ell)\Psi(C[g(\ell) - 1, g(k)]) > \frac{1}{\gamma}, \tag{6.2.99}$$

(2) *$g(k) > k + 1$, $k \geq m$, for some $m \in \mathbb{N}$, conditions (6.2.76) and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g(k)-1} \Psi^{-1}\left[\frac{\gamma}{c(\ell)} \sum_{j=k}^{\ell-1} q(j)\right] > 1, \tag{6.2.100}$$

(3) *$g(k) > k + 1$, $k \geq m$, for some $m \in \mathbb{N}$, conditions (6.1.105),*

$$\int^{+\infty} \frac{du}{f \circ \Psi^{-1}(u)} < \infty, \qquad \int^{-\infty} \frac{du}{f \circ \Psi^{-1}(u)} < \infty, \tag{6.2.101}$$

$$\sum^{\infty} q(j)f(C[g(j) - 1, j + 1]) = \infty, \tag{6.2.102}$$

where C is as in (6.2.54).

Then all unbounded solutions of (6.2.75) are oscillatory.

Proof. Let $\{x(k)\}$ be an unbounded nonoscillatory solution of inequality (6.2.75), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. By (I_2),

$$\Delta x(k) > 0 \quad \text{for } k \geq m_1 \text{ for some } m_1 \geq m. \tag{6.2.103}$$

Now, for $\mu - 1 \geq \tau \geq m_1$, we get

$$x(\mu) - x(\tau) = \sum_{j=\tau}^{\mu-1} \Delta x(j) = \sum_{j=\tau}^{\mu-1} \frac{\Psi^{-1}(c(j))\Delta x(j)}{\Psi^{-1}(c(j))}. \tag{6.2.104}$$

Since $\{\Psi^{-1}(c(k))\Delta x(k)\}$ is an increasing sequence, we can easily obtain

$$x(\mu) \geq C[\mu - 1, \tau]\Psi^{-1}(c(\tau))\Delta x(\tau) \quad \text{for } \mu - 1 \geq \tau \geq m_1. \tag{6.2.105}$$

Suppose that (1) holds. Set $\mu = g(\ell)$ and $\tau = g(k)$ in (6.2.105). Then we get

$$x[g(\ell)] \geq C[g(\ell) - 1, g(k)]y[g(k)] \quad \text{for } \ell - 1 \geq k \geq m_2 \geq m_1, \tag{6.2.106}$$

where $y(k) = \Psi^{-1}(c(k))\Delta x(k) > 0$ for $k \geq m_2$. Using (6.2.103), (6.2.76), and (6.2.106) in inequality (6.2.75), we obtain

$$\begin{aligned} \Delta\Psi(y(\ell)) &\geq \gamma q(\ell)\Psi(x[g(\ell)]) \\ &\geq \gamma q(\ell)\Psi(C[g(\ell) - 1, g(k)])\Psi(y[g(k)]) \end{aligned} \tag{6.2.107}$$

for $\ell - 1 \geq k \geq m_2$. Summing (6.2.107) from k to $g(k) - 1$ (noting that $g(k) > k$), we get

$$\Psi(y[g(k)]) - \Psi(y(k)) \geq \Psi(y[g(k)]) \sum_{\ell=k}^{g(k)-1} \gamma q(\ell)\Psi(C[g(\ell) - 1, g(k)]), \tag{6.2.108}$$

which implies

$$\Psi(y[g(k)])\left\{1 - \gamma \sum_{\ell=k}^{g(k)-1} q(\ell)\Psi(C[g(\ell) - 1, g(k)])\right\} \geq 0 \quad \text{for } k \geq m_2, \tag{6.2.109}$$

which is a contradiction to condition (6.2.99).

Suppose that (2) holds. Summing (6.2.75) from k to $\mu - 1$, we obtain

$$c(\mu)\Psi(\Delta x(\mu)) - c(k)\Psi(\Delta x(k)) \geq \sum_{\ell=k}^{\mu-1} q(\ell)f(x[g(\ell)]) \quad \text{for } \mu \geq k + 1 \geq m_1 + 1. \tag{6.2.110}$$

In view of (6.2.103), it is clear from (6.2.110) that

$$\Delta x(\mu) \geq \Psi^{-1}\left(\frac{\gamma}{c(\mu)}\sum_{\ell=k}^{\mu-1} q(\ell)\Psi(x[g(\ell)])\right) \quad \text{for } \mu \geq k+1 \geq m_1+1. \tag{6.2.111}$$

We write

$$x(\ell) = x(k) + \sum_{\mu=k}^{\ell-1} \Delta x(\mu) \quad \text{for } \ell \geq k+1 \geq m_1+1, \tag{6.2.112}$$

so that a substitution in (6.2.111) leads to

$$x(\ell) \geq \sum_{\mu=k}^{\ell-1} \Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=k}^{\mu-1} q(j)\Psi(x[g(j)])\right] \quad \text{for } \ell \geq k+1 \geq m_1+1. \tag{6.2.113}$$

Since $g(k) > k+1$, in (6.2.113) we set $\ell = g(k)$ to get for $k \geq m_2+1 \geq m_1$,

$$\begin{aligned} x[g(k)] &\geq \sum_{\mu=k}^{g(k)-1} \Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=k}^{\mu-1} q(j)\Psi(x[g(j)])\right] \\ &\geq x[g(k)]\sum_{\mu=k}^{g(k)-1} \Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=k}^{\mu-1} q(j)\right]. \end{aligned} \tag{6.2.114}$$

This inequality is equivalent to

$$x[g(k)]\left\{1 - \sum_{\mu=k}^{g(k)-1} \Psi^{-1}\left[\frac{\gamma}{c(\mu)}\sum_{j=k}^{\mu-1} q(j)\right]\right\} \geq 0 \quad \text{for } k \geq m_2+1, \tag{6.2.115}$$

and this contradicts condition (6.2.100).

Next, suppose that (3) holds. In inequality (6.2.105), set $\mu = g(k)$ and $\tau = k+1$ to obtain

$$x[g(k)] \geq C[g(k)-1, k+1]y(k+1) \quad \text{for } k \geq m_2, \tag{6.2.116}$$

where $y(k)$ is as defined above. Using (6.1.105) and (6.2.103) in (6.2.75), we obtain

$$\begin{aligned} \Delta\Psi(y(k)) &\geq q(k)f(x[g(k)]) \\ &\geq q(k)f(C[g(k)-1, k+1])f(y(k+1)), \end{aligned} \tag{6.2.117}$$

so

$$\frac{\Delta z(k)}{f(\Psi^{-1}(z(k+1)))} \geq q(k)f(C[g(k)-1, k+1]) \quad \text{for } k \geq m_2, \tag{6.2.118}$$

where $z(k) = \Psi(y(k))$ for $k \geq m_2$. The rest of the proof is similar to that of Theorem 6.2.14 and hence is omitted. This completes the proof. □

Next, we obtain oscillation criteria for equation (6.2.2) via applying Theorems 6.2.15 and 6.2.16. The results are immediate and the proofs are omitted.

Theorem 6.2.17. *Suppose that $g(k) < k$, $k \geq m$, for some $m \in \mathbb{N}$, condition (6.2.6) holds, the function Ψ satisfies either (i_1) or (i_2), the function f satisfies condition (6.2.76), and the function f_1 satisfies*

$$\frac{f_1(x)}{\Psi(x)} \geq \gamma_1 > 0 \quad \text{for } x \neq 0, \text{ where } \gamma_1 \text{ is a constant.} \tag{6.2.119}$$

Equation (6.2.2) is oscillatory if either (6.2.77) or (6.2.78) and one of the following holds:

(a_1) *$g_1(k) > k$ for $k \geq m$ and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g(k)-1} q_1(\ell)\Psi(C[g_1(\ell)-1, g_1(k)]) \geq \frac{1}{\gamma_1}, \tag{6.2.120}$$

(a_2) *$g_1(k) > k+1$ for $k \geq m$ and*

$$\limsup_{k\to\infty} \sum_{\ell=k}^{g_1(k)-1} \Psi\left[\frac{\gamma_1}{c(\ell)} \sum_{j=k}^{\ell-1} q_1(j)\right] > 1. \tag{6.2.121}$$

Theorem 6.2.18. *Suppose that $g(k) < k$ for $k \geq m$ for some $m \in \mathbb{N}$, conditions (6.2.6), (6.1.105), and (6.2.62) hold, and the function Ψ satisfies (i_1) or (i_2). Equation (6.2.2) is oscillatory if condition (6.2.79) and either Theorem 6.2.17(a_1) or (a_2) hold.*

Theorem 6.2.19. *Suppose that the hypotheses of Theorem 6.2.17 hold and either one of the conditions (6.2.77), (6.2.78), or (6.2.79) holds. If in addition $g_1(k) > k+1$ for $k \geq m$, condition (6.1.105) holds,*

$$\begin{gathered} \int^{\infty} \frac{du}{f_1 \circ \Psi^{-1}(u)} < \infty, \qquad \int^{-\infty} \frac{du}{f_1 \circ \Psi^{-1}(u)} < \infty, \\ \sum^{\infty} q_1(j) f_1(C[g_1(j)-1, j+1]) = \infty, \end{gathered} \tag{6.2.122}$$

then equation (6.2.2) is oscillatory.

The following example illustrates the methods presented above.

Example 6.2.20. Consider the mixed-type equation

$$\Delta(\Delta x(k))^{\alpha} = q(k)x^{\beta}[\tau(k)] + p(k)x^{\gamma}[\sigma(k)], \tag{6.2.123}$$

where each of the numbers α, β, and γ is a ratio of two positive odd integers, $\{p(k)\}$ and $\{q(k)\}$ are sequences of positive real numbers, and $\{\tau(k)\}$ and $\{\sigma(k)\}$ are increasing sequences of positive integers. Equation (6.2.123) is oscillatory if one of the conditions of (I_1) (below) and one of the conditions of (I_2) (below) hold:

(I_1) $\alpha \geq 1$,

(i_1) $\beta = 1$, $\tau(k) < k$, $k \geq m \in \mathbb{N}$, and either

$$\limsup_{k\to\infty} \sum_{\ell=\tau(k)}^{k-1} q(\ell)\left[\tau(k) + 1 - \tau(\ell)\right]^{\alpha} > 1 \tag{6.2.124}$$

or

$$\limsup_{k\to\infty} \sum_{\ell=\tau(k)}^{k-1} \left[\sum_{j=\ell}^{k} q(j)\right]^{1/\alpha} > 1, \tag{6.2.125}$$

(i_2) $0 < \beta < \alpha$, $\tau(k) < k$ for $k \geq m$, and

$$\sum^{\infty} q(j)\left[j - \tau(j)\right]^{\beta} = \infty, \tag{6.2.126}$$

(I_2) $\alpha \geq 1$,

(i_3) $\gamma = 1$, $\sigma(k) > k$, $k \geq m \in \mathbb{N}$, and

$$\limsup_{k\to\infty} \sum_{\ell=k}^{\sigma(k)-1} p(\ell)\left[\sigma(\ell) - \sigma(k)\right]^{\alpha} > 1, \tag{6.2.127}$$

(i_4) $\gamma = 1$, $\sigma(k) > k + 1$, $k \in \mathbb{N}$, and

$$\limsup_{k\to\infty} \sum_{\ell=k}^{\sigma(k)-1} \left[\sum_{j=k}^{\ell-1} p(j)\right]^{1/\alpha} > 1, \tag{6.2.128}$$

(i_5) $\beta > \alpha$, $\sigma(k) > k + 1$, $k \geq m \in \mathbb{N}$, and

$$\sum^{\infty} p(j)\left[\sigma(j) - j - 1\right]^{\alpha} = \infty. \tag{6.2.129}$$

6.2.3. Asymptotic behavior

Here, we will study some asymptotic properties of solutions of a special case of inequality (6.2.75), namely, the equation

$$\Delta\Psi(\Delta x(k)) = q(k)\Psi(x[g(k)]). \tag{6.2.130}$$

We will assume that the function Ψ satisfies either (i_1) or (i_2).

Now we present the following result.

Theorem 6.2.21. *Equation (6.2.130) has an unbounded nonoscillatory solution* $\{x(k)\}$ *such that*

$$\lim_{k\to\infty}\frac{x(k)}{k} = \text{constant} \neq 0 \tag{6.2.131}$$

if and only if

$$\sum^{\infty} q(j)\Psi(g(j)) < \infty. \tag{6.2.132}$$

Proof. First, suppose that equation (6.2.130) has an unbounded nonoscillatory solution $\{x(k)\}$ satisfying (6.2.131). Without loss of generality, we assume that $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. It is easy to see that $\Delta x(k) > 0$ for $k \geq m$. Further, (6.2.131) implies that $L = \lim_{k\to\infty}\Delta x(k)$ (a constant) is finite. On summing (6.2.130) from $\ell \geq m$ to $u \geq \ell$ and letting $u \to \infty$, we get

$$\Delta x(\ell) = \Psi^{-1}\left[\Psi(L) - \sum_{j=\ell}^{\infty} q(j)\Psi(x[g(j)])\right] \quad \text{for } \ell \geq m_1, \tag{6.2.133}$$

where $m_1 > m$ satisfies

$$\min_{k\geq m} g(k) \geq m. \tag{6.2.134}$$

Summing (6.2.133) from m_1 to $k-1$ provides

$$x(k) = x(m_1) + \sum_{\ell=m_1}^{k-1}\Psi^{-1}\left[\Psi(L) - \sum_{j=\ell}^{\infty} q(j)\Psi(x[g(j)])\right]. \tag{6.2.135}$$

It is clear from (6.2.135) that we must have

$$\sum^{\infty} q(j)\Psi(x[g(j)]) < \infty, \tag{6.2.136}$$

for otherwise, the right-hand side of (6.2.135) tends to $-\infty$ as $k \to \infty$, which contradicts the fact that $x(k)$ is eventually positive. In view of (6.2.131) we have

$$\lim_{j\to\infty}\frac{x[g(j)]}{g(j)} = \text{constant}, \tag{6.2.137}$$

and hence (6.2.136) implies that condition (6.2.132) holds.

Next, suppose that (6.2.132) holds. Let $K > 0$ be an arbitrary but fixed number. We choose $m_2 \geq m^* \geq m$ so large that

$$m_3 = \min_{k\geq m_2} g(k) \geq m^*, \tag{6.2.138}$$

and in view of (6.2.132),

$$\sum_{j=m_2+1}^{\infty} q(j-1)\Psi(g(j-1)) < 1-\Psi\left(\frac{1}{4}\right). \tag{6.2.139}$$

Let $\overline{m} = \min\{m_2, m_3\}$,

$$S = \{x(k) : x(k) \text{ is defined for } k \geq \overline{m}\}, \tag{6.2.140}$$

and let X be the set of all $x \in S$ that satisfy

$$\begin{gathered} \frac{K}{4}(k-m_2) \leq x(k) \leq K(k-m_2) \quad \text{if } k \geq m_2, \\ x(k) = 0 \quad \text{if } \overline{m} \leq k \leq m_2. \end{gathered} \tag{6.2.141}$$

Define $T : X \to S$ by

$$(Tx)(k) = \sum_{\ell=m_2}^{k-1} \Psi^{-1}\left[\Psi(K) - \sum_{j=\ell+1}^{\infty} q(j-1)\Psi(x[g(j-1)])\right] \quad \text{for } k \geq \overline{m}. \tag{6.2.142}$$

Let $x \in X$. If $\overline{m} \leq k \leq m_2$, then it is clear that $(Tx)(k) = 0$. For $k \geq m_2$, we have

$$(Tx)(k) \leq \sum_{\ell=m_2}^{k-1} \Psi^{-1}[\Psi(K) - 0] = K[k-m_2], \tag{6.2.143}$$

and on using (6.2.139),

$$\begin{aligned} (Tx)(k) &\geq \sum_{\ell=m_2}^{k-1} \Psi^{-1}\left[\Psi(K) - \sum_{j=\ell+1}^{\infty} q(j-1)\Psi(Kg(j-1))\right] \\ &\geq \sum_{\ell=m_2}^{k-1} \Psi^{-1}\left[\Psi(K) - \sum_{j=m_2+1}^{\infty} q(j-1)\Psi(Kg(j-1))\right] \\ &\geq \sum_{\ell=m_2}^{k-1} \Psi^{-1}\left[\Psi(K) - \left(1-\Psi\left(\frac{1}{4}\right)\right)\Psi(K)\right] \\ &= \frac{1}{4}K[k-m_2]. \end{aligned} \tag{6.2.144}$$

Hence $T(X) \subseteq X$. It is clear that X is a closed, convex, and compact subset of S and $T(X)$ is relatively compact in S. Therefore, by Schauder's fixed point theorem, T has a fixed point in X given by

$$x(k) = \sum_{\ell=m_2}^{k-1} \Psi^{-1}\left[\Psi(K) - \sum_{j=\ell+1}^{\infty} q(j-1)\Psi(x[g(j-1)])\right] \quad \text{for } k \geq \overline{m}. \tag{6.2.145}$$

It is easy to see that this $x(k)$ satisfies equation (6.2.130). To see that (6.2.131) is also fulfilled, in view of the discrete L'Hôpital rule, we have

$$\begin{aligned}\lim_{k\to\infty}\frac{x(k)}{k} &= \lim_{k\to\infty}\Delta x(k)\\ &= \lim_{k\to\infty}\Psi^{-1}\left[\Psi(K)-\sum_{j=k+1}^{\infty}q(j-1)\Psi(x[g(j-1)])\right]\\ &= K.\end{aligned} \tag{6.2.146}$$

Thus $x(k)$ given in (6.2.145) is an unbounded nonoscillatory solution of equation (6.2.130) such that (6.2.131) holds. □

Theorem 6.2.22. *Equation (6.2.130) has a bounded nonoscillatory solution $\{x(k)\}$ such that*

$$\lim_{k\to\infty}x(k) = \text{constant} \neq 0 \tag{6.2.147}$$

if and only if

$$\sum^{\infty}\Psi^{-1}\left[\sum_{j=k}^{\infty}q(j)\right] < \infty. \tag{6.2.148}$$

Proof. First, suppose that equation (6.2.130) has a bounded nonoscillatory solution $\{x(k)\}$ satisfying (6.2.147). Let $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. It is easy to see that $\Delta x(k) < 0$ for $k \geq m$. Next, (6.2.147) implies that $\lim_{k\to\infty}\Delta x(k) = 0$ and $\lim_{k\to\infty}x(k) = C$ is finite. Summing equation (6.2.130) from ℓ to u and letting $u \to \infty$, we have

$$\Psi[-\Delta x(\ell)] = \sum_{j=\ell}^{\infty}q(j)\Psi(x[g(j)]) \quad \text{for } \ell > m_1, \tag{6.2.149}$$

where m_1 is defined in (6.2.134). Now, on summing of (6.2.149) from k to u and letting $u \to \infty$, we obtain

$$x(k) = C + \sum_{\ell=k}^{\infty}\Psi^{-1}\left[\sum_{j=\ell}^{\infty}q(j)\Psi(x[g(j)])\right] \quad \text{for } k \geq m_1. \tag{6.2.150}$$

In view of the fact that $x(k)$ is bounded, condition (6.2.148) readily follows from the equality (6.2.150).

Next, suppose that (6.2.148) holds. Let $K > 0$ be an arbitrary but fixed number. We choose $m_2 \geq m^*$ so large that (6.2.138) holds, and also, in view of (6.2.148),

$$\sum_{\ell=m_2+1}^{\infty}\Psi^{-1}\left[\sum_{j=\ell}^{\infty}q(j-1)\right] \leq \frac{3}{4}. \tag{6.2.151}$$

Let S be as in (6.2.140) and

$$X = \{x(k) \in S : K \le x(k) \le 4K,\ k \ge \overline{m}\}. \tag{6.2.152}$$

We define $T : X \to S$ by

$$(Tx)(k) = \begin{cases} K + \sum\limits_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum\limits_{j=\ell}^{\infty} q(j-1)\Psi(x[g(j-1)])\right] & \text{for } k \ge m_2, \\ (Tx)(m_2) & \text{for } \overline{m} \le k \le m_2. \end{cases} \tag{6.2.153}$$

Let $x \in X$. For $k \ge \overline{m}$, obviously we have $(Tx)(k) \ge K$, and in view of (6.2.151),

$$\begin{aligned} (Tx)(k) &\le K + \sum_{\ell=m_2+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi(4K)\right] \\ &\le K + (4K)\left(\frac{3}{4}\right) \\ &= 4K. \end{aligned} \tag{6.2.154}$$

Thus $T(X) \subseteq X$. It is clear that X is a closed, convex, and compact subset of S and $T(X)$ is relatively compact in S. Therefore, by Schauder's fixed point theorem, T has a fixed point in X given by

$$x(k) = K + \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi(x[g(j-1)])\right] \quad \text{for } k \ge m_2. \tag{6.2.155}$$

Clearly this $x(k)$ satisfies (6.2.147). Further, since

$$\Delta x(k) = -\Psi^{-1}\left[\sum_{j=k+1}^{\infty} q(j-1)\Psi(x[g(j-1)])\right] < 0, \tag{6.2.156}$$

we find

$$\Psi(\Delta x(k)) = -\Psi(-\Delta x(k)) = -\sum_{j=k+1}^{\infty} q(j-1)\Psi(x[g(j-1)]), \tag{6.2.157}$$

which provides

$$\Delta\Psi(\Delta x(k)) = q(k)\Psi(x[g(k)]). \tag{6.2.158}$$

Hence $x(k)$ given in (6.2.155) is a bounded nonoscillatory solution of equation (6.2.130) such that (6.2.147) holds. □

Example 6.2.23. The difference equation

$$\Delta^2 x(k) = \frac{2}{k(k+2)(k^2+2k+2)} x(k+1) \quad \text{for } k \in \mathbb{N} \tag{6.2.159}$$

has an unbounded nonoscillatory solution $x(k) = k + (1/k)$ satisfying

$$\lim_{k\to\infty} \frac{x(k)}{k} = \lim_{k\to\infty} \left[1 + \frac{1}{k^2}\right] = 1. \tag{6.2.160}$$

All conditions of Theorem 6.2.21 are satisfied.

Example 6.2.24. The difference equation

$$\Delta\big[\,|\Delta x(k)|\,\Delta x(k)\big] = \frac{4}{(k+1)^5} \big|x[k(k+2)]\big|\,x[k(k+2)] \quad \text{for } k \geq 1 \tag{6.2.161}$$

has a bounded nonoscillatory solution $x(k) = 1 + (1/k)$ satisfying

$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right) = 1. \tag{6.2.162}$$

All conditions of Theorem 6.2.22 are satisfied.

6.2.4. Decaying nonoscillatory solutions of equation (6.2.130)

We will present some conditions under which equation (6.2.130) has a decaying nonoscillatory solution $\{x(k)\}$ such that

$$\lim_{k\to\infty} x(k) = 0. \tag{6.2.163}$$

Theorem 6.2.25. *Suppose that $g(k) < k$ and $q(k) > 0$ for $k \geq m \in \mathbb{N}$. Further suppose that there exists a positive decreasing sequence $\{h(k)\}$ such that*

$$h(k) \geq \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi\big(h[g(j-1)]\big)\right] \quad \textit{for } k \geq m^*, \tag{6.2.164}$$

where $m^* > m$ satisfies

$$\min_{k \geq m^*} g(k) \geq m. \tag{6.2.165}$$

Then equation (6.2.130) has a decaying nonoscillatory solution $\{y(k)\}$ such that (6.2.163) holds.

Proof. Let

$$S = \{x : x(k) \text{ is defined for } k \geq m^*\},$$
$$X = \{x \in S : 0 \leq x(k) \leq h(k),\ k \geq m^*\}. \tag{6.2.166}$$

For each $x \in X$, we define

$$\tilde{x}(k) = \begin{cases} x(k) & \text{for } k \geq m^*, \\ x(m^*) + h(k) - h(m^*) & \text{for } m \leq k \leq m^*. \end{cases} \tag{6.2.167}$$

We also note that for each $x \in X$,

$$\tilde{x}(k) \leq h(k) \quad \text{for } k \geq m. \tag{6.2.168}$$

Define $T : X \to S$ by

$$(Tx)(k) = \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi(\tilde{x}[g(j-1)])\right] \quad \text{for } k \geq m^*. \tag{6.2.169}$$

Clearly, for $x \in X$, we have $(Tx)(k) \geq 0$, and on using (6.2.164) and (6.2.168),

$$(Tx)(k) \leq \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi(h[g(j-1)])\right] \leq h(k). \tag{6.2.170}$$

Hence $T(X) \subseteq X$. Since X is a closed and compact subset of S and $T(X)$ is relatively compact in S, it follows from the Schauder fixed point theorem that T has a fixed point in X given by

$$x(k) = \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi(\tilde{x}[g(j-1)])\right] \quad \text{for } k \geq m^*. \tag{6.2.171}$$

Relation (6.2.171) readily yields

$$\Delta x(k) = -\Psi^{-1}\left[\sum_{j=k+1}^{\infty} q(j-1)\Psi(\tilde{x}[g(j-1)])\right], \tag{6.2.172}$$

and consequently for all sufficiently large k,

$$\Delta\Psi(\Delta x(k)) = -\Delta\Psi(-\Delta x(k)) = q(k)\Psi(x[g(k)]). \tag{6.2.173}$$

Thus, for all sufficiently large k, $x(k)$ given in (6.2.171) is a solution of equation (6.2.130) satisfying (6.2.163).

Finally, we will show that $x(k)$ given in (6.2.171) is nonoscillatory. For this, it suffices to prove that $x(k) > 0$ for $k \ge m^*$. Suppose that $x(m^*) = 0$. Since $x(k)$ is nonnegative and decreasing for $k \ge m^*$, it follows that $x(k)$ is identically zero for $k \ge m^*$. Hence, we get

$$\Delta\Psi(\Delta x(m^*)) = 0. \tag{6.2.174}$$

Now, using the fact that $\tilde{x}(k) > 0$ for $m \le k \le m^* - 1$ (by definition) and the fact that $g(k) < k$ and $q(k) > 0$ for $k \ge m$ in (6.2.173), we find $\Delta\Psi(\Delta x(k)) > 0$ for $m \le k \le m^*$. This inequality particularly gives $\Delta\Psi(\Delta x(m^*)) > 0$, which is a contradiction to (6.2.174). Hence $x(m^*) > 0$.

Next, let $m_1 > m^*$ be the first zero of $x(k)$. Then by definition $\tilde{x}(k) > 0$ for $m \le k \le m_1 - 1$. Using a similar argument as before (replacing m^* by m_1), from (6.2.173) we get

$$\Delta\Psi(\Delta x(m_1)) > 0. \tag{6.2.175}$$

On the other hand, $x(k)$ is identically zero for $k \ge m_1$. Hence it follows that $\Delta\Psi(\Delta x(m_1)) = 0$, which contradicts (6.2.175). By induction we see that $x(k) > 0$ for $k \ge m^*$. This completes the proof. □

Theorem 6.2.26. *Let condition (6.2.164) in Theorem 6.2.25 be replaced by condition (6.2.148). Then the conclusion of Theorem 6.2.25 holds.*

Proof. We will show that there exists a positive decreasing sequence $\{h(k)\}$ such that (6.2.164) is satisfied. Then the result follows immediately from Theorem 6.2.25. For this, let $m^* > m \in \mathbb{N}$ be sufficiently large so that

$$\min_{k \ge m^*} g(k) \ge \max\{m, 1\}, \tag{6.2.176}$$

and in view of condition (6.2.148),

$$\sum_{\ell=m^*+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell-1}^{\infty} q(j)\right] \le \frac{1}{2}. \tag{6.2.177}$$

Let

$$h(k) = 1 + \frac{1}{k}. \tag{6.2.178}$$

Clearly, $h(k)$ is positive and decreasing. To see that (6.2.164) holds for $k \ge m^*$, we have, on using the fact that $g(k) < k$, $q(k) > 0$ for $k \geq m$, (6.2.176), and (6.2.177),

$$\begin{aligned}
\sum_{\ell=k+1}^{\infty} \Psi^{-1}&\left[\sum_{j=\ell}^{\infty} q(j-1)\Psi(h[g(j-1)])\right] \\
&= \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell-1}^{\infty} q(j)\Psi\left(1+\frac{1}{g(j)}\right)\right] \\
&\leq 2 \sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell-1}^{\infty} q(j)\right] \\
&\leq 2 \sum_{\ell=m+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell-1}^{\infty} q(j)\right] \\
&\leq 2\left(\frac{1}{2}\right) \\
&= 1 < h(k).
\end{aligned} \tag{6.2.179}$$

The proof is complete. □

Next, we only state the following interesting results.

Theorem 6.2.27. *Let condition (6.2.164) in Theorem 6.2.25 be replaced by*

$$\limsup_{k\to\infty} \sum_{\ell=g(k)}^{k} \Psi^{-1}\left[\sum_{j=\ell}^{\infty} q(j)\right] < \frac{1}{e}. \tag{6.2.180}$$

Then the conclusion of Theorem 6.2.25 holds.

Theorem 6.2.28. *Let condition (6.2.164) in Theorem 6.2.25 be replaced by*

$$\sup_{k\geq m^*} \sum_{\ell=g(k)}^{k} q(\ell) < \infty, \tag{6.2.181}$$

where $m^* > m+1$ with $m \in \mathbb{N}$ such that $\min_{k\geq m^*} g(k) \geq a$ and

$$\sum_{\ell=k+1}^{\infty} \Psi^{-1}\left[\sum_{j=\ell-1}^{\infty} q(j)\right] < \exp\left(-\frac{\alpha+1}{\alpha}\right). \tag{6.2.182}$$

Then the conclusion of Theorem 6.2.25 holds.

Example 6.2.29. The linear difference equation

$$\Delta^2 x(k) = \left(1-\frac{1}{e}\right)^2 e^{-k/2} x\left[\frac{k}{2}\right] \quad \text{for } k \in \{0,2,4,\dots\} \tag{6.2.183}$$

has a nonoscillatory solution $x(k) = e^{-k/2}$ satisfying $\lim_{k\to\infty} x(k) = 0$. All conditions of Theorem 6.2.26 are satisfied.

We note that most of the results of this section are written in a form to be expandable to equations of type (6.2.2) and (6.2.130), where the function Ψ satisfies (i_3). The details are left to the reader.

Finally, we consider equation (6.2.1) when f need not be a monotonic function, for example, $f(x) = e^{-|x|}|x|^\gamma \operatorname{sgn} x$, $[|x|^\gamma/(1+x^2)] \operatorname{sgn} x$ with $\gamma > 0$, and so forth. To obtain some results regarding this case, we need the following notation and lemma.

$$\mathbb{R}_n = \begin{cases} (-\infty, -n] \cup [n, \infty) & \text{if } n > 0, \\ (-\infty, 0) \cup (0, \infty) & \text{if } n = 0, \end{cases}$$
$$C(\mathbb{R}) = \{f : \mathbb{R} \longrightarrow \mathbb{R} : f \text{ is continuous and } xf(x) > 0 \text{ for } x \neq 0\},$$
$$C_p(\mathbb{R}_n) = \{f \in C(\mathbb{R}) : f \text{ is of bounded variation on every interval } [a,b] \subset \mathbb{R}\}. \tag{6.2.184}$$

Lemma 6.2.30. *Suppose $n \geq 0$ and $f \in C(\mathbb{R})$. Then $f \in C_p(\mathbb{R}_n)$ if and only if $f(x) = G(x)H(x)$ for all $x \in \mathbb{R}_n$, where $G : \mathbb{R}_n \to (0, \infty)$ is nondecreasing on $(-\infty, -n)$ and nonincreasing on (n, ∞) and $H : \mathbb{R}_n \to \mathbb{R}$ is nondecreasing in $\mathbb{R}_n$.*

Now we are ready to prove the following comparison result.

Theorem 6.2.31. *Suppose $f \in C(\mathbb{R}_n)$, $n \geq 0$, and let G and H be a pair of continuous components of f with H being the nondecreasing one. Further assume that condition (6.2.6) and (6.2.7) hold and $\{g(k)\}$ is an increasing sequence and the function Ψ satisfies (i_1) or (i_2). If for every constant $a > 0$ and all sufficiently large $m \in \mathbb{N}$ the equation*

$$\Delta(c(k)\Psi(\Delta x(k))) + q(k)G(aC[g(k)-1, m])H(x[g(k)]) = 0, \tag{6.2.185}$$

where C is as in (6.2.54), is oscillatory, then equation (6.2.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.2.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 6.2.3, we see that $\Delta x(k) > 0$ eventually and $\{c(k)\Psi(\Delta x(k))\}$ is decreasing eventually. There exist a constant $b_1 > 0$ and an integer $m_1 \geq m$ such that

$$c(k)\Psi(\Delta x(k)) \leq b_1 \quad \text{for } k \geq m_1, \tag{6.2.186}$$

and hence we conclude that there exist a constant $b > 0$ and an integer $m_2 \geq m_1$ such that

$$x[g(k)] \leq bC[g(k)-1, m_1] \quad \text{for } k \geq m_2. \tag{6.2.187}$$

Using the above inequality and the fact that $f(x) = G(x)H(x)$ in equation (6.2.1), we have for $k \geq m_2$,

$$\begin{aligned} 0 &= \Delta(c(k)\Psi(\Delta x(k))) + q(k)f(x[g(k)]) \\ &= \Delta(c(k)\Psi(\Delta x(k))) + q(k)G(x[g(k)])H(x[g(k)]) \\ &\geq \Delta(c(k)\Psi(\Delta x(k))) + q(k)G(bC[g(k)-1,m_1])H(x[g(k)]). \end{aligned} \tag{6.2.188}$$

By Lemma 6.2.4 we conclude that the equation

$$\Delta(c(k)\Psi(\Delta x(k))) + q(k)G(bC[g(k)-1,m_1])H(x[g(k)]) = 0 \tag{6.2.189}$$

has an eventually positive solution, which is a contradiction. This completes the proof. □

As an application of Theorem 6.2.31 we see that the difference equation

$$\Delta\Big(c(k)\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)\Big) + q(k)\,|\,x[k-\tau]\,|^{\gamma}e^{-|x[k-\tau]|}\operatorname{sgn} x[t-\tau] = 0, \tag{6.2.190}$$

where $\alpha \geq 1$, $\gamma > 0$, and $\tau > 0$ is an integer, $\{c(k)\}$ and $\{q(k)\}$ are positive sequences, and condition (6.2.6) holds, is oscillatory if the equation

$$\Delta\Big(c(k)\,|\,\Delta x(k)\,|^{\alpha-1}\Delta x(k)\Big) + q(k)e^{-aC[k-\tau-1,m]}\,|\,x[k-\tau]\,|^{\gamma}\operatorname{sgn} x[k-\tau] = 0 \tag{6.2.191}$$

is oscillatory for every constant $a > 0$ and all large $m \in \mathbb{N}$. Here we let $G(x) = e^{-|x|}$ and $H(x) = |x|^{\gamma}\operatorname{sgn} x$.

Similarly, we see that the equation

$$\Delta(c(k)\Delta x(k)) + q(k)\frac{|\,x[k-\tau]\,|^{\gamma}}{1+x^2[k-\tau]}\operatorname{sgn} x[k-\tau] = 0, \tag{6.2.192}$$

where $c(k)$, $q(k)$, τ, and γ are as in equation (6.2.190) and condition (6.1.3) holds, is oscillatory if for every constant $a > 0$ and all large $m \in \mathbb{N}$, the equation

$$\Delta(c(k)\Delta x(k)) + q(k)\left(\frac{1}{1+[aC(k-\tau-1,m)]^2}\right)|\,x[k-\tau]\,|^{\gamma}\operatorname{sgn} x[k-\tau] = 0 \tag{6.2.193}$$

is oscillatory, where C is as in (6.1.8).

When condition (6.2.6) is violated, that is,

$$\sum^{\infty}\Psi^{-1}\left(\frac{1}{c(j)}\right) < \infty, \tag{6.2.194}$$

we can obtain the following oscillatory and asymptotic behavior result for equation (6.2.1).

Theorem 6.2.32. *Suppose $f \in C(\mathbb{R}_n)$, $n \geq 0$ and let G and H be a pair of continuous components of f with H being the nondecreasing one. Moreover, assume that condition (6.2.194) holds. If*

$$\sum_{i=m}^{\infty} \Psi^{-1}\left[\frac{1}{c(i)} \sum_{j=m\in\mathbb{N}}^{i} q(j)\right] = \infty, \tag{6.2.195}$$

then every solution $\{x(k)\}$ of equation (6.2.1) is either oscillatory or $x(k) \to 0$ monotonically as $k \to \infty$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.2.1), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since $\{\Delta x(k)\}$ is eventually of one sign, we consider the following two cases:

(i) $\Delta x(k) > 0$ eventually,
(ii) $\Delta x(k) < 0$ eventually.

First we consider (i). If $\Delta x(k) > 0$ for $k \geq m_1$ for some $m_1 \geq m$, then it follows that (6.2.187) holds, which in view of (6.2.194) implies that $x[g(k)] \leq M$ for $k \geq m_2 \geq m_1$, where M is a constant such that $bC[g(k) - 1, m_1] \leq M$. Since $\{x(k)\}$ is increasing, there exist a constant $a > 0$ and $m_3 \geq m_2$ such that

$$a \leq x[g(k)] \leq M \quad \forall k \geq m_3. \tag{6.2.196}$$

Using (6.2.196) in equation (6.2.1) and summing from m_3 to $k-1 \geq m_3$, we obtain

$$c(k)\Psi(\Delta x(k)) - c(m_3)\Psi(\Delta x(m_3)) + G(b)H(a) \sum_{j=m_3}^{k-1} q(j) \leq 0. \tag{6.2.197}$$

Clearly from conditions (6.2.194) and (6.2.195) we conclude that

$$\sum^{\infty} q(j) = \infty. \tag{6.2.198}$$

Thus there exists an integer $m_4 \geq m_3$ such that

$$c(m_3)\Psi(\Delta x(m_3)) \leq \frac{1}{2} G(b)H(a) \sum_{j=m_3}^{k-1} q(j) \quad \text{for } k \geq m_4 + 1, \tag{6.2.199}$$

and hence inequality (6.2.197) yields

$$c(k)\Psi(\Delta x(k)) + \frac{1}{2} G(b)H(a) \sum_{j=m_3}^{k-1} q(j) \leq 0 \quad \text{for } k \geq m_4 + 1. \tag{6.2.200}$$

Now

$$\Delta x(k) + \Psi^{-1}\left(\frac{1}{2}G(b)H(a)\right)\Psi^{-1}\left[\frac{1}{c(k)}\sum_{j=m_3}^{k-1} q(j)\right] \le 0 \quad \text{for } k \ge m_4. \tag{6.2.201}$$

Summing (6.2.201) from m_4 to $k-1$ and using condition (6.2.195), we arrive at the desired contradiction.

Finally we consider (ii). If $\Delta x(k) < 0$ for $k \ge m_1$ for some $m_1 \ge m$, then $0 < x[g(k)] < b_1$ for some constant $b_1 > 0$. We claim that $\lim_{k\to\infty} x(k) = 0$. Otherwise $\lim_{k\to\infty} x(k) = a_1 > 0$, and hence the decreasing nature of $\{x[g(k)]\}$ implies that $x[g(k)] \ge a_1$ for all $k \ge m_1$. Thus

$$a_1 \le x[g(k)] \le b_1 \quad \text{for } k \ge m_1. \tag{6.2.202}$$

Using (6.2.202) in equation (6.2.1) and as in (i) above, one can easily obtain a contradiction to the fact that $x(k) > 0$ for $k \ge m$. Hence we conclude that $a_1 = 0$, and this completes the proof. □

Example 6.2.33. As an example, we see that the difference equation

$$\Delta(k^2\Delta x(k)) + \left(\frac{(k+1)^2}{1+(k+1)^2}\right)\left(\frac{(k+1)^2}{k+2}\right)\frac{x^3(k+1)}{x^2(k+1)+1} = 0 \tag{6.2.203}$$

has a nonoscillatory solution $x(k) = 1/k \to 0$ as $k \to \infty$. All conditions of Theorem 6.2.32 are satisfied for equation (6.2.203) with $f(x) = x^3/(1+x^2)$, and its components are $G(x) = 1/(1+x^2)$ and $H(x) = x^3$.

6.3. Oscillation via characteristic equations

In this section we will consider the retarded difference equation

$$-\Delta^2 x(k) + \sum_{j=0}^{\infty} q(j)x[k-\tau(j)] = 0, \tag{6.3.1}$$

where

(i$_1$) $\{q(k)\}$ is a sequence of positive real numbers,
(i$_2$) $\{\tau(k)\}$ is a sequence of integers with $0 \le \tau(0) < \tau(1) < \tau(2) < \cdots$,

and the advanced difference equation

$$\Delta^2 x(k) - \sum_{j\in J} q(j)x[k+\sigma(j)] = 0, \tag{6.3.2}$$

where

(ii$_1$) J is a nonempty (finite or infinite) subset of $\mathbb{N}_0$,
(ii$_2$) $q(j)$ for $j \in J$ are positive real numbers,
(ii$_3$) $\sigma(j)$ for $j \in J$ are nonnegative integers such that $\sigma(j_1) \ne \sigma(j_2)$ if $j_1, j_2 \in J$ with $j_1 \ne j_2$.

The *characteristic equations* of (6.3.1) and (6.3.2) are

$$-(1-\lambda)^2 + \sum_{j=0}^{\infty} q(j)\lambda^{-\tau(j)} = 0, \tag{6.3.3}$$

$$(\lambda-1)^2 - \sum_{j\in J} q(j)\lambda^{\sigma(j)} = 0, \tag{6.3.4}$$

respectively.

The main purpose of this section is to establish necessary and sufficient conditions for oscillation of solutions of equations (6.3.1) and (6.3.2). These conditions are in terms of the associated characteristic equations (6.3.3) and (6.3.4).

6.3.1. Positive solutions of equation (6.3.1)

The following lemma is needed.

Lemma 6.3.1. *Let $x = \{x(k)\}$ be a positive solution of equation (6.3.1) which is bounded at infinity. Then*

$$\Delta x(k) < 0 \quad \forall k \in \mathbb{Z}. \tag{6.3.5}$$

Proof. From equation (6.3.1) we obtain for $k \in \mathbb{Z}$

$$\Delta^2 x(k) = \sum_{j=0}^{\infty} q(j)x[k-\tau(j)], \tag{6.3.6}$$

and consequently

$$\Delta^2 x(k) > 0 \quad \text{for every } k \in \mathbb{Z}. \tag{6.3.7}$$

We claim that (6.3.5) holds. Otherwise there exists an integer m with $\Delta x(m) \geq 0$. From (6.3.7) it follows that the sequence $\{\Delta x(k)\}$ is strictly increasing. Hence, if we choose an integer $m_1 > m$, then for $k \geq m_1$, $\Delta x(k) \geq \Delta x(m_1) > \Delta x(m) \geq 0$. Therefore

$$\Delta x(k) \geq a \quad \forall k \geq m_1, \tag{6.3.8}$$

where $a = \Delta x(m_1)$. Summing inequality (6.3.8) from m_1 to $k-1$, we can conclude $x(k) - x(m_1) \geq a(k-m_1)$. Thus $\lim_{k\to\infty} x(k) = \infty$, which contradicts the fact that $\{x(k)\}$ is bounded at infinity. □

Now we prove the following result.

Theorem 6.3.2. *Equation (6.3.1) has a positive solution which is bounded at infinity if and only if equation (6.3.3) has a root in* $(0,1)$.

Proof. Assume that (6.3.3) has a root in $(0,1)$. Then we set $x(k) = \lambda^k$ for $k \in \mathbb{Z}$ and obtain

$$\begin{aligned}
-\Delta^2 x(k) + \sum_{j=0}^{\infty} q(j)x[k-\tau(j)] &= -(\lambda-1)^2\lambda^k + \sum_{j=0}^{\infty} q(j)\lambda^{k-\tau(j)} \\
&= \left[-(1-\lambda)^2 + \sum_{j=0}^{\infty} q(j)\lambda^{-\tau(j)} \right]\lambda^k \\
&= 0
\end{aligned} \tag{6.3.9}$$

for all $k \in \mathbb{Z}$. Thus $\{x(k)\}$ is a positive solution of equation (6.3.1) which obviously is bounded at infinity.

Suppose, conversely, that there is a positive solution $\{x(k)\}$, $k \in \mathbb{Z}$, of equation (6.3.1) which is bounded at infinity. Also, assume for the sake of contradiction, that the characteristic equation (6.3.3) has no roots in $(0,1)$. From Lemma 6.3.1 it follows that $\Delta x(k) < 0$ for all $k \in \mathbb{Z}$ and consequently $\{x(k)\}$ is a strictly decreasing sequence. So from (6.3.1) we obtain for every $k \in \mathbb{Z}$,

$$0 = -\Delta^2 x(k) + \sum_{j=0}^{\infty} q(j)x[k-\tau(j)] > -\Delta^2 x(k) + \left(\sum_{j=0}^{\infty} q(j)\right)x(k), \tag{6.3.10}$$

and therefore

$$0 < \sum_{j=0}^{\infty} q(j) < \infty. \tag{6.3.11}$$

Set

$$F(\lambda) = -(1-\lambda)^2 + \sum_{j=0}^{\infty} q(j)\lambda^{-\tau(j)} \quad \text{for every } \lambda \in (0,1]. \tag{6.3.12}$$

Then $F(1) = \sum_{j=0}^{\infty} q(j) \in \mathbb{R}^+$. Moreover, we have

$$F(\lambda) > -(1-\lambda)^2 + q(1)\lambda^{-\tau(1)} \quad \text{for every } \lambda \in (0,1), \tag{6.3.13}$$

and so $F(0^+) = \infty$. Hence, as $F(\lambda) = 0$ has no roots in $(0,1)$, there exists a positive number μ such that

$$-(1-\lambda)^2 + \sum_{j=0}^{\infty} q(j)\lambda^{-\tau(j)} \geq \mu \quad \forall \lambda \in (0,1). \tag{6.3.14}$$

Next, by taking into account (6.3.11), we set $\lambda_0 = 1 - \sum_{j=0}^{\infty} q(j)$ and

$$\lambda_r = 1 - \left[\left(1-\lambda_{r-1}\right)^2 + \mu \right]^{1/2} \quad \text{for } r \in \mathbb{N}. \tag{6.3.15}$$

Furthermore we define $x(k,0) = x(k)$ for $k \in \mathbb{Z}$ and

$$x(k,r) = (1-\lambda_{r-1})x(k,r-1) - \Delta x(k,r-1) \quad \text{for } k \in \mathbb{Z},\ r \in \mathbb{N}. \tag{6.3.16}$$

Thus, for any $r \in \mathbb{N}_0$, $\{x(k,r)\}$ is a positive solution of (6.3.1) which is bounded at infinity. Indeed, consider a positive solution $\{\tilde{x}(k)\}$ of (6.3.1) which is bounded at infinity. By Lemma 6.3.1, we have $\Delta^2\tilde{x}(k) > 0$, $\Delta\tilde{x}(k) < 0$, and $\tilde{x}(k) > 0$ for $k \in \mathbb{Z}$. If $i \in \{0,1\}$, then $\{(-1)^i\Delta^i\tilde{x}(k)\}$ is a positive sequence which is strictly decreasing (and therefore bounded at infinity). Moreover, since equation (6.3.1) is linear and the coefficient $q(j)$ $(j \in \mathbb{N}_0)$ and the indices $\tau(j)$ $(j \in \mathbb{N}_0)$ are independent of k, it follows that for each $i \in \{0,1\}$, the sequence $\{(-1)^i\Delta^i\tilde{x}(k)\}$ is a solution of equation (6.3.1). Hence, each one of the sequences $\{(-1)^i\Delta^i\tilde{x}(k)\}$ $(i \in \{0,1\})$ is a positive solution of (6.3.1) which is bounded at infinity. Therefore, because of the linearity of (6.3.1), it follows that if c_0 and c_1 are positive constants, then the sequence $\{c_0\tilde{x}(k) - c_1\Delta\tilde{x}(k)\}$ is a positive solution of (6.3.1) which is bounded at infinity. Now, we can easily see that $1-\lambda_r > 0$ $(r \in \mathbb{N}_0)$. So by the above particular result and by mathematical induction, we can show that if $r \in \mathbb{N}_0$, then $\{x(k,r)\}$ is a positive solution of (6.3.1) which is bounded at infinity.

We have for $k \in \mathbb{Z}$,

$$-\Delta^2 x(k,r) + (1-\lambda_r)^2 x(k,r) = x(k+1,r+1) - \lambda_r x(k,r+1) \quad \text{for } r \in \mathbb{N}_0. \tag{6.3.17}$$

In fact, for any $r \in \mathbb{N}_0$ and every $k \in \mathbb{Z}$, we obtain

$$\begin{aligned}
x(k+1,&r+1) - \lambda_r x(k,r+1) \\
&= (1-\lambda_r)x(k+1,r) - \Delta x(k+1,r) - \lambda_r\big[(1-\lambda_r)x(k,r) - \Delta x(k,r)\big] \\
&= (1-\lambda_r)\big[x(k+1,r) - x(k,r)\big] - \big[\Delta x(k+1,r) - \Delta x(k,r)\big] \\
&\quad + (1-\lambda_r)\big[(1-\lambda_r)x(k,r) - \Delta x(k,r)\big] \\
&= (1-\lambda_r)\Delta x(k,r) - \Delta^2 x(k,r) + (1-\lambda_r)^2 x(k,r) - (1-\lambda_r)\Delta x(k,r) \\
&= -\Delta^2 x(k,r) + (1-\lambda_r)^2 x(k,r).
\end{aligned} \tag{6.3.18}$$

Now we will prove that

$$x(k+1,r+1) - \lambda_r x(k,r+1) < 0 \quad \forall k \in \mathbb{Z},\ r \in \mathbb{N}_0. \tag{6.3.19}$$

Indeed, in view of Lemma 6.3.1, the sequence $\{x(k)\}$ is strictly decreasing.

Therefore we obtain for every $k \in \mathbb{Z}$,

$$
\begin{aligned}
-\Delta^2 x(k,0) + (1-\lambda_0)^2 x(k,0) &= -\Delta^2 x(k) + \left(\sum_{j=0}^{\infty} q(j)\right) x(k) \\
&< -\Delta^2 x(k) + \sum_{j=0}^{\infty} q(j) x[k-\tau(j)] \\
&= 0,
\end{aligned}
\tag{6.3.20}
$$

and hence by (6.3.17) we conclude that (6.3.19) is true for $r = 0$. Next, assuming that (6.3.17) holds for some $r \in \mathbb{N}_0$, we should prove that it is also true for $r+1$. By the inductive assumption, $x(k+1, r+1) - \lambda_r x(k, r+1) < 0$ for every $k \in \mathbb{Z}$. This implies in particular that $\lambda_r > 0$. On the other hand, $\lambda_r < 1$. So, we must have $0 < \lambda_r < 1$. Furthermore, we have

$$
x(k, r+1) > \frac{1}{\lambda_r} x(k+1, r+1) \quad \text{for } k \in \mathbb{Z}. \tag{6.3.21}
$$

By applying (6.3.21), we can verify that

$$
\begin{aligned}
&x(k-\tau(0), r+1) \geq \lambda_r^{-\tau(0)} x(k, r+1) \quad \forall k \in \mathbb{Z}, \\
&x(k-\tau(j), r+1) > \lambda_r^{-\tau(j)} x(k, r+1) \quad \forall k \in \mathbb{Z},\ j \in \mathbb{N}.
\end{aligned}
\tag{6.3.22}
$$

Hence, from equation (6.3.1) we obtain for $k \in \mathbb{Z}$,

$$
\begin{aligned}
0 &= -\Delta^2 x(k, r+1) + \sum_{j=0}^{\infty} q(j) x(k-\tau(j), r+1) \\
&> -\Delta^2 x(k, r+1) + \left(\sum_{j=0}^{\infty} q(j) \lambda_r^{-\tau(j)}\right) x(k, r+1).
\end{aligned}
\tag{6.3.23}
$$

But (6.3.14) ensures that

$$
\sum_{j=0}^{\infty} q(j) \lambda_r^{-\tau(j)} \geq (1-\lambda_r)^2 + \mu = (1-\lambda_{r+1})^2. \tag{6.3.24}
$$

So in view of (6.3.17), we have for every $k \in \mathbb{Z}$,

$$
\begin{aligned}
0 &> -\Delta^2 x(k, r+1) + (1-\lambda_{r+1})^2 x(k, r+1) \\
&= x(k+1, r+2) - \lambda_{r+1} x(k, r+2),
\end{aligned}
\tag{6.3.25}
$$

that is, (6.3.19) is also satisfied for $r+1$.

Finally, since $x(k,r+1) > 0$ for all $k \in \mathbb{Z}$ ($r \in \mathbb{N}_0$), from (6.3.19) it follows that $\lambda_r > 0$ ($r \in \mathbb{N}_0$). On the other hand, it is easy to verify that the sequence $\{\lambda_r\}_{r\in\mathbb{N}_0}$ is strictly decreasing. So $L = \lim_{r\to\infty} \lambda_r$ exists and $0 \le L < \lambda_0 < 1$.

Since $\lambda_r = 1 - [(1-\lambda_{r-1})^2 + \mu]^{1/2}$ ($r \in \mathbb{N}$), we obtain $L = 1 - [(1-L)^2 + \mu]^{1/2}$, which gives $\mu = 0$, which is a contradiction. This completes the proof. □

It is easy to verify that

$$\sup_{\lambda\in(0,1)} \left[(1-\lambda)^2\lambda^\tau\right] = \frac{4\tau^\tau}{(2+\tau)^{2+\tau}} \quad \text{for } \tau \in \mathbb{N}_0. \tag{6.3.26}$$

(Here we use the convention that $0^0 = 1$.) Hence, for every $\lambda \in (0,1)$,

$$\begin{aligned} -(1-\lambda)^2 + \sum_{j=0}^{\infty} q(j)\lambda^{-\tau(j)} &= (1-\lambda)^2\left[-1 + \sum_{j=0}^{\infty} q(j)(1-\lambda)^{-2}\lambda^{-\tau(j)}\right] \\ &\ge (1-\lambda)^2\left[-1 + \sum_{j=0}^{\infty} q(j)\frac{\big(2+\tau(j)\big)^{2+\tau(j)}}{4\big(\tau(j)\big)^{\tau(j)}}\right], \end{aligned} \tag{6.3.27}$$

and so by assumption

$$\sum_{j=0}^{\infty} q(j)\frac{\big(2+\tau(j)\big)^{2+\tau(j)}}{4\big(\tau(j)\big)^{\tau(j)}} > 1 \tag{6.3.28}$$

implies that equation (6.3.3) has no roots in $(0,1)$. Therefore Theorem 6.3.2 leads to the following corollary.

Corollary 6.3.3. *Suppose that condition (6.3.28) holds. Then there is no positive solution of equation (6.3.1) which is bounded at infinity.*

Theorem 6.3.2 can be restated as follows.

Theorem 6.3.4. *Every bounded solution of equation (6.3.1) oscillates if and only if equation (6.3.3) has no roots in* $(0,1)$.

6.3.2. Oscillation of unbounded solutions of equation (6.3.2)

To prove the main result of this subsection we need the following lemma.

Lemma 6.3.5. *Let* $\{x(k)\}$ *be an eventually positive solution of equation (6.3.2) which is unbounded. Then* $\Delta^2 x(k) \ge a$ *eventually, where* a *is a positive constant, and* $\lim_{k\to\infty} \Delta^i x(k) = \infty$ *for* $i \in \{0,1\}$.

Proof. From equation (6.3.2) it follows that $\Delta^2 x(k) > 0$ eventually. Now the sequence $\{\Delta x(k)\}$ is either eventually positive or eventually negative. In particular, since $\{x(k)\}$ is unbounded, we have $\Delta x(k) > 0$ eventually. So there exists $m \in \mathbb{N}$ such that the sequence $\{x(k)\}$ is positive and increasing. From equation (6.3.2) we obtain

$$\Delta^2 x(k) \geq a \quad \text{for every } k \geq m, \tag{6.3.29}$$

where $a = (\sum_{j\in J} q(j))x(m)$ is a positive real number. Summing (6.3.29) from m to $k-1$, we have $\Delta x(k) \geq \Delta x(m) + a(k-m)$, which implies that

$$\lim_{k\to\infty} \Delta x(k) = \infty. \tag{6.3.30}$$

From (6.3.30) there exist an integer $m_1 \geq m$ and a positive constant b such that $\Delta x(k) \geq b$ for $k \geq m_1$. Summing this inequality from m_1 to $k-1$, we obtain $x(k) \geq x(m) + b(k-m_1) \to \infty$ as $k \to \infty$. This completes the proof. □

Now we prove the following result.

Theorem 6.3.6. *All unbounded solutions of equation (6.3.2) are oscillatory if and only if equation (6.3.4) has no roots in* $(1,\infty)$.

Remark 6.3.7. We note that the set J is not assumed to be necessarily finite. If J is finite, then equation (6.3.4) has a finite number of roots, and hence in this case a proof of Theorem 6.3.6 can be established by a detailed analysis of the representation of the solutions of equation (6.3.2) in terms of the roots of the characteristic equation (6.3.4). For infinite J no representation of solutions in terms of the roots of (6.3.4) is known, since then the characteristic equation has an infinite number of roots.

Proof of Theorem 6.3.6. This result will be proved in the contrapositive form: there is an unbounded nonoscillatory solution of equation (6.3.2) if and only if the characteristic equation (6.3.4) has a root in $(1,\infty)$. Assume first that (6.3.4) has a root $\lambda_0 \in (1,\infty)$. Then we can see that equation (6.3.2) has the unbounded nonoscillatory solution $x(k) = \lambda_0^k$ for $k \in \mathbb{N}$.

Assume conversely that (6.3.2) has an unbounded nonoscillatory solution $x = \{x(k)\}$. Moreover, for the sake of contradiction, suppose that the characteristic equation (6.3.4) has no roots in $(1,\infty)$. As $-x$ is also a solution of the same equation, we may (and do) assume that the solution x is eventually positive. From Lemma 6.3.5 it follows that the sequence $\{\Delta x(k)\}$ is eventually positive. So we can choose $m \in \mathbb{N}$ such that

$$\Delta^i x(k) > 0 \quad \text{for every } k \geq m,\ i \in \{0,1,2\}. \tag{6.3.31}$$

From equation (6.3.2) we obtain for all $k \geq m$,

$$\Delta^2 x(k) = \sum_{j\in J} q(j)x[k+\sigma(j)] \geq \left(\sum_{j\in J} q(j)\right)x(k), \tag{6.3.32}$$

and hence $\sum_{j\in J} q(j)$ is a positive real number. Now we set

$$F(\lambda) = (\lambda - 1)^2 - \sum_{j\in J} q(j)\lambda^{\sigma(j)} \quad \text{for } \lambda \geq 1. \tag{6.3.33}$$

We observe that $F(1) = -\sum_{j\in J} q(j) < 0$. Assume that $\sigma(j) \leq 2$ for all $j \in J$. If $\sigma(j) < 2$ for every $j \in J$, then obviously $F(\infty) = \infty$, which is impossible since $F(1) < 0$ and the equation $F(\lambda) = 0$ has no roots in $(1, \infty)$. Suppose next that there exists an index $\ell \in J$ such that $\sigma(\ell) = 2$. Then (6.3.2) gives

$$\Delta^2 x(k) \geq q(\ell)x[k+2] \quad \forall k \geq m. \tag{6.3.34}$$

Moreover, one has

$$\Delta^2 x(k) < x[k+2] \quad \text{for every } k \geq m. \tag{6.3.35}$$

Indeed, we have for $k \geq m$, $\Delta x(k) = x(k+1) - x(k) < x(k+1)$, and

$$\Delta^2 x(k) = \Delta\big(\Delta x(k)\big) = \Delta x(k+1) - \Delta x(k) < \Delta x(k+1) < x(k+2). \tag{6.3.36}$$

Combining (6.3.34) and (6.3.35), we conclude that $g(\ell) < 1$. By using this fact, we can see that $F(\infty) = \infty$. As above, this is a contradiction. We have thus proved that the statement that $\sigma(j) \leq 2$ for all $j \in J$ is not true. Hence there exists an index $j_0 \in J$ such that

$$\sigma(j_0) > 2. \tag{6.3.37}$$

Now, using (6.3.37) we obtain $F(\infty) = -\infty$. As $F(1) < 0$, $F(\infty) = -\infty$, and the equation $F(\lambda) = 0$ has no roots in $(1, \infty)$, there exists a positive number μ such that $F(\lambda) \leq -\mu$ for $\lambda \geq 1$, that is,

$$(\lambda - 1)^2 - \sum_{j\in J} q(j)\lambda^{\sigma(j)} \leq -\mu \quad \forall \lambda \geq 1. \tag{6.3.38}$$

Next we put

$$\lambda_0 = 1 + \left(\sum_{j\in J} q(j)\right)^{1/2}, \qquad \lambda_r = 1 + \left[(\lambda_{r-1} - 1)^2 + \mu\right]^{1/2} \quad \text{for } r \in \mathbb{N}. \tag{6.3.39}$$

Also, we define $x(k,0) = x(k)$ for $k \in \mathbb{N}$, and for $r \in \mathbb{N}$,

$$x(k,r) = (\lambda_{r-1} - 1)x(k,r-1) + \Delta x(k,r-1) \quad \text{for } k \in \mathbb{N}. \tag{6.3.40}$$

We see that for each $r \in \mathbb{N}_0$, $\{x(k,r)\}$ is an eventually positive solution of (6.3.2) which is unbounded. Indeed, let $\{\tilde{x}(k)\}$ be an eventually positive solution of (6.3.2) which is unbounded. In view of Lemma 6.3.5, we have $\Delta^i\tilde{x}(k) > 0$ for $i \in \{0,1\}$ eventually. Hence, for any $i \in \{0,1\}$, $\{\Delta^i\tilde{x}(k)\}$ is an eventually positive sequence. Moreover, if $i \in \{0,1\}$, then $\{\Delta^i\tilde{x}(k)\}$ is a solution of (6.3.2), since (6.3.2) is linear and the coefficient $q(j)$, $j \in J$, and the indices $\sigma(j)$, $j \in J$, are independent of k. Thus each one of the sequences $\{\Delta^i\tilde{x}(k)\}$, $i \in \{0,1\}$, is an eventually positive solution of (6.3.2). So since (6.3.2) is linear, the sequence $\{c_0\tilde{x}(k) + c_1\Delta\tilde{x}(k)\}$, where c_0 and c_1 are arbitrary positive numbers, is an eventually positive solution of (6.3.2) which is unbounded, since $c_0\tilde{x}(k) + c_1\Delta\tilde{x}(k) \geq c_0\tilde{x}(k)$ for all large k. Now we can see that $\lambda_r - 1 > 0$ for $r \in \mathbb{N}_0$. So we can conclude that if $r \in \mathbb{N}_0$, then $\{x(k,r)\}$ is an eventually positive solution of (6.3.2) which is unbounded.

Now, we will show that for $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$,

$$\Delta^2 x(k,r) - (\lambda_r - 1)^2 x(k,r) = x(k+1,r+1) - \lambda_r x(k,r+1). \tag{6.3.41}$$

Indeed, for any $r \in \mathbb{N}_0$ and every $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
x(k&+1,r+1) - \lambda_r x(k,r+1) \\
&= (\lambda_r - 1)x(k+1,r) + \Delta x(k+1,r) - \lambda_r\left[(\lambda_r - 1)x(k,r) + \Delta x(k,r)\right] \\
&= (\lambda_r - 1)\left[x(k+1,r) - x(k,r)\right] + \left(\Delta x(k+1,r) - \Delta x(k,r)\right) \\
&\quad - (\lambda_r - 1)\left[(\lambda_r - 1)x(k,r) + \Delta x(k,r)\right] \\
&= (\lambda_r - 1)\Delta x(k,r) + \Delta^2 x(k,r) - \left[(\lambda_r - 1)^2 x(k,r) + (\lambda_r - 1)\Delta x(k,r)\right] \\
&= \Delta^2 x(k,r) - (\lambda_r - 1)^2 x(k,r).
\end{aligned} \tag{6.3.42}$$

Furthermore, we will prove that

$$x(k+1,r+1) - \lambda_r x(k,r+1) \geq 0 \quad \forall k \geq m \in \mathbb{N},\ r \in \mathbb{N}_0. \tag{6.3.43}$$

In fact, from (6.3.31) it follows that the sequence $\{x(k)\}$ is strictly increasing. So we obtain for every $k \geq m$,

$$\begin{aligned}
\Delta^2 x(k,0) - (\lambda_0 - 1)^2 x(k,m) &= \Delta^2 x(k) - \left(\sum_{j\in J} q(j)\right)x(k) \\
&\geq \Delta^2 x(k) - \sum_{j\in J} q(j)x[k+\sigma(j)] \\
&= 0
\end{aligned} \tag{6.3.44}$$

and so, because of (6.3.41), we conclude that (6.3.43) holds when $r = 0$. Next we assume that (6.3.43) is true for some $r \in \mathbb{N}_0$, that is, $x(k+1,r+1) - \lambda_r x(k,r+1) \geq 0$ for every $k \geq m$, or $x(k+1,r+1) \geq \lambda_r x(k,r+1)$ for all $k \geq m$. Therefore we have

$$x(k+\sigma(j),r+1) \geq \lambda_r^{\sigma(j)} x(k,r+1) \quad \text{for any } j \in J \text{ and every } k \geq m. \tag{6.3.45}$$

Hence from (6.3.2) we obtain for $k \geq m$,

$$\begin{aligned} 0 &= \Delta^2 x(k, r+1) - \sum_{j \in J} q(j) x(k + \sigma(j), r+1) \\ &\leq \Delta^2 x(k, r+1) - \left(\sum_{j \in J} q(j) \lambda_r^{\sigma(j)} \right) x(k, r+1). \end{aligned} \tag{6.3.46}$$

But (6.3.38) gives

$$\sum_{j \in J} q(j) \lambda_r^{\sigma(j)} \geq (\lambda_r - 1)^2 + \mu = (\lambda_{r+1} - 1)^2. \tag{6.3.47}$$

Thus by (6.3.41) we get for every $k \geq m$,

$$\begin{aligned} 0 &\leq \Delta^2 x(k, r+1) - (\lambda_{r+1} - 1)^2 x(k, r+1) \\ &= x(k+1, r+2) - \lambda_{r+1} x(k, r+2), \end{aligned} \tag{6.3.48}$$

which means that (6.3.43) is also true for $r + 1$. So by mathematical induction, (6.3.43) is satisfied for any arbitrary $r \in \mathbb{N}_0$.

Next, it is easy to verify that $\{\lambda_r\}_{r \in \mathbb{N}_0}$ is a strictly increasing sequence of numbers in the interval $(1, \infty)$. Moreover, we will establish that the sequence $\{\lambda_r\}_{r \in \mathbb{N}_0}$ is bounded from above. In fact, we consider an arbitrary index $r \in \mathbb{N}_0$. The sequence $\{x(k, r+1)\}$ is an eventually positive solution of (6.3.2) which is unbounded. Hence, Lemma 6.3.5 ensures that for each $i \in \{0, 1, 2\}$, $\Delta^i x(k, r+1)$ is eventually positive. So we can choose an integer $m_1 \geq m$ such that $\Delta^i x(k, r+1) > 0$ for all $k \geq m_1$, $i \in \{0, 1, 2\}$. In particular, one has $\Delta x(k, r+1) > 0$ for all $k \geq m_1$ which means that the sequence $\{x(k, r+1)\}$ is strictly increasing. But we have for $k \geq m \in \mathbb{N}_0$,

$$\Delta^2 x(k, r+1) = \sum_{j \in J} q(j) x(k + \sigma(j), r+1) \geq q(j_0) x(k + \sigma(j_0), r+1). \tag{6.3.49}$$

Thus, by using (6.3.37), we obtain $\Delta^2 x(k, r+1) \geq q(j_0) x(k+3, r+1)$ for $k \geq m_1$. Further, we have for $k \geq m_1 + 1$,

$$\Delta x(k, r+1) = \Delta x(k-1, r+1) + \Delta^2 x(k-1, r+1) > \Delta^2 x(k-1, r+1), \tag{6.3.50}$$

and hence we get

$$\Delta x(k, r+1) > q(j_0) x(k+2, r+1) \quad \forall k \geq m_1 + 1. \tag{6.3.51}$$

Repeating the above procedure, we find

$$x(k, r+1) > q(j_0)x(k+1, r+1) \quad \forall k \geq m_1 + 2, \tag{6.3.52}$$

that is,

$$x(k+1, r+1) - \frac{1}{q(j_0)}x(k, r+1) < 0 \quad \text{for every } k \geq m_1 + 2. \tag{6.3.53}$$

Combining (6.3.43) and (6.3.53), we find that $\lambda_r < 1/q(j_0)$. Since r is an arbitrary number in the set $\mathbb{N}_0$, it follows that the number $1/q(j_0)$ is an upper bound of the sequence $\{\lambda_r\}_{r\in\mathbb{N}_0}$.

Finally, since $\{\lambda_r\}$ is a strictly increasing sequence which is bounded, the limit $L = \lim_{r\to\infty} \lambda_r$ exists and is a positive real number. But we have

$$\lambda_r = 1 + \left[(\lambda_{r-1} - 1)^2 + \mu\right]^{1/2} \quad \text{for } r \in \mathbb{N}. \tag{6.3.54}$$

Hence one has

$$L = 1 + \left[(L-1)^2 + \mu\right]^{1/2}, \tag{6.3.55}$$

which gives $\mu = 0$, which is a contradiction. This completes the proof. □

6.4. Oscillation of damped difference equations

In this section we will begin by recalling some results given in Section 3.9 which are also applicable to damped difference equations with deviating arguments of the form

$$\Delta(c(k)\Delta x(k)) + p(k)\Delta x(k) + q(k)f(x[g(k)]) = 0, \tag{6.4.1}$$

$$\Delta(c(k)\Psi(\Delta x(k))) + p(k)\Psi(\Delta x(k)) + q(k)f(x[q(k)]) = 0, \tag{6.4.2}$$

where $\{c(k)\}$, $\{g(k)\}$, $\{q(k)\}$, f, and Ψ are as in equations (6.1.1) and (6.2.2) and $\{p(k)\}$ is a sequence of nonnegative real numbers. Clearly equation (6.4.2) includes equation (6.4.1) as a special case (take $\Psi(x) = x$).

From Lemma 3.9.1 we see that if $\{x(k)\}$ is a nonoscillatory solution of (6.4.2),

$$c(k) > p(k) \quad \text{for } k \geq m \in \mathbb{N}, \tag{6.4.3}$$

$$\sum_{k=m}^{\infty} \Psi^{-1}\left[\frac{1}{c(k)} \prod_{j=m}^{k-1} \left(1 - \frac{p(j)}{c(j)}\right)\right] = \infty, \tag{6.4.4}$$

then $x(k)\Delta x(k) > 0$ eventually.

Also we can easily see that the following comparison result holds.

Theorem 6.4.1. *Let conditions (6.2.6), (6.4.3), and (6.4.4) hold. If the equation (6.2.1) is oscillatory, then equation (6.4.2) is oscillatory.*

Next we consider the damped difference equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + p(k)\Psi(\Delta x(k)) + F\big(k, x[k-\tau], \Delta x[k-\sigma]\big) = 0, \tag{6.4.5}$$

where $\{c(k)\}$, $\{p(k)\}$, Ψ are as in (6.4.2), $F \in C(\mathbb{N} \times \mathbb{R}^2, \mathbb{R})$, and $\tau, \sigma \in \mathbb{N}_0$.

We assume that there exist an eventually positive sequence $\{q(k)\}$ and real numbers $\beta > 0$ and $\gamma \geq 0$ such that

$$F(k, x, y)\operatorname{sgn} x \geq q(k)|x|^{\beta}|y|^{\gamma} \quad \text{for } k \in \mathbb{N},\ xy \neq 0. \tag{6.4.6}$$

Now we are ready to present the following result.

Theorem 6.4.2. *Let conditions (6.4.3), (6.4.4), and (6.4.6) hold. Suppose that Ψ satisfies either (i_1) or (i_2). If, for every constant $\lambda > 0$, the equation*

$$\Delta y(k) + \lambda q(k)\big(c[k-\sigma]\big)^{-\gamma/\alpha}\big|y[k-\sigma]\big|^{\gamma/\alpha}\operatorname{sgn} y[k-\sigma] = 0 \tag{6.4.7}$$

is oscillatory, then equation (6.4.5) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.4.5), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. It is easy to see that conditions (6.4.3) and (6.4.4) imply that $\Delta x(k) > 0$ for $k \geq m_1$ for some $m_1 \geq m$. There exist a constant $a > 0$ and an integer $m_2 \geq m_1$ such that

$$x[k-\tau] \geq a \quad \text{for } k \geq m_2. \tag{6.4.8}$$

Using condition (6.4.6) and (6.4.8) in equation (6.4.5), we have

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + a^{\beta}q(k)\big(\Delta x[k-\sigma]\big)^{\gamma} \leq 0 \quad \text{for } k \geq m_2. \tag{6.4.9}$$

Set $z(k) = c(k)\Psi(\Delta x(k))$ for $k \geq m_2$ to obtain

$$\Delta z(k) + a^{\beta}q(k)\left(\frac{1}{c[k-\sigma]}\right)^{\gamma/\alpha} z^{\gamma/\alpha}[k-\sigma] \leq 0 \quad \text{for } k \geq m_2. \tag{6.4.10}$$

Therefore, by Lemma 6.2.2, equation (6.4.7) has an eventually positive solution, which is a contradiction. This completes the proof. □

Theorem 6.4.3. *Let conditions (6.4.3), (6.4.4), and (6.4.6) hold. Suppose that* Ψ *satisfies either* (i_1) *or* (i_2). *If, for all large* $m \in \mathbb{N}$, *the equation*

$$\Delta y(k) + q(k)C^{\beta}[k-\tau-1, m]c^{-\gamma/\alpha}[k-\sigma]\,|\,y[k-\mu]\,|^{(\beta+\gamma)/\alpha}\operatorname{sgn} y[k-\mu] = 0, \tag{6.4.11}$$

where C is given in (6.2.54) and $\mu = \min\{\tau, \sigma\}$, *is oscillatory, then equation (6.4.5) is oscillatory.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.4.5), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 6.4.2, we obtain $\Delta x(k) > 0$ for $k \geq m_1 \geq m$. Also, as in the proof of Theorem 6.2.9, we obtain the inequality (6.2.57) which takes the form

$$x[k-\tau] \geq C[k-\tau-1, m^*]\Psi^{-1}(c[k-\tau])\Delta x(k-\tau] \quad \text{for } k \geq m^* + \tau + 1 \tag{6.4.12}$$

for some large $m^* \geq m_1$. Using the fact that $\{\Psi^{-1}(c(k))\Delta x(k)\}$ is nonincreasing for $k \geq m^*$ in (6.4.12), we obtain for $k \geq m^* + \tau + 1$,

$$x[k-\tau] \geq C[k-\tau-1, m^*]\Psi^{-1}(c[k-\mu])\Delta x[k-\mu], \tag{6.4.13}$$

$$\Delta x[k-\sigma] \geq \frac{1}{\Psi^{-1}(c[k-\sigma])}\Psi^{-1}(c[k-\mu])\Delta x[k-\mu] \tag{6.4.14}$$

for $k \geq m^* + \tau + 1$. Using condition (6.4.6), (6.4.13), and (6.4.14) in the inequality

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + F(k, x[k-\tau], \Delta x[k-\sigma]) \leq 0, \tag{6.4.15}$$

and setting $z(k) = c(k)\Psi(\Delta x(k))$ for $k \geq m^* + \tau + 1$, we have

$$\Delta z(k) + q(k)C^{\beta}[k-\tau-1, m^*]c^{-\gamma/\alpha}[k-\sigma]z^{(\beta+\gamma)/\alpha}[k-\mu] \leq 0 \tag{6.4.16}$$

for $k \geq m^* + \tau + 1$. The rest of the proof is similar to that of Theorem 6.4.2 and hence is omitted. □

We note that some immediate results from Theorems 6.4.2 and 6.4.3 similar to those in Corollary 6.2.13 can be formulated. The details are left to the reader.

Theorem 6.4.4. *Let $\Delta p(k) \le 0$, $g(k) = k - \tau$ for $k \ge m \in \mathbb{N}$ and $\tau \in \mathbb{N} \setminus \{1\}$. Suppose that condition (6.4.3) holds and*

$$\frac{f(x)}{x} \ge \lambda > 0 \quad \text{for } x \neq 0, \text{ where } \lambda \text{ is a constant.} \tag{6.4.17}$$

If, for all sufficiently large k, the equation

$$\Delta y(k) + Q(k)y[k-\tau] = 0, \tag{6.4.18}$$

where

$$Q(k) = \min\left\{\left(\frac{1}{c(k)}\left[\sum_{j=k-\tau}^{k-1} \lambda q(j) - p[k-\tau]\right]\right), (\lambda C(k-\tau-1, m)q(k))\right\} \tag{6.4.19}$$

and C is given in (6.1.8), is oscillatory, then equation (6.4.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.4.1), say, $x(k) > 0$ for $k \ge m$ for some $m \in \mathbb{N}$. It is easy to see that condition (6.4.3) implies that the sequence $\{\Delta x(k)\}$ is eventually of one sign. Next we consider the following two cases:

(I) $\Delta x(k) < 0$ eventually,
(II) $\Delta x(k) > 0$ eventually.

If (I) holds, then we suppose that $\Delta x(k) < 0$ for $k \ge m_1 \ge m$. Summing equation (6.4.1) from $k-\tau$ to $k-1$ and using condition (6.4.17), we have

$$c(k)\Delta x(k) - c[k-\tau]\Delta x[k-\tau] + \sum_{j=k-\tau}^{k-1} p(j)\Delta x(j) + \lambda \sum_{j=k-\tau}^{k-1} q(j)x[j-\tau] \le 0, \tag{6.4.20}$$

so for $k \ge m$,

$$\begin{aligned} c(k)\Delta x(k) + \Bigg[& p(k)x(k) - p[k-\tau]x[k-\tau] \\ & - \sum_{j=k-\tau}^{k-1} x(j+1)\Delta p(j) + \lambda x[k-\tau] \sum_{j=k-\tau}^{k-1} q(j)\Bigg] \le 0. \end{aligned} \tag{6.4.21}$$

Since $\Delta p(k) \le 0$ for $k \ge m$, we obtain

$$\Delta x(k) + \frac{1}{c(k)}\left[\lambda \sum_{j=k-\tau}^{k-1} q(j) - p[k-\tau]\right]x[k-\tau] \le 0 \quad \text{for } k \ge m_1, \tag{6.4.22}$$

and hence by (6.4.19) we find $\Delta x(k) + Q(k)x[k-\tau] \le 0$ for $k \ge m_1$. The rest of the proof in this case is similar to that of Theorem 6.4.2 and hence is omitted.

If (II) holds, then we suppose that $\Delta x(k) > 0$ for $k \geq m_1 \geq m$. By Lemma 6.1.2, there exists an integer $m_2 \geq m_1$ such that

$$x[k-\tau] \geq C(k-\tau-1, m_1)c[k-\tau]\Delta x[k-\tau] \quad \text{for } k \geq m_2. \tag{6.4.23}$$

Using (6.4.17) and (6.4.23) in equation (6.4.1), we find

$$\Delta z(k) + \lambda C(k-\tau-1, m_1)z[k-\tau] \leq 0 \quad \text{for } k \geq m_2, \tag{6.4.24}$$

where $z(k) = c(k)\Delta x(k)$ for $k \geq m$. By (6.4.19), we obtain $\Delta z(k)+Q(k)z[k-\tau] \leq 0$ for $k \geq m_2$. The rest of the proof in this case is similar to the proof of Theorem 6.4.2 and hence we omit the details. □

Next we present results for a forced difference equation of the form

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + q(k)f\big(x[k-\tau]\big) = e(k), \tag{6.4.25}$$

where $\{c(k)\}$, $\{p(k)\}$, $\{q(k)\}$, τ, and f are as in equation (6.4.1) and $\{e(k)\}$ is a sequence of real numbers.

Theorem 6.4.5. *Let the conditions of Theorem 6.4.4 hold with condition (6.4.17) replaced by conditions (6.1.11) and (6.1.16). If $\{x(k)\}$ and $\{y(k)\}$ are eventually positive solutions of equation (6.4.25), then $\{x(k)-y(k)\}$ is oscillatory.*

Proof. Let $\{x(k)\}$ and $\{y(k)\}$ be two positive solutions of equation (6.4.25) for $k \geq m$ for some $m \in \mathbb{N}$ and let $w(k) = x(k) - y(k)$ for $k \geq m$. From equation (6.4.25), we find

$$\Delta\big(c(k)\Delta w(k)\big) + p(k)\Delta w(k) + q(k)\big[f\big(x[k-\tau]\big) - f\big(y[k-\tau]\big)\big] = 0. \tag{6.4.26}$$

To show that $\{w(k)\}$ is oscillatory, we will assume that $\{w(k)\}$ is eventually positive. The negative case follows analogously.

Now let $w(k) > 0$ for $k \geq m$. Conditions (6.1.11) and (6.1.16) imply that

$$\Delta\big(c(k)\Delta w(k)\big) + p(k)\Delta w(k) + \lambda q(k)w[k-\tau] \leq 0 \quad \text{for } k \geq m. \tag{6.4.27}$$

The rest of the proof is similar to the proof of Theorem 6.4.4 and hence we omit it here. □

In the case when conditions (6.4.3) and (6.4.4) are satisfied, we have the following immediate result.

Theorem 6.4.6. *Let conditions (6.1.11), (6.1.16), (6.4.3), and (6.4.4) hold. If for all sufficiently large $m \in \mathbb{N}$ the equation*

$$\Delta y(k) + \lambda C(k - \tau - 1, m) q(k) y[k - \tau] = 0, \tag{6.4.28}$$

where C is as in (6.1.8), is oscillatory, then equation (6.4.25) is oscillatory.

Remark 6.4.7. The result presented above remains valid when $p(k) \equiv 0$. On the other hand, if $c(k) \equiv 1$, $p(k) \equiv p \in (0, 1)$, and $\Psi(x) = x$, then the series in condition (6.4.4) is a convergent geometric series, and hence condition (6.4.4) is violated. In this case we are able to describe the oscillatory behavior of the special case of equation (6.4.1) via applying Theorem 6.4.4.

As an application, we present the following criteria for oscillation of a special case of equation (6.4.1), namely, the linear damped equation

$$\Delta^2 x(k) + p\Delta x(k) + qx[k - \tau] = 0, \tag{6.4.29}$$

where $p \in [0, 1)$ and $q > 0$ are real constants, and $\tau \in \mathbb{N} \setminus \{1\}$.

Corollary 6.4.8. *If*

$$q\tau - p > \left(\frac{\tau}{\tau + 1}\right)^{\tau+1}, \tag{6.4.30}$$

then equation (6.4.29) is oscillatory.

Corollary 6.4.9. *If condition (6.4.30) holds, $\{x(k)\}$ and $\{y(k)\}$ are two eventually positive solutions of the forced equation*

$$\Delta^2 x(k) + p\Delta x(k) + qx[k - \tau] = e(k), \tag{6.4.31}$$

where p, q, τ are as in equation (6.4.29), and $\{e(k)\}$ is a sequence of real numbers, then $\{x(k) - y(k)\}$ is oscillatory.

Remark 6.4.10. From Corollary 6.4.8 we see that the characteristic equation associated with equation (6.4.29), namely,

$$(\mu - 1)^2 + p(\mu - 1) + q\mu^{-\tau} = 0 \tag{6.4.32}$$

has no positive roots provided that condition (6.4.30) holds.

6.4.1. Almost oscillation

Here, we will consider certain difference equations of second order of the form

$$\Delta^2 x(k) + p(k)\Delta x[k-\sigma] + q(k)\,|\,x[g(k)]\,|^{\gamma}\operatorname{sgn} x[g(k)] = 0, \tag{6.4.33}$$
$$\Delta^2 x(k) - p(k)\Delta x[k-\sigma] + q(k)\,|\,x[g(k)]\,|^{\gamma}\operatorname{sgn} x[g(k)] = 0, \tag{6.4.34}$$
$$\Delta^2 x(k) + p(k)\Delta x[k-\sigma] - q(k)\,|\,x[g(k)]\,|^{\gamma}\operatorname{sgn} x[g(k)] = 0, \tag{6.4.35}$$
$$\Delta^2 x(k) - p(k)\Delta x[k+\sigma] - q(k)\,|\,x[g(k)]\,|^{\gamma}\operatorname{sgn} x[g(k)] = 0, \tag{6.4.36}$$

where $\gamma > 0$ is a constant, σ is any integer, $\{p(k)\}$ and $\{q(k)\}$ are sequences of nonnegative real numbers, and $\{g(k)\}$ is a nondecreasing sequence of nonnegative integers with $\lim_{k\to\infty} g(k) = \infty$.

Any of the equations (6.4.33)–(6.4.36) is said to be *almost oscillatory* if for every solution $\{x(k)\}$ either $\{x(k)\}$ is oscillatory or $\{\Delta x(k)\}$ is oscillatory. The main goal here is to obtain some sufficient conditions for equations (6.4.33)–(6.4.36) to be almost oscillatory.

First, we are concerned with the oscillatory and asymptotic behavior of equation (6.4.33).

Theorem 6.4.11. *Let $\sigma \in \mathbb{N}$ and suppose*

$$\lim_{k\to\infty} \sum_{j=k-\sigma}^{k-1} p(j) > \left(\frac{\sigma}{1+\sigma}\right)^{1+\sigma}. \tag{6.4.37}$$

Further, assume that there exists a sequence $\{\rho(k)\}$ of real numbers such that

$$\rho(k) > 0, \quad \Delta\rho(k) \ge 0, \quad \Delta\big(\rho(k)p(k)\big) \le 0 \quad \text{for } k \ge m \in \mathbb{N}. \tag{6.4.38}$$

If

$$\sum^{\infty} \frac{1}{\rho(j)} = \infty, \tag{6.4.39}$$
$$\sum^{\infty} \rho(j)q(j) = \infty, \tag{6.4.40}$$

then every solution $\{x(k)\}$ of equation (6.4.33) is oscillatory, or $\{\Delta x(k)\}$ is oscillatory or else $x(k) \to 0$ monotonically as $k \to \infty$.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.4.33). There exists $m \in \mathbb{N}$ such that $x(k) > 0$ and $x[g(k)] > 0$ for $k \ge m$. Next, we consider the following two cases:

(i) $\Delta x(k) > 0$ eventually,
(ii) $\Delta x(k) < 0$ eventually.

Assume that (i) holds, that is, $\Delta x(k) > 0$ eventually. From equation (6.4.33) we see that

$$\Delta^2 x(k) + p(k)\Delta x[k-\sigma] = -q(k)x^\gamma[g(k)] \le 0 \quad \text{eventually.} \tag{6.4.41}$$

Then $y(k) := \Delta x(k) > 0$ eventually. Thus

$$\Delta y(k) + q(k)y[k-\sigma] \le 0 \quad \text{eventually.} \tag{6.4.42}$$

In view of Lemma 6.1.6 and condition (6.4.37), inequality (6.4.42) has no eventually positive solution, which is a contradiction.

Now assume that (ii) holds, that is, $\Delta x(k) < 0$ for $k \ge m_1$ for some $m_1 \ge m$. Thus we have $x(k) \to b \ge 0$ as $k \to \infty$. Suppose that $b > 0$ and consider the sequence $\{w(k)\}$ defined by $w(k) = \rho(k-1)\Delta x(k)$ for $k \ge m_1$. Then for $k \ge m_1$,

$$\begin{aligned} \Delta w(k) &= \Delta\big(\rho(k-1)\Delta x(k)\big) = \rho(k)\Delta^2 x(k) + \Delta\rho(k-1)\Delta x(k) \\ &\le -b^\gamma \rho(k)q(k) - \rho(k)p(k)\Delta x[k-\sigma] + \Delta\rho(k-1)\Delta x(k) \\ &\le -b^\gamma \rho(k)q(k) - \rho(k)p(k)\Delta x[k-\sigma]. \end{aligned} \tag{6.4.43}$$

Summing both sides of (6.4.43) from m_1 to $k-1 \ge m_1+\sigma$, we get

$$\begin{aligned} w(k) - w(m_1) &\le -b^\gamma \sum_{j=m_1}^{k-1} \rho(j)q(j) - \sum_{j=m_1}^{k-1} \rho(j)p(j)\Delta x[j-\sigma] \\ &= -b^\gamma \sum_{j=m_1}^{k-1} \rho(j)q(j) - \Bigg[\rho(k)p(k)x[k-\sigma] - \rho(m_1)p(m_1)x[m_1-\sigma] \\ &\qquad - \sum_{j=m_1}^{k-1} x[j-\sigma+1]\Delta\big(\rho(j)p(j)\big)\Bigg]. \end{aligned} \tag{6.4.44}$$

Using condition (6.4.38) in (6.4.44), we get

$$w(k) \le C - b^\gamma \sum_{j=m_1}^{k-1} \rho(j)q(j), \tag{6.4.45}$$

where $C = \rho(m_1)p(m_1)x[m_1-\sigma] > 0$. By (6.4.40), there exist $m_2 \ge m_1$ and $b^* > 0$ such that $w(k) = \rho(k-1)\Delta x(k) \le -b^*$, or $\Delta x(k) \le -b^*/\rho(k-1)$ for $k \ge m_2$. Summing both sides of this inequality from m_2 to $n \ge m_2+1$, letting $n \to \infty$, and using condition (6.4.39), we obtain a contradiction to the fact that $x(k) > 0$ eventually. This shows that $b = 0$ and completes the proof. $\square$

Theorem 6.4.12. *Let σ be any integer and $\Delta p(k) \leq 0$ for $k \geq m \in \mathbb{N}$. If*

$$\sum^{\infty} q(j) = \infty, \tag{6.4.46}$$

then the conclusion of Theorem 6.4.11 holds.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.4.33), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Next we consider the following two cases:

(i) $\Delta x(k) > 0$ eventually,
(ii) $\Delta x(k) < 0$ eventually.

If (i) holds, then suppose $\Delta x(k) > 0$ for $k \geq m_1$ for some $m_1 \geq m$. There exist a constant $b > 0$ and an integer $m_2 \geq m_1$ such that

$$x[g(k)] \geq b \quad \text{for } k \geq m_2. \tag{6.4.47}$$

Now, from equation (6.4.33) we have

$$\Delta^2 x(k) + b^* q(k) \leq 0 \quad \text{for } k \geq m_2, \tag{6.4.48}$$

where $b^* = b^\gamma$. Summing both sides of (6.4.48) from m_2 to $n \geq m_2 + 1$, letting $n \to \infty$, and using condition (6.4.46), we obtain a contradiction to the fact that $\Delta x(k) > 0$ for $k \geq m_2$.

Now assume that (ii) holds, that is, $\Delta x(k) < 0$ eventually. The proof of this case is similar to that of Theorem 6.4.11(ii) with $\rho(k) = 1$, and hence is omitted. ☐

Example 6.4.13. The difference equation

$$\Delta^2 x(k) + \Delta x[k-\sigma] + e^{-\tau}\left(1 - \frac{1}{e}\right)\left(e^{\sigma} + \frac{1}{e} - 1\right)x[k-\tau] = 0 \quad \text{for } k \geq m \in \mathbb{N}, \tag{6.4.49}$$

where $\tau, \sigma \in \mathbb{N}$, has a nonoscillatory solution $\{x(k)\}$ such that $x(k) = e^{-k} \to 0$ monotonically as $k \to \infty$. All conditions of Theorem 6.4.12 are satisfied.

It is easy to check that the undamped equation associated with (6.4.49), that is, the equation

$$\Delta^2 x(k) + e^{-\tau}\left(1 - \frac{1}{e}\right)\left(e^{\sigma} + \frac{1}{e} - 1\right)x[k-\tau] = 0, \tag{6.4.50}$$

is oscillatory for any $\sigma \geq 0$. Thus we conclude that the presence of the damped term in (6.4.49) disrupts the oscillatory properties of (6.4.50).

Next we present the following almost oscillation result for equation (6.4.33).

Theorem 6.4.14. *Let σ be a nonpositive integer, $\gamma = 1$, and assume $\Delta p(k) \leq 0$ for $k \geq m \in \mathbb{N}$. Assume that condition (6.4.46) holds. Moreover, assume that there exists a sequence $\{d(k)\}$ of positive integers such that $g(k) \leq k - d(k)$ for $k \geq m$ and $\{k - d(k)\}$ is increasing. If*

$$\liminf_{k\to\infty}\left[\frac{1}{d(k)}\sum_{j=k-d(k)}^{k-1} Q(j)\right] > \limsup_{k\to\infty}\frac{(d(k))^{d(k)}}{(1+d(k))^{1+d(k)}}, \tag{6.4.51}$$

where

$$Q(k) = \sum_{j=k-d(k)}^{k-1} q(j) - p[k-d(k)] > 0 \quad \text{for } k \geq m + d(k), \tag{6.4.52}$$

then equation (6.4.33) is almost oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.4.33), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Next, we consider the following two cases:

(i) $\Delta x(k) > 0$ eventually,

(ii) $\Delta x(k) < 0$ eventually.

Assume (i), that is, $\Delta x(k) > 0$ eventually. The proof of this case is similar to that of Theorem 6.4.12(i) and hence is omitted.

Assume (ii), that is, $\Delta x(k) < 0$ for $k \geq m_1$ for some $m_1 \geq m$. From equation (6.4.33) and the fact that $g(k) \leq k - d(k)$ for $k \geq m_1$, we have

$$\Delta^2 x(k) + p(k)\Delta x[k-\sigma] + q(k)x[k-d(k)] \leq 0 \quad \text{for } k \geq m_1. \tag{6.4.53}$$

Summing both sides of (6.4.53) from $k - d(k)$ to $k - 1$, we obtain

$$\Delta x(k) - \Delta x[k-d(k)] + \sum_{j=k-d(k)}^{k-1} p(j)\Delta x[j-\sigma] + \sum_{j=k-d(k)}^{k-1} q(j)x[j-d(j)] \leq 0, \tag{6.4.54}$$

so, for $k \geq m_1$,

$$\begin{aligned}\Delta x(k) &+ \left[p(k)x[k-\sigma] - p[k-d(k)]x[k-d(k)-\sigma] - \sum_{j=k-d(k)}^{k-1} x[j-\sigma+1]\Delta p(j)\right]\\ &+ \sum_{j=k-d(k)}^{k-1} q(j)x[j-d(j)] \leq 0.\end{aligned} \tag{6.4.55}$$

Using the fact that $\sigma \leq 0$ and $\Delta p(k) \leq 0$ for $k \geq m_1$, we have

$$\Delta x(k) - p[k-d(k)]x[k-d(k)] + x[k-d(k)] \sum_{j=k-d(k)}^{k-1} q(j) \leq 0, \qquad (6.4.56)$$

so

$$\Delta x(k) + Q(k)x[k-d(k)] \leq 0 \quad \text{for } k \geq m_1. \qquad (6.4.57)$$

But Lemma 6.1.6 and condition (6.4.51) imply that inequality (6.4.57) has no eventually positive solution, which is a contradiction. The proof is therefore complete. □

Example 6.4.15. Consider the difference equation (6.4.49) with $\sigma \leq 0$ and $\tau \geq 1$. In this case we see that all the hypotheses of Theorem 6.4.14 are satisfied except for condition (6.4.51).

Remark 6.4.16. (i) In equation (6.4.33) if $\sigma = 0$, then we see that the hypotheses of Theorem 6.4.14 are satisfied, and in this case we can conclude that equation (6.4.33) with $\sigma = 0$ is oscillatory. In fact, this result is also similar to Theorem 6.4.4.

(ii) If we set $p(k) \equiv 0$ in Theorems 6.4.11 and 6.4.12, then we can easily see that the conclusion of both theorems are replaced by "equation (6.4.33) with $p(k) \equiv 0$ is oscillatory" (see the obtained results in Section 6.1).

We note that the presence of the term $p(x)\Delta x[k-\sigma]$ makes the coexistence of oscillatory and monotonically decreasing positive (increasing negative) solutions for equation (6.4.33) possible.

Next we will present some sufficient conditions for equation (6.4.34) to be almost oscillatory.

Theorem 6.4.17. *Let $\sigma \in \mathbb{N}_0$ and $\Delta p(k) \geq 0$ for $k \geq m \in \mathbb{N}$. If condition (6.4.46) holds and*

$$\sum_{j=m}^{\infty} a(j+1) \sum_{i=m}^{j-1} q(i) = \infty, \qquad (6.4.58)$$

where

$$a(j+1) = \prod_{i=m}^{j} \left(1+p(i)\right)^{-1} \quad \textit{for } j \in \mathbb{N}, \qquad (6.4.59)$$

then equation (6.4.34) is almost oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.4.34), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. We consider the following two cases:

(i) $\Delta x(k) < 0$ eventually,

(ii) $\Delta x(k) > 0$ eventually.

For (i), assume $\Delta x(k) < 0$ eventually. From equation (6.4.34), we observe that $\Delta^2 x(k) \leq 0$ eventually and hence one can easily see that $x(k) \to -\infty$ as $k \to \infty$, which is a contradiction.

For (ii), assume $\Delta x(k) > 0$ for $k \geq m_1 \geq m$. There exist an integer $m_2 \geq m_1 + \sigma$ and a constant $b > 0$ such that (6.4.47) holds. Using (6.4.47) in equation (6.4.34), we have

$$\Delta^2 x(k) - p(k)\Delta x[k-\sigma] + b^\gamma q(k) \leq 0 \quad \text{for } k \geq m_2. \tag{6.4.60}$$

Summing both sides of (6.4.60) from m_2 to $k-1 \geq m_2$, we obtain

$$\Delta x(k) - \Delta x(m_2) - \sum_{j=m_2}^{k-1} p(j)\Delta x[j-\sigma] + b^\gamma \sum_{j=m_2}^{k-1} q(j) \leq 0, \tag{6.4.61}$$

so

$$\begin{aligned} \Delta x(k) - \Delta x(m_2) - p(k)x[k-\sigma] + p(m_2)x[m_2-\sigma] \\ + \sum_{j=m_2}^{k-1} x[j-\sigma+1]\Delta p(j) + b^\gamma \sum_{j=m_2}^{k-1} q(j) \leq 0. \end{aligned} \tag{6.4.62}$$

Using the fact that $\Delta p(k) \geq 0$ for $k \geq m$ in (6.4.62), we have

$$\Delta x(k) - \Delta x(m_2) - p(k)x(k) + b^\gamma \sum_{j=m_2}^{k-1} q(j) \leq 0 \quad \text{for } k \geq m_2. \tag{6.4.63}$$

From condition (6.4.46), there exists an integer $m_3 \geq m_2 + 1$ such that

$$\Delta x(m_2) \leq \frac{1}{2} b^\gamma \sum_{j=m_2}^{k-1} q(j) \quad \text{for } k \geq m_3 + 1. \tag{6.4.64}$$

Thus

$$\Delta x(k) - p(k)x(k) + \frac{1}{2} b^\gamma \sum_{j=m_2}^{k-1} q(j) \leq 0 \quad \text{for } k \geq m_3 + 1. \tag{6.4.65}$$

Define a sequence $\{r(k)\}$ by the recurrence relation

$$r(k+1) = \frac{r(k)}{1+p(k)} \quad \text{for } k \geq m \in \mathbb{N}, \qquad r(m) > 0. \tag{6.4.66}$$

Next, we multiply (6.4.65) by $r(k+1)$ to obtain

$$\Delta\big(r(k)x(k)\big) + \frac{1}{2}b^{\gamma}r(k+1)\sum_{j=m_2}^{k-1}q(j) \le 0 \quad \text{for } k \ge m_2+1. \tag{6.4.67}$$

Summing both sides of (6.4.67) from m_3+1 to $k \ge m_3+1$, we have

$$\begin{aligned} 0 &< r(k+1)x(k+1) \\ &\le r(m_3+1)x(m_3+1) \\ &\quad - \frac{1}{2}b^{\gamma}\sum_{n=m_3+1}^{k} r(n+1)\sum_{j=m_2}^{n-1}q(j) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{6.4.68}$$

which is a contradiction. This completes the proof. □

The following result deals with the almost oscillation of equation (6.4.34) when $g(k) \ge k+2$ for $k \ge m \in \mathbb{N}$ and $\gamma > 1$.

Theorem 6.4.18. *Let $\sigma \in \mathbb{N}_0$, $\gamma > 1$, $g(k) \ge k+2$ for $k \ge m \in \mathbb{N}$, and assume that there exists a sequence $\{\rho(k)\}$ of real numbers such that conditions (6.4.38) and (6.4.40) hold and that $\Delta^2\rho(k) \le 0$ for $k \ge m$. Then equation (6.4.34) is almost oscillatory.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.4.34), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \ge m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 6.4.17, we consider the cases (i) and (ii) and observe that case (i) is impossible. Next, we consider case (ii). Assume $\Delta x(k) > 0$ for $k \ge m_1 \ge m+\sigma$. Set

$$w(k) = \rho(k)\frac{\Delta x(k)}{x^{\gamma}(k+1)} \quad \text{for } k \ge m_1. \tag{6.4.69}$$

Then

$$\begin{aligned} \Delta w(k) &= \rho(k+1)\left(\frac{\Delta x(k+1)}{x^{\gamma}(k+2)}\right) - \rho(k)\frac{\Delta x(k)}{x^{\gamma}(k+1)} \\ &= -\rho(k)q(k)\left(\frac{x[g(k)]}{x(k+2)}\right)^{\gamma} + \rho(k)p(k)\left(\frac{\Delta x[k-\sigma]}{x^{\gamma}[k+2]}\right) \\ &\quad + \rho(k)\Delta x(k+1)\left[x^{-\gamma}(k+2) - x^{-\gamma}(k+1)\right] + \Delta\rho(k)\left(\frac{\Delta x(k+1)}{x^{\gamma}(k+2)}\right), \end{aligned} \tag{6.4.70}$$

and hence we see that for $k \ge m_1$,

$$\Delta w(k) \le -\rho(k)q(k) + \rho(k)p(k)\left(\frac{\Delta x[k-\sigma]}{x^{\gamma}(k+2)}\right) + \Delta\rho(k)\left(\frac{\Delta x(k+1)}{x^{\gamma}(k+2)}\right). \tag{6.4.71}$$

Summing both sides of (6.4.71) from m_1 to $k-1 \geq m_1$, using condition (6.4.38), and the fact that $x(k+2) \geq x[k-\sigma+1]$ for $k \geq m_1$, we have

$$\begin{aligned} w(k) - w(m_1) \leq -\sum_{j=m_1}^{k-1} \rho(j)q(j) + \rho(m_1)p(m_1) \sum_{j=m_1}^{k-1} \frac{\Delta x[j-\sigma]}{x^{\gamma}[j-\sigma+1]} \\ + \Delta\rho(m_1) \sum_{j=m_1}^{k-1} \frac{\Delta x(j+1)}{x^{\gamma}(j+2)}. \end{aligned} \tag{6.4.72}$$

As in the proof of Theorem 4.2.1 we have

$$\sum^{\infty} \frac{\Delta x(j)}{x^{\gamma}(j+1)} < \infty, \tag{6.4.73}$$

and hence, by condition (6.4.40), it follows that

$$0 < w(k) \leq C - \sum_{j=m_1}^{k-1} \rho(j)q(j) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{6.4.74}$$

where C is a constant, which is a contradiction. This completes the proof. □

Remark 6.4.19. One can easily observe that Theorems 6.4.17 and 6.4.18 are applicable to equation (6.4.34) when $\sigma = 0$ or $p(k) \equiv 0$ for $k \geq m \in \mathbb{N}$.

The following results are concerned with almost oscillation of equation (6.4.35).

Theorem 6.4.20. *Suppose that $\sigma \in \mathbb{N}$, $\gamma > 1$, $\Delta p(k) \geq 0$, $0 < p(k) < 1$, and $g(k) \geq k+1$ for $k \geq m \in \mathbb{N}$. If*

$$\liminf_{k\to\infty} \sum_{j=k-\sigma}^{k-1} p(j) > \left(\frac{\sigma}{1+\sigma}\right)^{1+\sigma}, \tag{6.4.75}$$

$$\sum_{j=m}^{\infty} P[j+1, g(j)-1]q(j) = \infty, \tag{6.4.76}$$

where

$$P[k+1, g(k)-1] = \sum_{j=k+1}^{g(k)-1} \left(\prod_{i=m}^{j} \frac{1}{1-p(i)}\right)^{1-\gamma}, \tag{6.4.77}$$

then equation (6.4.35) is almost oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.4.35), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As before, we consider the following two cases:

(i) $\Delta x(k) < 0$ eventually,

(ii) $\Delta x(k) > 0$ eventually.

For (i), assume $\Delta x(k) < 0$ eventually. From equation (6.4.35) we see that

$$\Delta^2 x(k) + p(k)\Delta x[k-\sigma] = q(k)x^{\gamma}[q(k)] \geq 0 \quad \text{eventually.} \tag{6.4.78}$$

Now $y(k) := \Delta x(k) < 0$ eventually. Then

$$\Delta y(k) + p(k)y[k-\sigma] \geq 0 \quad \text{eventually.} \tag{6.4.79}$$

But in view of Lemma 6.1.6 and condition (6.4.75), inequality (6.4.79) has no eventually negative solution, which is a contradiction.

For (ii), assume $\Delta x(k) > 0$ for $k \geq m_1 \geq m+\sigma$. Summing equation (6.4.35) from m_1 to $k-1$, we obtain

$$\Delta x(k) - \Delta x(m_1) + \sum_{j=m_1}^{k-1} p(j)\Delta x[j-\sigma] = \sum_{j=m_1}^{k-1} q(j)x^{\gamma}[g(j)], \tag{6.4.80}$$

and since

$$\begin{aligned}\sum_{j=m_1}^{k-1} p(j)\Delta x[j-\sigma] &= p(k)x[k-\sigma] - p(m_1)x[m_1-\sigma] - \sum_{j=m_1}^{k-1} x[j+1-\sigma]\Delta p(j)\\ &\leq p(k)x[k-\sigma] \leq p(k)x(k),\end{aligned} \tag{6.4.81}$$

we have

$$\Delta x(k) + p(k)x(k) \geq \sum_{j=m_1}^{k-1} q(j)x^{\gamma}[g(j)] \quad \text{for } k \geq m_1. \tag{6.4.82}$$

Define the sequence $\{r(k)\}$ for $k \geq m \in \mathbb{N}$ by the recurrence relation

$$r(m) > 0, \quad r(k+1) = \frac{r(k)}{1-p(k)} \quad \text{for } k \geq m \in \mathbb{N}. \tag{6.4.83}$$

Next, multiply (6.4.82) by $r(k+1)$ to obtain

$$\Delta\big(r(k)x(k)\big) \geq r(k+1)\sum_{j=m_1}^{k-1} q(j)x^{\gamma}[g(j)] \quad \text{for } k \geq m_1. \tag{6.4.84}$$

Choose $m_2 \geq m_1$ and define $m^* = \max\{m_2, \max_{m_1 \leq k \leq m_2} g(k)\}$. Dividing (6.4.84) by $(r(k+1)x(k+1))^\gamma$ and summing from m_1+1 to m^*, we obtain

$$\begin{aligned}\sum_{j=m_1+1}^{m^*} \frac{\Delta(r(j)x(j))}{(r(j+1)x(j+1))^\gamma} &\geq \sum_{j=m_1+1}^{m^*} (r(j+1))^{1-\gamma} \sum_{i=m_1}^{j-1} q(i)\left(\frac{x[g(i)]}{x(j+1)}\right)^\gamma \\ &\geq \sum_{i=m_1}^{m^*} q(i) \sum_{j=i+1}^{g(i)-1} (r(j+1))^{1-\gamma}\left(\frac{x[g(i)]}{x(j+1)}\right)^\gamma.\end{aligned} \tag{6.4.85}$$

Since $x[g(i)] \geq x(j+1)$ for $i+1 \leq j < g(i)-1$, we have

$$\sum_{j=m_1+1}^{m^*} \frac{\Delta(r(j)x(j))}{(r(j+1)x(j+1))^\gamma} \geq \sum_{i=m_1}^{m^*} q(i)\left[\sum_{j=i+1}^{g(i)-1}\left(\prod_{n\geq m}^{j} \frac{r(m)}{(1-p(n))}\right)^{1-\gamma}\right]. \tag{6.4.86}$$

As in the proof of Theorem 6.4.18, it follows that

$$\sum^{\infty} \frac{\Delta z(j)}{z^\gamma(j+1)} < \infty, \tag{6.4.87}$$

which is a contradiction to condition (6.4.76). This completes the proof. □

Theorem 6.4.21. *Suppose that $\gamma = 1$, $\Delta p(k) \geq 0$, $g(k) \geq k+1$, and $0 < p(k) < 1$ for $k \geq m \in \mathbb{N}$. If condition (6.4.75) holds and*

$$\limsup_{k\to\infty} \sum_{j=k}^{g(k)-1} B[j, g(k)-1]q(j) > 1, \tag{6.4.88}$$

where

$$B[j, g(k)-1] = \sum_{i=j}^{g(k)-1}\left[\prod_{s=i+1}^{g(k)-1}(1-p(s))\right] \quad \text{for } k \leq j \leq g(k)-1, \tag{6.4.89}$$

then equation (6.4.35) is almost oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.4.35), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 6.4.20, we consider the following two cases:

(i) $\Delta x(k) < 0$ eventually,
(ii) $\Delta x(k) > 0$ eventually.

Assume (i), that is, $\Delta x(k) < 0$ eventually. The proof of this case is similar to that of Theorem 6.4.20(i) and hence is omitted.

Now assume (ii), that is, $\Delta x(k) > 0$ for $k \geq m_1 \geq m + \sigma$. We proceed as in the proof of Theorem 6.4.20(ii) and define the sequence $\{r(k)\}$ as in (6.4.83) to obtain

$$\Delta\big(r(s)x(s)\big) \geq r(s+1)\sum_{j=k}^{s-1} q(j)x[g(j)] \quad \text{for } s \geq k \geq m_1. \tag{6.4.90}$$

Summing both sides of (6.4.90) from k to $g(k)-1$, we get

$$\begin{aligned} r[g(k)]x[g(k)] &\geq r[g(k)]x[g(k)] - r(k)x(k) \\ &\geq \sum_{s=k}^{g(k)-1} r(s+1)\sum_{j=k}^{s-1} q(j)x[g(j)], \end{aligned} \tag{6.4.91}$$

so

$$\begin{aligned} 1 &\geq \sum_{s=k}^{g(k)-1}\left(\frac{r(s+1)}{r[g(k)]}\right)\sum_{j=k}^{s-1} q(j)\left(\frac{x[g(j)]}{x[g(k)]}\right) \\ &\geq \sum_{j=k}^{g(k)-1} q(j)\left(\frac{x[g(j)]}{x[g(k)]}\right)\left[\sum_{s=j}^{g(k)-1}\frac{r(s+1)}{r[g(k)]}\right]. \end{aligned} \tag{6.4.92}$$

Since $x[g(j)] \geq x[g(k)]$ for $k \leq j \leq g(k)-1$, we obtain

$$1 \geq \sum_{j=k}^{g(k)-1} q(j)\left[\sum_{s=j}^{g(k)-1}\prod_{i=s+1}^{g(k)-1}\big(1-p(i)\big)\right], \tag{6.4.93}$$

which contradicts condition (6.4.88). This completes the proof. □

The following criterion deals with the almost oscillation of all bounded solutions of equation (6.4.35) for any $\gamma > 0$.

Theorem 6.4.22. *Suppose that $\Delta p(k) \geq 0$, $g(k) \geq k+1$ and $0 < p(k) < 1$ for $k \geq m \in \mathbb{N}$. If condition (6.4.75) holds and*

$$\limsup_{k\to\infty}\sum_{s=m_1\geq m}^{k}\left[\prod_{i=s+1}^{k}\frac{1}{1-p(i)}\right]\sum_{j=m_1}^{s-1} q(j) = \infty, \tag{6.4.94}$$

then every bounded solution $\{x(k)\}$ of equation (6.4.35) is oscillatory or $\{\Delta x(k)\}$ is oscillatory.

Proof. Let $\{x(k)\}$ be a bounded and eventually positive solution of equation (6.4.35), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Proceeding as in the proof of Theorem 6.4.20, we see that case (i) is impossible. Next we assume (ii), that is, $\Delta x(k) > 0$ for $k \geq m_1 \geq m$. There exist a constant $b > 0$ and an integer $m_2 \geq m_1$ such that (6.4.47) holds. As in the proof of Theorem 6.4.20(ii) we obtain (6.4.82) and then define the sequence $\{r(k)\}$ as in (6.4.83) and obtain (6.4.84) which takes the form

$$\Delta(r(k)x(k)) \geq b^{\gamma} r(k+1) \sum_{j=m_2}^{k-1} q(j) \quad \text{for } k \geq m_2. \tag{6.4.95}$$

Summing both sides of (6.4.95) from m_2 to $m^* \geq m_2$, we obtain

$$\begin{aligned} r(m^*+1)x(m^*+1) &\geq r(m^*+1)x(m^*+1) - r(m_2)x(m_2) \\ &\geq b^{\gamma} \sum_{i=m_2}^{m^*} r(i+1) \sum_{j=m_2}^{i-1} q(j), \end{aligned} \tag{6.4.96}$$

so

$$\begin{aligned} x(m^*+1) &\geq b^{\gamma} \sum_{i=m_2}^{m^*} \left(\frac{r(i+1)}{r(m^*+1)}\right) \sum_{j=m_2}^{i-1} q(j) \\ &= b^{\gamma} \sum_{i=m_2}^{m^*} \left[\sum_{s=i+1}^{m^*} \left(\frac{1}{1-p(s)}\right)\right] \sum_{j=m_2}^{i-1} q(j) \\ &\longrightarrow \infty \quad \text{as } m^* \longrightarrow \infty, \end{aligned} \tag{6.4.97}$$

which contradicts the fact that $\{x(k)\}$ is bounded. This completes the proof. □

Next we present two criteria for the almost oscillation of equation (6.4.36) when $0 < \gamma \leq 1$.

Theorem 6.4.23. *Suppose that $\gamma = 1$, $g(k) \leq k$, and $\Delta p(k) \leq 0$ for $k \geq m \in \mathbb{N}$. If*

$$\liminf_{k\to\infty} \sum_{j=k+1}^{k+\sigma-1} p(j) > \left(\frac{\sigma-1}{\sigma}\right)^{\sigma}, \tag{6.4.98}$$

$$\limsup_{k\to\infty} \sum_{j=g(k)}^{k-1} C[g(k), j]q(j) > 1, \tag{6.4.99}$$

where

$$C[g(k), j] = \sum_{s=g(k)}^{j} \left[\prod_{j=g(k)+1}^{s} \left(\frac{1}{1+p(j)}\right)\right] \quad \text{for } g(k) \leq j \leq k-1, \tag{6.4.100}$$

then equation (6.4.36) is almost oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.4.36), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. There are two cases to consider:

(i) $\Delta x(k) > 0$ eventually,

(ii) $\Delta x(k) < 0$ eventually.

For (i), assume $\Delta x(k) > 0$ eventually. From equation (6.4.36) we see that

$$\Delta y(k) - p(k)y[k+\sigma] = q(k)x[g(k)] \geq 0 \quad \text{eventually,} \tag{6.4.101}$$

where $y(k) = \Delta x(k) > 0$ eventually. But in view of Lemma 6.1.7 and condition (6.4.98), inequality (6.4.101) has no eventually positive solution, which is a contradiction.

For (ii), suppose $\Delta x(k) < 0$ for $k \geq m_1 \geq m+1$. Summing equation (6.4.36) from s to $k-1 \geq s \geq m_1$ provides

$$\Delta x(k) - \Delta x(s) = \sum_{j=s}^{k-1} p(j)\Delta x[j+\sigma] + \sum_{j=s}^{k-1} q(j)x[g(j)] \tag{6.4.102}$$

for $k \geq s+1 \geq m_1+1$. Since

$$\sum_{j=s}^{k-1} p(j)\Delta x[j+\sigma] = p(k)x[k+\sigma] - p(s)x[s+\sigma] - \sum_{j=s}^{k-1} x[j+1+\sigma]\Delta p(j) \tag{6.4.103}$$

and $\Delta p(k) \leq 0$ and $x(k)$ is increasing for $k \geq m_1$, we have

$$\sum_{j=s}^{k-1} p(j)\Delta x[j+\sigma] \geq -p(s)x(s) \quad \text{for } k \geq s \geq m_1. \tag{6.4.104}$$

Now (6.4.102) implies

$$-\big(\Delta x(s) - p(s)x(s)\big) \geq \sum_{j=s}^{k-1} q(j)x[g(j)] \quad \text{for } k-1 \geq s \geq m_1. \tag{6.4.105}$$

Define the sequence $\{r(k)\}$ by

$$r(m) > 0, \quad r(k+1) = \frac{r(k)}{1+p(k)} \quad \text{for } k \geq m \in \mathbb{N}. \tag{6.4.106}$$

Multiplying (6.4.105) by $r(s+1)$, we get

$$-\Delta\big(r(s)x(s)\big) \geq r(s+1)\sum_{j=s}^{k-1} q(j)x[g(j)] \quad \text{for } k-1 \geq s \geq m_1. \tag{6.4.107}$$

Summing both sides of (6.4.107) from $g(k)$ to $k-1 \geq g(k)$, we have

$$\begin{aligned} r[g(k)]x[g(k)] &\geq r[g(k)]x[g(k)] - r(k)x(k) \\ &\geq \sum_{s=g(k)}^{k-1} r(s+1) \sum_{j=s}^{k-1} q(j)x[g(j)]. \end{aligned} \tag{6.4.108}$$

Now,

$$\begin{aligned} 1 &\geq \sum_{s=g(k)}^{k-1} \left(\frac{r(s+1)}{r[g(k)]}\right) \sum_{j=s}^{k-1} q(j) \left(\frac{x[g(j)]}{x[g(k)]}\right) \\ &\geq \sum_{j=g(k)}^{k-1} q(j) \left(\frac{x[g(j)]}{x[g(k)]}\right) \left[\sum_{s=g(k)}^{j} \frac{r(s+1)}{r[g(k)]}\right]. \end{aligned} \tag{6.4.109}$$

Since $x[g(j)] \geq x[g(k)]$ for $g(k) \leq j \leq k-1 \leq k$, we see that

$$\begin{aligned} 1 &\geq \sum_{j=g(k)}^{k-1} q(j) \sum_{s=g(k)}^{j} \left(\frac{r(s+1)}{r[g(k)]}\right) \\ &= \sum_{j=g(k)}^{k-1} \left[\sum_{s=g(k)}^{j} \prod_{i=g(k)+1}^{s} \left(\frac{1}{1+p(i)}\right)\right] q(j). \end{aligned} \tag{6.4.110}$$

Taking lim sup on both sides of (6.4.110) as $k \to \infty$, we obtain a contradiction to condition (6.4.99). This completes the proof. □

Theorem 6.4.24. *Suppose that* $0 < \gamma < 1$, $\Delta p(k) \leq 0$, *and* $g(k) < k$ *for* $k \geq m \in \mathbb{N}$, *and let condition (6.4.98) hold. If*

$$\sum_{j=m}^{\infty} A[g(j), j]q(j) = \infty, \tag{6.4.111}$$

where

$$A[g(k), k] = \sum_{j=g(k)}^{k} \left(\frac{1}{1+p(j)}\right) \left[\prod_{s=1}^{j-1} \left(\frac{1}{1+p(s)}\right)\right]^{1-\gamma}, \tag{6.4.112}$$

then equation (6.4.36) is almost oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.4.36), say, $x(k) > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorem 6.4.23, we see that case (i) is impossible. Next, we consider (ii), that is, $\Delta x(k) < 0$ for $k \geq m_1 \geq m+1$. Define the sequence $\{r(k)\}$ as in (6.4.106) and proceed as in the proof of Theorem 6.4.23(ii) to obtain (6.4.107) which takes the form

$$-\Delta(r(s)x(s)) \geq r(s+1)\sum_{j=s}^{k-1} q(j)x^{\gamma}[g(j)] \quad \text{for } k-1 \geq s \geq m_1. \tag{6.4.113}$$

Choose $m^* \geq m_1$ so that $g(s) \geq m_1$ for $s \geq m^*$ and let $n > m^*$ be fixed. We see that

$$-\Delta(r(s)x(s)) \geq r(s+1)\sum_{j=s}^{n} q(j)x^{\gamma}[g(j)] \quad \text{for } n \geq s \geq m_1. \tag{6.4.114}$$

Dividing (6.4.114) by $(r(s)x(s))^{\gamma}$ and summing from m_1 to n, we have whenever $m^* \geq m_1$,

$$\begin{aligned}
\sum_{s=m_1}^{n} -\frac{\Delta(r(s)x(s))}{(r(s)x(s))^{\gamma}} &\geq \sum_{s=m_1}^{n}\left(\frac{r(s+1)}{r^{\gamma}(s)}\right)\sum_{j=s}^{n} q(j)\left(\frac{x[g(j)]}{x(s)}\right)^{\gamma} \\
&= \sum_{s=m_1}^{n}\frac{r^{1-\gamma}(s)}{1+p(s)}\sum_{j=s}^{n} q(j)\left(\frac{x[g(j)]}{x(s)}\right)^{\gamma} \\
&\geq \sum_{j=m^*}^{n} q(j)\sum_{s=g(j)}^{j}\frac{r^{1-\gamma}(s)}{1+p(s)}\left(\frac{x[g(j)]}{x(s)}\right)^{\gamma}.
\end{aligned} \tag{6.4.115}$$

Since $x[g(j)] \geq x(s)$ for $g(j) \leq s \leq j$, $n \geq j \geq m^*$, we get

$$\sum_{s=m_1}^{n} -\frac{\Delta(r(s)x(s))}{(r(s)x(s))^{\gamma}} \geq \sum_{j=m^*}^{n} q(j)\sum_{s=g(j)}^{j}\frac{r^{1-\gamma}(s)}{1+p(s)}. \tag{6.4.116}$$

As in the proof of Theorem 6.1.26 we see that

$$\sum_{s=m_1}^{n} -\frac{\Delta(r(s)x(s))}{(r(s)x(s))^{\gamma}} \quad \text{is bounded below for } n \geq m_1, \tag{6.4.117}$$

which contradicts condition (6.4.111). This completes the proof. □

As an application of Theorems 6.4.21 and 6.4.23, we consider special cases of equations (6.4.35) and (6.4.36), namely, the constant coefficients equations

$$\Delta^2 x(k) + p\Delta x[k-\sigma] - qx[k+\tau] = 0, \tag{6.4.118}$$

$$\Delta^2 x(k) - p\Delta x[k+\sigma] - qx[k-\tau] = 0, \tag{6.4.119}$$

where $\sigma, \tau \in \mathbb{N}$, and p and q are positive real numbers. We state the following two almost oscillation results for equations (6.4.118) and (6.4.119).

Corollary 6.4.25. *Let $\tau \geq 1$ and $0 < p < 1$. If*

$$p > \frac{\sigma^\sigma}{(1+\sigma)^{1+\sigma}}, \qquad \left(\frac{q}{p}\right)\left[\tau + \frac{1-p}{p}\left((1-p)^\tau - 1\right)\right] > 1, \tag{6.4.120}$$

then equation (6.4.118) is almost oscillatory.

Corollary 6.4.26. *If*

$$p > \frac{(\sigma-1)^{\sigma-1}}{\sigma^\sigma}, \qquad \left(\frac{q}{p}\right)(1+p)\left[\tau - \frac{1}{p}\left(1 - \frac{1}{(1+p)^\tau}\right)\right] > 1, \tag{6.4.121}$$

then equation (6.4.119) is almost oscillatory.

Example 6.4.27. As an illustration, we see that the damped difference equations

$$\begin{aligned} &\Delta^2 x(k) + \frac{1}{2}\Delta x[k-3] - q_1 x[k+3] = 0, \\ &\Delta^2 x(k) - \Delta x[k+4] - q_2 x[k-4] = 0 \end{aligned} \tag{6.4.122}$$

are almost oscillatory if $q_1 > 4/17$ and $q_2 > 8/49$. This follows by Corollaries 6.4.25 and 6.4.26, respectively.

Remark 6.4.28. If we let $p(k) \equiv 0$ in Theorems 6.4.20–6.4.24, then the remaining conditions in these results are not enough to describe the oscillatory character of the equation

$$\Delta^2 x(k) - q(k)\left|x[g(k)]\right|^\gamma \operatorname{sgn} x[g(k)] = 0 \quad \text{with } \gamma > 0, \tag{6.4.123}$$

and hence Theorems 6.4.20–6.4.24 are not applicable to equation (6.4.123).

6.5. Oscillation of forced difference equations

In this section we will be concerned with the oscillatory behavior of forced difference equations of the form

$$\Delta\big(c(k)\Delta x(k)\big) + q(k) f\big(x[g(k)]\big) = e(k), \tag{6.5.1}$$

where $\{c(k)\}$, $\{g(k)\}$, $\{q(k)\}$, and f are as in equation (6.1.1), $\{e(k)\}$ is a sequence of real numbers, and $\{c(k)\}$ satisfies condition (6.1.3).

We will assume the following hypothesis on the forcing term:

$$\begin{aligned} &\text{there exists a sequence } \{\eta(k)\} \text{ of real numbers such that} \\ &\Delta\big(c(k)\Delta\eta(k)\big) = e(k) \text{ and that } \{\eta(k)\} \text{ is oscillatory.} \end{aligned} \tag{6.5.2}$$

Let

$$x(k) = y(k) + \eta(k). \tag{6.5.3}$$

Then equation (6.5.1) can be rewritten as a homogeneous equation

$$\Delta\big(c(k)\Delta y(k)\big) + q(k)f\big(y[g(k)] + \eta[g(k)]\big) = 0. \tag{6.5.4}$$

To prove that equation (6.5.1) is oscillatory, it suffices to assume the existence of an eventually positive solution $x(k)$ and deduce a contradiction by applying the various hypotheses to equation (6.5.4). Suppose $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Since $q(k) \geq 0$ eventually, from equation (6.5.4) we note that $\Delta(c(k)\Delta y(k)) \leq 0$ eventually. We claim that $\Delta y(k) \geq 0$ eventually. If not, then $\Delta y(m_1) < 0$ for some $m_1 \geq m$. Since $\Delta(c(k)\Delta y(k)) \leq 0$, we see that $c(k)\Delta y(k) \leq c(m_1)\Delta y(m_1) < 0$ for all $k \geq m_1$. Hence, by condition (6.1.3), we see that $y(k) \to -\infty$ as $k \to \infty$, but this together with $\eta(k)$ being oscillatory contradicts the assumption that $x(k) > 0$ for $k \geq m_1$. In fact, one can easily see that $\Delta y(k) > 0$ for $k \geq m_1$.

Next we will show that $y(k) > 0$ eventually. If not, then $y(k) \leq 0$ eventually. From (6.5.3), we find $x(k) + \eta(k) \leq 0$ eventually, and hence $0 < x(k) \leq -\eta(k)$ eventually, which contradicts the fact that $\{\eta(k)\}$ is oscillatory. Hence we must have $y(k) > 0$ eventually. Thus, for simplicity, we conclude that

$$y(k) > 0, \qquad \Delta y(k) > 0, \qquad \Delta\big(c(k)\Delta y(k)\big) \leq 0 \quad \text{eventually.} \tag{6.5.5}$$

Next, from (6.5.5) one can easily see that there exist a constant $d^* > 0$ and an integer $m^* \geq m$ such that $c(k)\Delta y(k) \leq d^*$ for $k \geq m^*$, or

$$\Delta y(k) \leq \frac{d^*}{c(k)} \quad \text{for } k \geq m^*. \tag{6.5.6}$$

Summing (6.5.6) from m^* to $k-1$, we have

$$y(k) \leq y(m^*) + d^* \sum_{j=m^*}^{k-1} \frac{1}{c(j)}. \tag{6.5.7}$$

Now there exist a constant $d > 0$ and an integer $M \geq m^*$ such that

$$y(k) \leq dC(k-1, m^*) \quad \text{for } k \geq M, \tag{6.5.8}$$

where C is as in (6.1.8).

We will repeatedly use the conclusions (6.5.5) and (6.5.8) in proving some various results in this section.

Theorem 6.5.1. *Assume that conditions (6.1.3) and (6.5.2) hold and that $\{\eta(k)\}$ satisfies in addition*

$$\liminf_{k\to\infty} \frac{\eta(k)}{C(k-1,m)} = -\infty, \qquad \limsup_{k\to\infty} \frac{\eta(k)}{C(k-1,m)} = \infty \tag{6.5.9}$$

for $k \geq m+1$ for some $m \in \mathbb{N}$, where C is given in (6.1.8). Then equation (6.5.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.5.1). Let $x(k)$ be as in (6.5.3) and obtain equation (6.5.4). Now, under the given hypotheses, we find that (6.5.5) holds for $k \geq m$ for some $m \in \mathbb{N}$ and that there exists a constant $d > 0$ such that

$$\limsup_{k\to\infty} \frac{y(k)}{C(k-1,m)} \leq d. \tag{6.5.10}$$

On the other hand, from (6.5.3) we have that $y(k)+\eta(k) = x(k) > 0$, and therefore $y(k) > -\eta(k)$ eventually. Thus,

$$\begin{aligned} \limsup_{k\to\infty}\left[\frac{y(k)}{C(k-1,m)}\right] &> \limsup_{k\to\infty}\left[-\frac{\eta(k)}{C(k-1,m)}\right] \\ &= -\liminf_{k\to\infty}\left[\frac{\eta(k)}{C(k-1,m)}\right] \\ &= \infty, \end{aligned} \tag{6.5.11}$$

which contradicts (6.5.10). The first part of condition (6.5.9) is required when we assume the nonoscillatory solution $\{x(k)\}$ to be eventually negative. Here we omit the details. □

Example 6.5.2. Consider the forced difference equation

$$\Delta^2 x(k) + q(k)f(x[g(k)]) = 4(-1)^k(k+1)^2 \quad \text{for } k \in \mathbb{N}, \tag{6.5.12}$$

where g, q, and f are as in (6.5.1), $c(k) \equiv 1$, and $e(k) = 4(-1)^k(k+1)^2$. It is easy to check that there exists a sequence $\{\eta(k)\}$ with $\eta(k) = (-1)^k k^2$ for $k \in \mathbb{N}$ such that $\Delta^2\eta(k) = 4(-1)^k(k+1)^2$ and $\{\eta(k)\}$ is oscillatory. Clearly, $C(k-1,m) = O(k)$ as $k \to \infty$ for $m \in \mathbb{N}$. Now the hypotheses of Theorem 6.5.1 are satisfied, and hence we conclude that equation (6.5.12) is oscillatory.

We note that the forcing term in equation (6.5.12) generates oscillation even if the associated unforced equation, namely,

$$\Delta^2 x(k) + q(k)f(x[g(k)]) = 0 \tag{6.5.13}$$

is nonoscillatory.

Theorem 6.5.3. *Assume that conditions (6.1.3) and (6.5.2) hold. Moreover, assume that for every constant $\lambda > 0$,*

$$\begin{aligned} &\limsup_{k\to\infty} \sum_{j=m\in\mathbb{N}}^{k} q(j)f(\lambda + \eta[g(j)]) = \infty, \\ &\liminf_{k\to\infty} \sum_{j=m}^{k} q(j)f(-\lambda + \eta[g(j)]) = -\infty. \end{aligned} \tag{6.5.14}$$

Then equation (6.5.1) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (6.5.1). Set $x(k)$ as in (6.5.3) and obtain the equation (6.5.4). Next, we can have (6.5.5). Since $\Delta y(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$, there exist a constant $b > 0$ and an integer $m_1 \geq m$ such that

$$y[g(k)] \geq b \quad \text{for } k \geq m_1. \tag{6.5.15}$$

Summing equation (6.5.4) from m_1 to $k-1 \geq m_1$, we have

$$c(k)\Delta y(k) = c(m_1)\Delta y(m_1) - \sum_{j=m_1}^{k-1} q(j)f(y[g(j)] + \eta[g(j)]). \tag{6.5.16}$$

Using in (6.5.16) the fact that f is nondecreasing, we get

$$c(k)\Delta y(k) \leq c(m_1)\Delta y(m_1) - \sum_{j=m_1}^{k-1} q(j)f(b + \eta[g(j)]). \tag{6.5.17}$$

This easily implies that $\liminf_{k\to\infty} c(k)\Delta y(k) = -\infty$, which is a contradiction to (6.5.5). A similar situation appears in the case when $x(k) < 0$ eventually. □

Theorem 6.5.4. *Assume that conditions (6.1.3) and (6.5.2) hold and in addition* $\{\eta(k)\}$ *satisfies*

$$\sum^{\infty} q(j)f(\eta^{+}[g(j)]) = \infty, \tag{6.5.18}$$

$$\sum^{\infty} q(j)f(\eta^{-}[g(j)]) = \infty, \tag{6.5.19}$$

where $\eta^{+}(k) = \max\{\eta(k), 0\}$ *and* $\eta^{-}(k) = \max\{-\eta(k), 0\}$. *Then equation (6.5.1) is oscillatory.*

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.5.1). As in the proof of Theorem 6.5.3, we set $x(k)$ as in (6.5.3) and obtain equation (6.5.4) and (6.5.5). Also, we obtain the equality (6.5.16). Clearly, $\lim_{k\to\infty} c(k)\Delta y(k)$ exists and is finite, and hence the sum in (6.5.16) converges as $t \to \infty$.

We note that for all $k \geq m_1$ for some $m_1 \in \mathbb{N}$, $y(k) + \eta(k) > \eta^{+}(k)$. To see this, we write $y(k) + \eta(k) = y(k) + \eta^{+}(k) - \eta^{-}(k)$ and observe that

(i) for $\eta^{+}(k) = 0$, $y(k) + \eta(k) = y(k) - \eta^{-}(k) = x(k) > 0 = \eta^{+}(k)$,

(ii) for $\eta^{-}(k) = 0$, $y(k) + \eta(k) = y(k) + \eta^{+}(k) > \eta^{+}(k)$ (since $y(k) > 0$).

Since f is nondecreasing, we have that $f(y[g(k)] + \eta[g(k)]) \geq f(\eta^{+}[g(k)])$. With the fact that $q(k) \geq 0$ eventually, we can estimate as follows:

$$\sum_{j=m_1}^{k-1} q(j)f(\eta^{+}[g(j)]) \leq \sum_{j=m_1}^{k-1} q(j)f(y[g(j)] + \eta[g(j)]) < \infty. \tag{6.5.20}$$

By applying condition (6.5.18), we obtain the desired contradiction. □

Theorem 6.5.5. *Assume that conditions (6.1.3), (6.1.112), and (6.5.2) hold, and* $\{\eta(k)\}$ *satisfies*

$$\left|\eta(k)\right| \leq a, \quad \text{where } a > 0 \text{ is a constant, and } \lim_{k\to\infty} \eta(k) \text{ does not exist.} \tag{6.5.21}$$

Then the difference of every solution of equation (6.5.1) is oscillatory. Furthermore, all unbounded solutions of equation (6.5.1) are oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (6.5.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. We first show that if $\{x(k)\}$ is unbounded, then it must be oscillatory. Following the same argument as before, we may assume that (6.5.5) holds for $k \geq m$. By condition (6.5.21), $|\eta(k)| \leq a$. Suppose that $\{x(k)\}$ is unbounded. Then $\{y(k)\}$ must also be unbounded. Otherwise, $x(k) = y(k) + \eta(k)$ becomes bounded. Let $m_1 \geq m$ satisfy $y[g(m_1)] = 2a$, then $y[g(k)] \geq 2a$ for all $k \geq m_1$. Now we have $y[g(k)] + \eta[g(k)] \geq 2a - \eta^-[g(k)] \geq a$ and hence $f(y[g(k)] + \eta[g(k)]) \geq f(a)$ for all $k \geq m_1$, which upon substituting in (6.5.16) and using condition (6.1.112), we obtain the desired contradiction.

We now suppose that $\{x(k)\}$ is bounded and show that $\{\Delta x(k)\}$ must be oscillatory. Since $\{\eta(k)\}$ is bounded, so is $\{y(k)\}$. Note that $\Delta(c(k)\Delta y(k)) \leq 0$ implies that $c(k)\Delta y(k) \to 0$ and $y(k) \to b$ as $k \to \infty$, where b is some positive constant. If $\{\Delta x(k)\}$ is eventually of one sign, it cannot be $\Delta x(k) < 0$ because $\Delta y(k)+\Delta\eta(k) < 0$ would contradict $\Delta y(k) > 0$ when we set k equal to any zero of $\Delta\eta(k)$. On the other hand if $\Delta x(k) > 0$, then $\lim_{k\to\infty} x(k) = b_1$ for some constant b_1. Hence $\eta(k) = x(k) - y(k) \to b_1 - b$ as $k \to \infty$. This clearly contradicts condition (6.5.21). The proof is now complete. □

Next we will compare the oscillatory behavior of the equation

$$\Delta\big(c(k)\Delta u(k)\big) + F\big(k, u[g(k)]\big) = 0 \tag{6.5.22}$$

with that of the damped-forced equation

$$\Delta\big(c(k)\Delta x(k)\big) + p(k)\Delta x(k) + F\big(k, x[g(k)]\big) = e(k), \tag{6.5.23}$$

where $\{c(k)\}$, $\{g(k)\}$, and F are as in equation (6.2.21), $\{p(k)\}$ is as in equation (6.4.1), and $\{e(k)\}$ is as in equation (6.5.1).

We assume the following hypothesis on the forcing term:

$$\begin{gathered}\text{there exists a sequence } \{\eta(k)\} \text{ of real numbers such that}\\ \Delta\big(c(k)\Delta\eta(k)\big) + p(k)\Delta\eta(k) = e(k), \quad \{\eta(k)\} \text{ is oscillatory,}\\ \text{and } \lim_{k\to\infty}\eta(k) = 0.\end{gathered} \tag{6.5.24}$$

Theorem 6.5.6. *Assume that condition (6.5.24) holds,*

$$c(k) > p(k) \quad \textit{for } k \geq m \in \mathbb{N}, \qquad \sum_{i=m}^{\infty} \frac{1}{c(i)} \prod_{j=m}^{i-1} \left(1 - \frac{p(j)}{c(j)}\right) = \infty. \tag{6.5.25}$$

If equation (6.5.22) is oscillatory, then equation (6.5.23) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.5.23), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Let $x(k)$ be as in (6.5.3). Then equation (6.5.23) can be rewritten as the homogeneous equation

$$\Delta\big(c(k)\Delta y(k)\big) + p(k)\Delta y(k) + F\big(k, y[g(k)] + \eta[g(k)]\big) = 0. \tag{6.5.26}$$

As before, condition (6.5.2) implies that $\Delta y(k) > 0$ eventually, and hence we have

$$\Delta\big(c(k)\Delta y(k)\big) + F\big(k, y[g(k)] + \eta[g(k)]\big) \leq 0 \quad \text{eventually.} \tag{6.5.27}$$

Since $y(k) > 0$ is eventually increasing, there exists $\varepsilon > 0$ such that $y[g(k)] - \varepsilon > 0$ for $k \geq m_1$ for some $m_1 \geq m$. Since $\lim_{k\to\infty} \eta(k) = 0$, there exists an integer $m_2 \geq m_1$ such that $|\eta[g(k)]| < \varepsilon$ for $k \geq m_2$. It follows that

$$y[g(k)] + \eta[g(k)] > y[g(k)] - \varepsilon > 0 \quad \text{for } k \geq m_2. \tag{6.5.28}$$

Using (6.5.28) in inequality (6.5.27), we have

$$\Delta\big(c(k)\Delta y(k)\big) + F\big(k, y[g(k)] - \varepsilon\big) \leq 0 \quad \text{for } k \geq m_2. \tag{6.5.29}$$

Letting $v(k) = y(k) - \varepsilon > 0$ for $k \geq m_2$ in (6.5.29), we get

$$\Delta\big(c(k)\Delta v(k)\big) + F\big(k, v[g(k)]\big) \leq 0 \quad \text{for } k \geq m_2. \tag{6.5.30}$$

By Lemma 6.2.4 (with $\Psi(x) = x$), we see that equation (6.5.22) also has an eventually positive solution, which is a contradiction. The proof is now complete. □

Example 6.5.7. Consider the damped-forced difference equation

$$\begin{aligned}\Delta\big((k+1)\Delta x(k)\big) + \frac{1}{k+1}\Delta x(k) + q(k)\,|\,x[g(k)]\,|^{\gamma}\operatorname{sgn} x[g(k)] \\ = (-1)^k\left[\frac{4k^2+6k+1}{k(k+1)} - \frac{2k+1}{k(k+1)^2}\right] \quad \text{for } k \in \mathbb{N},\end{aligned} \tag{6.5.31}$$

where $\gamma > 0$ is a constant, $\{q(k)\}$ is a sequence of positive real numbers, and $\{g(k)\}$ is as in equation (6.5.1). Here we take $c(k) = k+1$ and $p(k) = 1/(k+1)$. It is easy to see that condition (6.5.25) is satisfied and that there exists a sequence $\{\eta(k)\}$ with $\eta(k) = (-1)^k/k$ satisfying condition (6.5.24). Now, by Theorem 6.5.9, we see that equation (6.5.31) is oscillatory if the associated undamped-unforced equation

$$\Delta\big((k+1)\Delta u(k)\big) + q(k)\,|\,u[g(k)]\,|^{\gamma}\operatorname{sgn} u[g(k)] = 0 \tag{6.5.32}$$

is oscillatory.

Finally, we will provide sufficient conditions which ensure that all nonoscillatory solutions of a special case of equation (6.5.1), namely, the equation

$$\Delta^2 x(k-1) + q(k)x[g(k)] = e(k), \tag{6.5.33}$$

tend to zero as $k \to \infty$. We will need the following lemma.

Lemma 6.5.8. *Consider the nonhomogeneous problem*

$$\Delta u(k) - \frac{1}{k}u(k) + \frac{1}{k}\phi(k) = 0 \quad \textit{for } k \geq m \in \mathbb{N},\ u(m) = 0, \tag{6.5.34}$$

where $\{\phi(k)\}$ is a sequence of nonzero real numbers. Then

$$u(k) = -k\sum_{j=m}^{k-1}\frac{\phi(j)}{j(j+1)} \quad \textit{for } k \geq m+1. \tag{6.5.35}$$

Proof. We rewrite the equation in (6.5.34) as

$$u(k+1) - \left(1+\frac{1}{k}\right)u(k) + \frac{1}{k}\phi(k) = 0 \quad \text{for } k \geq m. \tag{6.5.36}$$

The corresponding homogeneous linear difference equation is

$$w(k+1) = \left(1+\frac{1}{k}\right)w(k) \quad \text{for } k \geq m. \tag{6.5.37}$$

Equation (6.5.37) has a solution of the form

$$w(k) = \left(1+\frac{1}{k-1}\right)\left(1+\frac{1}{k-2}\right)\cdots\left(1+\frac{1}{m}\right)w(m) = \frac{k}{m}w(m). \tag{6.5.38}$$

It is well known that

$$u(k) = -w(k)\sum_{j=m}^{k-1}\frac{\phi(j)}{jw(j+1)} = -k\sum_{j=m}^{k-1}\frac{\phi(j)}{j(j+1)} \quad \text{for } k \geq m+1 \tag{6.5.39}$$

is a solution of equation (6.5.34). This completes the proof. □

Now we present the following result.

Theorem 6.5.9. *Suppose*

$$\sum_{j=m\in\mathbb{N}}^{\infty} jq(j) = \infty, \tag{6.5.40}$$

$$\sum_{j=m}^{\infty} j\left|e(j)\right| < \infty. \tag{6.5.41}$$

Then all nonoscillatory solutions of equation (6.5.33) tend to zero as $k \to \infty$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (6.5.33), say, $x(k) > 0$ for $k \geq m-1$ for some $m \in \mathbb{N}$. From equation (6.5.33) it follows that

$$\sum_{j=m}^{k} j\Delta^2 x(j-1) + \sum_{j=m}^{k} jq(j)x[g(j)] = \sum_{j=m}^{k} je(j). \tag{6.5.42}$$

Now we consider the following two cases:

(i) $\sum_{j=m}^{\infty} jq(j)x[g(j)] = \infty$,

(ii) $\sum_{j=m}^{\infty} jq(j)x[g(j)] < \infty$.

For (i), assume that $\sum_{j=m}^{k} jq(j)x[g(j)] \to \infty$ as $k \to \infty$, which implies that

$$\sum_{j=k}^{k} j\Delta^2 x(j-1) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{6.5.43}$$

since condition (6.5.41) holds. Now one can easily see that

$$\sum_{j=m}^{k} j\Delta^2 x(j-1) = k\Delta x(k) - (m-1)\Delta x(m-1) - \sum_{j=m}^{k} \Delta x(j-1). \tag{6.5.44}$$

Set $u(k) = x(k) - x(m)$. Then $u(m) = 0$, $\Delta u(k) = \Delta x(k)$, and thus

$$\sum_{j=m}^{k} j\Delta^2 x(j-1) = k\Delta u(k) - u(k) - m\Delta x(m-1). \tag{6.5.45}$$

Now we let

$$\phi(k) = -\sum_{j=m}^{k} j\Delta^2 x(j-1) - m\Delta x(m-1). \tag{6.5.46}$$

Then (6.5.45) becomes

$$\Delta u(k) - \frac{1}{k}u(k) + \frac{1}{k}\phi(k) = 0 \tag{6.5.47}$$

with $u(m) = 0$ and $\phi(k) \to \infty$ as $k \to \infty$. Without loss of generality, we assume that $\phi(j) \geq 1$ for $j \geq m$. By (6.5.35) we have

$$u(k) = -k\sum_{j=m}^{k-1} \frac{\phi(j)}{j(j+1)} \leq -k\sum_{j=m}^{k-1} \frac{1}{j(j+1)} = -k\left(\frac{1}{m} - \frac{1}{k}\right) = 1 - \frac{k}{m}. \tag{6.5.48}$$

Hence $u(k) \to -\infty$ as $k \to \infty$, that is, $x(k) \to -\infty$ as $k \to \infty$, which is a contradiction.

For (ii), assume that

$$\sum_{j=m}^{\infty} jq(j)x[g(j)] < \infty. \tag{6.5.49}$$

From equation (6.5.42), we have $\sum_{j=m}^{\infty} j\Delta^2 x(j-1) < \infty$. As in the proof of (i) above, we see that $\lim_{k\to\infty} \phi(k)$ exists and is finite. By (6.5.35),

$$u(k) = -k\sum_{j=m}^{k-1} \frac{\phi(j)}{j(j+1)}, \tag{6.5.50}$$

and hence $\lim_{k\to\infty} u(k)$ exists and is either infinite or finite.

If $\lim_{k\to\infty} u(k) = -\infty$, then $\lim_{k\to\infty} x(k) = -\infty$, which is a contradiction. If $\lim_{k\to\infty} u(k) = \infty$, then $\lim_{k\to\infty} x(k) = \infty$. By (6.5.49), we have $\sum_{j=m}^{\infty} jq(j) < \infty$, which contradicts condition (6.5.40). If $\lim_{k\to\infty} u(k) = a$ and a is a finite number, then $x(k) \to b \geq 0$ as $k \to \infty$, where b is a constant. We claim that $b = 0$. If $b > 0$, then there exists an integer $m_1 \geq m$ such that $x[g(k)] \geq b/2$ for all $k \geq m_1$. Thus,

$$\infty > \sum_{j=m_1}^{\infty} jq(j)x[g(k)] \geq \left(\frac{b}{2}\right)\sum_{j=m_1}^{\infty} jq(j), \tag{6.5.51}$$

which again contradicts condition (6.5.40). Therefore $b = 0$ which proves the theorem. □

Example 6.5.10. Consider the forced difference equation

$$\Delta^2 x(k-1) + \frac{1}{k^2} x[g(k)] = \frac{1}{k^2 g(k)} + \frac{2}{(k-1)k(k+1)}, \tag{6.5.52}$$

where $\{g(k)\}$ is a sequence of positive integers with $\lim_{k\to\infty} g(k) = \infty$ for $k > 2$. It is easy to check that the assumptions of Theorem 6.5.9 are satisfied. Therefore, all nonoscillatory solutions of equation (6.5.52) tend to zero as $k \to \infty$. In fact, $\{x(k)\}$ with $x(k) = 1/k$ is such a solution of equation (6.5.52).

6.6. Notes and general discussions

(1) Lemmas 6.1.1–6.1.4 and Theorems 6.1.9–6.1.12 are extensions of the results due to Zhang and Zhang [293], while Lemma 6.5.8 and Theorems 6.1.13 and 6.1.14 are generalizations of the results due to Zhang and Li [292]. Lemmas 6.1.6 and 6.1.7 are taken from Philos [228], and Lemma 6.1.8 is due to Ladas and Qian [192]. Theorems 6.1.16–6.1.26 are extensions of results obtained by Grace and Lalli [146, 147, 195] and Szafrański and Szmanda [260], while Theorems 6.1.27 and 6.1.28 are related to Wang and Yu [273]. Lemmas 6.1.30 and 6.1.31 and Theorem 6.1.32 are due to Zhang and Cheng [288]. Theorems 6.1.34 and 6.1.35 are extracted from Agarwal and Wong [25]. Parts of Theorems 6.1.40–6.1.44 are extracted from Agarwal and Wong [25], while other parts are new.

(2) Lemma 6.2.1 can be found in [153], and Lemma 6.2.2 is extracted from Ladas and Qian [192]. Theorem 6.2.3 is due to Agarwal et al. [22]. Lemma 6.2.4 and Theorems 6.2.5–6.2.7 are the discrete analogues of some special cases of results due to Kartsatos and Toro [170, 172] and Mahfoud [202]. These results are also related to those of Thandapani et al. [271]. Theorems 6.2.9 and 6.2.10 are consequences of results given in Section 3.7. Theorems 6.2.12 and 6.2.14 are new and extend those of Agarwal et al. [22]. Parts of Theorems 6.2.15–6.2.19 are extensions of some of the results given in [25], while the other parts are new. Theorems 6.1.32–6.1.42 are generalizations of results due to Agarwal and Wong [25]. Lemma 6.2.30 is due to Mahfoud [201] and Theorem 6.2.31 is new.

(3) The results of Section 6.3 are extracted from those obtained by Philos [230] and Philos and Sficas [234].

(4) Theorem 6.4.1 is a consequence of a result given in Section 3.9. Theorems 6.4.2–6.4.6 are taken from Grace and El-Morshedy [144], while Theorems 6.4.11–6.4.18 are due to Grace [138] and also Agarwal et al. [19]. Theorems 6.4.20–6.4.24 are taken from Grace [139]. It would be interesting to obtain oscillation criteria for the equations (6.4.33)–(6.4.36) rather than almost oscillation. We believe that the assumptions imposed are enough to obtain such claims for the considered equations.

(5) Theorems 6.5.1–6.5.5 are taken from Grace and Lalli [146, 195]. These criteria are the discrete analogues of the results due to Wong [281] and Kartsatos and Manougian [171]. Theorem 6.5.6 is new and in fact, it is a discrete analogue of a special case of a result due to Kosmala [181]. Theorem 6.5.9 when $g(k) = k$ is due to Zhang [289]. It would be interesting to obtain similar oscillation criteria for the more general equation

$$\Delta\big(c(k)\Psi(\Delta x(k))\big) + q(k)f\big(x[g(k)]\big) = e(k), \tag{6.6.1}$$

where c, g, q, f, and Ψ are as in equation (6.2.1) and e is as in equation (6.5.1).

7 Oscillation theory for neutral difference equations

It is known that rearrangement of terms of some recurrence relations may produce so-called difference equations. If the difference equations take the form of the discrete counterpart of neutral differential equations, then the related equations are called neutral difference equations.

It is a well-known fact that there is a similarity between the qualitative theories (including oscillation theory) of neutral differential equations and neutral difference equations. Therefore, the purpose of this chapter is to investigate some qualitative properties of certain neutral difference equations.

In Section 7.1 we establish some oscillation criteria for neutral second-order difference equations with and without forcing term via comparison with some oscillatory equations of the same order. Nonoscillation criteria for neutral equations are given in Section 7.2 while nonoscillation results for neutral equations with positive and negative coefficients are presented in Section 7.3.

Section 7.4 is devoted to the study of the classification of nonoscillatory solutions of certain neutral difference equations of second order. More criteria for the oscillation of certain nonlinear neutral difference equations are presented in Section 7.5. In Section 7.6 we establish oscillation criteria for neutral equations of mixed type with constant coefficients as well as periodic coefficients. Also, we present oscillation criteria for neutral equations of mixed type via their associated characteristic equations.

7.1. Oscillation criteria via comparison

In this section we will consider neutral difference equations of the form

$$\Delta\big(c(k-1)\Delta[x(k-1)+p(k)x[\sigma(k)]]\big)+q(k)f\big(x[g(k)]\big)=0, \tag{7.1.1}$$

$$\Delta\big(c(k-1)\Psi(\Delta[x(k-1)+p(k)x[\sigma(k)]])\big)+q(k)f\big(x[g(k)]\big)=0, \tag{7.1.2}$$

where

(i) $\{c(k)\}$ is a sequence of positive real numbers,

(ii) $\{p(k)\}$ and $\{q(k)\}$ are sequences of eventually nonnegative real numbers,

(iii) $\{g(k)\}$ and $\{\sigma(k)\}$ are sequences of integers with $\lim_{k\to\infty} g(k) = \infty$ and $\lim_{k\to\infty} \sigma(k) = \infty$,

(iv) $f \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$ for $x \neq 0$,

(v) $\Psi \in C(\mathbb{R}, \mathbb{R})$ and satisfies either one of the following conditions:

  (i$_1$) $\Psi(x) = |x|^{\alpha-1}x$ with $\alpha \geq 1$,

  (i$_2$) $\Psi(x) = x^\alpha$, where $\alpha \geq 1$ is the ratio of two positive odd integers.

By a solution of (7.1.1) (or (7.1.2)) we mean a sequence $\{x(k)\}$ which satisfies (7.1.1) (or (7.1.2)) for all $k \geq \min\{0, \inf_{j\geq 0}\sigma(j), \inf_{j\geq 0} g(j)\}$.

First we will compare the oscillatory property of equation (7.1.1) with that of some related nonneutral difference equations. This enables us to apply the results of Chapter 6. Throughout this section, we will consider $f \in C(\mathbb{R})$ and hence, by Lemma 6.2.30, f has a pair of continuous components G and H. Therefore, we will impose the following condition on f.

Suppose $f \in C(\mathbb{R}_n)$ for $n \geq 0$ and let G and H be a pair of continuous components of f with H being the nondecreasing one, that is,

$$f(x) = G(x)H(x) \quad \forall x \in \mathbb{R}_n. \tag{7.1.3}$$

Now we present the following results.

Theorem 7.1.1. *Suppose that condition (7.1.3) holds,*

$$\sum^{\infty} \frac{1}{c(j)} = \infty, \tag{7.1.4}$$

$$0 \leq p(k) < 1 \quad \textit{for } k \geq m \in \mathbb{N}, \tag{7.1.5}$$

$$\sigma(k) \leq k \quad \textit{for } k \geq m, \tag{7.1.6}$$

$$\tau(k) = \min\{k, g(k)\}, \quad \Delta\tau(k) \geq 0 \quad \textit{for } k \geq m, \qquad \lim_{k\to\infty} \tau(k) = \infty, \tag{7.1.7}$$

$$-H(-xy) \geq H(xy) \geq H(x)H(y) \quad \textit{for } x, y > 0. \tag{7.1.8}$$

If for every constant $a \geq 1$ and all large integers m^* with $\tau(k) > m + 1$ for $k \geq m^*$ the equation

$$\begin{aligned}\Delta\big(&c(k-1)\Delta y(k-1)\big)\\ &+ q(k)G\big(aC(g(k)-1, m)\big)H\big(1 - p[g(k)+1]\big)H\big(y[\tau(k)]\big) = 0\end{aligned} \tag{7.1.9}$$

is oscillatory, where

$$C(k, \ell) = \sum_{j=\ell}^{k} \frac{1}{c(j)} \quad \textit{for } k \geq \ell \geq m, \tag{7.1.10}$$

then equation (7.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.1), say, $x(k) > 0$, $x[\sigma(k)] > 0$, and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define

$$y(k-1) = x(k-1) + p(k)x[\sigma(k)]. \tag{7.1.11}$$

Then equation (7.1.1) takes the form

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)f\big(x[g(k)]\big) = 0. \tag{7.1.12}$$

As in most of the results of Chapter 6, one can easily see that

$$y(k) > 0, \quad \Delta y(k) > 0, \quad \Delta\big(c(k-1)\Delta y(k-1)\big) \leq 0 \quad \text{eventually.} \tag{7.1.13}$$

Now, there exist a constant $b \geq 1$ and an integer $m_1 \geq m$ such that $g(k) \geq m+1$ for $k \geq m_1$ and

$$x[g(k)] \leq y[g(k)] \leq b \sum_{j=m}^{g(k)-1} \frac{1}{c(j)} = bC(g(k)-1, m) \quad \text{for } k \geq m_1. \tag{7.1.14}$$

Using conditions (7.1.5)–(7.1.7) and (7.1.13) in (7.1.11), we obtain

$$\begin{aligned} x[k-1] &= y[k-1] - p(k)x[\sigma(k)] \\ &= y[k-1] - p(k)\big[y[\sigma(k)-1] - p[\sigma(k)]x[\sigma\circ\sigma(k)]\big] \\ &\geq y[k-1] - p(k)y[\sigma(k)-1] \\ &\geq \big(1-p(k)\big)y[k-1], \end{aligned} \tag{7.1.15}$$

that is,

$$x[k-1] \geq \big(1-p(k)\big)y[k-1] \quad \text{for } k \geq m_1. \tag{7.1.16}$$

There exists an integer $m_2 \geq m_1$ such that

$$\begin{aligned} x[g(k)] &\geq \big(1-p[g(k)+1]\big)y[g(k)] \\ &\geq \big(1-p[g(k)+1]\big)y[\tau(k)] \quad \text{for } k \geq m_2. \end{aligned} \tag{7.1.17}$$

Using (7.1.3), (7.1.8), (7.1.14), and (7.1.17) in (7.1.12), we have for $k \geq m_2$,

$$\begin{aligned} 0 &= \Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)G\big(x[g(k)]\big)H\big(x[g(k)]\big) \\ &\geq \Delta\big(c(k-1)\Delta y(k-1)\big) \\ &\quad + q(k)G\big(bC(g(k)-1,m)\big)H\big(\big(1-p[\tau(k)+1]\big)y[\tau(k)]\big) \\ &\geq \Delta\big(c(k-1)\Delta y(k-1)\big) \\ &\quad + q(k)G\big(bC(g(k)-1,m)\big)H\big(1-p[\tau(k)+1]\big)H\big(y[\tau(k)]\big). \end{aligned} \tag{7.1.18}$$

By applying Lemma 6.2.4 with $\Psi(x) = x$, we conclude that the equation

$$\Delta\big(c(k-1)\Delta y(k-1)\big)+q(k)G\big(bC(g(k)-1,m)\big)H\big(1-p[\tau(k)+1]\big)H\big(y[\tau(k)]\big) = 0 \tag{7.1.19}$$

has an eventually positive solution which contradicts the hypotheses of the theorem and completes the proof. □

Theorem 7.1.2. *Suppose that conditions (7.1.3), (7.1.4), and (7.1.8) hold,*

$$\{\sigma(k)\} \quad \text{is an increasing sequence with} \quad \sigma(k) > k \quad \text{for } k \geq m \in \mathbb{N}. \tag{7.1.20}$$

Moreover, assume that

$$p^*(k) := \frac{1}{p[\sigma^{-1}(k)]}\left[1 - \frac{1}{p[\sigma^{-1}(\sigma^{-1}(k)-1)]}\right] \geq 0 \quad \text{for } k \geq m, \tag{7.1.21}$$

$$\tau^*(k) := \min\{k, \sigma^{-1}(g(k)-1)\} \quad \text{satisfies} \quad \Delta\tau^*(k) \geq 0 \quad \text{for } k \geq m. \tag{7.1.22}$$

If for every constant $a \geq 1$ and all large $m^* \geq m$ with $\tau^*(k) \geq m+1$ for all $k \geq m^*$ the equation

$$\Delta\big(c(k-1)\Delta y(k)\big) + q(k)G\big(aC(g(k)-1,m)\big)H\big(p^*[g(k)]\big)H\big(y[\tau^*(k)]\big) = 0 \tag{7.1.23}$$

is oscillatory, where C is as in (7.1.10), then equation (7.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $y(k)$ as in (7.1.11) and proceed as in the proof of Theorem 7.1.1 to obtain (7.1.13) and (7.1.14) for $k \geq m_1$. Using conditions (7.1.20), (7.1.21), and (7.1.13) in (7.1.11), we find for $k \geq m_1$ that

$$x[\sigma(k)] = \frac{y(k-1) - x(k-1)}{p(k)}, \tag{7.1.24}$$

so

$$\begin{aligned}
x(k) &= \frac{1}{p[\sigma^{-1}(k)]}\left[y[\sigma^{-1}(k)-1] - x[\sigma^{-1}(k)-1]\right] \\
&= \frac{1}{p[\sigma^{-1}(k)]}\Bigg[y[\sigma^{-1}(k)-1] - \frac{1}{p[\sigma^{-1}(\sigma^{-1}(k)-1)]}\big(y[\sigma^{-1}(\sigma^{-1}(k)-1)-1] \\
&\qquad - x([\sigma^{-1}(\sigma^{-1}(k)-1)-1])\big)\Bigg] \\
&\geq \frac{1}{p[\sigma^{-1}(k)]}\left[y[\sigma^{-1}(k)-1] - \frac{1}{p[\sigma^{-1}(\sigma^{-1}(k)-1)]}y[\sigma^{-1}(\sigma^{-1}(k)-1)-1]\right] \\
&\geq \frac{1}{p[\sigma^{-1}(k)]}\left[1 - \frac{1}{p[\sigma^{-1}(\sigma^{-1}(k)-1)]}\right]y[\sigma^{-1}(k)-1],
\end{aligned} \tag{7.1.25}$$

that is,

$$x(k) \geq p^*(k)y[\sigma^{-1}(k)-1] \quad \text{for } k \geq m_1. \tag{7.1.26}$$

There exists an integer $m_2 \geq m_1$ such that

$$x[g(k)] \geq p^*[g(k)]y[\sigma^{-1}(g(k)-1)] \geq p^*[g(k)]y[\tau^*(k)] \quad \text{for } k \geq m_2. \tag{7.1.27}$$

The rest of the proof is similar to the proof of Theorem 7.1.1 and hence we omit it here. □

The following results are immediate.

Corollary 7.1.3. *Let conditions (7.1.4)–(7.1.7) hold,*

$$f'(x) \geq 0 \quad \textit{for } x \neq 0, \tag{7.1.28}$$

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \textit{for } x, y > 0. \tag{7.1.29}$$

If the equation

$$\Delta(c(k-1)\Delta y(k-1)) + q(k)f(1 - p[g(k)+1])f(y[\tau(k)]) = 0 \tag{7.1.30}$$

is oscillatory, then equation (7.1.1) is oscillatory.

Corollary 7.1.4. *Let conditions (7.1.4), (7.1.20)–(7.1.22), (7.1.28), and (7.1.29) hold. If the equation*

$$\Delta(c(k-1)\Delta y(k-1)) + q(k)f(p^*[g(k)])f(y[\tau^*(k)]) = 0 \tag{7.1.31}$$

is oscillatory, then equation (7.1.1) is oscillatory.

Corollary 7.1.5. *Let condition (7.1.5) of Theorem 7.1.1 be replaced by*

$$0 \leq p(k) \leq p_0 < 1, \quad k \in \mathbb{N}, \text{ where } p_0 \text{ is a constant}, \tag{7.1.32}$$

and let equation (7.1.30) be replaced by the equation

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)f(1-p_0)f\big(y[\tau(k)]\big) = 0. \tag{7.1.33}$$

Then the conclusion of Theorem 7.1.1 holds.

Corollary 7.1.6. *Let condition (7.1.21) of Theorem 7.1.2 be replaced by*

$$1 < p_1 \leq p(k) \leq p_2, \quad k \in \mathbb{N}, \text{ where } p_1 \text{ and } p_2 \text{ are constants}, \tag{7.1.34}$$

and let equation (7.1.31) be replaced by the equation

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)f\left(\frac{p_1-1}{p_1p_2}\right)f\big(y[\tau^*(k)]\big) = 0. \tag{7.1.35}$$

Then the conclusion of Theorem 7.1.2 holds.

For equation (7.1.2) we present the following results.

Theorem 7.1.7. *Suppose conditions (6.2.6), (7.1.3), (7.1.5)–(7.1.8) hold. If, for every constant $a \geq 1$ and all large integers $m^* \geq m \in \mathbb{N}$ with $\tau(k) > m+1$ for $k \geq m^*$, the difference equation*

$$\begin{aligned}\Delta\big(c(k-1)\Psi(\Delta y(k-1))\big)&\\ + q(k)G\big(aC[g(k)-1,m]\big)H\big(1-p[g(k)+1]\big)H\big(y[\tau(k)]\big) &= 0\end{aligned} \tag{7.1.36}$$

is oscillatory, where

$$C[k,\ell] = \sum_{j=\ell}^{k}\Psi^{-1}\left(\frac{1}{c(j)}\right) \quad \text{for } k \geq \ell \geq m \in \mathbb{N}, \tag{7.1.37}$$

then equation (7.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.2), say, $x(k) > 0$, $x[\sigma(k)] > 0$, and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $y(k)$ as in (7.1.11). Then equation (7.1.2) takes the form

$$\Delta\big(c(k-1)\Psi(\Delta y(k-1))\big) + q(k)f\big(x[g(k)]\big) = 0. \tag{7.1.38}$$

As in Chapter 6, one can easily obtain

$$y(k) > 0, \quad \Delta y(k) > 0, \quad \Delta\big(c(k-1)\Psi(\Delta y(k-1))\big) \le 0 \quad \text{eventually.} \tag{7.1.39}$$

Now there exist a constant $b > 0$ and an integer $m_1 \ge m$ such that $g(k) \ge m+1$ for $k \ge m_1$ and for $k \ge m_1$,

$$x[g(k)] \le y[g(k)] \le b \sum_{j=m}^{g(k)-1} \Psi^{-1}\left(\frac{1}{c(j)}\right) = bC[g(k)-1, m]. \tag{7.1.40}$$

The rest of the proof is similar to the proof of Theorem 7.1.1 and hence we omit it here. □

Theorem 7.1.8. *Suppose (6.2.6), (7.1.3), (7.1.8), and (7.1.20)–(7.1.22) hold. If for every constant $a \ge 1$ and all large integers $m^* \ge m \in \mathbb{N}$ with $\tau^*(k) \ge m+1$ for $k \ge m^*$ the equation*

$$\Delta\big(c(k-1)\Psi(\Delta y(k-1))\big)+q(k)G(aC[g(k)-1,m])H(p^*[g(k)])H(y[\tau^*(k)]) = 0 \tag{7.1.41}$$

is oscillatory, then equation (7.1.2) is oscillatory.

Proof. The proof is similar to that of Theorems 7.1.7 and 7.1.2 and is hence omitted. □

The following examples illustrate the methods presented above.

Example 7.1.9. Consider the neutral difference equation

$$\Delta\Psi(\Delta[x(k-1)+px[k-\sigma]]) + q(k)\frac{|x[k-\tau]|^\gamma}{1+x^2[k-\tau]}\operatorname{sgn} x[k-\tau] = 0, \tag{7.1.42}$$

where the function Ψ is as in equation (7.1.2) and satisfies either (i_1) or (i_2), $\gamma > 0$ is a constant, $p \in [0,1)$ is a constant, τ and σ are positive integers, and $\{q(k)\}$ is a sequence of eventually nonnegative real numbers. Here we let

$$f(x) = \frac{|x|^\gamma}{1+x^2}\operatorname{sgn} x. \tag{7.1.43}$$

Then we take $G(x) = 1/(1+x^2)$ and $H(x) = |x|^\gamma \operatorname{sgn} x$. Now, by applying Theorem 7.1.7 to equation (7.1.42), we see that equation (7.1.42) is oscillatory if for every constant $a \geq 1$ the equation

$$\Delta\Psi(\Delta y(k-1)) + (1-p)^\gamma \frac{q(k)}{1+a^2k^2} \left| y[k-\tau] \right|^\gamma \operatorname{sgn} y[k-\tau] = 0 \tag{7.1.44}$$

is oscillatory. We note that equation (7.1.44) is oscillatory by Theorem 6.2.10 if for every constant $a \geq 1$ one of the following conditions holds:

$$\limsup_{k\to\infty} (k-\tau)^\alpha \sum_{j=k+1}^{\infty} \frac{q(j)}{1+a^2 j^2} > (1-p)^{-\gamma}, \tag{7.1.45}$$

so

$$\liminf_{k\to\infty} (k-\tau)^\alpha \sum_{j=k+1}^{\infty} \frac{q(j)}{1+a^2 j^2} > (1-p)^{-\gamma} \left[\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \right]. \tag{7.1.46}$$

Example 7.1.10. Consider the equation

$$\Delta\Psi(\Delta[x(k-1) + px[k+\sigma]]) + q(k)e^{-|x[k-\tau]|} \left| x[k-\tau] \right|^\gamma \operatorname{sgn} x[k-\tau] = 0, \tag{7.1.47}$$

where Ψ, γ, σ, τ, and $\{q(k)\}$ are as in equation (7.1.42) and $p > 1$ is a constant. Here $f(x) = e^{-|x|}|x|^\gamma \operatorname{sgn} x$, and so the components of f are $G(x) = e^{-|x|}$ and $H(x) = |x|^\gamma \operatorname{sgn} x$. Next, by applying Theorem 7.1.8 to equation (7.1.47), one can conclude that equation (7.1.47) is oscillatory if for all constants $a \geq 1$ the equation

$$\Delta\Psi(\Delta y(k-1)) + \left(\frac{p-1}{p}\right)^\gamma e^{-ak} q(k) \left| y[k-\tau-\sigma-1] \right|^\gamma \operatorname{sgn} y[k-\tau-\sigma-1] = 0 \tag{7.1.48}$$

is oscillatory. As in Example 7.1.9, equation (7.1.48) is oscillatory if one of the following conditions holds:

$$\limsup_{k\to\infty} (k-\sigma-\tau-1)^\alpha \sum_{j=k+1}^{\infty} e^{-aj} q(j) > \left(\frac{p-1}{p^2}\right)^{-\gamma} \tag{7.1.49}$$

or

$$\liminf_{k\to\infty} (k-\sigma-\tau-1)^\alpha \sum_{j=k+1}^{\infty} e^{-aj} q(j) > \left(\frac{p-1}{p^2}\right)^{-\gamma} \left[\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \right]. \tag{7.1.50}$$

7.1.1. Linearization theorems

Our interest here is to relate oscillation of equations (7.1.1) and (7.1.2) to some linear and half-linear difference equations, respectively. This enables us to employ numerous known results for linear and half-linear difference equations; see, for example, the results of Chapters 1, 2, and 3.

Now we present the following results.

Theorem 7.1.11. *Suppose that conditions (7.1.3)–(7.1.7) hold and $\Delta c(k) \geq 0$ for $k \geq m \in \mathbb{N}$. If for all constants a^* and b^* such that $a^*b^* > 0$ and for all large integers m^* with $\tau(k) > m+1$ for $k \geq m^*$ the equation*

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)Q_1(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.51}$$

where

$$Q_1(k) = \frac{G\big(a^*C\big(g(k)-1,m\big)\big)}{a^*C\big(\tau(k)-1,m\big)}H\big(b^*\big(1-p\big[g(k)+1\big]\big)\big) \tag{7.1.52}$$

is oscillatory, then equation (7.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.1), say, $x(k) > 0$, $x[\sigma(k)] > 0$, and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $y(k)$ as in (7.1.11) and obtain (7.1.12)–(7.1.16) for $k \geq m_1 \geq m$. Define the sequence $\{w(k)\}$ by

$$w(k) = \frac{c(k-1)\Delta y(k-1)}{y\big[\tau(k)-1\big]} \quad \text{for } k \geq m_1. \tag{7.1.53}$$

Then

$$\Delta w(k) = \frac{\Delta\big(c(k-1)\Delta y(k-1)\big)}{y\big[\tau(k)\big]} - \frac{c(k-1)\Delta y(k-1)\Delta y\big[\tau(k)-1\big]}{y\big[\tau(k)-1\big]y\big[\tau(k)\big]}, \tag{7.1.54}$$

and hence

$$\begin{aligned}\Delta w(k) = {} & -q(k)\frac{G\big(x\big[g(k)\big]\big)H\big(x\big[g(k)\big]\big)}{y\big[\tau(k)\big]} \\ & - \frac{c(k-1)\Delta y(k-1)\Delta y\big[\tau(k)-1\big]}{y\big[\tau(k)-1\big]y\big[\tau(k)\big]} \quad \text{for } k \geq m_1.\end{aligned} \tag{7.1.55}$$

Since $\tau(k) \leq k$ and $\Delta c(k) \geq 0$ for $k \geq m$, it follows from (7.1.13) that

$$c(k-1)\Delta y(k-1) \leq c\big[\tau(k)-1\big]\Delta y\big[\tau(k)-1\big] \leq c(k-1)\Delta y\big[\tau(k)-1\big], \tag{7.1.56}$$

and so we have

$$\Delta y\big[\tau(k)-1\big] \geq \Delta y(k-1) \quad \text{for } k \geq m_1. \tag{7.1.57}$$

Observing that the term $\Delta y(k-1)/[\Delta y(k-1)+y[\tau(k)-1]]$ is increasing in $\Delta y(k-1)$, we obtain

$$\begin{aligned}\frac{w^2(k)}{w(k)+c(k-1)} &= \frac{[c(k-1)\Delta y(k-1)]^2}{(y[\tau(k)-1])^2(c(k-1)(\Delta y(k-1)/y[\tau(k)-1])+c(k-1))}\\ &= \frac{c(k-1)\Delta y(k-1)}{y[\tau(k)-1]}\frac{\Delta y(k-1)}{\Delta y(k-1)+y[\tau(k)-1]}\\ &\le \frac{c(k-1)\Delta y(k-1)}{y[\tau(k)-1]}\frac{\Delta y[\tau(k)-1]}{\Delta y[\tau(k)-1]+y[\tau(k)-1]},\end{aligned} \tag{7.1.58}$$

that is,

$$\frac{w^2(k)}{w(k)+c(k-1)} \le \frac{c(k-1)\Delta y(k-1)\Delta y[\tau(k)-1]}{y[\tau(k)-1]y[\tau(k)]} \quad \text{for } k \ge m_1. \tag{7.1.59}$$

Using (7.1.14) and (7.1.59) in (7.1.55) for $k \ge m_2 \ge m_1$, we obtain

$$\Delta w(k)+\frac{w^2(k)}{w(k)+c(k-1)} \le -q(k)G(bC(g(k)-1,m))\frac{H(x[g(k)])}{y[\tau(k)]}. \tag{7.1.60}$$

Using (7.1.17) and the fact that the function H is nondecreasing in (7.1.60) for $k \ge m_2$, we get

$$\begin{aligned}\Delta w(k)&+\frac{w^2(k)}{w(k)+c(k-1)}\\ &\le -q(k)G(bC(g(k)-1,m))\frac{H(y[\tau(k)](1-p[g(k)+1]))}{y[\tau(k)]}.\end{aligned} \tag{7.1.61}$$

By (7.1.13), there exist a constant $a>0$ and an integer $m_3 \ge m_2$ such that

$$y[\tau(k)] \ge a \quad \text{for } k \ge m_3. \tag{7.1.62}$$

Using (7.1.14) with $\tau(k)$ instead of $g(k)$ and (7.1.62) in (7.1.61) for $k \ge m_3$, we have

$$\Delta w(k)+\frac{w^2(k)}{w(k)+c(k-1)} \le -q(k)G(bC(g(k)-1,m))\frac{H(a(1-p[g(k)+1]))}{bC(\tau(k)-1,m)}. \tag{7.1.63}$$

Now, by Lemma 1.7.1 we see that equation (7.1.51) with a^* and b^* replaced by b and a, respectively, is nonoscillatory. This contradiction completes the proof. □

For the special case of equation (7.1.1) with $p(k) \equiv 0$, that is, for the equation

$$\Delta(c(k-1)\Delta y(k-1))+q(k)f(x[g(k)])=0, \tag{7.1.64}$$

Theorem 7.1.11 reduces to the following corollary.

Corollary 7.1.12. *Suppose that $\Delta c(k) \geq 0$ for $k \geq m \in \mathbb{N}$ and conditions (7.1.3), (7.1.4), (7.1.6), and (7.1.7) hold. If for all constants a^* and b^* with $a^*b^* > 0$ and all large integers $m^* \geq m \in \mathbb{N}$ with $\tau(k) > m+1$ for $k \geq m^*$ the linear difference equation*

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)\frac{G\big(a^*C\big(g(k)-1,m\big)\big)}{a^*C\big(\tau(k)-1,m\big)}H(b^*)y(k) = 0 \quad \text{for } k \geq m^* \tag{7.1.65}$$

is oscillatory, then equation (7.1.64) is oscillatory.

Theorem 7.1.13. *Suppose that (7.1.3)–(7.1.7) hold, $\Delta c(k) \geq 0$ for $k \geq m \in \mathbb{N}$, and*

$$H(x)\operatorname{sgn} x \geq |x|^\gamma \quad \text{for } x \neq 0, \tag{7.1.66}$$

where $\gamma > 0$ is a constant. If for all constants a^* and b^* with $a^*b^* > 0$ and all large integers $m^* \geq m$ with $\tau(k) > m+1$ for $k \geq m^*$ the equation

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)Q_2(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.67}$$

where

$$Q_2(k) = \begin{cases} |b^*|^{\gamma-1}\big(1-p[g(k)+1]\big)^\gamma G\big(a^*C\big(g(k)-1,m\big)\big) & \text{when } \gamma \geq 1, \\ \dfrac{\big(1-p[g(k)+1]\big)^\gamma G\big(a^*C\big(g(k)-1,m\big)\big)}{\big(|a^*|C\big(\tau(k)-1,m\big)\big)^{1-\gamma}} & \text{when } \gamma < 1, \end{cases} \tag{7.1.68}$$

is oscillatory, then equation (7.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.1), say, $x(k) > 0$, $x[\sigma(k)] > 0$, and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Proceeding as in the proof of Theorem 7.1.11, we obtain the inequality (7.1.61) for $k \geq m_2$. Using (7.1.62) and (7.1.66) in (7.1.61) for $k \geq m_3 \geq m_2$, we have

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} \leq -a^{\gamma-1}q(k)G\big(bC\big(g(k)-1,m\big)\big)\big(1-p[g(k)+1]\big)^\gamma \tag{7.1.69}$$

if $\gamma \geq 1$, and

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} \leq -\frac{q(k)G\big(bC\big(g(k)-1,m\big)\big)\big(1-p[g(k)+1]\big)^\gamma}{\big(bC\big(\tau(k)-1,m\big)\big)^{1-\gamma}} \tag{7.1.70}$$

if $\gamma < 1$. Once again, by applying Lemma 1.7.1 we arrive at the desired contradiction. This completes the proof. □

Putting $p(k) \equiv 0$ in Theorem 7.1.13 we obtain the following corollary.

Corollary 7.1.14. *Suppose that $\Delta c(k) \geq 0$ for $k \geq m \in \mathbb{N}$ and (7.1.3), (7.1.4), (7.1.6), (7.1.7), and (7.1.66) hold. If for all constants a^* and b^* with $a^*b^* > 0$ and all large integers $m^* \geq m$ with $\tau(k) > m+1$ for all $k \geq m^*$ the equation*

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)Q_2^*(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.71}$$

where

$$Q_2^*(k) = \begin{cases} |b^*|^{\gamma-1}G\big(a^*C(g(k)-1,m)\big) & \text{when } \gamma \geq 1, \\ G\big(a^*C(g(k)-1)\big)\big(|a^*|C(\tau(k)-1,m)\big)^{\gamma-1} & \text{when } \gamma < 1, \end{cases} \tag{7.1.72}$$

is oscillatory, then equation (7.1.64) is oscillatory.

Theorem 7.1.15. *Suppose that conditions (7.1.3), (7.1.4), (7.1.20)–(7.1.22) are satisfied and $c(k) \geq c[\tau^*(k)]$ for $k \geq m \in \mathbb{N}$. If for all constants a^* and b^* with $a^*b^* > 0$ and all large integers $m^* \geq m$ with $\tau^*(k) > m+1$ for $k \geq m^*$ the equation*

$$\Delta\big(c(k-1)\Delta y(k-1)\big) + q(k)Q_3(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.73}$$

where

$$Q_3(k) = \frac{G\big(a^*C(g(k)-1,m)\big)}{a^*C(\tau^*(k)-1,m)}H\big(b^*p^*[g(k)]\big), \tag{7.1.74}$$

is oscillatory, then equation (7.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.1), say, $x(k) > 0$, $x[\sigma(k)] > 0$ and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $y(k)$ as in (7.1.11) and proceed as in the proof of Theorem 7.1.2 to obtain (7.1.12), (7.1.13), (7.1.14), and (7.1.26) for all $k \geq m_1 \geq m$.

Next, we proceed as in the proof of Theorem 7.1.11 with $\tau(k)$ replaced by $\tau^*(k)$, and for $k \geq m_2 \geq m_1$ we obtain

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} \leq -q(k)\frac{G(x[g(k)])H(x[g(k)])}{y[\tau^*(k)]}. \tag{7.1.75}$$

By (7.1.13), there exist a constant $a > 0$ and an integer $m_3 \geq m_2$ such that

$$y[\tau^*(k)] \geq a \quad \text{for } k \geq m_3. \tag{7.1.76}$$

Using (7.1.14), (7.1.27), and (7.1.76) in (7.1.75) for $k \geq m_3$, we get

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} \leq -q(k)\frac{G\big(bC(g(k)-1,m)\big)}{bC(\tau^*(k)-1,m)}H\big(ap^*[g(k)]\big). \tag{7.1.77}$$

The rest of the proof is similar to the proof of Theorem 7.1.11 and hence we omit it here. □

Theorem 7.1.16. *Suppose that conditions (7.1.3), (7.1.4), (7.1.20)–(7.1.22), and (7.1.66) are satisfied and* $c(k) \geq c[\tau^*(k)]$ *for* $k \geq m \in \mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^* > 0$ *and all large integers* $m^* \geq m$ *with* $\tau^*(k) > m+1$ *for* $k \geq m^*$ *the equation*

$$\Delta(c(k-1)\Delta y(k-1)) + q(k)Q_4(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.78}$$

where

$$Q_4(k) = \begin{cases} |b^*|^{\gamma-1}(p^*[g(k)])^{\gamma}G(a^*C(g(k)-1,m)) & \text{when } \gamma \geq 1, \\ \dfrac{(p^*[g(k)])^{\gamma}G(a^*C(g(k)-1,m))}{(|a^*|C(\tau^*(k)-1,m))^{1-\gamma}} & \text{when } \gamma < 1, \end{cases} \tag{7.1.79}$$

is oscillatory, then equation (7.1.1) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.1), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Proceeding as in the proof of Theorem 7.1.15, we obtain the inequality (7.1.75) for $k \geq m_2$. Using (7.1.14), (7.1.66), and (7.1.76) in inequality (7.1.75) for $k \geq m_3 \geq m_2$ and $\gamma \geq 1$, we obtain

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} \leq -a^{\gamma-1}q(k)G(bC(g(k)-1,m))(p^*[g(k)])^{\gamma}, \tag{7.1.80}$$

so for $k \geq m_3$ and $\gamma < 1$,

$$\Delta w(k) + \frac{w^2(k)}{w(k)+c(k-1)} \leq -(bC(\tau^*(k)-1,m))^{\gamma-1}q(k)G(bC(g(k)-1,m))(p^*[g(k)])^{\gamma}. \tag{7.1.81}$$

The rest of the proof is similar to the proof of Theorem 7.1.13 and hence we omit it here. □

Next we consider equation (7.1.2) and present the following results.

Theorem 7.1.17. *Let conditions (6.2.6), (7.1.3), (7.1.5)–(7.1.7) hold and* $\Delta c(k) \geq 0$ *for* $k \geq m \in \mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^* > 0$ *and all large integers* $m^* \geq m$ *with* $\tau(k) > m+1$ *for* $k \geq m^*$ *the equation*

$$\Delta(c(k-1)\Psi(\Delta y(k-1))) + q(k)Q_5(k)\Psi(y(k)) = 0 \quad \text{for } k \geq m^*, \tag{7.1.82}$$

where

$$Q_5(k) = \frac{G(a^*C[g(k)-1,m])}{\Psi(a^*C[\tau(k)-1,m])}H(b^*(1-p[g(k)+1])) \tag{7.1.83}$$

and C is as in (7.1.37), is oscillatory, then equation (7.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.2), say, $x(k) > 0$, $x[\sigma(k)] > 0$, and $x[g(k)] > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $y(k)$ as in (7.1.11) and proceed as in the proof of Theorem 7.1.7 to obtain (7.1.38), (7.1.39), and (7.1.40) for $k \geq m_1 \geq m$. Next, define the sequence $\{w(k)\}$ by

$$w(k) = \frac{c(k-1)\Psi(\Delta y(k-1))}{\Psi(y[\tau(k)-1])} \quad \text{for } k \geq m_1. \tag{7.1.84}$$

Then for $k \geq m_1$, we obtain

$$\begin{aligned}\Delta w(k) &= \frac{\Delta(c(k-1)\Psi(\Delta y(k-1)))}{\Psi(y[\tau(k)])} \\ &\quad - \frac{c(k-1)\Psi(\Delta y(k-1))}{\Psi(y[\tau(k)-1])}\left(1 - \frac{\Psi(y[\tau(k)-1])}{\Psi(y[\tau(k)])}\right) \\ &= -q(k)\frac{G(x[g(k)])H(x[g(k)])}{\Psi(y[\tau(k)])} - w(k)\left[1 - \left(\frac{y[\tau(k)-1]}{y[\tau(k)]}\right)^{\alpha}\right].\end{aligned} \tag{7.1.85}$$

Set $\Phi(w(k), c(k-1))$ as in (3.2.17), that is, for $k \geq m_1$,

$$\begin{aligned}\Phi(w(k), c(k-1)) &= w(k)\left[1 - \frac{c(k-1)}{\Psi[\Psi^{-1}(c(k-1)) + \Psi^{-1}(w(k))]}\right] \\ &= w(k)\left[1 - \left(1 + \left(\frac{w(k)}{c(k-1)}\right)^{1/\alpha}\right)^{-\alpha}\right] \\ &= w(k)\left[1 - \left(\frac{y[\tau(k)-1]}{y[\tau(k)-1] + \Delta y(k)}\right)^{\alpha}\right].\end{aligned} \tag{7.1.86}$$

By (7.1.39) and the fact that $c(k) \geq c[\tau(k)]$ for $k \geq m_1$, one can easily see that $\Delta y[\tau(k)-1] \geq \Delta y(k)$ for $k \geq m_2 \geq m_1$ and hence

$$\Phi(w(k), c(k-1)) \leq w(k)\left[1 - \left(\frac{y[\tau(k)-1]}{y[\tau(k)]}\right)^{\alpha}\right] \quad \text{for } k \geq m_2. \tag{7.1.87}$$

Thus, for $k \geq m_2$, we obtain

$$\Delta w(k) + \Phi(w(k), c(k-1)) \leq -q(k)\frac{G(bC[g(k)-1, m])}{\Psi(bC[\tau(k)-1, m])}H(x[g(k)]). \tag{7.1.88}$$

As in the proof of Theorems 7.1.1 and 7.1.11, we get (7.1.17), (7.1.62), and for $k \geq m_2$,

$$\Delta w(k) + \Phi(w(k), c(k-1)) \leq -q(k)\frac{G(bC[g(k)-1, m])}{\Psi(bC[\tau(k)-1, m])}H(a(1-p[g(k)+1])). \tag{7.1.89}$$

Now, by Lemma 3.4.2 we see that equation (7.1.82) with a^* and b^* replaced by b and a, respectively, is nonoscillatory, which is a contradiction and completes the proof of the theorem. □

Theorem 7.1.18. *Let conditions (6.2.6), (7.1.3), (7.1.5)–(7.1.7), and (7.1.66) hold and $\Delta c(k) \geq 0$ for $k \geq m \in \mathbb{N}$. If for all constants a^* and b^* with $a^* b^* > 0$ and all large integers $m^* \geq m$ with $\tau(k) > m+1$ for $k \geq m^*$ the difference equation*

$$\Delta(c(k-1)\Psi(\Delta y(k-1))) + q(k)Q_6(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.90}$$

where

$$Q_6(k) = \begin{cases} |b^*|^{\gamma-a}(1-p[g(k)+1])^\gamma G(a^*C[g(k)-1, m]) & \text{when } \gamma \geq \alpha, \\ \dfrac{(1-p[g(k)+1])^\gamma G(a^*C[g(k)-1, m])}{(|a^*|C[\tau(k)-1, m])^{\alpha-\gamma}} & \text{when } \gamma < \alpha, \end{cases} \tag{7.1.91}$$

is oscillatory, then equation (7.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. As in the proof of Theorems 7.1.7, 7.1.11, and 7.1.17 we obtain (7.1.17), (7.1.39), (7.1.40), (7.1.62) for $k \geq m_3$, say, and

$$\Delta w(k) + \Phi(w(k), c(k-1)) \leq -q(k)\frac{G(x[g(k)])}{\Psi(y[\tau(k)])}H(x[g(k)]). \tag{7.1.92}$$

Thus one can easily see that if $\gamma \geq \alpha$, then

$$\Delta w(k) + \Phi(w(k), c(k-1)) \leq -a^{\gamma-\alpha}q(k)G(bC[g(k)-1, m])(1-p[g(k)+1])^\gamma \tag{7.1.93}$$

or if $\gamma < \alpha$, then

$$\Delta w(k) + \Phi(w(k), c(k-1)) \leq -\frac{q(k)G(bC[g(k)-1, m])(1-p[g(k)+1])^\gamma}{(bC[\tau(k)-1, m])^{\alpha-\gamma}}. \tag{7.1.94}$$

Once again, by Lemma 3.4.2 we see that equation (7.1.90) with a^* and b^* replaced by b and a, respectively, is oscillatory, which is a contradiction and completes the proof. □

Theorem 7.1.19. *Assume (6.2.6), (7.1.3), (7.1.20)–(7.1.22) and* $c(k) \geq c[\tau^*(k)]$ *for* $k \geq m \in \mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^* > 0$ *and all large integers* $m^* \geq m$ *with* $\tau^*(k) > m+1$ *for* $k \geq m^*$ *the equation*

$$\Delta\big(c(k-1)\Psi(\Delta y(k-1))\big) + q(k)Q_7(k)y(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.95}$$

where

$$Q_7(k) = \frac{G(a^*C[g(k)-1,m])}{\Psi(a^*C[\tau^*(k)-1,m])}H(b^*p^*[g(k)]), \tag{7.1.96}$$

is oscillatory, then equation (7.1.2) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of (7.1.2), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $y(k)$ as in (7.1.11) and proceed as in the proof of Theorems 7.1.2 and 7.1.7 to obtain (7.1.39), (7.1.40), and (7.1.27) for $k \geq m_2$, say. Next, as in the proof of Theorems 7.1.15 and 7.1.18 we get (7.1.76) and the inequality (7.1.92). Thus for $k \geq m_3 \geq m_2$,

$$\Delta w(k) + \Phi(w(k) + c(k-1)) \leq -q(k)\frac{G(bC[g(k)-1,m])}{\Psi(bC[\tau^*(k)-1,m])}H(ap^*[g(k)]). \tag{7.1.97}$$

The rest of the proof is similar to the proof of Theorem 7.1.17 and hence we omit it here. □

Theorem 7.1.20. *Let conditions (6.2.6), (7.1.3), (7.1.20)–(7.1.22), and (7.1.66) hold and* $c(k) \geq c[\tau^*(k)]$ *for* $k \geq m \in \mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^* > 0$ *and all large integers* $m^* \geq m$ *with* $\tau^*(k) > m+1$ *for* $k \geq m^*$ *the equation*

$$\Delta\big(c(k-1)\Psi(\Delta y(k-1))\big) + q(k)Q_8(k)y(k) = 0, \tag{7.1.98}$$

where

$$Q_8(k) = \begin{cases} |b^*|^{\gamma-\alpha}(p^*[g(k)])^{\gamma}G(a^*C[g(k)-1,m]) & \text{when } \gamma \geq \alpha, \\ \dfrac{(p^*[g(k)])^{\gamma}G(a^*C[g(k)-1,m])}{(|a^*|C[\tau^*(k)-1,m])^{\alpha-\gamma}} & \text{when } \gamma < \alpha, \end{cases} \tag{7.1.99}$$

is oscillatory, then equation (7.1.2) is oscillatory.

Proof. The proof is similar to the proofs of Theorems 7.1.16 and 7.1.19 and hence is omitted. □

7.1.2. Comparison results for forced neutral equations

Here we will derive some sufficient conditions of comparison type for the oscillation of the forced neutral difference equation

$$\Delta\big(c(k)\Delta[x(k-1)+p(k)x[\sigma(k)]]\big)+q(k)f\big(x[g(k)]\big)=e(k), \tag{7.1.100}$$

where $\{e(k)\}$ is a sequence of real numbers. We will assume that

$$\begin{aligned}&\text{there exists a sequence } \{\eta(k)\} \text{ of real numbers such that}\\ &\Delta\big(c(k-1)\Delta\eta(k)\big)=e(k) \text{ and } \{\eta(k)\} \text{ is oscillatory}\end{aligned} \tag{7.1.101}$$

and for every constant $d>0$,

$$P(k)=1-p(k)-\frac{1}{d}\left|p(k)\eta[\sigma(k)+1]-\eta(k)\right|\geq 0 \quad \text{eventually.} \tag{7.1.102}$$

Now we present the following result.

Theorem 7.1.21. *Assume that conditions (7.1.3), (7.1.4), (7.1.6), (7.1.7), (7.1.101), and (7.1.102) hold and* $\Delta c(k)\geq 0$ *for* $k\geq m\in\mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^*>0$ *and all large integers* $m^*\geq m$ *with* $\tau(k)>m+1$ *for* $k\geq m^*$ *the equation*

$$\Delta\big(c(k-1)\Delta u(k-1)\big)+q(k)Q_9(k)u(k)=0 \quad \text{for } k\geq m^*, \tag{7.1.103}$$

where

$$Q_9(k)=\frac{G\big(a^*C(g(k)-1,m)+\eta[g(k)+1]\big)}{a^*C(\tau(k)-1,m)}H\big(b^*P[g(k)+1]\big), \tag{7.1.104}$$

is oscillatory, then equation (7.1.100) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (7.1.100), say, $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. Define the sequence $\{y(k)\}$ by

$$x(k-1)+p(k)x[\sigma(k)]=y(k-1)+\eta(k). \tag{7.1.105}$$

Then equation (7.1.100) takes the form

$$\Delta\big(c(k-1)\Delta y(k-1)\big)+q(k)f\big(x[g(k)]\big)=0, \tag{7.1.106}$$

and hence $\Delta(c(k-1)\Delta y(k-1))\leq 0$ eventually, which implies that $y(k)$ and $\Delta y(k)$ are eventually of one sign. From (7.1.105) we see that $x(k-1)\leq y(k-1)=\eta(k)$ eventually.

Clearly, $\{y(k)\}$ is eventually positive. Otherwise, we obtain $0 < x(k-1) \le \eta(k)$ eventually, which contradicts the fact that $\{\eta(k)\}$ is oscillatory. As before, one can easily find

$$y(k) > 0, \quad \Delta y(k) > 0 \quad \text{for } k \ge m_1 \text{ for some integer } m_1 \ge m, \tag{7.1.107}$$

and there exist a constant $a_1 > 0$ and an integer $m_2 \ge m_1$ such that

$$y(k) \le a_1 C(k-1, m) \quad \text{for } k \ge m_2. \tag{7.1.108}$$

From (7.1.105) and the increasing nature of the sequence $\{y(k)\}$, we see that

$$\begin{aligned} x(k-1) + p(k)x[\sigma(k)] &\le x(k-1) + p(k)[y[\sigma(k)] + \eta[\sigma(k)+1]] \\ &\le x(k-1) + p(k)y(k-1) + p(k)\eta[\sigma(k)+1], \end{aligned} \tag{7.1.109}$$

so

$$y(k-1) + \eta(k) \le x(k-1) + p(k)y(k-1) + p(k)\eta[\sigma(k)+1] \quad \text{for } k \ge m_1. \tag{7.1.110}$$

Thus

$$(1 - p(k))y(k-1) \le x(k-1) + p(k)\eta[\sigma(k)+1] - \eta(k) \quad \text{for } k \ge m_1. \tag{7.1.111}$$

Let $\{v(k)\}$ be a sequence defined by

$$p(k)\eta[\sigma(k)+1] - \eta(k) = v(k)y(k-1) \quad \text{for } k \ge m_1. \tag{7.1.112}$$

Then, by substituting into (7.1.111), we have

$$(1 - p(k) - v(k))y(k-1) \le x(k-1) \quad \text{for } k \ge m_1. \tag{7.1.113}$$

By (7.1.107), there exist a constant $b_1 > 0$ and an integer $m_3 \ge \max\{m_1, m_2\}$ such that

$$y(k-1) \ge b_1 \quad \text{for } k \ge m_3. \tag{7.1.114}$$

Now, from the definition of $\{v(k)\}$ it follows that

$$|p(k)\eta[\sigma(k)+1]-\eta(k)| = |v(k)|y(k-1) \geq b_1|v(k)| \geq b_1 v(k), \tag{7.1.115}$$

and so

$$v(k) \leq \frac{1}{b_1}|p(k)\eta[\sigma(k)+1]-\eta(k)| \quad \text{for } k \geq m_3. \tag{7.1.116}$$

It follows from (7.1.116) and (7.1.113) that $P(k)y(k-1) \leq x(k-1)$ for $k \geq m_3$. Thus

$$P[g(k)+1]y[\tau(k)] \leq P[g(k)+1]y[g(k)] \leq x[g(k)] \quad \text{for } k \geq m_3. \tag{7.1.117}$$

Now define

$$w(k) = \frac{\Delta(c(k-1)\Delta y(k-1))}{y[\tau(k)-1]} \quad \text{for } k \geq m_3. \tag{7.1.118}$$

As in the proof of Theorem 7.1.11, it is easy to deduce from equation (7.1.106) and (7.1.117) and the inequality

$$x[g(k)] \leq a_1 C(g(k)-1,m) + \eta[g(k)+1], \tag{7.1.119}$$

the inequality

$$\begin{aligned}\Delta w(k) &+ \frac{w^2(k)}{w(k)+c(k-1)}\\ &\leq -q(k)\frac{G(a_1C(g(k)-1,m)+\eta[g(k)+1])H(P[g(k)+1]y[\tau(k)])}{y[\tau(k)]}\end{aligned} \tag{7.1.120}$$

for $k \geq m_3$. Using the inequality

$$b_1 \leq y(k) \leq a_1 C(k-1,m) \quad \text{for } k \geq m_3, \tag{7.1.121}$$

and the fact that the function H is nondecreasing in (7.1.120), we have

$$\begin{aligned}\Delta w(k) &+ \frac{w^2(k)}{w(k)+c(k-1)}\\ &\leq -q(k)\frac{G(a_1C(g(k)-1,m)+\eta[g(k)+1])}{a_1C(\tau(k)-1,m)}H(b_1P[g(k)+1])\end{aligned} \tag{7.1.122}$$

for $k \geq m_3$. Applying Lemma 1.7.1, the above inequality implies that equation (7.1.103) is nonoscillatory, which contradicts the assumption of the theorem and completes the proof. □

Theorem 7.1.22. *Assume that conditions (7.1.3), (7.1.4), (7.1.6), (7.1.7), (7.1.66), (7.1.101), (7.1.102) hold and* $\Delta c(k) \geq 0$ *for* $k \geq m \in \mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^* > 0$ *and all large integers* $m^* \geq m$ *with* $\tau(k) > m+1$ *for* $k \geq m^*$ *the equation*

$$\Delta\big(c(k-1)\Delta u(k-1)\big) + q(k)Q_{10}(k)u(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.123}$$

where

$$Q_{10}(k) = \begin{cases} |b^*|^{\gamma-1}\big(P[g(k)+1]\big)^{\gamma} G\big(a^*C(g(k)-1,m) + \eta[g(k)+1]\big) & \text{if } \gamma \geq 1, \\ \dfrac{\big(P[g(k)+1]\big)^{\gamma} G\big(a^*C(g(k)-1,m) + \eta[g(k)+1]\big)}{\big(a^*C(\tau(k)-1,m)\big)^{1-\gamma}} & \text{if } \gamma < 1, \end{cases} \tag{7.1.124}$$

is oscillatory, then equation (7.1.100) is oscillatory.

Proof. The proof can be modeled according to the proofs of Theorems 7.1.21 and 7.1.13 and hence is omitted. □

In the following results, we consider equation (7.1.100) with $p(k) \equiv 0$, that is, we consider the equation

$$\Delta\big(c(k-1)\Delta x(k-1)\big) + q(k)f\big(x[g(k)]\big) = e(k). \tag{7.1.125}$$

Theorem 7.1.23. *Suppose that conditions (7.1.3), (7.1.4), (7.1.6), (7.1.7), and (7.1.101) hold and* $\Delta c(k) \geq 0$ *for* $k \geq m \in \mathbb{N}$. *If for all constants* a^* *and* b^* *with* $a^*b^* > 0$ *and all large integers* $m^* \geq m$ *with* $\tau(k) > m+1$ *for* $k \geq m^*$ *the equation*

$$\Delta\big(c(k-1)\Delta u(k-1)\big) + q(k)Q_{11}(k)u(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.126}$$

where

$$Q_{11}(k) = \frac{G\big(a^*C(g(k)-1,m) + \eta[g(k)+1]\big)}{a^*C(\tau(k)-1,m)} H\left(b\frac{\big((1/a)\eta[g(k)+1]\big)^{+}}{C(g(k)-1,m)}\right), \tag{7.1.127}$$

is oscillatory, then equation (7.1.125) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (7.1.126), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $x(k-1) = y(k-1) + \eta(k)$ and the sequence $\{v(k)\}$ by $x(k) = v(k)y(k)$ for $k \geq m$. Thus

$$v(k)y(k) = y(k) + \eta(k+1) > \eta(k+1), \tag{7.1.128}$$

and by (7.1.108) we obtain

$$v(k) > \frac{\eta(k+1)}{a_1C(k-1,m)} \quad \text{for } k \geq m_1 \text{ for some } m_1 \geq m, \tag{7.1.129}$$

so

$$x(k) > \frac{\eta(k+1)}{a_1 C(k-1,m)} y(k) \quad \text{for } k \geq m_1. \tag{7.1.130}$$

From (7.1.130), we find

$$x[g(k)] > \frac{(\eta[g(k)+1]/a_1)^+}{C(g(k)-1,m)} y[g(k)] \geq \frac{(\eta[g(k)+1]/a_1)^+}{C(g(k)-1,m)} y[\tau(k)] \tag{7.1.131}$$

for $k \geq m_1$. The rest of the proof is similar to that of Theorem 7.1.22 and hence is omitted. □

Theorem 7.1.24. *Suppose that conditions (7.1.3), (7.1.4), (7.1.6), (7.1.7), (7.1.66), and (7.1.101) hold and $\Delta c(k) \geq 0$ for $k \geq m \in \mathbb{N}$. If for all constants a^* and b^* with $a^* b^* > 0$ and all large integers $m^* \geq m$ with $\tau(k) > m+1$ for $k \geq m^*$ the equation*

$$\Delta\big(c(k-1)\Delta u(k-1)\big) + q(k)Q_{12}(k)u(k) = 0 \quad \text{for } k \geq m^*, \tag{7.1.132}$$

where

$$Q_{12}(k) = \begin{cases} |b^*|^{\gamma-1} G(a^* C(g(k)-1,m) + \eta[g(k)+1]) \\ \times \left[\dfrac{1}{C(g(k)-1,m)} \left(\dfrac{\eta[g(k)+1]}{a^*} \right)^+ \right]^\gamma, & \gamma \geq 1, \\ G(a^* C(g(k)-1,m) + \eta[g(k)+1]) \\ \times \left[\dfrac{1}{C(g(k)-1,m)} \left(\dfrac{\eta[g(k)+1]}{a^*} \right)^\gamma (|a^*| C(\tau(k)-1,m))^{\gamma-1} \right], & \gamma < 1, \end{cases} \tag{7.1.133}$$

is oscillatory, then equation (7.1.126) is oscillatory.

Example 7.1.25. As an example, we consider the forced difference equation

$$\Delta^2 x(k) + k^2 \frac{x[k-\tau]}{1+|x[k-\tau]|} = 4(-1)^k \quad \text{for } k \in \mathbb{N}. \tag{7.1.134}$$

Here we take $\eta(k) = (-1)^k$. Now one can easily check that all conditions of Theorems 7.1.21 and 7.1.20 are satisfied, and hence equation (7.1.134) is oscillatory.

7.2. Nonoscillation criteria

In this section we will present some nonoscillation results for neutral difference equations of the form

$$\Delta\big(c(k)\Delta[x(k)+px[k-\tau]]\big)+F\big(k+1,x[k+1-\sigma]\big)=0 \quad \text{for } k\in\mathbb{N}, \tag{7.2.1}$$

where

- (i_1) $\tau,\sigma\in\mathbb{N}_0$ are fixed,
- (i_2) $\{c(k)\}$ is a sequence of positive real constants,
- (i_3) $p\in\mathbb{R}$,
- (i_4) $F:\mathbb{N}\times(0,\infty)\to[0,\infty)$ is continuous, that is, it is continuous as a map from the topological space $\mathbb{N}\times(0,\infty)$ into the topological space $[0,\infty)$; the topology on $\mathbb{N}$ is the discrete topology.

Our results rely on a nonlinear alternative of Leray-Schauder type (to be found in [113]) and on a compactness criterion [24, 114, 238] in $B(\mathbb{N})$ (the Banach space of all continuous, bounded mappings from $\mathbb{N}$ to $\mathbb{R}$, endowed with the usual supremum norm, i.e., $|u|_\infty=\sup_{i\in\mathbb{N}}|u(i)|$ for $u\in B(\mathbb{N})$). In the following, $\overline{U}$ and ∂U denote the closure and the boundary of a set U, respectively.

Theorem 7.2.1 (Leray-Schauder alternative). *Let C be a closed convex subset of a Banach space E. Suppose U is an open subset of C with $p^*\in U$. Also $N:\overline{U}\to C$ is a continuous, condensing map such that $N(\overline{U})$ is bounded. Then one of the following hold:*

- (a_1) *N has a fixed point in $\overline{U}$,*
- (a_2) *there is $y\in\partial U$ and $\lambda\in(0,1)$ with $y=(1-\lambda)p^*+\lambda Ny$.*

Theorem 7.2.2. *Let E be a uniformly bounded subset of the Banach space $B(\mathbb{N})$. If E is equiconvergent at ∞, it is also relatively compact.*

Remark 7.2.3. We note that the results of this section could be established using Krasnoselskii's fixed point theorem instead of Theorem 7.2.1.

Now we prove the following result.

Theorem 7.2.4. *Suppose that conditions (i_1)–(i_4) hold. Moreover, assume that the following two conditions are satisfied:*

$$|p|\neq 1, \tag{7.2.2}$$

and there exists $K>0$ and $m\in\mathbb{N}$ with

$$\sum_{k=m}^{\infty}\frac{1}{c(k)}\sum_{i=k}^{\infty}\sup_{w\in[K/2,K]}F(i+1,w)<\infty. \tag{7.2.3}$$

Then equation (7.2.1) has a bounded nonoscillatory solution.

Proof. Let $\nu = \max\{\tau, \sigma\}$. We will distinguish the following two cases:

(I) $|p| < 1$,
(II) $|p| > 1$.

For case (I), assume $|p| < 1$. Choose a positive integer $T > \max\{\nu, m\}$ sufficiently large so that

$$\sum_{k=T}^{\infty} \frac{1}{c(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i+1, w) < \frac{1}{4}(1 - |p|)K. \tag{7.2.4}$$

Then there exists $\varepsilon > 0$ with $\varepsilon < K/2$ and

$$\sum_{k=T}^{\infty} \frac{1}{c(k)} \sum_{i=k}^{\infty} \sup_{w \in [K/2, K]} F(i+1, w) \leq \frac{1}{4}(1 - |p|)K - \varepsilon. \tag{7.2.5}$$

We wish to apply Theorem 7.2.1. For notational purposes let

$$\mathbb{N}(T - \nu) = \{T - \nu, T - \nu + 1, \ldots\}. \tag{7.2.6}$$

We will apply Theorem 7.2.1 with $E = (B(\mathbb{N}(T - \nu)), |\cdot|_\infty)$,

$$\begin{gathered} C = \left\{x \in B(\mathbb{N}(T-\nu)) : x(i) \geq \frac{K}{2} \text{ for } i \in \mathbb{N}(T-\nu)\right\}, \\ U = \{x \in C : |x|_\infty < K\}, \end{gathered} \tag{7.2.7}$$

and with $p^* = K - \varepsilon$,

$$\begin{gathered} N_1x(i) = \begin{cases} \dfrac{3}{4}(1+p)K - px[T-\tau] & \text{for } i \in \{T-\nu, \ldots, T\}, \\ \dfrac{3}{4}(1+p)K - px[i-\tau] & \text{for } i \in \{T+1, T+2, \ldots\}, \end{cases} \\ N_2x(i) = \begin{cases} 0 & \text{for } i \in \{T-\nu, \ldots, T\}, \\ \displaystyle\sum_{k=T}^{i-1} \frac{1}{c(k)} \sum_{j=k}^{\infty} F\big(j+1, x[j+1-\sigma]\big) & \text{for } i \in \{T+1, \ldots\}. \end{cases} \end{gathered} \tag{7.2.8}$$

Notice $p^* \in U$ since $0 < \varepsilon < K/2$. First we show

$$N = N_1 + N_2 : \overline{U} \longrightarrow C. \tag{7.2.9}$$

To see this take $x \in \overline{U}$, so in particular $K/2 \le x(i) \le K$ for $i \in \mathbb{N}(T-\nu)$. Now we consider the following two subcases:

(I_1) $0 \le p < 1$,
(I_2) $-1 < p < 0$.

Subcase (I_1). $0 \le p < 1$. If $i \in \{T+1, T+2, \dots\}$, then we have

$$\begin{aligned} N_1x(i) + N_2x(i) &\ge \frac{3}{4}(1+p)K - px[i-\tau] \\ &\ge \frac{3}{4}(1+p)K - pK = \left(\frac{3}{4} - \frac{p}{4}\right)K \\ &\ge \frac{K}{2}, \end{aligned} \tag{7.2.10}$$

whereas if $i \in \{T-\nu, \dots, T\}$, then we have

$$\begin{aligned} N_1x(i) + N_2x(i) &= \frac{3}{4}(1+p)K - px[T-\tau] \\ &\ge \frac{3}{4}(1+p)K - pK \\ &\ge \frac{K}{2}. \end{aligned} \tag{7.2.11}$$

As a result

$$\frac{1}{2}K \le N_1x(i) + N_2x(i) \quad \text{for } i \in \mathbb{N}(T-\nu) \text{ for every } x \in \overline{U}. \tag{7.2.12}$$

Thus (7.2.9) holds in this case.

Subcase (I_2). $-1 < p < 0$. If $i \in \{T+1, T+2, \dots\}$, then we have

$$N_1x(i) + N_2x(i) \ge \frac{3}{4}(1+p)K - \frac{1}{2}pK = \left(\frac{3}{4} + \frac{1}{4}p\right)K \ge \frac{1}{2}K, \tag{7.2.13}$$

whereas if $i \in \{T-\nu, \dots, T\}$, then we have

$$N_1x(i) + N_2x(i) = \frac{3}{4}(1+p)K - px[T-\tau] \ge \left(\frac{3}{4} + \frac{1}{4}p\right)K \ge \frac{1}{2}K. \tag{7.2.14}$$

Thus (7.2.9) holds in this case also.

Next, we show

$$N_2 : \overline{U} \longrightarrow E \quad \text{is a continuous, compact map.} \tag{7.2.15}$$

The continuity of N_2 is immediate from (i_4). To see that $N_2\overline{U}$ is relatively compact, we will use Theorem 7.2.2. Clearly, $X = \{N_2 x : x \in \overline{U}\}$ is a uniformly bounded subset of $B(\mathbb{N}(T-\nu))$. Also, if $x \in \overline{U}$ and $i \in \{T+1, T+2, \dots\}$, then we have

$$\left| N_2 x(\infty) - N_2 x(i) \right| \le \sum_{k=i}^{\infty} \frac{1}{a(k)} \sum_{j=k}^{\infty} \sup_{w \in [K/2, K]} F(j+1, w), \tag{7.2.16}$$

so X is equiconvergent at ∞. Theorem 7.2.2 guarantees that $N_2\overline{U}$ is a relatively compact subset of $B(\mathbb{N}(T-\nu))$. Next, we claim that

$$N_1 : \overline{U} \longrightarrow E \quad \text{is a contractive map.} \tag{7.2.17}$$

To see this notice if $x_1, x_2 \in \overline{U}$ and $i \in \{T-\nu, \dots, T\}$, then we have

$$\left| N_1 x_1(i) - N_1 x_2(i) \right| = \left| p\{x_1[T-\tau] - x_2[T-\tau]\} \right| \le |p| \left| x_1 - x_2 \right|_{\infty}, \tag{7.2.18}$$

whereas if $i \in \{T+1, T+2, \dots\}$, then we have

$$\left| N_1 x_1(i) - N_1 x_2(i) \right| = \left| p\{x_1[i-\tau] - x_2[i-\tau]\} \right| \le |p| \left| x_1 - x_2 \right|_{\infty}. \tag{7.2.19}$$

Combining (7.2.18) and (7.2.19) gives

$$\left| N_1 x_1 - N_1 x_2 \right|_{\infty} \le |p| \left| x_1 - x_2 \right|_{\infty}, \tag{7.2.20}$$

so (7.2.17) is true since $|p| < 1$. Now (7.2.15) and (7.2.17) guarantee that

$$N : \overline{U} \longrightarrow C \quad \text{is a continuous, condensing map.} \tag{7.2.21}$$

Next, we show that Theorem 7.2.1(a_2) cannot occur. Suppose $x \in B(\mathbb{N}(T-\nu))$ is a solution of

$$x = (1-\lambda)p^* + \lambda N x \tag{7.2.22}$$

for some $\lambda \in (0,1)$ with $x \in \partial U$. Notice $K/2 \le x(i) \le K$ for $i \in \mathbb{N}(T-\nu)$. We will distinguish the following two subcases:

(I_1) $0 \le p < 1$,

(I_2) $-1 < p < 0$.

Subcase (I_1). $0 \le p < 1$. If $i \in \{T+1, T+2, \dots\}$, then we have

$$\begin{aligned} x(i) &= (1-\lambda)p^* + \lambda \left| N_1 x(i) + N_2 x(i) \right| \\ &\le (1-\lambda)[K-\varepsilon] \\ &\quad + \lambda\left[\frac{3}{4}(1+p)K - px[i-\tau] + \sum_{k=T}^{\infty} \frac{1}{c(k)} \sum_{j=k}^{\infty} \sup_{w\in[K/2,K]} F(j+1,w)\right], \end{aligned} \tag{7.2.23}$$

and so (7.2.5) implies

$$\begin{aligned} \sup_{i\in\{T+1,\dots\}} x(i) &\le (1-\lambda)[k-\varepsilon] + \lambda\left[\frac{3}{4}(1+p)K - \frac{1}{2}pK + \left\{\frac{1}{4}(1-p)K - \varepsilon\right\}\right] \\ &= (1-\lambda)[K-\varepsilon] + \lambda[K-\varepsilon] \\ &= K - \varepsilon \\ &< K. \end{aligned} \tag{7.2.24}$$

Thus

$$\sup_{i\in\{T+1,T+2,\dots\}} x(i) < K. \tag{7.2.25}$$

Now if $i \in \{T-\nu, \dots, T\}$, then we have

$$x(i) = (1-\lambda)p^* + \lambda N_1 x(i) \le (1-\lambda)[K-\varepsilon] + \lambda\left[\frac{3}{4}(1+p)K - \frac{1}{2}pK\right], \tag{7.2.26}$$

and so

$$\begin{aligned} \sup_{i\in\{T-\nu,\dots,T\}} x(i) &\le (1-\lambda)[K-\varepsilon] + \lambda\left[\frac{3}{4} + \frac{1}{4}p\right]K \\ &< (1-\lambda)[K-\varepsilon] + \lambda K \\ &< K. \end{aligned} \tag{7.2.27}$$

Thus

$$\sup_{i\in\{T-v,\ldots,T\}} x(i) < K. \tag{7.2.28}$$

Combining (7.2.25) and (7.2.28) gives

$$\sup_{i\in\mathbb{N}(T-v)} x(i) < K. \tag{7.2.29}$$

This is a contradiction since $K = |x|_\infty = \sup_{i\in\mathbb{N}(T-v)} x(i)$.

Subcase (I₂). $-1 < p < 0$. If $i \in \{T+1, T+2, \ldots\}$, then we have

$$x(i) \le (1-\lambda)[K-\varepsilon] + \lambda\left[\frac{3}{4}(1+p)K - pK + \sum_{k=T}^{\infty}\frac{1}{c(k)}\sum_{j=k}^{\infty}\sup_{w\in[K/2,K]} F(j+1,w)\right]. \tag{7.2.30}$$

As a result

$$\begin{aligned}\sup_{i\in\{T+1,T+2,\ldots\}} x(i) &\le (1-\lambda)[K-\varepsilon] + \lambda\left[\frac{3}{4}(1+p)K - pK + \left(\frac{1}{4}(1+p)K - \varepsilon\right)\right]\\ &= (1-\lambda)[K-\varepsilon] + \lambda[K-\varepsilon]\\ &= K - \varepsilon\\ &< K.\end{aligned} \tag{7.2.31}$$

Thus

$$\sup_{i\in\{T+1,T+2,\ldots\}} x(i) < K. \tag{7.2.32}$$

Now if $i \in \{T-v,\ldots,T\}$, then we have

$$x(i) \le (1-\lambda)[K-\varepsilon] + \lambda\left[\frac{3}{4}(1+p)K - pK\right], \tag{7.2.33}$$

and so

$$\begin{aligned}\sup_{i\in\{T-v,\ldots,T\}} x(i) &\le (1-\lambda)[K-\varepsilon] + \lambda\left[\frac{3}{4} - \frac{1}{4}p\right]K\\ &< (1-\lambda)[K-\varepsilon] + \lambda K\\ &< K.\end{aligned} \tag{7.2.34}$$

Thus

$$\sup_{i\in\{T-v,\ldots,T\}} x(i) < K. \tag{7.2.35}$$

Combining (7.2.32) and (7.2.35) gives $\sup_{i\in\mathbb{N}(T-\nu)} x(i) < K$, which is a contradiction.

Theorem 7.2.1 implies that there exists $x \in \overline{U}$ with $x = N_1x + N_2x$. Hence, for $i \in \{T+1, T+2, \dots\}$ we have

$$x(i) = \frac{3}{4}(1+p)K - px[i-\tau] + \sum_{k=T}^{i-1} \frac{1}{c(k)} \sum_{j=k}^{\infty} F(j+1, x[j+1-\sigma]), \qquad (7.2.36)$$

so the proof is complete in this case.

For case (II), assume $|p| > 1$. Choose a positive integer $T > \max\{\nu, m\}$ sufficiently large so that

$$\sum_{k=T}^{\infty} \frac{1}{c(k)} \sum_{i=k}^{\infty} \sup_{w\in[K/2,K]} F(i+1, w) < \frac{1}{4}(|p|-1)K. \qquad (7.2.37)$$

Then there exists $\varepsilon > 0$ with $\varepsilon < K/2$ and

$$\sum_{k=T}^{\infty} \frac{1}{c(k)} \sum_{i=k}^{\infty} \sup_{w\in[K/2,K]} F(i+1, w) \le \frac{1}{4}(|p|-1)K - \varepsilon. \qquad (7.2.38)$$

Let E, C, U, and p^* be as in case (I) with

$$N_1x(i) = \begin{cases} \dfrac{3}{4}\left(\dfrac{1+p}{p}\right)K - \dfrac{1}{p}x[T+\tau] & \text{if } i \in \{T-\nu, \dots, T\}, \\ \dfrac{3}{4}\left(\dfrac{1+p}{p}\right)K - \dfrac{1}{p}x[i+\tau] & \text{if } i \in \{T+1, T+2, \dots\}, \end{cases}$$
$$N_2x(i) = \begin{cases} 0 & \text{if } i \in \{T-\nu, \dots, T\}, \\ \dfrac{1}{p}\displaystyle\sum_{k=T}^{i+\tau-1} \frac{1}{c(k)} \sum_{j=k}^{\infty} F(j+1, x[j+1-\sigma]) & \text{if } i \in \{T+1, \dots\}. \end{cases} \qquad (7.2.39)$$

A slight modification of the argument in case (I) shows that $N = N_1 + N_2 : \overline{U} \to C$ is a continuous, condensing map, and any solution x to (7.2.22) satisfies $|x|_\infty \ne K$. Now apply Theorem 7.2.1. □

In Theorem 7.2.4 it is possible to replace condition (7.2.3) with the following less restrictive condition. There exist a constant $K > 0$ and $m \in \mathbb{N}$ with

$$\sum_{k=m}^{\infty} \frac{1}{c(k)} \sum_{i=m}^{k-1} \sup_{w\in[K/2,K]} F(i+1, w) < \infty. \qquad (7.2.40)$$

The proof is essentially the same as the proof of Theorem 7.2.4; the only difference is that we write N_2 in case (I) as

$$N_2x(i) = \begin{cases} 0 & \text{if } i \in \{T-\nu,\dots,T\}, \\ \displaystyle\sum_{k=i}^{\infty} \frac{1}{c(k)} \sum_{j=T}^{k-1} F(j+1, x[j+1-\sigma]) & \text{if } i \in \{T+1, T+2,\dots\}, \end{cases} \tag{7.2.41}$$

and N_2 in case (II) as

$$N_2x(i) = \begin{cases} 0 & \text{if } i \in \{T-\nu,\dots,T\}, \\ \displaystyle\frac{1}{p} \sum_{k=i+\tau}^{\infty} \frac{1}{c(k)} \sum_{j=T}^{k-1} F(j+1, x[j+1-\sigma]) & \text{if } i \in \{T+1,\dots\}. \end{cases} \tag{7.2.42}$$

Thus we have shown the following result.

Theorem 7.2.5. *Suppose that (i_1)–(i_4) are satisfied. Moreover, assume there exist* $K > 0$ *and* $m \in \mathbb{N}$ *with (7.2.40). Then equation (7.2.1) has a bounded nonoscillatory solution.*

Remark 7.2.6. Only minor adjustments are necessary to discuss equations of the form

$$\Delta\big(c(k)\Psi(\Delta[x(k) + px[k-\tau]])\big) + F(k+1, x[k+1-\sigma]) = 0 \quad \text{for } k \in \mathbb{N}, \tag{7.2.43}$$

where the function Ψ is as in equation (7.1.2) while $\{c(k)\}$, p, τ, σ, and F are as in equation (7.2.1). In fact, (7.2.3) and (7.2.40) take the form

$$\begin{aligned} &\sum_{k=m}^{\infty} \Psi^{-1}\left[\frac{1}{c(k)} \sum_{i=k}^{\infty} \sup_{w\in[K/2,K]} F(i+1, w)\right] < \infty, \\ &\sum_{k=m}^{\infty} \Psi^{-1}\left[\frac{1}{c(k)} \sum_{i=m}^{k-1} \sup_{w\in[K/2,K]} F(i+1, w)\right] < \infty, \end{aligned} \tag{7.2.44}$$

respectively. We leave the details to the reader.

Remark 7.2.7. The results of this section can be obtained with minor modification for higher-order equations. Again the details are left to the reader.

7.3. Existence of nonoscillatory solutions

Consider the neutral difference equation with positive and negative coefficients

$$\Delta^2[x(k)+px[k-\tau]]+q_1(k)x[k-\sigma_1]-q_2(k)x[k-\sigma_2]=0, \tag{7.3.1}$$

where

(i) $p\in\mathbb{R}\setminus\{-1,1\}$,
(ii) $\tau>0$, $\sigma_1,\sigma_2\geq 0$ are integers,
(iii) $\{q_i(k)\}$, $i\in\{1,2\}$, are sequences of nonnegative real numbers.

Here we will present some sufficient conditions for the existence of a nonoscillatory solution of equation (7.3.1).

Now we prove the following result.

Theorem 7.3.1. *Let conditions (i)–(iii) hold. If*

$$\sum_{j=m\in\mathbb{N}}^{\infty} jq_i(j)<\infty \quad \textit{for } i\in\{1,2\}, \tag{7.3.2}$$

and for all sufficiently large $m_1\geq m\in\mathbb{N}$ and a constant $a>0$,

$$q_1(k)-aq_2(k)\geq 0 \quad \textit{for } k\geq m_1, \tag{7.3.3}$$

then equation (7.3.1) has a nonoscillatory solution.

Proof. The proof of this theorem will be divided into four cases depending on the four different ranges of the parameter p.

Case 1. $p\in[0,1)$. Choose an integer $m_2>m\in\mathbb{N}$ sufficiently large such that

$$m_2\geq\max\{m_1,m+\sigma\}, \quad \text{where } \sigma=\max\{\tau,\sigma_1,\sigma_2\}, \tag{7.3.4}$$

$$\sum_{j=m_2}^{\infty} j[q_1(j)+q_2(j)]<1-p, \tag{7.3.5}$$

$$0\leq\sum_{j=m_2}^{\infty} j[b_2q_1(j)-b_1q_2(j)]\leq p-1+b_2, \tag{7.3.6}$$

$$\sum_{j=m_1}^{\infty} j[b_1q_1(j)-b_2q_2(j)]\geq 0 \tag{7.3.7}$$

hold, where $b_1, b_2 > 0$ are constants satisfying

$$1 - b_2 < p \leq \frac{1 - b_1}{1 + b_2}. \tag{7.3.8}$$

Consider the Banach space $\ell_\infty(m)$ of all real sequences $x = \{x(k)\}$ with the norm $\|x\| = \sup_{k \geq m} |x(k)|$. Set

$$X = \{x \in \ell_\infty(m) : b_1 \leq x(k) \leq b_2,\ k \geq m\}. \tag{7.3.9}$$

Define a mapping $T : X \to \ell_\infty(m)$ by

$$(Tx)(k) = \begin{cases} 1 - p - px[k-\tau] \\ +(k-1)\sum_{j=k}^{\infty}[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]] \\ +\sum_{j=m_2}^{k-1} j[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]] & \text{for } k \geq m_2, \\ (Tx)(m_2) & \text{for } m \leq k \leq m_2. \end{cases} \tag{7.3.10}$$

We will show that $TX \subset X$. For every $x \in X$ and $k \geq m_2$, using (7.3.3) and (7.3.6) we get

$$\begin{aligned} (Tx)(k) &= 1 - p - px[k-\tau] + (k-1)\sum_{j=k}^{\infty}[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]] \\ &\quad + \sum_{j=m_2}^{k-1} j[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]] \\ &\leq 1 - p + (k-1)\sum_{j=k}^{\infty}[b_2q_1(j) - b_1q_2(j)] + \sum_{j=m_2}^{k-1} j[b_2q_1(j) - b_1q_2(j)] \\ &\leq 1 - p + \sum_{j=k}^{\infty} j[b_2q_1(j) - b_1q_2(j)] + \sum_{j=m_2}^{k-1} j[b_2q_1(j) - b_1q_2(j)] \\ &= 1 - p + \sum_{j=m_2}^{\infty} j[b_2q_1(j) - b_1q_2(j)] \\ &\leq b_2. \end{aligned} \tag{7.3.11}$$

Furthermore, in view of (7.3.3) and (7.3.7) we have

$$
\begin{aligned}
(Tx)(t) &= 1 - p - px[k-\tau] + (k-1)\sum_{j=k}^{\infty}\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\
&\quad + \sum_{j=m_2}^{k-1} j\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\
&\geq 1-p-b_2p+(k-1)\sum_{j=k}^{\infty}\left[b_1q_1(j)-b_2q_2(j)\right]+\sum_{j=m_2}^{k-1} j\left[b_1q_1(j)-b_2q_2(j)\right] \\
&\geq 1-p-b_2p \\
&\geq b_1.
\end{aligned}
\tag{7.3.12}
$$

Thus we have proved that $TX \subset X$. Since X is a bounded, closed, and convex subset of $\ell_\infty(m)$, we have to prove that T is a contraction mapping on X to apply the contraction principle. Now for $x_1, x_2 \in X$ and $k \geq m_2$ we have

$$
\begin{aligned}
&\left|(Tx_1)(k) - (Tx_2)(k)\right| \\
&\quad \leq p\left|x_1[k-\tau] - x_2[k-\tau]\right| + (k-1)\sum_{j=k}^{\infty} q_1(j)\left|x_1[j-\sigma_1] - x_2[j-\sigma_1]\right| \\
&\qquad + (k-1)\sum_{j=k}^{\infty} q_2(j)\left|x_1[j-\sigma_2] - x_2[j-\sigma_2]\right| \\
&\qquad + \sum_{j=m_2}^{k-1} jq_1(j)\left|x_1[j-\sigma_1] - x_2[j-\sigma_1]\right| \\
&\qquad + \sum_{j=m_2}^{k-1} jq_2(j)\left|x_1[j-\sigma_2] - x_2[j-\sigma_2]\right| \\
&\quad \leq p\|x_1-x_2\|+\|x_1-x_2\|\left[\sum_{j=k}^{\infty}(k-1)\left[q_1(j)+q_2(j)\right]+\sum_{j=m_2}^{k-1} j\left[q_1(j)+q_2(j)\right]\right] \\
&\quad \leq \left[p + \sum_{j=m_2}^{\infty} j\left[q_1(j)+q_2(j)\right]\right]\|x_1-x_2\| \\
&\quad = \lambda_1\|x_1-x_2\|.
\end{aligned}
\tag{7.3.13}
$$

This implies that $\|Tx_1 - Tx_2\| \le \lambda_1\|x_1 - x_2\|$, where in view of (7.3.5), $\lambda_1 < 1$. This proves that T is a contraction mapping on T. By Theorem 4.4.15, T has a unique fixed point x in X, which is obviously a positive solution of equation (7.3.1). This completes the proof in Case 1.

Case 2. $p \in (1,\infty)$. Choose an integer $m_2 > m_1 > m \in \mathbb{N}$ sufficiently large such that

$$m_2 + \tau \ge m + \max\{\sigma_1, \sigma_2\}, \tag{7.3.14}$$

$$\sum_{j=m_2}^{\infty} j[q_1(j) + q_2(j)] < p - 1, \tag{7.3.15}$$

$$0 \le \sum_{j=m_2}^{\infty} j[c_2q_1(j) - c_1q_2(j)] \le 1 - p + pc_2, \tag{7.3.16}$$

$$\sum_{j=m_2}^{\infty} j[c_1q_1(j) - c_2q_2(j)] \ge 0, \tag{7.3.17}$$

where $c_1, c_2 > 0$ are constants such that

$$(1 - c_1)p \ge 1 + c_2, \qquad p(1 - c_2) < 1. \tag{7.3.18}$$

Let $\ell_\infty(m)$ be the set as in the proof of Case 1. Set

$$X = \{x \in \ell_\infty(m) : c_1 \le x(k) \le c_2,\ k \ge m\}. \tag{7.3.19}$$

Define a mapping $T : X \to \ell_\infty(m)$ by

$$(Tx)(k) = \begin{cases} 1 - \dfrac{1}{p} - \dfrac{1}{p}x[k+\tau] \\ +\dfrac{k+\tau-1}{p}\displaystyle\sum_{j=k+\tau}^{\infty}[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]] \\ +\dfrac{1}{p}\displaystyle\sum_{j=m_2}^{k+\tau-1} j[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]] & \text{for } k \ge m_2, \\ (Tx)(m_2) & \text{for } m \le k \le m_2. \end{cases} \tag{7.3.20}$$

We will show that $TX \subset X$. For every $x \in X$ and $k \geq m_2$, using (7.3.14) and (7.3.16), we get

$$\begin{aligned}
(Tx)(k) &= 1 - \frac{1}{p} - \frac{1}{p}x[k+\tau] \\
&\quad + \frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\
&\quad + \frac{1}{p}\sum_{j=m_2}^{k+\tau-1} j\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\
&\leq 1 - \frac{1}{p} + \frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\left[c_2q_1(j) - c_1q_2(j)\right] \\
&\quad + \frac{1}{p}\sum_{j=m_2}^{k+\tau-1} j\left[c_2q_1(j) - c_1q_2(j)\right] \\
&\leq 1 - \frac{1}{p} + \frac{1}{p}\left[\sum_{j=k+\tau}^{\infty} j\left[c_2q_1(j) - c_1q_2(j)\right] + \sum_{j=m_2}^{k+\tau-1} j\left[c_2q_1(j) - c_1q_2(j)\right]\right] \\
&= 1 - \frac{1}{p} + \frac{1}{p}\sum_{j=m_2}^{\infty} j\left[c_2q_1(j) - c_1q_2(j)\right] \\
&\leq c_2.
\end{aligned} \tag{7.3.21}$$

Furthermore, in view of (7.3.17) we have

$$\begin{aligned}
(Tx)(k) &= 1 - \frac{1}{p} - \frac{1}{p}x[k+\tau] \\
&\quad + \frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\
&\quad + \frac{1}{p}\sum_{j=m_2}^{k+\tau-1} j\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\
&\geq 1 - \frac{1}{p} - \frac{c_2}{p} + \frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\left[c_1q_1(j) - c_2q_2(j)\right] \\
&\quad + \frac{1}{p}\sum_{j=m_2}^{k+\tau-1} j\left[c_1q_1(j) - c_2q_2(j)\right] \\
&\geq 1 - \frac{1}{p} - \frac{c_2}{p} \\
&\geq c_1.
\end{aligned} \tag{7.3.22}$$

Thus, we have proved that $TX \subset X$. Now as in the proof of Case 1, we only need to show that T is a contraction mapping on X. For $x_1, x_2 \in X$ and $k \geq m_2$ we have

$$
\begin{aligned}
&\left|(Tx_1)(k) - (Tx_2)(k)\right| \\
&\quad \leq \frac{1}{p}\left|x_1[k+\tau] - x_2[k+\tau]\right| \\
&\qquad + \frac{k+\tau-1}{p}\left[\sum_{j=k+\tau}^{\infty} q_1(j)\left|x_1[j-\sigma_1] - x_2[j-\sigma_1]\right|\right. \\
&\qquad\qquad \left. + \sum_{j=k+\tau}^{\infty} q_2(j)\left|x_1[j-\sigma_2] - x_2[j-\sigma_2]\right|\right] \\
&\qquad + \frac{1}{p}\left[\sum_{j=m_2}^{k+\tau-1} jq_1(j)\left|x_1[j-\sigma_1] - x_2[j-\sigma_1]\right|\right. \\
&\qquad\qquad \left. + \sum_{j=m_2}^{k+\tau-1} jq_2(j)\left|x_1[j-\sigma_2] - x_2[j-\sigma_2]\right|\right] \\
&\quad \leq \frac{1}{p}\|x_1 - x_2\| \\
&\qquad + \frac{1}{p}\|x_1 - x_2\|\left[\sum_{j=k+\tau}^{\infty} j\left[q_1(j) + q_2(j)\right] + \sum_{j=m_2}^{k+\tau-1} j\left[q_1(j) + q_2(j)\right]\right] \\
&\quad = \frac{1}{p}\left[1 + \sum_{j=m_2}^{\infty} j\left[q_1(j) + q_2(j)\right]\right]\|x_1 - x_2\| \\
&\quad = \lambda_2\|x_1 - x_2\|.
\end{aligned}
\tag{7.3.23}
$$

This implies that $\|Tx_1 - Tx_2\| \leq \lambda_2\|x_1 - x_2\|$, where in view of (7.3.15), $\lambda_2 < 1$. This proves that T is a contraction mapping. Consequently, T has a unique fixed point x, which is obviously a positive solution of equation (7.3.1). This completes the proof in Case 2.

Case 3. $p \in (-1, 0)$. Choose an integer $m_2 > m_1 > m \in \mathbb{N}$ sufficiently large so that (7.3.4) and the inequalities

$$\sum_{j=m_2}^{\infty} j\left[q_1(j) + q_2(j)\right] < p + 1, \tag{7.3.24}$$

$$0 \leq \sum_{j=m_2}^{\infty} j\left[b_4 q_1(j) - b_3 q_2(j)\right] \leq (p+1)(b_4 - 1) \tag{7.3.25}$$

hold, where the positive constants b_3 and b_4 satisfy

$$0 < b_3 \leq 1 < b_4. \tag{7.3.26}$$

Let $\ell_\infty(m)$ be the set as in Case 1. Set

$$X = \{x \in \ell_\infty(m) : b_3 \le x(k) \le b_4,\ k \ge m\}. \tag{7.3.27}$$

Define a mapping $T : X \to \ell_\infty(m)$ by

$$(Tx)(k) = \begin{cases} 1 + p - px[k-\tau] & \\ +(k-1)\displaystyle\sum_{j=k}^{\infty}\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] & \\ +\displaystyle\sum_{j=m_2}^{k-1} j\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] & \text{for } k \ge m_2, \\ (Tx)(m_2) & \text{for } m \le k \le m_2. \end{cases} \tag{7.3.28}$$

As in the proof of Case 1, for every $x \in X$ and $k \ge m_2$, using (7.3.25), we get

$$\begin{aligned} (Tx)(k) &= 1 + p - px[k-\tau] + (k-1)\sum_{j=k}^{\infty}\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\ &\quad + \sum_{j=m_2}^{k-1} j\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\ &\le 1+p-pb_4+(k-1)\sum_{j=k}^{\infty}\left[b_4q_1(j)-b_3q_2(j)\right]+\sum_{j=m_2}^{k-1} j\left[b_4q_1(j)-b_3q_2(j)\right] \\ &\le 1 + p - pb_4 + \sum_{j=m_2}^{\infty} j\left[b_4q_1(j) - b_3q_2(j)\right] \\ &= b_4. \end{aligned} \tag{7.3.29}$$

Furthermore, in view of (7.3.26) we have

$$\begin{aligned} (Tx)(k) &= 1 + p - px[k-\tau] + (k-1)\sum_{j=k}^{\infty}\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\ &\quad + \sum_{j=m_2}^{k-1} j\left[q_1(j)x[j-\sigma_1] - q_2(j)x[j-\sigma_2]\right] \\ &\ge 1+p-pb_3+(k-1)\sum_{j=k}^{\infty}\left[b_3q_1(j)-b_4q_2(j)\right]+\sum_{j=m_2}^{k-1} j\left[b_3q_1(j)-b_4q_2(j)\right] \\ &\ge 1 + p - pb_3 \\ &\ge b_3. \end{aligned} \tag{7.3.30}$$

Thus we have proved that $TX \subset X$. As before, we will show that T is a contraction mapping. For $x_1, x_2 \in X$ and $k \geq m_2$, we have

$$\begin{aligned}
&\big|(Tx_1)(k) - (Tx_2)(k)\big| \\
&\quad \leq -p\big|x_1[k-\tau] - x_2[k-\tau]\big| + (k-1)\sum_{j=k}^{\infty} q_1(j)\big|x_1[j-\sigma_1] - x_2[j-\sigma_1]\big| \\
&\qquad + (k-1)\sum_{j=k}^{\infty} q_2(j)\big|x_1[j-\sigma_2] - x_2[j-\sigma_2]\big| \\
&\qquad + \sum_{j=m_2}^{k-1} jq_1(j)\big|x_1[j-\sigma_1] - x_2[j-\sigma_1]\big| \\
&\qquad + \sum_{j=m_2}^{k-1} jq_2(j)\big|x_1[j-\sigma_2] - x_2[j-\sigma_2]\big| \\
&\quad \leq -p||x_1 - x_2|| + ||x_1 - x_2||\left[\sum_{j=k}^{\infty} j[q_1(j) + q_2(j)] + \sum_{j=m_2}^{k-1} j[q_1(j) + q_2(j)]\right] \\
&\quad = \left[-p + \sum_{j=m_2}^{\infty} j[q_1(j) + q_2(j)]\right]||x_1 - x_2|| \\
&\quad = \lambda_3||x_1 - x_2||.
\end{aligned} \tag{7.3.31}$$

This implies that $\|Tx_1 - Tx_2\| \leq \lambda_3\|x_1 - x_2\|$, where in view of (7.3.24), $\lambda_3 < 1$. This proves that T is a contraction mapping. Consequently, T has a unique fixed point x, which is obviously a positive solution of equation (7.3.1). This completes the proof in Case 3.

Case 4. $p \in (-\infty, -1)$. Choose an integer $m_2 > m_1 > m \in \mathbb{N}$ sufficiently large such that (7.3.14) and the inequalities

$$\sum_{j=m_2}^{\infty} j[q_1(j) + q_2(j)] < -p - 1, \tag{7.3.32}$$

$$0 \leq \sum_{j=m_2}^{\infty} j[c_4q_1(j) - c_3q_2(j)] \leq (p+1)(c_3 - 1) \tag{7.3.33}$$

hold, where the positive constants c_3 and c_4 satisfy

$$0 < c_3 < 1 < c_4. \tag{7.3.34}$$

Let $\ell_\infty(m)$ be the set as in Case 1. Set

$$X = \{x \in X : c_3 \le x(k) \le c_4,\ k \ge m\}. \tag{7.3.35}$$

Define a mapping $T : X \to \ell_\infty(m)$ by

$$(Tx)(k) = \begin{cases} 1 + \dfrac{1}{p} - \dfrac{1}{p}x[k+\tau] \\ +\dfrac{k+\tau-1}{p}\displaystyle\sum_{j=k+\tau}^{\infty}\big[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]\big] \\ +\dfrac{1}{p}\displaystyle\sum_{j=m_2}^{k+\tau-1} j\big[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]\big] & \text{for } k \ge m_2, \\ (Tx)(m_2) & \text{for } m \le k \le m_2. \end{cases} \tag{7.3.36}$$

Now, for every $x \in X$ and $k \ge m_2$, using (7.3.34), we get

$$\begin{aligned} (Tx)(k) &= 1 + \frac{1}{p} - \frac{1}{p}x[k+\tau] \\ &\quad + \frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\big[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]\big] \\ &\quad + \frac{1}{p}\sum_{j=m_2}^{k+\tau-1} j\big[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]\big] \\ &\le 1 + \frac{1}{p} - \frac{c_4}{p} + \frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\big[c_3q_1(j)-c_4q_2(j)\big] \\ &\quad + \frac{1}{p}\sum_{j=m_2}^{k+\tau-1} j\big[c_3q_1(j)-c_4q_2(j)\big] \\ &\le 1 + \frac{1}{p} - \frac{c_4}{p} \\ &\le c_4. \end{aligned} \tag{7.3.37}$$

Furthermore, in view of (7.3.33) we have

$$
\begin{aligned}
(Tx)(k) &= 1+\frac{1}{p}-\frac{1}{p}x[k+\tau] \\
&\quad +\frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\left[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]\right] \\
&\quad +\frac{1}{p}\sum_{j=m_2}^{k+\tau-1}j\left[q_1(j)x[j-\sigma_1]-q_2(j)x[j-\sigma_2]\right] \\
&\geq 1+\frac{1}{p}-\frac{c_3}{p}+\frac{k+\tau-1}{p}\sum_{j=k+\tau}^{\infty}\left[c_4q_1(j)-c_3q_2(j)\right] \\
&\quad +\frac{1}{p}\sum_{j=m_2}^{k+\tau-1}j\left[c_4q_1(j)-c_3q_2(j)\right] \\
&\geq 1+\frac{1}{p}-\frac{c_3}{p}+\frac{1}{p}\sum_{j=m_2}^{\infty}j\left[c_4q_1(j)-c_3q_2(j)\right] \\
&\geq 1+\frac{1}{p}-\frac{c_3}{p}+\frac{1}{p}(p+1)(c_3-1) \\
&= c_3.
\end{aligned}
\tag{7.3.38}
$$

Thus we have proved that $TX\subset X$. We only need to show that T is a contraction mapping. For $x_1,x_2\in X$ and $k\geq m_2$ we have

$$
\begin{aligned}
&\left|(Tx_1)(k)-(Tx_2)(k)\right| \\
&\quad\leq -\frac{1}{p}\left|x_1[k+\tau]-x_2[k+\tau]\right| \\
&\qquad -\frac{k+\tau-1}{p}\Bigg[\sum_{j=k+\tau}^{\infty}q_1(j)\left|x_1[j-\sigma_1]-x_2[j-\sigma_1]\right| \\
&\qquad\qquad +\sum_{j=k+\tau}^{\infty}q_2(j)\left|x_1[j-\sigma_2]-x_2[j-\sigma_2]\right|\Bigg] \\
&\qquad -\frac{1}{p}\Bigg[\sum_{j=m_2}^{k+\tau-1}jq_1(j)\left|x_1[j-\sigma_1]-x_2[j-\sigma_1]\right| \\
&\qquad\qquad +\sum_{j=m_2}^{k+\tau-1}jq_2(j)\left|x_1[j-\sigma_2]-x_2[j-\sigma_2]\right|\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\frac{1}{p}||x_1 - x_2|| \\
&\quad -\frac{1}{p}||x_1 - x_2||\left[\sum_{j=k+\tau}^{\infty} j[q_1(j)+q_2(j)] + \sum_{j=m_2}^{k+\tau-1} j[q_1(j)+q_2(j)]\right] \\
&= -\frac{1}{p}\left[1 + \sum_{j=m_2}^{\infty} j[q_1(j)+q_2(j)]\right] \\
&= \lambda_4||x_1 - x_2||.
\end{aligned}
\tag{7.3.39}
$$

This immediately implies that $\|Tx_1 - Tx_2\| \leq \lambda_4\|x_1 - x_2\|$. By (7.3.32), $\lambda_4 < 1$. This proves that T is a contraction mapping. Consequently, T has a unique fixed point x, which is obviously a positive solution of equation (7.3.1). This completes the proof in Case 4.

The proof of the theorem is now complete. □

Remark 7.3.2. The condition (7.3.3), which implies that $q_1(k)$ dominates $q_2(k)$, may look too restrictive. This condition is actually affected by the choice of the constants b_i and c_i for $i \in \{1,2,3,4\}$. Choosing those constants in an appropriate way, we can specify that this condition holds for a single value of a. In this case, this condition becomes very easy to check and to use. For instance, if $b_2 = \alpha b_1$, $b_4 = \alpha b_3$, $c_2 = \alpha c_1$, and $c_4 = \alpha c_3$, then $a = \alpha$ in (7.3.3), where $\alpha > 1$ is a given number. Choosing α to be as close to 1 as we please, we get very precise asymptotic behavior for the nonoscillatory solution we constructed, since in all cases we have $b_1 \leq x(k) \leq \alpha b_1$ and $b_3 \leq x(k) \leq \alpha b_3$, or $c_1 \leq x(k) \leq \alpha c_1$ and $c_3 \leq x(k) \leq \alpha c_3$.

We can also specify our choice of constants by choosing $b_1 = b_3 = c_1 = c_3$ and $b_2 = b_4 = c_2 = c_4$, which can be achieved by taking b_1 and b_2 to satisfy $0 < b_1 < b_2$ and $b_2^2 > b_1$. In this case in all four cases we will have the same asymptotic behavior of nonoscillatory solutions as $b_1 \leq x(k) \leq b_2$ with the same value of $a = b_2/b_1$.

Combining the last two choices of constants we get $b_1 \leq x(k) \leq \alpha b_1$ and $a = \alpha$.

Once again and more precisely, if there exist $a > 0$ and $m_1 \geq m \in \mathbb{N}$ such that condition (7.3.3) holds, one can take a as follows:

(a_{I}) $a > 1/(1-p)$ when $0 \leq p < 1$. In this case, we choose the positive constants b_1 and b_2 in the proof of Case 1 to be b_1 and $b_2 = ab_1$ such that $(1-p)/a < b_1 < (1-p)/(1+pa)$. Here, we note that (7.3.7) is disregarded;

(a_{II}) $a > p/(p-1)$ when $1 < p < \infty$. In this case, we choose the positive constants c_1 and c_2 in the proof of Case 2 to be c_1 and $c_2 = ac_1$ such that $(p-1)/(ap) < c_1 < (p-1)/(p+a)$. Also, we see that (7.3.17) is disregarded;

(a_{III}) $a > 1$ when $-1 < p < 0$. In this case, we choose the positive constants b_3 and b_4 in the proof of Case 3 to be b_3 and $b_4 = ab_3$ such that, replacing (7.3.26), $1/a < b_3 < 1$;

(a_{IV}) $a > 1$ when $-\infty < p < -1$. In this case, we choose the positive constants c_3 and c_4 in the proof of Case 4 to be c_3 and $c_4 = ac_3$ such that, replacing (7.3.34), $1/a \le c_3 < 1$.

Remark 7.3.3. The results of this section can be easily extended to more general equations of the form

$$\Delta\big(c(k)\Delta[x(k) + px[k-\tau]]\big) + q_1(k)x[k-\sigma_1] - q_2(k)x[k-\sigma_2] = 0, \tag{7.3.40}$$

where $\{c(k)\}$ is a sequence of positive real numbers.

Similar results for equation (7.3.40) will require conditions of the form

$$\sum^{\infty} \frac{1}{c(k)} \sum_{j=k}^{\infty} q_i(j) < \infty \quad \text{for } i \in \{1,2\}, \tag{7.3.41}$$

or

$$\sum^{\infty} \frac{1}{c(k)} \sum_{j=m\in\mathbb{N}}^{k-1} q_i(j) < \infty \quad \text{for } i \in \{1,2\}. \tag{7.3.42}$$

Here we omit the details.

Remark 7.3.4. In the special case when $q_2(k) \equiv 0$, condition (7.3.3) is redundant and Theorem 7.3.1 holds under condition (7.3.2). This result is included in Theorem 7.2.1.

Finally, it will be interesting to obtain linearized oscillation results and some oscillation criteria for the equation

$$\Delta\big(c(k)\Delta[x(k) + px[k-\tau]]\big) + q_1(k)f\big(x[k-\sigma_1]\big) - q_2(k)f\big(x[k-\sigma_2]\big) = 0 \tag{7.3.43}$$

for all values of parameters $|p| \ne 1$ and $f \in C(\mathbb{R},\mathbb{R})$.

7.4. Classification of nonoscillatory solutions

We will consider second-order nonlinear neutral difference equations of the form

$$\Delta^2\big(x(k) - p(k)x[k-\tau]\big) + \delta F\big(k, x[g(k)]\big) = 0, \quad \text{where } \delta = \pm 1, \tag{7.4.1}$$

more precisely,

$$\Delta^2\big(x(k) - p(k)x[k-\tau]\big) + F\big(k, x[g(k)]\big) = 0, \tag{7.4.1$^+$}$$

$$\Delta^2\big(x(k) - p(k)x[k-\tau]\big) - F\big(k, x[g(k)]\big) = 0, \tag{7.4.1$^-$}$$

and throughout we will assume the following:

(i) $\tau \in \mathbb{N}$,

(ii) $\{p(k)\}$ is a sequence of nonnegative real numbers and there is $\sigma \in (0,1]$ such that $p(k) \leq 1-\sigma$ for $k \geq m$ for some $m \in \mathbb{N}$,

(iii) $g : \mathbb{N}(m) \to \mathbb{N}(m) = \{m, m+1, \ldots\}$ and $\lim_{k\to\infty} g(k) = \infty$,

(iv) $F : \mathbb{N}(m) \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to the last argument, and $xF(k,x) > 0$ for $x \neq 0$. Moreover, $|F(k,x)| \geq |F(k,y)|$ when $|x| \geq |y|$ and $xy > 0$.

In this section, we will study the existence and asymptotic behavior of nonoscillatory solutions of equation (7.4.1). More precisely, we give the classification of nonoscillatory solutions of equation (7.4.1) according to its asymptotic behavior. Moreover, we establish some existence results for each kind of nonoscillatory solution of equation (7.4.1). In particular, we present some necessary and sufficient conditions for the existence of nonoscillatory solutions of equation (7.4.1).

7.4.1. Nonoscillatory solutions of equation (7.4.1$^+$)

In this subsection we will present the following two lemmas which are useful in proving the main upcoming results.

Lemma 7.4.1. *Let $\{x(k)\}$ be an eventually positive (negative) solution of equation (7.4.1$^+$). If $\lim_{k\to\infty} x(k) = 0$, then $\{y(k)\}$, where*

$$y(k) = x(k) - p(k)x[k-\tau], \tag{7.4.2}$$

is eventually negative (positive) and $\lim_{k\to\infty} y(k) = 0$. Otherwise, $\{y(k)\}$ is eventually positive (negative).

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.4.1$^+$). Set $y(k)$ as in (7.4.2). Then

$$\Delta^2 y(k) = -F(k, x[g(k)]) < 0 \quad \text{eventually.} \tag{7.4.3}$$

Thus $\Delta y(k)$ is decreasing and $\Delta y(k) > 0$ or $\Delta y(k) < 0$ eventually. Also, we see that $y(k) > 0$ or $y(k) < 0$ eventually. If $\lim_{k\to\infty} x(k) = 0$, then from (7.4.2) we have $\lim_{k\to\infty} y(k) = 0$. Since $\{x(k)\}$ is monotonic, we have $\lim_{k\to\infty} \Delta y(k) = 0$, which implies that $\Delta y(k) > 0$ eventually. Thus $y(k) < 0$ eventually. If $\lim_{k\to\infty} x(k) = 0$ fails to hold, then $\limsup_{k\to\infty} x(k) > 0$. We claim that $y(k) > 0$ eventually. If not, then $y(k) < 0$ eventually. If $\{x(k)\}$ is unbounded, then there exists a sequence $\{k_n\}$ such that $\lim_{n\to\infty} k_n = \infty$, $x(k_n) = \max_{m \leq k \leq k_n}\{x(k)\}$, and $\lim_{n\to\infty} x(k_n) = \infty$. From (7.4.2), we have

$$y(k_n) = x(k_n) - p(k_n)x[k_n - \tau] \geq x(k_n)(1 - p(k_n)). \tag{7.4.4}$$

Thus $\lim_{n\to\infty} y(k_n) = \infty$, which is a contradiction. If $\{x(k)\}$ is bounded, then there exists a sequence $\{k_n\}$ with $\lim_{n\to\infty} k_n = \infty$ and $\lim_{n\to\infty} x(k_n) = \limsup_{k\to\infty} x(k)$.

Since the sequences $\{p(k_n)\}$ and $\{x[k_n-\tau]\}$ are bounded, there exist convergent subsequences. Without loss of generality, we may assume that $\lim_{n\to\infty} x[k_n-\tau]$ and $\lim_{n\to\infty} p(k_n)$ exist. Hence

$$\begin{aligned} 0 &\geq \lim_{n\to\infty} y(k_n) \\ &= \lim_{n\to\infty} \left(x(k_n) - p(k_n)x[k_n-\tau]\right) \\ &\geq \limsup_{k\to\infty} x(k)\Big(1 - \lim_{n\to\infty} p(k_n)\Big) \\ &> 0, \end{aligned} \tag{7.4.5}$$

which is a contradiction. Therefore, $y(k) > 0$ eventually. A similar proof can be presented if $x(k) < 0$ eventually. □

Lemma 7.4.2. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0,1)$ *and* $\{x(k)\}$ *is an eventually positive (negative) solution of equation (7.4.1⁺). If* $\lim_{k\to\infty} y(k) = a \in \mathbb{R}$, *then* $\lim_{k\to\infty} x(k) = a/(1-p)$. *If* $\lim_{k\to\infty} y(k) = \infty\ (-\infty)$, *then* $\lim_{k\to\infty} x(k) = \infty\ (-\infty)$.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation $(7.4.1^+)$. From (7.4.2), we see that $x(k) \geq y(k)$ eventually. If $\lim_{k\to\infty} y(k) = \infty$, then $\lim_{k\to\infty} x(k) = \infty$. Now we consider the case that $\lim_{k\to\infty} y(k) = a \in \mathbb{R}$. Thus $\{y(k)\}$ is bounded which implies that $\{x(k)\}$ is bounded. Therefore there exists a sequence $\{k_n\}$ such that $\lim_{n\to\infty} k_n = \infty$ and $\lim_{n\to\infty} x(k_n) = \limsup_{k\to\infty} x(k)$. As before, without loss of generality, we may assume that $\lim_{n\to\infty} p(k_n)$ and $\lim_{n\to\infty} x[k_n-\tau]$ exist. Hence

$$\begin{aligned} a &= \lim_{n\to\infty} y(k_n) \\ &= \lim_{n\to\infty} x(k_n) - \lim_{n\to\infty} p(k_n) \lim_{n\to\infty} x[k_n-\tau] \\ &\geq \limsup_{k\to\infty} x(k)(1-p), \end{aligned} \tag{7.4.6}$$

that is,

$$\frac{a}{1-p} \geq \limsup_{k\to\infty} x(k). \tag{7.4.7}$$

On the other hand there exists a sequence $\{\overline{k}_n\}$ with $\lim_{n\to\infty} x(\overline{k}_n) = \liminf_{k\to\infty} x(k)$. Without loss of generality, we assume that $\lim_{n\to\infty} p(\overline{k}_n)$ and $\lim_{n\to\infty} x[\overline{k}_n-\tau]$ exist. Hence

$$\begin{aligned} a &= \lim_{n\to\infty} y(\overline{k}_n) \\ &= \lim_{n\to\infty} x(\overline{k}_n) - \lim_{n\to\infty} p(\overline{k}_n) \lim_{n\to\infty} x[\overline{k}_n-\tau] \\ &\leq \liminf_{k\to\infty} x(k)(1-p), \end{aligned} \tag{7.4.8}$$

so

$$\frac{a}{1-p} \le \liminf_{k\to\infty} x(k). \tag{7.4.9}$$

Combining (7.4.7) and (7.4.9), we obtain $\lim_{k\to\infty} x(k) = a/(1-p)$. A similar proof can be given if $x(k) < 0$ eventually. □

Now we present the following result.

Theorem 7.4.3. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0,1)$. *Let* $\{x(k)\}$ *be a nonoscillatory solution of equation (7.4.1⁺). Let S denote the set of all nonoscillatory solutions of equation (7.4.1⁺) and define*

$$\begin{aligned}
S(0,0,0) &= \Big\{x \in S : \lim_{k\to\infty} x(k) = 0,\ \lim_{k\to\infty} y(k) = 0,\ \lim_{k\to\infty} \Delta y(k) = 0\Big\},\\
S(b,a,0) &= \Big\{x \in S : \lim_{k\to\infty} x(k) = b = \frac{a}{1-p},\ \lim_{k\to\infty} y(k) = a,\ \lim_{k\to\infty} \Delta y(k) = 0\Big\},\\
S(\infty,\infty,0) &= \Big\{x \in S : \lim_{k\to\infty} x(k) = \infty,\ \lim_{k\to\infty} y(k) = \infty,\ \lim_{k\to\infty} y(k) = 0\Big\},\\
S(\infty,\infty,d) &= \Big\{x \in S : \lim_{k\to\infty} x(k) = \infty,\ \lim_{k\to\infty} y(k) = \infty,\ \lim_{k\to\infty} \Delta y(k) = d \ne 0\Big\}.
\end{aligned} \tag{7.4.10}$$

Then

$$S = S(0,0,0) \cup S(b,a,0) \cup S(\infty,\infty,0) \cup S(\infty,\infty,d). \tag{7.4.11}$$

Proof. Without loss of generality, let $\{x(k)\}$ be an eventually positive solution of (7.4.1⁺). If $\lim_{k\to\infty} x(k) = 0$, then by Lemma 7.4.1, $\lim_{k\to\infty} y(k) = 0$ and $\lim_{k\to\infty} \Delta y(k) = 0$, that is, $\{x(k)\} \in S(0,0,0)$. If $\lim_{k\to\infty} x(k) = 0$ fails to hold, then by Lemma 7.4.2, $y(k) > 0$ eventually. Here, one can easily see that $\Delta y(k) > 0$ and $\Delta^2 y(k) < 0$ eventually. If $\lim_{k\to\infty} y(k) = a > 0$, then $\lim_{k\to\infty} \Delta y(k) = 0$, and by Lemma 7.4.2, we have that $\lim_{k\to\infty} x(k) = a/(1-p) = b$, that is, $x \in S(b,a,0)$. If $\lim_{k\to\infty} y(k) = \infty$, then by Lemma 7.4.2, $\lim_{k\to\infty} x(k) = \infty$. Since $\Delta^2 y(k) < 0$ and $\Delta y(k) > 0$ eventually, we have $\lim_{k\to\infty} \Delta y(k) = d$, where $d = 0$ or $d > 0$. Thus, either $x \in S(\infty,\infty,0)$ or $x \in S(\infty,\infty,d)$. □

Next, we will study some existence results for each kind of nonoscillatory solution of equation (7.4.1⁺).

Theorem 7.4.4. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0,1)$. *Then equation (7.4.1⁺) has a nonoscillatory solution* $x \in S(b,a,0)$ $(b \ne 0,\ a \ne 0)$ *if and only if*

$$\sum_{j=m\in\mathbb{N}}^{\infty} j\,|F(j,\alpha)| < \infty \quad \textit{for some constant } \alpha \ne 0. \tag{7.4.12}$$

Proof. We first show necessity. Without loss of generality, let $x \in S(b,a,0)$ be an eventually positive solution of equation (7.4.1$^+$). By Theorem 7.4.3 we see that $b > 0$ and $a > 0$. From equation (7.4.1$^+$) and (7.4.2) we have

$$\Delta^2 y(k) = -F(k, x[g(k)]). \tag{7.4.13}$$

Summing both sides of (7.4.13) from s to u for $k \geq m$ for some $m \in \mathbb{N}$ and letting $u \to \infty$, we obtain

$$\Delta y(s) = \sum_{i=s}^{\infty} F(i, x[g(i)]). \tag{7.4.14}$$

Next, summing (7.4.14) from $m_1 \geq m$ sufficiently large to $k-1$, we get

$$y(k) = y(m_1) + \sum_{j=m_1}^{k-1} (j - m_1 + 1) F(j, x[g(j)]) + \sum_{j=k}^{\infty} (k - m_1) F(j, x[g(j)]). \tag{7.4.15}$$

Since $\lim_{j\to\infty} x[g(j)] = b > 0$, there exists an integer $m_1 \geq m$ with $x[g(j)] \geq b/2$ for $j \geq m_1$. Hence from (7.4.15), we have

$$\sum_{j=m_1}^{k-1} (j - m_1 + 1) F\left(j, \frac{b}{2}\right) < y(k) - y(m_1), \tag{7.4.16}$$

which implies that condition (7.4.12) holds.

Now we show sufficiency. Let the constants $\alpha > 0$ and $\beta > 0$ be such that $\beta < (1-p)\alpha$. From (7.4.12) there exists a sufficiently large $m_1 \geq m$ so that for all $k \geq m_1$ we have $k - \tau \geq m$ and $g(k) \geq m$ and

$$\frac{\beta}{\alpha} + p(k) + \frac{1}{\alpha} \sum_{j=m_1}^{\infty} jF(j, \alpha) \leq 1. \tag{7.4.17}$$

Let B denote the Banach space $\ell^\infty(m)$ of all bounded real sequences $x = \{x(k)\}_{k=m}^{\infty}$ with the norm $\|x\| = \sup_{k \geq m} |x(k)|$. Define a set X by

$$X = \{x \in B : 0 \leq x(k) \leq \alpha,\ k \geq m\}. \tag{7.4.18}$$

Next define an operator T on X by

$$(Tx)(k) = \begin{cases} \beta + p(k)x[k-\tau] + \displaystyle\sum_{j=m_1}^{k-1} jF(j, x[g(j)]) & \\ + \displaystyle\sum_{j=k}^{\infty} (k-1) F(j, x[g(j)]) & \text{for } k \geq m_1, \\ Tx(m_1) & \text{for } m \leq k \leq m_1, \end{cases} \tag{7.4.19}$$

and set for $i \in \mathbb{N}_0$ and $k \in \mathbb{N}(m)$,

$$x(k, i+1) = \begin{cases} 0 & \text{if } i = 0, \\ Tx(k, i) & \text{if } i \in \mathbb{N}. \end{cases} \tag{7.4.20}$$

By the hypotheses and induction, it is easy to see that

$$0 \le x(k, i) \le x(k, i+1) \le \alpha \quad \text{for } k \in \mathbb{N}(m_1),\ i \in \mathbb{N}. \tag{7.4.21}$$

Let $x(k) = \lim_{i\to\infty} x(k, i)$. By Lebesgue's convergence theorem, $x(k) = (Tx)(k)$ for $k \in \mathbb{N}(m)$, that is,

$$x(k) = \begin{cases} \beta + p(k)x[k-\tau] + \sum_{j=m_1}^{k-1} jF(j, x[g(j)]) & \\ + \sum_{j=k}^{\infty}(k-1)F(j, x[g(j)]) & \text{for } k \in \mathbb{N}(m_1), \\ x(m_1) & \text{for } m \le k \le m_1. \end{cases} \tag{7.4.22}$$

Clearly, $x(k) > 0$ for $k \in \mathbb{N}(m)$. It is easy to check that $\{x(k)\}$ is a positive solution of equation (7.4.1$^+$). Since $0 < \beta \le x(k) \le \alpha$, from Theorem 7.4.3, $x \in S(b, a, 0)$. This completes the proof. □

Similarly, we can prove the following results.

Theorem 7.4.5. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0, 1)$. *Then equation (7.4.1$^+$) has a nonoscillatory solution* $x \in S(\infty, \infty, d)$ $(d \ne 0)$ *if and only if*

$$\sum_{j=m\in\mathbb{N}}^{\infty} |F(j, \gamma g(j))| < \infty \quad \textit{for some } \gamma \ne 0. \tag{7.4.23}$$

Sketch of the proof. First, let $x \in S(\infty, \infty, d)$ be an eventually positive solution of (7.4.1$^+$). From Theorem 7.4.3, we have $d > 0$. Since $\Delta y(k) = d > 0$, there exist a constant $d_1 > 0$ and an integer $m_1 \ge m$ for some $m \in \mathbb{N}$ such that $x[g(k)] \ge y[g(k)] \ge d_1 g(k)$ for $k \ge m_1$ and the necessity of Theorem 7.4.5 follows. For the sufficiency of Theorem 7.4.5, set $\gamma > 0$. Let $d > 0$ and $\beta > 0$. From (7.4.23) there exists a sufficiently large $m_1 \ge m$ for some $m \in \mathbb{N}$ so that for $k \ge m_1$ we have $k - \tau \ge m$, $g(k) \ge m$, and

$$\frac{d}{\gamma} + \frac{\beta}{\gamma k} + p(k) + \frac{1}{\gamma k}\sum_{j=m_1}^{\infty} F(j, \gamma g(j)) < 1. \tag{7.4.24}$$

Define the Banach space B as in the proof of Theorem 7.4.4, $X \subset B$ by

$$X = \{x \in B : d \le x(k) \le \gamma,\ k \ge m\}, \tag{7.4.25}$$

and an operator T on X by

$$(Tx)(k) = \begin{cases} d + \dfrac{\beta}{k} + \dfrac{k-\tau}{k} p(k)x[k-\tau] \\ + \dfrac{1}{k} \displaystyle\sum_{j=m_1}^{k-1} jF(j, g(j)x[g(j)]) \\ + \dfrac{k-1}{k} \displaystyle\sum_{j=k}^{\infty} F(j, g(j)x[g(j)]) & \text{for } k \geq m_1, \\ Tx(m_1) & \text{for } m \leq k \leq m_1. \end{cases} \tag{7.4.26}$$

The rest of the proof is similar to that of Theorem 7.4.4 and the details are left to the reader. □

Theorem 7.4.6. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0,1)$. *Further assume (7.4.23) and*

$$\sum_{j=m\in\mathbb{N}}^{\infty} j\left|F(j,\alpha)\right| = \infty \quad \textit{for some } \alpha \neq 0, \tag{7.4.27}$$

where $\alpha\gamma > 0$. *Then (7.4.1⁺) has a nonoscillatory solution* $x \in S(\infty, \infty, 0)$.

Sketch of the proof. To show that equation (7.4.1$^+$) has a nonoscillatory solution $x \in S(\infty,\infty,0)$, we assume without loss of generality that $\alpha > 0$ and $\gamma > 0$. There exists a sufficiently large integer $m_1 \geq m$ for some $m \in \mathbb{N}$ so that for $k \geq m_1$ we have $k - \tau \geq m$, $g(k) \geq m$, and

$$\frac{\alpha}{\gamma k} + p(k) + \frac{1}{\gamma} \sum_{j=m_1}^{\infty} F(j, \gamma g(j)) < 1. \tag{7.4.28}$$

Next, define B as above,

$$X = \{x \in B : 0 \leq x(k) \leq \gamma,\ k \geq m\}, \tag{7.4.29}$$

and an operator T on X by

$$(Tx)(k) = \begin{cases} \dfrac{\alpha}{k} + \dfrac{k-\tau}{k} p(k)x[k-\tau] \\ + \dfrac{1}{k} \displaystyle\sum_{j=m_1}^{k-1} jF(j, g(j)x[g(j)]) \\ + \dfrac{1}{k} \displaystyle\sum_{j=k}^{\infty} (k-1)F(j, g(j)x[g(j)]) & \text{for } k \geq m_1, \\ Tx(m_1) & \text{for } m \leq k \leq m_1. \end{cases} \tag{7.4.30}$$

Once again the rest of the proof is similar to that of Theorem 7.4.4 and the details are left to the reader. □

The following examples illustrate the theory presented above.

Example 7.4.7. The difference equation

$$\Delta^2\left(x(k) - \frac{1}{2}x[k-2]\right) + \frac{2^{2k-5}}{\left(2^{k-1}+1\right)^3}x^3[k-1] = 0, \quad k \in \mathbb{N}_0 \tag{7.4.31}$$

has a nonoscillatory solution $x(k) = 1 + 2^{-k}$ which is in $S(1, 1/2, 0)$. All conditions of Theorem 7.4.4 are satisfied.

Example 7.4.8. The difference equation

$$\Delta^2\left(x(k) - \frac{1}{2}x[k-1]\right) + \left(\frac{e}{2} - 1\right)\left(1 - \frac{1}{e}\right)^2 \frac{e^{-k}}{\left(k + e^{-k}\right)^{1/3}}x^{1/3}(k) = 0 \tag{7.4.32}$$

for $k \in \mathbb{N}$ has a nonoscillatory solution $x(k) = k + e^{-k}$ which is in $S(\infty, \infty, 1/2)$. All hypotheses of Theorem 7.4.5 are satisfied.

7.4.2. Nonoscillatory solutions of equation ($7.4.1^-$)

We will need the following two lemmas.

Lemma 7.4.9. *Let $\{x(k)\}$ be an eventually positive (negative) solution of equation ($7.4.1^-$). If $\lim_{k\to\infty} x(k) = 0$, then $\{y(k)\}$, where $y(k)$ is as in (7.4.2), is eventually positive (negative) and $\lim_{k\to\infty} y(k) = 0$, $\lim_{k\to\infty} \Delta y(k) = 0$. If $\lim_{k\to\infty} x(k) = 0$ fails to hold, then $\{y(k)\}$ is also eventually positive (negative).*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation ($7.4.1^-$). From equation ($7.4.1^-$) and (7.4.2), $\Delta^2 y(k) > 0$ eventually. Thus $\Delta y(k)$ is increasing and $\Delta y(k) > 0$ or $\Delta y(k) < 0$ eventually. Also, $y(k) > 0$ or $y(k) < 0$ eventually. If $\lim_{k\to\infty} x(k) = 0$, then from (7.4.2) we have $\lim_{k\to\infty} y(k) = 0$ which implies that $\lim_{k\to\infty} \Delta y(k) = 0$. Since $\Delta y(k)$ is increasing, $\Delta y(k) < 0$ eventually. So, $\{y(k)\}$ is decreasing eventually. Therefore, $y(k) > 0$ eventually. If $\lim_{k\to\infty} x(k) = 0$ fails, then $\limsup_{k\to\infty} x(k) > 0$. We claim that $y(k) > 0$ eventually. If not, then $y(k) < 0$ eventually. The rest of the proof is exactly the same as that of Lemma 7.4.1 and hence we omit the details here. □

Lemma 7.4.10. *Assume that $\lim_{k\to\infty} p(k) = p \in [0, 1)$ and $\{x(k)\}$ is an eventually positive (negative) solution of equation ($7.4.1^-$). Then a necessary and sufficient condition for $\lim_{k\to\infty} y(k) = \infty$ $(-\infty)$ is $\lim_{k\to\infty} x(k) = \infty$ $(-\infty)$, and a necessary and sufficient condition for $\lim_{k\to\infty} y(k) = a \in \mathbb{R}$ is $\lim_{k\to\infty} x(k) = a/(1 - p)$.*

Proof. Let $\{x(k)\}$ be an eventually positive (negative) solution of equation ($7.4.1^-$). Then $x(k) \geq (\leq) y(k)$ eventually. Hence, if $\lim_{k\to\infty} y(k) = \infty$ $(-\infty)$, then $\lim_{k\to\infty} x(k) = \infty$ $(-\infty)$. On the contrary, if $\lim_{k\to\infty} x(k) = \infty$ $(-\infty)$, then there exists a sequence $\{k_n\}$ such that $\lim_{n\to\infty} k_n = \infty$, $x(k_n) = \max_{m\leq k\leq k_n} x(k)$ $(\min_{m\leq k\leq k_n} x(k))$ for some $m \in \mathbb{N}$, and $\lim_{n\to\infty} x(k_n) = \infty$ $(-\infty)$. From (7.4.2) we obtain

$$y(k_n) = x(k_n) - p(k_n)x[k_n - \tau] \geq (\leq) x(k_n)(1 - p(k_n)). \tag{7.4.33}$$

From this, one can easily see that $\lim_{k\to\infty} y(k) = \infty\ (-\infty)$.

If $\lim_{k\to\infty} x(k) = a/(1-p)$, then by (7.4.2), $\lim_{k\to\infty} y(k) = a \in \mathbb{R}$. Now we consider the case that $\lim_{k\to\infty} y(k) = a \in \mathbb{R}$. The proof of this case is the same as that of Lemma 7.4.2 and hence we omit the details. □

Theorem 7.4.11. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0,1)$. *Let* $\{x(k)\}$ *be a nonoscillatory solution of equation (7.4.1⁻), let* S *denote the set of all nonoscillatory solutions of equation (7.4.1⁻), and define the subsets* $S(0,0,0)$, $S(b,a,0)$, *and* $S(\infty,\infty,d)$ *as in Theorem 7.4.3 and*

$$S(\infty,\infty,\infty) = \left\{x \in S : \lim_{k\to\infty} x(k) = \lim_{k\to\infty} y(k) = \lim_{k\to\infty} \Delta y(k) = \infty\right\}. \tag{7.4.34}$$

Then

$$S = S(0,0,0) \cup S(b,a,0) \cup S(\infty,\infty,d) \cup S(\infty,\infty,\infty). \tag{7.4.35}$$

Proof. Without loss of generality, let $\{x(k)\}$ be an eventually positive solution of equation (7.4.1⁻). Then $\Delta^2 y(k) > 0$ eventually and $\{\Delta y(k)\}$ is increasing eventually.

If $\lim_{k\to\infty} x(k) = 0$, then by Lemma 7.4.9, we have $\lim_{k\to\infty} y(k) = 0$ and $\lim_{k\to\infty} \Delta y(k) = 0$, that is, $x \in S(0,0,0)$.

If $\lim_{k\to\infty} x(k) = 0$ fails to hold, then by Lemma 7.4.9, $y(k) > 0$ eventually. Now we consider the following two possible cases:

(i) $\Delta y(k) < 0$ eventually,
(ii) $\Delta y(k) > 0$ eventually.

Assume (i), that is, $\Delta y(k) < 0$ eventually. So $\lim_{k\to\infty} \Delta y(k) = d \leq 0$. If $d < 0$, then there exists an integer $m_1 \geq m$ for some $m \in \mathbb{N}$ such that $\Delta y(k) \leq d/2$ for $k \geq m_1$. Summing from m_1 to $k-1$ and letting $k \to \infty$ yields $y(k) \to -\infty$ as $k \to \infty$, which is a contradiction. So $d = 0$. That $\Delta y(k) < 0$ holds eventually implies that $\{y(k)\}$ is decreasing eventually, which together with $y(k) > 0$ eventually leads to $\lim_{k\to\infty} y(k) = a \geq 0$. If $a = 0$, then by Lemma 7.4.10, $\lim_{k\to\infty} x(k) = 0$, which contradicts that $\lim_{k\to\infty} x(k) = 0$ fails. So $a > 0$. It is clear from Lemma 7.4.10 that $\lim_{k\to\infty} x(k) = a/(1-p) = b$. Thus $x \in S(b,a,0)$.

Assume (ii), that is, $\Delta y(k) > 0$ eventually. So either $\lim_{k\to\infty} \Delta y(k) = d < \infty$ or $\lim_{k\to\infty} \Delta y(k) = \infty$, where $d > 0$. If $\lim_{k\to\infty} \Delta y(k) = d < \infty$, then it is easily shown that $\lim_{k\to\infty} y(k) = \infty$ and by Lemma 7.4.10, $\lim_{k\to\infty} x(k) = \infty$. So $x \in S(\infty,\infty,d)$. If $\lim_{k\to\infty} \Delta y(k) = \infty$, then $\lim_{k\to\infty} y(k) = \infty$ and so $\lim_{k\to\infty} x(k) = \infty$. Therefore, $x \in S(\infty,\infty,\infty)$.

The case when $\{x(k)\}$ is eventually negative is similar and the details are omitted. This completes the proof. □

Theorem 7.4.12. *Theorem 7.4.4 remains valid for equation (7.4.1$^-$).*

Proof. We first show necessity. Without loss of generality, let $x \in S(b,a,0)$ be an eventually positive solution of equation (7.4.1$^-$). By Theorem 7.4.11, $b > 0$ and $a > 0$. From (7.4.1$^-$) and (7.4.2), we obtain $\Delta^2 y(k) = F(k, x[g(k)])$. Summing both sides of this equation from $s \geq m$ for some $m \in \mathbb{N}$ to u and letting $u \to \infty$, we obtain

$$\Delta y(s) = -\sum_{i=s}^{\infty} F(i, x[g(i)]) \tag{7.4.36}$$

and once again, summing both sides of (7.4.36) from $m_1 \geq m$ sufficiently large to $k-1$ yields

$$y(k) = y(m_1) - \sum_{j=m_1}^{k-1} (j - m_1 + 1) F(j, x[g(j)]) - \sum_{j=k}^{\infty} (k - m_1) F(j, x[g(j)]). \tag{7.4.37}$$

Since $\lim_{j\to\infty} x[g(j)] = b > 0$, there exists an $m_1 \geq m$ such that $x[g(j)] \geq b/2$ for $j \geq m_1$. Hence, from (7.4.37) we have

$$\sum_{j=m_1}^{k-1} (j - m_1 + 1) \left| F\left(j, \frac{b}{2}\right) \right| < y(m_1) - y(k), \tag{7.4.38}$$

which implies that (7.4.12) holds.

Now we show sufficiency. Set $\alpha > 0$. Choose $\beta > 0$ so that $\beta < (1-p)\alpha$. As in the proof of Theorem 7.4.4, there exists an integer $m_1 \geq m$ such that (7.4.17) holds. Define $X \subseteq B$ as in the proof of Theorem 7.4.4 and an operator T on X by

$$(Tx)(k) = \begin{cases} \beta + p(k)x[k-\tau] + \displaystyle\sum_{j=k}^{\infty} (j - k + 1) F(j, x[g(j)]) & \text{if } k \geq m_1, \\ (Tx)(m_1) & \text{if } m \leq k \leq m_1. \end{cases} \tag{7.4.39}$$

The rest of the proof is similar to that of Theorem 7.4.4 and the details are therefore omitted. ☐

Theorem 7.4.13. *Assume that* $\lim_{k\to\infty} p(k) = p \in [0,1)$. *Then the following statements are true.*

(I_1) *If equation (7.4.1$^-$) has a nonoscillatory solution* $x \in S(\infty,\infty,d)$, $d \neq 0$, *then condition (7.4.23) holds.*

(I_2) *If*

$$\sum_{j=m\in\mathbb{N}}^{\infty} j\left|F(j,\gamma g(j))\right| < \infty, \tag{7.4.40}$$

then (7.4.1$^-$) has a nonoscillatory solution $x \in S(\infty,\infty,d)$, $d \neq 0$.

Proof. We first show (I_1). Without loss of generality, let $x \in S(\infty,\infty,d)$ be an eventually positive solution of equation (7.4.1$^-$). From Theorem 7.4.11, $d > 0$. From equation (7.4.1$^-$) and (7.4.2) we have $\Delta^2 y(k) - F(k,x[g(k)]) = 0$. Summing both sides of this equation from $m_1 \geq m$ for some $m \in \mathbb{N}$ to $k-1$, we have

$$\Delta y(k) - \Delta y(m_1) - \sum_{i=m_1}^{k-1} F(i,x[g(i)]) = 0. \tag{7.4.41}$$

Since $\lim_{k\to\infty} \Delta y(k) = d > 0$, we obtain

$$\sum_{i=m_1}^{\infty} F(i,x[g(i)]) < \infty, \tag{7.4.42}$$

and there exist $c > 0$ and an integer $m_2 \geq m_1$ such that

$$x[g(i)] \geq y[g(i)] \geq cg(i) \quad \text{for } i \geq m_2. \tag{7.4.43}$$

Combining (7.4.42) and (7.4.43), we see that condition (7.4.23) holds.

Now we show (I_2). Set $\gamma > 0$. Choose $d > 0$ and $\beta > 0$ such that $d < (1-p)\gamma$. From condition (7.4.40), there exists a sufficiently large $m_1 \geq m \in \mathbb{N}$ such that for $k \geq m_1$ we have $k - \tau \geq m$, $g(k) \geq m$, and

$$\frac{d}{\gamma} + \frac{\beta}{\gamma k} + p(k) + \frac{1}{\gamma k}\sum_{j=m_1}^{\infty} jF(j,\gamma g(j)) < 1. \tag{7.4.44}$$

Define a set $X \subseteq B$ by

$$X = \{x \in B : d \leq x(k) \leq \gamma,\ k \geq m\}, \tag{7.4.45}$$

and an operator T on X by

$$(Tx)(k) = \begin{cases} d + \dfrac{\beta}{k} + \dfrac{k-\tau}{k} p(k)x[k-\tau] & \\ + \dfrac{1}{k} \displaystyle\sum_{j=k}^{\infty} (j-k+1)F(j, g(j)x[g(j)]) & \text{if } k \geq m_1, \\ (Tx)(m_1) & \text{if } m \leq k \leq m_1. \end{cases} \tag{7.4.46}$$

The rest of the proof is similar to that of Theorem 7.4.4 and the details are left to the reader. □

The following examples illustrate the methods presented above.

Example 7.4.14. Consider the difference equation

$$\Delta^2 \left(x(k) - \frac{1}{2} x[k-2] \right) = \frac{2^{2k-5}}{(2^{k-1}-1)^3} x^3[k-1] \quad \text{for } k \geq 2. \tag{7.4.47}$$

All conditions of Theorem 7.4.12 are satisfied, and hence equation (7.4.47) has a nonoscillatory solution $x \in S(b, a, 0)$ $(b \neq 0,\ a \neq 0)$. One such solution is $x(k) = 1 - 2^{-k}$ which is in $S(1, 1/2, 0)$.

Example 7.4.15. Consider the difference equation

$$\Delta^2 \left(x(k) - \frac{1}{4} x[k-1] \right) = \left(1 - \frac{e}{4} \right) \left(1 - \frac{1}{e} \right)^2 \frac{e^{-k}}{(k + e^{-k})^{1/3}} x^{1/3}(k) \tag{7.4.48}$$

for $k \in \mathbb{N}$. All the hypotheses of Theorem 7.4.13 are satisfied, and hence equation (7.4.48) has a nonoscillatory solution $x \in S(\infty, \infty, d)$ $(d \neq 0)$. In fact, equation (7.4.48) has the nonoscillatory solution $x(k) = k + e^{-k}$ which is in $S(\infty, \infty, 3/4)$.

7.4.3. Oscillation and asymptotic behavior

We will present the following result.

Theorem 7.4.16. *Assume that* $\lim_{k \to \infty} p(k) = p \in [0, 1)$ *and*

$$\sum_{j=m \in \mathbb{N}}^{\infty} F(j, \gamma) = \infty \quad \textit{for every constant } \gamma > 0. \tag{7.4.49}$$

Then

(i) *every solution x of equation (7.4.1$^+$) is oscillatory or belongs to* $S(0, 0, 0)$,
(ii) *every solution x of equation (7.4.1$^-$) is oscillatory or belongs to either* $S(0, 0, 0)$ *or* $S(\infty, \infty, \infty)$.

Proof. We first show (i). Let $\{x(k)\}$ be an eventually positive solution of equation (7.4.1$^+$). By Lemma 7.4.1 if $\lim_{k\to\infty} x(k) = 0$, then $\lim_{k\to\infty} y(k) = 0$ and so $\lim_{k\to\infty} \Delta y(k) = 0$. Hence $x \in S(0,0,0)$. If $\lim_{k\to\infty} x(k) = 0$ fails to hold, then $y(k) > 0$ eventually. Since $\Delta^2 y(k) < 0$ eventually, we have $\Delta y(k) > 0$ eventually. There exist a constant $c > 0$ and an integer $m_1 \geq m$ for some $m \in \mathbb{N}$ such that

$$x[g(k)] \geq y[g(k)] \geq c \quad \forall k \geq m_1. \tag{7.4.50}$$

From equation (7.4.1$^+$) and (7.4.2), we have

$$\Delta^2 y(k) = -F(k, x[g(k)]). \tag{7.4.51}$$

Using (7.4.50) in equation (7.4.51), we have

$$\Delta^2 y(k) \leq -F(k,c) \quad \text{for } k \geq m_1. \tag{7.4.52}$$

Summing (7.4.52) from m_1 to $k-1$ and letting $k \to \infty$, we have $\sum_{j=m_1}^{\infty} F(j,c) < \infty$, which contradicts (7.4.49).

Now we prove (ii). Let $\{x(k)\}$ be an eventually positive solution of equation (7.4.1$^-$). By Lemma 7.4.9 if $\lim_{k\to\infty} x(k) = 0$, then $\lim_{k\to\infty} y(k) = 0$ and so $\lim_{k\to\infty} \Delta y(k) = 0$. Hence $x \in S(0,0,0)$. If $\lim_{k\to\infty} x(k) = 0$ fails to hold, then $y(k) > 0$ eventually. From equation (7.4.1$^-$) and (7.4.2) we have

$$\Delta^2 y(k) = F(k, x[g(k)]) \geq 0 \quad \text{eventually}. \tag{7.4.53}$$

Now we consider the following two cases:

(i_1) $\Delta y(k) < 0$ eventually,

(i_2) $\Delta y(k) > 0$ eventually.

Assume (i_1), that is, let $\Delta y(k) < 0$ eventually. Then from the proof of Theorem 7.4.11, we see that $x \in S(b,a,0)$ with $b > 0$ and $a > 0$. Thus there exist a constant $c \in (0,a]$ and an integer $m_1 \geq m$ for some $m \in \mathbb{N}$ such that

$$x[g(k)] \geq y[g(k)] \geq c \quad \text{for } k \geq m_1. \tag{7.4.54}$$

Using (7.4.54) in equation (7.4.53) and summing both sides of the resulting inequality from m_1 to $k-1$, we obtain

$$\Delta y(k) - \Delta y(m_1) \geq \sum_{j=m_1}^{k-1} F(j,c). \tag{7.4.55}$$

Letting $k \to \infty$ in (7.4.55), we get $\sum_{j=m_1}^{\infty} F(j,c) < \infty$, which contradicts (7.4.49).

Assume (i_2), that is, let $\Delta y(k) > 0$ eventually. Then from Theorem 7.4.11 we see that $x \in S(\infty,\infty,d)$ $(d > 0)$ or $x \in S(\infty,\infty,\infty)$. In both cases, there exist a constant $c > 0$ and an integer $m_1 \geq m$ such that (7.4.54) holds. Using (7.4.54) in equation (7.4.53) we obtain (7.4.55) which leads to the desired contradiction. This completes the proof. □

The following examples illustrate the methods presented above.

Example 7.4.17. The difference equation

$$\Delta^2\left(x(k) - \frac{1}{2}x[k-1]\right) + \left(\frac{e}{2} - 1\right)\left(\frac{1}{e} - 1\right)^2 x(k) = 0 \tag{7.4.56}$$

has a nonoscillatory solution $x(k) = e^{-k}$ which is in $S(0,0,0)$. We note that the hypotheses of Theorem 7.4.16(i) are satisfied.

Example 7.4.18. The difference equation

$$\Delta^2\left(x(k) - \frac{1}{4}x[k-1]\right) = \left(1 - \frac{e}{4}\right)\left(\frac{1}{e} - 1\right)x(k) \tag{7.4.57}$$

has a nonoscillatory solution $x(k) = e^{-k}$ which is in $S(0,0,0)$. All the conditions of Theorem 7.4.16(ii) are satisfied.

Example 7.4.19. The difference equation

$$\Delta^2\left(x(k) - \frac{1}{2}x[k-1]\right) = (e-1)^2\left(1 - \frac{1}{2e}\right)x(k) \tag{7.4.58}$$

has a nonoscillatory solution $x(k) = e^{k}$ which is in $S(\infty,\infty,\infty)$. We note that all conditions of Theorem 7.4.16(ii) are satisfied.

Example 7.4.20. The difference equations

$$\Delta^2\left(x(k) - \frac{1}{2}x[k-1]\right) + 6x[k-3] = 0, \tag{7.4.59}$$

$$\Delta^2\left(x(k) - \frac{1}{2}x[k-1]\right) = 6x[k-2] \tag{7.4.60}$$

have an oscillatory solution $x(k) = (-1)^k$. All conditions of Theorem 7.4.16(i) and (ii) are satisfied for equations (7.4.59) and (7.4.60), respectively.

7.5. Further oscillation criteria

Here, we deal with the oscillatory behavior of solutions of neutral difference equations of the form

$$\Delta\Big(c(k)\big(\Delta(x(k) - p(k)x[\tau(k)])\big)^{\alpha}\Big) + \delta q(k)f(x[g(k)]) = 0, \quad \text{where } \delta = \pm 1, \tag{7.5.1}$$

more precisely,

$$\Delta\Big(c(k)\big(\Delta(x(k) - p(k)x[\tau(k)])\big)^{\alpha}\Big) + q(k)f(x[g(k)]) = 0, \tag{7.5.1$^+$}$$

$$\Delta\Big(c(k)\big(\Delta(x(k) - p(k)x[\tau(k)])\big)^{\alpha}\Big) - q(k)f(x[g(k)]) = 0, \tag{7.5.1$^-$}$$

where $\alpha \geq 1$ is the ratio of two odd positive integers and

(i) $\{c(k)\}$ is a sequence of positive real numbers with $\sum^{\infty} c^{-1/\alpha}(j) = \infty$,
(ii) $\{p(k)\}$ and $\{q(k)\}$ are sequences of nonnegative real numbers,
(iii) $\{\tau(k)\}$ and $\{g(k)\}$ are sequences of increasing nonnegative integers, $\lim_{k\to\infty} \tau(k) = \infty$ and $\lim_{k\to\infty} g(k) = \infty$,
(iv) $f \in C(\mathbb{R},\mathbb{R})$, $f'(x) \geq 0$ and $xf(x) > 0$ for $x \neq 0$, and f satisfies $-f(-xy) \geq f(xy) \geq f(x)f(y)$ for $xy > 0$.

Now we present the following lemma.

Lemma 7.5.1. *Let* $0 < p(k) < 1$ *and* $\tau(k) < k$ *for* $k \geq m \in \mathbb{N}$. *If the difference equation*

$$\Delta\Big(c(k)(\Delta u(k))^{\alpha}\Big) + q(k)f(u[g(k)]) = 0 \tag{7.5.2}$$

is oscillatory, then any nonoscillatory solution $\{x(k)\}$ *of equation (7.5.1*$^+$*) tends to zero as* $k \to \infty$.

Proof. Without loss of generality, let $\{x(k)\}$ be an eventually positive solution of equation (7.5.1$^+$) and define

$$y(k) = x(k) - p(k)x[\tau(k)]. \tag{7.5.3}$$

From equation (7.5.1$^+$) we have

$$\Delta\Big(c(k)(\Delta y(k))^{\alpha}\Big) = -q(k)f(x[g(k)]) \leq 0 \quad \text{eventually.} \tag{7.5.4}$$

If $\Delta y(k) < 0$ eventually, then $\lim_{k\to\infty} y(k) = -\infty$. But $y(k) < 0$ eventually implies that $\lim_{k\to\infty} x(k) = 0$, which contradicts the fact that $\lim_{k\to\infty} y(k) = -\infty$. Therefore $\Delta y(k) > 0$ eventually. There are two cases to consider:

(i_1) $y(k) > 0$ eventually,
(i_2) $y(k) < 0$ eventually.

Assume that (i_1) holds, that is, $y(k) > 0$ eventually. From (7.5.3) we have $0 < y(k) \leq x(k)$ eventually. Thus

$$\Delta\Big(c(k)(\Delta y(k))^{\alpha}\Big) + q(k)f(y[g(k)]) \leq 0 \quad \text{eventually.} \tag{7.5.5}$$

Using Lemma 6.2.4, we see that equation (7.5.2) has an eventually positive solution. This contradicts the assumption.

Assume that (i_2) holds, that is, $y(k) < 0$ eventually. Then $x(k) < px[\tau(k)]$ for $k \geq m_1 \geq m \in \mathbb{N}$, which implies that $\lim_{k\to\infty} x(k) = 0$. This completes the proof. $\square$

Example 7.5.2. Consider the neutral difference equation

$$\Delta^2(x(k) - px[k-\tau]) + (pe^\tau - 1)\left(1 - \frac{1}{e}\right)^2 e^{-g}x[k-g] = 0, \tag{7.5.6}$$

where $\tau \geq 1$ and $g \geq 0$ are integers, $p \in (0,1)$ with $pe^\tau > 1$. Since the equation

$$\Delta^2 u(k) + (pe^\tau - 1)\left(1 - \frac{1}{e}\right)^2 e^{-g}u[k-g] = 0 \tag{7.5.7}$$

is obviously oscillatory, we see that the conditions of Lemma 7.5.1 are satisfied for equation (7.5.6), and hence we conclude that the nonoscillatory solutions of equation (7.5.6) are bounded. One such solution is $x(k) = e^{-k}$ which tends to 0 as $k \to \infty$.

Next, we present the following oscillation criterion for (7.5.1$^+$).

Theorem 7.5.3. *In addition to the assumptions of Lemma 7.5.1 assume further that $\tau^{-1} \circ g(k) < k$ for $k \geq m \in \mathbb{N}$ and one of the following conditions hold:*

(a$_1$) *$x^{-\alpha}f(x) \geq \gamma > 0$ for $x \neq 0$, where γ is a constant, and*

$$\limsup_{k\to\infty} \sum_{\ell=\tau^{-1}\circ g(k)}^{k-1} \overline{q}(\ell)C^\alpha[\tau^{-1}\circ g(k), \tau^{-1}\circ g(\ell)] > \frac{1}{\gamma}, \tag{7.5.8}$$

(a$_2$) *$x^{-\alpha}f(x) \geq \gamma > 0$ for $x \neq 0$, where γ is a constant, and*

$$\limsup_{k\to\infty} \sum_{\ell=\tau^{-1}\circ g(k)}^{k} \left(\frac{\gamma}{c(\ell)} \sum_{j=\ell}^{k} \overline{q}(j)\right)^{1/\alpha} > 1, \tag{7.5.9}$$

(a$_3$) *$\int^{+0}(du/f(u^{1/\alpha})) < \infty$ and $\int^{-0}(du/f(u^{1/\alpha})) < \infty$, and*

$$\sum^{\infty} \overline{q}(j)f(C[j, \tau^{-1}\circ g(j)]) = \infty, \tag{7.5.10}$$

where C is as in (6.2.54) and $\overline{q}(k) = q(k)f(1/p[\tau^{-1}\circ g(k)])$. Then equation (7.5.1$^+$) is oscillatory.

Proof. As in the proof of Lemma 7.5.1 it suffices to show that $y(k) < 0$ eventually is impossible under the assumptions. Suppose $x(k) > 0$, $\Delta(c(k)(\Delta y(k))^\alpha) \leq 0$, $\Delta y(k) > 0$, and $y(k) < 0$ eventually. Then

$$y[\tau^{-1}(k)] > -p[\tau^{-1}(k)]x(k) \quad \text{eventually}, \tag{7.5.11}$$

so

$$0 < x[g(k)] < \frac{z[\tau^{-1}\circ g(k)]}{p[\tau^{-1}\circ g(k)]} \quad \text{eventually}, \tag{7.5.12}$$

where $z(k) = -y(k)$. Substituting this into equation (7.5.1$^+$), we have

$$\Delta\Big(c(k)(\Delta z(k))^{\alpha}\Big) - q(k)f\left(\frac{z[\tau^{-1}\circ g(k)]}{p[\tau^{-1}\circ g(k)]}\right) \geq 0 \quad \text{eventually,} \tag{7.5.13}$$

so

$$\Delta\Big(c(k)(\Delta z(k))^{\alpha}\Big) - \overline{q}(k)f\big(z[\tau^{-1}\circ g(k)]\big) \geq 0 \quad \text{eventually.} \tag{7.5.14}$$

By applying Theorem 6.2.15 to (7.5.14) we arrive at the desired conclusion. □

Lemma 7.5.4. *Let $p(k) \equiv 1$ and $\tau(k) < k$ for $k \geq m \in \mathbb{N}$. If equation (7.5.2) is oscillatory, then all nonoscillatory solutions $\{x(k)\}$ of equation (7.5.1$^+$) are bounded.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.5.1$^+$) and $y(k) = x(k) - x[\tau(k)]$. Then $\Delta(c(k)(\Delta y(k))^{\alpha}) \leq 0$ for $k \geq m$ for some $m \in \mathbb{N}$. If $\Delta y(k) < 0$ for $k \geq m$, then we have $\lim_{k\to\infty} y(k) = -\infty$. Thus

$$x(k) < x[\tau(k)] \quad \text{for all sufficiently large } k, \tag{7.5.15}$$

which implies that $\{x(k)\}$ is bounded, which is a contradiction. Therefore $\Delta y(k) > 0$ for $k \geq m_1 \geq m$. Next, we consider the following two cases:

(i$_1$) $y(k) > 0$ for $k \geq m_2 \geq m_1$,

(i$_2$) $y(k) < 0$ for $k \geq m_2 \geq m_1$.

Assume (i$_1$), that is, $y(k) > 0$ for $k \geq m_2 \geq m_1$. Clearly, $0 < y(k) \leq x(k)$ for $k \geq m_2$. Using this in equation (7.5.1$^+$) we have

$$\Delta\Big(c(k)(\Delta y(k))^{\alpha}\Big) + q(k)f\big(y[g(k)]\big) \leq 0 \quad \text{for } k \geq m_2. \tag{7.5.16}$$

The rest of the proof is similar to that of Lemma 7.5.1 case (i$_1$) and the details are therefore omitted.

Now assume (i$_2$), that is, $y(k) < 0$ for $k \geq m_2$. Then inequality (7.5.15) holds for $k \geq m_2$, which implies that $\{x(k)\}$ is bounded. This completes the proof. □

Example 7.5.5. Consider the neutral difference equation

$$\Delta^2\big(x(k) - x[k-\tau]\big) + e^{-g}\left(1 - \frac{1}{e}\right)^2 (e^{\tau} - 1)x[k-g] = 0, \tag{7.5.17}$$

where $\tau \geq 1$ and $g \geq 0$ are integers. It is easy to see that the equation

$$\Delta^2 u(k) + e^{-g}\left(1 - \frac{1}{e}\right)^2 (e^{\tau} - 1)u[k-g] = 0 \tag{7.5.18}$$

is oscillatory and so all conditions of Lemma 7.5.4 are satisfied for equation (7.5.17). Thus we conclude that all nonoscillatory solutions of equation (7.5.17) are bounded. One such solution is $x(k) = e^{-k}$.

The following result is immediate.

Theorem 7.5.6. *Let $p(k) \equiv 1$ and the assumptions of Theorem 7.5.3 hold with Lemma 7.5.1 replaced by Lemma 7.5.4 and $\overline{q}(k) = q(k)$. Then equation (7.5.1$^+$) is oscillatory.*

Next we present the following two results for equation (7.5.1$^-$).

Theorem 7.5.7. *Let $\Delta p(k) \geq 0$ for $k \geq m \in \mathbb{N}$ and let the hypotheses of Theorem 7.5.3 hold with $\tau^{-1} \circ g(k)$ and $\overline{q}(k)$ replaced by $g(k)$ and $q(k)$, respectively, and without assuming that equation (7.5.2) is oscillatory. Then every bounded solution of equation (7.5.1$^-$) is oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive bounded solution of (7.5.1$^-$). Define $y(k)$ by (7.5.3). Then we have

$$\Delta\Big(c(k)\big(\Delta y(k)\big)^{\alpha}\Big) = q(k)f\big(x[g(k)]\big) \geq 0 \quad \text{eventually.} \tag{7.5.19}$$

If $\Delta y(k) > 0$ eventually, then $\lim_{k\to\infty} y(k) = \infty$, which contradicts the boundedness of $\{x(k)\}$. Therefore $\Delta y(k) \leq 0$ eventually. There are two possibilities for $y(k)$:

(i_1) $y(k) > 0$ for $k \geq m_1$ for some $m_1 \geq m \in \mathbb{N}$,
(i_2) $y(k) < 0$ for $k \geq m_1$.

Assume (i_1), that is, $y(k) > 0$ for $k \geq m_1$. Clearly, from (7.5.3) there exists an integer $m_2 \geq m_1$ such that $x[g(k)] \geq y[g(k)]$ for $k \geq m_2$. Using this in equation (7.5.19), we get

$$\Delta\Big(c(k)\big(\Delta y(k)\big)^{\alpha}\Big) \geq q(k)f\big(y[g(k)]\big) \quad \text{for } k \geq m_2. \tag{7.5.20}$$

The rest of the proof is similar to the proofs of Theorems 6.2.15 and 7.5.3 and hence we omit the details.

Now assume (i_2), that is, $y(k) < 0$ for $k \geq m_1$. Then,

$$\begin{aligned} x(k) &< p(k)x[\tau(k)] < p(k)p[\tau(k)]x[\tau\circ\tau(k)] < \cdots \\ &< p(k)p[\tau(k)]\cdots p[\tau\circ\tau\circ\cdots\circ\tau(k)]x[\tau\circ\tau\circ\cdots\circ\tau(k)], \end{aligned} \tag{7.5.21}$$

so

$$x(k) < p(k)x[\tau(k)] < \cdots < p^n(k)x[\tau\circ\tau\circ\cdots\circ\tau(k)] \tag{7.5.22}$$

for $k \geq m_2 + n\tau(m)$, which implies that $\lim_{k\to\infty} x(k) = 0$. Thus $\lim_{k\to\infty} y(k) = 0$, which is a contradiction. □

Theorem 7.5.8. *Let $p(k) \equiv 1$, $\tau(k) < k$, and $g(k) \leq k$ for $k \geq m \in \mathbb{N}$. If*

$$\sum^{\infty}\left[\frac{1}{c(j)}\sum_{i=j}^{\infty}q(i)\right]^{1/\alpha} = \infty, \tag{7.5.23}$$

then every bounded solution of equation (7.5.1⁻) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually bounded solution of equation $(7.5.1^-)$ and let $y(k)$ be defined as in (7.5.3). There are two cases to consider:

(I_1) $\Delta(c(k)(\Delta y(k))^\alpha) \geq 0$, $\Delta y(k) \leq 0$, and $y(k) < 0$ for $k \geq m_1$ for some $m_1 \geq m \in \mathbb{N}$,

(I_2) $\Delta(c(k)(\Delta y(k))^\alpha) \geq 0$, $\Delta y(k) \leq 0$, and $y(k) > 0$ for $k \geq m_1 \geq m$.

In case (I_1), there exists a finite number $\beta > 0$ such that $\lim_{k\to\infty} y(k) = -\beta$. Thus there exists an integer $m_2 \geq m_1$ such that $-\beta < y(k) < -\beta/2$ for $k \geq m_2$, that is, $-\beta < x(k) - x[\tau(k)] < -\beta/2$ for $k \geq m_2$. Hence $x[\tau(k)] \geq \beta/2$ for $k \geq m_2$. Then there exists an integer $m_3 \geq m_2$ such that

$$x[g(k)] \geq \frac{\beta}{2} \quad \text{for } k \geq m_3. \tag{7.5.24}$$

Using (7.5.24) in equation (7.5.19), we obtain

$$\Delta\Big(c(k)(\Delta y(k))^\alpha\Big) \geq q(k)f\left(\frac{\beta}{2}\right) \quad \text{for } k \geq m_3. \tag{7.5.25}$$

In case (I_2), we have $x(k) > x[\tau(k)]$ for $k \geq m_1$. Thus there exist a constant $\gamma > 0$ and an integer $m_2 \geq m_1$ such that $x[g(k)] \geq \gamma$ for $k \geq m_3$. Hence

$$\Delta\Big(c(k)(\Delta y(k))^\alpha\Big) \geq q(k)f(\gamma). \tag{7.5.26}$$

In both cases we are led to the same inequality (7.5.26). Summing both sides of (7.5.26) from k to $N-1$ for $N-1 \geq k \geq m_3$, we obtain

$$c(N)(\Delta y(N))^\alpha - c(k)(\Delta y(k))^\alpha \geq f(\gamma)\sum_{j=k}^{N-1}q(j) \quad \text{for } N-1 \geq k \geq m_3. \tag{7.5.27}$$

Hence

$$-c(k)(\Delta y(k))^\alpha \geq f(\gamma)\sum_{j=k}^{N-1}q(j) \quad \text{for } N-1 \geq k \geq m_3, \tag{7.5.28}$$

which implies that $\sum^\infty q(j) < \infty$, and so

$$-\Delta y(k) \geq (f(\gamma))^{1/\alpha}\left[\frac{1}{c(k)}\sum_{j=k}^{\infty}q(j)\right]^{1/\alpha}. \tag{7.5.29}$$

Summing (7.5.29) from k to $N-1 \geq k \geq m_2$, we have

$$y(k) \geq y(N) + \big(f(\gamma)\big)^{1/\alpha} \sum_{i=k}^{N-1} \left[\frac{1}{c(i)} \sum_{j=i}^{\infty} q(j) \right]^{1/\alpha} \quad \text{for } k \geq m_3, \tag{7.5.30}$$

which in view of (7.5.23) leads to a contradiction to the boundedness of $\{x(k)\}$. This completes the proof. □

From inequality (7.5.28) in the proof of Theorem 7.5.8, the following result is immediate.

Theorem 7.5.9. *Let condition (7.5.23) of Theorem 7.5.8 be replaced by*

$$\sum^{\infty} q(j) = \infty. \tag{7.5.31}$$

Then the conclusion of Theorem 7.5.8 holds.

Example 7.5.10. The difference equation

$$\Delta\big[\Delta\big(x(k) - x[k-3]\big)\big]^{\alpha} = 2^{2\alpha+1} x^{\alpha}[k-2], \tag{7.5.32}$$

where α is as in equation (7.5.1), has a bounded oscillatory solution $x(k) = (-1)^k$. All conditions of Theorem 7.5.9 are fulfilled for equation (7.5.32), and hence all bounded solutions of equation (7.5.32) are oscillatory.

Remark 7.5.11. Theorem 7.5.7 is also true for $p(k) \equiv 0$.

Next, we will consider nonlinear neutral difference equations with a nonlinear neutral term of the form

$$\Delta\Big(c(k)\big(\Delta\big(x(k) - p(k)x^{\beta}[\tau(k)]\big)\big)^{\alpha}\Big) + \delta q(k) f\big(x[g(k)]\big) = 0, \quad \text{where } \delta = \pm 1, \tag{7.5.33}$$

more precisely

$$\Delta\Big(c(k)\big(\Delta\big(x(k) - p(k)x^{\beta}[\tau(k)]\big)\big)^{\alpha}\Big) + q(k) f\big(x[g(k)]\big) = 0, \tag{7.5.33$^+$}$$

$$\Delta\Big(c(k)\big(\Delta\big(x(k) - p(k)x^{\beta}[\tau(k)]\big)\big)^{\alpha}\Big) - q(k) f\big(x[g(k)]\big) = 0, \tag{7.5.33$^-$}$$

where $\{c(k)\}$, $\{p(k)\}$, $\{q(k)\}$, $\{\tau(k)\}$, $\{g(k)\}$, and f are as in equation (7.5.1), α and β are quotients of odd positive integers.

Now we present the following results.

Theorem 7.5.12. *Let $p(k) > 0$, $\tau(k) < k$, $g(k) < k$, and $\tau^{-1} \circ g(k) = \sigma(k) < k$ for $k \geq m \in \mathbb{N}$. Moreover, suppose that equation (7.5.2) is oscillatory and all bounded solutions of the inequality*

$$\left\{\Delta\left(c(k)(\Delta u(k))^{\alpha}\right) - q(k)f\left(p^{-1/\beta}[\sigma(k)]\right)f\left(u^{1/\beta}[\sigma(k)]\right)\right\} \operatorname{sgn} u[\sigma(k)] \geq 0 \tag{7.5.34}$$

are oscillatory, where $0 < p(k) < 1$ for $\beta = 1$ and $p(k) > 0$ for $\beta \in (0,1)$. Then equation (7.5.33$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.5.33$^+$) and define

$$y(k) = x(k) - p(k)x^{\beta}[\tau(k)]. \tag{7.5.35}$$

From equation (7.5.33$^+$), we have

$$\Delta\left(c(k)(\Delta y(k))^{\alpha}\right) = -q(k)f\left(x[g(k)]\right) \leq 0 \quad \text{eventually.} \tag{7.5.36}$$

If $\Delta y(k) < 0$ eventually, then $\lim_{k\to\infty} y(k) = -\infty$. Thus $\lim_{k\to\infty} x(k) = \infty$ and there exists a sequence $\{k_n\}$ such that $\lim_{n\to\infty} k_n = \infty$ and $x(k_n) = \max_{m\leq k\leq k_n} x(k) \to \infty$ as $n \to \infty$, $m \in \mathbb{N}$. Hence

$$\begin{aligned} y(k_n) &= x(k_n) - p(k_n)x^{\beta}[\tau(k_n)] \geq x(k_n) - p(k_n)x^{\beta}(k_n) \\ &= x(k_n)\left[1 - p(k_n)x^{\beta-1}(k_n)\right] \\ &\longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{7.5.37}$$

which is a contradiction. Therefore, $\Delta y(k) > 0$ eventually. Now we consider the following two cases:

(i$_1$) $y(k) < 0$ eventually,

(i$_2$) $y(k) > 0$ eventually.

Assume (i$_1$), that is, $y(k) < 0$ eventually. Then

$$y(k) > -p(k)x^{\beta}[\tau(k)] \quad \text{eventually,} \tag{7.5.38}$$

so

$$x[g(k)] > \left(\frac{z[\tau^{-1} \circ g(k)]}{p[\tau^{-1} \circ g(k)]}\right)^{1/\beta} \quad \text{eventually,} \tag{7.5.39}$$

where $z(k) = -y(k) > 0$ eventually. Using (7.5.39) in equation (7.5.36), we obtain

$$\Delta\Big(c(k)(\Delta z(k))^{\alpha}\Big) - q(k)f(p^{-1/\beta}[\sigma(k)])f(z^{1/\beta}[\sigma(k)]) \geq 0 \quad \text{eventually.} \tag{7.5.40}$$

This inequality has an eventually positive solution, which contradicts the hypothesis of the theorem.

Now assume (i_2), that is, $y(k) > 0$ eventually. The rest of the proof is similar to that of Lemma 7.5.1 and hence is omitted. This completes the proof. □

Theorem 7.5.13. *Let* $0 < p(k) < \infty$, $\tau(k) < k$, $g(k) < k$, $\tau^{-1} \circ g(k) = \sigma(k) < k$ *for* $k \geq m \in \mathbb{N}$. *If all bounded solutions of the inequality*

$$\Big\{\Delta\Big(c(k)(\Delta v(k))^{\alpha}\Big) - q(k)f(v[g(k)])\Big\} \operatorname{sgn} v[g(k)] \geq 0 \tag{7.5.41}$$

are oscillatory and the equation

$$\Delta\Big(c(k)(\Delta w(k))^{\alpha}\Big) + q(k)f(p^{1/\beta}[\sigma(k)])f(w^{1/\beta}[\sigma(k)]) = 0 \tag{7.5.42}$$

is oscillatory, then every bounded solution of equation ($7.5.33^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be a bounded eventually positive solution of equation ($7.5.33^-$) and define $y(k)$ by (7.5.35). Then

$$\Delta\Big(c(k)(\Delta y(k))^{\alpha}\Big) = q(k)f(x[g(k)]) \geq 0 \quad \text{eventually.} \tag{7.5.43}$$

By the boundedness of $y(k)$, we have $\Delta y(k) < 0$ eventually. Now we consider the following two cases:

(i_1) $y(k) < 0$ eventually,
(i_2) $y(k) > 0$ eventually.

Assume (i_1), that is, $y(k) < 0$ eventually. Then eventually

$$0 < z(k) = -y(k) = p(k)x^{\beta}[\tau(k)] - x(k) \leq p(k)x^{\beta}[\tau(k)], \tag{7.5.44}$$

so

$$x[g(k)] \geq \left(\frac{z[\sigma(k)]}{p[\sigma(k)]}\right)^{1/\beta} \quad \text{eventually.} \tag{7.5.45}$$

Using (7.5.45) in equation (7.5.43), we obtain

$$\Delta\Big(c(k)(\Delta z(k))^{\alpha}\Big) + q(k)f(p^{-1/\beta}[\sigma(k)])f(z^{1/\beta}[\sigma(k)]) \leq 0 \quad \text{eventually.} \tag{7.5.46}$$

The rest of the proof in this case is similar to that of Lemma 7.5.1 and hence is omitted.

Now assume (i_2), that is, $y(k) > 0$ eventually. From (7.5.35) we see that

$$x[g(k)] \geq y[g(k)] \quad \text{eventually.} \tag{7.5.47}$$

Using (7.5.47) in equation (7.5.43), we have

$$\Delta\Big(c(k)\big(\Delta y(k)\big)^{\alpha}\Big) \geq q(k)f\big(y[g(k)]\big) \quad \text{eventually.} \tag{7.5.48}$$

The rest of the proof in this case is similar to that of Theorem 7.5.12 case (i_1) and hence is omitted. □

Next, we consider a special case of equation (7.5.1), namely, the equation

$$\Delta\big(c(k)\Delta\big(x(k) + px[k-\tau]\big)\big) + \delta q(k)x[k-\sigma] = 0, \quad \text{where } \delta = \pm 1, \tag{7.5.49}$$

that is,

$$\Delta\big(c(k)\Delta\big(x(k) + px[k-\tau]\big)\big) + q(k)x[k-\sigma] = 0, \tag{7.5.49$^+$}$$

$$\Delta\big(c(k)\Delta\big(x(k) + px[k-\tau]\big)\big) - q(k)x[k-\sigma] = 0, \tag{7.5.49$^-$}$$

where $\tau, \sigma \in \mathbb{N}$ and p is a constant. We present the following interesting results.

Theorem 7.5.14. *Let $p \geq 0$, let $c(k)$ be periodic of period τ, and*

$$q^*(k) = \min\{q(k), q[k-\tau]\}, \tag{7.5.50}$$

hence $q^*(k) = q(k)$ if $q(k)$ is periodic of period τ. If the equation

$$\Delta\big(c(k)\Delta u(k)\big) + q^*(k)u[k-\sigma] = 0 \tag{7.5.51}$$

is oscillatory, then equation (7.5.49$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (7.5.49$^+$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set

$$y(k) = x(k) + px[k-\tau], \tag{7.5.52}$$

$$z(k) = y(k) + py[k-\tau]. \tag{7.5.53}$$

Then $y(k) > 0$ for $k \geq m + \tau$ and $\Delta(c(k)\Delta y(k)) \leq 0$ for $k \geq m_1 = m + \max\{\tau, \sigma\}$. Therefore $\Delta y(k) > 0$ for $k \geq m_1$. Clearly,

$$\begin{aligned}
\Delta\big(c(k)\Delta z(k)\big) &= \Delta\big(c(k)\Delta y(k)\big) + p\Delta\big(c(k)\Delta y[k-\tau]\big) \\
&= \Delta\big(c(k)\Delta y(k)\big) + p\Delta\big(c[k-\tau]\Delta y[k-\tau]\big) \\
&= -q(k)x[k-\sigma] - pq[t-\tau]x[k-\tau-\sigma] \\
&\leq -q^*(k)\big(x[k-\sigma] + px[k-\tau-\sigma]\big) \\
&= -q^*(k)y[k-\sigma].
\end{aligned} \tag{7.5.54}$$

Using this and

$$y(k) \geq \frac{z(k)}{1+p} \quad \text{for } k \geq m_1 \tag{7.5.55}$$

(which follows from (7.5.53) and the fact that $\Delta y(k) > 0$ for $k \geq m_1$), we have

$$\Delta\big(c(k)\Delta z(k)\big) \leq -\frac{q^*(k)}{1+p} z[k-\sigma] \quad \text{for } k \geq m_1. \tag{7.5.56}$$

The rest of the proof is similar to that of Lemma 7.5.1 and hence is omitted. □

Theorem 7.5.15. *Let $p \geq 0$, let $c(k)$ be periodic of period $\tau < \sigma$, and let $q^*(k)$ be as in (7.5.50). If every bounded solution of the inequality*

$$\big\{\Delta\big(c(k)\Delta v(k)\big) - q^*(k)v[k-(\sigma-\tau)]\big\} \operatorname{sgn} v[k-(\sigma-\tau)] \geq 0 \tag{7.5.57}$$

is oscillatory, then every bounded solution of equation (7.5.49$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (7.5.49$^-$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set $y(k)$ and $z(k)$ as in (7.5.52) and (7.5.53), respectively. Then $y(k) > 0$ for $k \geq m+\tau$ and $\Delta(c(k)\Delta y(k)) > 0$ for $k \geq m_1 = m+\sigma$. Therefore $\Delta y(k) < 0$ for $k \geq m_1$. Now for $k \geq m_1$,

$$\begin{aligned}
\Delta\big(c(k)\Delta z(k)\big) &= \Delta\big(c(k)\Delta y(k)\big) + p\Delta\big(c(k)\Delta y[k-\tau]\big)\\
&= \Delta\big(c(k)\Delta y(k)\big) + p\Delta\big(c[k-\tau]\Delta y[k-\tau]\big)\\
&= q(k)x[k-\sigma] + pq[k-\tau]x[k-\tau-\sigma]\\
&\geq q^*(k)\big(x[k-\sigma] + px[k-\tau-\sigma]\big)\\
&= q^*(k)y[k-\sigma].
\end{aligned} \tag{7.5.58}$$

Using this and

$$y(k) \geq \frac{q^*(k)}{1+p} z[k+\tau] \quad \text{for } k \geq m_1 \tag{7.5.59}$$

(which follows from (7.5.53) and the fact that $\Delta y(k) < 0$ for $k \geq m_1$), we find

$$\Delta\big(c(k)\Delta z(k)\big) \geq \frac{q^*(k)}{1+p} z[k-(\sigma-\tau)] \quad \text{for } k \geq m_1. \tag{7.5.60}$$

The rest of the proof is similar to that of Theorem 7.5.12, and hence we omit it here. □

Theorem 7.5.16. *Let $c(k) \equiv 1$, $p > 0$, and $\sigma > \tau$. If there exists a constant $\gamma > 0$ such that*

$$\limsup_{k\to\infty} \frac{q(k)}{q[k-\tau]} = \gamma, \tag{7.5.61}$$

and one of the conditions

$$\limsup_{k\to\infty} \sum_{\ell=k-(\sigma-\tau)}^{k-1} [k-\ell+1]q(\ell) > 1+\gamma p, \tag{7.5.62}$$

$$\limsup_{k\to\infty} \sum_{\ell=k-(\sigma-\tau)}^{k} \left[\sum_{i=\ell}^{k} q(i)\right] > 1+\gamma p, \tag{7.5.63}$$

holds, then every bounded solution of equation (7.5.49$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be a bounded, eventually positive solution of equation (7.5.49$^-$) and let $y(k)$ be defined as in (7.5.52). As shown before, $\Delta^2 y(k) > 0$, $\Delta y(k) < 0$, and $y(k) > 0$ eventually.

From (7.5.61), (7.5.62), and (7.5.63) there exists a constant $\lambda > 1$ such that

$$\frac{q(k)}{q[k-\tau]} < \lambda\gamma \quad \text{for } k \geq m_1 \text{ for some } m_1 \geq m \in \mathbb{N}, \tag{7.5.64}$$

$$\limsup_{k\to\infty} \sum_{\ell=k-(\sigma-\tau)}^{k-1} (k-\ell+1)q(\ell) > 1+\gamma\lambda p, \tag{7.5.65}$$

$$\limsup_{k\to\infty} \sum_{\ell=k-(\sigma-\tau)}^{k} \left[\sum_{i=\ell}^{k} q(i)\right] > 1+\gamma\lambda p. \tag{7.5.66}$$

We rewrite equation (7.5.49$^-$) in the form

$$\Delta^2 y(k) + p\frac{q(k)}{q[k-\tau]}\Delta^2 y[k-\tau] = q(k)y[k-\sigma]. \tag{7.5.67}$$

Using (7.5.64) in equation (7.5.67), we obtain

$$\Delta^2 y(k) + \gamma\lambda p\Delta^2 y[k-\tau] \geq q(k)y[k-\sigma] \quad \text{for } k \geq m_1. \tag{7.5.68}$$

Set

$$w(k) = y(k) + \gamma\lambda p y[k-\tau]. \tag{7.5.69}$$

Then

$$\Delta^2 w(k) \geq q(k)y[k-\sigma] \quad \text{for } k \geq m_1. \tag{7.5.70}$$

From (7.5.69) and the fact that $\Delta y(k) < 0$ for $k \geq m_1$, we have

$$y(k) \geq \frac{q(k)}{1+\gamma\lambda p} w[k+\tau]. \tag{7.5.71}$$

Using this in (7.5.70), we obtain

$$\Delta^2 w(k) \geq \frac{q(k)}{1+\gamma\lambda p} w[k-(\sigma-\tau)] \quad \text{for } k \geq m_1. \tag{7.5.72}$$

The rest of the proof follows by applying Theorem 6.2.15, and hence we omit the details. □

7.6. Oscillation criteria for mixed neutral difference equations

In this section we will consider neutral difference equations with mixed arguments of the form

$$\begin{aligned}&\Delta^2\big(x(k) + ax[k-\tau] - bx[k+\sigma]\big)\\ &\qquad + \delta\big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \quad \delta = \pm 1,\end{aligned} \tag{7.6.1}$$

$$\begin{aligned}&\Delta^2\big(x(k) - ax[k-\tau] + bx[k+\sigma]\big)\\ &\qquad + \delta\big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \quad \delta = \pm 1,\end{aligned} \tag{7.6.2}$$

$$\begin{aligned}&\Delta^2\big(x(k) + ax[k-\tau] + bx[k+\sigma]\big)\\ &\qquad + \delta\big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \quad \delta = \pm 1,\end{aligned} \tag{7.6.3}$$

$$\begin{aligned}&\Delta^2\big(x(k) - ax[k-\tau] - bx[k+\sigma]\big)\\ &\qquad + \delta\big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \quad \delta = \pm 1,\end{aligned} \tag{7.6.4}$$

more precisely

$$\Delta^2\big(x(k) + ax[k-\tau] - bx[k+\sigma]\big) + q(k)x[k-g] + p(k)x[k+h] = 0, \tag{7.6.1$^+$}$$

$$\Delta^2\big(x(k) - ax[k-\tau] + bx[k+\sigma]\big) + q(k)x[k-g] + p(k)x[k+h] = 0, \tag{7.6.2$^+$}$$

$$\Delta^2\big(x(k) + ax[k-\tau] + bx[k+\sigma]\big) + q(k)x[k-g] + p(k)x[k+h] = 0, \tag{7.6.3$^+$}$$

$$\Delta^2\big(x(k) - ax[k-\tau] - bx[k+\sigma]\big) + q(k)x[k-g] + p(k)x[k+h] = 0 \tag{7.6.4$^+$}$$

and

$$\Delta^2\big(x(k) + ax[k-\tau] - bx[k+\sigma]\big) - \big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \tag{7.6.1$^-$}$$

$$\Delta^2\big(x(k) - ax[k-\tau] + bx[k+\sigma]\big) - \big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \tag{7.6.2$^-$}$$

$$\Delta^2\big(x(k) + ax[k-\tau] + bx[k+\sigma]\big) - \big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \tag{7.6.3$^-$}$$

$$\Delta^2\big(x(k) - ax[k-\tau] - bx[k+\sigma]\big) - \big(q(k)x[k-g] + p(k)x[k+h]\big) = 0, \tag{7.6.4$^-$}$$

where

(i) $a, b \geq 0$ are constants,
(ii) $g, h, \tau, \sigma \in \mathbb{N}_0$,
(iii) $\{p(k)\}$ and $\{q(k)\}$ are sequences of nonnegative real numbers.

The characteristic equations associated to (7.6.1)–(7.6.4) when $p(k) \equiv p$ and $q(k) \equiv q$ with constants $p, q \geq 0$ are

$$F_1(\lambda;\delta) = (\lambda-1)^2\big[1 + a\lambda^{-\tau} - b\lambda^{\sigma}\big] + \delta\big[q\lambda^{-g} + p\lambda^{h}\big] = 0, \tag{7.6.5}$$

$$F_2(\lambda;\delta) = (\lambda-1)^2\big[1 - a\lambda^{-\tau} + b\lambda^{\sigma}\big] + \delta\big[q\lambda^{-g} + p\lambda^{h}\big] = 0, \tag{7.6.6}$$

$$F_3(\lambda;\delta) = (\lambda-1)^2\big[1 + a\lambda^{-\tau} + b\lambda^{\sigma}\big] + \delta\big[q\lambda^{-g} + p\lambda^{h}\big] = 0, \tag{7.6.7}$$

$$F_4(\lambda;\delta) = (\lambda-1)^2\big[1 - a\lambda^{-\tau} - b\lambda^{\sigma}\big] + \delta\big[q\lambda^{-g} + p\lambda^{h}\big] = 0, \tag{7.6.8}$$

respectively.

Here we will establish some sufficient (easily verifiable) conditions involving only the coefficients and the arguments under which equations (7.6.1)–(7.6.4) oscillate. We will employ two different techniques. In the first one, the equations (7.6.1)–(7.6.4) are reduced to certain inequalities and then by applying some known results to these inequalities, we obtain the desired conclusions, while the other technique involves certain behavior of the functions $F_i(\lambda)$, $i \in \{1,2,3,4\}$, which determine the fact that the equations (7.6.5)–(7.6.8) have no positive roots.

7.6.1. Oscillation of equations with constant coefficients

In this subsection we discuss oscillation of equations (7.6.1)–(7.6.4) with constant coefficients, that is, we let $p(k) \equiv p$ and $q(k) \equiv q$, where p and q are nonnegative real numbers.

The following lemma is needed in the proof of the results of this subsection.

Lemma 7.6.1. *Assume that q is a positive real number and $\tau \in \mathbb{N}$ is even. Then the following statements hold.*

(I_1) *If*

$$q > \frac{4(\tau-2)^{\tau-2}}{\tau^\tau} \quad \text{for } \tau > 2, \tag{7.6.9}$$

then the advanced difference inequality

$$\Delta^2 x(k) \geq qx[k+\tau] \tag{7.6.10}$$

has no eventually positive solution $\{x(k)\}$ which satisfies $\Delta^i x(k) > 0$ eventually for $i \in \{0,1\}$.

(I_2) *If*

$$q > \frac{4\tau^\tau}{(\tau+2)^{\tau+2}} \quad \text{for } \tau > 1, \tag{7.6.11}$$

then the delay difference inequality

$$\Delta^2 x(k) \geq qx[k-\tau] \tag{7.6.12}$$

has no eventually positive solution $\{x(k)\}$ which satisfies $(-1)^i\Delta^i x(k) > 0$ eventually for $i \in \{0,1\}$.

Proof. We first show (I_1). Let $\{x(k)\}$ be an eventually positive solution of inequality (7.6.10) which satisfies $\Delta^i x(k) > 0$ eventually for $i \in \{0,1\}$. Set

$$y(k) = \Delta x(k) + \sqrt{q}x\left[k+\frac{\tau}{2}\right]. \tag{7.6.13}$$

Then eventually $y(k) > 0$. Also observe that in view of inequality (7.6.10) and (7.6.13)

$$\Delta y(k) - \sqrt{q}y\left[k+\frac{\tau}{2}\right] \geq 0, \tag{7.6.14}$$

and because of (7.6.9) and Lemma 6.1.7(I), inequality (7.6.14) cannot have an eventually positive solution. This contradicts the fact that $y(k) > 0$ eventually and the proof of (I_1) is complete.

Now we show (I_2). Let $\{x(k)\}$ be an eventually positive solution of inequality (7.6.12) which satisfies $(-1)^i\Delta^i x(k) > 0$ eventually for $i \in \{0,1\}$. Set

$$y(k) = \Delta x(k) - \sqrt{q}x\left[k-\frac{\tau}{2}\right]. \tag{7.6.15}$$

Then eventually $y(k) < 0$. Also observe that in view of inequality (7.6.12) and (7.6.15)

$$\Delta y(k) + \sqrt{q}\, y\left[k - \frac{\tau}{2}\right] \geq 0, \tag{7.6.16}$$

and because of (7.6.11) and Lemma 6.1.6(ii), inequality (7.6.16) cannot have an eventually negative solution. This contradicts the fact that $y(k) < 0$ eventually and completes the proof. □

The following two criteria are concerned with the oscillation of (7.6.1).

Theorem 7.6.2. *Suppose that $h > 2$ and $g - \tau > 0$ are even integers and $b > 0$. If*

$$\frac{p}{1+a} > \frac{4(h-2)^{h-2}}{h^h}, \tag{7.6.17}$$

$$\frac{q}{1+a} > \frac{4(g-\tau)^{g-\tau}}{(2+g-\tau)^{2+g-\tau}}, \tag{7.6.18}$$

then equation (7.6.1$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.1$^-$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set

$$z(k) = x(k) + ax[k-\tau] - bx[k+\sigma]. \tag{7.6.19}$$

Then

$$\Delta^2 z(k) = qx[k-g] + px[k+h] \geq 0 \quad \text{eventually}, \tag{7.6.20}$$

and hence we see that $\Delta^i z(k)$ for $i \in \{0, 1\}$ are eventually of one sign. There are two possibilities to consider:

(i_1) $z(k) < 0$ eventually,
(i_2) $z(k) > 0$ eventually.

Assume (i_1), that is, $z(k) < 0$ for $k \geq m_1$ for some $m_1 \geq m$. In this case for $k \geq m_1$, we let

$$0 < u(k) = -z(k) = bx[k+\sigma] - ax[k-\tau] - x(k) \leq bx[k+\sigma]. \tag{7.6.21}$$

Thus $x[k+\sigma] \geq u(k)/b$ or

$$x(k) \geq \frac{1}{b}u[k-\sigma] \quad \text{for } k \geq m_2 \geq m_1. \tag{7.6.22}$$

From equation (7.6.20), we have

$$\Delta^2 u(k) + qx[k-g] \leq 0 \quad \text{for } k \geq m_2. \tag{7.6.23}$$

Using (7.6.22) in inequality (7.6.23), we obtain

$$\Delta^2 u(k) + \frac{q}{b} u[k - \sigma - g] \le 0 \quad \text{for } k \ge m_2. \tag{7.6.24}$$

It is easy to check that $\Delta u(k) > 0$ for $k \ge m_3 \ge m_2$. There exist an integer $m_4 \ge m_3$ and a constant $\alpha > 0$ such that

$$u[k - \sigma - g] \ge \alpha \quad \text{for } k \ge m_4. \tag{7.6.25}$$

Using this in (7.6.24) and summing both sides of the resulting inequality from m_4 to $k \ge m_4$, we get

$$0 < \Delta u(k) \le \Delta u(m_4) - \frac{q}{b}\alpha(k - m_4) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{7.6.26}$$

which is a contradiction.

Now assume (i_2), that is, $z(k) > 0$ for $k \ge m_1$ for some $m_1 \ge m$. Set

$$w(k) = z(k) + az[k - \tau] - bz[k + \sigma]. \tag{7.6.27}$$

Then

$$\Delta^2 w(k) = qz[k - g] + pz[k + h], \tag{7.6.28}$$

and since $w(k)$ satisfies equation ($7.6.1^-$), we have

$$\Delta^2\big(w(k) + aw[k - \tau] - bw[k + \sigma]\big) = qw[k - g] + pw[k + h]. \tag{7.6.29}$$

Using the procedure of case (i_1), we observe that $w(k) > 0$ for $k \ge m_2 \ge m_1$. Next, we have two subcases to consider:

(I_1) $\Delta z(k) > 0$ for $k \ge m_3 \ge m_2$,
(I_2) $\Delta z(k) < 0$ for $k \ge m_3 \ge m_2$.

Assume (I_1), that is, $\Delta z(k) > 0$ for $k \ge m_3$. From equation (7.6.28), we have $\Delta^i w(k) > 0$ for $k \ge m_3$ and $i \in \{2, 3\}$, and hence we see that

$$\Delta^i w(k) > 0 \quad \text{for } k \ge m_3,\ i \in \{0, 1, 2, 3\}. \tag{7.6.30}$$

Now, using the fact that $\Delta^2 w(k)$ is eventually increasing in (7.6.29) for $k \ge m_3$, we obtain

$$\begin{aligned}(1 + a)\Delta^2 w(k) &\ge \Delta^2 w(k) + a\Delta^2 w[k - \tau] - b\Delta^2 w[k + \sigma] \\ &= qw[k - g] + pw[k + h] \\ &\ge pw[k + h],\end{aligned} \tag{7.6.31}$$

and hence

$$\Delta^2 w(k) \geq \frac{p}{1+a} w[k+h] \quad \text{for } k \geq m_3. \tag{7.6.32}$$

But this inequality, in view of Lemma 7.6.1(I_1) and condition (7.6.17), has no solution such that (7.6.30) holds, which is a contradiction.

Now assume (I_2), that is, $\Delta z(k) < 0$ for $k \geq m_3$. We claim that $\Delta w(k) < 0$ for $k \geq m_4 \geq m_3$. To prove this assume that $\Delta w(k) > 0$ for $k \geq m_4$. Then from equation (7.6.28) we see that $\Delta^2 w(k) > 0$ and $\Delta^3 w(k) < 0$ for $k \geq m_4$. Using this fact in equation (7.6.29), one can easily see that

$$(1+a)\Delta^2 w[k-\tau] \geq p w[k+h], \tag{7.6.33}$$

so

$$\Delta^2 w(k) \geq \frac{p}{1+a} w[k+h+\tau] \quad \text{for } k \geq m_4. \tag{7.6.34}$$

Since $\{w(k)\}$ is increasing, we have $\Delta^i w(k) > 0$ for $k \geq m_4$ and $i \in \{0,1,2\}$ and

$$\Delta^2 w(k) \geq \frac{p}{1+a} w[k+h] \quad \text{for } k \geq m_4, \tag{7.6.35}$$

and again we are led to a contradiction. Thus $\Delta w(k) < 0$ for $k \geq m_4$, and from equation (7.6.28) we have

$$(-1)^i \Delta^i w(k) > 0 \quad \text{for } k \geq m_4,\ i \in \{0,1,2,3\}. \tag{7.6.36}$$

Now, using the fact that $\Delta^2 w(k)$ is decreasing for $k \geq m_4$ in (7.6.29) for $k \geq m_4$, we obtain

$$\begin{aligned}(1+a)\Delta^2 w[k-\tau] &\geq \Delta^2 w(k) + a\Delta^2 w[k-\tau] - b\Delta^2 w[k+\sigma] \\ &= q w[k-g] + p w[k+h] \\ &\geq q w[k-g],\end{aligned} \tag{7.6.37}$$

and hence

$$\Delta^2 w(k) \geq \frac{q}{1+a} w[k-(g-\tau)] \quad \text{for } k \geq m_4. \tag{7.6.38}$$

But this inequality, in view of Lemma 7.6.1(I_2) and condition (7.6.18), has no solutions such that (7.6.36) holds, which is a contradiction. This completes the proof. $\square$

Theorem 7.6.3. *Let $h-\sigma>2$ and $g+\sigma>0$ be even integers and $b>0$. If*

$$\frac{p}{b}>\frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \tag{7.6.39}$$

$$\frac{q}{b}>\frac{4(g+\sigma)^{g+\sigma}}{(g+\sigma+2)^{g+\sigma+2}}, \tag{7.6.40}$$

then equation (7.6.1$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.1$^+$), say, $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. Define $z(k)$ by (7.6.19). Then

$$\Delta^2 z(k)=-qx[k-g]-px[k+h]\leq 0 \quad \text{eventually}, \tag{7.6.41}$$

and hence we see that $\Delta^i z(k)$ for $i\in\{0,1\}$ are eventually of one sign. Now we consider the following two cases:

(i_1) $z(k)<0$ eventually,
(i_2) $z(k)>0$ eventually.

First assume (i_1), that is, $z(k)<0$ for $k\geq m_1$ for some $m_1\geq m$. As in the proof of Theorem 7.6.2 case (i_1), we obtain (7.6.22). Using (7.6.22) and $-z(k)=u(k)$ in equation (7.6.41) for $k\geq m_2\geq m_1$, we have

$$\begin{aligned}\Delta^2 u(k)&=qx[k-g]+px[k+h]\\ &\geq \frac{q}{b}u[k-(g+\sigma)]+\frac{p}{b}u[k+(h-\sigma)].\end{aligned} \tag{7.6.42}$$

Next, we consider the following two subcases:

(i_3) $\Delta u(k)>0$ for $k\geq m_3\geq m_2$,
(i_4) $\Delta u(k)<0$ for $k\geq m_3\geq m_2$.

Assume (i_3), that is, $\Delta u(k)>0$ for $k\geq m_3$. It is easy to see that $\Delta^i u(k)>0$ for $k\geq m_3$ and $i\in\{0,1,2\}$. Thus

$$\Delta^2 u(k)\geq \frac{p}{b}u[k+(h-\sigma)] \quad \text{for } k\geq m_3. \tag{7.6.43}$$

The rest of the proof is similar to that of case (i_2), (I_1) in Theorem 7.6.2 and hence is omitted.

Now assume (i_4), that is, $\Delta u(k)<0$ for $k\geq m_3$. Then we see that

$$(-1)^i\Delta^i u(k)>0 \quad \text{for } k\geq m_3,\ i\in\{0,1,2\}. \tag{7.6.44}$$

Now

$$\Delta^2 u(k)\geq \frac{q}{b}u[k-(g+\sigma)] \quad \text{for } k\geq m_3. \tag{7.6.45}$$

But this inequality, in view of Lemma 7.6.1(I_2) and condition (7.6.40), has no solution such that (7.6.44) holds, which is a contradiction.

Finally, assume (i_2), that is, $z(k) > 0$ for $k \geq m_1$. Define $w(k)$ by (7.6.27) and obtain

$$\Delta^2 w(k) + qz[k-g] + pz[k+h] = 0, \tag{7.6.46}$$

$$\Delta^2\big(w(k) + aw[k-\tau] - bw[k+\sigma]\big) + qw[k-g] + pw[k+h] = 0. \tag{7.6.47}$$

Using the procedure of case (i_1), we see that $w(k) > 0$ eventually. From (7.6.41) it is easy to see that $\Delta z(k) > 0$ for $k \geq m_2 \geq m_1$. Using the facts that $z(k) > 0$ and $\Delta z(k) > 0$ for $k \geq m_2$ in equation (7.6.46), we obtain $\Delta^i w(k) \leq 0$ for $k \geq m_3$ and $i \in \{2,3\}$, and hence we see that $w(k) < 0$ for $k \geq m_3$, which is a contradiction. This completes the proof. □

Next we present the following two theorems for the oscillation of equation (7.6.2).

Theorem 7.6.4. *Suppose that $h-\sigma > 2$ and $g > 0$ are even integers and $a > 0$. If*

$$\frac{p}{1+b} > \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \qquad \frac{q}{1+b} > \frac{4g^g}{(g+2)^{g+2}}, \tag{7.6.48}$$

then equation (7.6.2$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.2$^-$), say $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set

$$z(k) = x(k) - ax[k-\tau] + bx[k+\sigma]. \tag{7.6.49}$$

Then

$$\Delta^2 z(k) = qx[k-g] + px[k+h] \geq 0 \quad \text{eventually}. \tag{7.6.50}$$

Clearly, $\Delta^i z(k)$ for $i \in \{0,1\}$ are eventually of one sign and the two cases (i_1) and (i_2) from the proof of Theorem 7.6.2 are considered.

(i_1) $z(k) < 0$ for $k \geq m_1$ for some $m_1 \geq m$. For $k \geq m_2 \geq m_1$, set

$$0 < u(k) = -z(k) = ax[k-\tau] - bx[k+\sigma] - x(k) \tag{7.6.51}$$

or

$$x(k) \geq \frac{1}{a}u[k+\tau] \quad \text{for } k \geq m_2. \tag{7.6.52}$$

The rest of the proof is similar to that of case (i_1) in Theorem 7.6.2 and hence we omit it here.

(i_2) Assume $z(k) > 0$ for $k \geq m_1 \geq m$. We let

$$w(k) = z(k) - az[k-\tau] + bz[k+\sigma]. \tag{7.6.53}$$

Then

$$\Delta^2 w(k) = qz[k-g] + pz[k+h], \tag{7.6.54}$$

$$\Delta^2\big(w(k) - aw[k-\tau] + bw[k+\sigma]\big) = qw[k-g] + pw[k+h]. \tag{7.6.55}$$

As in the proof of case (i_2) in Theorem 7.6.2 the two subcases (I_1) and (I_2) are considered.

(I_1) Assume $\Delta z(k) > 0$ for $k \geq m_2 \geq m_1$. From equation (7.6.54), it is easy to check that $\Delta^i w(k) > 0$ for $k \geq m_2$ and $i \in \{2,3\}$, and hence we see that $\Delta^i w(k) > 0$ for $k \geq m_2$ and $i \in \{0,1,2,3\}$. Using this fact in equation (7.6.55), we obtain

$$\Delta^2 w(k) \geq \frac{p}{1+b} w[k+(h-\sigma)] \quad \text{for } k \geq m_3 \geq m_2, \tag{7.6.56}$$

and again by applying Lemma 7.6.1(I_1) to the above inequality, we are led to a contradiction.

(I_2) Assume $\Delta z(k) < 0$ for $k \geq m_2$. From equation (7.6.54), one can easily see that $\Delta^2 w(k) > 0$ and $\Delta^3 w(k) < 0$ for $k \geq m_2$. Next, we claim that $\Delta w(k) < 0$ for $k \geq m_3 \geq m_2$. Otherwise, $\Delta w(k) > 0$ for $k \geq m_3$. From equation (7.6.55) and the fact that $\Delta^i w(k)$ for $i \in \{0,1\}$ are increasing and $\Delta^2 w(k)$ is decreasing, we have

$$(1+b)\Delta^2 w(k) \geq pw[k+h] \geq pw[k+h-\sigma] \quad \text{for } k \geq m_3, \tag{7.6.57}$$

and as in the above case, we are led to a contradiction. Thus $\Delta w(k) < 0$ for $k \geq m_3$ and hence we conclude that $(-1)^i \Delta^i w(k) > 0$ for $k \geq m_3$ and $i \in \{0,1,2,3\}$. From equation (7.6.55), one can easily obtain

$$\Delta^2 w(k) \geq \frac{q}{1+b} w[k-g] \quad \text{for } k \geq m_3. \tag{7.6.58}$$

The rest of the proof is similar to that of case (i_2), (I_2) in Theorem 7.6.2 and hence is omitted.

This completes the proof. □

Theorem 7.6.5. *Let $h+\tau > 2$ and $g-\tau > 0$ be even integers and $a > 0$. If*

$$\frac{p}{a} > \frac{4(h+\tau-2)^{h+\tau-2}}{(h+\tau)^{h+\tau}}, \tag{7.6.59}$$

$$\frac{q}{a} > \frac{4(g-\tau)^{g-\tau}}{(2+g-\tau)^{2+g-\tau}}, \tag{7.6.60}$$

then equation (7.6.2$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.2$^+$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $z(k)$ by (7.6.49) and obtain

$$\Delta^2 z(k) = -qx[k-g] - px[k+h] \leq 0 \quad \text{eventually,} \tag{7.6.61}$$

and hence we conclude that $\Delta^i z(k)$ for $i \in \{0,1,2\}$ are eventually of one sign. As in the proof of Theorem 7.6.2 we consider the two cases (i_1) and (i_2).

(i_1) Assume $z(k) < 0$ for $k \geq m_1 \geq m$. Define $u(k)$ by (7.6.51) and obtain (7.6.52), and

$$\Delta^2 u(k) \geq \frac{q}{a} u[k-(g-\tau)] + \frac{p}{a} u[k+(h+\tau)] \quad \text{for } k \geq m_2 \geq m_1. \tag{7.6.62}$$

Now we consider the following two subcases:

(i_3) $\Delta u(k) > 0$ for $k \geq m_3 \geq m_2$,
(i_4) $\Delta u(k) < 0$ for $k \geq m_3$.

(i_3) If $\Delta u(k) > 0$ for $k \geq m_2$, then we see that $\Delta^i u(k) > 0$ for $k \geq m_3$ and $i \in \{0,1,2\}$. Thus

$$\Delta^2 u(k) \geq \frac{p}{a} u[k+(h+\tau)] \quad \text{for } k \geq m_3, \tag{7.6.63}$$

and by Lemma 7.6.1(I_1) and condition (7.6.59) we are led to a contradiction.

(i_4) If $\Delta u(k) < 0$ for $k \geq m_3$, then one can easily conclude that $(-1)^i \Delta^i u(k) > 0$ for $k \geq m_3$ and $i \in \{0,1,2\}$. Thus

$$\Delta^2 u(k) \geq \frac{q}{a} u[k-(g-\tau)] \quad \text{for } k \geq m_3, \tag{7.6.64}$$

and again by Lemma 7.6.1(I_2) and condition (7.6.60) we arrive at a contradiction.

(i_2) Assume $z(k) > 0$ for $k \geq m_1$. Define $w(k)$ by (7.6.53) and obtain

$$\begin{gathered} \Delta^2 w(k) + qz[k-g] + pz[k+h] = 0, \\ \Delta^2\big(w(k) - aw[k-\tau] + bw[k+\sigma]\big) + qw[k-g] + pw[k+h] = 0. \end{gathered} \tag{7.6.65}$$

Under the procedure of the proof of case (i_1) above, one can easily see that $w(k) > 0$ for $k \geq m_2 \geq m_1$. From equation (7.6.61) we see that $\Delta z(k) > 0$ for $k \geq m_1$. Now it is easy to check that $\Delta^2 w(k) < 0$ and $\Delta^3 w(k) < 0$ for $k \geq m_2$, and hence $w(k) < 0$ for $k \geq m_2$, which is a contradiction.

This completes the proof. □

The following two results deal with the oscillation of equation (7.6.3).

Theorem 7.6.6. *Let $h-\sigma>2$ and $g-\tau>0$ be even integers. If*

$$\frac{p}{1+a+b} > \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \tag{7.6.66}$$

$$\frac{q}{1+a+b} > \frac{4(g-\tau)^{g-\tau}}{(2+g-\tau)^{2+g-\tau}}, \tag{7.6.67}$$

then equation (7.6.3$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.3$^-$), say $x(k)>0$ for $k\ge m$ for some $m\in\mathbb{N}$. Set

$$z(k)=x(k)+ax[k-\tau]+bx[k+\sigma]. \tag{7.6.68}$$

Then

$$\Delta^2 z(k)=qx[k-g]+px[k+h]\ge 0 \quad \text{eventually}, \tag{7.6.69}$$

and hence we see that $z(k)>0$ and $\Delta z(k)$ is eventually of one sign. Next, we let

$$w(k)=z(k)+az[k-\tau]+bz[k+\sigma]. \tag{7.6.70}$$

Then

$$\Delta^2 w(k)=qz[k-g]+pz[k+h], \tag{7.6.71}$$

$$\Delta^2\big(w(k)+aw[k-\tau]+bw[k+\sigma]\big)=qw[k-g]+pw[k+h]. \tag{7.6.72}$$

Clearly $w(k)>0$ eventually. Now, we consider the two cases:

(I_1) $\Delta z(k)>0$ eventually,
(I_2) $\Delta z(k)<0$ eventually.

First assume (I_1), that is, $\Delta z(k)>0$ for $k\ge m_1\ge m$. Then $\Delta^i w(k)>0$ for $k\ge m_1$ and $i\in\{2,3\}$ and hence we conclude that $\Delta^i w(k)>0$ for $k\ge m_1$ and $i\in\{0,1,2,3\}$. From equation (7.6.72) we have

$$\Delta^2 w(k)\ge \frac{p}{1+a+b}w[k+(h-\sigma)] \quad \text{for } k\ge m_1, \tag{7.6.73}$$

and by Lemma 7.6.1(I_1) and condition (7.6.66) we arrive at a contradiction.

Next assume (I_2), that is, $\Delta z(k)<0$ for $k\ge m_1$. Then one can easily see that $\Delta^2 w(k)>0$ and $\Delta^3 w(k)<0$ for $k\ge m_1$. We claim that $\Delta w(k)<0$ for $k\ge m_2\ge m_1$. Otherwise, $\Delta w(k)>0$ for $k\ge m_2$, and hence we see that $\Delta^i w(k)>0$

for $k \geq m_2$ and $i \in \{0,1,2\}$ and $\Delta^2 w(k)$ is decreasing. Using these facts in equation (7.6.72), we obtain for $k \geq m_2$,

$$\begin{aligned} \Delta^2 w(k) &\geq \frac{p}{1+a+b} w[k+h+\tau] \\ &\geq \frac{p}{1+a+b} w[k+h] \\ &> \frac{p}{1+a+b} w[k+h-\sigma] \end{aligned} \tag{7.6.74}$$

and again we are led to a contradiction. Thus $\Delta w(k) < 0$ for $k \geq m_2$, and hence we conclude that $(-1)^i \Delta^i w(k) > 0$ for $k \geq m_2$ and $i \in \{0,1,2,3\}$. From equation (7.6.72), we have

$$\Delta^2 w(k) \geq \frac{q}{1+a+b} w[k-(g-\tau)] \quad \text{for } k \geq m_2, \tag{7.6.75}$$

and again we arrive at a contradiction. □

Theorem 7.6.7. *The equation (7.6.3⁺) is oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.3⁺), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $z(k)$ by (7.6.68) and obtain

$$\Delta^2 z(k) = -qx[k-g] - px[k+h] \leq 0 \quad \text{eventually.} \tag{7.6.76}$$

Clearly, $z(k) > 0$ and $\Delta z(k) > 0$ eventually. Next, define $w(k)$ by (7.6.70) and obtain

$$\begin{gathered} \Delta^2 w(k) + qz[k-g] + pz[k+h] = 0, \\ \Delta^2 \big(w(k) + aw[k-\tau] + bw[k+\sigma]\big) + qw[k-g] + pw[k+h] = 0. \end{gathered} \tag{7.6.77}$$

Here, $w(k) > 0$ and $\Delta z(k) > 0$ for $k \geq m_1 \geq m$. Since $\Delta z(k) > 0$ for $k \geq m_1$, we see that $\Delta^i w(k) < 0$ for $k \geq m_1$ and $i \in \{2,3\}$, and hence we have $w(k) < 0$ for $k \geq m_1$, a contradiction which completes the proof. □

Finally, we give the following two criteria for the oscillation of equation (7.6.4).

Theorem 7.6.8. *Let $h > 2$ and $g > 0$ be even integers. If*

$$p > \frac{4(h-2)^{h-2}}{h^h}, \tag{7.6.78}$$

$$q > \frac{4g^g}{(2+g)^{2+g}}, \tag{7.6.79}$$

then equation (7.6.4⁻) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation ($7.6.4^-$), say $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set

$$z(k) = x(k) - ax[k-\tau] - bx[k+\sigma]. \tag{7.6.80}$$

Then

$$\Delta^2 z(k) = qx[k-g] + px[k+h] \geq 0 \quad \text{eventually}, \tag{7.6.81}$$

and hence we see that $\Delta^i z(k)$ for $i \in \{0,1,2\}$ are eventually of one sign. There are two cases to consider:

(i_1) $z(k) > 0$ eventually,

(i_2) $z(k) < 0$ eventually.

First we assume (i_1), that is, $z(k) > 0$ for $k \geq m_1$ for some $m_1 \geq m$. Clearly $x(k) \geq z(k)$ for $k \geq m_1$. We consider the following two subcases:

(i_3) $\Delta z(k) > 0$ for $k \geq m_2 \geq m_1$,

(i_4) $\Delta z(k) < 0$ for $k \geq m_3$.

(i_3) If $\Delta z(k) > 0$ for $k \geq m_2$, then $\Delta^i z(k) > 0$ for $k \geq m_2$ and $i \in \{0,1,2\}$ and $\Delta^2 z(k) \geq pz[k+h]$ for $k \geq m_2$. By Lemma 7.6.1(I_1) and condition (7.6.78), we arrive at a contradiction.

(i_4) If $\Delta z(k) < 0$ for $k \geq m_2$, then we have $(-1)^i \Delta^i z(k) > 0$ for $k \geq m_2$ and $i \in \{0,1,2\}$. Also, we have $\Delta^2 z(k) \geq qz[k-g]$ for $k \geq m_2$. Again by Lemma 7.6.1(I_2) and condition (7.6.79), we obtain the desired contradiction.

Now we assume (i_2), that is, $z(k) < 0$ for $k \geq m_1$. Set

$$0 < u(k) = -z(k) = ax[k-\tau] + bx[k+\sigma] - x(k). \tag{7.6.82}$$

Then

$$\Delta^2 u(k) + qx[k-g] + px[k+h] = 0. \tag{7.6.83}$$

Define

$$w(k) = au[k-\tau] + bu[k+\sigma] - u(k). \tag{7.6.84}$$

Then

$$\Delta^2 w(k) + qu[k-g] + pu[k+h] = 0, \tag{7.6.85}$$

$$\Delta^2\big(aw[k-\tau] + bw[k+\sigma] - w(k)\big) + qw[k-g] + pw[k+h] = 0. \tag{7.6.86}$$

Using the procedure of case (i_1) above, one can easily see that $w(k) > 0$ for $k \geq m_1$. Next, from equation (7.6.83) we have $\Delta u(k) > 0$ for $k \geq m_2 \geq m_1$. Using the fact that $\Delta^i u(k) > 0$ for $k \geq m_2$ and $i \in \{0,1\}$ in equation (7.6.85), one can easily see that $\Delta^i w(k) < 0$ for $k \geq m_2$ and $i \in \{2,3\}$. Thus $w(k) < 0$ for $k \geq m_2$, which is a contradiction. This completes the proof. □

Theorem 7.6.9. *Let $h-\sigma>2$ and $g-\tau>0$ be even integers and $a+b>0$. If*

$$\frac{p}{a+b}>\frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \tag{7.6.87}$$

$$\frac{q}{a+b}>\frac{4(g-\tau)^{g-\tau}}{(2+g-\tau)^{2+g-\tau}}, \tag{7.6.88}$$

then equation (7.6.4$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.4$^+$), say, $x(k)>0$ for $k\geq m$ for some $m\in\mathbb{N}$. Define $z(k)$ by (7.6.80) and obtain

$$\Delta^2 z(k)=-qx[k-g]-px[k+h]\leq 0 \quad \text{eventually}, \tag{7.6.89}$$

and hence we conclude that $\Delta^i z(k)$ for $i\in\{0,1\}$ are eventually of one sign. As in the proof of Theorem 7.6.8, the two cases (i$_1$) and (i$_2$) are considered.

(i$_1$) Assume $z(k)>0$ for $k\geq m_1\geq m$. Then $x(k)\geq z(k)$ for $k\geq m_1$ and $\Delta^2 z(k)+qz[k-g]\leq 0$ for $k\geq m_1$. Clearly $\Delta z(k)>0$ for $k\geq m_3\geq m_2$, and hence one can easily see that $0<\Delta z(k)\to-\infty$ as $k\to\infty$, which is a contradiction.

(i$_2$) Assume $z(k)<0$ for $k\geq m_1$. Define $u(k)$ by (7.6.82) and obtain

$$\Delta^2 u(k)=qx[k-g]+px[k+h]. \tag{7.6.90}$$

Next, define $w(k)$ by (7.6.84) and obtain

$$\Delta^2 w(k)=qu[k-g]+pu[k+h], \tag{7.6.91}$$

$$\Delta^2\left(aw[k-\tau]+bw[k+\sigma]-w(k)\right)=qw[k-g]+pw[k+h]. \tag{7.6.92}$$

It is easy to check that $w(k)>0$ for $k\geq m_2\geq m_1$. Next, we consider the following two subcases:

(i$_3$) $\Delta u(k)>0$ for $k\geq m_3\geq m_2$,
(i$_4$) $\Delta u(k)<0$ for $k\geq m_3$.

(i$_3$) Suppose $\Delta u(k)>0$ for $k\geq m_3$. From equation (7.6.91), we see that $\Delta^i w(k)>0$ for $k\geq m_3$ and $i\in\{2,3\}$, and hence one can easily conclude that $\Delta^i w(k)>0$ for $k\geq m_3$ and $i\in\{0,1,2,3\}$. From equation (7.6.92) we have

$$(a+b)\Delta^2 w[k+\sigma]>pw[k+h], \tag{7.6.93}$$

so

$$\Delta^2 w(k)>\frac{p}{a+b}w[k+(h-\sigma)] \quad \text{for } k\geq m_3. \tag{7.6.94}$$

By Lemma 7.6.1(I$_1$) and condition (7.6.87), we obtain the desired contradiction.

(i$_4$) Suppose $\Delta u(k)<0$ for $k\geq m_3$. It is easy to see that $\Delta^2 w(k)>0$ and $\Delta^3 w(k)<0$ for $k\geq m_3$. We claim $\Delta w(k)<0$ for $k\geq m_3$. Otherwise, $\Delta w(k)>0$

for $k \geq m_3$ and hence one can easily conclude that $\Delta^i w(k) > 0$ for $k \geq m_3$ and $i \in \{0,1,2\}$. From (7.6.92), one can easily see that $(a+b)\Delta^2 w[k-\tau] \geq pw[k+h]$ or

$$(a+b)\Delta^2 w(k) \geq pw[k+h+\tau] \geq pw[k+h-\sigma], \tag{7.6.95}$$

and again we are led to a contradiction. Thus $\Delta w(k) < 0$ for $k \geq m_3$, and hence we conclude that $(-1)^i \Delta^i w(k) > 0$ for $k \geq m_3$ and $i \in \{0,1,2\}$. From equation (7.6.92), we have $(a+b)\Delta^2 w[k-\tau] \geq qw[k-g]$ or

$$\Delta^2 w(k) \geq \frac{q}{a+b} w[k-(g-\tau)] \quad \text{for } k \geq m_3. \tag{7.6.96}$$

By Lemma 7.6.1(I_2) and condition (7.6.88) we arrive at the desired contradiction.
This completes the proof. □

Remark 7.6.10. From the proofs of the results presented in this subsection, we observe that when the coefficient $p = 0$ or when the conditions on p are violated, the conclusions of the theorems given in this subsection may be replaced by "every solution $\{x(k)\}$ of each of (7.6.1)–(7.6.4) is oscillatory or $\Delta^j x(k) \to \infty$ monotonically as $k \to \infty$ for $j \in \{0,1\}$."

Example 7.6.11. As an example, we see that the equations

$$\begin{aligned}
&\Delta^2\big(x(k) + e^{\tau}x[k-\tau] - e^{-\sigma}x[k+\sigma]\big)\\
&\quad = \frac{(e-1)^2}{2}\big(e^{g}x[k-g] + e^{-h}x[k+h]\big),\\
&\Delta^2\big(x(k) + e^{\tau}x[k-\tau] - e^{-\sigma}x[k+\sigma]\big) = (e-1)^2 e^{g}x[k-g]
\end{aligned} \tag{7.6.97}$$

have a nonoscillatory solution $\{x(k)\}$ with $x(k) = e^k \to \infty$ as $k \to \infty$.

Remark 7.6.12. Once again from the proofs of the theorems presented in this subsection, we see that if $q = 0$ or if the conditions on q are not satisfied, then the conclusions of these theorems may be replaced by "every solution $\{x(k)\}$ of each of the equations (7.6.1)–(7.6.4) is oscillatory or $\Delta^j x(k) \to 0$ monotonically as $k \to \infty$ for $j \in \{0,1\}$."

Example 7.6.13. As an example, we observe that the equations

$$\begin{aligned}
&\Delta^2\big(x(k) + e^{-\tau}x[k-\tau] - e^{\sigma}x[k+\sigma]\big)\\
&\quad = \frac{1}{2}\left(\frac{1}{e}-1\right)^2\big(e^{-g}x[k-g] + e^{h}x[k+h]\big),\\
&\Delta^2\big(x(k) + e^{-\tau}x[k-\tau] - e^{\sigma}x[k+\sigma]\big) = \left(\frac{1}{e}-1\right)^2 e^{h}x[k+h]
\end{aligned} \tag{7.6.98}$$

have a nonoscillatory solution $\{x(k)\}$ with $x(k) = e^{-k} \to 0$ as $k \to \infty$.

Remark 7.6.14. (i) We note that the results of this subsection are not applicable to equations (7.6.1)–(7.6.4) when $p = 0$ or $q = 0$. It is remarkable that these results are possibly valid if $p = 0$ or $q = 0$ (but not $p = q = 0$) provided that either $a = 0$, $b = 0$, or $a = b = 0$. The details are left to the reader.

(ii) By using the same technique as presented in this subsection, one can easily obtain similar criteria for equations (7.6.1)–(7.6.4) for different signs on τ and σ and also different signs on a and b. The details are left to the reader.

7.6.2. Oscillation of equations with periodic coefficients

In this subsection we discuss oscillation of equations (7.6.1)–(7.6.4) with periodic coefficients, that is, we let $\sigma = \tau$ and assume that $p(k)$ and $q(k)$ are periodic of period τ, that is, $p(k \pm \tau) = p(k)$ and $q(k \pm \tau) = q(k)$.

We will need the following lemma which is extracted from Theorems 6.1.40 and 6.1.41.

Lemma 7.6.15. *Assume that $\{q(k)\}$ is a sequence of nonnegative real numbers and $\tau \in \mathbb{N}$. Then the following statements hold.*

(I_1) *If $\tau > 2$ and*

$$\limsup_{k\to\infty} \sum_{j=k}^{k+\tau-2} (k + \tau - j - 1)q(j) > 1, \tag{7.6.99}$$

then the advanced difference inequality

$$\Delta^2 x(k) \geq q(k)x[k + \tau] \tag{7.6.100}$$

has no eventually positive solution which satisfies $\Delta^i x(k) > 0$ eventually for $i \in \{0, 1\}$.

(I_2) *If*

$$\limsup_{k\to\infty} \sum_{j=k-\tau}^{k} (k - j + 1)q(j) > 1, \tag{7.6.101}$$

then the delay difference inequality

$$\Delta^2 x(k) \geq q(k)x[k - \tau] \tag{7.6.102}$$

has no eventually positive solution $\{x(k)\}$ which satisfies $(-1)^i \Delta^i x(k) > 0$ eventually for $i \in \{0, 1\}$.

The following two criteria are concerned with the oscillation of equation (7.6.1).

Theorem 7.6.16. *Let $b > 0$, $h > 2$, and $g > \tau$. If*

$$\limsup_{k\to\infty} \sum_{j=k}^{k+h-2} (k+h-j-1)p(j) > 1+a, \tag{7.6.103}$$

$$\limsup_{k\to\infty} \sum_{j=k-(g-\tau)}^{k} (k-j+1)q(j) > 1+a, \tag{7.6.104}$$

then equation (7.6.1$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.1$^-$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set

$$z(k) = x(k) + ax[k-\tau] - bx[k+\tau]. \tag{7.6.105}$$

Then

$$\Delta^2 z(k) = q(k)x[k-g] + p(k)x[k+h] \geq 0 \quad \text{for } k \geq m_1 \geq m, \tag{7.6.106}$$

which implies that $\Delta^i z(k)$ for $i \in \{0,1\}$ are eventually of one sign. Now there are two possibilities to consider:

(i$_1$) $z(k) < 0$ eventually,

(i$_2$) $z(k) > 0$ eventually.

First we assume (i$_1$), that is, $z(k) < 0$ for $k \geq m_2 \geq m_1$. Set

$$0 < u(k) = -z(k) = bx[k+\tau] - ax[k-\tau] - x(k) \leq bx[k+\tau]. \tag{7.6.107}$$

There exists an integer $m_3 \geq m_2$ such that

$$x(k) \geq \frac{1}{b}u[k-\tau] \quad \text{for } k \geq m_3. \tag{7.6.108}$$

Using (7.6.108) in equation (7.6.106), we have

$$\Delta^2 u(k) + \frac{1}{b}q(k)u[k-(g+\tau)] \leq 0 \quad \text{for } k \geq m_3. \tag{7.6.109}$$

Since $\Delta^2 u(k) \leq 0$ and $u(k) > 0$ for $k \geq m_3$, we have $\Delta u(k) > 0$ for $k \geq m_4 \geq m_3$. There exist an integer $m_5 \geq m_4$ and a constant $\alpha > 0$ such that $u[k-(g+\tau)] \geq \alpha$ for $k \geq m_5$. Thus

$$\Delta^2 u(k) + \frac{\alpha}{b}q(k) \leq 0 \quad \text{for } k \geq m_5, \tag{7.6.110}$$

and hence

$$0 < \Delta u(k) \leq \Delta u(m_5) - \frac{\alpha}{b}\sum_{j=m_5}^{k-1} q(j) \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{7.6.111}$$

which is a contradiction.

Now we assume (i_2), that is, $z(k) > 0$ for $k \ge m_1 \ge m$. Set

$$w(k) = z(k) + az[k-\tau] - bz[k+\tau]. \tag{7.6.112}$$

Then

$$\Delta^2 w(k) = q(k)z[k-g] + p(k)z[k+h], \tag{7.6.113}$$

$$\Delta^2\big(w(k) + aw[k-\tau] - bw[k+\tau]\big) = q(k)w[k-g] + p(k)w[k+h]. \tag{7.6.114}$$

Using the procedure of case (i_1) we see that $w(k) > 0$ eventually. Next, there are two subcases to consider:

(I_1) $\Delta z(k) > 0$ eventually,

(I_2) $\Delta z(k) < 0$ eventually.

(I_1) Suppose $\Delta z(k) > 0$ for $k \ge m_2 \ge m_1$. There exist positive constants α_1, α_2, and an integer $m_2 \ge m_1$ such that $z[k-g] \ge \alpha_1$ and $z[k+h] \ge \alpha_2$ for $k \ge m_2$. Thus $\Delta^2 w(k) \ge \alpha_1 q(k) + \alpha_2 p(k)$ for $k \ge m_2$ and hence $\Delta w(k) \to \infty$ as $w(k) \to \infty$ as $k \to \infty$. Therefore we conclude that $\Delta^i w(k) > 0$ for $k \ge m_2$ and $i \in \{0,1,2\}$. From equation (7.6.113) and the fact that $\{z(k)\}$ is an increasing sequence, and $p(k)$ and $q(k)$ are periodic of period τ, we obtain for $k \ge m_2$,

$$\begin{aligned}
\Delta^2 w[k-\tau] &= q[k-\tau]z[k-\tau-g] + p[k-\tau]z[k-\tau+h] \\
&= q(k)z[k-\tau-g] + p(k)z[k-\tau+h] \\
&\le q(k)z[k-g] + p(k)z[k+h] \\
&= \Delta^2 w(k).
\end{aligned} \tag{7.6.115}$$

Using this fact in equation (7.6.114), we obtain $(1+a)\Delta^2 w(k) \ge p(k)w[k+h]$, so

$$\Delta^2 w(k) \ge \frac{p(k)}{1+a} w[k+h] \quad \text{for } k \ge m_3 \ge m_2. \tag{7.6.116}$$

But this inequality, in view of Lemma 7.6.15(I_1) and condition (7.6.103), has no eventually positive solution $\{w(k)\}$ such that $\Delta^i w(k) > 0$ eventually for $i \in \{0,1\}$, which is a contradiction.

(I_2) Assume $\Delta z(k) > 0$ for $k \ge m_2$. First we claim that $z(k) \to 0$ monotonically as $k \to \infty$. Otherwise, $z(k) \to \beta > 0$ as $k \to \infty$. There exists an integer $m_3 \ge m_2$ such that $x[k-g] \ge \beta/2$ and $x[k+h] \ge \beta/2$ for $k \ge m_3$. Thus

$$\Delta^2 w(k) \ge \frac{\beta}{2}\big[q(k) + p(k)\big] \quad \text{for } k \ge m_3, \tag{7.6.117}$$

and hence $\Delta w(k) \to \infty$ and $w(k) \to \infty$ as $k \to \infty$. From (7.6.112), we see that $w(k) < \infty$ for all $k \ge m_3$, which is a contradiction. Therefore we conclude that $z(k) \to 0$, $w(k) \to 0$, and $\Delta^2 w(k) \to 0$ monotonically as $k \to \infty$, and so, one can easily see that

$$\Delta w(k) < 0 \quad \text{for } k \ge m_3. \tag{7.6.118}$$

From equation (7.6.113) and the fact that $z(k)$ is decreasing and $p(k)$, $q(k)$ are periodic of period τ, we have for $k \geq m_3$,

$$\begin{aligned}
\Delta^2 w[k-\tau] &= q[k-\tau]z[k-\tau-g] + p[k-\tau]z[k-\tau+h] \\
&= q(k)z[k-\tau-g] + p(k)z[k-\tau+h] \\
&\geq q(k)z[k-g] + p(k)z[k+h] \\
&= \Delta^2 w(k).
\end{aligned} \tag{7.6.119}$$

Using this fact in equation (7.6.114), we obtain $(1+a)\Delta^2 w(k) \geq q(k)w[k-g]$ or

$$\Delta^2 w(k) \geq \frac{q(k)}{1+a} w[k-g] \quad \text{for } k \geq m_4 \geq m_3. \tag{7.6.120}$$

But this inequality, in view of Lemma 7.6.15(I_2) and condition (7.6.104), has no eventually positive solution $\{w(k)\}$ with $(-1)^i \Delta^i w(k) > 0$ eventually for $i \in \{0, 1\}$, which is a contradiction.

This completes the proof. □

Theorem 7.6.17. *Let $b > 0$ and $h > \tau + 2$. If*

$$\begin{aligned}
&\limsup_{k\to\infty} \sum_{j=k}^{k+h-\tau-2} (k+h-\tau-j-1)p(j) > b, \\
&\limsup_{k\to\infty} \sum_{j=k-(g+\tau)}^{k} (k-j+1)q(j) > b,
\end{aligned} \tag{7.6.121}$$

then equation (7.6.1$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.1$^+$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $z(k)$ by (7.6.105) and obtain

$$\Delta^2 z(k) = -q(k)x[k-g] - p(k)x[k+h] \leq 0 \quad \text{for } k \geq m_2 \geq m_1. \tag{7.6.122}$$

As in the proof of Theorem 7.6.16, we consider the two cases (i_1) and (i_2).

(i_1) Assume $z(k) < 0$ for $k \geq m_1$. Set $u(k)$ as in (7.6.107) and obtain (7.6.108). Using (7.6.108) in equation (7.6.122), we have

$$\Delta^2 u(k) \geq \frac{p(k)}{b} u[k+(h-\tau)] \quad \text{for } k \geq m_2 \geq m_1, \tag{7.6.123}$$

so

$$\Delta^2 u(k) \geq \frac{q(k)}{b} u[k-(g+\tau)] \quad \text{for } k \geq m_2. \tag{7.6.124}$$

The rest of the proof is similar to that of cases (i_2), (I_1) and (i_2), (I_2), respectively, from Theorem 7.6.16. Hence we omit the details.

(i_2) Assume $z(k) > 0$ for $k \geq m_1$. Set $w(k)$ as in (7.6.112) and obtain

$$\Delta^2 w(k) + q(k)z[k-g] + p(k)z[k+h] = 0,$$
$$\Delta^2\big(w(k) + aw[k-\tau] - bw[k+\tau]\big) + q(k)w[k-g] + p(k)w[k+h] = 0. \tag{7.6.125}$$

It is easy to check that $w(k) > 0$, $\Delta w(k) > 0$, and $\Delta z(k) > 0$ for $k \geq m_2 \geq m_1$. Thus there exist positive constants α_1 and α_2 and an integer $m_3 \geq m_2$ such that $z[k-g] \geq \alpha_1$ and $z[k+h] \geq \alpha_2$ for $k \geq m_3$ and hence

$$\Delta^2 w(k) + \alpha_1 q(k) + \alpha_2 p(k) \leq 0 \quad \text{for } k \geq m_3. \tag{7.6.126}$$

Thus

$$0 < \Delta w(k) \leq \Delta w(m_3) - \sum_{j=m_3}^{k-1}\big[\alpha_1 q(j) + \alpha_2 p(j)\big] \longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \tag{7.6.127}$$

which is a contradiction.

This completes the proof. □

Next, we give the following two criteria for the oscillation of equation (7.6.2). The proofs of these two results can be modelled according to those of Theorems 7.6.16 and 7.6.17, and hence we omit the details.

Theorem 7.6.18. *Let $a > 0$ and $h > \tau + 2$. If*

$$\limsup_{k\to\infty} \sum_{j=k}^{k+(h-\tau)-2} (k+h-\tau-j-1)p(j) > 1+b,$$
$$\limsup_{k\to\infty} \sum_{j=k-g}^{k} (k-j+1)q(j) > 1+b, \tag{7.6.128}$$

then equation (7.6.2$^-$) is oscillatory.

Theorem 7.6.19. *Let $a > 0$, $g > \tau$, and $h + \tau > 2$. If*

$$\limsup_{k\to\infty} \sum_{j=k}^{k+h+\tau-2} (k+h+\tau-j-1)p(j) > a,$$
$$\limsup_{k\to\infty} \sum_{j=k-(g-\tau)}^{k} (k-j+1)q(j) > a, \tag{7.6.129}$$

then equation (7.6.2$^+$) is oscillatory.

The following two theorems are concerned with the oscillatory behavior of equation (7.6.3).

Theorem 7.6.20. *Let $g > \tau$ and $h > \tau + 2$. If*

$$\limsup_{k\to\infty} \sum_{j=k}^{k+h-\tau-2} (k+h-\tau-j-1)p(j) > 1+a+b,$$
$$\limsup_{k\to\infty} \sum_{j=k-(g-\tau)}^{k} (k-j+1)q(j) > 1+a+b, \tag{7.6.130}$$

then equation (7.6.3$^-$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.3$^-$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Set

$$z(k) = x(k) + ax[k-\tau] + bx[k+\tau]. \tag{7.6.131}$$

Then $z(k) > 0$ for $k \geq m_1 \geq m$ and satisfies the equation

$$\Delta^2 z(k) = q(k)x[k-g] + p(k)x[k+h] \geq 0 \quad \text{for } k \geq m_2 \geq m_1, \tag{7.6.132}$$

which implies that $\Delta^i z(k)$ for $i \in \{0,1\}$ are eventually of one sign. Next, we set

$$w(k) = z(k) + az[k-\tau] + bz[k+\tau]. \tag{7.6.133}$$

Then

$$\Delta^2 w(k) = q(k)z[k-g] + p(k)z[k+h], \tag{7.6.134}$$
$$\Delta^2\big(w(k) + aw[k-\tau] + bw[k+\tau]\big) = q(k)w[k-g] + p(k)w[k+h]. \tag{7.6.135}$$

Now we consider the two subcases:

(I_1) $\Delta z(k) > 0$ for $k \geq m_3 \geq m_2$,
(I_2) $\Delta z(k) < 0$ for $k \geq m_3$.

Assume (I_1), that is, $\Delta z(k) > 0$ for $k \geq m_3$. Then $\Delta w(k) > 0$ for $k \geq m_4 \geq m_3$ and for $k \geq m_5 \geq m_4$,

$$\begin{aligned}\Delta^2 w[k-\tau] &= q[k-\tau]z[k-\tau-g] + p[k-\tau]z[k-\tau+h] \\ &= q(k)z[k-\tau-g] + p(k)z[k-\tau+h] \\ &\leq q(k)z[k-g] + p(k)z[k+h] \\ &= \Delta^2 w(k),\end{aligned} \tag{7.6.136}$$

and for $k \geq m_5$,

$$\begin{aligned}
\Delta^2 w[k+\tau] &= q[k+\tau]z[k+\tau-g] + p[k+\tau]z[k+\tau+h] \\
&= q(k)z[k+\tau-g] + p(k)z[k+\tau+h] \\
&\geq q(k)z[k-g] + p(k)z[k+h] \\
&= \Delta^2 w(k).
\end{aligned} \tag{7.6.137}$$

From equation (7.6.135), we see that

$$(1+a+b)\Delta^2 w[k+\tau] \geq p(k)w[k+h] \quad \text{for } k \geq m_5, \tag{7.6.138}$$

so

$$\Delta^2 w(k) \geq \frac{p(k)}{1+a+b} w[k+(h-\tau)] \quad \text{for } k \geq m_5. \tag{7.6.139}$$

The rest of the proof is similar to that of case (i_2), (I_1) from Theorem 7.6.16, and hence we omit it.

Now assume (I_2), that is, $\Delta z(k) < 0$ for $k \geq m_2$. So $\Delta w(k) < 0$ for $k \geq m_3 \geq m_2$ and for $k \geq m_4 \geq m_3$ one can easily see that

$$\Delta^2 w[k-\tau] \geq \Delta^2 w(k) \geq \Delta^2 w[k+\tau], \tag{7.6.140}$$

and from equation (7.6.135) we have

$$(1+a+b)\Delta^2 w[k-\tau] \geq q(k)w[k-g] \quad \text{for } k \geq m_4, \tag{7.6.141}$$

so

$$\Delta^2 w(k) \geq \frac{q(k)}{1+a+b} w[k-(g-\tau)] \quad \text{for } k \geq m_4. \tag{7.6.142}$$

The rest of the proof is similar to that of case (i_2), (I_2) from Theorem 7.6.16, and hence we omit the details. This completes the proof. □

Theorem 7.6.21. *The equation (7.6.3⁺) is oscillatory.*

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.3⁺), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $z(k)$ by (7.6.131) and obtain

$$\Delta^2 z(k) + q(k)x[k-g] + p(k)x[k+h] = 0. \tag{7.6.143}$$

Next, define $w(k)$ by (7.6.133) and obtain

$$\Delta^2 w(k) + q(k)z[k-g] + p(k)z[k+h] = 0. \tag{7.6.144}$$

Clearly, $z(k) > 0$, $\Delta z(k) > 0$, $w(k) > 0$, and $\Delta w(k) > 0$ for $k \geq m_2 \geq m_1$. The rest of the proof is similar to that of case (i_1) from Theorem 7.6.17, and hence we omit the details. This completes the proof. □

Finally, we present the following two theorems for the oscillation of equation (7.6.4).

Theorem 7.6.22. *If* $h > 2$,

$$\limsup_{k\to\infty} \sum_{j=k}^{k+h-2} (k+h-j-1)p(j) > 1, \tag{7.6.145}$$

$$\limsup_{k\to\infty} \sum_{j=k-g}^{k} (k-j+1)q(j) > 1, \tag{7.6.146}$$

then equation (7.6.4⁻) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.4⁻), say, $x(k) > 0$ for $k \ge m$ for some $m \in \mathbb{N}$. Set

$$z(k) = x(k) - ax[k-\tau] - bx[k+\tau]. \tag{7.6.147}$$

Then

$$\Delta^2 z(k) = q(k)x[k-g] + p(k)x[k+h] \ge 0 \quad \text{eventually}, \tag{7.6.148}$$

and hence we see that $\Delta^i z(k)$ for $i \in \{0,1\}$ are eventually of one sign. There are two possible cases to consider:

(i_1) $z(k) > 0$ eventually,
(i_2) $z(k) < 0$ eventually.

Assume (i_1), that is, $z(k) > 0$ for $k \ge m_1 \ge m$. Clearly $x(k) \ge z(k)$ for $k \ge m_1$. Now we consider the following two subcases:

(i_3) $\Delta z(k) > 0$ for $k \ge m_2 \ge m_1$,
(i_4) $\Delta z(k) < 0$ for $k \ge m_2$.

(i_3) If $\Delta z(k) > 0$ for $k \ge m_2$, then $\Delta^i z(k) > 0$ for $k \ge m_2$ and $i \in \{0,1,2\}$ and $\Delta^2 z(k) \ge p(k)z[k+h]$ for $k \ge m_2$. By Lemma 7.6.15(I_1) and condition (7.6.145), we arrive at a contradiction.

(i_4) If $\Delta z(k) < 0$ for $k \ge m_2$, then we have $(-1)^i\Delta^i z(k) > 0$ for $k \ge m_2$ and $i \in \{0,1,2\}$. Also, we have

$$\Delta^2 z(k) \ge q(k)z[k-g] \quad \text{for } k \ge m_2. \tag{7.6.149}$$

Again, by Lemma 7.6.15(I_2) and condition (7.6.146), we arrive at the desired contradiction.

Now assume (i_2), that is, $z(k) < 0$ for $k \ge m_1$. Set

$$0 < u(k) = -z(k) = ax[k-\tau] + bx[k+\tau] - x(k). \tag{7.6.150}$$

Then

$$\Delta^2 u(k) + q(k)x[k-g] + p(k)x[k+h] = 0, \tag{7.6.151}$$

and hence we conclude that $\Delta u(k) > 0$ for $k \geq m_2 \geq m_1$. Define

$$w(k) = au[k-\tau] + bu[k+\tau] - u(k). \tag{7.6.152}$$

Then we have

$$\Delta^2 w(k) + q(k)u[k-g] + p(k)u[k+h] = 0, \tag{7.6.153}$$

$$\Delta^2\big(aw[k-\tau] + bw[k+\tau] - w(k)\big) + q(k)w[k-g] + p(k)w[k+h] = 0. \tag{7.6.154}$$

Now it is easy to check that $w(k) > 0$ and hence $\Delta w(k) > 0$ for $k \geq m_3 \geq m_2$. There exist positive constants α_1 and α_2 and an integer $m_4 \geq m_3$ such that

$$u[k-g] \geq \alpha_1, \quad u[k+h] \geq \alpha_2 \quad \text{for } k \geq m_4. \tag{7.6.155}$$

Using this in equation (7.6.153), we obtain

$$\Delta^2 w(k) \leq -\alpha_1 q(k) - \alpha_2 p(k) \quad \text{for } k \geq m_4. \tag{7.6.156}$$

Summing both sides of (7.6.156) from m_4 to $k-1$, we get

$$\begin{aligned} 0 &< \Delta w(k) \\ &\leq \Delta w(m_4) - \sum_{j=m_4}^{k-1}\big[\alpha_1 q(j) + \alpha_2 p(j)\big] \\ &\longrightarrow -\infty \quad \text{as } k \longrightarrow \infty, \end{aligned} \tag{7.6.157}$$

which is a contradiction and completes the proof. □

Theorem 7.6.23. *Let $a+b>0$, $g>\tau$, and $h>\tau+2$. If*

$$\begin{gathered} \limsup_{k\to\infty} \sum_{j=k}^{k+h-\tau-2} (k+h-\tau-j-1)p(j) > a+b, \\ \limsup_{k\to\infty} \sum_{j=k-(g-\tau)}^{k} (k-j+1)q(j) > a+b, \end{gathered} \tag{7.6.158}$$

then equation (7.6.4$^+$) is oscillatory.

Proof. Let $\{x(k)\}$ be an eventually positive solution of equation (7.6.4$^+$), say, $x(k) > 0$ for $k \geq m$ for some $m \in \mathbb{N}$. Define $z(k)$ by (7.6.147) and get

$$\Delta^2 z(k) = -q(k)x[k-g] - px[k+h] \leq 0 \quad \text{for } k \geq m_1 \geq m, \tag{7.6.159}$$

which implies that $\Delta^i z(k)$ for $i \in \{0,1\}$ are eventually of one sign. As in the proof of Theorem 7.6.22, we consider the cases (i$_1$) and (i$_2$).

(i$_1$) Suppose $z(k) > 0$ for $k \geq m_2 \geq m_1$. The proof of this case is similar to that of case (i$_2$) from Theorem 7.6.22, and hence we omit the details.

(i$_2$) Suppose $z(k) < 0$ for $k \geq m_2 \geq m_1$. Define $u(k)$ by (7.6.150) and obtain

$$\Delta^2 u(k) = q(k)x[k-g] + p(k)x[k+h] \geq 0 \quad \text{for } k \geq m_2, \tag{7.6.160}$$

which implies that $\Delta u(k)$ is eventually of one sign. Next, we consider the following two subcases:

(I$_1$) $\Delta u(k) > 0$ for $k \geq m_3 \geq m_2$,

(I$_2$) $\Delta u(k) < 0$ for $k \geq m_3$.

(I$_1$) Assume $\Delta u(k) > 0$ for $k \geq m_3$. Define $w(k)$ by (7.6.152) and obtain

$$\Delta^2 w(k) = q(k)u[k-g] + p(k)u[k+h], \tag{7.6.161}$$

$$\Delta^2\big(aw[k-\tau] + bw[k+\tau] - w(k)\big) = q(k)w[k-g] + p(k)w[k+h]. \tag{7.6.162}$$

As in the proof of case (i$_2$), (I$_1$) in Theorem 7.6.16, we see that $w(k) > 0$ and $\Delta w(k) > 0$ for $k \geq m_4 \geq m_3$ and

$$(a+b)\Delta^2 w[k+\tau] \geq p(k)w[k+h], \tag{7.6.163}$$

so

$$\Delta^2 w(k) \geq \frac{p(k)}{a+b} w[k+(h-\tau)] \quad \text{for } k \geq m_5 \geq m_4. \tag{7.6.164}$$

The rest of the proof is similar to that of case (i$_2$), (I$_1$) from Theorem 7.6.16 and hence is omitted.

(I$_2$) Assume $\Delta u(k) < 0$ for $k \geq m_3$. As in the above case we obtain equation (7.6.162), and as in the proof of case (i$_2$), (I$_2$) from Theorem 7.6.16, we observe that $w(k) > 0$ and $\Delta w(k) < 0$ for $k \geq m_4 \geq m_3$, and hence we have

$$(a+b)\Delta^2 w[k-\tau] \geq q(k)w[k-g] \quad \text{for } k \geq m_5 \geq m_4, \tag{7.6.165}$$

or

$$\Delta^2 w(k) \geq \frac{q(k)}{a+b} w[k-(g-\tau)] \quad \text{for } k \geq m_5. \tag{7.6.166}$$

Once again, the rest of the proof is similar to that of case (i$_2$), (I$_2$) in Theorem 7.6.16, and hence we omit the details.

This completes the proof. □

Remark 7.6.24. The discussions and observations given in Remarks 7.6.10–7.6.14 hold for the obtained results of this subsection too, and so we omit this here.

7.6.3. Oscillation via characteristic equations

In this section we discuss oscillation of the mixed neutral equations (7.6.1)–(7.6.4) with constant coefficients via its associated characteristic equations. We consider the equations (7.6.1)–(7.6.4) when $p(k) \equiv p$ and $q(k) \equiv q$, where p and q are positive real numbers.

To obtain the results of this subsection, we need the following lemma.

Lemma 7.6.25. *Consider the linear difference equation*

$$x[k+n] + \sum_{j=1}^{n} q(j)x[k+n-j] = 0, \; k \in \mathbb{N}_0, \tag{7.6.167}$$

where $n \in \mathbb{N}_0$ and $q(j) \in \mathbb{R}$ for $j \in \{1, 2, \ldots, n\}$. Then the following statements are equivalent.

(i) *Every solution of equation (7.6.167) oscillates.*

(ii) *The characteristic equation associated with the equation (7.6.167), that is,*

$$\lambda^n + \sum_{j=1}^{n} q(j)\lambda^{n-j} = 0, \tag{7.6.168}$$

has no positive roots.

The following two theorems are concerned with the oscillation of equations (7.6.1)–(7.6.4).

Theorem 7.6.26. *Let $a+1 > b > 1$, $h > 2$, and $g > \tau$. If*

$$\begin{aligned} \frac{p}{1+a-b} &> \frac{4(h-2)^{h-2}}{h^h}, \\ \frac{q}{1+a} &> \frac{4(g-\tau)^{g-\tau}}{(2+g-\tau)^{2+g-\tau}}, \end{aligned} \tag{7.6.169}$$

then equation (7.6.1$^-$) is oscillatory.

Proof. Our strategy is to prove that under the hypotheses given above the equation (7.6.5) with $\delta = -1$ has no positive roots. There are two possible cases to consider:

(i_1) $\lambda > 1$,

(i_2) $0 < \lambda < 1$.

(i_1) Assume $\lambda > 1$. From equation (7.6.5) with $\delta = -1$, we see that

$$
\begin{aligned}
-\frac{F_1(\lambda;-1)}{(\lambda-1)^2} &= \frac{q\lambda^{-g}+p\lambda^h}{(\lambda-1)^2} - 1 - a\lambda^{-\tau} + b\lambda^{\sigma} \\
&\geq p\frac{\lambda^h}{(\lambda-1)^2} - 1 - a + b.
\end{aligned}
\tag{7.6.170}
$$

Now, since the local minimum of the function $x^{\alpha}/(x-1)^{\beta}$, $x > 1$ and $\alpha > \beta$, occurs at $x = \alpha/(\alpha-\beta)$, we find

$$
-\frac{F_1(\lambda;-1)}{(\lambda-1)^2} \geq p\frac{\left[h/(h-2)\right]^h}{\left[2/(h-2)\right]^2} - 1 - a + b > 0.
\tag{7.6.171}
$$

(i_2) Assume $0 < \lambda < 1$. In this case we obtain

$$
\begin{aligned}
-\frac{F_1(\lambda;-1)\lambda^{\tau}}{(\lambda-1)^2} &= -\frac{F_1(\lambda;-1)\lambda^{\tau}}{(1-\lambda)^2} = -\lambda^{\tau} - a + b\lambda^{\tau+\sigma} + \frac{q\lambda^{-(g-\tau)}+p\lambda^{h+\tau}}{(1-\lambda)^2} \\
&\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^2} - 1 - a,
\end{aligned}
\tag{7.6.172}
$$

and since the function $x^{-\alpha}/(1-x)^{\beta}$, $0 < x < 1$, $\alpha, \beta > 0$, has a local minimum at $x = \alpha/(\alpha+\beta)$, we conclude that

$$
-\frac{F_1(\lambda;-1)\lambda^{\tau}}{(1-\lambda)^2} \geq q\frac{\left[(g-\tau)/(2+g-\tau)\right]^{-(g-\tau)}}{\left[2/(2+g-\tau)\right]^2} - 1 - a > 0.
\tag{7.6.173}
$$

Cases (i_1) and (i_2) imply $F_1(\lambda;-1) < 0$ for $\lambda \in \mathbb{R}^+$, that is, equation (7.6.5) with $\delta = -1$ has no positive roots. Thus by Lemma 7.6.25, we conclude that equation (7.6.1$^-$) is oscillatory. This completes the proof. □

Theorem 7.6.27. *Let $b > a+1$, $h > \sigma+2$, and $g > \tau$. If*

$$
\begin{aligned}
\frac{p}{b} &> \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \\
\frac{q}{b-a-1} &> \frac{4(g+\sigma)^{g+\sigma}}{(g+\sigma+2)^{g+\sigma+2}},
\end{aligned}
\tag{7.6.174}
$$

then equation (7.6.1$^+$) is oscillatory.

Proof. As in the proof of Theorem 7.6.26, we consider the two cases (i_1) and (i_2).

(i_1) Suppose $\lambda > 1$. From equation (7.6.5) with $\delta = 1$ we obtain

$$\begin{aligned}\frac{F_1(\lambda;1)\lambda^{-\sigma}}{(\lambda-1)^2} &= \frac{q\lambda^{-(g+\sigma)} + p\lambda^{h-\sigma}}{(\lambda-1)^2} + \lambda^{-\sigma} + a\lambda^{-(\tau+\sigma)} - b \\ &\geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^2} - b.\end{aligned} \tag{7.6.175}$$

As in the proof of case (i_1) in Theorem 7.6.26 we find

$$\frac{F_1(\lambda;1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq p\frac{\left[(h-\sigma)/(h-\sigma-2)\right]^{h-\sigma}}{\left[2/(h-\sigma-2)\right]^2} - b > 0. \tag{7.6.176}$$

(i_2) Suppose $0 < \lambda < 1$. It follows from (7.6.5) with $\delta = 1$ that

$$\frac{F_1(\lambda;1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq q\frac{\lambda^{-(g+\sigma)}}{(1-\lambda)^2} + \lambda^{-\sigma} + a\lambda^{-(\tau+\sigma)} - b. \tag{7.6.177}$$

As in the proof of case (i_2) in Theorem 7.6.26 we see that

$$\frac{F_1(\lambda;1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq q\frac{\left[(g+\sigma)/(g+\sigma+2)\right]^{-(g+\sigma)}}{\left[2/(g+\sigma+2)\right]^2} + 1 + a - b > 0. \tag{7.6.178}$$

Cases (i_1) and (i_2) imply that $F_1(\lambda;1) > 0$ for $\lambda \in (0,1) \cup (1,\infty)$, and since $F_1(1;1) > 0$, we have that $F_1(\lambda;1) > 0$ for all $\lambda \in \mathbb{R}^+$, that is, equation (7.6.5) with $\delta = 1$ has no positive roots. By Lemma 7.6.25 we conclude that equation ($7.6.1^+$) is oscillatory. This completes the proof. □

The following two theorems deal with the oscillation of equation (7.6.2).

Theorem 7.6.28. *Suppose that* $1 + b > a > 0$, $g > \tau$, *and* $h > \sigma + 2$. *If*

$$\begin{aligned}\frac{p}{1+b} &> \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \\ \frac{q}{1+b-a} &> \frac{4g^g}{(g+2)^{g+2}},\end{aligned} \tag{7.6.179}$$

then equation ($7.6.2^-$) is oscillatory.

Proof. For $\lambda \neq -1$, we find

$$-\frac{F_2(\lambda;-1)\lambda^{-\sigma}}{(\lambda-1)^2} = \frac{q\lambda^{-(g+\sigma)} + p\lambda^{h-\sigma}}{(\lambda-1)^2} - \lambda^{-\sigma} + a\lambda^{-(\tau+\sigma)} - b. \tag{7.6.180}$$

As in the proof of Theorem 7.6.26, we consider the two cases (i_1) and (i_2).

(i_1) Suppose $\lambda > 1$. From (7.6.180) we obtain

$$-\frac{F_2(\lambda;-1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^2} - 1 - b. \tag{7.6.181}$$

As in the proof of case (i_1) in Theorem 7.6.26 we see that

$$-\frac{F_2(\lambda;-1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq p\frac{\left[(h-\sigma)/(h-\sigma-2)\right]^{h-\sigma}}{\left[2/(h-\sigma-2)\right]^2} - 1 - b > 0. \tag{7.6.182}$$

(i_2) Suppose $0 < \lambda < 1$. In this case we have

$$\begin{aligned}-\frac{F_2(\lambda;-1)}{(\lambda-1)^2} = -\frac{F_2(\lambda;-1)}{(1-\lambda)^2} &= \frac{q\lambda^{-g} + p\lambda^{h}}{(1-\lambda)^2} - 1 + a\lambda^{-\tau} - b\lambda^{\sigma} \\ &\geq q\frac{\lambda^{-g}}{(1-\lambda)^2} - 1 + a - b,\end{aligned} \tag{7.6.183}$$

and as in the proof of case (i_2) in Theorem 7.6.26 we have

$$-\frac{F_2(\lambda;-1)}{(1-\lambda)^2} \geq q\frac{\left[g/(g+2)\right]^{-g}}{\left[2/(g+2)\right]^2} - 1 + a - b > 0. \tag{7.6.184}$$

The rest of the proof is similar to that of Theorem 7.6.26, and hence we omit the details. □

Theorem 7.6.29. *Suppose that $a > 0$, $\tau + h > 2$, and $g > \tau$. If*

$$\begin{aligned}\frac{p}{a} &> \frac{4(h+\tau-2)^{h+\tau-2}}{(h+\tau)^{h+\tau}}, \\ \frac{q}{a} &> \frac{4(g-\tau)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}},\end{aligned} \tag{7.6.185}$$

then equation (7.6.2$^+$) is oscillatory.

Proof. For $\lambda \neq 1$, we have

$$\frac{F_2(\lambda;1)\lambda^{\tau}}{(\lambda-1)^2} = \lambda^{\tau} - a + b\lambda^{\tau+\sigma} + \frac{q\lambda^{\tau-g} + p\lambda^{\tau+h}}{(\lambda-1)^2}. \tag{7.6.186}$$

Now we consider the same two cases as in Theorem 7.6.28.

(i_1) Assume $\lambda > 1$. From (7.6.186) we find

$$\frac{F_2(\lambda;1)\lambda^\tau}{(\lambda-1)^2} \geq p\frac{\lambda^{\tau+h}}{(\lambda-1)^2} - a, \tag{7.6.187}$$

and since the function $x^\alpha/(x-1)^\beta$, $x > 1$ and $\alpha > \beta$, has its local minimum value at $x = \alpha/(\alpha-\beta)$, we see that

$$\frac{F_2(\lambda;1)\lambda^\tau}{(\lambda-1)^2} \geq p\frac{\left[(\tau+h)/(\tau+h-2)\right]^{\tau+h}}{\left[2/(\tau+h-2)\right]^2} - a > 0. \tag{7.6.188}$$

(i_2) Assume $0 < \lambda < 1$. Once again, from (7.6.186), we see that

$$\begin{aligned}\frac{F_2(\lambda;1)\lambda^\tau}{(\lambda-1)^2} &= \frac{F_2(\lambda;1)\lambda^\tau}{(1-\lambda)^2}\\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^2} + \lambda^\tau - a + b\lambda^{\tau+\sigma}\\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^2} - a.\end{aligned} \tag{7.6.189}$$

Since the minimum of the function $x^{-\alpha}/(1-x)^\beta$, $0 < x < 1$, occurs at $x = \alpha/(\alpha+\beta)$, we conclude that

$$\frac{F_2(\lambda;1)\lambda^\tau}{(1-\lambda)^2} \geq q\frac{\left[(g-\tau)/(g-\tau+2)\right]^{-(g-\tau)}}{\left[2/(g-\tau+2)\right]^2} - a > 0. \tag{7.6.190}$$

From the cases (i_1) and (i_2) and the fact that $F_2(1;1) > 0$, we conclude that $F_2(\lambda;1) > 0$ for all $\lambda \in \mathbb{R}^+$, that is, equation (7.6.5) with $\delta = 1$ has no positive roots. Thus the conclusion of the theorem follows by applying Lemma 7.6.25. This completes the proof. □

Next we present the following two criteria for the oscillation of equation (7.6.3).

Theorem 7.6.30. *Suppose that $g > \tau$ and $h > \sigma + 2$. If*

$$\frac{p}{1+a+b} > \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \tag{7.6.191}$$

$$\frac{q}{1+a+b} > \frac{4(g-\tau)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}}, \tag{7.6.192}$$

then equation ($7.6.3^-$) is oscillatory.

Proof. For $\lambda \neq 1$, we have

$$-\frac{F_3(\lambda;-1)\lambda^{-\sigma}}{(\lambda-1)^2} = \frac{q\lambda^{-g-\sigma}+p\lambda^{h-\sigma}}{(\lambda-1)^2} - \lambda^{-\sigma} - a\lambda^{-\tau-\sigma} - b. \tag{7.6.193}$$

We consider the same two possible cases as in Theorem 7.6.29.

(i_1) Assume $\lambda > 1$. From (7.6.193) we see that

$$-\frac{F_3(\lambda;-1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^2} - 1 - a - b. \tag{7.6.194}$$

As in the proof of case (i_1) from Theorem 7.6.26 and in view of condition (7.6.191), we find $-F_3(\lambda;-1)/(\lambda-1)^2 > 0$ for all $\lambda > 1$.

(i_2) Assume $0 < \lambda < 1$. In this case we find

$$\begin{aligned} -\frac{F_3(\lambda;-1)\lambda^{\tau}}{(1-\lambda)^2} &= \frac{q\lambda^{-(g-\tau)}+p\lambda^{h+\tau}}{(1-\lambda)^2} - \lambda^{\tau} - a - b\lambda^{\tau+\sigma} \\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^2} - 1 - a - b. \end{aligned} \tag{7.6.195}$$

Once again and as in the proof of case (i_2) from Theorem 7.6.29 one can easily see that

$$-\frac{F_3(\lambda;-1)\lambda^{\tau}}{(1-\lambda)^2} > 0 \quad \text{for } 0 < \lambda < 1. \tag{7.6.196}$$

The rest of the proof is similar to that of Theorem 7.6.26 and hence we omit the details. □

Theorem 7.6.31. *The equation (7.6.3$^+$) is oscillatory.*

Proof. The proof is obvious since $F_3(\lambda;1) > 0$ for all $\lambda \in \mathbb{R}^+$. □

The following two theorems are concerned with the oscillation of equation (7.6.4).

Theorem 7.6.32. *Assume that $a < 1$, $b < 1$, and $h > 2$. If*

$$\frac{p}{1-a} > \frac{4(h-2)^{h-2}}{h^h}, \qquad \frac{q}{1-a} > \frac{4g^g}{(g+2)^{g+2}}, \tag{7.6.197}$$

then equation (7.6.4$^-$) is oscillatory.

Proof. Again we consider the same two cases as before.

(i_1) Suppose $\lambda > 1$. It follows from (7.6.8) with $\delta = -1$ that

$$-\frac{F_4(\lambda;-1)}{(\lambda-1)^2} = \frac{q\lambda^{-g} + p\lambda^h}{(\lambda-1)^2} - 1 + a\lambda^{-\tau} + b\lambda^{\sigma} \geq p\frac{\lambda^h}{(\lambda-1)^2} - 1 + b. \tag{7.6.198}$$

Proceeding as in case (i_1) of Theorem 7.6.26, we obtain $-F_4(\lambda;-1)/(\lambda-1)^2 > 0$ for all $\lambda > 1$.

(i_2) Suppose $0 < \lambda < 1$. In this case we have

$$-\frac{F_4(\lambda;-1)}{(1-\lambda)^2} \geq \frac{q\lambda^{-g}}{(1-\lambda)^2} - 1 + a. \tag{7.6.199}$$

As in the proof of case (i_2) from Theorem 7.6.29, we obtain $-F_4(\lambda;-1)/(1-\lambda)^2 > 0$ for $0 < \lambda < 1$.

The rest of the proof is similar to that of Theorem 7.6.26 and hence we omit the details. □

Theorem 7.6.33. *Assume that $a + b > 0$, $g > \tau$, and $h > \sigma + 2$. If*

$$\frac{p}{a+b} > \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \qquad \frac{q}{a+b} > \frac{4(g-\tau)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}}, \tag{7.6.200}$$

then equation (7.6.4$^+$) is oscillatory.

Proof. For $\lambda \neq 1$, we have

$$\frac{F_4(\lambda;-1)\lambda^{-\sigma}}{(\lambda-1)^2} = \frac{q\lambda^{-(g+\sigma)} + p\lambda^{h-\sigma}}{(\lambda-1)^2} + \lambda^{-\sigma} - a\lambda^{-(\tau+\sigma)} - b. \tag{7.6.201}$$

Next, we will consider the same two cases as before.

(i_1) Suppose $\lambda > 1$. From (7.6.201) it follows that

$$\frac{F_4(\lambda;1)\lambda^{-\sigma}}{(\lambda-1)^2} \geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^2} + \lambda^{-\sigma} - a\lambda^{-(\tau+\sigma)} - b \geq p\frac{\lambda^{h-\sigma}}{(\lambda-1)^2} - a - b. \tag{7.6.202}$$

As in the proof of case (i_1) from Theorem 7.6.26, we find

$$\frac{F_4(\lambda;1)\lambda^{-\sigma}}{(\lambda-1)^2} > 0 \quad \forall \lambda > 1. \tag{7.6.203}$$

(i_2) Suppose $0 < \lambda < 1$. In this case we see that

$$\begin{aligned} \frac{F_4(\lambda;1)\lambda^\tau}{(\lambda-1)^2} = \frac{F_4(\lambda;1)\lambda^\tau}{(1-\lambda)^2} &= \frac{q\lambda^{-(g-\tau)} + p\lambda^{h+\tau}}{(1-\lambda)^2} + \lambda^\tau - a - b\lambda^{\sigma+\tau} \\ &\geq q\frac{\lambda^{-(g-\tau)}}{(1-\lambda)^2} - a - b. \end{aligned} \tag{7.6.204}$$

Proceeding as in the proof of case (i_2) from Theorem 7.6.29, we find

$$\frac{F_4(\lambda;1)\lambda^\tau}{(1-\lambda)^2} > 0 \quad \forall \lambda \in (0,1). \tag{7.6.205}$$

The rest of the proof is similar to that of Theorem 7.6.29 and hence we omit the details. □

By employing the same technique as given in this subsection one can easily provide sufficient conditions for oscillation of equations of the form

$$\Delta^2\big(x(k) + ax[k-\tau] - bx[k-\sigma]\big) + \delta\big(qx[k-g] + px[k+h]\big) = 0, \tag{7.6.206}$$
$$\Delta^2\big(x(k) - ax[k+\tau] + bx[k+\sigma]\big) + \delta\big(qx[k-g] + px[k+h]\big) = 0, \tag{7.6.207}$$
$$\Delta^2\big(x(k) + ax[k-\tau] + bx[k-\sigma]\big) + \delta\big(qx[k-g] + px[k+h]\big) = 0, \tag{7.6.208}$$
$$\Delta^2\big(x(k) + ax[k+\tau] + bx[k+\sigma]\big) + \delta\big(qx[k-g] + px[k+h]\big) = 0, \tag{7.6.209}$$
$$\Delta^2\big(x(k) - ax[k-\tau] - bx[k-\sigma]\big) + \delta\big(qx[k-g] + px[k+h]\big) = 0, \tag{7.6.210}$$
$$\Delta^2\big(x(k) - ax[k+\tau] - bx[k+\sigma]\big) + \delta\big(qx[k-g] + px[k+h]\big) = 0 \tag{7.6.211}$$

via the associated characteristic equations of these equations, provided that the coefficients of these equations are defined as those given above.

Here we will only state the following interesting criteria.

Theorem 7.6.34. *Assume that $1 + a > b > 0$, $h > 2$, and $g > \tau$. If*

$$\frac{p}{1+a} > \frac{4(h-2)^{h-2}}{h^h}, \qquad \frac{q}{1+a-b} > \frac{4(g-\tau)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}}, \tag{7.6.212}$$

then equation (7.6.206) with $\delta = -1$ is oscillatory.

Theorem 7.6.35. *Assume that $b > a + 1$, $h + \sigma > 2$, and $g > \sigma$. If*

$$\frac{p}{b-a-1} > \frac{4(h+\sigma-2)^{h+\sigma-2}}{(h+\sigma)^{h+\sigma}}, \qquad \frac{q}{b} > \frac{4(g-\sigma)^{g-\sigma}}{(g-\sigma+2)^{g-\sigma+2}}, \tag{7.6.213}$$

then equation (7.6.206) with $\delta = 1$ is oscillatory.

Theorem 7.6.36. *Assume that $b+1>a>0$, $g>0$, $\tau\geq\sigma$, and $h-\sigma>2$. If*

$$\frac{p}{b-a+1}>\frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}},\qquad \frac{q}{b+1}>\frac{4g^g}{(g+2)^{g+2}},\tag{7.6.214}$$

then equation (7.6.207) with $\delta=-1$ is oscillatory.

Theorem 7.6.37. *Assume that $a>b$, $a>1$, $\sigma>\tau$, and $h-\tau>2$. If*

$$\frac{p}{a-b}>\frac{4(h-\tau-2)^{h-\tau-2}}{(h-\tau)^{h-\tau}},\qquad \frac{q}{a-1}>\frac{4(g+h)^{g+h}}{(g+\tau+2)^{g+\tau+2}},\tag{7.6.215}$$

then equation (7.6.207) with $\delta=1$ is oscillatory.

Theorem 7.6.38. *Suppose that $h>2$ and $g>\tau\geq\sigma$. If*

$$\frac{p}{1+a+b}>\frac{4(h-2)^{h-2}}{h^h},\qquad \frac{q}{1+a+b}>\frac{4(g-h)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}},\tag{7.6.216}$$

then equation (7.6.208) with $\delta=-1$ is oscillatory.

Theorem 7.6.39. *The equation (7.6.208) with $\delta=1$ is oscillatory.*

Theorem 7.6.40. *Suppose that $g>0$, $h>\sigma+2$, and $\sigma\geq\tau$. If*

$$\frac{p}{1+a+b}>\frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}},\qquad \frac{q}{1+a+b}>\frac{4g^g}{(g+2)^{g+2}},\tag{7.6.217}$$

then equation (7.6.209) with $\delta=-1$ is oscillatory.

Theorem 7.6.41. *The equation (7.6.209) with $\delta=1$ is oscillatory.*

Theorem 7.6.42. *Suppose that $a+b<1$ and $h>2$. If*

$$p>\frac{4(h-2)^{h-2}}{h^h},\qquad \frac{q}{1-a-b}>\frac{4g^g}{(g+2)^{g+2}},\tag{7.6.218}$$

then equation (7.6.210) with $\delta=-1$ is oscillatory.

Theorem 7.6.43. *Suppose that $a+b>1$, $g>\tau\geq\sigma$, and $h+\sigma>2$. If*

$$\frac{p}{a+b-1}>\frac{4(h+\sigma-2)^{h+\sigma-2}}{(h+\sigma)^{h+\sigma}},\qquad \frac{q}{a+b}>\frac{4(g-\tau)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}},\tag{7.6.219}$$

then equation (7.6.210) with $\delta=1$ is oscillatory.

Theorem 7.6.44. *Suppose that $g > 0$ and $h > 2$. If*

$$p > \frac{4(h-2)^{h-2}}{h^h}, \qquad q > \frac{4g^g}{(g+2)^{g+2}}, \tag{7.6.220}$$

then equation (7.6.211) with $\delta = -1$ is oscillatory.

Theorem 7.6.45. *Suppose that $a + b > 0$, $h > \sigma + 2$, $\sigma \geq \tau$, and $g > \tau$. If*

$$\frac{p}{a+b} > \frac{4(h-\sigma-2)^{h-\sigma-2}}{(h-\sigma)^{h-\sigma}}, \qquad \frac{q}{a+b} > \frac{4(g-\tau)^{g-\tau}}{(g-\tau+2)^{g-\tau+2}}, \tag{7.6.221}$$

then equation (7.6.211) with $\delta = 1$ is oscillatory.

Remark 7.6.46. We note that the results of this subsection either improve or are similar to those presented in the previous subsections. Further improvements of the results of this subsection can also be achieved by studying the local minimum of the reduced functions obtained from the characteristic equations under consideration. Here, we omit the details.

7.7. Notes and general discussions

(1) Some of the results from Section 7.1 are taken from Grace and El-Morshedy [144], while other results are new.
(2) The results of Section 7.2 are due to Agarwal et al. [21]. In fact, these results extend and correct the results in [25, Section 21]. Also, it is possible to use the idea in [25, Section 21] to discuss when the solution x in Theorem 7.2.4 (or Theorem 7.2.5) lies in M^+ and so forth (see [25] for the appropriate definitions). We leave the details to the reader.
(3) The results of Section 7.3 are the discrete analogs of the modified results due to Kulenović and Hadžiomerspahić [185].
(4) The results of Section 7.4 are the discrete analogs of some of the results presented in Erbe et al. [119]. It would be interesting to obtain such classification for equations of type (7.5.1).
(5) Some of the results of Section 7.5 are new while other results are the discrete analogs of results due to Erbe et al. [119].
(6) The results of Section 7.6.1 are taken from Grace [137] and also extracted from Agarwal and Grace [13]. The results of Section 7.6.2 are taken from Agarwal and Grace [17], while the results presented in Section 7.6.3 are new. Lemma 7.6.25 can be found in [150, 176]. We note that further improvements of the results of Section 7.6.3 can be obtained rather easily. The details are left to the reader.

8 Stability and oscillation theory for differential equations with piecewise constant arguments

Differential equations with piecewise constant arguments represent a hybrid of continuous and discrete dynamical systems and therefore combine properties of both differential and difference equations. These equations have applications in control theory and in certain biomedical models [79].

In Section 8.1, we provide many criteria for oscillation and nonoscillation of first-order differential equations with piecewise constant arguments. Stability, oscillation, and nonoscillation of certain second-order differential equations with piecewise constant arguments are established in Section 8.2. Section 8.3 is devoted to the study of stability and oscillation of systems with piecewise constant arguments. In Section 8.4 and as an application of the techniques presented, necessary and sufficient conditions for all positive solutions of the so-called logistic equation with quadratic nonlinearity with piecewise constant arguments to oscillate about its positive equilibrium are obtained.

8.1. Oscillation of linear first-order delay equations

Consider the linear delay differential equation with piecewise constant argument of the form

$$x'(t) + p(t)x(t) + q(t)x([t-1]) = 0 \quad \text{for } t \geq 0, \tag{8.1.1}$$

where $p \in C([-1,\infty),\mathbb{R})$, $q \in C([-1,\infty),\mathbb{R}^+)$, and $[\cdot]$ denotes the greatest integer function.

By a solution of equation (8.1.1) we mean a function x which is defined on the set $(-1,0) \cup \mathbb{R}^+$ and which satisfies the following conditions:

(i) x is continuous on $\mathbb{R}^+$,
(ii) the derivative $x'(t)$ exists at each point $t \in \mathbb{R}_0^+ = [0,\infty)$, with the possible exception of the point $[t] \in \mathbb{R}_0^+$, where one-sided derivatives exist,
(iii) equation (8.1.1) is satisfied on each interval $[k, k+1)$ for $k \in \mathbb{N}_0$.

Let $a(-1)$ and $a(0)$ be any given real numbers. Then, as a result of Lemma 8.1.3 below, equation (8.1.1) has a unique solution x satisfying

$$x(-1) = a(-1), \qquad x(0) = a(0). \tag{8.1.2}$$

8.1.1. Oscillation criteria

Now we present the following two oscillation results.

Theorem 8.1.1. *If*

$$\limsup_{k\to\infty}\int_{k-1}^{k} q(s)\exp\Big(\int_{k-2}^{s} p(u)du\Big)ds > 1, \tag{8.1.3}$$

then equation (8.1.1) is oscillatory.

Proof. Assume that equation (8.1.1) has a nonoscillatory solution x. Then there exists an integer $m \geq 0$ such that $x(t) > 0$ for $t \geq m$. Therefore, for $k \in \mathbb{N}(m)$ and for $t \in [k, k+1)$ we have

$$x'(t) + p(t)x(t) = -a(k-1)q(t) \quad \text{for } k \leq t < k+1, \tag{8.1.4}$$

where we use the notation $a(k) = x(k)$ for $k \in \mathbb{N}(-1)$. Equation (8.1.4) can be rewritten for $k \leq t < k+1$ as

$$\Big(x(t)\exp\Big(\int_{k}^{t} p(s)ds\Big)\Big)' + q(t)\exp\Big(\int_{k}^{t} p(s)ds\Big)a(k-1) = 0. \tag{8.1.5}$$

Integrating (8.1.5) from k to $t \in [k, k+1)$ provides

$$x(t)\exp\Big(\int_{k}^{t} p(u)du\Big) - a(k) + \Big(\int_{k}^{t} q(s)\exp\Big(\int_{k}^{s} p(u)du\Big)ds\Big)a(k-1) = 0. \tag{8.1.6}$$

By the continuity of x, letting $t \to k+1$ and replacing k by $k-1$, we obtain

$$a(k)\exp\Big(\int_{k-1}^{k} p(u)du\Big) = a(k-1) - a(k-2)\int_{k-1}^{k} q(s)\exp\Big(\int_{k-1}^{s} p(u)du\Big)ds. \tag{8.1.7}$$

It follows from (8.1.7) that

$$a(k-1)\exp\Big(\int_{k-2}^{k-1} p(u)du\Big) < a(k-2) \quad \text{for } k \geq m+2. \tag{8.1.8}$$

Using (8.1.8) in (8.1.7) we find for $k \geq m+2$ that

$$a(k)\exp\Big(\int_{k-1}^{k} p(u)du\Big) < a(k-1)\Big[1 - \int_{k-1}^{k} q(s)\exp\Big(\int_{k-2}^{s} p(u)du\Big)ds\Big], \tag{8.1.9}$$

and since $a(k), a(k-1) \in \mathbb{R}^+$ for $k \geq m$, we conclude that

$$\int_{k-1}^{k} q(s) \exp\left(\int_{k-2}^{s} p(u)du\right) ds < 1. \tag{8.1.10}$$

Hence

$$\limsup_{k\to\infty} \int_{k-1}^{k} q(s) \exp\left(\int_{k-2}^{s} p(u)du\right) ds \leq 1, \tag{8.1.11}$$

which contradicts condition (8.1.3) and therefore completes the proof of the theorem. □

Theorem 8.1.2. *If*

$$\limsup_{k\to\infty} \int_{k-1}^{k} p(s)ds > -\infty, \tag{8.1.12}$$

$$\liminf_{k\to\infty} \left(\exp\left(\int_{k-1}^{k} p(s)ds\right)\right) \liminf_{k\to\infty} \left(\int_{k-1}^{k} q(s) \exp\left(\int_{k-1}^{s} p(u)du\right) ds\right) > \frac{1}{4}, \tag{8.1.13}$$

then equation (8.1.1) is oscillatory.

Proof. Let x be a nonoscillatory solution of equation (8.1.1), say, $x(t) > 0$ for $t \geq m$ for some $m \in \mathbb{N}_0$. As in the proof of Theorem 8.1.1, we obtain (8.1.7) and so, for $k \geq m+2$,

$$\begin{aligned}\left(\frac{a(k)}{a(k-1)}\right)\left(\frac{a(k-1)}{a(k-2)}\right) \exp\left(\int_{k-1}^{k} p(s)ds\right) + \int_{k-1}^{k} q(s) \exp\left(\int_{k-1}^{s} p(u)du\right) ds \\ = \frac{a(k-1)}{a(k-2)}.\end{aligned} \tag{8.1.14}$$

Set $w(k) = a(k-1)/a(k-2)$. Then $w(k) > 0$ for $k \geq m+2$. There are two cases to consider.

Case 1. $\liminf_{k\to\infty} w(k) = \rho < \infty$. From (8.1.14) we get

$$\begin{aligned}&\left(\liminf_{k\to\infty} w(k+1)\right)\left(\liminf_{k\to\infty} w(k)\right)\left(\liminf_{k\to\infty} \exp\left(\int_{k-1}^{k} p(u)du\right)\right) \\ &\qquad + \liminf_{k\to\infty} \int_{k-1}^{k} q(s) \exp\left(\int_{k-1}^{s} p(u)du\right) ds \\ &\quad \leq \liminf_{k\to\infty} w(k).\end{aligned} \tag{8.1.15}$$

Set

$$
\begin{aligned}
X &= \liminf_{k\to\infty} \exp\left(\int_{k-1}^{k} p(u)du\right), \\
Y &= \liminf_{k\to\infty} \int_{k-1}^{k} q(s)\exp\left(\int_{k-1}^{s} p(u)du\right)ds.
\end{aligned}
\tag{8.1.16}
$$

Then (8.1.15) implies that $X\rho^2 + Y \le \rho$ and by completing the square, we obtain

$$
X\left[\left(\rho - \frac{1}{2X}\right)^2 + \frac{4XY-1}{4X^2}\right] \le 0. \tag{8.1.17}
$$

Hence $XY \le 1/4$, which contradicts condition (8.1.13).

Case 2. $\lim_{k\to\infty} w(k) = \infty$. Then (8.1.7) implies that

$$
0 < \exp\left(\int_{k-1}^{k} p(u)du\right) < \frac{1}{w(k+1)}, \tag{8.1.18}
$$

and so

$$
\lim_{k\to\infty} \exp\left(\int_{k-1}^{k} p(u)du\right) = 0, \tag{8.1.19}
$$

which contradicts condition (8.1.12).

This completes the proof. □

We note that condition (8.1.13) is the "best possible" in the sense that when $p(t)$ and $q(t)$ are equal to the constants p and q, respectively, condition (8.1.13) reduces to

$$
q > p - \frac{e^{-p}}{4(e^p - 1)}, \tag{8.1.20}
$$

which is a necessary and sufficient condition. However, in the case when $p(t)$ and $q(t)$ are not constants, condition (8.1.13) can be improved. To obtain such improved results we need the following lemmas.

Lemma 8.1.3. (i) *Let $a(-1)$ and $a(0)$ be given. Then the initial value problem (IVP) (8.1.1) and (8.1.2) has a unique solution x given on $[k, k+1)$, $k \in \mathbb{N}_0$, by*

$$
x(t) = a(k)\exp\left(-\int_{k}^{t} p(s)ds\right) - a(k-1)\int_{k}^{t} q(s)\exp\left(-\int_{s}^{t} p(u)du\right)ds, \tag{8.1.21}
$$

where the sequence $\{a(k)\}$ satisfies the difference equation

$$\begin{aligned} a(k-1) = a(k) \exp\Big(\int_{k-1}^{k} p(s)ds\Big) \\ + a(k-2)\int_{k-1}^{k} q(s)\exp\Big(\int_{k-1}^{s} p(u)du\Big)ds \quad \textit{for } k \in \mathbb{N}. \end{aligned} \tag{8.1.22}$$

(ii) *Equation (8.1.1) has a nonoscillatory solution if and only if the difference equation (8.1.22) has a nonoscillatory solution.*

Proof. We first show (i). Let x be a solution of IVP (8.1.1) and (8.1.2). Then in the interval $k \le t < k+1$ for $k \in \mathbb{N}_0$, equation (8.1.1) can be written in the form (8.1.4). As in the proof of Theorem 8.1.1, we get (8.1.6) which implies (8.1.21). From (8.1.21) and by continuity, letting $t \to k+1$ and replacing k by $k-1$, we obtain (8.1.22). Conversely, let $\{a(k)\}$ be the solution of (8.1.22) and define x on $(-1,0) \cup \mathbb{R}^+$ by (8.1.2) and (8.1.21). Then, clearly for every $k \in \mathbb{N}_0$ and $k \le t < k+1$, (8.1.21) implies (8.1.4) and, in turn, (8.1.4) is equivalent to equation (8.1.1) in the interval $k \le t < k+1$.

Now we address (ii). Assume that x is a nonoscillatory solution of equation (8.1.1). Then $\{a(k)\}$, $(a(k) = x(k))$ is a nonoscillatory solution of equation (8.1.22). Conversely, assume that $\{a(k)\}$ is a nonoscillatory solution of equation (8.1.22) such that $a(k) > 0$ eventually (the case when $x(k) < 0$ eventually is similar and is omitted). From (8.1.6), letting $t \to k+1$ and by continuity for k sufficiently large,

$$a(k+1)\exp\Big(\int_{k}^{k+1} p(u)du\Big) = a(k) - a(k-1)\int_{k}^{k+1} q(s)\exp\Big(\int_{k}^{s} p(u)du\Big)ds > 0. \tag{8.1.23}$$

Then, by (8.1.6) we obtain for $k \le t < k+1$ with k sufficiently large,

$$\begin{aligned} x(t)\exp\Big(\int_{k}^{t} p(u)du\Big) &= a(k) - a(k-1)\int_{k}^{t} q(s)\exp\Big(\int_{k}^{s} p(u)du\Big)ds \\ &\ge a(k) - a(k-1)\int_{k}^{k+1} q(s)\exp\Big(\int_{k}^{s} p(u)du\Big)ds \\ &> 0. \end{aligned} \tag{8.1.24}$$

This shows that $x(t) > 0$ eventually and so x is a nonoscillatory solution of equation (8.1.1). This completes the proof. □

In the following, for convenience, we let for any $k \in \mathbb{N}_0$,

$$P(k) = \exp\Big(\int_{k-1}^{k} p(s)ds\Big), \qquad Q(k) = \int_{k-1}^{k} q(s)\exp\Big(\int_{k-1}^{s} p(u)du\Big)ds. \tag{8.1.25}$$

Then

$$Q(k)P(k-1) = \int_{k-1}^{k} q(s)\exp\left(\int_{k-2}^{s} p(u)du\right)ds. \tag{8.1.26}$$

Observe that by (8.1.25) the difference equation (8.1.22) can be rewritten as

$$a(k-1) = P(k)a(k) + Q(k)a(k-2), \tag{8.1.27}$$

and by Lemma 8.1.3, if x is a solution of equation (8.1.1), then $a(k) = x(k)$ satisfies (8.1.27).

In the following, for convenience, we will assume that inequalities about values of functions or sequences are satisfied eventually for all large t or k.

Lemma 8.1.4. *Assume that there exists a constant $h \in [0, 1/4]$ such that*

$$Q(k)P(k-1) \geq h \quad \textit{for all large } k \in \mathbb{N}. \tag{8.1.28}$$

Let $\{a(k)\}$ be an eventually positive solution of equation (8.1.27). Set for $k \in \mathbb{N}$,

$$W(k,1) = \frac{a(k-1)}{a(k-2)}P(k-1), \qquad W(k,2) = \frac{a(k-2)}{a(k-1)}Q(k). \tag{8.1.29}$$

Then

$$\limsup_{k\to\infty} W(k,i) \leq \frac{1+\sqrt{1-4h}}{2} \quad \textit{for } i \in \{1,2\}. \tag{8.1.30}$$

Proof. We first prove (8.1.30) for $i = 1$. From (8.1.27) we have

$$a(k-2) \geq P(k-1)a(k-1). \tag{8.1.31}$$

This implies $\limsup_{k\to\infty} W(k,1) \leq 1$ and so (8.1.30) holds for $h = 0$. We now consider the case when $0 < h \leq 1/4$. From (8.1.31) it follows that

$$\frac{a(k-1)}{a(k-2)}P(k-1) \leq 1 = \lambda_1, \tag{8.1.32}$$

so

$$\frac{a(k-2)}{a(k-1)} \geq \frac{1}{\lambda_1}P(k-1). \tag{8.1.33}$$

Dividing both sides of (8.1.27) by $a(k-1)$, then using (8.1.33) and next (8.1.28), we obtain

$$1 \geq P(k)\frac{a(k)}{a(k-1)} + \frac{1}{\lambda_1}Q(k)P(k-1) \geq P(k)\frac{a(k)}{a(k-1)} + \frac{h}{\lambda_1}. \tag{8.1.34}$$

This yields

$$W(k,1) \leq \frac{\lambda_1 - h}{\lambda_1} = \lambda_2. \tag{8.1.35}$$

Following this iterative procedure, we have

$$W(k,1) \leq \frac{\lambda_n - h}{\lambda_n} = \lambda_{n+1} \quad \text{for } n \in \mathbb{N}. \tag{8.1.36}$$

It is not difficult to see that

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} > 0 \quad \text{for } n \in \mathbb{N}. \tag{8.1.37}$$

Hence the limit $\lim_{n\to\infty} \lambda_n = \lambda$ exists and satisfies $\lambda^2 - \lambda + h = 0$. Therefore we have

$$\limsup_{k\to\infty} W(k,1) \leq \frac{1+\sqrt{1-4h}}{2}. \tag{8.1.38}$$

This shows (8.1.30) for $i = 1$.

Next, we prove (8.1.30) for $i = 2$. From (8.1.27) we have

$$a(k-1) \geq Q(k)a(k-2), \tag{8.1.39}$$

$$a(k-2) = P(k-1)a(k-1) + Q(k-1)a(k-3). \tag{8.1.40}$$

Inequality (8.1.39) yields

$$W(k,2) = \frac{a(k-2)}{a(k-1)}Q(k) \leq 1 = \lambda_1. \tag{8.1.41}$$

Thus (8.1.30) holds for $h = 0$. In the case when $0 < h \leq 1/4$, from (8.1.40) and (8.1.41) we have

$$\begin{aligned} 1 &= P(k-1)\frac{a(k-1)}{a(k-2)} + Q(k-1)\frac{a(k-3)}{a(k-2)} \\ &\geq P(k-1)\frac{Q(k)}{\lambda_1} + Q(k-1)\frac{a(k-3)}{a(k-2)} \\ &\geq \frac{h}{\lambda_1} + Q(k-1)\frac{a(k-3)}{a(k-2)}. \end{aligned} \tag{8.1.42}$$

This leads to $W(k,2) \le (\lambda_1 - h)/\lambda_1 = \lambda_2$. Following this iterative procedure, we obtain

$$W(k,2) \le \frac{\lambda_n - h}{\lambda_n} = \lambda_{n+1} \quad \text{for } n \in \mathbb{N}. \tag{8.1.43}$$

Now the conclusion follows from the above inequalities and by the same arguments as in the case when $i = 1$. This completes the proof. ☐

Lemma 8.1.5. *Let $a(k)$ satisfy equation (8.1.27). Then the following equation holds for any integer $n \in \mathbb{N}_0$:*

$$\begin{aligned} a(k-2) = {} & Q(k-1)a(k-1) + a(k-n-4)\prod_{j=0}^{n+1} Q(k-j-1) \\ & + \sum_{i=0}^{n}\left(P(k-i-2)\prod_{j=0}^{i} Q(k-j-1)\right)a(k-i-2). \end{aligned} \tag{8.1.44}$$

Proof. From equation (8.1.27), it follows that

$$a(k-2) = P(k-1)a(k-1) + Q(k-1)a(k-3). \tag{8.1.45}$$

For any $i \in \mathbb{N}$, induction gives

$$a(k-i-1) = P(k-i)a(k-i) + Q(k-i)a(k-i-2). \tag{8.1.46}$$

Using (8.1.46) with $i = 2$ and next $i = 3$ in (8.1.45), we obtain

$$\begin{aligned} a(k-2) &= P(k-1)a(k-1) + Q(k-1)P(k-2)a(k-2) \\ &\quad + Q(k-1)Q(k-2)a(k-4) \\ &= P(k-1)a(k-1) + Q(k-1)P(k-2)a(k-2) \\ &\quad + Q(k-1)Q(k-2)P(k-3)a(k-3) \\ &\quad + Q(k-1)Q(k-2)Q(k-3)a(k-5) \\ &= P(k-1)a(k-1) + a(k-5)\prod_{j=0}^{2} Q(k-j-1) \\ &\quad + \sum_{i=0}^{1} P(k-i-2)\prod_{j=0}^{i} Q(k-j-1)a(k-i-2). \end{aligned} \tag{8.1.47}$$

By induction, it is not difficult to see that for $n \in \mathbb{N}$,

$$
\begin{aligned}
a(k-2) = P(k-1)a(k-1) + a(k-n-4)\prod_{j=0}^{n+1} Q(k-j-1) \\
+ \sum_{i=0}^{n} P(k-i-2)\prod_{j=0}^{i} Q(k-j-1)a(k-i-2).
\end{aligned}
\tag{8.1.48}
$$

This together with (8.1.45) and (8.1.46) imply that (8.1.44) holds for $j \in \mathbb{N}$. This completes the proof. □

From the proof of Lemma 8.1.3, we can also see that if all solutions of equation (8.1.22) are nonoscillatory, then all solutions of equation (8.1.1) are nonoscillatory. Since equation (8.1.22) is a second-order linear difference equation, by Section 1.2 (see also [126]), we see also that if one solution of equation (8.1.22) is nonoscillatory, then all its solutions are nonoscillatory. On the basis of this discussion and by Lemma 8.1.3 and a simple analysis, we see that the following result is true.

Theorem 8.1.6. *Equation (8.1.1) is nonoscillatory if and only if it has a nonoscillatory solution. This also implies that equation (8.1.1) is oscillatory if and only if it has an oscillatory solution.*

Now we present the following result.

Theorem 8.1.7. *Equation (8.1.1) is oscillatory if*

$$
\liminf_{k\to\infty} Q(k)P(k-1) > \frac{1}{4}, \tag{8.1.49}
$$

and nonoscillatory if

$$
Q(k)P(k-1) \le \frac{1}{4} \quad \text{for all large } k \in \mathbb{N}. \tag{8.1.50}
$$

Proof. First we prove that if (8.1.49) holds, then equation (8.1.1) is oscillatory. Suppose to the contrary that equation (8.1.1) has an eventually positive solution x. Then, by (8.1.25) and Lemma 8.1.3, $a(k) = x(k)$, $k \in \mathbb{N}$, satisfies equation (8.1.27). Let $R(k) = a(k-1)/[P(k)a(k)]$ for $k \in \mathbb{N}$. Then equation (8.1.27) reduces to

$$
1 = \frac{1}{R(k)} + Q(k)P(k-1)R(k-1), \tag{8.1.51}
$$

so

$$
Q(k+1)P(k)R(k) = \frac{Q(k+1)P(k)}{1-Q(k)P(k-1)R(k-1)}. \tag{8.1.52}
$$

Let

$$\alpha(k) = Q(k)P(k-1)R(k-1). \tag{8.1.53}$$

It is clear that $0 < \alpha(k) < 1$. This implies $\alpha(k)(1-\alpha(k)) \le 1/4$ because of $\max_{0\le y\le 1} y(1-y) = 1/4$. From (8.1.52) and (8.1.53), we have

$$\begin{aligned}
Q(k+1)P(k) &= \alpha(k+1)\big(1-\alpha(k)\big) \\
&= \frac{\alpha(k+1)}{\alpha(k)}\alpha(k)\big(1-\alpha(k)\big) \\
&\le \frac{1}{4}\left(\frac{\alpha(k+1)}{\alpha(k)}\right) \\
&= \frac{Q(k+1)P(k)R(k)}{4Q(k)P(k-1)R(k-1)} \\
&= \frac{1}{4Q(k)P(k-1)}\left(\frac{a(k-1)}{a(k)}Q(k+1)\right)\left(\frac{a(k-1)}{a(k-2)}P(k-1)\right).
\end{aligned} \tag{8.1.54}$$

By condition (8.1.49), there exits a number c such that $Q(k)P(k-1) \ge c > 1/4$ for $k \in \mathbb{N}$. Let the number h in Lemma 8.1.4 be $1/4$. Then, by Lemma 8.1.4, for any number $\varepsilon \in (0, 1/4)$ we have for all large k,

$$\frac{a(k-1)}{a(k)}Q(k+1) \le \frac{1}{2-\varepsilon}, \qquad \frac{a(k-1)}{a(k-2)}P(k-1) \le \frac{1}{2-\varepsilon}. \tag{8.1.55}$$

We choose such an ε so that $1/(2-\varepsilon)^2 < c$. Thus we obtain

$$Q(k+1)P(k) \le \frac{1}{4Q(k)P(k-1)}\frac{1}{(2-\varepsilon)^2} \le \frac{1}{4(2-\varepsilon)^2 c} < \frac{1}{4}. \tag{8.1.56}$$

This contradicts condition (8.1.49). Thus equation (8.1.1) is oscillatory.

Next we prove that (8.1.50) implies that equation (8.1.1) is nonoscillatory. To this end, we first show that the difference equation

$$x(k) = \frac{1}{1-b(k)x(k-1)} \quad \text{for } k \in \mathbb{N} \tag{8.1.57}$$

has an eventually positive solution $\{x(k)\}$, where

$$b(k) = Q(k)P(k-1) \quad \text{for } k \in \mathbb{N}. \tag{8.1.58}$$

By condition (8.1.50), without loss of generality, we may assume that

$$0 \le b(k) \le \frac{1}{4} \quad \text{for } k \in \mathbb{N}. \tag{8.1.59}$$

Set

$$\gamma = \begin{cases} \dfrac{1-\sqrt{1-4b(1)}}{2b(1)} & \text{if } b(1) > 0, \\ 1 & \text{if } b(1) = 0. \end{cases} \tag{8.1.60}$$

Then γ satisfies

$$\gamma = \frac{1}{1-b(1)\gamma}. \tag{8.1.61}$$

We claim that

$$1 \leq \gamma \leq 2. \tag{8.1.62}$$

Indeed, let

$$f(y) = \frac{1-\sqrt{1-4y}}{2y} \quad \text{for } 0 < y \leq \frac{1}{4}. \tag{8.1.63}$$

Then

$$f'(y) = \frac{1-2y-\sqrt{1-4y}}{2y^2\sqrt{1-4y}} \quad \text{for } 0 < y < \frac{1}{4}. \tag{8.1.64}$$

Set $F(y) = 1-2y-\sqrt{1-4y}$ for $0 \leq y \leq 1/4$. Then

$$F'(y) = 2\left(\frac{1}{\sqrt{1-4y}} - 1\right) > 0 \quad \text{for } 0 < y < \frac{1}{4}. \tag{8.1.65}$$

Thus F is strictly increasing on $(0, 1/4)$. Since $F(0) = 0$, it follows that $F(y) > 0$ for $0 < y \leq 1/4$. Therefore $f'(y) > 0$ for $0 < y < 1/4$. Notice that $f(1/4) = 2$ and

$$\lim_{y\to 0^+} f(x) = \lim_{y\to 0} \frac{1-\sqrt{1-4y}}{2y} = \lim_{y\to 0} \frac{1}{\sqrt{1-4y}} = 1, \tag{8.1.66}$$

and we have $1 < f(y) \leq 2$ for $0 < y \leq 1/4$. This and (8.1.60) leads to (8.1.62). Now, we define a sequence $\{x(k)\}$ by

$$x(0) = \gamma, \qquad x(k) = \frac{1}{1-b(k)x(k-1)} \quad \text{for } k \in \mathbb{N}. \tag{8.1.67}$$

It is clear that $1 \leq x(0) \leq 2$ and

$$1 \leq x(1) = \frac{1}{1-b(1)x(0)} = \frac{1}{1-b(1)\gamma} = \gamma \leq 2. \tag{8.1.68}$$

Thus, by (8.1.59), we have

$$1 \le x(2) = \frac{1}{1 - b(2)x(1)} \le \frac{4}{4-2} = 2. \tag{8.1.69}$$

By induction, we have $1 \le x(k) \le 2$ for $k \in \mathbb{N}$. This shows that equation (8.1.57) has a solution $\{x(k)\}$ such that $x(k) > 0$ for $k \in \mathbb{N}$ and $\{x(k)\}$ satisfies

$$x(k) = \frac{1}{1 - Q(k)P(k-1)x(k-1)} \quad \text{for } k \in \mathbb{N}. \tag{8.1.70}$$

Next, we define

$$a(-1) = 1, \qquad a(k) = \frac{a(k-1)}{P(k)x(k)} \quad \text{for } k \in \mathbb{N}_0. \tag{8.1.71}$$

Clearly $a(k) > 0$ for $k \in \mathbb{N}(-1)$. Substituting $x(k) = a(k-1)/[P(k)a(k)]$ into (8.1.70), we obtain

$$\frac{a(k-1)}{P(k)a(k)}\left(1 - Q(k)P(k)\frac{a(k-2)}{P(k-1)a(k-1)}\right) = 1, \tag{8.1.72}$$

that is,

$$a(k-1) = P(k)a(k) + Q(k)a(k-2) \quad \text{for } k \in \mathbb{N}. \tag{8.1.73}$$

This proves that $\{a(k)\}$ is a nonoscillatory solution of equation (8.1.27) (or (8.1.22)). By Lemma 8.1.3 and Theorem 8.1.6 we see that equation (8.1.1) is nonoscillatory. This completes the proof. □

Remark 8.1.8. By the inequality

$$\liminf(AB) \ge (\liminf A)(\liminf B), \quad \text{where } A, B \ge 0, \tag{8.1.74}$$

it is easy to see that condition (8.1.49) improves condition (8.1.13). It should be noted that condition (8.1.12) is no longer required in Theorem 8.1.7. On the other hand, both conditions (8.1.49) and (8.1.50) are "best possible" for the oscillation and nonoscillation, respectively. Because in the case when $p(t)$ and $q(t)$ are constants, condition (8.1.49) reduces to condition (8.1.20) and condition (8.1.50) reduces to

$$0 < q \le p\frac{e^{-p}}{4(e^{p}-1)}. \tag{8.1.75}$$

The following theorem improves condition (8.1.3).

Theorem 8.1.9. *Assume that for some $n \in \mathbb{N}_0$,*

$$\limsup_{k\to\infty}\left(Q(k)P(k-1)+\sum_{i=0}^{n}\prod_{j=0}^{i}Q(k-j-1)P(k-j-2)\right)>1. \tag{8.1.76}$$

Then equation (8.1.1) is oscillatory.

Proof. Assume that equation (8.1.1) has an eventually positive solution x. Then, by Lemma 8.1.3, $a(k)=x(k)$ satisfies (8.1.27). Thus $a(k-1)\geq P(k)a(k)$. By induction, the following iterative formula holds:

$$a(k-i)\geq a(k)\prod_{j=0}^{i-1}P(k-j)\quad \text{for } i\in\mathbb{N}. \tag{8.1.77}$$

By Lemma 8.1.5, $a(k)$ satisfies (8.1.44). Now, using $a(k-1)\geq Q(k)a(k-2)$ and (8.1.77) in (8.1.44), we obtain

$$a(k-2)\geq Q(k)P(k-1)a(k-2)+\sum_{i=0}^{j}\prod_{\ell=0}^{i}Q(k-\ell-1)P(k-\ell-2)a(k-2). \tag{8.1.78}$$

Dividing both sides of (8.1.78) by $a(k-2)$ and taking lim sup on both sides as $k\to\infty$, we find

$$1\geq\limsup_{k\to\infty}\left(Q(k)P(k-1)+\sum_{i=0}^{j}\prod_{\ell=0}^{i}Q(k-\ell-1)P(k-\ell-2)\right). \tag{8.1.79}$$

This contradicts condition (8.1.76) and completes the proof. □

In the following we present another (also "best possible") oscillation criteria for equation (8.1.1) in the case when conditions (8.1.49), (8.1.3), or even condition (8.1.76) are not satisfied.

The obtained results are formulated in terms of the numbers μ and M defined by

$$\mu=\liminf_{k\to\infty}Q(k)P(k-1),\qquad M=\limsup_{k\to\infty}Q(k)P(k-1). \tag{8.1.80}$$

Theorem 8.1.10. *Assume that $0\leq\mu\leq 1/4$ and that for some $n\in\mathbb{N}_0$,*

$$\limsup_{k\to\infty}\left(LQ(k)P(k-1)+\sum_{i=0}^{n}L^i\prod_{j=0}^{i}Q(k-j-1)P(k-j-2)\right)>1, \tag{8.1.81}$$

where

$$L = \left(\frac{1+\sqrt{1-4\mu}}{2}\right)^{-1}. \tag{8.1.82}$$

Then equation (8.1.1) is oscillatory.

Proof. By Theorem 8.1.9, the conclusion holds when $\mu = 0$. To prove the conclusion when $0 < \mu \leq 1/4$, suppose to the contrary that equation (8.1.1) has an eventually positive solution x. Then by Lemma 8.1.3, $a(k) = x(k)$ satisfies equation (8.1.27). Since for any $\eta \in (0,\mu)$ we have $Q(k)P(k-1) \geq \mu - \eta$ for all large k, by Lemma 8.1.4 we have

$$\begin{aligned} &\limsup_{k\to\infty} \frac{a(k-1)}{a(k-2)}P(k-1) \leq \frac{1+\sqrt{1-4(\mu-\eta)}}{2}, \\ &\limsup_{k\to\infty} \frac{a(k-2)}{a(k-1)}Q(k) \leq \frac{1+\sqrt{1-4(\mu-\eta)}}{2}. \end{aligned} \tag{8.1.83}$$

Letting $\eta \to 0$, we see that the two inequalities (8.1.83) hold for $\eta = 0$. Thus, for any sufficiently small $\varepsilon > 0$, the following inequalities hold for k sufficiently large:

$$a(k-1) \geq L(\varepsilon)P(k)a(k), \tag{8.1.84}$$

$$a(k-1) \geq L(\varepsilon)Q(k)a(k-2), \tag{8.1.85}$$

where

$$L(\varepsilon) = \left(\frac{1+\sqrt{1-4\mu}}{2} + \varepsilon\right)^{-1}. \tag{8.1.86}$$

From (8.1.84), by induction, we have the iterative formula

$$a(k-i) \geq L^i(\varepsilon)\prod_{j=0}^{i-1} P(k-j)a(k) \quad \text{for } i \in \mathbb{N}. \tag{8.1.87}$$

By Lemma 8.1.5, $a(k)$ satisfies (8.1.44). Now, using (8.1.85) and (8.1.87) in (8.1.44), we have

$$\begin{aligned} a(k-2) \geq{}& L(\varepsilon)Q(k)P(k-1)a(k-2) \\ &+ a(k-2)\sum_{i=0}^{n} P(k-i-2)L^i(\varepsilon)\prod_{j=0}^{i-1} P(k-j-2)\prod_{j=0}^{i} Q(k-j-1) \\ ={}& L(\varepsilon)Q(k)P(k-1)a(k-2) \\ &+ a(k-2)\sum_{i=0}^{n} L^i(\varepsilon)\prod_{j=0}^{i} Q(k-j-1)P(k-j-2). \end{aligned} \tag{8.1.88}$$

Dividing both sides of (8.1.88) by $a(k-2)$ and taking lim sup as $k \to \infty$, we have

$$1 \geq \limsup_{k\to\infty}\left(L(\varepsilon)Q(k)P(k-1)+\sum_{i=0}^{n}L^{i}(\varepsilon)\prod_{j=0}^{i}Q(k-j-1)P(k-j-2)\right). \tag{8.1.89}$$

Letting $\varepsilon \to 0$, we have $L(\varepsilon) \to L$ so that (8.1.81) and (8.1.89) lead to the contradiction

$$1 \geq \limsup_{k\to\infty}\left(LQ(k)P(k-1)+\sum_{i=0}^{n}L^{i}\prod_{j=0}^{i}Q(k-j-1)P(k-j-2)\right) > 1. \tag{8.1.90}$$

This completes the proof. □

Corollary 8.1.11. *Assume that* $0 \leq \mu \leq 1$ *and*

$$M > \left(\frac{1+\sqrt{1-4\mu}}{2}\right)^{2}. \tag{8.1.91}$$

Then equation (8.1.1) is oscillatory.

Proof. By (8.1.3), the conclusion holds when $\mu = 0$. Let $0 < \mu \leq 1/4$. It suffices to prove that (8.1.91) implies (8.1.81). Indeed, notice

$$\frac{1+\sqrt{1-4\mu}}{2} = 1-\frac{\mu}{1-L\mu}, \tag{8.1.92}$$

and by condition (8.1.91) there exists $\varepsilon \in (0,\mu)$ such that $Q(k)P(k-1) \geq \mu-\varepsilon$ for all large $k \in \mathbb{N}$ and

$$L\limsup_{k\to\infty} Q(k)P(k-1) > 1-\frac{\mu-\varepsilon}{1-L(\mu-\varepsilon)}. \tag{8.1.93}$$

From this and the fact that $[L(\mu-\varepsilon)]^{m} \to 0$ as $m \to \infty$, we have

$$\begin{aligned} L\limsup_{k\to\infty} Q(k)P(k-1) &> 1-\frac{(\mu-\varepsilon)\left(1-\left[L(\mu-\varepsilon)\right]^{n+1}\right)}{1-L(\mu-\varepsilon)} \\ &= 1-\left[(\mu-\varepsilon)+L(\mu-\varepsilon)^{2}+\cdots+L^{n}(\mu-\varepsilon)^{n+1}\right], \end{aligned} \tag{8.1.94}$$

where $n \in \mathbb{N}$ is sufficiently large integer. Inequality (8.1.94) leads to condition (8.1.81) because

$$\sum_{i=0}^{n} L^i \prod_{j=0}^{i} Q(k-j-1)P(k-j-2) \geq (\mu-\varepsilon) + L(\mu-\varepsilon)^2 + \cdots + L^n(\mu-\varepsilon)^{n+1}. \tag{8.1.95}$$

This completes the proof. □

Remark 8.1.12. Observe that $0 \leq \mu \leq 1/4$ implies $L \geq 1$ and that $L = 1$ if and only if $\mu = 0$. Also note that when $\mu \to 0$, condition (8.1.81) reduces to condition (8.1.76). However, it is clear that (8.1.81) improves (8.1.76) when $0 < \mu \leq 1/4$. It is interesting to observe that when $\mu \to 1/4$, (8.1.91) reduces to $M > 1/4$, which cannot be improved in the sense that the lower bound 1/4 cannot be replaced by a smaller number (cf. condition (8.1.50)).

The following is an illustrative example.

Example 8.1.13. Consider the equation

$$x'(t) + \frac{1}{2+t}x(t) + b\left(1+\cos\frac{\pi}{2}t\right)x([t-1]) = 0 \quad \text{for } t \geq 0, \tag{8.1.96}$$

where $b = \pi/(5(\pi-2))$. It is not difficult to see that

$$\begin{aligned}
&\int_{k-1}^{k} q(t)\exp\left(\int_{k-2}^{t} p(s)ds\right)dt \\
&\quad = b\int_{k-1}^{k}\left(1+\cos\frac{\pi}{2}t\right)\left(\frac{t+2}{k}\right)dt \\
&\quad = \frac{b}{k}\left\{\frac{2k+3}{2} + \frac{2}{\pi}\left[(2+k)\sin\frac{k\pi}{2} - (1+k)\sin\frac{(k-1)\pi}{2}\right]\right\} \\
&\qquad + \frac{4b}{k\pi^2}\left(\cos\frac{k\pi}{2} - \cos\frac{(k-1)\pi}{2}\right),
\end{aligned} \tag{8.1.97}$$

and so

$$\begin{aligned}
\mu &= \liminf_{k\to\infty}\int_{k-1}^{k} q(t)\exp\left(\int_{k-2}^{t} p(s)ds\right)dt = \frac{b(\pi-2)}{\pi} = \frac{1}{5} < \frac{1}{4}, \\
M &= \limsup_{k\to\infty}\int_{k-1}^{k} q(t)\exp\left(\int_{k-2}^{t} p(s)ds\right)dt = \frac{b(\pi+2)}{\pi} = \frac{\pi+2}{5(\pi-2)} < 1.
\end{aligned} \tag{8.1.98}$$

Thus, none of the conditions (8.1.3), (8.1.4) and (8.1.49) is satisfied. However, it is easy to see that condition (8.1.91) is satisfied. Therefore, by Corollary 8.1.11, equation (8.1.96) is oscillatory.

8.1.2. First-order equations of alternately retarded and advanced type

Here we will consider differential equations with piecewise constant arguments of the form

$$x'(t) + p(t)x(t) + q(t)x\left(2\left[\frac{t+1}{2}\right]\right) = 0, \tag{8.1.99}$$

$$x'(t) + p(t)x(t) + q(t)x\left(\left[t+\frac{1}{2}\right]\right) = 0, \tag{8.1.100}$$

where $p, q \in C(\mathbb{R}_0^+, \mathbb{R})$ and $[\cdot]$ denotes the greatest integer function. The argument deviation of equation (8.1.99) is given by $\tau(t) = t - 2[(t+1)/2]$, while the argument deviation of equation (8.1.100) is given by $\sigma(t) = t - [t + 1/2]$. The arguments $\tau(t)$ and $\sigma(t)$ are piecewise linear periodic functions with period 2 and 1, respectively. More precisely, for every integer k,

$$\begin{aligned} \tau(k) &= t - 2k \quad \text{for } 2k-1 \le t < 2k+1, \\ \sigma(k) &= t - k \quad \text{for } k - \frac{1}{2} \le t < k + \frac{1}{2}. \end{aligned} \tag{8.1.101}$$

Also,

$$\begin{aligned} &-1 \le \tau(k) < 1 \quad \text{for } 2k-1 \le t < 2k+1, \\ &-\frac{1}{2} \le \sigma(t) < \frac{1}{2} \quad \text{for } k - \frac{1}{2} \le t < k + \frac{1}{2}. \end{aligned} \tag{8.1.102}$$

Therefore, in each interval $[2k-1, 2k+1)$, equation (8.1.99) is of alternately advanced and retarded type. It is of advanced type in $[2k-1, 2k]$ and of retarded type in $(2k, 2k+1)$. Similarly, in each interval $[k-(1/2), k+(1/2)]$, equation (8.1.100) is of alternately advanced and retarded type. It is of advanced type in $[k-(1/2), k)$ and of retarded type in $(k, k+(1/2))$.

By a solution of equation (8.1.99) (resp., equation (8.1.100)) we mean a function x which satisfies the following properties:

(i_1) x is continuous on $\mathbb{R}_0^+$,
(i_2) the derivative $x'(t)$ exists at each point $t \in \mathbb{R}_0^+$ with the possible exception of the point $t = 2k+1$ (resp., $t = k+1$) for $k \in \mathbb{N}$, where one-sided derivatives exist,
(i_3) equation (8.1.99) (resp., equation (8.1.100)) is satisfied on each interval of the form $[2k-1, 2k+1)$ (resp., $[k-(1/2), k+(1/2))$) for $k \in \mathbb{N}$.

With each of these two equations, we associate an initial condition of the form

$$x(0) = a(0), \tag{8.1.103}$$

where $a(0)$ is a given real number. Let

$$u(\alpha, \beta) = \exp \int_\alpha^\beta p(s)\,ds \quad \text{for } \alpha, \beta \in \mathbb{R}. \tag{8.1.104}$$

Now we prove the following result.

Lemma 8.1.14. *Assume that*

$$u(2k-1,2k) \neq -\int_{2k-1}^{2k} q(s)u(2k-1,s)ds. \tag{8.1.105}$$

Then the IVP (8.1.99) and (8.1.103) has a unique solution x which is given by

$$x(t) = \frac{1}{u(2k,t)}\left(1-\int_{2k}^{t} q(s)u(2k,s)ds\right)a(2k) \quad \text{for } t \in [2k-1,2k+1) \tag{8.1.106}$$

and $k \in \mathbb{N}$, where the sequence $\{a(k)\}$ satisfies the difference equations

$$\begin{aligned} a(2k+1) &= \frac{1}{u(2k,2k+1)}\left(1-\int_{2k}^{2k+1} q(s)u(2k,s)ds\right)a(2k) \quad \text{for } k \in \mathbb{N}_0, \\ a(2k-1) &= u(2k-1,2k)\left(1+\int_{2k-1}^{2k} q(s)u(2k,s)ds\right)a(2k) \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{8.1.107}$$

Proof. Let x be a solution of the IVP (8.1.99) and (8.1.103). Then, in the interval $[2k-1,2k+1)$ and for every $k \in \mathbb{N}$, equation (8.1.99) becomes

$$x'(t) + p(t)x(t) + q(t)a(2k) = 0, \tag{8.1.108}$$

where we use the notation $x(2k) = a(2k)$ for $k \in \mathbb{N}$. Equation (8.1.108) can be rewritten as

$$\left(x(t)u(2k,t)\right)' + q(t)u(2k,t)a(2k) = 0 \quad \text{for } t \in [2k-1,2k+1),\ k \in \mathbb{N}. \tag{8.1.109}$$

Integrating equation (8.1.109) from $2k$ to $t \in [2k-1,2k+1)$, we have

$$x(t)u(2k,t) - a(2k) + \left(\int_{2k}^{t} q(s)u(2k,s)ds\right)a(2k) = 0. \tag{8.1.110}$$

This implies (8.1.106). From (8.1.106) and by continuity, letting $t \to 2k+1$ and $t \to 2k-1$, we obtain (8.1.107). This completes the proof. □

For a special case of equation (8.1.99), namely, the equation

$$x'(t) + px(t) + qx\left(2\left[\frac{t+1}{2}\right]\right) = 0, \tag{8.1.111}$$

where p and q are real numbers, Lemma 8.1.14 takes the following form.

Corollary 8.1.15. *Assume that*

$$p \neq 0, \qquad e^{p} + \frac{q}{p}\left(e^{p} - 1\right) \neq 0. \tag{8.1.112}$$

Then the IVP (8.1.111) and (8.1.103) has a unique solution x which is given by

$$x(t) = \left\{e^{-p(t-2k)} + \frac{q}{p}\left(e^{-p(t-2k)} - 1\right)\right\}a(2k) \quad \text{for } t \in [2k-1, 2k+1),\ k \in \mathbb{N}, \tag{8.1.113}$$

where the sequence $\{a(k)\}$ satisfies the difference equations

$$\begin{aligned} a(2k+1) &= \left\{e^{-p} + \frac{q}{p}\left(e^{-p} - 1\right)\right\}a(2k) \quad \text{for } k \in \mathbb{N}_0, \\ a(2k-1) &= \left\{e^{p} + \frac{q}{p}\left(e^{p} - 1\right)\right\}a(2k) \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{8.1.114}$$

When $p = 0$ in equation (8.1.111), that is, for the equation

$$x'(t) + qx\left(2\left[\frac{t+1}{2}\right]\right) = 0, \tag{8.1.115}$$

Corollary 8.1.15 takes the following form.

Corollary 8.1.16. *Assume that $q \neq -1$. Then the IVP (8.1.115) and (8.1.103) has a unique solution x which is given by*

$$x(t) = \{1 - q(t-2k)\}a(2k) \quad \text{for } t \in [2k-1, 2k+1),\ k \in \mathbb{N}, \tag{8.1.116}$$

where the sequence $\{a(k)\}$ satisfies the difference equations

$$\begin{aligned} a(2k+1) &= (1-q)a(2k) \quad \text{for } k \in \mathbb{N}_0, \\ a(2k-1) &= (1+q)a(2k) \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{8.1.117}$$

Proof. Let $p \to 0$ in Corollary 8.1.15. □

The following theorem provides a necessary and sufficient condition for the oscillation of equation (8.1.115).

Theorem 8.1.17. *Assume that $q \in \mathbb{R}$. Then every solution of equation (8.1.115) oscillates if and only if*

$$q \in (-\infty, -1) \cup [1, \infty). \tag{8.1.118}$$

Proof. Assume that (8.1.118) holds. Then either $q < -1$ or $q \geq 1$, and in either case it follows from (8.1.117) that the sequence $\{a(k)\}$ is oscillating. As $x(k) = a(k)$ for $k \in \mathbb{N}$, $x(t)$ also oscillates.

Conversely, assume that every solution x of equation (8.1.115) oscillates, and for the sake of contradiction assume that

$$|q| < 1. \tag{8.1.119}$$

Let x be the solution of (8.1.115) with $x(0) = a(0) = 1$. Then from (8.1.117) and because of (8.1.119), $a(k) > 0$ for $k \in \mathbb{N}_0$. Hence, for $t \in [2k-1, 2k+1)$ and $k \in \mathbb{N}$, $|2k - t| \leq 1$, so (8.1.116) yields

$$\begin{aligned} x(t) &= \{1 - q(t - 2k)\}a(2k) \\ &\geq \{1 - |q||t - 2k|\}a(2k) \\ &\geq (1 - |q|)a(2k) > 0. \end{aligned} \tag{8.1.120}$$

This contradicts the assumption that $x(t)$ oscillates, and therefore the proof is complete.

□

Next we present two sufficient conditions for the oscillation of equation (8.1.99).

Theorem 8.1.18. *If $q(t) > 0$ for $t \geq 0$ and*

$$\limsup_{k\to\infty} \int_{2k}^{2k+1} q(s)u(2k, s)ds > 1, \tag{8.1.121}$$

then equation (8.1.99) is oscillatory.

Proof. The proof is similar to the proof of Theorem 8.1.1 and hence is omitted. □

Theorem 8.1.19. *If $q(t) \geq 0$ for $t \geq 0$ and*

$$\liminf_{k\to\infty} \int_{2k-1}^{2k} q(s)u(2k, s)ds < -1, \tag{8.1.122}$$

then equation (8.1.99) is oscillatory.

Proof. Suppose that x is a solution of equation (8.1.99) such that $x(t) < 0$ for $t \geq 2k - 1$, where $k \in \mathbb{N}$ is sufficiently large. Integrating equation (8.1.109) from $2k - 1$ to $2k$ gives

$$x(2k)\left\{1 + \int_{2k-1}^{2k} q(s)u(2k, s)ds\right\} = x(2k-1)u(2k-1, 2k), \tag{8.1.123}$$

and since $x(t) < 0$ for $t \geq 2k-1$, we have

$$1 + \int_{2k-1}^{2k} q(s)u(2k,s)ds > 0, \tag{8.1.124}$$

so

$$\liminf_{k\to\infty} \int_{2k-1}^{2k} q(s)u(2k,s)ds \geq -1, \tag{8.1.125}$$

which contradicts condition (8.1.122). This completes the proof. □

By employing the same technique as presented above, one can easily obtain the following results for equation (8.1.100).

Lemma 8.1.20. *Assume that*

$$u\left(k-\frac{1}{2},k\right) \neq -\int_{k-(1/2)}^{k} q(s)u\left(k-\frac{1}{2},s\right)ds. \tag{8.1.126}$$

Then the IVP (8.1.100) and (8.1.103) has a unique solution x, which is given by

$$x(t) = \frac{1}{u(k,t)}\left(1-\int_k^t q(s)u(k,s)ds\right)a(k) \quad \textit{for } t \in \left[k-\frac{1}{2},k+\frac{1}{2}\right),\ k \in \mathbb{N}, \tag{8.1.127}$$

where the sequence $\{a(k)\}$ satisfies the difference equation

$$a(k+1) = \frac{1}{u(k-(1/2),k+(1/2))}\left(\frac{1-\int_k^{k+(1/2)} q(s)u(k,s)ds}{1+\int_{k-(1/2)}^{k} q(s)u(k,s)ds}\right) \quad \textit{for } k \in \mathbb{N}_0 \tag{8.1.128}$$

with $u(\alpha,\beta)$ given by (8.1.104).

For equation (8.1.100) with constant coefficients, that is, for the equation

$$x'(t) + px(t) + qx\left(\left[t+\frac{1}{2}\right]\right) = 0, \tag{8.1.129}$$

where $p, q \in \mathbb{R}$, Lemma 8.1.20 takes the following form.

Corollary 8.1.21. *Assume that*

$$e^{p/2} + \frac{q}{p}\left(e^{p/2}-1\right) \neq 0. \tag{8.1.130}$$

Then the IVP (8.1.129) and (8.1.103) has a unique solution x which is given by

$$x(t) = \left\{e^{-p(t-k)} + \frac{q}{p}\left(e^{-p(t-k)}-1\right)\right\}a(k) \quad \textit{for } t \in \left[k-\frac{1}{2},k+\frac{1}{2}\right),\ k \in \mathbb{N}, \tag{8.1.131}$$

where the sequence $\{a(k)\}$ *satisfies the difference equation*

$$a(k+1) = \frac{e^{-p/2} + (q/p)(e^{-p/2} - 1)}{e^{p/2} + (q/p)(e^{p/2} - 1)} a(k) \quad \text{for } k \in \mathbb{N}_0. \tag{8.1.132}$$

When $p = 0$ in equation (8.1.129), that is, for the equation

$$x'(t) + qx\left(\left[t + \frac{1}{2}\right]\right) = 0, \tag{8.1.133}$$

Corollary 8.1.21 can be restated as follows.

Corollary 8.1.22. *Assume that* $a(0), q \in \mathbb{R}$ *and* $q \neq -2$. *Then the IVP (8.1.133) and (8.1.103) has a unique solution* x *which is given by*

$$x(t) = \{1 - q(t-k)\}a(k) \quad \text{for } t \in \left[k - \frac{1}{2}, k + \frac{1}{2}\right), \ k \in \mathbb{N}, \tag{8.1.134}$$

where the sequence $\{a(k)\}$ *satisfies the difference equation*

$$a(k+1) = \frac{2-q}{2+q} a(k) \quad \text{for } k \in \mathbb{N}_0. \tag{8.1.135}$$

Similar to Theorem 8.1.17, we give the following result for equation (8.1.133).

Theorem 8.1.23. *Assume that* $q \in \mathbb{R}$. *Then every solution of equation (8.1.133) oscillates if and only if* $q \in (-\infty, 2) \cup [2, \infty)$.

Finally, we state the following result for equation (8.1.100) which is similar to Theorems 8.1.18 and 8.1.19.

Theorem 8.1.24. *If* $q(t) \geq 0$, $t \geq 0$, *and either*

$$\limsup_{k \to \infty} \int_k^{k+(1/2)} q(s)u(k,s)\,ds > 1 \tag{8.1.136}$$

or

$$\liminf_{k \to \infty} \int_{k-(1/2)}^{k} q(s)u(k,s)\,ds < -1, \tag{8.1.137}$$

then equation (8.1.100) is oscillatory.

8.1.3. Characteristic equations

Consider the equation with continuous and piecewise constant arguments

$$x'(t) + px(t-\tau) + qx([t-\sigma]) = 0, \tag{8.1.138}$$

where $p, q, \tau \in \mathbb{R}^+$, and $\sigma \in \mathbb{N}_0$.

When $q = 0$, equation (8.1.138) reduces to the delay equation

$$x'(t) + px(t-\tau) = 0, \tag{8.1.139}$$

whose characteristic equation is the well-known equation

$$\lambda + pe^{-\lambda\tau} = 0. \tag{8.1.140}$$

When $p = 0$, equation (8.1.138) reduces to the equation with piecewise constant argument

$$x'(t) + qx([t-\sigma]) = 0, \tag{8.1.141}$$

whose characteristic equation is

$$\lambda - 1 + q\lambda^{-\sigma} = 0. \tag{8.1.142}$$

Here we introduce the concept of *characteristic equation* for equation (8.1.138), namely the integral equation

$$\lambda(t) = p\exp\left(\int_{t-\tau}^{t}\lambda(s)ds\right) + q\exp\left(\int_{[t-\sigma]}^{t}\lambda(s)ds\right) \quad \text{for } t \geq 0. \tag{8.1.143}$$

We will show that equations (8.1.140) and (8.1.142) can be derived from equation (8.1.143) by looking for special solutions of equation (8.1.143). Also, we will prove a necessary and sufficient condition for oscillation of all solutions of equation (8.1.138) in terms of its characteristic equation (8.1.143).

By a solution of equation (8.1.138) we mean a function x which is defined on the set $\{-\sigma,\dots,-1,0\} \cup [-\tau,\infty)$ and which satisfies the following properties:

(i_1) x is continuous on $[-\tau,\infty)$,

(i_2) the derivative $x'(t)$ exists at each point $t \in [0,\infty)$ with the possible exception of the points $t \in \mathbb{N}$, where finite one-sided derivatives exist,

(i_3) equation (8.1.138) is satisfied on each interval $[k, k+1)$ for $k \in \mathbb{N}$.

Let $\phi \in C([-\tau,0],\mathbb{R})$ and $a(-\sigma),\dots,a(-1),a(0)$ be such that $a(-j) = \phi(-j)$ for $j \leq \tau$ and $j \in \{0,1,\dots,\sigma\}$. Then one can show that equation (8.1.138) has a unique solution x satisfying $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$ and $x(-j) = a(-j)$ for $j \in \{0,1,\dots,\sigma\}$.

By a solution of equation (8.1.143) we mean a function λ which is defined on $[-m,\infty)$, where $m = \max\{\tau,\sigma\}$, and which satisfies the following properties:

(I_1) λ is continuous on $[-m,\infty)$ with the possible exception of the points $t \in \{-\sigma,\dots,0,1,\dots\}$, where it has finite one-sided limits and where $\lambda(t) = \lambda(t+)$,

(I_2) λ satisfies equation (8.1.143) for $t \geq 0$.

The following theorem is the analogue of a well-known result in delay differential equations which states that "equation (8.1.139) has nonoscillatory solutions if and only if its characteristic equation (8.1.140) has a real root."

Theorem 8.1.25. *Assume that $p, q \in \mathbb{R}$, $\tau \in \mathbb{R}^+$, and $\sigma \in \mathbb{N}$. Then the following statements are equivalent:*

(a) *equation (8.1.138) has a nonoscillatory solution,*
(b) *equation (8.1.143) has a solution.*

Proof. We first show (a)⇒(b). Let x be a nonoscillatory solution of equation (8.1.138). Then there exists $t_0 \geq 0$ such that $x(t) \neq 0$ for $t \geq t_0$. As equation (8.1.138) is autonomous, $y(t) = x(t + t_0 + m)$, where $m = \max\{\sigma, \tau\}$, is also a solution of equation (8.1.138), and $y(t) \neq 0$ for $t \geq -m$. Set

$$\lambda(t) = -\frac{y'(t)}{y(t)} \quad \text{for } t \geq -m \tag{8.1.144}$$

with the convention that for $t \in \{-\sigma, \dots, 0, 1, \dots\}$, the derivative $y'(t)$ in (8.1.144) is the right-hand side derivative. It follows from (8.1.144) that for any $t_1 \geq -m$,

$$y(t) = y(t_1)\exp\left(-\int_{t_1}^{t} \lambda(s)ds\right) \quad \text{for } t \geq -m, \tag{8.1.145}$$

and in particular for $t \geq 0$

$$\begin{aligned} y(t-\tau) &= y(t)\exp\left(\int_{t-\tau}^{t} \lambda(s)\right)ds, \\ y([t-\sigma]) &= y(t)\exp\left(\int_{[t-\sigma]}^{t} \lambda(s)\right)ds. \end{aligned} \tag{8.1.146}$$

From this, (8.1.144), and the fact that y is a solution of equation (8.1.138), we see that

$$-\lambda(t)y(t) + py(t)\exp\left(\int_{t-\tau}^{t} \lambda(s)ds\right) + qy(t)\exp\left(\int_{[t-\sigma]}^{t} \lambda(s)ds\right) = 0 \quad \text{for } t \geq 0. \tag{8.1.147}$$

As $y(t) \neq 0$ for $t \geq 0$, this implies that λ satisfies equation (8.1.143).

Now we address (b)⇒(a). Assume that equation (8.1.143) has a solution λ. Then one can easily show that

$$x(t) = \exp\left(-\int_{-m}^{t} \lambda(s)ds\right) \quad \text{for } t \geq -m \tag{8.1.148}$$

is a nonoscillatory solution of (8.1.138). This completes the proof. □

Remark 8.1.26. In the special case when $q = 0$, equation (8.1.143) reduces to

$$\lambda(t) = p \exp\left(\int_{t-\tau}^{t} \lambda(s)ds\right). \tag{8.1.149}$$

By looking for constant solutions $\lambda(t) \equiv \lambda$ of equation (8.1.149), we see that the constant $-\lambda$ is a real root of equation (8.1.140). In fact, equation (8.1.149) admits a constant solution if and only if equation (8.1.140) has a real root.

Remark 8.1.27. In the special case when $p = 0$, equation (8.1.143) reduces to

$$\lambda(t) = q \exp\left(\int_{[t-\sigma]}^{t} \lambda(s)ds\right). \tag{8.1.150}$$

It is interesting to observe that by looking for a periodic solution of equation (8.1.150) of the form

$$\lambda(t) = \frac{1}{b-(t-k)} \quad \text{for } k \le t < k+1 \text{ with } b > 1, \tag{8.1.151}$$

we see that this is possible provided that

$$q\frac{b^{\sigma+1}}{(b-1)^{\sigma}} = 1. \tag{8.1.152}$$

It follows that when $b = 1/(1-\lambda(0))$ with $\lambda(0) \in (0,1)$, then $\lambda(0)$ is a real root of equation (8.1.142). In fact, we can show that equation (8.1.150) admits a solution of the form

$$\lambda(t) = \frac{1}{1/(1-\lambda(0))-(t-k)} \quad \text{with } \lambda(0) \in (0,1),\ k \le t < k+1, \tag{8.1.153}$$

if and only if $\lambda(0)$ is a root of equation (8.1.142) in the interval $(0,1)$.

8.2. Oscillation of linear second-order delay equations

In this section, we will be concerned with stability and oscillation properties of second-order delay differential equations of the form

$$x''(t) + \omega^2 x(t) = bx([t-1]), \tag{8.2.1}$$

where $b \in \mathbb{R} \setminus \{0\}$, $\omega \in \mathbb{R}$, and $[\cdot]$ signifies the greatest integer function.

Here, equation (8.2.1) can be described in brief by two properties. First, for certain values of the coefficients some or all its solutions are monotone although the corresponding homogeneous equation is clearly oscillatory. Second, for a specific relation between ω and b, there exist periodic solutions with different periods.

By a solution of equation (8.2.1) we mean a function x that satisfies the following conditions:

(i) x is continuously differentiable on $\mathbb{R}_0^+ = [0, \infty)$,
(ii) $x''(t)$ exists at each point $t \in \mathbb{R}_0^+$ with the possible exception of the points $[t] \in \mathbb{R}_0^+$, where it has one-sided limits,
(iii) equation (8.2.1) is satisfied on each interval $[k, k+1)$ with integral endpoints.

A typical equation with piecewise constant arguments contains arguments that are constant on certain intervals. Continuity of a solution at a point joining any two consecutive intervals leads to recursion relations for the solution at such points. Hence the solutions are determined by a finite set of initial data rather than by an initial function as in the case of general functional differential equations. Therefore, underlying each equation with piecewise constant arguments is a dynamical system governed by a difference equation of a discrete argument which describes its stability, oscillation, and periodic properties.

8.2.1. Existence of solutions

Now, we will study the existence of solutions of equation (8.2.1).

Theorem 8.2.1. *Equation (8.2.1) has a solution in $\mathbb{R}_0^+$.*

Proof. Denote by $x_k(t)$ the solution of (8.2.1) on the interval $k \leq t \leq k+1$ and let $x(k) = c(k)$ and $x(k-1) = c(k-1)$. Then

$$x_k''(t) + \omega^2 x_k(t) = bc(k-1), \tag{8.2.2}$$

whence

$$x_k(t) = A_k \cos \omega(t-k) + B_k \sin \omega(t-k) + \frac{b}{\omega^2} c(k-1) \tag{8.2.3}$$

with arbitrary constants A_k and B_k. Putting $t = k$ in (8.2.3) gives

$$c(k) = A_k + \frac{b}{\omega^2} c(k-1) \quad \text{or} \quad A_k = c(k) - \frac{b}{\omega^2} c(k-1). \tag{8.2.4}$$

Differentiating $x_k(t)$ at $t = k$ yields $B_k = d(k)/\omega$, where $d(k) = x_k'(k)$. Hence

$$x_k(t) = \left(c(k) - \frac{b}{\omega^2} c(k-1)\right) \cos \omega(t-k) + \frac{1}{\omega} d(k) \sin \omega(t-k) + \frac{b}{\omega^2} c(k-1), \tag{8.2.5}$$

$$x_k'(t) = -\omega\left(c(k) - \frac{b}{\omega^2} c(k-1)\right) \sin \omega(t-k) + d(k) \cos \omega(t-k). \tag{8.2.6}$$

At $t = k + 1$ it follows from (8.2.5) and (8.2.6) that

$$c(k+1) = c(k)\cos\omega + \frac{1}{\omega}d(k)\sin\omega + \frac{b}{\omega^2}(1-\cos\omega)c(k-1), \tag{8.2.7}$$

$$d(k+1) = -\omega c(k)\sin\omega + d(k)\cos\omega + \frac{b}{\omega}c(k-1)\sin\omega. \tag{8.2.8}$$

Now we introduce the vector $v(k) = \binom{c(k)}{d(k)}$ and the matrices

$$A = \begin{pmatrix} \cos\omega & \dfrac{\sin\omega}{\omega} \\ -\omega\sin\omega & \cos\omega \end{pmatrix}, \qquad B = \begin{pmatrix} b\dfrac{1-\cos\omega}{\omega^2} & 0 \\ b\dfrac{\sin\omega}{\omega} & 0 \end{pmatrix} \tag{8.2.9}$$

and write

$$v(k+1) = Av(k) + Bv(k-1). \tag{8.2.10}$$

We look for a nonzero solution of equation (8.2.10) in the form

$$v(k) = n\lambda^k \quad \text{with a constant vector } n \tag{8.2.11}$$

and conclude that λ satisfies the equation

$$\det\left(\lambda^2 I - \lambda A - B\right) = 0, \tag{8.2.12}$$

which is rewritten as

$$\det\begin{pmatrix} \lambda^2 - \lambda\cos\omega - b\left(\dfrac{1-\cos\omega}{\omega^2}\right) & -\lambda\dfrac{\sin\omega}{\omega} \\ \lambda\omega\sin\omega - b\dfrac{\sin\omega}{\omega} & \lambda^2 - \lambda\cos\omega \end{pmatrix} = 0 \tag{8.2.13}$$

or

$$\lambda^3 - 2\lambda^2\cos\omega + \left(1 - \frac{(1-\cos\omega)b}{\omega^2}\right)\lambda - \frac{b(1-\cos\omega)}{\omega^2} = 0, \tag{8.2.14}$$

that has three nontrivial solutions if

$$1 - \cos\omega \neq 0. \tag{8.2.15}$$

Assuming that these roots are simple, we write the general solution of equation (8.2.10) as

$$v(k) = n_1\lambda_1^k + n_2\lambda_2^k + n_3\lambda_3^k \tag{8.2.16}$$

with constant vectors n_j each of which depends on the corresponding value λ_j, $j \in \{1,2,3\}$, and contains one arbitrary scalar factor. These factors can be found from adequate initial or boundary conditions. If some λ_j is a multiple zero of equation (8.2.14), then the expression for $v(k)$ also includes products of λ^k by k or k^2. Finally, the solution $x_k(t)$ is obtained by substituting the appropriate components of the vectors $v(k)$ and $v(k-1)$ in (8.1.3). □

Remark 8.2.2. Note that we can eliminate $d(k)$ and $d(k+1)$ from equations (8.2.7) and (8.2.8) and derive the equation

$$c(k+2) - 2c(k+1)\cos\omega + \left(1 - \frac{(1-\cos\omega)b}{\omega^2}\right)c(k) - \frac{b(1-\cos\omega)}{\omega^2}c(k-1) = 0. \tag{8.2.17}$$

Theorem 8.2.3. *The three-point boundary value problem*

$$x(-1) = c(-1), \qquad x(0) = c(0), \qquad x(N-1) = c(N-1) \tag{8.2.18}$$

for equation (8.2.1) has a unique solution on $\mathbb{R}_0^+$ *if* $N \in \mathbb{N} \setminus \{1\}$ *and the following conditions are satisfied.*

(i$_1$) *The characteristic roots* λ_j, $j \in \{1,2,3\}$ *of (8.2.14) are nontrivial and distinct.*

(i$_2$) $(\lambda_2^N - \lambda_1^N)/(\lambda_2 - \lambda_1) \neq (\lambda_3^N - \lambda_1^N)/(\lambda_3 - \lambda_1)$.

(i$_3$) $\cos\omega \neq 1$.

Proof. Formula (8.2.16) furnishes for the components $c(k)$ of the vectors $v(k)$ the representation

$$c(k) = n_{11}\lambda_1^k + n_{21}\lambda_2^k + n_{31}\lambda_3^k \tag{8.2.19}$$

with arbitrary constants n_{ij}, $i \in \{1,2,3\}$, $j = 1$. If the values $c(-1)$, $c(0)$, and $c(N-1)$ are given, then the coefficients n_{ij} satisfy the system of equations

$$\begin{aligned} n_{11}\lambda_1^{-1} + n_{21}\lambda_2^{-1} + n_{31}\lambda_3^{-1} &= c(-1), \\ n_{11} + n_{21} + n_{31} &= c(0), \\ n_{11}\lambda_1^{N-1} + n_{21}\lambda_2^{N-1} + n_{31}\lambda_3^{N-1} &= c(N-1). \end{aligned} \tag{8.2.20}$$

By virtue of the hypothesis (i$_2$), the determinant

$$\det\begin{pmatrix} \lambda_1^{-1} & \lambda_2^{-1} & \lambda_3^{-1} \\ 1 & 1 & 1 \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \lambda_3^{N-1} \end{pmatrix} \neq 0. \tag{8.2.21}$$

Hence we can find the coefficients n_{ij} and the components $c(k)$ uniquely. Condition (i$_3$) merely restates (8.2.15) which ensures that the roots λ_j, $j \in \{1,2,3\}$, are not zero. Furthermore, once the values $c(k)$ have been found, we calculate the components $d(k)$ from equation (8.2.7) and then substitute both $c(k)$ and $d(k)$ in equation (8.2.5). For $N = 2$, hypothesis (i$_2$) is part of (i$_1$). □

The following results are immediate.

Theorem 8.2.4. *If the characteristic roots λ_j, $j \in \{1,2,3\}$, of equation (8.2.14) are nontrivial, $\lambda_1 = \lambda_3$, and*

$$\frac{\lambda_2^N - \lambda_1^N}{\lambda_2 - \lambda_1} \neq N\lambda_1^{N-1}, \tag{8.2.22}$$

then the boundary value problem (8.2.1) and (8.2.18) has a unique solution on $\mathbb{R}_0^+$.

Theorem 8.2.5. *If the roots of equation (8.2.14) are equal, that is, $\lambda_1 = \lambda_2 = \lambda_3$, then the boundary value problem (8.2.1) and (8.2.18) has a unique solution on $\mathbb{R}_0^+$.*

Remark 8.2.6. If $\omega = 2\pi j$ for $j \neq 0$, then the characteristic equation (8.2.14) has only two nonzero roots $\lambda_1 = \lambda_2 = 1$, and in this case a two-point boundary value problem is posed for equation (8.2.1).

Theorem 8.2.7. *If $\omega = 2\pi j$, where $j \neq 0$ is an integer, then the problem*

$$x(-1) = c(-1), \qquad x(0) = c(0) \tag{8.2.23}$$

for equation (8.2.1) has infinitely many solutions on $\mathbb{R}_0^+$, and each solution is a periodic function with period 1 for $1 \leq t < \infty$.

Proof. Equation (8.2.1) on the interval $0 \leq t < 1$ becomes

$$x_0''(t) + \omega^2 x_0(t) = bc(-1), \tag{8.2.24}$$

whence

$$x_0(t) = \left(c(0) - \frac{b}{\omega^2}c(-1)\right)\cos\omega t + \frac{d(0)}{\omega}\sin\omega t + \frac{b}{\omega^2}c(-1), \tag{8.2.25}$$

where $d(0) = x_0'(0)$. For $1 \leq t < 2$ we have the equation

$$x_1''(t) + \omega^2 x_1(t) = bc(0), \tag{8.2.26}$$

and find the solution

$$x_1(t) = \left(c(0) - \frac{b}{\omega^2}c(0)\right)\cos\omega(t-1) + \frac{d(0)}{\omega}\sin\omega(t-1) + \frac{b}{\omega^2}c(0) \tag{8.2.27}$$

satisfying the conditions $x_1(1) = x_0(1) = c(0)$ and $x_1'(1) = x_0'(1) = d(0)$. In general, for $\cos\omega = 1$ and $k \geq 0$, it follows from equations (8.2.7) and (8.2.8) that $c(k+1) = c(k)$ and $d(k+1) = d(k)$, and so $c(k) = c(0)$ and $d(k) = d(0)$ for $k \geq 0$.

Substituting these values in equation (8.2.5) yields the solution

$$x_k(t) = c(0)\left(1 - \frac{b}{\omega^2}\right)\cos\omega(t-k) + \frac{d(0)}{\omega}\sin\omega(t-k) + \frac{b}{\omega^2}c(0) \qquad (8.2.28)$$

on the interval $[k, k+1)$ with $k \geq 1$. This formula shows that the solution includes an arbitrary constant $d(0)$. □

Remark 8.2.8. For large ω, the term containing $d(0)$ is small and the dominant terms in the solution formula (8.2.28) depend on $c(0)$.

Theorem 8.2.9. *The boundary value problem (8.1.1) and*

$$x(-1) = c(-1), \qquad x(0) = c(0), \qquad x'(0) = d(0) \qquad (8.2.29)$$

has a unique solution on $\mathbb{R}_0^+$.

8.2.2. The case $\omega = 0$

Next, we will be concerned with oscillation and stability of equation

$$x''(t) = bx([t-1]) \quad \text{with } b \neq 0, \qquad (8.2.30)$$

that is, equation (8.2.1) with $\omega = 0$.

Letting $\omega \to 0$ in equation (8.2.14) yields the characteristic equation

$$\lambda^3 - 2\lambda^2 + \left(1 - \frac{b}{2}\right)\lambda - \frac{b}{2} = 0 \qquad (8.2.31)$$

for equation (8.2.30). It is interesting to consider problem (8.2.30) and (8.2.18). Note that formula (8.2.5) for the solution of equation (8.2.1) was derived with the implicit condition $\omega \neq 0$. Writing equation (8.2.5) as

$$\begin{aligned} x_k(t) &= c(k)\cos\omega(t-k) + \frac{b}{\omega^2}\big(1 - \cos\omega(t-k)\big)c(k) \\ &\quad + \frac{1}{\omega}d(k)\sin\omega(t-k), \end{aligned} \qquad (8.2.32)$$

and letting $\omega \to 0$ yields the solution

$$x_k(t) = \frac{b}{2}c(k-1)(t-k)^2 + d(k)(t-n) + c(k) \qquad (8.2.33)$$

of equation (8.2.30).

Theorem 8.2.10. *If $b < 0$, then every solution of equation (8.2.30) oscillates in $\mathbb{R}_0^+$ and is either unbounded or tends to zero as $t \to \infty$.*

Before we present the proof of Theorem 8.2.10, we state the following auxiliary result which will be employed in the proof.

Lemma 8.2.11. *All solutions of equation (8.2.1) oscillate if and only if the corresponding characteristic equation (8.2.14) has no positive roots.*

Proof of Theorem 8.2.10. Lemma 8.2.11 is true for equation (8.2.31) when $b < 0$ since it can be written as

$$F(\lambda) = \lambda(\lambda - 1)^2 - \frac{b}{2}(\lambda + 1) = 0. \tag{8.2.34}$$

The inequalities $F(-1) < 0$ and $F(0) > 0$ show that equation (8.2.34) has a root $\lambda_1 \in (-1, 0)$. Furthermore, the Descartes rule of signs confirms that λ_1 is the only real root of equation (8.2.31). Next, from the equation $\lambda_1 + \lambda_2 + \lambda_3 = 2$, we conclude that $\lambda_2 + \lambda_3 > 2$, $\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3 > 1$, and $|\lambda_2| = |\lambda_3| > 1$. With regard to equation (8.2.19), it means that $\lim_{k\to\infty} c(k) = \infty$ provided the boundary values $c(-1)$, $c(0)$, and $c(N-1)$ are chosen so that in the solution of system (8.2.20), the values n_{21} and n_{31} are not zero simultaneously. On the other hand, taking any $c(0) \neq 0$ and selecting $c(-1) = c(0)/\lambda_1$ and $c(N-1) = c(0)/\lambda_1^{N-1}$ implies that $n_{11} = c_0$ and $n_{21} = n_{31} = 0$. Since $|\lambda_1| < 1$, we have in this case $\lim_{t\to\infty} x_k(t) = 0$. This completes the proof. □

Theorem 8.2.12. *For*

$$0 < b < \frac{1}{4}\left(71 - (17)^{3/2}\right), \tag{8.2.35}$$

each solution of equation (8.2.30) is nonoscillatory in $\mathbb{R}_0^+$ and is either unbounded or tends to zero as $t \to \infty$.

Proof. Solving equation (8.2.34) with respect to the parameter b produces the function

$$b(\lambda) = \frac{2\lambda(\lambda - 1)^2}{\lambda + 1}, \tag{8.2.36}$$

with the derivatives

$$b'(\lambda) = \frac{2(\lambda - 1)(2\lambda^2 + 3\lambda - 1)}{(\lambda + 1)^2}, \tag{8.2.37}$$

whose zeros

$$\lambda_1' = \frac{-(3 + (17)^{1/2})}{4}, \qquad \lambda_2' = 1, \qquad \lambda_3' = \frac{-3 + (17)^{1/2}}{4} \tag{8.2.38}$$

yield two local minima and a maximum, respectively, of $b(\lambda)$. Calculations show that

$$b(\lambda_3') = \frac{71 - (17)^{3/2}}{4} \simeq 0.2268. \tag{8.2.39}$$

Each line b = constant in the upper half-plane (λ, b) satisfying inequalities (8.2.35) intersects the curve (8.2.36) at three points with two abscissas in the interval $(0, 1)$ and one in $\lambda > 1$. Hence, under condition (8.2.35), equation (8.2.31) has three positive roots λ_1, λ_2, and λ_3 such that $0 < \lambda_1, \lambda_2 < 1$, and $\lambda_3 > 1$ which proves that $c(k) = x_k(t)$ retains its sign for all large k.

Hence the solution (parabola) $x_k(t)$ given by (8.2.33) is nonoscillatory for all large k if it does not intersect the interval $(k, k+1)$ twice. Assuming the opposite, we must conclude that the derivatives $d(k) = x_k'(k)$ and $d(k+1) = x_k'(k+1)$ have different signs. On the other hand, $x_k'(t) = bc(k-1)(t-k) + d(k)$ at $t = k+1$ gives $d(k+1) - d(k) = bc(k-1)$, hence

$$d(k+1) = d(0) + \sum_{i=0}^{k} bc(i-1). \tag{8.2.40}$$

Since the sum on the right becomes monotone, starting with some k, it has a limit (finite or infinite) which implies that $d(k)$ preserves its sign for all large k. This proves that all solutions of equation (8.2.30) are nonoscillatory.

Depending on the boundary conditions it may happen that $n_{31} \neq 0$, and in this case the corresponding solutions of equation (8.2.30) are unbounded. On the contrary, the case $n_{31} = 0$ generates solutions that go to zero as $t \to \infty$. This completes the proof. □

Theorem 8.2.13. *For*

$$\frac{71 - (17)^{3/2}}{4} < b < 6, \tag{8.2.41}$$

each solution of equation (8.2.30) is either unbounded nonoscillatory, or oscillatory and approaching zero as $t \to \infty$.

Proof. For the values of b satisfying (8.2.41), equation (8.2.31) has only one real root $\lambda_1 > 1$ and two complex roots λ_2 and λ_3. Let $|\lambda_2| = |\lambda_3| = r$. From the equation $\lambda_1\lambda_2\lambda_3 = b/2$ we have $r^2 = b/(2\lambda)$, and therefore we consider the function

$$r^2(\lambda) = \frac{(\lambda - 1)^2}{\lambda + 1} \quad \text{for } \lambda > 0 \tag{8.2.42}$$

generated by equation (8.2.36). Over the interval $0 < \lambda < 3$, the graph of $r^2(\lambda)$ in the plane (λ, r^2) lies below the line $r^2 = 1$, with values $r^2(0^+) = 1$ and $r^2(3) = 1$ at the endpoints. For $\lambda > 3$, the graph grows to infinity as $\lambda \to \infty$ approaching the slant asymptotes $r^2 = \lambda - 3$. The line $b = b_0 = (71 - (17)^{3/2})/4$ in the plane (λ, b) meets the graph of $b(\lambda)$ at two points with abscissas $0 < \lambda_3' < 1$ and $1 < \lambda_0 < 2$. In the domain $\lambda > \lambda_0$, $b > b_0$, the function $b(\lambda)$ has an increasing inverse $\lambda(b)$ and, furthermore, $r(\lambda) < 1$ for $\lambda_0 < \lambda < 3$. Also, from equation (8.2.36), we see that $b(3) = 6$. Therefore, λ increases from λ_0 to 3 as b runs through the interval (8.2.41) and in this case $r < 1$. Under the appropriate boundary conditions, the characteristic root $\lambda_1 > 1$ implies the existence of monotone unbounded solutions to equation (8.2.30), whereas the complex roots λ_2 and λ_3 lead to oscillatory solutions. The latter tend to zero since λ_2 and λ_3 lie in the unit disk $|\lambda| < 1$. This completes the proof. □

Theorem 8.2.14. *For $b = 6$, each solution of equation (8.2.30) is either unbounded nonoscillatory, or periodic with period 3. The number 6 is the only value of the parameter b for which the equation (8.2.30) has periodic solutions.*

Proof. In the given case, equation (8.2.31) becomes $\lambda^3 - 2\lambda^2 - 2\lambda - 3 = 0$, which can be written as $(\lambda - 3)(\lambda^2 + \lambda + 1) = 0$. The root $\lambda_1 = 3$ is a source of unbounded nonoscillatory solutions for equation (8.2.30) while the roots λ_2 and λ_3 satisfy the relations $\lambda_2^3 = \lambda_3^3 = 1$ and generate, under appropriate boundary conditions, solutions of period 3 for equation (8.2.30). Conversely, assume that equation (8.2.30) has a periodic solution. Since $\lambda = 1$ and $\lambda = -1$ do not satisfy (8.2.31), there exist two complex roots λ_2 and λ_3 such that $|\lambda_2| = |\lambda_3| = 1$. From the equality $\lambda_1\lambda_2\lambda_3 = b/2$ it follows that $|\lambda_1| = |b|/2$. The value $\lambda = -b/2$ does not satisfy equation (8.2.31) and so $\lambda_1 = b/2$. Substituting this number into equation (8.2.31) gives the identity

$$\frac{b^3}{8} - \frac{b^2}{2} + \frac{b}{2} - \frac{b^2}{4} - \frac{b}{2} = 0, \tag{8.2.43}$$

thus $b = 6$. □

Theorem 8.2.15. *For $b > 6$, the solutions of equation (8.2.30) are unbounded. Depending on the boundary conditions (8.2.18), they are either oscillatory or nonoscillatory.*

Proof. For $b > 6$, equation (8.2.31) has a positive root $\lambda_1 > 3$ and two roots λ_2 and λ_3 which are either negative or complex. If λ_2 and λ_3 are complex, then from equation (8.2.42) it follows that $r^2(\lambda) > 1$, which implies the unboundedness of the solutions to equation (8.2.31). We have noted in the proof of Theorem 8.2.12 that the function $b(\lambda)$ defined by equation (8.2.36) attains a local minimum at $\lambda_1' = -(3 + (17)^{1/2})/4$. Some calculations show that

$$b(\lambda_1') = \frac{71 + (17)^{3/2}}{4} \simeq 35.2732. \tag{8.2.44}$$

Therefore λ_2 and λ_3 are complex for $6 < b < b(\lambda_1')$.

On the other hand, λ_2 and λ_3 are negative if $b > b(\lambda_1')$, and in this case $\lambda_2 < -1$ and $\lambda_3 < -1$ which shows that all solutions of equation (8.2.30) are unbounded. Furthermore, the positive root is a source of nonoscillatory solutions, and the negative or complex roots generate oscillatory solutions. This completes the proof. □

8.2.3. The case $\omega \neq 0$

Next we will modify the technique of the preceding part since the characteristic equation (8.2.14) contains two parameters b and ω.

For equation (8.2.1) with $\omega \neq 0$ we will present the following results.

Theorem 8.2.16. *For $b < 0$ all solutions of equation (8.2.1) oscillate in $\mathbb{R}_0^+$.*

Proof. The rule of signs confirms that the characteristic polynomial

$$P(\lambda) = \lambda^3 - 2\lambda^2 \cos\omega + \left(1 - \frac{b(1-\cos\omega)}{\omega^2}\right)\lambda - b\frac{(1-\cos\omega)}{\omega^2} \tag{8.2.45}$$

has no positive roots when $\cos\omega < 0$, and therefore all solutions of equation (8.2.1) oscillate. This conclusion remains valid if $\cos\omega > 0$ since the parabola

$$f(\lambda) = \lambda^2 - 2\lambda\cos\omega + 1 - \frac{b(1-\cos\omega)}{\omega^2} \tag{8.2.46}$$

attaining the positive minimum

$$f(\cos\omega) = (1-\cos\omega)\left(1+\cos\omega - \frac{b}{\omega^2}\right) \tag{8.2.47}$$

intersects the hyperbola $g(\lambda) = b(1-\cos\omega)/(\lambda\omega^2)$ at a single point, with a negative abscissa. This completes the proof. □

Theorem 8.2.17. *For $b < 0$ all solutions of equation (8.2.1) tend to zero as $t \to \infty$ if and only if*

$$\cos\omega < -\frac{1}{2}, \qquad \frac{b}{\omega^2} > \frac{1+2\cos\omega}{1-\cos\omega}. \tag{8.2.48}$$

Proof. For $b < 0$ the polynomial (8.2.45) has a zero $\lambda_1 \in (-1, 0)$ since

$$P(0) = -\frac{b(1-\cos\omega)}{\omega^2} > 0, \qquad P(-1) = -2 + 2\cos\omega < 0. \tag{8.2.49}$$

On the other hand we have

$$\begin{aligned} P(-1-\varepsilon) &= -(1+\varepsilon)^3 - 2(1+\varepsilon)^2\cos\omega - (1+\varepsilon) + \frac{b\varepsilon(1-\cos\omega)}{\omega^2} \\ &< -(1+\varepsilon)^3 + 2(1+\varepsilon)^2 - (1+\varepsilon) \\ &= -\varepsilon^2(1+\varepsilon) \\ &< 0 \end{aligned} \tag{8.2.50}$$

for any $\varepsilon > 0$ which implies that $P(\lambda)$ has no zero $\lambda_i < -1$, $i \in \{1,2,3\}$. Further, the zeros λ_i satisfy the relation

$$\lambda_1\lambda_2\lambda_3 = \frac{b(1-\cos\omega)}{\omega^2}, \tag{8.2.51}$$

and from equation (8.2.14) we find

$$\frac{b(1-\cos\omega)}{\omega^2} = \frac{\lambda(\lambda^2 - 2\lambda\cos\omega + 1)}{\lambda+1}. \tag{8.2.52}$$

Hence

$$\lambda_2\lambda_3 = \frac{\lambda_1^2 - 2\lambda_1\cos\omega + 1}{\lambda_1 + 1}. \tag{8.2.53}$$

The condition $|\lambda_i| < 1$, $i \in \{1,2,3\}$, is necessary and sufficient for all solutions of equation (8.2.1) to go to zero as $t \to \infty$, which is equivalent to the inequality

$$\frac{\lambda_1^2 - 2\lambda_1\cos\omega + 1}{\lambda_1 + 1} < 1. \tag{8.2.54}$$

From here, $\lambda_1 > 1 + 2\cos\omega$, and now we use the inequality

$$\frac{b(1-\cos\omega)}{\lambda_1\omega^2} < 1 \tag{8.2.55}$$

in order to obtain

$$\frac{b}{\omega^2} > \frac{\lambda_1}{1-\cos\omega} > \frac{1+2\cos\omega}{1-\cos\omega}. \tag{8.2.56}$$

This completes the proof. □

Theorem 8.2.18. *All solutions of equation (8.2.1) are unbounded as $t \to \infty$ if*

$$\frac{b(1-\cos\omega)}{\omega^2} > 1, \quad \cos\omega < 0. \tag{8.2.57}$$

Proof. Inequalities (8.2.57) imply the existence of a single positive root λ_1 of the polynomial (8.2.45). In addition, since

$$P(0) < 0, \qquad P(1) = 2(1-\cos\omega)\left(1 - \frac{b}{\omega^2}\right) < 0, \tag{8.2.58}$$

we have $\lambda_1 > 1$. The inequality $\lambda_2\lambda_3 \le 1$ is impossible, since in this case equation (8.2.53) gives $\lambda_1 \le 1 + 2\cos\omega$, which contradicts $\lambda_1 > 1$. In turn, the inequality $\lambda_2\lambda_3 > 1$ implies that $|\lambda_2| > 1$ and $|\lambda_3| > 1$ if λ_2 and λ_3 are complex. Finally, if λ_2 and λ_3 are negative, then we must conclude that $\lambda_2 < -1$ and $\lambda_3 < -1$. Indeed, from the inequalities $P(0) < 0$ and $P(-1) = -2 - 2\cos\omega < 0$, it follows that either $-1 < \lambda_2 < 0$ and $-1 < \lambda_3 < 0$, or $\lambda_2 < -1$ and $\lambda_3 < -1$. However, the first case should be dismissed since $\lambda_2\lambda_3 > 1$. The inequalities $|\lambda_i| > 1$ for $i \in \{1,2,3\}$ confirm that each solution of equation (8.2.1) is unbounded as $t \to \infty$. This completes the proof. □

Theorem 8.2.19. *For $b > 0$ all solutions of equation (8.2.1) tend to zero as $t \to \infty$ if and only if either*

$$\cos\omega > 0, \qquad b < \omega^2 \tag{8.2.59}$$

or

$$-\frac{1}{2} < \cos\omega < 0, \qquad \frac{b}{\omega^2} < \frac{1 + 2\cos\omega}{1 - \cos\omega}. \tag{8.2.60}$$

Proof. By virtue of the inequality $\cos\omega > 0$, the polynomial $P(\lambda)$ defined by (8.2.45) has either one or three positive roots. Since

$$P(0) < 0, \qquad P(1) = 2(1 - \cos\omega)\left(1 - \frac{b}{\omega^2}\right), \tag{8.2.61}$$

the condition $b < \omega^2$ implies the existence of a positive root $\lambda_1 < 1$. Condition (8.2.59) also shows that $P(\lambda)$ has no negative roots. From $P(0) < 0$ and $P(1) > 0$ it follows that $P(\lambda)$ has either one or three zeros in $(0, 1)$, and if the latter case holds true, the first part of the theorem is proved. On the other hand, the equation $\lambda_1 + \lambda_2 + \lambda_3 = 2\cos\omega$ indicates that the inequalities $\lambda_2 > 1$ and $\lambda_3 > 1$ cannot occur simultaneously, and therefore it remains to consider the case when λ_2 and λ_3 are complex. From equation (8.2.53), we conclude that the inequality $|\lambda_2\lambda_3| < 1$ takes place for $\lambda_1 < 1 + 2\cos\omega$; hence it is also valid for $\lambda_1 < 1$.

For $\cos\omega < 0$ and $0 < b/\omega^2 < 1$, the polynomial $P(\lambda)$ has a single positive zero $\lambda_1 \in (0, 1)$. Next, we have

$$P(-1 + \varepsilon) = -(1 - \varepsilon)\{(1 - \varepsilon)^2 + 2(1 - \varepsilon)\cos\omega + 1\} - \frac{b\varepsilon(1 - \cos\omega)}{\omega^2}, \tag{8.2.62}$$

and since $(1 - \varepsilon)^2 + 2(1 - \varepsilon)\cos\omega + 1 > 0$, we find $P(-1 + \varepsilon) < 1$ for $0 < \varepsilon < 1$. In other words, $P(\lambda)$ has no zero in $(-1, 0)$. The case $\lambda_2 < -1$ and $\lambda_3 < -1$ is impossible since the inequality $\lambda_2\lambda_3 > 1$ implies

$$\frac{\lambda_1^2 - 2\lambda_1\cos\omega + 1}{\lambda_1 + 1} > 1, \tag{8.2.63}$$

that is, $\lambda_1 > 1 + 2\cos\omega$. Hence it follows from $b(1 - \cos\omega)/(\lambda_1\omega^2) > 1$ that

$$\frac{b}{\omega^2} > \frac{1 + 2\cos\omega}{1 - \cos\omega}, \tag{8.2.64}$$

which contradicts inequality (8.2.60).

Finally, for $\lambda_2, \lambda_3 \in \mathbb{C}$, the assumption $|\lambda_2\lambda_3| \geq 1$ leads to $\lambda_1 \geq 1 + 2\cos\omega$. On the other hand, from the inequality

$$|\lambda_2\lambda_3| = \frac{b(1 - \cos\omega)}{\lambda_1\omega^2} \geq 1, \tag{8.2.65}$$

it follows that

$$\frac{b(1-\cos\omega)}{\omega^2} \geq \lambda_1 \geq 1 + 2\cos\omega, \tag{8.2.66}$$

which again contradicts inequality (8.2.60). This completes the proof. □

Next we will discuss the oscillatory behavior of equation (8.2.1) when $b > 0$. In this case the polynomial $P(\lambda)$ given by (8.2.45) may have either one or three positive roots. The inequalities $b > 0$ and $\cos\omega < 0$ guarantee the existence of a single positive root which implies that some solutions of equation (8.2.1) (generated by the negative or complex roots of $P(\lambda)$) oscillate. It is therefore interesting to find out whether there exist values of $b > 0$ and $\cos\omega > 0$ such that all solutions of equation (8.2.1) are nonoscillatory.

Theorem 8.2.20. *If* $0 < \omega < \pi/2$, $4/5 < \cos\omega < 1$, *and* $0 < b < \omega^2$, *then each solution of equation (8.2.1) is nonoscillatory.*

Proof. Subject to assumption (8.2.59), the polynomial $P(\lambda)$ given by equation (8.2.45) has a positive root $\lambda_1 < 1$ and no negative root. If $P(\lambda)$ has three positive roots, then all of them lie in $(0,1)$, and in this case the derivative $P'(\lambda)$ has two zeros in the same interval. The roots of the equation

$$P'(\lambda) = 3\lambda^2 - 4\lambda\cos\omega + \left(1 - \frac{b(1-\cos\omega)}{\omega^2}\right) = 0 \tag{8.2.67}$$

are real and distinct if

$$\frac{b(1-\cos\omega)}{\omega^2} > 1 - \frac{4}{3}\cos^2\omega, \tag{8.2.68}$$

and since $b < \omega^2$, we have necessarily $1-(4/3)\cos^2\omega < 1-\cos\omega$, thus $\cos\omega > 3/4$. Furthermore, the zeros of $P'(\lambda)$ lie in $(0,1)$ if $b < 4\omega^2$, a condition weaker than $b < \omega^2$. Let $B(\lambda) = b(1-\cos\omega)/\omega^2$. Then it follows from equation (8.2.14) that

$$B(\lambda) = \frac{\lambda(\lambda^2 - 2\lambda\cos\omega + 1)}{\lambda + 1}. \tag{8.2.69}$$

Next, we explore the curve (8.2.69) in the (λ, B) plane. Clearly

$$B'(\lambda) = \frac{Q(\lambda)}{(\lambda+1)^2} \tag{8.2.70}$$

with

$$Q(\lambda) = 2\lambda^3 + (3 - 2\cos\omega)\lambda^2 - 4\lambda\cos\omega + 1. \tag{8.2.71}$$

The derivative

$$Q'(\lambda) = 6\lambda^2 + 2(3 - 2\cos\omega)\lambda - 4\cos\omega \tag{8.2.72}$$

shows that $Q(\lambda)$ has a maximum at $\lambda = -1$ and a minimum at $\lambda = (2/3)\cos\omega$, with values $Q(-1) = 2(1+\cos\omega)$ and

$$Q\left(\frac{2}{3}\cos\omega\right) = -\frac{8}{27}\cos^3\omega - \frac{4}{3}\cos^2\omega + 1. \tag{8.2.73}$$

If $Q((2/3)\cos\omega) < 0$, that is,

$$\frac{8}{27}\cos^3\omega + \frac{4}{3}\cos^2\omega - 1 > 0, \tag{8.2.74}$$

then the graph of $Q(\lambda)$ intersects the interval $(0,1)$ twice since $Q(0) = 1 > 0$ and $Q(1) = 6(1-\cos\omega) > 0$. This implies that $B(\lambda)$ has a maximum $B_M = B(\lambda_M)$ and a minimum $B_m = B(\lambda_m)$, where $0 < \lambda_M < \lambda_m < 1$. Because of the condition $b < \omega^2$, any horizontal line $B(\lambda) = C$ such that $B_m < C < B_M$ crosses the graph $B(\lambda)$ at three points with abscissas in $(0,1)$. This means that equation (8.2.14) has three roots in the interval $(0,1)$. The substitution $u = (2/3)\cos\omega$ changes (8.2.74) to

$$u^3 + 3u^2 - 1 > 0. \tag{8.2.75}$$

Clearly equation (8.2.75) has a single positive root u_0, and therefore (8.2.75) holds true for $u > u_0$. We know already that $\cos\omega > 3/4$ and so $u_0 > 1/2$. At $u_1 = 1/2$ and $u_2 = 7/12$, we have $(1/8)+(3/4)-1 < 0$ and $(343/1728)+(49/48)-1 > 0$. In fact, $u_0 = 0.5321$, and we conclude that inequality (8.2.74) is valid for $0.8 < \cos\omega < 1$. Since all roots of $P(\lambda)$ are positive, the variable $x_k(t) = c(k)$ retains its sign for all large k. We want to show that for such k the integral curve (8.2.5) does not intersect the interval $[k, k+1]$. Assuming the opposite implies that $x_k(t)$ crosses $[k, k+1]$ an even number of times since $c(k)c(k+1) > 0$. Hence there exist points $t_1, t_2 \in [k, k+1]$ such that $x_k(t_1) = 0$ and $x_k(t_2) = 0$. Keeping in mind that $0 < t_i - k < 1$ for $i \in \{1, 2\}$ and $0 < \omega < \pi/2$, we find $0 < \omega(t_i - k) < \pi/2$, and turning to equation (8.2.5), we see that the equation

$$\left(c(k) - \frac{b}{\omega^2}c(k-1)\right)\cos\theta + \frac{1}{\omega}d(k)\sin\theta + \frac{b}{\omega^2}c(k-1) = 0 \tag{8.2.76}$$

must have at least two solutions $\theta_i = \omega(t_i - k)$ in $(0, \pi/2)$. This is impossible, which proves that each solution of equation (8.2.1) is nonoscillatory. □

Corollary 8.2.21. *With the hypotheses of Theorem 8.2.20 each solution of equation (8.2.1) tends to zero monotonically as* $t \to \infty$.

Theorem 8.2.22. *If* $0 < \omega < \pi/2$ *and*

$$1 < \frac{b}{\omega^2} < \frac{1}{1-\cos\omega}, \tag{8.2.77}$$

then each solution of equation (8.2.1) is either eventually monotone unbounded or oscillating and approaching zero.

Corollary 8.2.23. *Assuming that* $0 < \omega < \pi/2$, *each solution of equation (8.2.1) is oscillatory if and only if* $b < 0$.

Finally, we will discuss the existence of periodic solutions to equation (8.2.1). Note that the functions

$$\cos\omega(t-k) = \cos\omega(t-[t]), \qquad \sin\omega(t-k) = \sin\omega(t-[t]) \tag{8.2.78}$$

are periodic with period 1 since $0 \le t-[t] < 1$. Further, the coefficients $c(k)$ and $d(k)$ in the solution formula (8.2.5) are the components of vectors (8.2.16) which are represented as linear combinations of the powers λ_i^k of the characteristic roots λ_i, $i \in \{1,2,3\}$. Since the coefficients in (8.2.16) depend only on the boundary conditions (8.2.18), equation (8.2.1) has a periodic solution if and only if there exists an eigenvalue λ_j which is a root of unity. We rewrite equation (8.2.14) in the form

$$\lambda^3 - 2\lambda^2\cos\omega + (1-B)\lambda - B = 0, \tag{8.2.79}$$

where $B = b(1-\cos\omega)/\omega^2$, and assume that (8.2.79) has two complex zeros λ_2 and λ_3 which are roots of unity. From the equation $\lambda_1\lambda_2\lambda_3 = B$ for the roots of equation (8.2.79), it follows that $\lambda_1 = B$ since $\lambda_2\lambda_3 = 1$. In other words, the parameter B is also a root of equation (8.2.79), that is, $B^3 - 2B^2\cos\omega + (1-B)B - B = 0$ or $B^3 - (1+2\cos\omega)B^2 = 0$. Hence

$$B = 1 + 2\cos\omega, \tag{8.2.80}$$

$$b = \frac{(1+2\cos\omega)\omega^2}{1-\cos\omega}. \tag{8.2.81}$$

In this case equation (8.2.79) becomes

$$(\lambda - B)(\lambda^2 + \lambda + 1) = 0, \tag{8.2.82}$$

and since the zeros of the second factor in equation (8.2.82) are the complex roots of the equation $\lambda^3 = 1$, we arrive at the following conclusion.

Theorem 8.2.24. *Condition (8.2.79) is necessary and sufficient for the existence of periodic solutions with period* 3 *to equation (8.2.1).*

Note that the condition $b = 6$ for the existence of periodic solutions with period 3 to equation (8.2.30) follows from equation (8.2.81) as $\omega \to 0$. Furthermore, the only real eigenvalues that generate periodic solutions of equation (8.2.1) are $\lambda = 1$ or $\lambda = -1$. If $B = 1$, then $\cos\omega = 0$ and

$$\omega = (2j-1)\frac{\pi}{2}, \quad b = \omega^2,\ j \in \mathbb{Z}\setminus\{0\}. \tag{8.2.83}$$

If $B = -1$, then $\cos\omega = -1$ and

$$\omega = (2j-1)\pi, \quad b = -\frac{\omega^2}{2}, \quad j \in \mathbb{Z} \setminus \{0\}. \tag{8.2.84}$$

Now we state the following immediate results.

Theorem 8.2.25. *If hypotheses (8.2.83) hold true, then each solution of equation (8.2.1) is periodic and is either constant or has period* 3.

Theorem 8.2.26. *If hypotheses (8.2.84) hold true, then each solution of equation (8.2.1) is periodic with period* 6*. There also exist solutions with period* 2 *or* 3.

Theorem 8.2.27. *The condition $b = \omega^2$ is necessary and sufficient for the existence of constant solutions to equation (8.2.1).*

Theorem 8.2.28. *The condition $\omega = (2j-1)\pi$, $j \in \mathbb{Z}\setminus\{0\}$ is necessary and sufficient for the existence of periodic solutions with period* 2 *to equation (8.2.1).*

8.2.4. Second-order equations of alternately retarded and advanced type

In this subsection we will consider the second-order equation

$$x''(t) + bx\left(2\left[\frac{t+1}{2}\right]\right) = 0, \tag{8.2.85}$$

where b is a real number and $[\cdot]$ denotes the greatest integer function. As before, the argument deviation is

$$\tau(t) = t - 2\left[\frac{t+1}{2}\right] = t - 2k \quad \text{for } 2k-1 \le t < 2k+1 \tag{8.2.86}$$

for any $k \in \mathbb{Z}$. Thus equation (8.2.85) is of advanced type on $[2k-1, 2k)$ and of retarded type on $[2k, 2k+1)$ for every $k \in \mathbb{Z}$.

A solution of equation (8.2.85) on $\mathbb{R}$ is a function x that satisfies the following conditions:

(i) x is continuously differentiable on $\mathbb{R}$,

(ii) $x''(t)$ exists at each $t \in \mathbb{R}$ except possibly at the point $2k-1$ for $k \in \mathbb{Z}$, where one-sided second derivatives exist,

(iii) $x(t)$ satisfies equation (8.2.85) on every interval $[2k-1, 2k+1)$ for $k \in \mathbb{Z}$.

Throughout we will employ the notation $c(k) = x(k)$ and $d(k) = x'(k)$ for $k \in \mathbb{Z}$. When $k \le (t+1)/2 < k+1$ for $k \in \mathbb{Z}$, that is, $t \in [2k-1, 2k+1)$, equation (8.2.85) reduces to $x''(t) = -bc(2k)$. By integrating from $2k$ to $t \in [2k-1, 2k+1)$, we obtain

$$x'(t) = -bc(2k)(t-2k) + d(2k), \tag{8.2.87}$$

$$x(t) = \left(1 - \frac{1}{2}b(t-2k)^2\right)c(2k) + (t-2k)d(2k). \tag{8.2.88}$$

By taking limits in (8.2.88) as $t \to (2k-1)^+$ and (8.2.87) as $t \to (2k+1)^-$ and by using the continuity of x and x', we obtain

$$c(2k-1) = \frac{2-b}{2}c(2k) - d(2k), \tag{8.2.89}$$
$$d(2k-1) = bc(2k) + d(2k), \tag{8.2.90}$$
$$c(2k+1) = \frac{2-b}{2}c(2k) + d(2k), \tag{8.2.91}$$
$$d(2k+1) = -bc(2k) + d(2k). \tag{8.2.92}$$

Direct substitution of these equations leads to

$$c(2k+1) = (2-b)c(2k) - c(2k-1), \tag{8.2.93}$$
$$c(2k+2) = \frac{6-5b}{b+2}c(2k) - \frac{4}{b+2}c(2k-1) \quad \text{with } b \neq -2. \tag{8.2.94}$$

From this and (8.2.93), we see that for $b \neq -2$,

$$\begin{aligned}\begin{pmatrix} c(2k+2) \\ c(2k+1) \end{pmatrix} &= \begin{pmatrix} \dfrac{6-5b}{b+2} & \dfrac{-4}{b+2} \\ 2-b & -1 \end{pmatrix} \begin{pmatrix} c(2k) \\ c(2k-1) \end{pmatrix} \\ &= B\begin{pmatrix} c(2k) \\ c(2k-1) \end{pmatrix}.\end{aligned} \tag{8.2.95}$$

From this equation we find for any $k \in \mathbb{Z}$,

$$\begin{pmatrix} c(2k) \\ c(2k-1) \end{pmatrix} = B^k \begin{pmatrix} c(0) \\ c(-1) \end{pmatrix}. \tag{8.2.96}$$

Therefore it is evident that the stability and the oscillatory behavior of solutions of equation (8.2.85), when $b \neq -2$, depend on the eigenvalues and the eigenvectors of the matrix B. In fact, the eigenvalues of B are given by

$$\lambda_1 = \frac{(2-3b) + 2\sqrt{2b(b-2)}}{b+2}, \qquad \lambda_2 = \frac{(2-3b) - 2\sqrt{2b(b-2)}}{b+2}. \tag{8.2.97}$$

Clearly λ_1 and λ_2 are distinct unless $b = 0$ in which case $\lambda_1 = \lambda_2 = 1$, or $b = 2$ in which case $\lambda_1 = \lambda_2 = -1$.

Case 1. Assume that $b = 0$. By using the Jordan canonical form of B one can see that in this case

$$B^k = \begin{pmatrix} 2k+1 & -2k \\ 2k & -2k+1 \end{pmatrix}, \tag{8.2.98}$$

and so by (8.2.96)

$$c(2k) = 2k\big(c(0) - c(-1)\big),$$
$$c(2k-1) = 2k\big(c(0) - c(-1)\big) + c(-1). \tag{8.2.99}$$

Using (8.2.89) in (8.2.87) and (8.2.88), we have

$$x(t) = \left\{1 + \frac{2-b}{2}(t-2k) - \frac{b}{2}(t-2k)^2\right\}c(2k) - (t-2k)c(2k-1). \tag{8.2.100}$$

Substituting (8.2.99) into (8.2.100) and simplifying, we find

$$x(t) = \big(c(0) - c(-1)\big)t + c(0), \tag{8.2.101}$$

which is a straight line through the points with coordinates $(0, c(0))$ and $(-1, c(-1))$. Hence in this case, equation (8.2.85) is unstable and every nontrivial solution is nonoscillatory.

Case 2. Assume that $b = 2$. By using the Jordan canonical form of B one can see that in this case

$$B^k = (-1)^k \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \tag{8.2.102}$$

and so by (8.2.96),

$$c(2k) = (-1)^k\big(c(0) + kc(-1)\big), \qquad c(k-1) = (-1)^k c(-1). \tag{8.2.103}$$

From these equalities, it is obvious that when $b = 2$, the solutions of equation (8.2.85) oscillate and unless $c(-1) = 0$, the solutions are unbounded. This follows from
(8.2.100) since $|x(t)| \leq 2|c(2k)| + |c(2k-1)|$ and so, if $c(2k)$ and $c(2k-1)$ are bounded, then the solution is also bounded. Also, if $c(2k)$ is unbounded, then so is the solution $x(t)$.

Case 3. Assume that $b \notin \{-2, 0, 2\}$. In this case, the eigenvalues of B are distinct and $B = PDP^{-1}$, where

$$P = \begin{pmatrix} \dfrac{2a_1}{4-b^2} & \dfrac{2a_2}{4-b^2} \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{8.2.104}$$

and $a_1 = 2 - b + \sqrt{2b(b-2)}$ and $a_2 = 2 - b - \sqrt{2b(b-2)}$. Thus $B^k = PD^kP^{-1}$ for $k \in \mathbb{Z}$, and from (8.2.96) we obtain

$$c(2k) = \frac{1}{2\gamma}\{(a_1c(0) - 2c(-1))\lambda_1^k + (2c(-1) - a_2c(0))\lambda_2^k\}, \tag{8.2.105}$$

$$c(2k-1) = \frac{1}{2\gamma}\left\{\left(\frac{4-b^2}{2}c(0) - a_2c(-1)\right)\lambda_1^k + \left(a_1c(-1) - \frac{4-b^2}{2}c(0)\right)\lambda_2^k\right\}, \tag{8.2.106}$$

where $\gamma = \sqrt{2b(b-2)}$.

Now we are ready to present the following stability result.

Theorem 8.2.29. (a) *The trivial solution of equation (8.2.85) is not asymptotically stable.*

(b) *Every solution of equation (8.2.85) is bounded if and only if*

$$0 < b < 2. \tag{8.2.107}$$

Proof. (a) From (8.2.100) we see that

$$|x(t)| \le \left\{1 + \frac{|2-b|}{2} + \frac{|b|}{2}\right\}|c(2k)| + |c(2k-1)| \tag{8.2.108}$$

for every $k \in \mathbb{Z}$ and $t \in [2k-1, 2k+1)$. Hence the trivial solution of equation (8.2.85) is asymptotically stable if and only if

$$\lim_{k\to\infty} |c(k)| = 0. \tag{8.2.109}$$

From (8.2.105) and (8.2.106) we also see that (8.2.109) holds if and only if $|\lambda_i| < 1$ for $i \in \{1,2\}$. But $\det B = \lambda_1\lambda_2 = 1$, from which the proof of (a) follows.

(b) From (8.2.108) we see that every solution of (8.2.85) is bounded if and only if the sequence $\{c(k)\}$ is bounded. From (8.2.105) and (8.2.106) it is obvious that the solutions of (8.2.85) are all bounded if and only if $|\lambda_i| < 1$ for $i \in \{1,2\}$. When (8.2.107) holds, $|\lambda_i| = 1$ for $i \in \{1,2\}$, while for $b \in (-\infty, 0) \cup (2,\infty)$ either $|\lambda_1| > 1$ or $|\lambda_2| > 1$. Also, for $b = 0$ or $b = 2$, we saw that the solutions of equation (8.2.85) are unbounded.

This completes the proof. □

The next result deals with the oscillatory behavior of equation (8.2.85).

Theorem 8.2.30. *Assume that $b \neq -2$. Then the following statements are equivalent:*

(a) $b \in (-\infty, -2) \cup (0,\infty)$,

(b) *every solution of equation (8.2.85) oscillates.*

Proof. We first show (a)⇒(b). If (a) holds, then the roots λ_1 and λ_2 are either real and negative, or complex conjugate. Hence, in view of (8.2.105) and (8.2.106), each of the sequences $\{c(2k)\}$ and $\{c(2k-1)\}$ oscillates. Clearly, if $\{c(k)\}$ oscillates, $x(t)$ also oscillates.

Now we address (b)⇒(a). Assume for the sake of contradiction $b \in (-2, 0]$. If $b = 0$, then (8.2.101) implies that there exists a nonoscillatory solution. Hence $b \in (-2, 0)$. But then for $2k - 1 \le t < 2k + 1$, the coefficient of $c(2k)$ in (8.2.100) is nonnegative and

$$0 < \lambda_2 < 1 < \lambda_1. \tag{8.2.110}$$

Choose $c(0)$ and $c(-1)$ in (8.2.105) and (8.2.106) so that

$$a_1 c(0) - 2c(-1) > \frac{4 - b^2}{2} c(0) - a_2 c(-1) > 0. \tag{8.2.111}$$

Then for $k \in \mathbb{N}$ sufficiently large, $c(2k) > c(2k - 1) > 0$, and so (8.2.100) yields

$$x(t) > \left\{1 - \frac{b}{2}(t - 2k) - \frac{b}{2}(t - 2k)^2\right\} c(2k - 1) > 0 \quad \text{for } b \in (-2, 0). \tag{8.2.112}$$

This contradicts the hypothesis that every solution of equation (8.2.85) oscillates. This completes the proof. □

8.2.5. Neutral differential equations with piecewise constant argument

In this subsection we will study the second-order neutral delay differential equations with piecewise constant arguments of the form

$$\frac{d^2}{dt^2}\left(x(t) + px(t - 1)\right) + qx\left(2\left[\frac{t + 1}{2}\right]\right) = 0, \tag{8.2.113}$$

where $p, q \in \mathbb{R}$, $t \in [-1, \infty)$, and $[\cdot]$ denotes the greatest integer function.

A function $x : [-1, \infty) \to \mathbb{R}$ is a solution of equation (8.2.113) if the following conditions are satisfied:

(i) x is continuous on $[-1, \infty)$,
(ii) $(d/dt)(x(t) + px(t - 1)) = g(t)$ exists on $\mathbb{R}_0^+$ and g is continuous on $\mathbb{R}_0^+$,
(iii) $(d^2/dt^2)(x(t) + px(t - 1))$ exists on $\mathbb{R}_0^+$ except possibly at the points $2k - 1$, $k \in \mathbb{N}$, where one-sided second derivatives exist,
(iv) x satisfies (8.2.113) on $[0, 1)$ and on each interval $[2k - 1, 2k + 1)$ for $k \in \mathbb{N}$.

Next we present the following result.

Theorem 8.2.31. *Let $q \neq -2$, $x_0 : [-1,0] \to \mathbb{R}$ be a continuous function, and $c(0), c(1) \in \mathbb{R}$. Then if $p \neq 0$ (resp., $p = 0$), equation (8.2.113) has a unique solution x which satisfies*

$$x(t) = x_0(t), \quad t \in [-1,0] \quad (\text{resp., } x(0) = a_0), \qquad x(1) = c(1). \tag{8.2.114}$$

Moreover, for $t = 2k - 1 + \theta$, $k \in \mathbb{N}_0$, $0 \leq \theta \leq 1$, the function x is given by

$$\begin{aligned} x(t) = {} & (-p)^{2k}\Big(x_0(\theta-1) + \Big(\frac{q}{2}(\theta^2-\theta) - \theta\Big)c(0) + (\theta-1)c(-1)\Big) \\ & + (1-\theta)c(2k+1) + \Big(\theta + \frac{q}{2}(\theta-\theta^2)\Big)c(2k) \\ & + \frac{q}{2}(\theta^2-\theta)(p-1)\sum_{j=0}^{k-1}(-p)^{2k-2j-1}c(2j), \end{aligned} \tag{8.2.115}$$

and if $t = 2k + \theta$, $k \in \mathbb{N}_0$, $0 \leq \theta \leq 1$,

$$\begin{aligned} x(t) = {} & (-1)^{2k+1}\Big(x_0(\theta-1) + \Big(\frac{q}{2}(\theta^2-\theta) - \theta\Big)c(0) + (\theta-1)c(-1)\Big) \\ & + \Big(1 - \theta + \frac{q}{2}(\theta^2-\theta)(p-1)\Big)c(2k) + \theta c(2k+1) \\ & + \frac{q}{2}(\theta^2-\theta)(p-1)\sum_{j=0}^{k-1}(-p)^{2k-2j}c(2j), \end{aligned} \tag{8.2.116}$$

where $c(-1) = x_0(-1)$, $c(0) = x_0(0)$, and $c(2k+1)$, $c(2k)$, $c(2k-1)$ are given by the difference equation

$$\begin{pmatrix} c(2k+3) \\ c(2k+2) \\ c(2k+1) \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c(2k+1) \\ c(2k) \\ c(2k-1) \end{pmatrix}, \tag{8.2.117}$$

where

$$\begin{gathered} b_{11} = \frac{2p^2 - 4p + 5q + 4pq + 6}{2+q}, \qquad b_{12} = \frac{8p - 4p^2 + q^2 - 4pq - 4}{2+q}, \\ b_{13} = \frac{2p(p+q-2)}{2+q}, \\ b_{21} = \frac{2(2-p)}{2+q}, \qquad b_{22} = \frac{4p-q-2}{2+q}, \qquad b_{23} = \frac{-2p}{2+q}. \end{gathered} \tag{8.2.118}$$

Proof. Let x be a solution of equation (8.2.113) such that (8.2.114) is satisfied. For each $t \in [-1, \infty)$ there exists $k \in \mathbb{N}_0$ such that $k \leq (t+1)/2 < k+1$. Then, $2k - 1 \leq t < 2k + 1$ for $k \in \mathbb{N}_0$. Set

$$x(k) = c(k) \quad \text{for } k \in \mathbb{N}(-1). \tag{8.2.119}$$

Then from equation (8.2.113) and (8.2.119) it follows that

$$\frac{d^2}{dt^2}\big(x(t) + px(t-1)\big) = -qc(2k), \tag{8.2.120}$$

where $0 \leq t < 1$ for $k = 0$ or $2k - 1 \leq t < 2k + 1$ for $k \in \mathbb{N}$. If

$$\beta(k) = \frac{d}{dt}\big(x(t) + px(t-1)\big) \quad \text{at } t = k \in \mathbb{N}_0, \tag{8.2.121}$$

then by integrating equation (8.2.120) from $2k$ to t, where $t \in [0, 1)$ for $k = 0$ or $t \in [2k - 1, 2k + 1)$ for $k \in \mathbb{N}$ we have

$$\frac{d}{dt}\big(x(t) + px(t-1)\big) = \beta(2k) - q(t - 2k)c(2k). \tag{8.2.122}$$

Hence, by integrating (8.2.122) from $2k$ to t we have

$$\begin{aligned} x(t) + px(t-1) = (t - 2k)\beta(2k) + c(2k) + pc(2k - 1) \\ - \frac{q}{2}(t - 2k)^2 c(2k). \end{aligned} \tag{8.2.123}$$

From (8.2.123) and by the continuity of x on $[-1, \infty)$, letting $t \to 2k-1$, $t \to 2k+1$ in (8.2.123), and using (8.2.119) yields

$$c(2k-1) + pc(2k-2) = c(2k) + pc(2k-1) - \beta(2k) - \frac{q}{2}c(2k) \quad \text{for } k \in \mathbb{N}, \tag{8.2.124}$$

$$c(2k+1) + pc(2k) = c(2k) + pc(2k-1) + \beta(2k) - \frac{q}{2}c(2k) \quad \text{for } k \in \mathbb{N}_0. \tag{8.2.125}$$

By the continuity of the function g defined in (ii) and if we take the limits as $t \to 2k - 1$, $t \to 2k + 1$ in (8.2.122), then we get from (8.2.121), respectively,

$$\beta(2k-1) = \beta(2k) + qc(2k) \quad \text{for } k \in \mathbb{N}, \tag{8.2.126}$$

$$\beta(2k+1) = \beta(2k) - qc(2k) \quad \text{for } k \in \mathbb{N}_0. \tag{8.2.127}$$

Using (8.2.124)–(8.2.127) and performing some algebraic calculations, we can prove that $c(2k+1)$, $c(2k)$, $c(2k-1)$ are given by the difference equation (8.2.117).

Now we will prove that x satisfies (8.2.115) and (8.2.116). By applying [215, Lemma 3, page 463] to (8.2.123) and using (8.2.125), for $t = 2k-1+\theta$, $k \in \mathbb{N}_0$, $0 \le \theta \le 1$,

$$\begin{aligned} x(t) = (-1)^{2k}x_0(\theta-1) &+ \sum_{j=0}^{k-1}(-1)^{2k-2j-1}z(2j+\theta) \\ &+ \sum_{j=1}^{k}(-1)^{2k-2j}z(2j-1+\theta), \end{aligned} \tag{8.2.128}$$

and for $t = 2k+\theta$, $k \in \mathbb{N}_0$, $0 \le \theta \le 1$,

$$\begin{aligned} x(t) = (-p)^{2k+1}x_0(\theta-1) &+ \sum_{j=0}^{k}(-1)^{2k-2j}z(2j+\theta) \\ &+ \sum_{j=1}^{k}(-p)^{2k-2j+1}z(2j-1+\theta), \end{aligned} \tag{8.2.129}$$

where

$$\begin{aligned} z(2j+\theta) &= \left(1-\frac{q}{2}\theta^2+\theta\left(p-1+\frac{q}{2}\right)\right)c(2j) \\ &\quad + \theta c(2j+1) + p(1-\theta)c(2j-1), \\ z(2j-1+\theta) &= \left(1-\frac{q}{2}(\theta-1)^2+(\theta-1)\left(p-1+\frac{q}{2}\right)\right)c(2j) \\ &\quad + (\theta-1)c(2j+1) + p(2-\theta)c(2j-1). \end{aligned} \tag{8.2.130}$$

By setting

$$\begin{aligned} a_1(k) &= \sum_{j=0}^{k-1}(-p)^{2k-2j-1}c(2j+1), \\ a_2(k) &= \sum_{j=0}^{k-1}(-p)^{2k-2j-1}c(2j), \\ a_3(k) &= \sum_{j=0}^{k}(-p)^{2k-2j}c(2j-1), \end{aligned} \tag{8.2.131}$$

we can easily obtain

$$a_1(k) + pa_3(k) = p(-p)^{2k}c(-1). \tag{8.2.132}$$

Moreover, from (8.2.128) and (8.2.131) we obtain for $t = 2k - 1 + \theta$, $k \in \mathbb{N}_0$, $0 \leq \theta \leq 1$,

$$\begin{aligned} x(t) = (-p)^{2k} x_0(\theta - 1) &+ (\theta + p - \theta p) a_1(k) \\ &+ \big(\lambda(\theta) - p\lambda(\theta - 1)\big) a_2(k) + (2p - p\theta + \theta - 1) a_3(k) \\ &+ (\theta - 1) c(2k+1) + \lambda(\theta - 1) c(2k) + (1 - \theta) c(2k - 1) \\ &- (-p)^{2k} \big(\lambda(\theta - 1) c(0) + (\theta - 1) c(1) + p(2 - \theta) c(-1)\big), \end{aligned} \tag{8.2.133}$$

where λ is a function defined by

$$\lambda(\theta) = 1 - \frac{q}{2}\theta^2 + \theta\left(p - 1 + \frac{q}{2}\right). \tag{8.2.134}$$

Also, from (8.2.129) and (8.2.131) for $t = 2k + \theta$, $k \in \mathbb{N}_0$, $0 \leq \theta \leq 1$,

$$\begin{aligned} x(t) = (-p)^{2k+1} x_0(\theta - 1) &+ (\theta p^2 - p^2 - \theta p) a_1(k) \\ &+ \big(p^2\lambda(\theta - 1) - p\lambda(\theta)\big) a_2(k) + (p - p\theta - 2p^2 + \theta p^2) a_3(k) \\ &+ (p + \theta - p\theta) c(2k+1) + \big(\lambda(\theta) - p\lambda(\theta - 1)\big) c(2k) \\ &- (-p)^{2k+1} \big(\lambda(\theta - 1) c(0) + (\theta - 1) c(1) + p(2 - \theta) c(-1)\big). \end{aligned} \tag{8.2.135}$$

Setting $\theta = 0$ in (8.2.133) and using (8.2.119), we get

$$\begin{aligned} p a_1(k) &+ \big((p-1)^2 + pq\big) a_2(k) + (2p - 1) a_3(k) \\ &= c(2k+1) + (p + q - 2) c(2k) \\ &\quad + (-p)^{2k} \big((2 - p - q) c(0) - c(1) + (2p - 1) c(-1)\big). \end{aligned} \tag{8.2.136}$$

This and relation (8.2.132) imply for $p \neq 1$ that

$$\begin{aligned} a_1(k) &= p\eta c(2k+1) + p\mu c(2k) + (-p)^{2k+1}\big(\mu c(0) + \eta c(1)\big) - (\eta q p^2 + p) a_2(k), \\ a_3(k) &= -\eta c(2k+1) - \mu c(2k) + (-p)^{2k}\big(\mu c(0) + \eta c(1) + c(-1)\big) + (\eta q p + 1) a_2(k), \end{aligned} \tag{8.2.137}$$

where $\eta = 1/(p-1)^2$ and $\mu = (p + q - 2)/(p - 1)^2$. From this and (8.2.133) (resp., (8.2.135)), we can show that $x(t)$ satisfies (8.2.115) for $t = 2k - 1 + \theta$ (resp., (8.2.116) for $t = 2k + \theta$), $k \in \mathbb{N}_0$, $0 \leq \theta \leq 1$.

Suppose $p = 1$. By adding (8.2.124) and (8.2.125), we obtain

$$\begin{aligned} c(2k+1) - c(1) &= \sum_{j=1}^{k} \big(c(2j+1) - c(2j-1)\big) \\ &= \sum_{j=1}^{k} \big((1 - q) c(2j) - c(2j - 2)\big), \end{aligned} \tag{8.2.138}$$

which together with (8.2.131) implies

$$qa_2(k) = (1-q)c(0) - c(1) + c(2k+1) + (q-1)c(2k). \tag{8.2.139}$$

Then using (8.2.133) (resp., (8.2.135)), (8.2.132), and (8.2.139), one can easily see that $x(t)$ satisfies (8.2.115) for $t = 2k - 1 + \theta$ (resp., (8.2.116) for $t = 2k + \theta$), $k \in \mathbb{N}_0$, $0 \le \theta \le 1$ in the case $p = 1$. Therefore we have proved that if x is a solution of equation (8.2.113) which satisfies (8.2.114), then x is defined by (8.2.115) and (8.2.116).

Conversely, let x be a function which satisfies (8.2.115) and (8.2.116). One can show that x is a continuous function that satisfies equations (8.2.113) and (8.2.114). Therefore x is the unique solution of equation (8.2.113) which satisfies (8.2.114). This completes the proof. □

The following result is concerned with asymptotic stability of equation (8.2.113).

Theorem 8.2.32. *Let $q \neq -2$. Then equation (8.2.113) is asymptotically stable if and only if the following conditions holds:*

$$0 < p < 1, \qquad 0 < q < 2p^2 + 2. \tag{8.2.140}$$

Proof. Suppose first that equation (8.2.113) is asymptotically stable. Then it is obvious that the difference equation (8.2.117) is also asymptotically stable. It is easy to check that the characteristic equation associated to (8.2.117) is

$$y^3 + \gamma_1 y^2 + \gamma_2 y + \gamma_3 = 0, \tag{8.2.141}$$

where

$$\gamma_1 = \frac{6q - 2p^2 - 4pq - 4}{2+q}, \qquad \gamma_2 = \frac{4p^2 - 4pq + q + 2}{2+q}, \qquad \gamma_3 = \frac{-2p^2}{2+q}. \tag{8.2.142}$$

Since (8.2.117) is asymptotically stable, we have that every root of (8.2.141) is of modulus less than 1. Then from [215, Lemma 4, page 467], the following conditions are satisfied:

(i_1) $(q+2)q(1-p) > 0$,
(i_2) $(q+2)(2p^2 - q + 2) > 0$,
(i_3) $q(q+2) > 0$,
(i_4) $(q+2)(-2p^2 + 2pq + 2 - q) > 0$,
(i_5) $pq(-4p + q + 2 + 2p^2) > 0$.

From (i_1) and (i_3), we take $p < 1$ and $q > 0$ or $q < -2$.

If $q > 0$, from (i_2), we take $q < 2p^2 + 2$ and from (i_5), we see that $p > 0$ holds. Thus, (8.2.140) is satisfied.

If $q < -2$, from (i_2), we have $2p^2 + 2 < q$ which is a contradiction. Hence, we have proved that if (8.2.113) is asymptotically stable, then (8.2.140) is satisfied.

Conversely, suppose that (8.2.140) holds. Then we can easily see that all the conditions (i_1)–(i_5) hold. So from [215, Lemma 4, page 467], we have that equation (8.2.117) is asymptotically stable. Hence there exist constants $K > 0$ and $p \in (0,1)$ such that

$$|c(2k)| \leq Kp^k \quad \text{for } k \in \mathbb{N}_0. \tag{8.2.143}$$

From this, (8.2.115), and (8.2.116) we can see that equation (8.2.113) is asymptotically stable. This completes the proof. □

In the following result the oscillatory behavior of equation (8.2.113) is presented.

Theorem 8.2.33. *Let $q \neq -2$. Then every solution of equation (8.2.113) oscillates if one of the following conditions is satisfied:*

$$p < \frac{1}{4}, \qquad q < \min\left\{\frac{-4p^2-2}{1-4p}, -2\right\}, \tag{8.2.144}$$

$$p = 0, \qquad q > 0, \tag{8.2.145}$$

$$p = 1, \qquad q > 0. \tag{8.2.146}$$

Proof. As is known every solution of (8.2.117) oscillates if the associated characteristic equation (8.2.141) has no positive roots.

First, we can easily see that if condition (8.2.144) holds, then the coefficients γ_i for $i \in \{1,2,3\}$ of equation (8.2.141) are nonnegative. This implies that equation (8.2.141) has no positive roots, and so every solution of equation (8.2.117) oscillates. Then it is obvious that every solution of equation (8.2.113) oscillates if condition (8.2.144) holds.

Suppose that condition (8.2.145) holds. The proof is similar to that of Theorem 8.2.30 and hence is omitted.

Finally, let condition (8.2.146) hold. Then from (8.2.124)–(8.2.127), we get

$$c(2k+4) + \frac{3q-4}{2+q}c(2k+2) + \frac{2}{2+q}c(2k) = 0. \tag{8.2.147}$$

Using [215, Corollary 1], we see that every solution of (8.2.147) oscillates. Therefore, equation (8.2.113) is oscillatory.

This completes the proof. □

8.3. Systems of alternately retarded and advanced type

Here we will investigate the global asymptotic behavior as well as oscillation of solutions of the system of equations with piecewise constant argument

$$x'(t) + Ax(t) + Bx\big(g(t)\big) = f(t) \quad \text{for } t \in \mathbb{R}^+ \tag{8.3.1}$$

subject to the initial condition

$$x(0) = x_0, \tag{8.3.2}$$

where

(i) A and B are $r \times r$-matrices,
(ii) x is an r-vector,
(iii) f is a locally integrable function on $\mathbb{R}_0^+ = [0, \infty)$,
(iv) g is a piecewise constant function, $g(t) = kp$ for $t \in [kp-\ell, (k+1)p-\ell]$ for all $k \in \mathbb{Z}$, where p and ℓ are positive constants satisfying $p > \ell$.

Since the argument deviation for system (8.3.1), namely,

$$\tau(t) = t - g(t), \tag{8.3.3}$$

is negative in $[kp - \ell, kp)$ and positive in $(kp, (k+1)p - \ell)$, system (8.3.1) is said to be of alternately advanced and retarded type.

A function $x : \mathbb{R}_0^+ \to \mathbb{R}^r$ is a solution of (8.3.1) and (8.3.2) if the following conditions hold:

(i_1) x is continuous on $\mathbb{R}_0^+$,
(i_2) x is differentiable in $\mathbb{R}_0^+$ except possibly at the points $t = kp - \ell$, $k \in \mathbb{N}$, where one-sided derivatives exist,
(i_3) $x(0) = x_0$ and x satisfies (8.3.1) in $(0, p - \ell)$ and in every interval of the form $[kp - \ell, (k+1)p - \ell)$ for $k \in \mathbb{N}$.

A solution of (8.3.1) and (8.3.2) is called oscillatory if each of its components has no last zero.

Let $|\cdot|$ denote both any vector norm in $\mathbb{R}^r$ and its induced matrix norm.

The following lemma is needed.

Lemma 8.3.1. *Let $n, r \in \mathbb{N}$ and $P_1, \ldots, P_n$ be $r \times r$-matrices. Every solution of equation*

$$a(k+n) + P_1 a(k+n-1) + \cdots + P_n a(k) = 0 \quad \text{for } k \in \mathbb{N}_0 \tag{8.3.4}$$

oscillates if and only if its characteristic equation

$$\det\left(\lambda^n I + \lambda^{n-1} P_1 + \cdots + \lambda P_{n-1} + P_n\right) = 0 \tag{8.3.5}$$

has no positive roots.

8.3.1. The case $A = 0$

In this case, (8.3.1) becomes

$$x'(t) + Bx(g(t)) = f(t) \quad \text{for } t \in \mathbb{R}^+. \tag{8.3.6}$$

To simplify the notation, we define

$$\begin{aligned} N(t) &= I - Bt, \qquad N_0 = (I + B\ell)^{-1}(I - B(p-\ell)), \\ x(kp) &= x_k, \quad I_k = [kp - \ell, (k+1)p - \ell) \quad \text{for } k \in \mathbb{N}_0. \end{aligned} \tag{8.3.7}$$

In the following result we will provide a closed formula for the solution of (8.3.6) and (8.3.2).

Theorem 8.3.2. *Let $I + B\ell$ and $I - B(p-\ell)$ be nonsingular matrices and let f be locally integrable on $\mathbb{R}_0^+$. Then the system (8.3.6) and (8.3.2) has a unique solution on $\mathbb{R}_0^+$ given by*

$$x(t) = N(\tau(t))N_0^{g(t)/p}\left(x_0 + \sum_{j=1}^{g(t)/p} N_0^{-j}N(-\ell)\int_{(j-1)p}^{jp} f(s)ds\right) + \int_{g(t)}^{t} f(s)ds, \tag{8.3.8}$$

where $\tau(t)$ is defined in (8.3.3).

In addition, if f is integrable on $\mathbb{R}_0^- = (-\infty, 0]$, then this solution can be continued backwards on $\mathbb{R}_0^-$ and is given by

$$x(t) = N(\tau(t))N_0^{g(t)/p}\left(x_0 + \sum_{j=1}^{-g(t)/p} N_0^{-j}N(p-\ell)\int_{-(j-1)p}^{-jp} f(s)ds\right) + \int_{g(t)}^{t} f(s)ds. \tag{8.3.9}$$

Proof. In each interval of the type I_k, (8.3.6) becomes $x'(t) + Bx(kp) = f(t)$, which has a unique solution whenever a preassigned value for $x(kp)$ is given. The solution of (8.3.6) with $x(kp) = x_k$ is

$$x(t) = N(t - kp)x_k + \int_{kp}^{t} f(s)ds \quad \text{for } t \in I_k, \tag{8.3.10}$$

and with $x((k+1)p) = x_{k+1}$ is

$$x(t) = N(t - (k+1)p)x_{k+1} + \int_{(k+1)p}^{t} f(s)ds \quad \text{for } t \in I_{k+1}. \tag{8.3.11}$$

Continuity of the solution at $t = (k+1)p - \ell$ requires

$$N(-\ell)x_{k+1} + \int_{(k+1)p}^{(k+1)p-\ell} f(s)ds = N(p-\ell)x_k + \int_{kp}^{(k+1)p-\ell} f(s)ds \tag{8.3.12}$$

so that $x_{k+1} = N_0x_k + f_{k+1}$, where

$$f_{k+1} = N^{-1}(-\ell)\int_{kp}^{(k+1)p} f(s)ds, \tag{8.3.13}$$

from which it follows that

$$x_k = N_0^k\left(x_0 + \sum_{j=1}^{k} N_0^{-j} f_j\right). \tag{8.3.14}$$

Substituting this into (8.3.10) yields (8.3.8). The continuation of (8.3.8) on $\mathbb{R}_0^-$ is obtained in a similar way. □

Corollary 8.3.3. (a) *If $I + B\ell$ and $I - B(p - \ell)$ are nonsingular matrices, then the problem*

$$x'(t) + Bx\left(p\left[\frac{t+\ell}{p}\right]\right) = 0, \qquad x(0) = x_0 \tag{8.3.15}$$

has a unique solution on $\mathbb{R}$ which is given by

$$x(t) = N(\tau(t))N_0^{[(t+\ell)/p]}x_0, \tag{8.3.16}$$

where $\tau(t) = t - p[(t+\ell)/p]$.

(b) *Let $j, n \in \mathbb{N}$ with $j < k$. If $I + (j/n)B$ and $I - [(n-j)/j]B$ are nonsingular matrices, then the problem*

$$x'(t) + Bx\left(\left[t + \frac{j}{n}\right]\right) = 0, \qquad x(0) = x_0 \tag{8.3.17}$$

has a unique solution on $\mathbb{R}$ which is given by

$$x(t) = N(\tau(t))N_0^{[t+j/n]}x_0, \tag{8.3.18}$$

where $\tau(t) = t - [t + (j/n)]$.

In the following result we show that, under some restrictions on the matrix B and the function f, every solution of (8.3.1) tends to zero as $t \to \infty$.

Theorem 8.3.4. *Assume that every eigenvalue λ_j of N_0 satisfies*

$$|\lambda_j| < 1 \quad \textit{for } j \in \{1, 2, \dots, r\}. \tag{8.3.19}$$

(a) *If $f(t) \equiv 0$, then the trivial solution of (8.3.6) is globally asymptotically stable if and only if (8.3.19) holds.*
(b) *If $\lim_{t\to\infty} f(t) = 0$, then every solution of (8.3.6) tends to zero as $t \to \infty$.*

Proof. We show that every term in (8.3.8) tends to zero as $t \to \infty$. By writing N_0 in Jordan canonical form, we observe that N_0^k tends to the $r \times r$-zero matrix as $k \to \infty$ if and only if (8.3.19) holds.

Note that if $t \in I_k$, then

$$\left|N(\tau(t))N_0^{g(t)/p}x_0\right| \le N^* \left|N_0^k\right| |x_0|, \tag{8.3.20}$$

where $N^* = \max\{|N(u)| : u \in \{0, \max\{\ell, p - \ell\}\}\}$. Therefore (a) is proved.

To prove (b) we observe that the remaining terms in (8.3.8) also tend to zero as $t \to \infty$. For $t \in I_k$,

$$\left| \int_{g(t)}^{t} f(s)ds \right| < \max\{\ell, p-\ell\} \max\{|f(t)| : t \in I_k\}. \tag{8.3.21}$$

Similarly $F_j = \int_{(j-1)p}^{jp} f(s)ds \to 0$ as $j \to \infty$. Hence, given $\varepsilon > 0$ choose m such that

$$|F_j| < \begin{cases} h \text{ for some constant } h & \text{if } j < m, \\ \dfrac{\varepsilon(1-|N_0|)}{2N^*|N(-\ell)|} & \text{if } j \geq m, \end{cases} \tag{8.3.22}$$

and choose m_0 so that if $k > m_0$, then

$$|N_0|^k < \frac{\varepsilon(1-|N_0|)|N_0|^m}{\left(2hN^*|N(-\ell)|\left(1-|N_0|^m\right)\right)}. \tag{8.3.23}$$

If $k > \max\{m, m_0\}$, then

$$\begin{aligned} \left| N_0^{g(t)/p} \sum_{j=1}^{g(t)/p} N_0^{-j} F_j \right| &< \sum_{j=1}^{m} |N_0|^{k-j} |F_j| + \sum_{j=m+1}^{k} |N_0|^{k-j} |F_j| \\ &< \frac{\varepsilon}{2N^*|N(-\ell)|}. \end{aligned} \tag{8.3.24}$$

This completes the proof. □

Remark 8.3.5. Note that the eigenvalues of N_0 have the form

$$\frac{1-(p-\ell)\lambda}{1+\ell\lambda}, \tag{8.3.25}$$

where λ is an eigenvalue of B. Therefore, if ± 1 and ± 2 are not eigenvalues of B, then the trivial solution of each system

$$x'(t) + Bx\left(2\left[\frac{t+1}{2}\right]\right) = 0, \tag{8.3.26}$$

$$x'(t) + Bx\left[t + \frac{1}{2}\right] = 0, \tag{8.3.27}$$

which are special cases of (8.3.6), is asymptotically stable if and only if every eigenvalue of B has positive real part.

The next theorem states a necessary and sufficient condition for the oscillation of (8.3.6) when $f(t) = 0$.

Theorem 8.3.6. *Every solution of the problem*

$$x'(t) + Bx\big(g(t)\big) = 0, \qquad x(0) = x_0 \tag{8.3.28}$$

is oscillatory if and only if B has no eigenvalues in $(-1/\ell, 1/(p-\ell))$.

Proof. Let x solve (8.3.28). The continuity of $x(t)$ at $t = (k+1)p - \ell$ in (8.3.10) gives $x((k+1)p-\ell) = N(p-\ell)x_k$. Again, using (8.3.10) with $t = kp-\ell$ and the fact that $N(p-\ell)$ commutes with $N^{-1}(-\ell)$ yields $x((k+1)p-\ell) = N_0 x(kp-\ell)$. From Lemma 8.3.1 (with $n = 1$ and $P_1 = -N_0$), the sequence $\{x(kp-\ell)\}$ oscillates if and only if N_0 has no positive eigenvalues, and in the light of (8.3.25) this condition is equivalent to B having no eigenvalues in $(-1/\ell, 1/(p-\ell))$ (provided that $-1/\ell$ is not an eigenvalue of B). The oscillation of x is tied to the oscillation of the sequence $\{x(kp-\ell)\}$. Indeed, (8.3.10) with $f(t) \equiv 0$ implies that in each pair of intervals of the form $[kp-\ell, kp]$, $[kp, (k+1)p-\ell]$, $x(t)$ is composed of two line segments joining $x(kp-\ell)$ to $x(k_p)$ and $x(k_p)$ to $x((k+1)p-\ell)$. Since $x(t)$ is differentiable at $t = kp$, $x(t)$ is simply a line segment joining $x(kp-\ell)$ and $x((k+1)p-\ell)$ which passes through $x(kp)$. Therefore x oscillates if and only if $\{x(kp-\ell)\}$ oscillates. □

Corollary 8.3.7. *Every solution of system (8.3.17) is oscillatory if and only if B has no eigenvalues in $(-n/j, n/(n-j))$.*

Proof. Use Theorem 8.3.6 with $p = 1$ and $\ell = j/n$. □

Remark 8.3.8. By Theorem 8.3.6, every solution of system (8.3.26) oscillates if and only if B has no eigenvalues in $(-1,1)$, and every solution of system (8.3.27) oscillates if and only if B has no eigenvalues in $(-2,2)$.

8.3.2. The case A nonsingular

To simplify the notation, we define

$$\begin{gathered} M(t) = e^{-At} + (e^{-At} - I)A^{-1}B, \qquad M_0 = M^{-1}(-\ell)M(p-\ell), \\ x(kp) = x_0, \quad I_k = \big[kp-\ell, (k+1)p-\ell\big] \quad \text{for } k \in \mathbb{N}_0. \end{gathered} \tag{8.3.29}$$

Now we state some results for system (8.3.1).

Theorem 8.3.9. *Let A, $M(-\ell)$, and $M(p-\ell)$ be nonsingular matrices.*

(a) *If f is locally integrable on $\mathbb{R}_0^+$, then the system (8.3.1) and (8.3.2) has a unique solution on $\mathbb{R}_0^+$ given by*

$$x(t) = M\big(\tau(t)\big)M_0^{g(t)/p}\left(x_0 + \sum_{j=1}^{g(t)/p} M_0^{-j} f_j\right) + \int_{g(t)}^{t} e^{-A(t-s)} f(s)ds, \tag{8.3.30}$$

where $\tau(t)$ is defined in (8.3.3) and

$$f_j = M^{-1}(-\ell)\int_{(j-1)p}^{jp} e^{-A(t-s)}f(s)ds \quad \text{for } j \ge 1. \tag{8.3.31}$$

(b) *If f is locally integrable on $\mathbb{R}_0^-$, then the system (8.3.1) and (8.3.2) has a unique solution on $\mathbb{R}_0^-$ given by*

$$x(t) = M(\tau(t))M_0^{g(t)/p}\left(x_0 + \sum_{j=1}^{-g(t)/p} M_0^{-j} f_{j-1}\right) + \int_{g(t)}^{t} e^{-A(t-s)}f(s)ds, \tag{8.3.32}$$

where $\tau(t)$ is defined in (8.3.3) and

$$f_j = M^{-1}(p-\ell)\int_{-jp}^{-(j+1)p} e^{-A(t-s)}f(s)ds \quad \text{for } j \ge 0. \tag{8.3.33}$$

Theorem 8.3.10. *Assume that A, $M(-\ell)$, and $M(p-\ell)$ are nonsingular matrices and that every eigenvalue λ_j of M_0 satisfies (8.3.19).*

(a) *If $\lim_{t\to\infty} f(t) = 0$, then every solution of (8.3.1) tends to zero as $t \to \infty$.*
(b) *If $f(t) \equiv 0$, then the trivial solution of (8.3.1) is globally asymptotically stable if and only if condition (8.3.19) holds.*

Theorem 8.3.11. *Assume that A, $M(-\ell)$, and $M(p-\ell)$ are nonsingular matrices and consider a special case of (8.3.1), namely, the system*

$$x'(t) + Ax(t) + Bx(g(t)) = 0 \quad \text{for } t \in \mathbb{R}^+. \tag{8.3.34}$$

(a) *If every eigenvalue of M_0 is negative, then each solution of (8.3.34) is oscillatory and has a zero in each interval of the form $[kp,(k+1)p]$ for sufficiently large k.*
(b) *If M_0 has a negative eigenvalue, then there exists an initial vector x_0 so that the corresponding solution of (8.3.34) with $x(0) = x_0$ is oscillatory.*
(c) *If M_0 has no positive eigenvalues and no eigenvalues with equal moduli (unless they are complex conjugates of each other), then each solution of (8.1.28) is oscillatory.*

The proofs of Theorems 8.3.9–8.3.11 are left to the reader.

The following example illustrates the methods presented above.

Example 8.3.12. The differential equation

$$x''(t) + 2x'(t) + x(t) = \beta x'\left(2\left[\frac{t+1}{2}\right]\right) \tag{8.3.35}$$

with $x(0) = u_0$ and $x'(0) = v_0$ represents a damped spring-mass system subject to the external piecewise constant force $\beta x'(2[(t+1)/2])$, where $x(t)$ is the displacement of an object of mass 1 from the rest position of the system. The external force is discontinuous on $\mathbb{R}_0^+$ and may only vary its magnitude or magnitude and direction simultaneously at a time t of the form $2k+1$, where k is an integer. System (8.3.35) can be written in the form (8.3.34) with

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & -\beta \end{pmatrix}, \qquad x_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \qquad g(t) = 2\left[\frac{t+1}{2}\right]. \tag{8.3.36}$$

If $\beta \notin \{e, e^{-1}\}$, then Theorem 8.3.9 asserts that (8.3.35) subject to $x(0) = u_0$ and $x'(0) = v_0$ has a unique solution on $\mathbb{R}$. The solution of the associated system is given by

$$x(t) = M(t-2k)M_0^k x_0 \quad \text{for } t \in [2k-1, 2k+1], \tag{8.3.37}$$

where

$$\begin{gathered} M(t) = e^{-t}\begin{pmatrix} 1+t & t-\beta(1+t-e^t) \\ -t & 1-t-\beta t \end{pmatrix}, \\ M_0 = \frac{1}{e^2 - e\beta}\begin{pmatrix} 3-2\beta+e^{-1} & \beta^2(2-e-e^{-1})+\beta(2e-4)+2 \\ -2 & \beta(2-e)-1 \end{pmatrix}. \end{gathered} \tag{8.3.38}$$

The eigenvalues of M_0 are the solutions of

$$\lambda^2 + A_1\lambda + A_0 = 0, \tag{8.3.39}$$

where

$$A_1 = \frac{e\beta - 2 - \beta e^{-1}}{e(\beta - e)}, \qquad A_0 = \frac{e^{-1}\beta + e^{-2}}{e(\beta - e)}. \tag{8.3.40}$$

The solutions of (8.3.39) are lying inside the unit disk if and only if $A_0 < 1$ and $1 + A_0 > |A_1|$, which in this case translates to

$$\beta < e, \qquad \beta > \frac{e^3 - e^{-1} - 2e}{2}. \tag{8.3.41}$$

Therefore, by Theorem 8.3.10, if $\beta \notin \{e, e^{-1}\}$, then the trivial solution of (8.3.35) is globally asymptotically stable if and only if (8.3.41) holds. Also, Theorem 8.3.11 asserts that every solution of (8.3.35) oscillates if $0 < \beta < e$ and that (8.3.35) has an oscillatory solution if $\beta < 0$ or $\beta > e$.

8.4. Applications

As an application of the techniques presented in this chapter, we consider the *logistic equation*

$$x'(t) = rx(t)\left(1 - \sum_{j=0}^{n}\left(a_j x([t-\tau_j]) + b_j x^2([t-\tau_j])\right)\right), \tag{8.4.1}$$

where $r \in \mathbb{R}^+$ and for $j \in \{0,1,\dots,n\}$, $\tau_j \in \mathbb{N}_0$ and $a_j, b_j \in \mathbb{R}_0^+$ with $a_j + b_j > 0$. When $\tau_j = 0$ for all j, we also assume $r \neq 1$. We let $\tau = \max\{\tau_1,\dots,\tau_n\}$.

By a solution of equation (8.4.1) we mean a continuous function x which is defined on the set $\{-\tau, -\tau+1,\dots,0\} \cup \mathbb{R}^+$ and which satisfies the following two conditions.

(i) The derivative $x'(t)$ exists at each $t \in \mathbb{R}_0^+$ with the possible exception of the points $t \in \mathbb{N}_0$, where one-sided derivatives exist and are finite.

(ii) x satisfies equation (8.4.1) in each interval of the form $[k, k+1)$, where $k \in \mathbb{N}_0$.

A solution x of equation (8.4.1) is said to oscillate about a real number x^* if the function $x(t) - x^*$ has arbitrarily large zeros. Otherwise the solution is called nonoscillatory about x^*.

Equation (8.4.1) has a unique positive equilibrium. If we denote this equilibrium by x^*, then

$$\sum_{j=0}^{n}\left(a_j x^* + b_j (x^*)^2\right) = 0. \tag{8.4.2}$$

The following lemma is used in the proof of the main result of this section.

Lemma 8.4.1. *Consider the nonlinear differential equation with piecewise constant arguments*

$$y'(t) + \sum_{j=0}^{n} q_j f_j(y[t-\tau_j]) = 0 \quad \textit{for } t \geq 0, \tag{8.4.3}$$

where for $j \in \{0,1,\dots,n\}$ the following conditions hold:

(i$_1$) $q_j \in \mathbb{R}^+$, $\tau_j \in \mathbb{N}_0$, *and* $\sum_{j=0}^{n}(q_j + \tau_j) \neq 1$,

(i$_2$) $f_j \in C(\mathbb{R},\mathbb{R})$, $uf_j(u) > 0$ *for* $u \neq 0$, *and* $\lim_{u\to 0}(f_j(u)/u) = 1$,

(i$_3$) *there exists a positive number δ such that for all $j \in \{0,1,\dots,n\}$ either $f_j(u) \leq u$ for $0 \leq u \leq \delta$ or $f_j(u) \geq u$ for $-\delta \leq u \leq 0$.*

Then every solution of equation (8.4.3) oscillates about zero if and only if

$$\lambda - 1 + \sum_{j=0}^{n} q_j \lambda^{-\tau_j} = 0 \tag{8.4.4}$$

has no roots in $(0,1)$.

To simplify the notation, we introduce the functions

$$g_j(u) = a_j u + b_j u^2 - \left(a_j x^* + b_j (x^*)^2\right) \quad \text{for } j \in \{0, 1, \dots, n\}. \tag{8.4.5}$$

The following lemma gives a useful expression for the solutions of equation (8.4.1).

Lemma 8.4.2. *Let $x(0) \in \mathbb{R}^+$ and $x(-j) \in \mathbb{R}$ for $j \in \{1, 2, \dots, \tau\}$. Then equation (8.4.1) has a unique positive solution x which is given by*

$$x(t) = x(k) \exp\left(-r \sum_{j=0}^{n} g_j(x(k-\tau_j))(t-k)\right) \quad \text{for } k \le t < k+1,\ k \in \mathbb{N}_0, \tag{8.4.6}$$

where the sequence $\{x(k)\}$ satisfies the difference equation

$$x(k+1) = x(k) \exp\left(-r \sum_{j=0}^{n} g_j(x(k-\tau_j))\right) \quad \text{for } k \in \mathbb{N}_0. \tag{8.4.7}$$

Proof. For $t \in [k, k+1)$, $k \in \{-\tau, -\tau+1, \dots\}$, equation (8.4.1) becomes

$$\begin{aligned} x'(t) &= rx(t)\left(1 - \sum_{j=0}^{n} \left(a_j x(k-\tau_j) + b_j x^2(k-\tau_j)\right)\right) \\ &= -rx(t) \sum_{j=0}^{n} g_j(x(k-\tau_j)). \end{aligned} \tag{8.4.8}$$

By integrating (8.4.8) from k to t we obtain (8.4.6). By taking limits as $t \to k+1$ on both sides of (8.4.6) and by invoking the continuity of x we obtain (8.4.7). The proof is complete. □

Let x be a positive solution of equation (8.4.1). Set $x(t) = x^* e^{y(t)}$ for $t \in \mathbb{R}_0^+$. Observe that $x(t)$ oscillates about x^* if and only if $y(t)$ oscillates about zero. Also, $y(t)$ satisfies

$$y'(t) = -r \sum_{j=0}^{n} g_j\left(x^* \exp\left(y([t-\tau_j])\right)\right), \tag{8.4.9}$$

so

$$y'(t) + \sum_{j=0}^{n} q_j f_j\left(y([t-\tau_j])\right) = 0, \tag{8.4.10}$$

where

$$q_j = rx^*(2b_j x^* + a_j), \qquad f_j(u) = \frac{g_j(x^* e^u)}{2b_j (x^*)^2 + a_j x^*}. \tag{8.4.11}$$

We claim that f_j satisfies the hypotheses of Lemma 8.4.1. In fact, if $0 < u$, then $x^*e^u > x^*$ and $f_j(u) > 0$. Similarly, if $u < 0$, then $0 < x^*e^u < x^*$ and $f_j(u) < 0$. Therefore, $uf_j(u) > 0$ for $u \neq 0$. Also, $f_j'(0) = 1$. Hence the condition Lemma 8.4.1(i_2) is satisfied. By using the mean value theorem, we will show that $f_j(u) > 0$ for $u < 0$. Indeed, for $u < 0$, there exists $\xi \in (u, 0)$ such that

$$f_j(u) = \frac{g_j(x^*e^u) - g_j(x^*e^0)}{2b_j(x^*)^2 + a_jx^*} = \frac{2b_j(x^*e^\xi)^2 + a_j(x^*e^\xi)}{2b_j(x^*)^2 + a_jx^*}(u - 0). \tag{8.4.12}$$

Hence $f_j(u) > u$. Since $f_j(0) = 0$, Lemma 8.4.1(i_3) is also satisfied.

Now the following result is obtained by applying Lemma 8.4.1 to (8.4.10).

Theorem 8.4.3. *Every positive solution of equation (8.4.1) oscillates about the positive equilibrium x^* if and only if the equation*

$$\lambda - 1 + rx^* \sum_{j=0}^{n} (2b_jx^* + a_j)\lambda^{-\tau_j} = 0 \tag{8.4.13}$$

has no roots in $(0, 1)$.

Remark 8.4.4. Note that if $n = 0$, then the condition $a_0 \in \mathbb{R}_0^+$ can be disregarded. We claim that the function f_0 as defined in (8.4.11) satisfies all the conditions of Lemma 8.4.1. In fact, it follows from the definition of x^* that $q_0 \in \mathbb{R}^+$.

Because of Remark 8.4.4, when $n = 0$, Theorem 8.4.3 is restated as follows.

Theorem 8.4.5. *Assume $r \in \mathbb{R}^+$, $b \in \mathbb{R}_0^+$, $a \in \mathbb{R}$ which is assumed positive if $b = 0$, and let $\tau \in \mathbb{N}_0$. If $\tau = 0$, also assume that $r \neq 1$. Then every positive solution of the equation*

$$x'(t) = ax(t)\big(1 - ax([t-\tau]) - bx^2([t-\tau])\big) \quad \text{for } t \geq 0 \tag{8.4.14}$$

oscillates about x^* if and only if one of the following conditions holds:

$$rx^*(2bx^* + a) > \frac{\tau^\tau}{(\tau+1)^{\tau+1}} \quad \text{if } \tau > 0 \tag{8.4.15}$$

or

$$rx^*(2bx^* + a) > 1 \quad \text{if } \tau = 0. \tag{8.4.16}$$

If $b = 0$, then $x^* = 1/a$; otherwise $x^* = (1/2b)(-a + \sqrt{a^2 + 4b})$.

The following two results are corollaries of Theorem 8.4.3.

Corollary 8.4.6. *Assume $a, b \in \mathbb{R}^+$, $a \in \mathbb{R}_0^+$, and let $\tau, \sigma \in \mathbb{N}$. Let $x(0) > 0$ and $x(-j) \geq 0$ for $1 \leq j \leq \max\{\tau, \sigma\}$ be given. Then the unique positive solution of the equation*

$$x'(t) = rx(t)\left(1 - ax([t-\tau]) - bx^2([t-\sigma])\right) \quad \text{for } t \geq 0 \tag{8.4.17}$$

oscillates about $x^* = (1/2b)(-a + \sqrt{a^2 + 4b})$ if and only if the equation

$$\lambda - 1 + arx^*\lambda^{-\tau} + 2br(x^*)^2\lambda^{-\sigma} = 0 \tag{8.4.18}$$

has no roots in $(0, 1)$.

Corollary 8.4.7. *If $r \geq 1$, then every positive solution of equation (8.4.1) oscillates about its positive equilibrium x^*.*

Proof. In view of Theorem 8.4.3 it suffices to show that

$$F(\lambda) = \lambda - 1 + rx^* \sum_{j=0}^{n} (2b_j x^* + a_j)\lambda^{-\tau_j} = 0 \tag{8.4.19}$$

has no roots in $(0, 1)$. If $\lambda \in (0, 1)$, then it follows from the facts that $r \geq 1$ and $\lambda^{-\tau_j} \geq 0$ and from (8.4.2) that

$$F(\lambda) \geq \lambda - 1 + x^* \sum_{j=0}^{n} (2b_j x^* + a_j) = \lambda + \sum_{j=0}^{n} b_j (x^*)^2 > 0. \tag{8.4.20}$$

This completes the proof. □

8.5. Notes and general discussions

(1) Theorems 8.1.1 and 8.1.2 are due to Aftabizadeh et al. [3]. Lemmas 8.1.3–8.1.5 and Theorems 8.1.6–8.1.10 are taken from Shen and Stavroulakis [256]. Lemmas 8.1.14 and 8.1.20 are extensions of [150, Lemmas 8.4.1 and 8.4.2] due to Győri and Ladas. Theorems 8.1.17 and 8.1.23 are taken from Győri and Ladas [150]. Theorems 8.1.18 and 8.1.19 are due to Cooke and Wiener [97]. Theorem 8.1.24 is new. The results of Section 8.1.3 are extracted from Gopalsamy et al. [132] and Grove et al. [148].

To obtain easily verifiable conditions for the oscillation of equation (8.1.138) we require the following lemma which is taken from Győri and Ladas [150, Lemma 8.5.1].

Lemma 8.5.1. *Assume that $p\tau e \leq 1$. Then equation (8.1.140) has a unique root $\lambda_0 \in [-1/\tau, 0]$. (When $\tau = 0$, we use the convention that $1/\tau$ stands for ∞.)*

When $p\tau e \le 1$, let λ_0 be as defined in Lemma 8.5.1 and set

$$A = \lambda_0(1+\lambda_0\tau), \qquad B = \lambda^{-\lambda_0\tau}(1+e^{-\lambda_0\tau}). \tag{8.5.1}$$

The following theorem is due to Győri and Ladas [150, Theorem 8.5.1].

Theorem 8.5.2. *Let $p, q, r \in \mathbb{R}^+$, $\sigma \in \mathbb{N}$, and let A and B be as defined by (8.5.1). Then each of the following four conditions implies that every solution of equation (8.1.138) oscillates:*

(I_1) *$p\tau e > 1$ and $q \ge 0$,*
(I_2) *$p\tau e = 1$ and $q > 0$,*
(I_3) *$p\tau e < 1$, $p > 0$, $q > 0$, and the equation $A(\lambda-1)+B\lambda^{-\sigma} = 0$ has no roots in $[0,1)$,*
(I_4) *$p \ge 0$ and*

$$\begin{aligned} &q > \frac{\sigma^\sigma}{(\sigma+1)^{\sigma+1}} \quad \text{if } \sigma \ge 1, \\ &p \ge 0, \quad q \ge 1 \quad \text{if } \sigma = 0. \end{aligned} \tag{8.5.2}$$

For more additional results concerning equation (8.1.138) and its characteristic equation (8.1.143), we introduce the following related inequalities and equation:

$$x'(t) + px(t-\tau) + qx([t-\sigma]) \le 0 \quad \text{for } t \ge 0, \tag{8.5.3}$$

$$\alpha(t) \ge p\exp\left(\int_{t-\tau}^{t} \alpha(s)ds\right) + q\exp\left(\int_{[t-\sigma]}^{t} \alpha(s)ds\right) \quad \text{for } t \ge 0, \tag{8.5.4}$$

$$x'(t) + (1-\varepsilon)px(t-\tau) + (1+\varepsilon)qx([t-\sigma]) = 0 \quad \text{for } t \ge 0,\ 0<\varepsilon<1. \tag{8.5.5}$$

We note that solutions of inequalities are defined in a manner similar to the definition of solutions of equations.

The following theorem is extracted from Grove et al. [148].

Theorem 8.5.3. *Suppose that $p, q, \tau \in \mathbb{R}_0^+$, $\sigma \in \mathbb{N}_0$, and $p\tau + q\sigma > 0$. Then the following five statements are equivalent.*

(i_1) *Equation (8.1.138) has an eventually positive solution.*
(i_2) *Equation (8.1.143) has a solution.*
(i_3) *Inequality (8.5.3) has an eventually positive solution.*
(i_4) *Inequality (8.5.4) has a solution.*
(i_5) *There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon \le \varepsilon_0$, then equation (8.5.5) has an eventually positive solution.*

(2) The results of Sections 8.2.1–8.2.3 are due to Wiener [275]. The results of Section 8.2.4 are taken from Ladas et al. [191], while the results of Section 8.2.5 are taken from Papaschinopoulos and Schinas [213].

(3) It would be interesting to study the oscillatory behavior of half-linear differential equations with piecewise constant arguments of the form

$$\left(|x'(t)|^{\alpha}\right)' + q(t)|x([t-\tau])|^{\alpha} = 0, \tag{8.5.6}$$

where $\alpha \geq 1$, $\tau \in \mathbb{R}^{+}$, and $q \in C(\mathbb{R}, \mathbb{R}^{+})$.

(4) The results of Section 8.3 are due to Rodrigues [253].

(5) The results of Section 8.4 are taken from Rodrigues [252].

(6) It would be interesting to discuss the techniques presented in this chapter for higher-order differential equations with piecewise constant arguments of the form

$$x^{(n)}(t) + p(t)x(t-\tau) + q(t)x([t-\sigma]) = 0 \quad \text{for } n > 0. \tag{8.5.7}$$

9 Miscellaneous topics

This chapter is concerned with other interesting results. Section 9.1 contains the concept of the generalized characteristic equation and deals with some existence and comparison results of positive solutions of first-order delay difference equations. Section 9.2 deals with the oscillation of some linear as well as neutral difference equations with periodic coefficients. Linearized oscillations for autonomous and nonautonomous delay difference equations are established in Section 9.3. In Section 9.4 we present a systematic study for the oscillation of various types of recursive sequences. Another systematic study for the global asymptotic stability of different types of recursive sequences is made in Section 9.5. In Section 9.6 we present results on oscillation of second-order nonlinear difference equations with continuous variables. Section 9.7 is devoted to the study of oscillation for systems of delay difference equations. Finally, Section 9.8 deals with the oscillatory behavior of linear functional equations of second order.

9.1. Generalized characteristic equations

Consider the delay difference equation with variable coefficients

$$x(k+1) - x(k) + \sum_{i=1}^{m} q_i(k)x(k-\tau_i) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.1.1}$$

where

$$q_i(k) \in \mathbb{R}, \quad \tau_i \in \mathbb{N}_0 \quad \text{for } i \in \{1,2,\dots,m\},\ k \in \mathbb{N}_0. \tag{9.1.2}$$

In this section we will introduce the concept of a generalized characteristic equation associated with equation (9.1.1) and investigate how it relates to the existence of positive solutions of equation (9.1.1).

Let $\tau = \max\{\tau_1,\dots,\tau_m\}$ and let $n_0 \geq 0$. By a solution of equation (9.1.1) for $k \geq n_0$, we mean a sequence $\{x(k)\}$ which is defined for $k \geq n_0 - \tau$ and which satisfies equation (9.1.1) for $k \geq n_0$.

With equation (9.1.1) and with a given initial point $n_0 \geq 0$, one associates an initial condition of the form

$$a(n_0 - \tau), a(n_0 - \tau + 1), \ldots, a(n_0). \tag{9.1.3}$$

Then by the method of steps it follows that the IVP (9.1.1) and (9.1.3) has a unique solution $\{x(k)\}$ valid for $k \geq n_0$.

In the following, for convenience we speak of $\{\cdot\}_{n_0}^N$ as a sequence without considering N to be finite or infinite.

Consider equation (9.1.1) and $n_0 \geq 0$. For every $k \geq n_0$ and $i \in \{1, 2, \ldots, m\}$ define the sequences

$$r_i(k) = \min\{n_0, k - \tau_i\}, \qquad s_i(k) = \max\{n_0, k - \tau_i\}. \tag{9.1.4}$$

We refer to the equation

$$\alpha(k) - 1 + \sum_{i=1}^{m} q_i(k)\mu_i(k, n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\alpha(j)} = 0 \quad \text{for } k \geq n_0, \tag{9.1.5}$$

where $\prod_{j=s_i(k)}^{k-1} 1/\alpha(j) \equiv 1$ if $k = s_i(k)$ and $\mu_i(k, n_0) = a(r_i(k))/a(n_0)$ as the *generalized characteristic equation* associated with equation (9.1.1).

The following lemma, which is interesting in its own right, will be needed in the upcoming results.

Lemma 9.1.1. *Assume that (9.1.2) holds and that (9.1.3) is given with $a(n_0) > 0$. Let $N > n_0 + 1$ be a positive integer or infinity. Then the following statements are equivalent.*

(i) *The solution of IVP (9.1.1) and (9.1.3) remains positive for $n_0 \leq k < N$.*
(ii) *There exists a sequence $\{\alpha(k)\}$ which is positive and satisfies equation (9.1.5) for $n_0 \leq k < N$.*
(iii) *There exist two positive sequences $\{\beta(k)\}$ and $\{\gamma(k)\}$ with $\beta(k) \leq \gamma(k)$ for $n_0 \leq k < N - 1$ and such that for any sequence $\{\delta(k)\}$ which satisfies $\beta(k) \leq \delta(k) \leq \gamma(k)$ for $n_0 \leq k < N - 1$, the following inequality holds:*

$$\beta(k) \leq 1 - \sum_{i=1}^{m} q_i(k)\mu_i(k, n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta(j)} \leq \gamma(k) \quad \textit{for } k \geq n_0. \tag{9.1.6}$$

Proof. We first show (i)⇒(ii). Let $\{x(k)\}$ be the solution of the IVP (9.1.1) and (9.1.3) which remains positive for $n_0 \le k < N$. We claim that the sequence $\{\alpha(k)\}$ defined by

$$\alpha(k) = \frac{x(k+1)}{x(k)} \tag{9.1.7}$$

satisfies equation (9.1.5) for $n_0 \le k < N-1$. In fact, for $n_0 \le k < N-1$,

$$\begin{aligned}
\alpha(k) - 1 + \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\alpha(j)}
&= \frac{x(k+1)}{x(k)} - 1 + \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0) \prod_{j=s_i(k)}^{k-1} \frac{x(j)}{x(j+1)} \\
&= \frac{x(k+1)}{x(k)} - 1 + \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0) \frac{x(s_i(k))}{x(k)} \\
&= \frac{1}{x(k)} \left[x(k+1) - x(k) + \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0)x(s_i(k)) \right].
\end{aligned} \tag{9.1.8}$$

Now we will show that for $n_0 \le k < N-1$ and $1 \le i \le m$,

$$\frac{x(k-\tau_i)}{x(s_i(k))} = \mu_i(k,n_0). \tag{9.1.9}$$

To this end observe that if $k - \tau_i \ge n_0$, then $r_i(k) = n_0$, $s_i(k) = k - \tau_i$, and

$$\frac{x(k-\tau_i)}{x(s_i(k))} = \frac{x(s_i(k))}{x(s_i(k))} = 1 = \mu_i(k,n_0). \tag{9.1.10}$$

If $k - \tau_i < n_0$, then $r_i(k) = k - \tau_i$ and $s_i(k) = n_0$, and so (9.1.9) holds obviously. Hence, from (9.1.7), (9.1.8), and (9.1.9) we see that $\{\alpha(k)\}$ is positive and satisfies equation (9.1.5) for $n_0 \le k < N-1$.

Next we address (ii)⇒(iii). If $\{\alpha(k)\}$ is positive and satisfies equation (9.1.5) for $n_0 \le k < N-1$, then take $\beta(k) = \gamma(k) = \alpha(k)$ and the proof is obvious.

Finally we prove (iii)⇒(i). First, by utilizing Banach's contraction mapping principle, Theorem 4.4.15, we will show that under the given hypotheses there exists a positive sequence $\{\alpha(k)\}$ which satisfies (9.1.5) for $n_0 \le k < N-1$. Consider the set of sequences

$$A = \left\{\delta = \{\delta(k)\}_{n_0}^{N} : \beta(k) \le \delta(k) \le \gamma(k) \text{ for } n_0 \le k \le N_1\right\}, \tag{9.1.11}$$

where $n_0 < N_1 \le N-1$ is a positive integer. For any $\delta^{(1)} = \{\delta^{(1)}(k)\} \in A$ and $\delta^2 = \{\delta^{(2)}(k)\} \in A$, define

$$d(\delta^{(1)},\delta^{(2)}) = \sup_{n_0 \le k \le N_1} \{ |\delta^{(1)}(k) - \delta^{(2)}(k)| (b\eta)^k \}, \tag{9.1.12}$$

where $b = \min_{n_0 \le k \le N_1}\{\beta(k)\}$ and

$$0 < \eta < \min_{n_0 \le k \le N_1}\left\{\frac{1}{2b}, \frac{1}{2}\left(\sum_{i=1}^{m}\frac{|q_i(k)|}{2b^{k-s_i(k)}}\mu_i(k,n_0)\right)^{-1}\right\}. \tag{9.1.13}$$

Then (A,d) is a complete metric space. Now define a mapping T on (A,d) by

$$(T\delta)(k) = 1 - \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0)\prod_{j=s_i(k)}^{k-1}\frac{1}{\delta(j)}. \tag{9.1.14}$$

Then by the hypotheses we see that T maps A into A. Furthermore, we claim that T is a contraction mapping. Indeed, for any $\delta^{(1)} = \{\delta^{(1)}(k)\}$, $\delta^{(2)} = \{\delta^{(2)}(k)\} \in A$,

$$\begin{aligned}
&|(T\delta^{(1)})(k) - (T\delta^{(2)})(k)| \\
&\quad \le \sum_{i=1}^{m}|q_i(k)||\mu_i(k,n_0)|\left|\prod_{j=s_i(k)}^{k-1}\frac{1}{\delta^{(1)}(j)} - \prod_{j=s_i(k)}^{k-1}\frac{1}{\delta^{(2)}(j)}\right| \\
&\quad = \sum_{i=1}^{m}|q_i(k)||\mu_i(k,n_0)|\prod_{j=s_i(k)}^{k-1}\frac{1}{\delta^{(1)}(j)\delta^{(2)}(j)}\left|\prod_{j=s_i(k)}^{k-1}\delta^{(1)}(j) - \prod_{j=s_i(k)}^{k-1}\delta^{(2)}(j)\right| \\
&\quad \le \sum_{i=1}^{m}|q_i(k)||\mu_i(k,n_0)| \\
&\qquad \times \prod_{j=s_i(k)}^{k-1}\frac{1}{\delta^{(1)}(j)\delta^{(2)}(j)}|(\delta^{(1)}(s_i(k)) - \delta^{(2)}(s_i(k))) \\
&\qquad\qquad \times \delta^{(1)}(s_i(k)+1)\cdots\delta^{(1)}(k-2)\delta^{(1)}(k-1) \\
&\qquad\qquad + (\delta^{(1)}(s_i(k)+1) - \delta^{(2)}(s_i(k)+1)) \\
&\qquad\qquad \times \delta^{(1)}(s_i(k)+2)\cdots\delta^{(1)}(k-1)\delta^{(2)}(s_i(k)) \\
&\qquad\qquad + \cdots + (\delta^{(1)}(k-2) - \delta^{(2)}(k-2)) \\
&\qquad\qquad \times \delta^{(1)}(k-1)\delta^{(2)}(s_i(k))\cdots\delta^{(2)}(k-3) \\
&\qquad\qquad + (\delta^{(1)}(k-1) - \delta^{(2)}(k-1)) \\
&\qquad\qquad \times \delta^{(2)}(s_i(k))\cdots\delta^{(2)}(k-2)| \\
&\quad \le \sum_{i=1}^{m}|q_i(k)||\mu_i(k,n_0)|b^{s_i(k)-k-1}\sum_{j=s_i(k)}^{k-1}|\delta^{(1)}(j) - \delta^{(2)}(j)|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,b^{s_i(k)-k-1} \prod_{j=s_i(k)}^{k-1} \left(|\delta^{(1)}(j) - \delta^{(2)}(j)|\right)(b\eta)^j (b\eta)^{-j} \\
&\le d(\delta^{(1)},\delta^{(2)}) \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,b^{s_i(k)-k-1} \sum_{j=s_i(k)}^{k-1} (b\eta)^{-j} \\
&\le d(\delta^{(1)},\delta^{(2)}) \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,\frac{\eta^{k-s_i(k)+1}}{(b\eta)^{k+1}} \sum_{j=0}^{k-s_i(k)-1} (b\eta)^{-j} \\
&= d(\delta^{(1)},\delta^{(2)}) \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,\frac{\eta^{k-s_i(k)+1}}{(b\eta)^{k+1}} \left(\frac{b\eta(1-(b\eta)^{k-s_i(k)})}{(b\eta)^{k-s_i(k)}(1-b\eta)}\right) \\
&= \frac{d(\delta^{(1)},\delta^{(2)})}{(b\eta)^k} \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,\frac{\eta(1-(b\eta)^{k-s_i(k)})}{b^{k-s_i(k)}(1-b\eta)} \\
&\le \frac{d(\delta^{(1)},\delta^{(2)})}{(b\eta)^k} \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,\frac{\eta}{b^{k-s_i(k)}(1-b\eta)},
\end{aligned}
\tag{9.1.15}
$$

and so

$$
\begin{aligned}
&|(T\delta^{(1)})(k) - (T\delta^{(2)})(k)|(b\eta)^k \\
&\qquad \le \sum_{i=1}^{m} |q_i(k)|\,|\mu_i(k,n_0)|\,\frac{\eta}{b^{k-s_i(k)}(1-b\eta)} d(\delta^{(1)},\delta^{(2)}).
\end{aligned}
\tag{9.1.16}
$$

From this and (9.1.13) we see that

$$
d(T\delta^{(1)}, T\delta^{(2)}) \le \frac{1}{2} d(\delta^{(1)},\delta^{(2)}). \tag{9.1.17}
$$

Therefore, by Banach's contraction mapping principle, there is $\delta^* = \{\delta^*(k)\} \in A$ such that

$$
(T\delta^*)(k) = \delta^*(k) \quad \text{for } k \in \{n_0, n_0+1, \dots, N_1\}. \tag{9.1.18}
$$

Now define the sequence

$$
x^*(k) = \begin{cases} a(k) & \text{for } n_0 - \tau \le k \le n_0, \\ a(0) \displaystyle\prod_{i=n_0}^{k-1} \delta^*(i) & \text{for } n_0 + 1 \le k \le N_1 + 1. \end{cases} \tag{9.1.19}
$$

Clearly

$$
x^*(k) > 0 \quad \text{for } n_0 \le k \le N_1 + 1. \tag{9.1.20}
$$

Furthermore, for $n_0 \le k \le N_1$,

$$x^*(k+1) - x^*(k) = x^*(k)(\delta^*(k) - 1), \tag{9.1.21}$$

$$\sum_{i=1}^{m} q_i(k)x^*(k-\tau_i) = \sum_{i=1}^{m} q_i(k)\frac{x^*(k)x^*(k-\tau_i)}{x^*(s_i(k))} \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta^*(j)}. \tag{9.1.22}$$

Hence, from (9.1.18), (9.1.21), (9.1.22), and $[x^*(k-\tau_i)/x^*(s_i(k))] = \mu_i(k, n_0)$, we see that for $n_0 \le k \le N_1$,

$$x^*(k+1) - x^*(k) + \sum_{i=1}^{m} q_i(k)x^*(k-\tau_i) = 0. \tag{9.1.23}$$

Since N_1 is arbitrary, it follows that there exists a sequence $\{\bar{x}(k)\}$ with $\bar{x}(k) = a(k)$ for $n_0 - \tau \le k \le n_0$, $\bar{x}(k) > 0$ for $n_0 \le k < N$, and

$$\bar{x}(k+1) - \bar{x}(k) + \sum_{i=1}^{m} q_i(k)\bar{x}(k-\tau_i) = 0 \quad \text{for } n_0 \le k < N-1. \tag{9.1.24}$$

Since the IVP (9.1.1) and (9.1.3) has a unique solution x, we see that $x(k) = \bar{x}(k)$ for $n_0 - \tau \le k < N$. This completes the proof. □

Remark 9.1.2. Assume that $N = \infty$. Then, for k sufficiently large, $r_i(k) = n_0$ and $s_i(k) = k - \tau_i$ for $i \in \{1, 2, \dots, m\}$, and so equation (9.1.5) reduces to

$$\alpha(k) - 1 + \sum_{i=1}^{m} q_i(k) \prod_{j=k-\tau_i}^{k-1} \frac{1}{\alpha(j)} = 0. \tag{9.1.25}$$

When the coefficients of (9.1.1), $q_i(k) \equiv q_i$, are constants for $i \in \{1, 2, \dots, m\}$, then equation (9.1.1) reduces to the linear autonomous delay equation

$$x(k+1) - x(k) + \sum_{i=1}^{m} q_i x(k-\tau_i) = 0 \tag{9.1.26}$$

whose characteristic equation is

$$\lambda - 1 + \sum_{i=1}^{m} q_i \lambda^{-\tau_i} = 0. \tag{9.1.27}$$

If we look for a solution of the form $\alpha(k) = \lambda$, then equation (9.1.25) gives precisely equation (9.1.27). This is the reason why we call equation (9.1.25) the generalized characteristic equation.

Remark 9.1.3. From the proof of Lemma 9.1.1 and by the uniqueness of the solution of the IVP (9.1.1) and (9.1.3) we see that if Lemma 9.1.1(iii) holds, then there exists a unique positive sequence $\{\alpha(k)\}$ which satisfies equation (9.1.5) for $n_0 \le k < N - 1$ and satisfies $\beta(k) \le \alpha(k) \le \gamma(k)$ for $n_0 \le k < N - 1$.

The following result is an immediate consequence of Lemma 9.1.1.

Corollary 9.1.4. *Assume that condition (9.1.2) holds and that equation (9.1.1) has an eventually positive solution. Then there exist $n_0 \in \mathbb{N}_0$ and a positive sequence $\{\alpha(k)\}$ for $k \geq n_0 - \tau$ such that (9.1.25) holds for $k \geq n_0$.*

Next, we will apply Lemma 9.1.1 to establish an explicit condition for the existence of positive solutions of equation (9.1.1).

Theorem 9.1.5. *Assume that condition (9.1.2) holds and that*

$$\sum_{i=1}^{m} q_i^+(k) \leq \frac{\tau^\tau}{(\tau+1)^{\tau+1}}, \tag{9.1.28}$$

where $q_i^+(k) = \max\{q_i(k), 0\}$ for $i \in \{1,2,\ldots,m\}$ and $n_0 - \tau \leq k < N$. Then the solution of the IVP (9.1.1) and (9.1.3) with

$$a(n_0) > 0, \qquad 0 \leq a(k) \leq a(n_0) \quad \text{for } k \in \{n_0 - \tau, \ldots, n_0 - 1\} \tag{9.1.29}$$

remains positive for $n_0 \leq k < N$.

Proof. For $n_0 \leq k < N$, define the sequences

$$\beta(k) = \frac{1}{1 + \left((\tau+1)/\tau\right)^{k+1} \sum_{i=1}^{m} q_i^+(k)}, \qquad \gamma(k) = 1 + \sum_{i=1}^{m} 2^{\tau_i} \left| q_i(k) \right|, \tag{9.1.30}$$

and $r_i(k)$, $s_i(k)$ by (9.1.4). Then by noting (9.1.29) we see that $0 \leq \mu_i(k, n_0) \leq 1$ for $i \in \{1,2,\ldots,m\}$ and $n_0 \leq k < N$. Let $\{\delta(k)\}$ be any sequence which satisfies $\beta(k) \leq \delta(k) \leq \gamma(k)$ for $n_0 \leq k < N$. Then, for $n_0 \leq k < N$,

$$1 - \sum_{i=1}^{m} q_i(k)\mu_i(k, n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta(j)} \leq 1 + \sum_{i=1}^{m} 2^{\tau_i} \left| q_i(k) \right| = \gamma(k). \tag{9.1.31}$$

Also,

$$\begin{aligned} 1 - \sum_{i=1}^{m} & q_i(k)\mu_i(k, n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta(j)} \\ & \geq 1 - \sum_{i=1}^{m} q_i^+(k) \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta(j)} \\ & \geq 1 - \sum_{i=1}^{m} q_i^+(k) \prod_{j=k-\tau_i}^{k-1} \left[1 + \left(\frac{\tau+1}{\tau} \right)^{\tau+1} \sum_{i=1}^{m} q_i^+(j) \right] \\ & \geq 1 - \sum_{i=1}^{m} q_i^+(k) \prod_{j=k-\tau}^{k-1} \left[1 + \left(\frac{\tau+1}{\tau} \right)^{\tau+1} \sum_{i=1}^{m} q_i^+(j) \right]. \end{aligned} \tag{9.1.32}$$

From (9.1.32) and by using the inequality between the arithmetic and geometric mean, it follows that

$$\begin{aligned}
1 - \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta(j)} \\
&\geq 1 - \sum_{i=1}^{m} q_i^+(k)\left[1 + \frac{1}{\tau}\left(\frac{\tau+1}{\tau}\right)^{\tau+1} \sum_{j=k-\tau}^{k-1}\left(\sum_{i=1}^{m} q_i^+(j)\right)\right]^{\tau} \\
&\geq 1 - \left(1+\frac{1}{\tau}\right)^{\tau} \sum_{i=1}^{m} q_i^+(k).
\end{aligned} \tag{9.1.33}$$

Now we show that the inequality

$$f(x) = \left(1 - \left(\frac{\tau+1}{\tau}\right)^{\tau} x\right)\left(1 + \left(\frac{\tau+1}{\tau}\right)^{\tau+1} x\right) \geq 1 \quad \text{for } 0 \leq x \leq \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}} \tag{9.1.34}$$

holds. In fact,

$$f(0) = 1, \qquad f\left(\frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}\right) = \left(1 - \frac{1}{\tau+1}\right)\left(1+\frac{1}{\tau}\right) = 1. \tag{9.1.35}$$

Let $f'(x) = 0$, that is,

$$-\left(\frac{\tau+1}{\tau}\right)^{\tau+1}\left(1 + \left(\frac{1+\tau}{\tau}\right)^{\tau+1} x\right) + \left(\frac{\tau+1}{\tau}\right)^{\tau+1}\left(1 - \left(\frac{\tau+1}{\tau}\right)^{\tau} x\right) = 0. \tag{9.1.36}$$

Then we find

$$\begin{gathered}
x = \frac{1}{2\tau}\left(\frac{\tau}{\tau+1}\right)^{\tau+1}, \\
f\left(\frac{1}{2\tau}\left(\frac{\tau}{\tau+1}\right)^{\tau+1}\right) = \left(1 - \frac{1}{2(\tau+1)}\right)\left(1 + \frac{1}{2\tau}\right) = \frac{(2\tau+1)^2}{4\tau(\tau+1)} > 1.
\end{gathered} \tag{9.1.37}$$

Therefore (9.1.34) holds, and it follows that

$$1 - \left(1+\frac{1}{\tau}\right)^{\tau} \sum_{i=1}^{m} q_i^+(k) \geq \frac{1}{1 + ((\tau+1)/\tau)^{\tau+1} \sum_{i=1}^{m} q_i^+(k)} = \beta(k), \tag{9.1.38}$$

which in view of (9.1.33) implies that

$$1 - \sum_{i=1}^{m} q_i(k)\mu_i(k,n_0) \prod_{j=s_i(k)}^{k-1} \frac{1}{\delta(j)} \geq \beta(k) \quad \text{for } n_0 \leq k < N. \tag{9.1.39}$$

Hence all the hypotheses of Lemma 9.1.1(iii) are satisfied. We see that the solution of the IVP (9.1.1) and (9.1.3) remains positive for $n_0 \le k < N$, and the proof is complete. □

When $m = 1$, equation (9.1.1) reduces to the equation

$$x(k+1) - x(k) + q(k)x(k-\tau) = 0, \tag{9.1.40}$$

which is the discrete analogue of the differential equation

$$x'(t) + q(t)x(t-\tau) = 0, \tag{9.1.41}$$

where $q \in C([t_0, \infty), \mathbb{R})$ and $\tau \in \mathbb{R}_0^+$. It has been shown that if $q(t) \ge 0$ for $t \ge t_0$ and

$$\int_{t-\tau}^{t} q(s)ds \le \frac{1}{e}, \tag{9.1.42}$$

then equation (9.1.41) has an eventually positive solution. Since

$$\sum_{i=k-\tau}^{k-1} q(i) \le \left(\frac{\tau}{\tau+1}\right)^{\tau+1} \tag{9.1.43}$$

is the discrete analogue of (9.1.42), the following problem arises: assume that $\{q(k)\}$ is a nonnegative sequence of real numbers and let $\tau \in \mathbb{N}$ such that (9.1.43) holds. Does equation (9.1.41) have an eventually positive solution? In fact, this is one of the known criteria for differential equations that has no discrete analogue. By utilizing Lemma 9.1.1 we present the following example which shows that (9.1.43) cannot guarantee that equation (9.1.40) has an eventually positive solution.

Example 9.1.6. The delay difference equation

$$x(k+1) - x(k) + q(k)x(k-2) = 0, \tag{9.1.44}$$

where

$$q(k) = \begin{cases} 0 & \text{if } k = 2m,\ m \in \mathbb{N}_0, \\ \left(\dfrac{2}{3}\right)^3 & \text{if } k = 2m+1,\ m \in \mathbb{N}_0 \end{cases} \tag{9.1.45}$$

cannot have an eventually positive solution.

Since $q(k-2) + q(k-1) = (2/3)^3$, we see that (9.1.43) cannot guarantee that equation (9.1.40) has an eventually positive solution.

Proof. We assume that equation (9.1.44) has an eventually positive solution. Then, by Lemma 9.1.1 there exists a positive sequence $\{\beta(k)\}$ such that for all k sufficiently large, say, $k \geq 2m_0$, $m_0 \geq 0$, the inequality

$$1 - \frac{q(k)}{\beta(k-2)\beta(k-1)} \geq \beta(k) \tag{9.1.46}$$

holds. From (9.1.46) we see that for $k \geq 2m_0$, $\beta(k) \leq 1$ and

$$\beta(k-2)\beta(k-1)\left(1-\beta(k)\right) \geq q(k). \tag{9.1.47}$$

From this and (9.1.45) it follows that $1-\beta(2m_0+1) \geq (2/3)^3$, that is,

$$\beta(2m_0+1) \leq 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27}. \tag{9.1.48}$$

Then from (9.1.45) and (9.1.47), we get

$$\beta(2m_0+1)\beta(2m_0+2)\left(1-\beta(2m_0+3)\right) \geq \left(\frac{2}{3}\right)^3, \tag{9.1.49}$$

which implies that

$$1-\beta(2m_0+3) \geq \left(\frac{2}{3}\right)^3 \frac{1}{\beta(2m_0+1)} \geq \left(\frac{2}{3}\right)^3\left(\frac{27}{19}\right) = \frac{8}{19}, \tag{9.1.50}$$

and so $\beta(2m_0+3) \leq 1-(8/19) = 11/19$. Then, by the same procedure, we obtain

$$\beta(2m_0+5) \leq 1 - \left(\frac{2}{3}\right)^3\left(\frac{19}{11}\right) = \frac{145}{297} \leq \frac{1}{2}, \tag{9.1.51}$$

and hence

$$\beta(2m_0+7) \leq 1 - \left(\frac{2}{3}\right)^3 2 = \frac{11}{27}. \tag{9.1.52}$$

Then it follows that

$$\begin{aligned} \beta(2m_0+9) &\leq 1 - \left(\frac{2}{3}\right)^3\left(\frac{27}{11}\right) = \frac{3}{11}, \\ \beta(2m_0+11) &\leq 1 - \left(\frac{2}{3}\right)^3\left(\frac{11}{3}\right) = -\frac{7}{81}. \end{aligned} \tag{9.1.53}$$

This contradicts the hypothesis that $\beta(k) > 0$ and completes the proof. □

Finally, we will apply Lemma 9.1.1 and its proof to establish some comparison results for positive solutions of equation (9.1.1) and the inequalities

$$y(k+1) - y(k) + \sum_{i=1}^{m} p_i(k)y(k-\tau_i) \le 0, \tag{9.1.54}$$

$$z(k+1) - z(k) + \sum_{i=1}^{m} h_i(k)z(k-\tau_i) \ge 0. \tag{9.1.55}$$

Theorem 9.1.7. *Assume that condition (9.1.2) holds and*

$$p_i(k) \ge q_i(k) \ge h_i(k) \quad \text{for } n_0 \le k < N,\ i \in \{1,2,\dots,m\}. \tag{9.1.56}$$

Assume that $\{x(k)\}$, $\{y(k)\}$, *and* $\{z(k)\}$ *are solutions of (9.1.1), (9.1.54), and (9.1.55), respectively, such that*

$$y(k) > 0 \quad \text{for } n_0 \le k < N, \tag{9.1.57}$$

$$z(n_0) \ge x(n_0) \ge y(n_0), \tag{9.1.58}$$

$$\frac{y(k)}{y(n_0)} \ge \frac{x(k)}{x(n_0)} \ge \frac{z(k)}{z(n_0)} \quad \text{for } n_0 - \tau \le k < n_0. \tag{9.1.59}$$

Then

$$z(k) \ge x(k) \ge y(k) \quad \text{for } n_0 \le k < N. \tag{9.1.60}$$

Proof. Assume that

$$N_1 = \max\{N^* : x(k) > 0 \text{ and } z(k) > 0 \text{ for } n_0 \le k < N^*\}. \tag{9.1.61}$$

We claim that $N_1 = N$. Otherwise, $x(k) > 0$ and $z(k) > 0$ for $n_0 \le k < N_1$ and

$$\text{either} \quad x(N_1) \le 0 \quad \text{or} \quad z(N_1) \le 0. \tag{9.1.62}$$

Set for $n_0 \le k < N_1$,

$$\beta(k) = \frac{y(k+1)}{y(k)}, \qquad \mu(k) = \frac{x(k+1)}{x(k)}, \qquad \eta(k) = \frac{z(k+1)}{z(k)}. \tag{9.1.63}$$

Then, by direct substitution and by noting the fact that for $n_0 \le k < N_1$,

$$\frac{y(k-\tau_i)}{y(s_i(k))} = \frac{y(r_i(k))}{y(n_0)}, \qquad \frac{x(k-\tau_i)}{x(s_i(k))} = \frac{x(r_i(k))}{x(n_0)}, \qquad \frac{z(k-\tau_i)}{z(s_i(k))} = \frac{z(r_i(k))}{z(n_0)}, \tag{9.1.64}$$

where $r_i(k)$ and $s_i(k)$ are defined by (9.1.4), we see that

$$\beta(k) - 1 + \sum_{i=1}^{m} p_i(k) \frac{y(r_i(k))}{y(n_0)} \prod_{j=k-\tau_i}^{k-1} \frac{1}{\beta(j)} \leq 0, \tag{9.1.65}$$

$$\mu(k) - 1 + \sum_{i=1}^{m} q_i(k) \frac{x(r_i(k))}{x(n_0)} \prod_{j=k-\tau_i}^{k-1} \frac{1}{\mu(j)} = 0, \tag{9.1.66}$$

$$\eta(k) - 1 + \sum_{i=1}^{m} h_i(k) \frac{z(r_i(k))}{z(n_0)} \prod_{j=k-\tau_i}^{k-1} \frac{1}{\eta(j)} \geq 0, \tag{9.1.67}$$

respectively. Now we claim that

$$\beta(k) \leq \mu(k) \leq \eta(k) \quad \text{for } n_0 \leq k < N_1. \tag{9.1.68}$$

We will show that $\beta(k) \leq \mu(k)$ while the proof that $\mu(k) \leq \eta(k)$ is similar and will be omitted.

Let $\{\delta(k)\}$ be an arbitrary sequence such that $\beta(k) \leq \delta(k)$ for $n_0 \leq k < N_1$. Then by (9.1.65), (9.1.56), and (9.1.66), we see that for $n_0 \leq k < N_1$,

$$\begin{aligned} \beta(k) &\leq 1 - \sum_{i=1}^{m} p_i(k) \frac{y(r_i(k))}{y(n_0)} \prod_{j=k-\tau_i}^{k-1} \frac{1}{\beta(j)} \\ &\leq 1 - \sum_{i=1}^{m} q_i(k) \frac{x(r_i(k))}{x(n_0)} \prod_{j=k-\tau_i}^{k-1} \frac{1}{\delta(j)} \\ &\leq 1, \end{aligned} \tag{9.1.69}$$

that is, Lemma 9.1.1(iii) is satisfied (with $\gamma(k) \equiv 1$), and so by Remark 9.1.3 the equation

$$\alpha(k) - 1 + \sum_{i=1}^{m} q_i(k) \frac{x(r_i(k))}{x(n_0)} \prod_{j=k-\tau_i}^{k-1} \frac{1}{\alpha(j)} = 0 \tag{9.1.70}$$

has exactly one solution $\{\alpha(k)\}$ for $n_0 \leq k < N_1$, and moreover, the solution satisfies $\beta(k) \leq \alpha(k) \leq 1$ for $n_0 \leq k < N_1$. But, by (9.1.66), $\{\mu(k)\}$ is a solution of (9.1.66) for $n_0 \leq k < N_1$. Therefore, $\beta(k) = \alpha(k) = \mu(k) \leq 1$ for $n_0 \leq k < N_1$, and (9.1.68) has been established.

Clearly, by using the definitions of $\{\beta(k)\}$, $\{\mu(k)\}$, and $\{\eta(k)\}$, we see that for $n_0 < k \le N_1$,

$$y(k) = y(n_0)\prod_{i=n_0}^{k-1}\beta(i), \qquad x(k) = x(n_0)\prod_{i=n_0}^{k-1}\mu(i), \qquad z(k) = z(n_0)\prod_{i=n_0}^{k-1}\eta(i). \tag{9.1.71}$$

Hence (9.1.58) and (9.1.68) imply that

$$z(k) \ge x(k) \ge y(k) \quad \text{for } n_0 \le k \le N. \tag{9.1.72}$$

As $y(N_1) > 0$, it follows from (9.1.72) that $z(N_1) \ge x(N_1) > 0$. This contradicts (9.1.62) and shows $N_1 = N$. Thus (9.1.60) holds and the proof is complete. □

The following corollary of Theorem 9.1.7 compares two positive solutions of equation (9.1.1) by means of their initial conditions.

Corollary 9.1.8. *Assume that condition (9.1.2) holds with $q_i(k) \ge 0$, $i \in \{1, 2, \dots, m\}$ and $k \ge n_0$. Let*

$$a(n_0 - \tau), a(n_0 - \tau + 1), \dots, a(n_0), \tag{9.1.73}$$

$$b(n_0 - \tau), b(n_0 - \tau + 1), \dots, b(n_0) \tag{9.1.74}$$

be two initial constants such that

$$b(n_0) \ge a(n_0), \qquad \frac{a(k)}{a(n_0)} \ge \frac{b(k)}{b(n_0)} \ge 0 \quad \text{for } n_0 - \tau \le k \le n_0. \tag{9.1.75}$$

Suppose that $\{x^{(1)}(k)\}$ and $\{x^{(2)}(k)\}$ are solutions of the IVP (9.1.1) and (9.1.73) and IVP (9.1.1) and (9.1.74), respectively, with $x^{(1)}(k) > 0$ for $n_0 \le k < N$. Then $x^{(2)}(k) \ge x^{(1)}(k)$ for $n_0 \le k < N$.

The following result is an immediate consequence of Theorem 9.1.7.

Corollary 9.1.9. *Assume that (9.1.2) holds with $q_i(k) \ge 0$ for $i \in \{1, 2, \dots, m\}$ and $k \ge n_0$. Then the difference inequality*

$$y(k+1) - y(k) + \sum_{i=1}^{m} q_i(k)y(k - \tau_i) \le 0 \quad \text{for } n \in \mathbb{N}_0 \tag{9.1.76}$$

has an eventually positive solution if and only if equation (9.1.1) has an eventually positive solution.

9.2. Difference equations with periodic coefficients

Consider the linear difference equation

$$x(k+1) - x(k) + a(k)\big[px(k-\tau) + qx(k-\sigma)\big] = 0, \tag{9.2.1}$$

where

(i) $\{a(k)\}$ is a sequence of nonnegative real numbers with $a(k) \neq 0$,
(ii) $p, q \in \mathbb{R}$,
(iii) $\tau, \sigma \in \mathbb{N}_0$,
(iv) $\{a(k)\}$ is periodic of period m (where $m \in \mathbb{N}$) and there exists $\nu, \rho \in \mathbb{N}_0$ such that $\tau = \nu m$ and $\sigma = m\rho$.

By a solution on $\mathbb{N}(n_0) = \{n_0, n_0+1, \dots\}$ of equation (9.2.1) we mean a sequence of real numbers $\{x(k)\}_{n \geq n_0 - \mu}$, where $\mu = \max\{\tau, \sigma\}$ which satisfies equation (9.2.1). With the equation (9.2.1), we associate the characteristic equation, namely,

$$\lambda^m - \prod_{r=0}^{m-1} \big(1 - a(r)\big[p\lambda^{-\tau} + q\lambda^{-\sigma}\big]\big) = 0. \tag{9.2.2}$$

We also consider the neutral difference equations

$$\Delta\big(x(k) + cx(k-\tau)\big) + \sum_{j=0}^{N} p_j(k)x(k-\sigma_j) = 0, \tag{9.2.3}$$

$$\Delta\big(x(k) + cx(k+\tau)\big) - q_0(k)x(k) - \sum_{j\in J} q_j(k)x(k+\sigma_j^*) = 0, \tag{9.2.4}$$

where

(i) $c \in \mathbb{R}$, $\tau \in \mathbb{N}$,
(ii) $\{p_j(k)\}$, $j \in \{0, 1, \dots, N\}$, $\{q_0(k)\}$ and $\{q_j(k)\}$, $j \in J \subseteq \mathbb{N}$, are sequences of real numbers,
(iii) $\sigma_j \in \mathbb{N}_0$ for $j \in \{0, 1, \dots, N\}$ such that $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N$,
(iv) $\sigma_j^* \in \mathbb{N}$ for $j \in J$ such that $\sigma_{j_1}^* \neq \sigma_{j_2}^*$ if $j_1, j_2 \in J$ with $j_1 \neq j_2$,
(v) $p_j(k) \not\equiv 0$, $j \in \{0, 1, \dots, N\}$, and $\{p_j(k)\}$, $j \in \{0, 1, \dots, N\}$, are periodic sequences with a common period $m \in \mathbb{N}$ and there exist u and $v_j \in \mathbb{N}$ for $j \in \{1, 2, \dots, N\}$ such that $\tau = um$ and $\sigma_j = v_j m$ for $j \in \{1, 2, \dots, N\}$,
(vi) $q_j(k) \not\equiv 0$ for $j \in J$ is such that $\sum_{j\in J} q_j(k) > 0$, $\{q_0(k)\}$ and $\{q_j(k)\}$, $j \in J$, are periodic sequences with a common period $m \in \mathbb{N}$ and there exist $u, w_j \in \mathbb{N}$ for $j \in J$ so that $\tau = um$ and $\sigma_j^* = w_j m$ for $j \in J$.

It is worth noting that the set J may be infinite. For any $n_0 \in \mathbb{N}_0$, by a solution on $\mathbb{N}(n_0)$ of equation (9.2.3), we mean a sequence $\{x(k)\}_{k \geq n_0 - \max\{\tau, \sigma_N\}}$ of real numbers, which satisfies equation (9.2.3) for all $k \geq n_0$. Also, by a solution on $\mathbb{N}(n_0)$ of equation (9.2.4) we mean a sequence $\{x(k)\}_{k \geq n_0}$ of real numbers, which satisfies equation (9.2.4) for all $k \geq n_0$.

With the difference equations (9.2.3) and (9.2.4), we associate the characteristic equations of (9.2.3) and (9.2.4), namely,

$$\lambda^m - \prod_{r=0}^{m-1}\left(1 - \frac{1}{1+c\lambda^{-\tau}}\sum_{j=0}^{N} p_j(r)\lambda^{-\sigma_j}\right) = 0, \tag{9.2.5}$$

$$\lambda^m - \prod_{r=0}^{m-1}\left(1 + \frac{1}{1+c\lambda^{\tau}}\left[q_0(r) + \sum_{j\in J} q_j(r)\lambda^{\sigma_j^*}\right]\right) = 0. \tag{9.2.6}$$

In this section we will be concerned with necessary conditions and also sufficient conditions for the oscillation of all solutions of the above equations. These conditions are in terms of the roots of the associated characteristic equations.

9.2.1. Oscillation of equation (9.2.1)

We will use the notation

$$A = \frac{1}{m}\sum_{r=0}^{m-1} a(r) \tag{9.2.7}$$

and use the convention that $\sum_{\theta}^{\theta-1}(\cdot) = 0$, $\prod_{\theta}^{\theta-1}(\cdot) = 1$, and $0^0 = 1$.

We will need the following lemmas.

Lemma 9.2.1. *Assume that $\tau \geq \sigma$. Let $\{x(k)\}_{k\geq n_0-\tau}$ be a solution (on $\mathbb{N}(n_0)$) of equation (9.2.1), where $n_0 \in \mathbb{N}_0$, and set*

$$z(k) = x(k) - p\sum_{j=k-\tau}^{k-1-\sigma} a(j)x(j) \quad \text{for } k \geq N_0 = \max\{n_0, \tau\}. \tag{9.2.8}$$

Then $\{z(k)\}_{k\geq N_0}$ is a solution (on $\mathbb{N}(N_0+\tau)$) of equation (9.2.1).

Proof. As the sequence $\{a(k)\}_{k\geq 0}$ is m-periodic and $\tau = \nu m$, $\sigma = \rho m$, from (9.2.8) and equation (9.2.1) we obtain for $k \geq N_0$,

$$\begin{aligned} z(k+1) - z(k) &= x(k+1) - x(k) - p\big[a(k-\sigma)x(k-\sigma) - a(k-\tau)x(k-\tau)\big] \\ &= -a(k)\big[px(k-\tau) + qx(k-\sigma)\big] - pa(k)\big[x(k-\sigma) - x(k-\tau)\big] \end{aligned} \tag{9.2.9}$$

and consequently,

$$z(k+1) - z(k) = -a(k)[p+q]x(k-\sigma) \quad \forall k \geq N_0. \tag{9.2.10}$$

Furthermore, by taking again into account the fact that $\{a(k)\}_{k\geq 0}$ is m-periodic and that $\tau = \nu m$ and $\sigma = \rho m$, from (9.2.8) and equation (9.2.1), we derive for any

$k \geq N_0 + \tau$,

$$\begin{aligned}
&pz(k-\tau) + qz(k-\sigma) \\
&\quad = p\left[x(k-\tau) - p\sum_{s=k-2\tau}^{k-1-\tau-\sigma} a(s)x(s)\right] + q\left[x(k-\sigma) - p\sum_{s=k-\tau-\sigma}^{k-1-2\sigma} a(s)x(s)\right] \\
&\quad = px(k-\tau) + qx(k-\sigma) - p\left[p\sum_{s=k-2\tau}^{k-1-\tau-\sigma} a(s)x(s) + q\sum_{s=k-\tau-\sigma}^{k-1-2\sigma} a(s)x(s)\right] \\
&\quad = px(k-\tau) + qx(k-\sigma) \\
&\qquad - p\left[p\sum_{s=k-\tau}^{k-1-\sigma} a(s-\tau)x(s-\tau) + q\sum_{s=k-\tau}^{k-1-\sigma} a(s-\sigma)x(s-\sigma)\right] \\
&\quad = px(k-\tau) + qx(k-\sigma) - p\left[p\sum_{s=k-\tau}^{k-1-\sigma} a(s)x(s-\tau) + q\sum_{s=k-\tau}^{k-1-\sigma} a(s)x(s-\sigma)\right] \\
&\quad = px(k-\tau) + qx(k-\sigma) - p\sum_{s=k-\tau}^{k-1-\sigma} a(s)\left[px(s-\tau) + qx(s-\sigma)\right] \\
&\quad = px(k-\tau) + qx(k-\sigma) + p\sum_{s=k-\tau}^{k-1-\sigma} \left[x(s+1) - x(s)\right] \\
&\quad = px(k-\tau) + qx(k-\sigma) + p\left[x(k-\sigma) - x(k-\tau)\right],
\end{aligned} \tag{9.2.11}$$

and so

$$pz(k-\tau) + qz(k-\sigma) = [p+q]x(k-\sigma) \quad \forall k \geq N_0 + \tau. \tag{9.2.12}$$

By combining (9.2.10) and (9.2.12), we see that the sequence $\{z(k)\}_{k \geq N_0}$ is a solution (on $\mathbb{N}(N_0 + \tau)$) of equation (9.2.1). □

Lemma 9.2.2. *Assume that* $p \neq 0$ *and*

$$\tau > \sigma > 0. \tag{9.2.13}$$

Then the inequalities

$$p + q > 0, \tag{9.2.14}$$

$$p > 0, \tag{9.2.15}$$

$$\theta = \min_{\lambda > 0}\left\{\lambda^m - \prod_{r=0}^{m-1}\left[1 - a(r)\left(p\lambda^{-\tau} + q\lambda^{-\sigma}\right)\right]\right\} > 0 \tag{9.2.16}$$

are necessary conditions for equation (9.2.2) to have no positive roots.

Proof. Suppose that the equation (9.2.2) has no positive roots and set

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1} \left[1 - a(r)\left(p\lambda^{-\tau} + q\lambda^{-\sigma}\right)\right] \quad \text{for } \lambda > 0. \tag{9.2.17}$$

By (9.2.13) we see that

$$F(\infty) = \infty, \tag{9.2.18}$$

$$F(\lambda) > 0 \quad \forall \lambda > 0. \tag{9.2.19}$$

This guarantees that $F(1) > 0$, namely,

$$\prod_{r=0}^{m-1} \left[1 - a(r)(p+q)\right] < 1. \tag{9.2.20}$$

Since $a(r) \geq 0$ for $r \in \{0, 1, \ldots, m-1\}$, from (9.2.20) it follows that $p+q$ must be positive, that is, (9.2.14) is satisfied. Now we define

$$I = \left\{r \in \{0, 1, \ldots, m-1\} : a(r) > 0\right\}. \tag{9.2.21}$$

Since the sequence $\{a(k)\}_{k\geq 0}$ is m-periodic and nonnegative and not identically zero, it is clear that $I \neq \emptyset$. Moreover, we have

$$F(\lambda) = \lambda^m - \prod_{r\in I} \left[1 - a(r)\left(p\lambda^{-\tau} + q\lambda^{-\sigma}\right)\right] \quad \text{for } \lambda > 0. \tag{9.2.22}$$

Because of (9.2.13), we can easily verify that for any $r \in I$,

$$\lim_{\lambda\to 0^+} \left[1 - a(r)\left(p\lambda^{-\tau} + q\lambda^{-\sigma}\right)\right] = \begin{cases} -\infty & \text{if } p > 0, \\ +\infty & \text{if } p < 0, \end{cases} \tag{9.2.23}$$

and hence from (9.2.22) it follows that

$$F(0+) = \begin{cases} +\infty & \text{if } p > 0, \\ -\infty & \text{if } p < 0. \end{cases} \tag{9.2.24}$$

But $F(0+) = -\infty$ contradicts (9.2.19). Thus we must have $p > 0$ and so (9.2.15) holds. Moreover,

$$F(0+) = \infty. \tag{9.2.25}$$

Finally, (9.2.18), (9.2.19), and (9.2.25) ensure that $\theta = \min_{\lambda>0} F(\lambda)$ exists and is positive, that is, (9.2.16) is true. □

The following lemma is interesting in its own right, and its first part will only be used in the proof of the main result of this subsection.

Lemma 9.2.3. *Suppose that the conditions (9.2.13), (9.2.14), and (9.2.15) hold. Let $\{x(k)\}_{k\geq n_0-\tau}$, where $n_0 \in \mathbb{N}_0$, be a solution (on $\mathbb{N}(n_0)$) of equation (9.2.1) which is positive. Also, let $\{z(k)\}_{k\geq N_0}$, where $N_0 = \max\{n_0,\tau\}$, be defined by (9.2.8). Then the following statements are true.*

(I_1) *Assume that*

$$Ap(\tau-\sigma) \leq 1. \tag{9.2.26}$$

Then $\{z(k)\}_{k\geq N_0}$ is a solution (on $\mathbb{N}(N_0+\tau)$) of equation (9.2.1) which is decreasing and eventually positive.

(I_2) *Assume that*

$$Ap(\tau-\sigma) > 1 \tag{9.2.27}$$

and set $w(k) = -z(k)$ for $k \geq N_0$. Then $\{w(k)\}_{k\geq N_0}$ is a solution (on $\mathbb{N}(N_0+\tau)$) of equation (9.2.1) which is increasing and eventually positive.

Proof. First, we will establish that

$$\sum_{s=k-\tau}^{k-1} a(s) = A\tau \quad \text{for } k \geq \tau, \tag{9.2.28}$$

$$\sum_{s=k-\sigma}^{k-1} a(s) = A\sigma \quad \text{for } k \geq \sigma. \tag{9.2.29}$$

Using the fact that the sequence $\{a(k)\}$ is m-periodic and $\tau = \nu m$, we obtain for $k \geq \tau$,

$$\sum_{s=k-\tau}^{k-1} a(s) = \sum_{s=0}^{\tau-1} a(s) = \left[\frac{1}{\tau}\sum_{s=0}^{\tau-1} a(s)\right]\tau = A\tau. \tag{9.2.30}$$

Similarly, one can prove (9.2.29).

Now we show (I_1). Lemma 9.2.1 ensures that the sequence $\{z(k)\}_{k\geq N_0}$ is a solution (on $\mathbb{N}(N_0+\tau)$) of equation (9.2.1). In view of (9.2.14) it follows from (9.2.10) that $\{z(k)\}_{k\geq N_0}$ is decreasing. So it remains to show that $\{z(k)\}_{k\geq N_0}$ is positive. Since $\{a(k)\}_{k\geq 0}$ is m-periodic and not identically zero, for any $\xi \in \mathbb{N}_0$, the sequence $\{a(k)\}_{k\geq\xi}$ is not identically zero. Thus, from (9.2.10) and (9.2.14), we see that $\{z(k)\}_{k\geq N_0}$ is not eventually constant. So, in order to show that the sequence $\{z(k)\}_{k\geq N_0}$ is positive, it suffices to establish that

$$\lim_{k\to\infty} z(k) = 0. \tag{9.2.31}$$

We first claim that $L = \lim_{k\to\infty} z(k)$ is finite. Otherwise, we have

$$\lim_{k\to\infty} z(k) = -\infty, \tag{9.2.32}$$

and consequently $\{z(k)\}_{k\geq N_0}$ is eventually negative. On the other hand, from (9.2.32) it follows that $\{x(k)\}_{k\geq n_0-\tau}$ is unbounded. In fact, in the opposite case, there exists a constant $b > 0$ with $x(k) \leq b$ for all $k \geq n_0 - \tau$. Then, by (9.2.13), (9.2.15), (9.2.28), and (9.2.29), from (9.2.8) we derive for $k \geq N_0$,

$$\begin{aligned}
x(k) &= z(k) + p\sum_{s=k-\tau}^{k-1-\sigma} a(s)x(s) \\
&\leq z(k) + pb\sum_{s=k-\tau}^{k-1-\sigma} a(s) \\
&= z(k) + pb\left(\sum_{s=k-\tau}^{k-1} a(s) - \sum_{s=k-\sigma}^{k-1} a(s)\right) \\
&= z(k) + pbA(\tau-\sigma) \\
&\longrightarrow -\infty \quad \text{as } k \longrightarrow \infty,
\end{aligned} \tag{9.2.33}$$

and consequently $\lim_{k\to\infty} x(k) = -\infty$, which is a contradiction. By using the fact that $\{z(k)\}_{k\geq N_0}$ is eventually negative and $\{x(k)\}_{k\geq n_0-\tau}$ is unbounded, we can choose an integer $n_1 \geq N_0$ so that

$$z(n_1) < 0, \qquad x(n_1) = \max\{x(s) : n_0 - \tau \leq s \leq n_1 - 1\}. \tag{9.2.34}$$

By using (9.2.13), (9.2.15), (9.2.28), (9.2.29), and (9.2.26), from (9.2.8) we obtain

$$\begin{aligned}
0 &> z(n_1) \\
&= x(n_1) - p\sum_{s=n_1-\tau}^{n_1-1-\sigma} a(s)x(s) \\
&\geq x(n_1)\left(1 - p\sum_{s=n_1-\tau}^{n_1-1-\sigma} a(s)\right) \\
&= x(n_1)\left[1 - p\left(\sum_{s=n_1-\tau}^{n_1-1} a(s) - \sum_{s=n_1-\sigma}^{n_1-1} a(s)\right)\right] \\
&= x(n_1)\left[1 - pA(\tau-\sigma)\right] \\
&\geq 0.
\end{aligned} \tag{9.2.35}$$

This contradiction proves the claim that L is finite. Next, (9.2.10) gives

$$z(k) - z(N_0) = -[p+q]\sum_{s=N_0}^{k-1} a(s)x(s-\sigma) \quad \forall k \geq N_0, \tag{9.2.36}$$

and so, letting $k \to \infty$, we get

$$L - z(N_0) = -[p+q] \sum_{s=N_0}^{\infty} a(s)x(s-\sigma). \tag{9.2.37}$$

Because of (9.2.14), this implies that

$$\sum_{s=N_0}^{\infty} a(s)x(s-\sigma) < \infty. \tag{9.2.38}$$

But, by using the fact that $\{a(k)\}_{k\geq 0}$ is m-periodic and that $\tau = \nu m$ and $\sigma = \rho m$, we obtain

$$\begin{aligned}
\sum_{s=N_0}^{\infty} a(s)x(s-\sigma) &= \sum_{s=N_0-\sigma}^{\infty} a(s+\sigma)x(s) = \sum_{s=N_0-\sigma}^{\infty} a(s)x(s) \\
&= \sum_{s=N_0-\sigma+\tau}^{\infty} a(s-\tau)x(s-\tau) \\
&= \sum_{s=N_0-\sigma+\tau}^{\infty} a(s)x(s-\tau).
\end{aligned} \tag{9.2.39}$$

Thus, in view of (9.2.38), we have

$$\sum_{s=N_0}^{\infty} a(s)x(s) < \infty, \tag{9.2.40}$$

$$\sum_{s=N_0}^{\infty} a(s)x(s-\tau) < \infty. \tag{9.2.41}$$

From equation (9.2.1), we obtain for $k \geq N_0$,

$$x(k) - x(N_0) + p\sum_{s=N_0}^{k-1} a(s)x(s-\tau) + q\sum_{s=N_0}^{k-1} a(s)x(s-\sigma) = 0 \tag{9.2.42}$$

and so, letting $k\to\infty$ and using (9.2.38) and (9.2.41), we conclude that $\lim_{k\to\infty} x(k)$ exists and is finite. This limit must be zero, that is,

$$\lim_{k\to\infty} x(k) = 0. \tag{9.2.43}$$

Otherwise, there exists a constant b_1 such that $x(k) \geq b_1$ for all $k \geq N_0 - \tau$ and so (9.2.41) gives $\sum_{s=N_0}^{\infty} a(s) < \infty$. This is a contradiction, since the sequence $\{a(k)\}_{k\geq 0}$

is m-periodic and not identically zero. Finally, we see that (9.2.40) implies that

$$\lim_{k\to\infty}\sum_{s=k-\tau}^{k-1-\sigma} a(s)x(s) = 0. \tag{9.2.44}$$

By (9.2.43) and (9.2.44) from (9.2.8) we obtain (9.2.31). Thus, the proof of statement (I_1) is complete.

Next we prove (I_2). Since equation (9.2.1) is linear, from Lemma 9.2.1 it follows that the sequence $\{w(k)\}_{k\geq N_0}$ is a solution (on $\mathbb{N}(N_0+\tau)$) of equation (9.2.1). Furthermore, (9.2.10) gives

$$w(k+1) - w(k) = a(k)[p+q]x(k-\sigma) \quad \forall k \geq N_0. \tag{9.2.45}$$

Thus, by (9.2.14) we conclude that $\{w(k)\}_{k\geq N_0}$ is increasing. Now, in order to show that $\{w(k)\}_{k\geq N_0}$ is eventually positive, it suffices to prove that

$$\lim_{k\to\infty} w(k) = \infty. \tag{9.2.46}$$

Suppose that (9.2.46) fails. Then $L=\lim_{k\to\infty} w(k)$ exists and is finite. From (9.2.45), we obtain

$$w(k) - w(N_0) = [p+q]\sum_{s=N_0}^{k-1} a(s)x(s-\sigma) \quad \text{for } k \geq N_0. \tag{9.2.47}$$

Letting $k \to \infty$, we get

$$L - w(N_0) = [p+q]\sum_{s=N_0}^{\infty} a(s)x(s-\sigma), \tag{9.2.48}$$

which by (9.2.14) implies that (9.2.38) is true. As in the proof of part (I_1), we can verify that (9.2.40) and (9.2.41) hold, and next we can see that (9.2.43) and (9.2.44) are also satisfied. Now, in view of (9.2.43) and (9.2.44) from (9.2.8), it follows that L must be zero. But, for any $\xi \in \mathbb{N}_0$, the sequence $\{a(k)\}_{k\geq\xi}$ is not identically zero. Hence (9.2.14) and (9.2.45) ensure that $\{w(k)\}_{k\geq N_0}$ is not eventually constant. Hence $\{w(k)\}_{k\geq N_0}$ is always negative. So we can consider an integer $n_1 \geq N_0$ such that $w(n_1) < 0$ and $x(n_1) = \min\{x(s) : n_0 - \tau \leq s \leq n_1 - 1\}$. Using

(9.2.13), (9.2.15), (9.2.28), (9.2.29), and (9.2.27), from (9.2.8) we derive

$$\begin{aligned}
0 &> w(n_1) \\
&= -x(n_1) + p\sum_{s=n_1-\tau}^{n_1-1-\sigma} a(s)x(s) \\
&\geq x(n_1)\left(-1+\sum_{s=n_1-\tau}^{n_1-1-\sigma} a(s)\right) \\
&= x(n_1)\left[-1+p\left(\sum_{s=n_1-\tau}^{n_1-1} a(s) - \sum_{s=n_1-\sigma}^{n_1-1} a(s)\right)\right] \\
&= x(n_1)\left[-1+pA(\tau-\sigma)\right] \\
&\geq 0.
\end{aligned} \tag{9.2.49}$$

This contradiction shows that (9.2.46) is true, and so the proof of part (I_2) is complete. □

Next we present the following result.

Theorem 9.2.4. (i_1) *A necessary condition for the oscillation of equation (9.2.1) is that there is no positive root λ_0 of equation (9.2.2) with the following property: if $m > 1$, then*

$$a(r)\left[p\lambda_0^{-\tau} + q\lambda_0^{-\sigma}\right] < 1 \quad \text{for } r \in \{1, 2, \ldots, m-1\}. \tag{9.2.50}$$

(i_2) *Assume that the following hypothesis fails to hold:*

$$\tau > \sigma > 0, \qquad p+q > 0, \qquad p > 0, \qquad Ap(\tau-\sigma) > 1. \tag{9.2.51}$$

Then a sufficient condition for the oscillation of equation (9.2.1) is that equation (9.2.2) has no positive roots.

Proof. For any $\lambda > 0$ we define

$$c(k,\lambda) = 1 - a(k)\left[p\lambda^{-\tau} + q\lambda^{-\sigma}\right] \quad \text{for } k \geq 0. \tag{9.2.52}$$

Then equation (9.2.2) becomes

$$\lambda^m - \prod_{r=0}^{m-1} c(r,\lambda) = 0. \tag{9.2.53}$$

Furthermore, we have for every $\lambda > 0$,

$$\prod_{r=k-\tau}^{k-1} c(r,\lambda) = \left[\prod_{r=0}^{m-1} c(r,\lambda)\right]^{\nu} \quad \forall k \geq \tau. \tag{9.2.54}$$

Indeed, consider an arbitrary number $\lambda > 0$. If $\tau = 0$, then $\nu = 0$ and hence (9.2.54) is true. So, we assume that $\tau > 0$. Then by using the fact that $\{a(k)\}_{k\geq 0}$ is m-periodic and $\tau = m\nu$, we obtain for every $k \geq \tau$,

$$\begin{aligned}\prod_{r=k-\tau}^{k-1} c(r,\lambda) &= \prod_{r=k-\tau}^{k-1} \left(1 - a(r)\left[p\lambda^{-\tau} + q\lambda^{-\sigma}\right]\right) \\ &= \prod_{r=0}^{\tau-1} \left(1 - a(r)\left[p\lambda^{-\tau} + q\lambda^{-\sigma}\right]\right) \\ &= \left[\prod_{r=0}^{m-1} \left(1 - a(r)\left[p\lambda^{-\tau} + q\lambda^{-\sigma}\right]\right)\right]^{\nu} \\ &= \left[\prod_{r=0}^{m-1} c(r,\lambda)\right]^{\nu}.\end{aligned} \tag{9.2.55}$$

By a parallel argument, we can show that for every $\lambda > 0$,

$$\prod_{r=k-\sigma}^{k-1} c(r,\lambda) = \left[\prod_{r=0}^{m-1} c(r,\lambda)\right]^{\rho} \quad \forall k \geq \sigma. \tag{9.2.56}$$

First we prove (i_1). Assume that equation (9.2.53) has a positive root λ_0 with the following property: if $m > 1$, then (9.2.50) is satisfied. Then, we get

$$\begin{aligned}\left[\prod_{r=0}^{m-1} c(r,\lambda_0)\right]^{\nu} &= \left(\lambda_0^m\right)^{\nu} = \lambda_0^{m\nu} = \lambda_0^{\tau}, \\ \left[\prod_{r=0}^{m-1} c(r,\lambda_0)\right]^{\rho} &= \left(\lambda_0^m\right)^{\rho} = \lambda_0^{m\rho} = \lambda_0^{\sigma}.\end{aligned} \tag{9.2.57}$$

So, (9.2.54) and (9.2.56) give, respectively,

$$\prod_{r=k-\tau}^{k-1} c(r,\lambda_0) = \lambda_0^{\tau} \quad \forall k \geq \tau, \tag{9.2.58}$$

$$\prod_{r=k-\sigma}^{k-1} c(r,\lambda_0) = \lambda_0^{\sigma} \quad \forall k \geq \sigma. \tag{9.2.59}$$

Next, we define $x(k) = \prod_{r=0}^{k-1} c(r,\lambda_0)$ for $k \geq 0$. Then we obtain for $k \geq \tau$,

$$\begin{aligned}x(k-\tau) = \prod_{r=0}^{k-\tau-1} c(r,\lambda_0) &= \left[\prod_{r=0}^{k-1} c(r,\lambda_0)\right]\left[\prod_{r=k-\tau}^{k-1} c(r,\lambda_0)\right]^{-1} \\ &= x(k)\left[\prod_{r=k-\tau}^{k-1} c(r,\lambda_0)\right]^{-1}.\end{aligned} \tag{9.2.60}$$

Thus, by using (9.2.58), we conclude that

$$x(k-\tau) = \lambda_0^{-\tau} x(k) \quad \text{for every } k \ge \tau. \tag{9.2.61}$$

In a similar way, we can use (9.2.59) to conclude that

$$x(k-\sigma) = \lambda_0^{-\sigma} x(k) \quad \text{for every } k \ge \sigma. \tag{9.2.62}$$

On the other hand, we derive

$$x(k+1) - x(k) = \left[c(k,\lambda_0) - 1\right] x(k) \quad \text{for } k \ge 0. \tag{9.2.63}$$

Thus, by (9.2.61) and (9.2.62), we get for every $k \ge \mu = \max\{\tau, \sigma\}$,

$$\begin{aligned} x(k+1) &- x(k) + a(k)\left[px(k-\tau) + qx(k-\sigma)\right] \\ &= \left[c(k,\lambda_0) - 1\right]x(k) + a(k)\left[p\lambda_0^{-\tau} + q\lambda_0^{-\sigma}\right]x(k) \\ &= \left[c(k,\lambda_0) - \left(1 - a(k)\left[p\lambda_0^{-\tau} + q\lambda_0^{-\sigma}\right]\right)\right]x(k) \\ &= 0, \end{aligned} \tag{9.2.64}$$

which means that the sequence $\{x(k)\}_{k\ge 0}$ is a solution on $\mathbb{N}(\mu)$ of equation (9.2.1). Now, we will show that $x(k) > 0$ for all $k \ge 0$. If $m = 1$, then equation (9.2.53) gives $c(0,\lambda_0) = \lambda_0 > 0$. If $m > 1$, then (9.2.50) guarantees that $c(r,\lambda_0) > 0$ for $r \in \{1, 2, \dots, m-1\}$, and hence from equation (9.2.53) it follows that

$$c(0,\lambda_0) = \lambda_0^m \left[\prod_{r=1}^{m-1} c(r,\lambda_0)\right]^{-1} > 0. \tag{9.2.65}$$

Thus, when $m = 1$ or $m > 1$, we have $c(r,\lambda_0) > 0$ for $r \in \{0, 1, \dots, m-1\}$. Since the sequence $\{a(k)\}_{k\ge 0}$ is m-periodic, we conclude that $c(r,\lambda_0) > 0$ for all $r \in \mathbb{N}_0$, which ensures that $x(k) > 0$ for every $k \ge 0$. We have thus proved that there exists a nonoscillatory solution of equation (9.2.1).

Next we show (i_2). Without loss of generality we may assume that $\tau \ge \sigma$. Suppose that equation (9.2.2) has no positive roots and set

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1} \left(1 - a(r)\left[p\lambda^{-\tau} + q\lambda^{-\sigma}\right]\right) \quad \text{for } \lambda > 0. \tag{9.2.66}$$

Also, assume for the sake of contradiction that equation (9.2.1) has a nonoscillatory solution $\{x(k)\}_{k\ge n_0-\tau}$ on $\mathbb{N}(n_0)$ with $n_0 \in \mathbb{N}_0$. As the negative of a solution of equation (9.2.1) is also a solution of the same equation, we can restrict ourselves to the case where $\{x(k)\}_{k\ge n_0-\tau}$ is eventually positive. Furthermore, we suppose that $x(k) > 0$ for all $k \ge n_0 - \tau$.

We first consider the particular case where $p = 0$. In this case equation (9.2.1) becomes

$$x(k+1) - x(k) + qa(k)x(k-\sigma) = 0, \tag{9.2.67}$$

and equation (9.2.2) takes the form

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1}\left[1 - qa(r)\lambda^{-\sigma}\right] = 0. \tag{9.2.68}$$

Now assume that $\sigma = 0$. Then from equation (9.2.67), we obtain

$$x(k+1) = \left[1 - qa(k)\right]x(k) \quad \text{for } k \geq n_0, \tag{9.2.69}$$

which ensures that $1-qa(k) > 0$ for every $k \geq n_0$. Thus, as $\{a(k)\}_{k\geq 0}$ is m-periodic, we must have $1 - qa(r) > 0$ for $r \in \{0, 1, \ldots, m-1\}$ and consequently,

$$\lambda_0 \equiv \left[\prod_{r=0}^{m-1}\left(1 - qa(r)\right)\right]^{1/m} > 0. \tag{9.2.70}$$

But λ_0 is a root of equation (9.2.68) and this is a contradiction.

Next, suppose that $\sigma > 0$. Since $F(\infty) = \infty$, we always have $F(\lambda) > 0$ for all $\lambda > 0$. In particular, we have $F(1) > 0$, that is, $\prod_{r=0}^{m-1}(1-qa(r)) < 1$. This guarantees that q is always positive. Thus, by applying the statement (ii) of the main theorem in [231] for equation (9.2.67) (or Theorem 9.2.8 with $c = 0$), we conclude that equation (9.2.67) is oscillatory, which is a contradiction.

Next, we examine the case where $\tau = \sigma$. Then equation (9.2.1) takes the form

$$x(k+1) - x(k) + [p+q]a(k)x(k-\tau) = 0, \tag{9.2.71}$$

and equation (9.2.2) reduces to the equation

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1}\left(1 - [p+q]a(r)\lambda^{-\tau}\right) = 0. \tag{9.2.72}$$

Suppose first that $\tau = 0$. Then from (9.2.71), it follows that $1 - [p+q]a(k) > 0$ for all $k \geq n_0$, and so by the fact that $\{a(k)\}_{k\geq 0}$ is m-periodic, we get $1-[p+q]a(r) > 0$ for $r \in \{0, 1, \ldots, m-1\}$. Thus equation (9.2.72) admits the positive root

$$\lambda_1 = \left[\prod_{r=0}^{m-1}\left(1 - [p+q]a(r)\right)\right]^{1/m}, \tag{9.2.73}$$

which is a contradiction.

Now let τ be positive. As $F(\infty) = \infty$, we always have $F(\lambda) > 0$ for $\lambda > 0$. So $F(1) > 0$ and consequently

$$\prod_{r=0}^{m-1} \left(1 - [p+q]a(r)\lambda^{-\tau}\right) < 1. \tag{9.2.74}$$

From this inequality it follows that $p + q$ must be positive, and as in the above proof we see that equation (9.2.71) is oscillatory, which is a contradiction.

Now, consider the case where $p \neq 0$ and $\tau > \sigma = 0$. In this case equation (9.2.1) has the form

$$x(k+1) - x(k) + qa(k)x(k) + pa(k)x(k-\tau) = 0, \tag{9.2.75}$$

and equation (9.2.2) becomes

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1} \left(1 - a(r)[p\lambda^{-\tau} + q]\right) = 0. \tag{9.2.76}$$

Assume that $p<0$. Define $I = \{r \in \{0, 1, \dots, m-1\} : a(r) > 0\}$. Since the sequence $\{a(k)\}_{k\geq 0}$ is m-periodic, nonnegative, and not identically zero, we have $I \neq \emptyset$. Moreover, one has

$$F(\lambda) = \lambda^m - \prod_{r\in I} \left(1 - a(r)[p\lambda^{-\tau} + q]\right) \quad \text{for } \lambda > 0. \tag{9.2.77}$$

Thus we see that $F(0+) = -\infty$. But this is impossible since as $F(\infty) = \infty$, we have $F(\lambda) > 0$ for all $\lambda > 0$. So we have proved that

$$p > 0. \tag{9.2.78}$$

Now, from equation (9.2.75) we obtain

$$x(k+1) \leq [1 - qa(k)]x(k) \quad \text{for every } k \geq n_0, \tag{9.2.79}$$

and consequently $1 - qa(k) > 0$ for $k \geq n_0$. As $\{a(k)\}_{k\geq 0}$ is m-periodic, we have

$$1 - qa(k) > 0 \quad \forall k \geq 0. \tag{9.2.80}$$

Hence

$$P = \left[\prod_{r=0}^{m-1} \left(1 - qa(r)\right)\right]^{1/m} > 0. \tag{9.2.81}$$

Moreover, define

$$y(k) = x(k)\left[\prod_{s=0}^{k-1} \left(1 - qa(s)\right)\right]^{-1} \quad \text{for } k \geq n_1 = \max\{0, n_0 - \tau\}. \tag{9.2.82}$$

Clearly $\{y(k)\}_{k\geq n_1}$ is a sequence of positive numbers. Furthermore, by using the fact that $\{x(k)\}_{k\geq n_0-\tau}$ satisfies equation (9.2.75) for $k \geq n_0$, we can easily verify that $\{y(k)\}_{k\geq n_1}$ satisfies

$$y(k+1)-y(k)+p\left[\prod_{s=0}^{k-1}(1-qa(s))\right]^{-1}(1-qa(k))^{-1}a(k)y(k-\tau)=0 \tag{9.2.83}$$

for $k \geq n_1+\tau$. But by using the fact that $\{a(k)\}_{k\geq 0}$ is m-periodic and that $\tau = \nu m$, we obtain for $k \geq \tau$,

$$\prod_{s=k-\tau}^{k-1}(1-qa(s))=\prod_{s=0}^{\tau-1}(1-qa(s))=\left[\prod_{r=0}^{m-1}(1-qa(r))\right]^{\nu}=(P^m)^{\nu}=P^{\tau}. \tag{9.2.84}$$

So we have verified that $\{y(k)\}_{k\geq n_1}$ is a positive solution (on $\mathbb{N}(n_1+\tau)$) of the equation

$$y(k+1)-y(k)+\left[pP^{-\tau}(1-qa(k))^{-1}a(k)\right]y(k-\tau)=0. \tag{9.2.85}$$

On the other hand, equation (9.2.76) can be written as

$$\lambda^m-\prod_{r=0}^{m-1}(1-qa(r))\left[1-p(1-qa(r))^{-1}a(r)\lambda^{-\tau}\right]=0, \tag{9.2.86}$$

so

$$\lambda^m-P^m\prod_{r=0}^{m-1}\left[1-p(1-qa(r))^{-1}a(r)\lambda^{-\tau}\right]=0. \tag{9.2.87}$$

Thus, by using the transformation $\bar{\lambda}=\lambda/P$, equation (9.2.87) becomes

$$\bar{\lambda}^m-\prod_{r=0}^{m-1}\left(1-\left[pP^{-\tau}(1-qa(r))^{-1}a(r)\right]\bar{\lambda}^{-\tau}\right)=0. \tag{9.2.88}$$

Clearly equation (9.2.88) has no positive roots. So, in view of (9.2.78), (9.2.80), as in the above proof, we see that equation (9.2.85) is oscillatory. This is a contradiction.

We have arrived at a contradiction in each one of the three particular cases considered above. So, in what follows, we may assume that $p \neq 0$ and $\tau > \sigma > 0$ (i.e., (9.2.13) holds). Then from Lemma 9.2.2 it follows that (9.2.14)–(9.2.16) are also true. Moreover, since hypothesis (9.2.51) fails to hold, we always have $Ap(\tau-\sigma) \leq 1$, that is, (9.2.26) is satisfied. In the rest of the proof, this inequality plays an important rôle. In the sequel, for convenience, we will suppose that inequalities about terms of sequences are satisfied eventually for all large k. From Lemma 9.2.3(I_1), it follows that the sequence $\{z(k)\}$ defined by

$$z(k) = x(k) - p \sum_{s=k-\tau}^{k-1-\sigma} a(s)x(s) \tag{9.2.89}$$

is a solution of equation (9.2.1) which is eventually positive and decreasing. Moreover, this solution satisfies (9.2.10), that is,

$$z(k+1) - z(k) + a(k)[p+q]x(k-\sigma) = 0. \tag{9.2.90}$$

Define

$$Z(k) = z(k) - p \sum_{s=k-\tau}^{k-1-\sigma} a(s)z(s). \tag{9.2.91}$$

Then, by Lemma 9.2.3(I_1), the sequence $\{Z(k)\}$ is a solution of equation (9.2.1) which is eventually positive and decreasing. Moreover, we have

$$Z(k+1) - Z(k) + a(k)[p+q]z(k-\sigma) = 0. \tag{9.2.92}$$

Next, we define

$$\Lambda = \{\lambda \in (0,1] : Z(k+1) - c(k,\lambda)Z(k) \leq 0\}. \tag{9.2.93}$$

The proof will be accomplished by proving that the set Λ has the following contradictory properties (where θ is the positive number defined by (9.2.16)):

(P_1) $1 \in \Lambda$ and all numbers of Λ are greater than $\theta^{1/m}$,

(P_2) for every $\lambda \in (0,1]$, $\lambda \in \Lambda$ implies $(\lambda^m - \theta)^{1/m} \in \Lambda$.

First, we establish (P_1). By taking into account (9.2.15), (9.2.91), and the fact that $\{z(k)\}$ is eventually decreasing, we obtain $Z(k) \leq z(k) \leq Z(k-\sigma)$ and so, in view of (9.2.14), from (9.2.92) it follows that

$$Z(k+1) - Z(k) + a(k)[p+q]Z(k) \leq 0. \tag{9.2.94}$$

The last inequality can be written as

$$Z(k+1) - (1 - a(k)[p+q])Z(k) \leq 0 \quad \text{or} \quad Z(k+1) - c(k,1)Z(k) \leq 0 \tag{9.2.95}$$

so that $1 \in \Lambda$. Now, let $\lambda \in \Lambda$ be arbitrary. Then $Z(k+1) \leq c(k,\lambda)Z(k)$ which ensures that $c(k,\lambda) > 0$ for all large k. Since $\{a(k)\}_{k\geq 0}$ is m-periodic, we must have

$$c(k,\lambda) > 0 \quad \forall k \in \mathbb{N}_0. \tag{9.2.96}$$

But (9.2.16) gives

$$\lambda^m - \theta \geq \prod_{r=0}^{m-1} c(r,\lambda). \tag{9.2.97}$$

From (9.2.96) and (9.2.97) it follows that $\lambda^m - \theta > 0$, that is, $\lambda > \theta^{1/m}$. We have thus proved (P_1).

Next, we will establish (P_2). Consider an arbitrary $\lambda \in \Lambda$. Then by the definition of Λ, we have $Z(k+1) \leq c(k,\lambda)Z(k)$ or $Z(k) \leq c(k-1,\lambda)Z(k-1)$. Using (9.2.96), we obtain

$$Z(k-1) \geq \frac{1}{c(k-1,\lambda)} Z(k). \tag{9.2.98}$$

By using (9.2.98), it is easy to verify that

$$\begin{aligned} Z(k-\tau) &\geq \left[\prod_{r=k-\tau}^{k-1-\sigma} c(r,\lambda)\right]^{-1} Z(k-\sigma), \\ Z(k-\sigma) &\geq \left[\prod_{r=k-\sigma}^{k-1} c(r,\lambda)\right]^{-1} Z(k), \end{aligned} \tag{9.2.99}$$

and hence, in view of (9.2.54) and (9.2.56),

$$\begin{aligned} Z(k-\tau) &\geq \left[\prod_{r=0}^{m-1} c(r,\lambda)\right]^{-(\nu-\rho)} Z(k-\sigma), \\ Z(k-\sigma) &\geq \left[\prod_{r=0}^{m-1} c(r,\lambda)\right]^{-\rho} Z(k). \end{aligned} \tag{9.2.100}$$

Therefore, by (9.2.97), we have

$$Z(k-\tau) \geq (\lambda^m - \theta)^{-(\nu-\rho)} Z(k-\sigma), \tag{9.2.101}$$

$$Z(k-\sigma) \geq (\lambda^m - \theta)^{-\rho} Z(k). \tag{9.2.102}$$

Since $\nu > \rho$ and, by (9.2.14), p is positive, we get

$$p(\lambda^m - \theta)^{-(\nu-\rho)} + q > p + q, \tag{9.2.103}$$

and so, by virtue of (9.2.15), we conclude that the number $p(\lambda^m - \theta)^{-(\nu-\rho)} + q$ must be positive. Combining the fact that $\{Z(k)\}$ is a solution of (9.2.1) with (9.2.101) and (9.2.102), we derive

$$\begin{aligned}
0 &= Z(k+1) - Z(k) + a(k)[pZ(k-\tau) + qZ(k-\sigma)] \\
&\geq Z(k+1) - Z(k) + a(k)\Big[p(\lambda^m - \theta)^{-(\nu-\rho)} + q\Big]Z(k-\sigma) \\
&\geq Z(k+1) - Z(k) + a(k)\Big[p(\lambda^m - \theta)^{-\nu} + q(\lambda^m - \theta)^{-\rho}\Big]Z(k) \\
&= Z(k+1) - \Big\{1 - a(k)\Big[p\Big((\lambda^m - \theta)^{1/m}\Big)^{-\tau} + q\Big((\lambda^m - \theta)^{1/m}\Big)^{-\sigma}\Big]\Big\}Z(k) \\
&= Z(k+1) - c\Big(k, (\lambda^m - \theta)^{1/m}\Big)Z(k),
\end{aligned} \tag{9.2.104}$$

which means that $(\lambda^m - \theta)^{1/m} \in \Lambda$, and so (P_2) has been established.

This completes the proof. □

9.2.2. Delay difference equations with positive and negative coefficients

Consider the linear difference equation

$$x(k+1) - x(k) + p(k)x(k-\tau) - q(k)x(k-\sigma) = 0, \tag{9.2.105}$$

where

(i_1) $\{p(k)\}$ and $\{q(k)\}$ are sequences of nonnegative real numbers which are not identically zero and m-periodic,

(i_2) $\tau, \sigma \in \mathbb{N}_0$ and there exist $\nu, \rho \in \mathbb{N}_0$ such that $\tau = \nu m$ and $\sigma = \rho m$.

Moreover, we will employ the notation

$$P = \frac{1}{m}\sum_{r=0}^{m-1} p(r), \qquad Q = \frac{1}{m}\sum_{r=0}^{m-1} q(r). \tag{9.2.106}$$

Here, we will obtain conditions under which every nonoscillatory solution of equation (9.2.105) tends to zero as $k \to \infty$. Moreover, we give sufficient conditions in order that all solutions of equation (9.2.105) are oscillatory.

Theorem 9.2.5. *Let $c > 1$ be a constant such that*

$$p(k) \geq cq(k) \quad \text{for } k \geq 0, \tag{9.2.107}$$

and assume that

$$0 \leq (\tau - \sigma)Q \leq 1. \tag{9.2.108}$$

Then every nonoscillatory solution of equation (9.2.105) tends to zero as $k \to \infty$.

Proof. Let $\{x(k)\}_{k\geq n_0-\tau}$ be a nonoscillatory solution of equation (9.2.105), where $n_0 \in \mathbb{N}_0$, say $x(k) > 0$ for $k \geq n_0 - \tau$. Set $N_0 = \max\{n_0, \tau\}$ and define

$$z(k) = x(k) - \sum_{s=k-\tau}^{k-1-\sigma} q(s)x(s) \quad \text{for } k \geq N_0. \tag{9.2.109}$$

From (9.2.109) and equation (9.2.105), we obtain for $k \geq N_0$,

$$\begin{aligned} z(k+1) - z(k) \\ &= \big(x(k+1) - x(k)\big) - \big(q(k-\sigma)x(k-\sigma) - q(k-\tau)x(k-\tau)\big) \\ &= -\big[p(k)x(k-\tau) - q(k)x(k-\sigma)\big] - \big[q(k)x(k-\sigma) - q(k)x(k-\tau)\big] \\ &= -\big[p(k) - q(k)\big]x(k-\tau), \end{aligned} \tag{9.2.110}$$

that is,

$$\Delta z(k) = -\big[p(k) - q(k)\big]x(k-\tau) \quad \text{for every } k \geq N_0. \tag{9.2.111}$$

But $p(k)-q(k) \geq (c-1)q(k)$ for $k \geq 0$, and hence the sequence $\{p(k)-q(k)\}_{k\geq 0}$ is nonnegative. Thus (9.2.111) guarantees that the sequence $\{z(k)\}_{k\geq N_0}$ is decreasing. We will show that $\{z(k)\}_{k\geq N_0}$ is bounded below. To this end, suppose that

$$\lim_{k\to\infty} z(k) = -\infty. \tag{9.2.112}$$

Then $\{z(k)\}_{k\geq N_0}$ is eventually negative. We claim that the sequence $\{x(k)\}_{k\geq n_0-\tau}$ is unbounded. Otherwise, there exists a positive constant b such that $x(k) \leq b$ for $k \geq n_0 - \tau$ and so, from (9.2.109), we find

$$x(k) \leq z(k) + b\sum_{s=k-\tau}^{k-1-\sigma} q(s) \quad \forall k \geq N_0. \tag{9.2.113}$$

But we have

$$\sum_{s=k-\tau}^{k-1} q(s) = Q\tau \quad \text{for } k \geq \tau, \tag{9.2.114}$$

$$\sum_{s=k-\sigma}^{k-1} q(s) = Q\sigma \quad \text{for } k \geq \sigma. \tag{9.2.115}$$

Indeed, since (9.2.114) is obvious when $\tau = 0$, we assume that $\tau > 0$. By using the fact that the sequence $\{q(k)\}_{k\geq 0}$ is m-periodic and $\tau = \nu m$, we obtain for $k \geq \tau$,

$$\sum_{s=k-\tau}^{k-1} q(s) = \sum_{s=0}^{\tau-1} q(s) = \left[\frac{1}{\tau}\sum_{s=0}^{\tau-1} q(s)\right]\tau = \left[\frac{1}{m}\sum_{s=0}^{m-1} q(s)\right]\tau = Q\tau. \tag{9.2.116}$$

By a similar argument, we can verify that (9.2.115) is also true. By taking into account (9.2.114) and (9.2.115), from (9.2.113) we obtain for $k \geq N_0$,

$$x(k) \leq z(k) + b\left(\sum_{s=k-\tau}^{k-1} q(s) - \sum_{s=k-\sigma}^{k-1} q(s) \right) = z(k) + bQ(\tau - \sigma), \tag{9.2.117}$$

and thus by (9.2.112) we arrive at the contradiction $\lim_{k\to\infty} x(k) = -\infty$, and the claim is proved. Now, since $\{z(k)\}_{k\geq N_0}$ is eventually negative and the solution $\{x(k)\}_{k\geq n_0-\tau}$ is unbounded, there exists an integer $n_1 > N_0$ such that $z(n_1) < 0$ and $x(n_1) = \max\{x(s) : n_0 - \tau \leq s \leq n_1 - 1\}$. By taking into account (9.2.114), (9.2.115), and condition (9.2.108), from (9.2.109) we find

$$\begin{aligned} 0 &> z(n_1) \\ &= x(n_1) - \sum_{s=n_1-\tau}^{n_1-1-\sigma} q(s)x(s) \\ &\geq x(n_1) - x(n_1) \sum_{s=n_1-\tau}^{n_1-1-\sigma} q(s) \\ &= x(n_1) - x(n_1)\left[\sum_{s=n_1-\tau}^{n_1-1} q(s) - \sum_{s=n_1-\sigma}^{n_1-1} q(s) \right] \\ &= x(n_1)\left[1 - Q(\tau - \sigma)\right] \\ &\geq 0. \end{aligned} \tag{9.2.118}$$

This contradiction completes the proof that the sequence $\{z(k)\}_{k\geq N_0}$ is bounded below. Thus $L = \lim_{k\to\infty} z(k)$ exists and is finite. Now, from (9.2.111) we derive

$$z(k) - z(N_0) = -\sum_{s=N_0}^{k-1} [p(s) - q(s)]x(s-\tau) \quad \text{for every } k \geq N_0, \tag{9.2.119}$$

and so, letting $k \to \infty$, we have

$$L - z(N_0) = -\sum_{s=N_0}^{\infty} [p(s) - q(s)]x(s-\tau). \tag{9.2.120}$$

This implies that

$$\sum_{s=N_0}^{\infty} [p(s) - q(s)]x(s-\tau) < \infty. \tag{9.2.121}$$

On the other hand, we have

$$\sum_{s=N_0}^{\infty} [p(s) - q(s)]x(s-\tau) \geq (c-1) \sum_{s=N_0}^{\infty} q(s)x(s-\tau). \tag{9.2.122}$$

Thus, from (9.2.121) it follows that

$$\sum_{s=N_0}^{\infty} q(s)x(s-\tau) < \infty. \tag{9.2.123}$$

By (9.2.121) and (9.2.123), we have

$$\sum_{s=N_0}^{\infty} p(s)x(s-\tau) < \infty. \tag{9.2.124}$$

Using the fact that $\{q(k)\}_{k\geq 0}$ is m-periodic and that $\tau = \nu m$ and $\sigma = \rho m$, we obtain

$$\sum_{s=N_0}^{\infty} q(s)x(s-\tau) = \sum_{s=N_0-\tau+\sigma}^{\infty} q(s-\sigma+\tau)x(s-\sigma) = \sum_{s=N_0-\tau+\sigma}^{\infty} q(s)x(s-\sigma), \tag{9.2.125}$$

and so (9.2.123) gives

$$\sum_{s=N_0}^{\infty} q(s)x(s-\sigma) < \infty. \tag{9.2.126}$$

But equation (9.2.105) ensures that

$$x(k) - x(N_0) + \sum_{s=N_0}^{k-1} p(s)x(s-\tau) - \sum_{s=N_0}^{k-1} q(s)x(s-\sigma) = 0 \quad \text{for } k \geq N_0. \tag{9.2.127}$$

Thus, letting $k \to \infty$ and taking into account (9.2.124) and (9.2.126), we conclude that $\lim_{k\to\infty} x(k)$ exists and is finite. If this limit is positive, then there exists a constant $b_1 > 0$ such that $x(k) \geq b_1$ for all $k \geq N_0 - \tau$, and hence (9.2.124) gives $\sum_{s=N_0}^{\infty} p(s) < \infty$. This is a contradiction since the sequence $\{p(k)\}_{k\geq 0}$ is m-periodic and not identically zero. Thus we have $\lim_{k\to\infty} x(k) = 0$, and this completes the proof. □

Next we present the following oscillation result for equation (9.2.105).

Theorem 9.2.6. *Let $c > 1$ be a constant such that $p(k) \geq cq(k)$ for $k \geq 0$ and assume that (9.2.108) holds. Moreover, assume that $\tau > 0$ and*

$$P - Q > \frac{\tau^{\tau}}{(\tau+1)^{\tau+1}}. \tag{9.2.128}$$

Then equation (9.2.105) is oscillatory.

Proof. Let $\{x(k)\}_{k\geq n_0-\tau}$ be a nonoscillatory solution of equation (9.2.105), say, $x(k) > 0$ for $k \geq n_0 - \tau$ with $n_0 \in \mathbb{N}_0$. Let $\{z(k)\}_{k\geq N_0}$ be defined by (9.2.109), where $N_0 = \max\{n_0, \tau\}$. Then (9.2.111) holds and hence $\{z(k)\}_{k\geq N_0}$ is not decreasing and not eventually constant. Furthermore, by Theorem 9.2.5, we have

$$\lim_{k\to\infty} x(k) = 0. \tag{9.2.129}$$

Now, from (9.2.87), we obtain for $k \geq N_0$,

$$\begin{aligned} z(k) &\geq x(k) - \Big[\max_{k-\tau\leq s\leq k-\sigma} x(s)\Big] \sum_{s=k-\tau}^{k-1-\sigma} q(s) \\ &= x(k) - \Big[\max_{k-\tau\leq s\leq k-\sigma} x(s)\Big]\left(\sum_{s=k-\tau}^{k-1} q(s) - \sum_{s=k-\sigma}^{k-1} q(s)\right) \end{aligned} \tag{9.2.130}$$

and so, in view of (9.2.114) and (9.2.115), we get

$$z(k) \geq x(k) - Q(\tau-\sigma)\Big[\max_{k-\tau\leq s\leq k-\sigma} x(s)\Big] \quad \text{for } k \geq N_0. \tag{9.2.131}$$

On the other hand, (9.2.109) gives

$$z(k) \leq x(k) \quad \text{for every } k \geq N_0. \tag{9.2.132}$$

From (9.2.129), (9.2.131), and (9.2.132) it follows that

$$\lim_{k\to\infty} z(k) = 0. \tag{9.2.133}$$

Since $\{z(k)\}_{k\geq N_0}$ is decreasing and not eventually constant, (9.2.133) guarantees that $z(k) > 0$ for all $k \geq N_0$. Furthermore, in view of (9.2.132), from (9.2.111) we obtain

$$z(k+1) - z(k) + \big[p(k) - q(k)\big]z(k-\tau) \leq 0 \tag{9.2.134}$$

for all $k \geq N_0 + \tau$. Now it is easy to see that

$$\lim_{k\to\infty} \frac{1}{\tau} \sum_{s=k-\tau}^{k-1} \big[p(s) - q(s)\big] = P - Q. \tag{9.2.135}$$

As before, in view of condition (9.2.128), inequality (9.2.134) has no eventually positive solutions, a contradiction which completes the proof. □

9.2.3. Oscillation of equation (9.2.3)

The following lemma is needed.

Lemma 9.2.7. *Assume that $c \geq -1$. Let $\{x(k)\}_{k \geq n_0 - \max\{\tau, \sigma_N\} = \bar{n}}$ be a positive solution of equation (9.2.3), where $n_0 \in \mathbb{N}_0$, and set*

$$y(k) = x(k) + cx(k-\tau) \quad \text{for } k \geq n_0 + \tau - \max\{\tau, \sigma_N\}. \tag{9.2.136}$$

Then $\{y(k)\}_{k \geq \bar{n}}$ is a solution on $\mathbb{N}(n_0+\tau)$ of equation (9.2.3), which is positive and decreasing on $\mathbb{N}(n_0)$.

Proof. It is easy to see that $\{y(k)\}_{k \geq \bar{n}}$ is a solution on $\mathbb{N}(n_0+\tau)$ of equation (9.2.3). Furthermore, from equation (9.2.3), it follows that

$$\Delta y(k) = -\sum_{j=0}^{N} p_j(k)x(k-\sigma_j) \quad \text{for } k \geq n_0. \tag{9.2.137}$$

Thus $\Delta y(k) \leq 0$ for every $k \geq n_0$, which means that the sequence $\{y(k)\}_{k \geq n_0}$ is decreasing. Next, we will show that $\{y(k)\}_{k \geq n_0}$ is positive. To this end, suppose for the sake of contradiction that there exists $n_1 > n_0$ such that $y(n_1) \leq 0$. If $y(n_1) = 0$, then we can choose $n_2 > n_0$ so that $y(n_2) < 0$. Indeed, in the opposite case, we always have $y(k) = 0$ for all $k \geq n_1$, and so $\Delta y(k) = 0$ for every $k \geq n_1$ which contradicts (9.2.137). Define $n_3 = n_1$ if $y(n_1) < 0$ and $n_3 = n_2$ if $y(n_1) = 0$. Then $n_3 \geq n_0$ and $y(n_3) < 0$, and consequently we have

$$y(k) \leq -\delta \quad \text{for every } k \geq n_3, \tag{9.2.138}$$

where $\delta = -y(n_3) > 0$. Next we define

$$z(k) = y(k) + cy(k-\tau) \quad \text{for } k \geq n_0 + 2\tau - \max\{\tau, \sigma_N\}, \tag{9.2.139}$$

and observe that

$$\Delta z(k) = -\sum_{j=0}^{N} p_j(k)y(k-\sigma_j) \quad \text{for } k \geq n_0 + \tau. \tag{9.2.140}$$

Thus, by using (9.2.138), we get

$$\Delta z(k) \geq \delta \sum_{j=0}^{N} p_j(k) \quad \text{for every } k \geq n_4, \tag{9.2.141}$$

where $n_4 = \max\{n_0+\tau, n_3+\sigma_N\}$. From this it follows easily that

$$z(k) \geq z(n_4) + \delta \sum_{i=n_4}^{k-1} \left[\sum_{j=0}^{N} p_j(i)\right] \quad \forall k > n_4. \tag{9.2.142}$$

Since the sequence $\{\sum_{j=0}^{N} p_j(k)\}_{k\geq 0}$ is m-periodic and nonnegative, one can prove that inequality (9.2.142) implies that $\lim_{k\to\infty} z(k) = \infty$. Therefore there exists an integer $n_5 \geq \max\{n_1+\tau, n_0+2\tau-\max\{\tau,\sigma_N\}\}$ such that $z(k) > 0$ for all $k \geq n_5$. Hence we have for $k \geq n_5$,

$$\begin{aligned} 0 &< z(k) \\ &= y(k) + cy(k-\tau) \\ &\leq y(k-\tau) + cy(k-\tau) \\ &= (1+c)y(k-\tau) \\ &\leq (1+c)y(n_1) \\ &\leq 0, \end{aligned} \tag{9.2.143}$$

since $1+c > 0$, which is a contradiction. □

Now we present the following result.

Theorem 9.2.8. (i) *A necessary condition for oscillation of equation (9.2.3) is that there is a root λ_0 of equation (9.2.5) with the following property: if $m > 1$, then*

$$\frac{1}{1+c\lambda_0^{-\tau}} \sum_{j=0}^{N} p_j(r)\lambda_0^{-\sigma_j} < 1 \quad \text{for } r \in \{1,2,\ldots,m-1\}. \tag{9.2.144}$$

(ii) *Assume that $-1 < c \leq 0$. Then a sufficient condition for the oscillation of equation (9.2.3) is that equation (9.2.5) has no roots in the interval $((-c)^{1/\tau}, 1)$.*

Proof. For any $\lambda > 0$ with $1 + c\lambda^{-\tau} \neq 0$, we define

$$P(k,\lambda) = 1 - \frac{1}{1+c\lambda^{-\tau}} \sum_{j=0}^{N} p_j(k)\lambda^{-\sigma_j} \quad \text{for } k \geq 0. \tag{9.2.145}$$

Thus equation (9.2.5) becomes

$$\lambda^m - \prod_{r=0}^{m-1} P(r,\lambda) = 0. \tag{9.2.146}$$

Furthermore, for every $\lambda > 0$ with $1 + c\lambda^{-\tau} \neq 0$, we have

$$\prod_{r=k-\tau}^{k-1} P(r,\lambda) = \left[\prod_{r=0}^{m-1} P(r,\lambda)\right]^u \quad \text{for } k \geq \tau, \tag{9.2.147}$$

$$\prod_{r=k-\sigma_j}^{k-1} P(r,\lambda) = \left[\prod_{r=0}^{m-1} P(r,\lambda)\right]^{v_j} \quad \text{for } k \geq \sigma_j,\ j \in \{1,2,\ldots,N\}. \tag{9.2.148}$$

To see (9.2.147), for any $\lambda > 0$, one can easily find

$$\begin{aligned}\prod_{r=k-\tau}^{k-1} P(r,\lambda) &= \prod_{r=k-\tau}^{k-1}\left(1 - \frac{1}{1+c\lambda^{-\tau}}\sum_{j=0}^{N} p_j(r)\lambda^{-\sigma_j}\right)\\ &= \prod_{r=0}^{\tau-1}\left(1 - \frac{1}{1+c\lambda^{-\tau}}\sum_{j=0}^{N} p_j(r)\lambda^{-\sigma_j}\right)\\ &= \left[\prod_{r=0}^{m-1}\left(1 - \frac{1}{1+c\lambda^{-\tau}}\sum_{j=0}^{N} p_j(r)\lambda^{-\sigma_j}\right)\right]^u\\ &= \left[\prod_{r=0}^{m-1} P(r,\lambda)\right]^u.\end{aligned} \tag{9.2.149}$$

In a similar way, one can easily establish (9.2.148).

We first show (i). Suppose that equation (9.2.5) admits a root $\lambda_0 > 0$ with the following property: if $m > 1$, then (9.2.144) holds. Since $\tau = um$, we get

$$\left[\prod_{r=0}^{m-1} P(r,\lambda_0)\right]^u = \left(\lambda_0^m\right)^u = \lambda_0^{mu} = \lambda_0^{\tau} \tag{9.2.150}$$

and so, because of (9.2.147), we have

$$\prod_{r=k-\tau}^{k-1} P(r,\lambda_0) = \lambda_0^{\tau} \quad \text{for } k \geq \tau. \tag{9.2.151}$$

Similarly, by using the fact that $\sigma_j = m\nu_j$ for $j \in \{1,2,\dots,N\}$ and (9.2.148), we obtain

$$\prod_{r=k-\sigma_j}^{k-1} P(r,\lambda_0) = \lambda_0^{\sigma_j} \quad \text{for } k \geq \sigma_j,\ j \in \{1,2,\dots,N\}. \tag{9.2.152}$$

Set

$$A(k) = \prod_{r=0}^{k-1} P(r,\lambda_0) \quad \text{for } k \geq 1. \tag{9.2.153}$$

Now, for every $k \geq \tau + 1$, we obtain

$$\begin{aligned}A(k-\tau) &= \prod_{r=0}^{k-\tau-1} P(r,\lambda_0) = \left[\prod_{r=0}^{k-1} P(r,\lambda_0)\right]\left[\prod_{r=k-\tau}^{k-1} P(r,\lambda_0)\right]^{-1}\\ &= A(k)\left[\prod_{r=k-\tau}^{k-1} P(r,\lambda_0)\right]^{-1}\end{aligned} \tag{9.2.154}$$

and thus, by using (9.2.151), we conclude that

$$A(k-\tau) = \lambda_0^{-\tau} A(k) \quad \text{for } k \geq \tau + 1. \tag{9.2.155}$$

By a parallel argument, via (9.2.152) one can see that

$$A(k-\sigma_j) = \lambda_0^{-\sigma_j} A(k) \quad \text{for } k \geq \sigma_j + 1,\ j \in \{1,2,\dots,N\}. \tag{9.2.156}$$

Furthermore, we have

$$\Delta A(k) = A(k+1) - A(k) = \big[P(k,\lambda_0) - 1\big] A(k) \quad \text{for } k \geq 1. \tag{9.2.157}$$

In view of (9.2.155) and (9.2.156), we derive for every $k \geq 1 + \max\{\tau, \sigma_j\}$,

$$\begin{aligned}
\Delta\big(A(k) + cA(k-\tau)\big) &+ \sum_{j=0}^{N} p_j(k) A(k-\sigma_j) \\
&= \big(1 + c\lambda_0^{-\tau}\big)\Delta A(k) + \left[\sum_{j=0}^{N} p_j(k)\lambda_0^{-\sigma_j}\right] A(k) \\
&= \left\{\big(1 + c\lambda_0^{-\tau}\big)\big[P(k,\lambda_0) - 1\big] + \sum_{j=0}^{N} p_j(k)\lambda_0^{-\sigma_j}\right\} A(k) \\
&= 0.
\end{aligned} \tag{9.2.158}$$

Thus, the sequence $\{A(k)\}_{k\geq 1}$ is a solution on $\mathbb{N}(\max\{\tau,\sigma_N\} + 1)$ of equation (9.2.3). Next, we will show that $A(k) > 0$ for $k \geq 1$. In the case where $m = 1$, equation (9.2.5) gives $P(0,\lambda_0) = \lambda_0 > 0$. When $m > 1$, (9.2.144) ensures that $P(r,\lambda_0) > 0$ for $r \in \{1,2,\dots,m-1\}$. From equation (9.2.5) it follows that

$$P(0,\lambda_0) = \lambda_0^m \left[\prod_{r=1}^{m-1} P(r,\lambda_0)\right]^{-1} > 0. \tag{9.2.159}$$

Hence, when $m = 1$ or $m > 1$, we have $P(r,\lambda_0) > 0$ for $r \in \{0,1,\dots,m-1\}$. Finally, since sequences $\{p_j(k)\}_{k\geq 0}$ for $j \in \{0,1,\dots,N\}$ are m-periodic, we conclude that $P(r,\lambda_0) > 0$ for all $r \geq 0$, which guarantees that all terms of the sequence $\{A(k)\}_{k\geq 1}$ are positive. We have thus proved that equation (9.2.3) has a nonoscillatory positive solution.

Now we prove (ii). Assume that $-1 < c \leq 0$ and that (9.2.5) has no roots in the interval $((-c)^{1/\tau}, 1)$. (It must be noted that $1 + c\lambda^{-\tau} > 0$ for all $\lambda \in ((-c)^{1/\tau}, 1)$.) Suppose that $\{x(k)\}_{k\geq n_0-\max\{\tau,\sigma_N\}}$ is an eventually positive solution of equation (9.2.3), $n_0 \in \mathbb{N}_0$. Moreover, without loss of generality, we assume that $x(k) > 0$ for all $k \geq n_0 - \max\{\tau,\sigma_N\}$. Define $\{y(k)\}$ as in (9.2.136). By Lemma 9.2.7, the sequence $\{y(k)\}_{k\geq n_0+\tau-\max\{\tau,\sigma_N\}}$ is a solution on $\mathbb{N}(n_0+\tau)$ of equation (9.2.3), which is positive and decreasing on $\mathbb{N}(n_0)$. Define

$$C(k,0) = y(k), \qquad C(k,\nu) = C(k,\nu-1) + cC(k-\tau,\nu-1) \quad \text{for } \nu \in \mathbb{N}. \tag{9.2.160}$$

Then Lemma 9.2.7 guarantees that for every $\nu \in \mathbb{N}$ the sequence $\{C(k,\nu)\}$ is a solution of equation (9.2.3), which is eventually positive and decreasing. Furthermore, for each $\nu \in \mathbb{N}$, we define

$$\Lambda(\nu) = \{\lambda \in \big((-c)^{1/\tau}, 1\big] : C(k+1,\nu) - P(k,\lambda)C(k,\nu) \le 0\}. \tag{9.2.161}$$

First we will show that $1 \in \Lambda(\nu)$ and so $A(\nu) \ne \emptyset$ for every $\nu \in \mathbb{N}$. Indeed, let $\nu \in \mathbb{N}$ be arbitrary. Then we obtain

$$\begin{aligned} C(k,\nu) &= C(k,\nu-1) + cC(k-\tau,\nu-1) \\ &\le C(k,\nu-1) + cC(k,\nu-1) \\ &= (1+c)C(k,\nu-1), \end{aligned} \tag{9.2.162}$$

and so $C(k,\nu-1) \ge C(k,\nu)/(1+c)$. Thus equation (9.2.3) yields

$$\begin{aligned} \Delta C(k,\nu) &= \Delta\big(C(k,\nu-1) + cC(k-\tau,\nu-1)\big) \\ &= -\sum_{j=0}^{N} p_j(k)C(k-\sigma_j,\nu-1) \\ &\le -\left[\sum_{j=0}^{N} p_j(k)\right]C(k,\nu-1) \\ &\le -\frac{1}{1+c}\left[\sum_{j=0}^{N} p_j(k)\right]C(k,\nu). \end{aligned} \tag{9.2.163}$$

Hence we have

$$C(k+1,\nu) - \left[1 - \frac{1}{1+c}\sum_{j=0}^{N} p_j(k)\right]C(k,\nu) \le 0, \tag{9.2.164}$$

that is, $C(k+1,\nu) - P(k,1)C(k,\nu) \le 0$, which means that $1 \in \Lambda(\nu)$. From this it follows that

$$1 - \frac{1}{1+c}\sum_{j=0}^{N} p_j(k) \quad \text{is eventually positive.} \tag{9.2.165}$$

Since $\{p_j(k)\}_{k\ge 0}$ for $j \in \{0,1,\dots,N\}$ are periodic with common period, we have

$$1 - \frac{1}{1+c}\sum_{j=0}^{N} p_j(k) > 0 \quad \forall k > 0, \tag{9.2.166}$$

and so

$$M = \prod_{r=0}^{m-1}\left(1 - \frac{1}{1+c}\sum_{j=0}^{N} p_j(r)\right) > 0. \tag{9.2.167}$$

Now we can easily see that $M < 1$. Furthermore, we define

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1} P(r,\lambda) \quad \text{for } \lambda \in \left((-c)^{1/\tau}, 1\right]. \tag{9.2.168}$$

Then

$$F(1) = 1 - \prod_{r=0}^{m-1}\left(1 - \frac{1}{1+c}\sum_{j=0}^{N} p_j(r)\right) = 1 - M, \tag{9.2.169}$$

and so, as $0 < M < 1$, we have

$$F(1) > 0. \tag{9.2.170}$$

Next we will show that

$$F(\lambda) \longrightarrow \infty \quad \text{as } \lambda \longrightarrow \left((-c)^{1/\tau}\right)^+. \tag{9.2.171}$$

Define $I = \{r \in \{0,1,\dots,m-1\} : P_j(r) > 0 \text{ for some } j \in \{1,2,\dots,N\}\}$ and $I^* = \{0,1,\dots,m-1\} \setminus I$. From the hypotheses of the theorem, we see that $I \neq \varnothing$. If $r \in I$, then $p_{j_0}(r) > 0$ for some $j_0 \in \{1,2,\dots,N\}$ and

$$\begin{aligned} P(r,\lambda) &= 1 - \frac{1}{1+c\lambda^{-\tau}}\sum_{j=0}^{N} p_j(r)\lambda^{-\sigma_j} \\ &\le 1 - \frac{1}{1+c\lambda^{-\tau}} p_{j_0}(r)\lambda^{-\sigma_{j_0}} \\ &\longrightarrow -\infty \quad \text{as } \lambda \longrightarrow \left((-c)^{1/\tau}\right)^+. \end{aligned} \tag{9.2.172}$$

Therefore

$$P(r,\lambda) \longrightarrow -\infty \quad \text{as } \lambda \longrightarrow \left((-c)^{1/\tau}\right)^+ \ \forall r \in I. \tag{9.2.173}$$

On the other hand if $I^* \neq \varnothing$ and $r \in I^*$, then $p_j(r) = 0$ for all $j \in \{1,2,\dots,N\}$ and hence for every $\lambda \in ((-c)^{1/\tau}, 1]$,

$$P(r,\lambda) = 1 - \frac{1}{1+c\lambda^{-\tau}}\sum_{j=0}^{N} p_j(r)\lambda^{-\sigma_j} = 1 - \frac{1}{1+c\lambda^{-\tau}} p_0(r), \tag{9.2.174}$$

that is,

$$
\begin{aligned}
P(r,\lambda) &= 1 \quad \text{if } p_0(r) = 0, \\
P(r,\lambda) &= 1 - p_0(r) = 1 - \sum_{j=0}^{N} p_j(r) > 0 \quad \text{if } c = 0,\ p_0(r) > 0, \\
P(r,\lambda) &\longrightarrow -\infty \quad \text{as } \lambda \longrightarrow \left((-c)^{1/\tau}\right)^+ \text{ if } -1 < c < 0,\ p_0(r) > 0.
\end{aligned}
\tag{9.2.175}
$$

We have established that if $I^* \neq \emptyset$ and $r \in I^*$, then as $\lambda \to ((-c)^{1/\tau})^+$, $P(r,\lambda)$ tends to a positive real number or to $-\infty$. Thus, by combining this fact and (9.2.173), we conclude that $F(\lambda) \to \pm\infty$ as $\lambda \to ((-c)^{1/\tau})^+$. If $F(\lambda) \to -\infty$ as $\lambda \to ((-c)^{1/\tau})^+$, then (9.2.170) implies that the equation $F(\lambda) = 0$ has a root in the interval $((-c)^{1/\tau}, 1)$, which is a contradiction. Hence (9.2.171) is always satisfied. From (9.2.170) and (9.2.171) and the hypothesis that equation (9.2.5) has no roots in the interval $((-c)^{1/\tau}, 1)$, it follows that $\mu = \min\{F(\lambda) : \lambda \in ((-c)^{1/\tau}, 1)\}$ exists and is positive. Obviously we have

$$
\prod_{r=0}^{m-1} P(r,\lambda) = \lambda^m - \mu \quad \forall \lambda \in ((-c)^{1/\tau}, 1). \tag{9.2.176}
$$

The proof can now be completed by proving that for any $\nu \in \mathbb{N} \setminus \{1\}$ it holds that $\lambda \in \Lambda(\nu - 1)$ implies $(\lambda^m - \mu)^{1/m} \in \Lambda(\nu)$. Let $\nu \in \mathbb{N} \setminus \{1\}$. Consider an arbitrary number $\lambda \in \Lambda(\nu - 1)$. Then $C(k+1, \nu - 1) - P(k,\lambda)C(k, \nu - 1) \leq 0$, which ensures that $P(k,\lambda) > 0$ (for all large k). Since the sequences $\{p_j(k)\}_{k\geq 0}$ for $j \in \{0, 1, \ldots, N\}$ are m-periodic, we must have

$$
P(r,\lambda) > 0 \quad \text{for } r = 0, 1, \ldots, m-1, \tag{9.2.177}
$$

and hence from (9.2.176) it follows that $\eta = (\lambda^m - \mu)^{1/m} > 0$. Obviously $\sigma < 1$. Now we have $C(k+1, \nu-1) - P(k,\lambda)C(k, \nu-1) \leq 0$. Thus, by taking into account (9.2.147), (9.2.176), and (9.2.177), we get

$$
\begin{aligned}
\frac{C(k-\tau, \nu-1)}{C(k, \nu-1)} &\geq \left[\prod_{r=k-\tau}^{k-1} P(r,\lambda)\right]^{-1} = \left[\prod_{r=0}^{m-1} P(r,\lambda)\right]^{-u} \\
&\geq (\lambda^m - \mu)^{-u} = \left[(\lambda^m - \mu)^{1/m}\right]^{-um} = \eta^{-\tau},
\end{aligned}
\tag{9.2.178}
$$

that is,

$$
C(k-\tau, \nu-1) \geq \eta^{-\tau} C(k, \nu-1). \tag{9.2.179}
$$

In a similar way, by using (9.2.148), (9.2.176), and (9.2.177), we obtain

$$C(k-\sigma_j, \nu-1) \geq \eta^{-\sigma_j} C(k, \nu-1) \quad \text{for } j \in \{0, 1, \dots, N\}. \tag{9.2.180}$$

In view of (9.2.179), we have

$$\begin{aligned} C(k, \nu) &= C(k, \nu-1) + cC(k-\tau, \nu-1) \\ &\leq C(k, \nu-1) + c\eta^{-\tau} C(k, \nu-1) \\ &= (1 + c\eta^{-\tau}) C(k, \nu-1), \end{aligned} \tag{9.2.181}$$

and hence $1 + c\eta^{-\tau} > 0$, that is, $\eta \in ((-c)^{1/\tau}, 1)$ and in addition

$$C(k, \nu-1) \geq \frac{1}{1 + c\eta^{-\tau}} C(k, \nu). \tag{9.2.182}$$

Finally, by virtue of (9.2.180) and (9.2.182), from equation (9.2.3) we obtain

$$\begin{aligned} 0 &= \Delta\big(C(k, \nu-1) + cC(k-\tau, \nu-1)\big) + \sum_{j=0}^{N} p_j(k) C(k-\sigma_j, \nu-1) \\ &= \Delta C(k, \nu) + \sum_{j=0}^{N} p_j(k) C(k-\sigma_j, \nu-1) \\ &\geq \Delta C(k, \nu) + \left[\sum_{j=0}^{N} p_j(k)\eta^{-\sigma_j}\right] C(k, \nu-1) \\ &\geq \Delta C(k, \nu) + \frac{1}{1 + c\eta^{-\tau}} \left[\sum_{j=0}^{N} p_j(k)\eta^{-\sigma_j}\right] C(k, \nu) \\ &= C(k+1, \nu) - \left(1 - \frac{1}{1 + c\eta^{-\tau}} \sum_{j=0}^{N} p_j(k)\eta^{-\sigma_j}\right) C(k, \nu), \end{aligned} \tag{9.2.183}$$

that is, $C(k+1, \nu) - P(k, \eta)C(k, \nu) \leq 0$, and hence $\eta \in \Lambda(\nu)$. This completes the proof. □

9.2.4. Oscillation of equation (9.2.4)

The following lemma is needed.

Lemma 9.2.9. *Assume that $c > -1$. Let $\{x(k)\}_{k \geq n_0}$ be a positive solution on $\mathbb{N}(n_0)$ of equation (9.2.4) where $n_0 \in \mathbb{N}_0$ and set*

$$y(k) = x(k) + cx(k+\tau) \quad \text{for } k \geq n_0. \tag{9.2.184}$$

Then $\{y(k)\}_{k \geq n_0}$ is an increasing solution on $\mathbb{N}(n_0)$ of equation (9.2.4), which is eventually positive.

Proof. It is obvious that $\{y(k)\}_{k\geq n_0}$ is a solution of equation (9.2.4). Moreover, from equation (9.2.4) it follows that

$$\Delta y(k) = q_0(k)x(k) + \sum_{j\in J} q_j(k)x(k+\sigma_j^*) \quad \text{for } k \geq n_0. \tag{9.2.185}$$

Thus $\Delta y(k) \geq 0$ for $k \geq n_0$, which means that the solution $\{y(k)\}_{k\geq n_0}$ is increasing. It now remains to show that $y(k) > 0$ for all large k. For this purpose, we will assume that

$$y(k) < 0 \quad \text{for every } k \geq n_0. \tag{9.2.186}$$

Set $z(k) = y(k) + cy(k+\tau)$ for $k \geq n_0$. Then we have

$$\Delta z(k) = q_0(k)y(k) + \sum_{j\in J} q_j(k)y(k+\sigma_j^*) \quad \text{for } k \geq n_0. \tag{9.2.187}$$

Thus, because of (9.2.186), $\Delta z(k) \leq 0$ for all $k \geq n_0$, and consequently the sequence $\{z(k)\}_{k\geq n_0}$ is decreasing. Furthermore, we obtain for $k \geq n_0$,

$$z(k) = y(k) + cy(k+\tau) \leq y(k+\tau) + cy(k+\tau) = (1+c)y(k+\tau), \tag{9.2.188}$$

and hence, in view of (9.2.186), we must have $z(k) < 0$ for $k \geq n_0$. Thus,

$$z(k) \leq -\varepsilon \quad \forall k \geq n_0, \tag{9.2.189}$$

where $\varepsilon = -z(n_0) > 0$. Now we put

$$h(k) = z(k) + cz(k+\tau) \quad \text{for } k \geq n_0, \tag{9.2.190}$$

and see that

$$\Delta h(k) = q_0(k)z(k) + \sum_{j\in J} q_j(k)z(k+\sigma_j^*) \quad \text{for } k \geq n_0. \tag{9.2.191}$$

From this it follows by (9.2.189) that

$$\Delta h(k) \leq -\varepsilon \sum_{j\in J} q_j(k) \quad \text{for every } k \geq n_0, \tag{9.2.192}$$

which gives

$$h(k) \leq h(n_0) - \varepsilon \sum_{i=n_0}^{k-1} \left[\sum_{j\in J} q_j(i)\right] \quad \text{for } k > n_0. \tag{9.2.193}$$

As the sequence $\{\sum_{j\in J} q_j(k)\}_{k\geq 0}$ is positive and m-periodic, we conclude that

$$\lim_{k\to\infty} h(k) = -\infty. \tag{9.2.194}$$

We see that $\beta = \lim_{k\to\infty} y(k)$ exists in $(-\infty, 0]$. We have

$$\begin{aligned}
&\lim_{k\to\infty} z(k) = \beta + c\beta = (1+c)\beta, \\
&\lim_{k\to\infty} h(k) = (1+c)\beta + c(1+c)\beta = (1+c)^2\beta \in (-\infty, 0],
\end{aligned} \tag{9.2.195}$$

which contradicts (9.2.194). This contradiction shows that (9.2.186) fails. Thus there exists an integer $n_1 \geq n_0$ such that $y(n_1) \geq 0$. Then it follows that $y(k) > 0$ for all $k \geq n_1$. If $y(k) = 0$ for every $k \geq n_1$, then $\Delta y(k) = 0$ for $k \geq n_1$, which contradicts (9.2.185) since the sequences $\{q_j(k)\}_{k\geq 0}$ for $j \in J$ are supposed to be not identically zero. Hence we always have $y(n_2) > 0$ for some $n_2 \geq n_1$. So $y(k) > 0$ for all $k \geq n_2$, and the proof is now complete. □

Next, we give the following oscillation criterion for equation (9.2.4).

Theorem 9.2.10. (i) *A necessary condition for the oscillation of equation (9.2.4) is that there is no positive root of equation (9.2.6) with* $1 + c\lambda_0^\tau > 0$.

(ii) *Assume that* $-1 < c \leq 0$. *Then a sufficient condition for the oscillation of equation (9.2.4) is that equation (9.2.6) has no roots in the interval* $(1, 1/(-c)^\tau)$ *(which is the interval* $(1, \infty)$ *when* $c = 0$*).*

Proof. For any $\lambda > 0$ with $1 + c\lambda^\tau \neq 0$, we define

$$G(k,\lambda) = 1 + \frac{1}{1+c\lambda^\tau}\left[q_0(k) + \sum_{j\in J} q_j(k)\lambda^{\sigma_j^*}\right] \quad \text{for } k \geq 0. \tag{9.2.196}$$

Then equation (9.2.6) becomes

$$\lambda^m - \prod_{r=0}^{m-1} G(r,\lambda) = 0. \tag{9.2.197}$$

Moreover, we have

$$\prod_{r=k}^{k+\tau-1} G(r,\lambda) = \left[\prod_{r=0}^{m-1} G(r,\lambda)\right]^u \quad \text{for } k \geq 0, \tag{9.2.198}$$

$$\prod_{r=k}^{k+\sigma_j^*-1} G(r,\lambda) = \left[\prod_{r=0}^{m-1} G(r,\lambda)\right]^{w_j} \quad \text{for } k \geq 0,\ j \in J. \tag{9.2.199}$$

Indeed, if $\lambda > 0$ is such that $1 + c\lambda^{-\tau} \neq 0$, then we take into account the fact that the sequences $\{q_0(k)\}_{k\geq 0}$ and $\{q_j(k)\}_{k\geq 0}$ for $j \in J$ are m-periodic and that $\tau = um$

to obtain for every $k \geq 0$,

$$\begin{aligned}\prod_{r=k}^{k+\tau-1} G(r,\lambda) &= \prod_{r=k}^{k+\tau-1}\left(1+\frac{1}{1+c\lambda^{\tau}}\left[q_0(r)+\sum_{j\in J}q_j(r)\lambda^{\sigma_j^*}\right]\right)\\ &= \prod_{r=0}^{\tau-1}\left(1+\frac{1}{1+c\lambda^{\tau}}\left[q_0(r)+\sum_{j\in J}q_j(r)\lambda^{\sigma_j^*}\right]\right)\\ &= \left\{\prod_{r=0}^{m-1}\left(1+\frac{1}{1+c\lambda^{\tau}}\left[q_0(r)+\sum_{j\in J}q_j(r)\lambda^{\sigma_j^*}\right]\right)\right\}^{u}\\ &= \left[\prod_{r=0}^{m-1} G(r,\lambda)\right]^{u}.\end{aligned} \tag{9.2.200}$$

By the same procedure, (9.2.199) can be proved.

Now we show (i). Assume that equation (9.2.6) has a positive root λ_0 such that $1+c\lambda_0^{\tau} > 0$. Then

$$\left[\prod_{r=0}^{m-1} G(r,\lambda_0)\right]^{u} = \left(\lambda_0^{m}\right)^{u} = \lambda_0^{um} = \lambda_0^{\tau}, \tag{9.2.201}$$

and for any $j \in J$,

$$\left[\prod_{r=0}^{m-1} G(r,\lambda_0)\right]^{w_j} = \left(\lambda_0^{m}\right)^{w_j} = \lambda_0^{w_j m} = \lambda_0^{\sigma_j^*}. \tag{9.2.202}$$

Thus (9.2.198) and (9.2.199) give, respectively,

$$\prod_{r=k}^{k+\tau-1} G(r,\lambda_0) = \lambda_0^{\tau} \quad \text{for } k \geq 0, \tag{9.2.203}$$

$$\prod_{r=k}^{k+\sigma_j^*-1} G(r,\lambda_0) = \lambda_0^{\sigma_j^*} \quad \text{for } k \geq 0,\ j \in J. \tag{9.2.204}$$

Define $A(k) = \prod_{r=0}^{k-1} G(r,\lambda_0)$ for $k \geq 0$. Clearly $A(k) > 0$ for $k \geq 1$. Furthermore, we obtain for $k \geq 1$,

$$\begin{aligned}A(k+\tau) &= \prod_{r=0}^{k+\tau-1} G(r,\lambda_0) = \left[\prod_{r=0}^{k-1} G(r,\lambda_0)\right]\left[\prod_{r=k}^{k+\tau-1} G(r,\lambda_0)\right]\\ &= A(k)\prod_{r=k}^{k+\tau-1} G(r,\lambda_0),\end{aligned} \tag{9.2.205}$$

and in view of (9.2.203), we have

$$A(k+\tau) = \lambda_0^{\tau} A(k) \quad \text{for } k \geq 1. \tag{9.2.206}$$

In a similar way, by using (9.2.204) we can conclude that

$$A(k+\sigma_j^*) = \lambda_0^{\sigma_j^*} A(k) \quad \text{for } k \geq 1,\ j \in J. \tag{9.2.207}$$

On the other hand it holds that

$$\Delta A(k) = A(k+1) - A(k) = \left[G(k,\lambda_0) - 1\right]A(k) \quad \text{for } k \geq 1. \tag{9.2.208}$$

Thus, by (9.2.206) and (9.2.207), we obtain for all $k \geq 1$,

$$\begin{aligned}
&\Delta\big(A(k) + cA(k+\tau)\big) - q_0(k)A(k) - \sum_{j\in J} q_j(k)A(k+\sigma_j^*) \\
&\quad = (1 + c\lambda_0^\tau)\Delta A(k) - q_0(k)A(k) - \left[\sum_{j\in J} q_j(k)\lambda_0^{\sigma_j^*}\right]A(k) \\
&\quad = \left\{(1+c\lambda_0^\tau)\left[G(k,\lambda_0) - 1\right] - q_0(k) - \sum_{j\in J} q_j(k)\lambda_0^{\sigma_j^*}\right\}A(k) \\
&\quad = 0,
\end{aligned} \tag{9.2.209}$$

which means that the sequence $\{A(k)\}_{k\geq 1}$ is a solution on $\mathbb{N}(1)$ of equation (9.2.4). So it has been established that equation (9.2.4) admits an eventually positive solution.

Next we prove (ii). Assume that $-1 < c \leq 0$ and that there exists a nonoscillatory solution $\{x(k)\}_{k\geq n_0}$ of equation (9.2.4), where $n_0 \in \mathbb{N}_0$. Moreover, suppose for the sake of contradiction that equation (9.2.6) has no roots in the interval $(1, 1/(-c)^{1/\tau})$. Without loss of generality, we assume that $x(k) > 0$ for $k \geq n_0$. By Lemma 9.2.9, the sequence $\{y(k)\}_{k\geq n_0}$ defined by (9.2.184) is an increasing solution on $\mathbb{N}(n_0)$ of (9.2.4), which is eventually positive. Set $C(k,0) = y(k)$ and $C(k,\nu) = C(k,\nu-1) + cC(k+\tau,\nu-1)$ for $\nu \in \mathbb{N}$. Then Lemma 9.2.9 ensures that for each $\nu \in \mathbb{N}$ the sequence $\{C(k,\nu)\}$ is a solution of equation (9.2.4), which is eventually positive and increasing. Furthermore, for any $\nu \in \mathbb{N}$ we define

$$\Lambda(\nu) = \left\{\lambda \in \left[1, 1/(-c)^{1/\tau}\right) : C(k+1,\nu) - G(k,\lambda)C(k,\nu) \geq 0\right\}. \tag{9.2.210}$$

(Note that $1/(-c)^{1/\tau} = \infty$ if $c = 0$.) For any $\nu \in \mathbb{N}$, we have $1 \in \Lambda(\nu)$ and so $\Lambda(\nu) \not\equiv \varnothing$. Indeed, consider an arbitrary $\nu \in \mathbb{N}$. Then we have

$$\begin{aligned}
C(k,\nu) &= C(k,\nu-1) + cC(k+\tau,\nu-1) \\
&\leq C(k,\nu-1) + cC(k,\nu-1) \\
&= (1+c)C(k,\nu-1),
\end{aligned} \tag{9.2.211}$$

and consequently $C(k,\nu-1) \geq C(k,\nu)/(1+c)$. Thus, from equation (9.2.4), we have

$$\begin{aligned}
\Delta C(k,\nu) &= \Delta\big(C(k,\nu-1)+cC(k+\tau,\nu-1)\big) \\
&= q_0(k)C(k,\nu-1)+\sum_{j\in J} q_j(k)C(k+\sigma_j^*,\nu-1) \\
&\geq q_0(k)C(k,\nu-1)+\left[\sum_{j\in J} q_j(k)\right]C(k,\nu-1) \\
&= \left[q_0(k)+\sum_{j\in J} q_j(k)\right]C(k,\nu-1) \\
&\geq \frac{1}{1+c}\left[q_0(k)+\sum_{j\in J} q_j(k)\right]C(k,\nu),
\end{aligned} \tag{9.2.212}$$

and hence

$$C(k+1,\nu)-\left(1+\frac{1}{1+c}\left[q_0(k)+\sum_{j\in J} q_j(k)\right]\right)C(k,\nu) \geq 0, \tag{9.2.213}$$

that is, $C(k+1,\nu)-G(k,1)G(k,\nu) \geq 0$. This means that $1 \in \Lambda(\nu)$. Set

$$F(\lambda) = \lambda^m - \prod_{r=0}^{m-1} G(r,\lambda) \quad \text{for } \lambda \in \big[1, 1/(-c)^{1/\tau}\big). \tag{9.2.214}$$

The sequence $\{q_0(k)\}_{k\geq 0}$ is nonnegative and the sequence $\{\sum_{j\in J} q_j(k)\}_{k\geq 0}$ is positive. Hence we have

$$F(1) = 1-\prod_{r=0}^{m-1}\left(1+\frac{1}{1+c}\left[q_0(r)+\sum_{j\in J} q_j(r)\right]\right) < 0, \tag{9.2.215}$$

that is,

$$F(1) < 0. \tag{9.2.216}$$

In the particular case where $c=0$, one can see that the following statement is true (see [233]).

$$\begin{aligned}
&\text{For any } \nu \in \mathbb{N}, \text{ the set } \Lambda(\nu) \text{ is bounded from above} \\
&\text{by a positive real number which is independent of } \nu.
\end{aligned} \tag{9.2.217}$$

If $-1 < c < 0$, then we obtain for $\lambda \in [1, 1/(-c)^{1/\tau})$,

$$\begin{aligned} F(\lambda) &= \lambda^m - \prod_{r=0}^{m-1}\left(1 + \frac{1}{1+c\lambda^\tau}\left[q_0(r) + \sum_{j\in J} q_j(r)\lambda^{\sigma_j^*}\right]\right) \\ &< \left[\frac{1}{(-c)^{1/\tau}}\right]^m - \prod_{r=0}^{m-1}\left(1 + \frac{1}{1+c\lambda^\tau}\left[\sum_{j\in J} q_j(r)\right]\right). \end{aligned} \tag{9.2.218}$$

Thus

$$F(\lambda) \longrightarrow \infty \quad \text{as } \lambda \longrightarrow \left(\frac{1}{(-c)^{1/\tau}}\right)^-. \tag{9.2.219}$$

Using (9.2.216) and (9.2.217) if $c = 0$ and (9.2.216) and (9.2.219) if $c \in (-1, 0)$ and by using the fact that the equation $F(\lambda) = 0$ has no roots in the interval $(1, 1/(-c)^{1/\tau})$, we conclude that there exists a positive real number γ such that $F(\lambda) \le -\gamma$ for all $\lambda \in \Lambda(\nu)$, $\nu \in \mathbb{N}$ (the number γ is independent of ν). Hence we have

$$\prod_{r=0}^{m-1} G(r,\lambda) \ge \lambda^m + \gamma \quad \forall \lambda \in \Lambda(\nu),\ \nu \in \mathbb{N}. \tag{9.2.220}$$

The proof will be accomplished by establishing that for all $\nu \in \mathbb{N}\setminus\{1\}$ it holds that

$$\lambda \in \Lambda(\nu - 1) \Longrightarrow (\lambda^m + \gamma)^{1/m} \in \Lambda(\nu). \tag{9.2.221}$$

Consider an arbitrary $\nu \in \mathbb{N}\setminus\{1\}$ and an arbitrary number $\lambda \in \Lambda(\nu - 1)$. We set $\theta = (\lambda^m + \gamma)^{1/m} > 1$ and we will show $\theta \in \Lambda(\nu)$. To this end, we observe that

$$C(k+1, \nu - 1) \ge G(k,\lambda)C(k, \nu - 1). \tag{9.2.222}$$

Thus, by (9.2.198) and (9.2.220), we obtain

$$\begin{aligned} \frac{C(k+\tau, \nu-1)}{C(k, \nu-1)} &\ge \prod_{r=k}^{k+\tau-1} G(r,\lambda) = \left[\prod_{r=0}^{m-1} G(r,\lambda)\right]^u \\ &\ge (\lambda^m + \gamma)^u = \left[(\lambda^m + \gamma)^{1/m}\right]^{um} = \theta^\tau, \end{aligned} \tag{9.2.223}$$

that is,

$$C(k+\tau, \nu - 1) \ge \theta^\tau C(k, \nu - 1). \tag{9.2.224}$$

Similarly, by using (9.2.199) and (9.2.220), we can derive

$$C(k+\sigma_j^*, \nu-1) \geq \theta^{\sigma_j^*} C(k, \nu-1) \quad \text{for } j \in J. \tag{9.2.225}$$

In view of (9.2.224), we get

$$\begin{aligned} C(k,\nu) &= C(k,\nu-1) + cC(k+\tau, \nu-1) \\ &\leq C(k,\nu-1) + c\theta^{\tau} C(k,\nu-1) \\ &= (1+c\theta^{\tau}) C(k,\nu-1), \end{aligned} \tag{9.2.226}$$

and consequently $1 + c\theta^{\tau} > 0$, that is, $\theta \in (1, 1/(-c)^{1/\tau})$, and in addition

$$C(k,\nu-1) \geq \frac{1}{1+c\theta^{\tau}} C(k,\nu). \tag{9.2.227}$$

Finally, taking into account (9.2.225) and (9.2.227), from equation (9.2.4) we obtain

$$\begin{aligned} 0 &= \Delta\big(C(k,\nu-1) + cC(k+\tau,\nu-1)\big) - q_0(k)C(k,\nu-1) \\ &\quad - \sum_{j\in J} q_j(k) C(k+\sigma_j^*, \nu-1) \\ &= \Delta C(k,\nu) - q_0(k)C(k,\nu-1) - \sum_{j\in J} q_j(k)C(k+\sigma_j^*,\nu-1) \\ &\leq \Delta C(k,\nu) - \left[q_0(k) + \sum_{j\in J} q_j(k)\theta^{\sigma_j^*}\right] C(k,\nu-1) \\ &\leq \Delta C(k,\nu) - \frac{1}{1+c\theta^{\tau}}\left[q_0(k) + \sum_{j\in J} q_j(k)\theta^{\sigma_j^*}\right] C(k,\nu) \\ &= C(k+1,\nu) - \left(1 + \frac{1}{1+c\theta^{\tau}}\left[q_0(k) + \sum_{j\in J} q_j(k)\theta^{\sigma_j^*}\right]\right) C(k,\nu), \end{aligned} \tag{9.2.228}$$

that is,

$$C(k+1,\nu) - G(k,\theta)C(k,\nu) \geq 0 \tag{9.2.229}$$

and hence $\theta \in \Lambda(\nu)$. This completes the proof. □

9.3. Linearized oscillations for difference equations

We will consider the nonlinear difference equations

$$x(k+1) - x(k) + f(x(k-\tau_1), \dots, x(k-\tau_m)) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.3.1}$$

$$x(k+1) - x(k) + F(k, x(k-\tau_1), \dots, x(k-\tau_m)) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.3.2}$$

where

(i) $\tau_1, \dots, \tau_m \in \mathbb{N}_0$, $\tau = \max\{\tau_1, \dots, \tau_m\}$,

(ii) $f : \mathbb{R}^m \to \mathbb{R}$ is continuous with

$$f(u_1, \dots, u_m) \begin{cases} \geq 0 & \text{for } u_1, \dots, u_m \geq 0, \\ \leq 0 & \text{for } u_1, \dots, u_m \leq 0, \end{cases} \qquad f(u, \dots, u) = 0 \Longleftrightarrow u = 0, \tag{9.3.3}$$

(iii) $F : \mathbb{N}_0 \times \mathbb{R}^m \to \mathbb{R}$ is continuous for every fixed k and satisfies

$$F(k, u_1, \dots, u_m) \begin{cases} \geq 0 & \text{for } u_1, \dots, u_m \geq 0, \\ \leq 0 & \text{for } u_1, \dots, u_m \leq 0. \end{cases} \tag{9.3.4}$$

In this section we will obtain necessary and sufficient conditions for the oscillation of all solutions of equation (9.3.1) in terms of the oscillation of all solutions of an associated linear difference equation. Also, we establish a linearized oscillation result for the more general equation (9.3.2).

By a solution of equation (9.3.1) (resp., (9.3.2)) we mean a sequence $\{x(k)\}$ which is defined for $k \geq -\tau$ and which satisfies equation (9.3.1) (resp., (9.3.2)) for $k \in \mathbb{N}_0$. Let $a(-\tau), a(-\tau+1), \dots, a(0)$ be given numbers. Then it is easily seen that equation (9.3.1) (resp., (9.3.2)) has a unique solution $\{x(k)\}$ which satisfies the initial conditions $x(i) = a(i)$ for $i \in \{-\tau, \dots, 0\}$.

9.3.1. Linearized oscillations for equation (9.3.1)

We will assume that the following hypothesis holds.

(iv) there exists $\delta > 0$ such that f has continuous first-order partial derivatives $D_i f$ for all $u_1, \dots, u_m \in [-\delta, \delta]$ satisfying

$$D_i f(0, \dots, 0) = q_i \quad \text{for } i \in \{1, 2, \dots, m\}, \tag{9.3.5}$$

with

$$q_1, \dots, q_m \in \mathbb{R}^+, \qquad \sum_{i=1}^{m} (q_i + \tau_i) \neq 1. \tag{9.3.6}$$

Furthermore, either

$$f(u_1,\dots,u_m) \le \sum_{i=1}^{m} q_i u_i \quad \text{for } u_1,\dots,u_m \in [0,\delta] \tag{9.3.7}$$

or

$$f(u_1,\dots,u_m) \ge \sum_{i=1}^{m} q_i u_i \quad \text{for } u_1,\dots,u_m \in [-\delta,0]. \tag{9.3.8}$$

The following two lemmas will be needed in the proofs of the main result of this subsection.

Lemma 9.3.1. *Consider the difference inequality*

$$x(k+1) - x(k) + \sum_{i=1}^{m} Q_i(k)x(k-\tau_i) \le 0 \quad \textit{for } k \in \mathbb{N}_0, \tag{9.3.9}$$

and the difference equation

$$y(k+1) - y(k) + \sum_{i=1}^{m} q_i y(k-\tau_i) = 0 \quad \textit{for } k \in \mathbb{N}_0, \tag{9.3.10}$$

where $\liminf_{k\to\infty} Q_i(k) \ge q_i > 0$ *and* $\tau_i \in \mathbb{N}_0$ *for* $i \in \{1,2,\dots,m\}$ *are such that* $\sum_{i=1}^{m}(q_i+\tau_i) \ne 1$. *Suppose that inequality (9.3.9) has an eventually positive solution. Then equation (9.3.10) also has an eventually positive solution.*

Lemma 9.3.2. *Let* $\{x(k)\}$ *be a solution of the difference inequality*

$$x(k+1) - x(k) + \sum_{i=1}^{m} q_i x(k-\tau_i) \ge 0 \quad \textit{for } k \in \{0,1,\dots,N_1-1\} \tag{9.3.11}$$

with the initial conditions $x(k) = \theta\lambda_0^k$ *for* $k \in \{-\tau,-\tau+1,\dots,0\}$, *where* $q_i \in \mathbb{R}^+$ *and* $\tau_i \in \mathbb{N}_0$ *for* $i \in \mathbb{N}$ *and such that* $\sum_{i=1}^{n}(q_i+\tau_i) \ne 1$, $N_1 \in \mathbb{N}$, $\theta \in \mathbb{R}^+$, *and* λ_0 *is a positive root of the equation*

$$\lambda - 1 + \sum_{i=1}^{m} q_i \lambda^{-\tau_i} = 0. \tag{9.3.12}$$

Then

$$x(k) \ge \theta\lambda_0^k \quad \textit{for } k \in \{1,2,\dots,N_1\}. \tag{9.3.13}$$

With equation (9.3.1) we associate the linear equation (9.3.10) and its corresponding characteristic equation (9.3.12). Now we prove the following result.

Theorem 9.3.3. *Assume that conditions (i), (ii), and (iv) hold. Then every solution of equation (9.3.1) oscillates if and only if every solution of the associated linear equation (9.3.10) oscillates.*

Proof. Suppose that every solution of equation (9.3.1) oscillates and that, for the sake of contradiction, equation (9.3.10) has a positive solution $\{y(k)\}$. Assume that (9.3.7) holds. The case where (9.3.8) holds is similar and will be omitted. Then equation (9.3.12) has a positive root λ_0. As $q_i > 0$ for $i \in \{1, 2, \dots, m\}$, it follows that $\lambda_0 \in (0, 1)$. Let $\{x(k)\}$ be the unique solution of equation (9.3.1) with initial conditions $x(k) = \theta\lambda_0^k$ for $k \in \{-\tau, -\tau+1, \dots, 0\}$, where $\tau = \max\{\tau_1, \dots, \tau_m\}$ and $\theta = \delta\lambda_0^\tau$. We claim that

$$x(k) > 0 \quad \text{for } k \in \mathbb{N}. \tag{9.3.14}$$

To this end, assume that this is false. Thus there exists $N_1 \in \mathbb{N}$ such that $x(k) > 0$ for $-\tau \le k \le N_1 - 1$ and $x(N_1) \le 0$. Then from equation (9.3.1) we see that $x(k+1) < x(k)$ for $0 \le k \le N_1 - 1$, and so

$$0 < x(k) < x(0) = \theta = \delta\lambda_0^\tau < \delta \quad \text{for } 0 \le k \le N_1 - 1. \tag{9.3.15}$$

By using (9.3.7), we obtain

$$x(k+1) - x(k) + \sum_{i=1}^{m} q_i x(k - \tau_i) \ge 0 \quad \text{for } k \in \{0, 1, \dots, N_1 - 1\}. \tag{9.3.16}$$

By Lemma 9.3.2, this implies that $x(N_1) \ge \theta\lambda_0^{N_1} > 0$, and this contradiction completes the proof of the first part of the theorem.

Conversely, assume that every solution of equation (9.3.10) oscillates. Otherwise, equation (9.3.1) has a nonoscillatory solution $\{x(k)\}$. We will assume that $\{x(k)\}$ is eventually positive. The case where $\{x(k)\}$ is eventually negative is similar and will be omitted. Then one can easily see that

$$\lim_{k\to\infty} x(k) = 0. \tag{9.3.17}$$

Now observe that in view of (ii) and the mean value theorem for functions of m variables

$$\begin{aligned} &f(x(k-\tau_1), \dots, x(k-\tau_m)) - f(0, \dots, 0) \\ &\quad = \sum_{i=1}^{m} D_i f(\theta x(k-\tau_1), \dots, \theta x(k-\tau_m)) x(k-\tau_i), \end{aligned} \tag{9.3.18}$$

where $\theta \in (0, 1)$. Set

$$Q_i(k) = D_i f(\theta x(k-\tau_1), \dots, \theta x(k-\tau_m)) \quad \text{for } i \in \{1, \dots, m\},\ k \in \mathbb{N}_0. \tag{9.3.19}$$

Then from (9.3.5) and (9.3.17) and by the continuity of the partial derivatives of f,

$$\lim_{k\to\infty} Q_i(k) = q_i \quad \text{for } i \in \{1,2,\dots,m\}. \tag{9.3.20}$$

Therefore equation (9.3.1) can be written in the form

$$x(k+1) - x(k) + \sum_{i=1}^{m} Q_i(k)x(k-\tau_i) = 0. \tag{9.3.21}$$

In view of (9.3.20), we see that the hypotheses of Lemma 9.3.1 are satisfied, and hence the associated linearized equation (9.3.10) has an eventually positive solution. This is a contradiction and the proof of the theorem is complete. □

From the second part of the proof of Theorem 9.3.3 it follows that a sufficient (but not necessary) condition for the oscillation of equation (9.3.1) is as given in the following corollary.

Corollary 9.3.4. *Assume that (i) and (ii) hold and that there exists $\delta > 0$ such that f has continuous first-order partial derivatives $D_i f$ for $u_1,\dots,u_m \in [-\delta,\delta]$ such that (9.3.5) and (9.3.6) are satisfied. Suppose also that the characteristic equation (9.3.12) of the associated linearized equation (9.3.10) has no positive roots. Then every solution of equation (9.3.1) oscillates.*

9.3.2. Linearized oscillations for equation (9.3.2)

The following lemma will be needed in the proof of the main result of this subsection.

Lemma 9.3.5. *Assume that there exist $N_1 \geq 0$ and $\delta > 0$ such that for $k \geq N_1$ and for $u_1,\dots,u_m \in [0,\delta]$,*

(v) *$F(k,u_1,\dots,u_m)$ is positive for $(u_1,\dots,u_m) \neq 0$ and F is increasing in $u_1,\dots,u_m$ in the sense that if $u_i \leq \overline{u}_i$ for all $i \in \{1,2,\dots,m\}$, then we have $F(k,u_1,\dots,u_m) \leq F(k,\overline{u}_1,\dots,\overline{u}_m)$.*

Suppose that for $k \geq N_1$ the difference inequality

$$y(k+1) - y(k) + F\big(k, y(k-\tau_1),\dots,y(k-\tau_m)\big) \leq 0 \quad \textit{for } k \in \mathbb{N}_0 \tag{9.3.22}$$

has an eventually positive solution $\{y(k)\}$ with $y(k) \leq \delta$. Then equation (9.3.2) has an eventually positive solution $\{x(k)\}$ with $x(k) \leq y(k)$ for all k sufficiently large.

Proof. Let $\tau = \max\{\tau_1, \dots, \tau_m\}$ and set $N = N_1 + \tau$. Then

$$\{y(k)\} \quad \text{is strictly decreasing for } k \ge N - \tau, \tag{9.3.23}$$

and so $\lim_{k\to\infty} y(k) = \ell \in [0, \infty)$ exists. Summing both sides of inequality (9.3.22) from k to s and letting $s \to \infty$, we see that

$$\ell + \sum_{j=k}^{\infty} F\big(j, y(j-\tau_1), \dots, y(j-\tau_m)\big) \le y(k) \quad \text{for } k \ge N. \tag{9.3.24}$$

Now we define the nonnegative sequences

$$A = \big\{\{z(k)\} : 0 \le z(k) \le y(k),\ k \ge N\big\}, \tag{9.3.25}$$

and for each $\{z(k)\} \in A$, define an associated sequence $\{Y(k)\}_{k=N-\tau}^{\infty}$ by

$$Y(k) = \begin{cases} z(k) & \text{for } k \ge N, \\ z(N) + y(k) - y(N) & \text{for } N - \tau \le k < N. \end{cases} \tag{9.3.26}$$

From (9.3.23) and (9.3.25), we see that

$$0 \le Y(k) \le y(k) \quad \text{for } k \ge N - \tau, \tag{9.3.27}$$

$$Y(k) > 0 \quad \text{for } N - \tau \le k < N. \tag{9.3.28}$$

Now we define the mapping T on A by

$$(Tz)(k) = \ell + \sum_{j=k}^{\infty} F\big(j, Y(j-\tau_1), \dots, Y(j-\tau_m)\big) \quad \text{for } k \ge N. \tag{9.3.29}$$

In view of (v), we see that if $\{z(k)\}, \{\bar{z}(k)\} \in A$ with $z(k) \le \bar{z}(k)$ for $k \ge N$, then $(Tz)(k) \le (T\bar{z})(k)$ for $k \ge N$. Note that by (9.3.24), $(Ty)(k) \le y(k)$ for $k \ge N$. Hence for any $\{z(k)\} \in A$, $(Tz)(k) \le (Ty)(k) \le y(k)$ for $k \ge N$, and so $T : A \to A$. Now consider the following sequences: $\{x(k,0)\} = \{y(k)\}$ and $\{x(k,i)\} = \{(Tx)(k, i-1)\}$ for $i \in \mathbb{N}$. Then one can see by induction that for $k \ge N$,

$$0 \le x(k, i+1) \le x(k,i) \le y(k) \quad \text{for } i \in \mathbb{N}. \tag{9.3.30}$$

Thus

$$x(k) = \lim_{i\to\infty} x(k,i) \quad \text{for } k \geq N \tag{9.3.31}$$

exists and $\{x(k)\} \in A$. Also,

$$x(k) = \ell + \sum_{j=k}^{\infty} F(j, x(j-\tau_1), \ldots, x(j-\tau_m)) \quad \text{for } k \geq N, \tag{9.3.32}$$

and so $\{x(k)\}$ satisfies equation (9.3.2) for $k \geq N$. Now we claim that

$$x(k) > 0 \quad \text{for } k \geq N. \tag{9.3.33}$$

First assume that $\ell > 0$. Then from (9.3.32) it is clear that (9.3.33) holds. Next assume that $\ell = 0$. In view of (9.3.28), if (9.3.33) were false, then there would exist some $N_2 \geq N$ such that $x(N_2) = 0$ and $x(k) > 0$ for $N - \tau \leq k < N_2$. But from (9.3.32) and (v),

$$x(N_2) = \sum_{j=N_2}^{\infty} F(j, x(j-\tau_1), \ldots, x(j-\tau_m)) > 0. \tag{9.3.34}$$

This contradiction implies that (9.3.33) holds. Clearly, $\{x(k)\}$ can be extended as a solution of equation (9.3.2). Finally, from (9.3.30) and (9.3.31), we see that $x(k) \leq y(k)$ for $k \geq N$. This completes the proof. □

Similarly, we can establish the following dual result of Lemma 9.3.5.

Lemma 9.3.6. *Let there exist $N_1 \geq 0$ and $\delta > 0$ such that for $u_1, \ldots, u_m \in [-\delta, 0]$, $F(k, u_1, \ldots, u_m)$ is negative for $(u_1, \ldots, u_m) \neq 0$ and increasing. Suppose also that for $k \geq N_1$ the difference inequality*

$$y(k+1) - y(k) + F(k, y(k-\tau_1), \ldots, y(k-\tau_m)) \geq 0 \quad \textit{for } k \in \mathbb{N}_0 \tag{9.3.35}$$

has an eventually negative solution $\{y(k)\}$ with $y(k) \geq -\delta$. Then equation (9.3.2) has an eventually negative solution $\{x(k)\}$ with $x(k) \geq y(k)$ for k sufficiently large.

Also, we will need the following lemma.

Lemma 9.3.7. *Consider the linear delay difference equations*

$$a(k+1) - a(k) + \sum_{i=1}^{m} p_i(k) a(k-\tau_i) = 0, \tag{9.3.36}$$

$$b(k+1) - b(k) + \sum_{i=1}^{m} q_i(k) b(k-\tau_i) = 0, \tag{9.3.37}$$

where for each $i \in \{1, 2, \ldots, m\}$, $\{p_i(k)\}$, $\{q_i(k)\}$ *are nonnegative sequences and* $\tau_i \in \mathbb{N}_0$. *Let*

$$Z_i = \{k \in \mathbb{N}_0 : p_i(k) = 0\}, \qquad \overline{Z}_i = \{k \in \mathbb{N}_0 : q_i(k) = 0\}. \tag{9.3.38}$$

Assume that for each $i \in \{1, 2, \ldots, m\}$ *the sets* Z_i *and* $\overline{Z}_i$ *are equal and that*

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}_0\setminus Z_i}} \frac{p_i(k)}{q_i(k)} = 1. \tag{9.3.39}$$

Then every solution of equation (9.3.36) oscillates if and only if every solution of equation (9.3.37) oscillates.

The following result gives sufficient conditions for the oscillation of equation (9.3.2) in terms of the oscillatory behavior of an associated linear difference equation with variable coefficients of the form

$$y(k+1) - y(k) + \sum_{i=1}^{m} q_i(k) y(k-\tau_i) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{9.3.40}$$

Theorem 9.3.8. *Assume that in every interval* $[a, b]$ *with* $ab > 0$,

$$\sum_{k=0}^{\infty} c_k = \infty, \quad \text{where } c(k) = \min_{u_1,\ldots,u_m \in [a,b]} \left| F(k, u_1, \ldots, u_m) \right|, \tag{9.3.41}$$

and that there exist $\delta > 0$ *and nonnegative sequences* $\{q_1(k)\}, \ldots, \{q_m(k)\}$ *such that one of the following hypotheses (9.3.42) or (9.3.43) holds:*

$$F(k, u_1, \ldots, u_m) \begin{cases} \geq \sum_{i=1}^{m} q_i(k) u_i > 0 & \text{for } u_1, \ldots, u_m \in (0, \delta], \\ \leq \sum_{i=1}^{m} q_i(k) u_i < 0 & \text{for } u_1, \ldots, u_m \in [-\delta, 0), \end{cases} \tag{9.3.42}$$

$$\lim_{\substack{(u_1,\ldots,u_m)\to 0 \\ u_i u_j > 0,\ i,j \in \{1,2,\ldots,m\}}} \frac{F(k, u_1, \ldots, u_m)}{q_1(k) u_1 + \cdots + q_m(k) u_m} \equiv 1. \tag{9.3.43}$$

Suppose that equation (9.3.40) is oscillatory. Then equation (9.3.2) is also oscillatory.

Proof. Suppose that equation (9.3.2) has a nonoscillatory solution $\{x(k)\}$. We assume that $\{x(k)\}$ is eventually positive. The proof when $\{x(k)\}$ is eventually negative is similar and will be omitted. Choose $N \geq 0$ such that $x(k-\tau) > 0$ for $k \geq N$, where $\tau = \max\{\tau_1, \ldots, \tau_m\}$. From equation (9.3.2) we see that $\{x(k)\}$ is decreasing for $k \geq N$, and so $\lim_{k\to\infty} x(k) = \ell \in [0, \infty)$ exists. We claim that $\ell = 0$.

Otherwise $\ell > 0$. Then by summing both sides of equation (9.3.2) from N to s and letting $s \to \infty$, we have

$$\ell - x(N) + \sum_{j=N}^{\infty} F(j, x(j-\tau_1), \dots, x(j-\tau_m)) = 0, \tag{9.3.44}$$

which clearly contradicts (9.3.41). Hence $\lim_{k\to\infty} x(k) = 0$.

Now we first assume that (9.3.42) holds. Then it follows from equation (9.3.2) and (9.3.42) that for $k \geq N$,

$$x(k+1) - x(k) + \sum_{i=1}^{m} q_i(k)x(k-\tau_i) \leq 0. \tag{9.3.45}$$

By Lemma 9.3.5 with $F(k, u_1, \dots, u_m) = \sum_{i=1}^{m} q_i(k)u_i$, we see that equation (9.3.40) has an eventually positive solution. This is a contradiction and so the proof is complete when (9.3.42) holds.

Next assume that (9.3.43) holds. Set

$$Q_i(k) = q_i(k)\frac{F(k, x(k-\tau_1), \dots, x(k-\tau_m))}{q_1(k)x(k-\tau_1) + \cdots + q_m(k)x(k-\tau_m)}. \tag{9.3.46}$$

Then, in view of the hypotheses, for each $i \in \{1, 2, \dots, m\}$ with $Q_i(k) \geq 0$ the sequences $\{q_i(k)\}$ and $\{Q_i(k)\}$ have the same set of zeros Z_i and

$$\lim_{\substack{k\to\infty \\ k\in\mathbb{N}_0\setminus Z_i}} \frac{q_i(k)}{Q_i(k)} = 1. \tag{9.3.47}$$

Observe that $\{x(k)\}$ is a nonoscillatory solution of the equation

$$z(k+1) - z(k) + \sum_{i=1}^{m} Q_i(k)z(k-\tau_i) = 0. \tag{9.3.48}$$

Then it follows by Lemma 9.3.7 that equation (9.3.40) has a nonoscillatory solution. This contradicts the hypothesis and therefore completes the proof of the theorem. □

Remark 9.3.9. The hypothesis (9.3.43) can be replaced by the following condition, which is stronger but easier to check.

$$\begin{gathered}\text{There is a positive number } \delta > 0 \text{ such that the partial derivatives} \\ D_iF(k, u_1, \dots, u_m),\ i \in \{1, 2, \dots, m\}, \text{ are continuous for } k \in \mathbb{N}_0 \\ \text{and } u_1, \dots, u_m \in [-\delta, \delta] \text{ and } D_iF(k, 0, \dots, 0) = q_i(k),\ i \in \{1, \dots, m\}.\end{gathered} \tag{9.3.49}$$

The following result gives sufficient conditions for the existence of a nonoscillatory solution of equation (9.3.2).

Theorem 9.3.10. *Assume that there exist $\delta > 0$ and nonnegative sequences $\{q_i(k)\}$, $i \in \{1, \dots, m\}$, such that*

$$\begin{aligned} 0 &< F(k, u_1, \dots, u_m) \\ &\leq \sum_{i=1}^{m} q_i(k) u_i \text{ and } F \text{ is increasing in } u_1, \dots, u_m \quad \text{for } u_1, \dots, u_m \in (0, \delta] \end{aligned} \tag{9.3.50}$$

or

$$\begin{aligned} 0 &> F(k, u_1, \dots, u_m) \\ &\geq \sum_{i=1}^{m} q_i(k) u_i \text{ and } F \text{ is increasing in } u_1, \dots, u_m \quad \text{for } u_1, \dots, u_m \in [-\delta, 0). \end{aligned} \tag{9.3.51}$$

Assume that equation (9.3.40) has a nonoscillatory solution. Then equation (9.3.2) also has a nonoscillatory solution.

Proof. Suppose that (9.3.50) holds. The proof when (9.3.51) holds is similar and will be omitted. As the negative of a solution of equation (9.3.40) is also a solution, we will assume that equation (9.3.40) has an eventually positive solution $\{\overline{y}(k)\}$. Choose $N \geq 0$ such that $\overline{y}(k - \tau) > 0$ for $k \geq N$. Then from equation (9.3.40), we see that $\{\overline{y}(k)\}$ is decreasing for $k \geq N$, and so $\{\overline{y}(k)\}$ is bounded. Therefore, for M sufficiently large, $y(k) = \overline{y}(k)/M \leq \delta$ for $k \geq N$. Clearly, $\{y(k)\}$ is also a positive solution of equation (9.3.40) for $k \geq N$. From equation (9.3.40) and (9.3.50), it follows that

$$y(k+1) - y(k) + F(k, y(k - \tau_1), \dots, y(k - \tau_m)) \leq 0 \quad \text{for } k \geq N. \tag{9.3.52}$$

Then, by Lemma 9.3.5, equation (9.3.2) has an eventually positive solution, and the proof is complete. □

Combining Theorems 9.3.8 and 9.3.10 we obtain the following linearized oscillation result for equation (9.3.2).

Theorem 9.3.11. *Assume that condition (9.3.41) holds and that there exist a positive number δ and nonnegative sequences $\{q_1(k)\}, \dots, \{q_m(k)\}$ such that condition (9.3.43) holds and such that either (9.3.50) or (9.3.51) is satisfied. Then equation (9.3.2) is oscillatory if and only if equation (9.3.40) is oscillatory.*

The following result about difference equations with separable delays is an immediate consequence of Theorem 9.3.11.

Corollary 9.3.12. *Consider the delay difference equation*

$$x(k+1) - x(k) + \sum_{i=1}^{m} q_i(k) f_i(x(k - \tau_i)) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.3.53}$$

where for $i \in \{1,2,\dots,m\}$, $\{q_i(k)\}$ are nonnegative sequences, $f_i \in C(\mathbb{R},\mathbb{R})$ such that $u f_i(u) > 0$ for $u \neq 0$ and $\lim_{u\to 0} f_i(u)/u = 1$. Assume that

$$\sum_{i=1}^{m}\sum_{k=0}^{\infty} q_i(k) = \infty, \quad \sum_{i=1}^{m} \tau_i q_i(k) > 0 \quad \text{for } k \in \mathbb{N}_0. \tag{9.3.54}$$

Furthermore, suppose that there exists $\delta > 0$ such that for all $i \in \{1,2,\dots,m\}$ either $f_i(u) \le u$ and f_i is increasing for $u \in (0,\delta)$, or $f_i(u) \ge u$ and f_i is increasing for $u \in (-\delta,0)$. Then equation (9.3.2) is oscillatory if and only if equation (9.3.40) is oscillatory.

9.4. Oscillation of recursive sequences

Consider the more general difference equation

$$x(k+1) = F\big(x(k), x(k-1), \dots, x(k-\tau)\big), \tag{9.4.1}$$

where $\tau \in \mathbb{N}$ and $F \in C(I^{\tau+1},\mathbb{R})$ with $I \subseteq \mathbb{R}$.

Definition 9.4.1. (a) A sequence $\{x(k)\}$ is said to *oscillate* if the terms are neither eventually all positive nor eventually all negative. Otherwise, the sequence is called *nonoscillatory*. A sequence $\{x(k)\}$ is called *strictly oscillatory* if for every $n_0 \ge 0$ there exist $n_1, n_2 \ge n_0$ such that $x(n_1)x(n_2) < 0$.

(b) A sequence $\{x(k)\}$ is said to oscillate about $\overline{x}$ if the sequence $\{x(k) - \overline{x}\}$ oscillates. The sequence $\{x(k)\}$ is called strictly oscillatory about $\overline{x}$ if the sequence $\{x(k) - \overline{x}\}$ is strictly oscillatory.

(c) A sequence $\{x(k)\}$ is said to oscillate about the sequence $\{y(k)\}$ if the sequence $\{x(k) - y(k)\}$ oscillates.

When we talk about the oscillation of a solution of equation (9.4.1) about $\overline{x}$, we will assume that $\overline{x}$ is an equilibrium point of (9.4.1), that is, $\overline{x} = F(\overline{x},\dots,\overline{x})$.

A sequence which oscillates about zero consists of a "string" of nonnegative terms followed by a string of negative terms, or vice versa, and so forth. We will call these strings positive and negative semicycles, respectively.

When we study the oscillation of a solution about $\overline{x}$, the *semicycles* are defined relative to $\overline{x}$ and consist of strings of terms greater than or equal to $\overline{x}$ followed by strings of terms below $\overline{x}$, and so forth. More precisely, we make the following definitions about the semicycles of a solution $\{x(k)\}$ of equation (9.4.1) relative to $\overline{x}$.

Definition 9.4.2 (semicycles). A *positive semicycle* of a solution $\{x(k)\}$ of equation (9.4.1) consists of a string of terms $\{x(\ell), x(\ell+1), \dots, x(m)\}$, all greater than or equal to $\overline{x}$ with $\ell \ge -\tau$ and $m \le \infty$, and such that either $\ell = -\tau$, or $\ell > -\tau$ and $x(\ell-1) < \overline{x}$; and either $m = \infty$, or $m < \infty$ and $x(m+1) < \overline{x}$.

A *negative semicycle* of a solution $\{x(k)\}$ of equation (9.4.1) consists of a string of terms $\{x(\ell), x(\ell+1), \dots, x(m)\}$, all less than $\overline{x}$ with $\ell \geq -\tau$ and $m \leq \infty$, and such that either $\ell = -\tau$, or $\ell > -\tau$ and $x(\ell - 1) \geq \overline{x}$; and either $m = \infty$, or $m < \infty$ and $x(m+1) \geq \overline{x}$.

The first semicycle of a solution of equation (9.4.1) starts with the term $x(-\tau)$ and is positive if $x(-k) \geq \overline{x}$ and is negative if $x(-k) < \overline{x}$. A solution may have a finite number of semicycles or infinitely many semicycles.

Let $\overline{x}$ be an equilibrium point of equation (9.4.1). Then the solution $\{x(k)\}$ of equation (9.4.1) with $x(k) = \overline{x}$ for all $k \geq -\tau$ is called a "trivial solution." A trivial solution has only one semicycle, namely, the positive semicycle $\{\overline{x}, \dots, \overline{x}\}$. This semicycle is called a "trivial semicycle."

We will employ the linearization results presented in Section 9.3 to establish some oscillation criteria for certain recursive sequences which are special cases of equation (9.4.1).

9.4.1. Oscillation of $x(k+1) = ax(k)/[1 + bx(k-\tau)]$

Here we will study the oscillation of all positive solutions of the delay difference equation

$$x(k+1) = \frac{ax(k)}{1 + bx(k-\tau)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.2}$$

where

$$a \in (1, \infty), \qquad b \in \mathbb{R}^+, \qquad \tau \in \mathbb{N}_0. \tag{9.4.3}$$

If $\alpha(-\tau), \dots, \alpha(0)$ are $\tau + 1$ given constants, then equation (9.4.2) has a unique solution satisfying the initial conditions

$$x(i) = \alpha(i) \quad \text{for } i \in \{-\tau, -\tau+1, \dots, 0\}. \tag{9.4.4}$$

If the initial values are such that

$$\alpha(i) \geq 0 \quad \text{for } i \in \{-\tau, -\tau+1, \dots, -1\}, \qquad \alpha(0) > 0, \tag{9.4.5}$$

then the unique solution of the IVP (9.4.2) and (9.4.4) is positive for $k \geq 0$.

The following is a necessary and sufficient condition for the oscillation of equation (9.4.2).

Theorem 9.4.3. *Assume that condition (9.4.2) holds. Then every positive solution of equation (9.4.2) oscillates about its positive equilibrium $\overline{x} = (a-1)/b$ if and only if*

$$\frac{a-1}{a} > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}. \tag{9.4.6}$$

Proof. The change of variable $x(k) = [(a-1)/b]e^{y(k)}$ transforms equation (9.4.2) into the difference equation

$$y(k+1) - y(k) + \frac{a-1}{a} f\big(y(k-\tau)\big) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.7}$$

where

$$f(u) = \frac{a}{a-1} \ln \frac{(a-1)e^u + 1}{a}. \tag{9.4.8}$$

Clearly, every solution of equation (9.4.7) oscillates about zero if and only if every solution of equation (9.4.2) oscillates about $\overline{x} = (a-1)/b$. One can easily see now that all the hypotheses of Theorem 9.3.3 are satisfied for the difference equation (9.4.7). In particular, note that $f(u) \geq u$ for $u < 0$. The linearized equation associated with equation (9.4.7) is

$$z(k+1) - z(k) + \frac{a-1}{a} z(k-\tau) = 0. \tag{9.4.9}$$

It is known that every solution of this equation oscillates if and only if (9.4.6) holds. The proof is now an elementary consequence of Theorem 9.3.3. □

Next, we will consider the more general difference equation

$$x(k+1) = \frac{ax(k)}{1 + \sum_{i=1}^m b_i x(k-\tau_i)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.10}$$

where

$$a \in (1, \infty), \qquad b_i \in \mathbb{R}^+, \quad \tau_i \in \mathbb{N}_0 \quad \text{for } i \in \{1, 2, \ldots, m\}. \tag{9.4.11}$$

We will obtain conditions for the oscillation of all positive solutions of equation (9.4.10) about its positive equilibrium $\overline{x} = (a-1)/\sum_{i=1}^m b_i$. We will need the following lemma which provides a useful inequality for the proof of Theorem 9.4.6 below.

Lemma 9.4.4. *Assume that $a > 0$ and let $c_i \in \mathbb{R}^+$ for $i \in \{1, 2, \ldots, m\}$ be such that $\sum_{i=1}^m c_i = 1$. Then*

$$\ln\left(\frac{1}{a}\left[1 + (a-1)\sum_{i=1}^m c_i e^{u_i}\right]\right) \geq \frac{a-1}{a} \sum_{i=1}^m c_i u_i \tag{9.4.12}$$

for all $u_i \in \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$.

Proof. The proof is by induction on m. First, we consider $m = 1$. Let f be defined for any $u \in \mathbb{R}$ by

$$f(u) = \ln\left(\frac{1}{a}\big[1 + (a-1)e^u\big]\right) - \frac{a-1}{a} u. \tag{9.4.13}$$

Then

$$f'(u) = \frac{(a-1)e^u}{1+(a-1)e^u} - \frac{a-1}{a}, \tag{9.4.14}$$

and so

$$f'(u) \geq 0 \iff u \geq 0. \tag{9.4.15}$$

From this we see since $f(0) = 0$ that $f(u) \geq 0$ for all $u \in \mathbb{R}$. Thus (9.4.12) holds for $m = 1$.

Next, we assume that the statement of Lemma 9.4.4 is true for $m \in \mathbb{N}$. We will show that it is also true for $m+1$. For $i \in \{1, 2, \ldots, m-1\}$, set $c_i^* = c_i$ and $u_i^* = u_i$. Also, let

$$c_m^* = c_m + c_{m+1}, \qquad u_m^* = \ln\left(\frac{1}{c_m^*}\left[c_m e^{u_m} + c_{m+1}e^{u_{m+1}}\right]\right). \tag{9.4.16}$$

Since we are assuming that $\sum_{i=1}^{m+1} c_i = 1$, we have $\sum_{i=1}^{m} c_i^* = 1$. Thus, by the induction hypothesis,

$$\ln\left(\frac{1}{a}\left[1 + (a-1)\sum_{i=1}^{m} c_i^* e^{u_i^*}\right]\right) \geq \frac{a-1}{a}\sum_{i=1}^{m} c_i^* u_i^*, \tag{9.4.17}$$

or equivalently,

$$\begin{aligned} &\ln\left(\frac{1}{a}\left[1 + (a-1)\sum_{i=1}^{m+1} c_i e^{u_i}\right]\right) \\ &\quad \geq \frac{a-1}{a}\left[\sum_{i=1}^{m-1} c_i u_i + c_m^* \ln\left(\frac{1}{c_m^*}\left[c_m e^{u_m} + c_{m+1}e^{u_{m+1}}\right]\right)\right]. \end{aligned} \tag{9.4.18}$$

Now, since ln is a concave function and

$$\frac{c_m}{c_m^*} + \frac{c_{m+1}}{c_m^*} = 1, \tag{9.4.19}$$

we have

$$\ln\left[\frac{c_m}{c_m^*}e^{u_m} + \frac{c_{m+1}}{c_m}e^{u_{m+1}}\right] \geq \frac{c_m}{c_m^*}u_m + \frac{c_{m+1}}{c_m^*}u_{m+1}. \tag{9.4.20}$$

Combining this and (9.4.18), we obtain

$$\ln\left(\frac{1}{a} + \left[1 + (a-1)\sum_{i=1}^{m+1} c_i e^{u_i}\right]\right) \geq \frac{a-1}{a}\sum_{i=1}^{m+1} c_i u_i, \tag{9.4.21}$$

and the proof is complete. $\square$

Also, we will need the following lemma.

Lemma 9.4.5. *Assume that $p, q \in \mathbb{R}^+$ and $\tau \in \mathbb{N}$. Then every solution of the difference equation*

$$x(k+1) - qx(k) + px(k-\tau) = 0 \tag{9.4.22}$$

oscillates if and only if

$$p > \frac{\tau^\tau}{(\tau+1)^{\tau+1}} q^{\tau+1}. \tag{9.4.23}$$

Proof. The characteristic equation associated with equation (9.4.22) is

$$F(\lambda) = \lambda^{\tau+1} - q\lambda^\tau + p = 0. \tag{9.4.24}$$

Since $F(0) > 0$, $F(\infty) = \infty$, and $F(\lambda)$ has only one critical point $\overline{\lambda} = q\tau/(\tau+1)$, it follows that

$$\min\{F(\lambda) : \lambda \in [0, \infty)\} = F(\overline{\lambda}). \tag{9.4.25}$$

Thus $F(\lambda)$ has no positive roots if and only if $F(\overline{\lambda}) > 0$. This is equivalent to condition (9.4.23). Now the proof is completed by applying Lemma 8.3.1. □

Now we present the following result.

Theorem 9.4.6. *Assume that condition (9.4.11) holds. Then every positive solution of equation (9.4.10) oscillates about its positive equilibrium $\overline{x} = (a-1)/\sum_{i=1}^m b_i$ if and only if every solution of the linear difference equation*

$$z(k+1) - z(k) + \frac{\overline{x}}{a}\sum_{i=1}^m b_i z(k-\tau_i) = 0 \tag{9.4.26}$$

oscillates.

Proof. The change of variable $x(k) = \overline{x}e^{y(k)}$ transforms equation (9.4.10) into the difference equation

$$y(k+1) - y(k) + f\big(y(k-\tau_1), \dots, y(k-\tau_m)\big) = 0, \tag{9.4.27}$$

where

$$f(u_1, \dots, u_m) = \ln\left(\frac{1}{a}\left[1 + \overline{x}\sum_{i=1}^m b_i e^{u_i}\right]\right) = \ln\left(\frac{1}{a}\left[1 + (a-1)\sum_{i=1}^m b_i^* e^{u_i}\right]\right) \tag{9.4.28}$$

with

$$b_i^* = \frac{b_i}{\sum_{i=1}^m b_i} \quad \text{for } i \in \{1, 2, \dots, m\}. \tag{9.4.29}$$

Observe that by Lemma 9.4.4,

$$f(u_1, \dots, u_m) \geq \frac{a-1}{a} \sum_{i=1}^{m} b_i^* u_i. \tag{9.4.30}$$

One can easily see that the hypotheses of Theorem 9.3.3 are satisfied and that the linear equation associated with equation (9.4.27) is equation (9.4.26). The proof of the theorem is therefore a consequence of Theorem 9.3.3. □

The following corollary extends Theorem 9.4.6.

Corollary 9.4.7. *Assume that $a \in (1, \infty)$, $b_1, b_2 \in \mathbb{R}^+$, and $\tau \in \mathbb{N}_0$. Then every positive solution of the difference equation*

$$x(k+1) = \frac{ax(k)}{1 + b_1 x(k) + b_2 x(k-\tau)} \quad \textit{for } k \in \mathbb{N}_0 \tag{9.4.31}$$

oscillates about its positive equilibrium $\overline{x} = (a-1)/(b_1+b_2)$ if and only if

$$\frac{a^\tau (a-1) b_2 \left(b_1 + b_2\right)^\tau}{\left(ab_2 + b_1\right)^{\tau+1}} > \frac{\tau^\tau}{(\tau+1)^{\tau+1}}. \tag{9.4.32}$$

The second corollary of Theorem 9.4.6 provides sufficient conditions for the oscillation of all positive solutions of equation (9.4.10). It is a direct application of Lemma 9.4.5 and Theorem 9.4.6.

Corollary 9.4.8. *Assume that (9.4.11) holds. Then every positive solution of equation (9.4.10) oscillates about its positive equilibrium $\overline{x}$ provided that one of the following conditions holds:*

$$\begin{gathered} \frac{\overline{x}}{a} \sum_{i=1}^{m} b_i \frac{\left(\tau_i + 1\right)^{\tau_i+1}}{\tau_i^{\tau_i}} > 1, \\ \frac{\overline{x}}{a} \left[\prod_{i=1}^{m} b_i\right]^{i/m} \frac{(\nu+1)^{\nu+1}}{\nu^\nu} > 1, \quad \textit{where } \nu = \frac{1}{m} \sum_{i=1}^{m} \tau_i. \end{gathered} \tag{9.4.33}$$

9.4.2. Oscillation of $x(k+1) = [a + bx(k)]/[c + x(k-\tau)]$

Here we will investigate the oscillatory behavior of positive solutions of the recursive sequence

$$x(k+1) = \frac{a + bx(k)}{c + x(k-\tau)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.34}$$

where

$$a, b, c \in \mathbb{R}^+, \qquad \tau \in \mathbb{N}. \tag{9.4.35}$$

Equation (9.4.34) has a unique positive equilibrium $\overline{x}$ which is the positive root of the equation

$$\overline{x} = \frac{a + b\overline{x}}{c + \overline{x}}, \tag{9.4.36}$$

and is given by

$$\overline{x} = \frac{b - c + \sqrt{(b-c)^2 + 4a}}{2}. \tag{9.4.37}$$

The next two theorems provide sufficient conditions for the oscillation of all positive solutions of equation (9.4.34) about the positive equilibrium $\overline{x}$.

Theorem 9.4.9. *Assume that (9.4.35) holds and that*

$$\overline{x}(c + \overline{x})^\tau > \frac{1}{\tau}\left(\frac{b\tau}{\tau + 1}\right)^{\tau+1}, \tag{9.4.38}$$

where $\overline{x}$ is given by (9.4.37). Then every positive solution of equation (9.4.34) oscillates about $\overline{x}$.

Proof. Assume for the sake of contradiction that equation (9.4.34) has a positive solution $\{x(k)\}$ which is not oscillatory about $\overline{x}$. Then for some $n_0 \geq 0$ either

$$x(k) > \overline{x} \quad \text{for } k \geq n_0 \tag{9.4.39}$$

or

$$x(k) < \overline{x} \quad \text{for } k \geq n_0, \tag{9.4.40}$$

and in either case

$$\lim_{k\to\infty} x(k) = \overline{x}. \tag{9.4.41}$$

We will assume that (9.4.39) holds. The case where (9.4.40) holds is similar and will be omitted. Set

$$x(k) = z(k) + \overline{x} \quad \text{for } k \geq -\tau. \tag{9.4.42}$$

Then equation (9.4.34) yields

$$z(k+1) - z(k) + \frac{c + \overline{x} - b}{c + x(k-\tau)}z(k) + \frac{x(k)}{c + x(k-\tau)}z(k-\tau) = 0, \tag{9.4.43}$$

which shows that $\{z(k)\}$ is an eventually positive solution of the equation

$$y(k+1) - y(k) + P(k)y(k) + Q(k)y(k-\tau) = 0, \tag{9.4.44}$$

where

$$P(k) = \frac{c + \overline{x} - b}{c + x(k-\tau)}, \qquad Q(k) = \frac{x(k)}{c + x(k-\tau)}. \tag{9.4.45}$$

Note from (9.4.37) that $\overline{x} > b - c$ and so, because of (9.4.41) the limits

$$\lim_{k\to\infty} P(k) = \frac{c + \overline{x} - b}{c + \overline{x}}, \qquad \lim_{k\to\infty} Q(k) = \frac{\overline{x}}{c + \overline{x}} \tag{9.4.46}$$

exist and are positive numbers. By applying Lemma 9.3.1 we see that the equation

$$\lambda - 1 + \frac{c + \overline{x} - b}{c + \overline{x}} + \frac{\overline{x}}{c + \overline{x}}\lambda^{-\tau} = 0, \tag{9.4.47}$$

or equivalently,

$$F(\lambda) = \lambda^{\tau+1} - \frac{b}{c + \overline{x}}\lambda^{\tau} + \frac{\overline{x}}{c + \overline{x}} = 0, \tag{9.4.48}$$

has a positive root. By computing the extreme of the polynomial F we see that $\min\{F(\lambda) : \lambda > 0\} > 0$ if and only if (9.4.38) holds. Then equation (9.4.48) cannot have a positive root. This is a contradiction and the proof is complete. $\square$

We will need the following lemma.

Lemma 9.4.10. *Assume that (9.4.35) holds and that $c > b$. The positive equilibrium of equation (9.4.34) is asymptotically stable.*

Theorem 9.4.11. *Assume that*

$$a > bc. \tag{9.4.49}$$

Then every nontrivial positive solution of equation (9.4.34) is strictly oscillatory about the positive equilibrium $\overline{x}$. Furthermore, a semicycle of such a solution has at most $2\tau + 1$ terms.

Proof. Assume for the sake of contradiction that equation (9.4.34) has a nontrivial positive solution $\{x(k)\}$ which is not strictly oscillatory about $\overline{x}$. Then for some $n_0 \geq 0$, either

$$x(k) \geq \overline{x} \quad \text{for } k \geq n_0 \tag{9.4.50}$$

or

$$x(k) \leq \overline{x} \quad \text{for } k \geq n_0. \tag{9.4.51}$$

Suppose that (9.4.50) holds. The case where (9.4.51) holds is similar and will be omitted. By Lemma 9.4.10, $\{x(k)\}$ is decreasing for $k \geq n_0 + \tau$. Also, because the solution is nontrivial, $x(n_0 + \tau) > \bar{x}$. Now observe that because of (9.4.49) the function $g(x) = [a + bx]/[c + x]$ is strictly decreasing. Therefore

$$\begin{aligned}\bar{x} &= g(\bar{x}) > g\big(x(n_0+\tau)\big)\\ &= \frac{a+bx(n_0+\tau)}{c+x(n_0+\tau)} \geq \frac{a+bx(n_0+2\tau)}{c+x(n_0+\tau)}\\ &= x(n_0+2\tau+1).\end{aligned} \tag{9.4.52}$$

This is a contradiction. Therefore, when (9.4.50) holds, every positive solution of equation (9.4.34) is strictly oscillatory.

The above analysis also shows that no semicycle may contain more than $2\tau+1$ terms. This completes the proof. □

Next, we will consider some special cases of the recursive sequence (9.4.34). First, we consider the equation

$$x(k+1) = \frac{bx(k)}{c+x(k-\tau)} \quad \text{for } k \in \mathbb{N}_0 \tag{9.4.53}$$

(i.e., equation (9.4.34) with $a = 0$), where $b, c \in \mathbb{R}^+$ and $\tau \in \mathbb{N}$. Equation (9.4.53) has the equilibrium points 0 and $b - c$.

Now we state the following known oscillation criterion for equation (9.4.53).

Theorem 9.4.12. *Assume that $b > c > 0$ and $\tau \in \mathbb{N}$. Then every positive solution of equation (9.4.53) oscillates about the equilibrium $b - c$ if and only if*

$$\frac{b}{c} > \frac{1}{1-\big[\tau/(\tau+1)\big]^{\tau+1}}. \tag{9.4.54}$$

Next we consider the recursive sequence

$$x(k+1) = \frac{bx(k)}{x(k-\tau)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.55}$$

where $b > 0$ and $\tau \in \mathbb{N}$. Equation (9.4.55) is the equation (9.4.34) with $a = c = 0$. Now, the change of variables $x(k) = be^{y(k)}$ reduces equation (9.4.55) to the linear difference equation

$$y(k+1) - y(k) + y(k-\tau) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{9.4.56}$$

The characteristic equation of equation (9.4.56) is

$$\lambda^{\tau+1} - \lambda^{\tau} + 1 = 0. \tag{9.4.57}$$

When $\tau = 1$, the characteristic roots of equation (9.4.56) are two 6th roots of unity. In this case, we give the following oscillation result for equation (9.4.55).

Theorem 9.4.13. *If $\tau = 1$, then every positive solution of equation (9.4.55) is periodic with period* 6 *and oscillates about the positive equilibrium $\overline{x} = b$ of equation (9.4.55).*

We note that if $\tau \geq 2$, then equation (9.4.57) has a root λ_1 with $|\lambda_1| > 1$, and in this case we see that equation (9.4.55) is unstable.

Another special case of equation (9.4.34) is the equation

$$x(k+1) = \frac{a}{x(k-\tau)} \quad \text{for } k \in \mathbb{N}_0. \tag{9.4.58}$$

Here, the change of variables $x(k) = \sqrt{a}e^{y(k)}$ reduces equation (9.4.58) to the linear difference equation

$$y(k+1) - y(k-\tau) = 0. \tag{9.4.59}$$

The characteristic equation of equation (9.4.59) is

$$\lambda^{\tau+1} + 1 = 0, \tag{9.4.60}$$

whose roots are the $(\tau+1)$st and $[2(\tau+1)]$st roots of 1. Therefore, we can give the following result.

Theorem 9.4.14. *Assume that $a \in \mathbb{R}^+$ and $\tau \in \mathbb{N}_0$. Then every positive solution of equation (9.4.58) is periodic with period $2(\tau+1)$ and oscillates about the positive equilibrium $\overline{x} = \sqrt{a}$.*

Next we consider the equation

$$x(k+1) = \frac{a}{c + x(k-\tau)} \quad \text{for } k \in \mathbb{N}_0 \tag{9.4.61}$$

(i.e., equation (9.4.34) with $b = 0$). The change of variables

$$x(k) = \frac{y(k+\tau+1)}{y(k)} - c \quad \text{for } k \in \{-\tau, -\tau+1, \dots\} \tag{9.4.62}$$

with $y(-\tau) = \cdots = y(0) = 1$ and $y(1) = x(-\tau) + c, \dots, y(\tau+1) = x(0) + c$ reduces the equation (9.4.61) to the linear difference equation

$$y(k+\tau+2) - cy(k+1) - ay(k-\tau) = 0. \tag{9.4.63}$$

The characteristic equation of (9.4.63) is

$$\lambda^{2(\tau+1)} - c\lambda^{\tau+1} - a = 0 \tag{9.4.64}$$

with roots satisfying

$$\lambda^{\tau} = \frac{c \pm \sqrt{c^2 + 4a}}{2}. \tag{9.4.65}$$

From this, one can derive the following result.

Theorem 9.4.15. *Assume that $a \in \mathbb{R}^+$ and $\tau \in \mathbb{N}_0$. Then every positive solution of equation (9.4.61) oscillates about the positive equilibrium $\overline{x}$.*

Another special case of equation (9.4.34) is

$$x(k+1) = \frac{a + bx(k)}{x(k-\tau)} \quad \text{for } k \in \mathbb{N}_0 \tag{9.4.66}$$

(take $c = 0$ in equation (9.4.34)), where $a, b \in \mathbb{R}^+$ and $\tau \in \mathbb{N}$. The change of variable $x(k) = by(k)$ reduces equation (9.4.66) to

$$y(k+1) = \frac{\alpha + y(k)}{y(k-\tau)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.67}$$

where $\alpha = a/b^2$. Equation (9.4.67) has a unique positive equilibrium $\overline{y}$ which is the positive root of the equation $\overline{y} = (\alpha/\overline{y}) + 1$, and it is given by

$$\overline{y} = \frac{1 + \sqrt{1 + 4\alpha}}{2}. \tag{9.4.68}$$

The linearized equation of equation (9.4.67) about $\overline{y}$ is

$$z(k+1) - \frac{1}{\overline{y}} z(k) + z(k-\tau) = 0 \tag{9.4.69}$$

with characteristic equation

$$\lambda^{\tau+1} - \frac{1}{\overline{y}} \lambda^{\tau} + 1 = 0. \tag{9.4.70}$$

One can prove that for $\tau \geq 2$ this equation has a root λ_1 with $|\lambda_1| > 1$. Hence, for $\tau \geq 2$, equation (9.4.66) is unstable. For the case $\tau = 1$ one can easily see that every positive solution of equation (9.4.66) is strictly oscillatory about the positive equilibrium $\overline{x}$ of equation (9.4.66). Also for $k \in \mathbb{N}$, we give the following interesting result.

Theorem 9.4.16. *Let $\{x^*(k)\}$ be a fixed positive solution of equation (9.4.66). Then every other positive solution of equation (9.4.66) oscillates about $\{x^*(k)\}$.*

Proof. Assume for the sake of contradiction that $\{x(k)\}$ does not oscillate about $\{x^*(k)\}$. Set

$$x^*(k+1)e^{z(k+1)} = \frac{a+bx^*(k)e^{z(k)}}{x^*(k-\tau)e^{z(k-\tau)}}. \tag{9.4.71}$$

Also

$$x^*(k+1) = \frac{a+bx^*(k)}{x^*(k-\tau)}, \tag{9.4.72}$$

and so one can easily find

$$z(k+1)+z(k-\tau) = \ln\frac{a+bx^*(k)e^{z(k)}}{a+bx^*(k)}. \tag{9.4.73}$$

As $\{z(k)\}$ is eventually positive, it follows that

$$\begin{aligned} z(k+1)-z(k)+z(k-\tau) &= \ln\frac{a+bx^*(k)e^{z(k)}}{a+bx^*(k)} - \ln e^{z(k)} \\ &= \ln\frac{ae^{-z(k)}+bx^*(k)}{a+bx^*(k)} \\ &\le 0. \end{aligned} \tag{9.4.74}$$

By applying Theorem 9.3.3, we arrive at the contradiction that $\lambda^{\tau+1}-\lambda^{\tau}+1=0$ has a positive root. This completes the proof. □

Next, we consider the more general recursive sequence

$$x(k+1) = \frac{a+\sum_{i=0}^{\tau-1} b(i)x(k-i)}{x(k-\tau)} \quad \text{for } k\in\mathbb{N}_0, \tag{9.4.75}$$

where

$$a, b(0), b(1),\dots, b(\tau-1)\in[0,\infty) \quad \text{with } a+\sum_{i=0}^{\tau-1} b(i)>0,\ \tau\in\mathbb{N}. \tag{9.4.76}$$

Now, the unique positive equilibrium of equation (9.4.75) is the positive root $\overline{x}$ of the equation

$$\overline{x} = \frac{a}{\overline{x}} + \sum_{i=0}^{\tau-1} b(i). \tag{9.4.77}$$

The following result deals with the oscillation of equation (9.4.75).

Theorem 9.4.17. *Assume that (9.4.76) holds. Then every nontrivial solution of equation (9.4.75) is strictly oscillatory about $\overline{x}$. Furthermore, every semicycle of a nontrivial solution contains no more than $2\tau+1$ terms.*

Proof. Let $\{x(k)\}$ be a nontrivial solution of equation (9.4.75) and assume, for the sake of contradiction, that $\{x(k)\}$ is not strictly oscillatory about $\overline{x}$. Then there exists $n_0 \geq \tau$ such that either

$$x(k) \geq \overline{x} \quad \text{for } k \geq n_0 - \tau - 1 \tag{9.4.78}$$

or

$$x(k) \leq \overline{x} \quad \text{for } k \geq n_0 - \tau - 1. \tag{9.4.79}$$

We will assume that (9.4.78) holds. The case where (9.4.79) holds is similar and will be omitted. Let A be the set of points $\{x(n_0 - \tau), \dots, x(n_0), \dots, x(n_0 + \tau)\}$ and let j be the smallest integer in the set $\{n_0 - \tau, \dots, n_0 + \tau\}$ with the property that $x(j) = \max A$. If $j < n_0$, then

$$x(j + \tau + 1) = \frac{a + \sum_{i=0}^{\tau-1} b(i)x(j + \tau - i)}{x(j)} < \frac{a}{\overline{x}} + \sum_{i=0}^{\tau-1} b(i) = \overline{x}, \tag{9.4.80}$$

while if $j \geq n_0$, then

$$x(j - \tau - 1) = \frac{a + \sum_{i=0}^{\tau-1} b(i)x(j - i - 1)}{x(j)} < \frac{a}{\overline{x}} + \sum_{i=0}^{\tau-1} b(i) = \overline{x}. \tag{9.4.81}$$

In either case, we obtain a contradiction, and so the proof that every nontrivial solution of equation (9.4.75) is strictly oscillatory is complete. The above analysis also shows that no semicycle may contain more than $2\tau + 1$ terms. The proof is complete. □

Finally, we consider the more general difference equation

$$x(k + 1) = \frac{ax(k)}{1 + \sum_{i=1}^{m} b_i x(k - \tau_i)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.82}$$

where

$$a \in (1, \infty), \qquad b_1, \dots, b_m \in \mathbb{R}^+, \qquad \tau_1, \dots, \tau_m \in \mathbb{N}. \tag{9.4.83}$$

Let $\tau = \max\{\tau_1, \dots, \tau_m\}$. If $\alpha(-\tau), \dots, \alpha(0)$ are $(\tau + 1)$ given constants such that

$$\alpha(k) \geq 0 \quad \text{for } k \in \{-\tau, \dots, -1\}, \qquad \alpha(0) > 0, \tag{9.4.84}$$

then equation (9.4.82) has a unique positive solution satisfying the initial conditions

$$x(k) = \alpha(k) \quad \text{for } k \in \{-\tau, -\tau + 1, \dots, 0\}. \tag{9.4.85}$$

The next theorem gives necessary and sufficient conditions for all positive solutions of equation (9.4.82) to oscillate about its unique positive equilibrium

$$\overline{x} = \frac{a-1}{\sum_{i=1}^{m} b_i}. \tag{9.4.86}$$

Theorem 9.4.18. *Assume that (9.4.83) holds. Then every solution of equation (9.4.82) oscillates about its positive equilibrium $\overline{x}$ if and only if every solution of the linear difference equation*

$$y(k+1) - y(k) + \frac{\overline{x}}{a}\sum_{i=1}^{m} b_i y(k-\tau_i) = 0 \tag{9.4.87}$$

oscillates.

Proof. The change of variables $x(k) = \overline{x}e^{-z(k)}$ for $k \in \mathbb{N}_0$ transforms equation (9.4.82) to the difference equation

$$z(k+1) - z(k) + f\big(z(k-\tau_1),\dots,z(k-\tau_m)\big) = 0, \tag{9.4.88}$$

where

$$f(u_1,u_2,\dots,u_m) = \ln\left(\frac{1}{a} + \frac{\overline{x}}{a}\sum_{i=1}^{m} b_i e^{u_i}\right). \tag{9.4.89}$$

Consider the function $\psi(x) = \ln 1/x$. Since ψ is a convex function in $\mathbb{R}^+$, it satisfies the well-known Jensen inequality for convex functions

$$\psi\left(\sum_{i=0}^{m} q_i t_i\right) \le \sum_{i=0}^{m} q_i \psi(t_i), \tag{9.4.90}$$

where $q_0, q_1, \dots, q_m \in \mathbb{R}^+$ with $\sum_{i=0}^{m} q_i = 1$ and $t_0, t_1, \dots, t_m \in \mathbb{R}^+$ are arbitrary. By taking $t_0 = 1$ and $t_i = e^{u_i}$ for $i \in \{1,2,\dots,m\}$ and $q_0 = 1/a$ and $q_i = \overline{x}b_i/a$ for $i \in \{1,2,\dots,m\}$, we obtain

$$\psi\left(\frac{1}{a} + \frac{\overline{x}}{a}\sum_{i=1}^{m} b_i e^{u_i}\right) \le \frac{\psi(1)}{a} + \frac{\overline{x}}{a}\sum_{i=1}^{m} b_i \psi(e^{u_i}). \tag{9.4.91}$$

Since $\psi(1) = 0$, $\psi(e^{u_i}) = -u_i$, and

$$\psi\left(\frac{1}{a} + \frac{\overline{x}}{a}\sum_{i=1}^{m} b_i e^{u_i}\right) = -f(u_1,u_2,\dots,u_m), \tag{9.4.92}$$

we have

$$f(u_1, u_2, \dots, u_m) \geq \frac{\overline{x}}{a} \sum_{i=1}^{m} b_i e^{u_i}. \tag{9.4.93}$$

One can now see that all the hypotheses of Theorem 9.3.3 are satisfied and that the linear equation associated with (9.4.88) is equation (9.4.87). The proof of the theorem is therefore a consequence of Theorem 9.3.3. □

9.4.3. Oscillation of the discrete delay logistic equation

Consider the nonautonomous, discrete *delay logistic equation*

$$x(k+1) = x(k)\left[A(k) - \sum_{i=0}^{m} B_i(k)x(k-i)\right], \tag{9.4.94}$$

where $\{A(k)\}, \{B_0(k)\}, \dots, \{B_m(k)\}$ are positive sequences.

The following result gives a characterization of the oscillation of all positive solutions of equation (9.4.94) about a fixed positive solution $\{x^*(k)\}$ of the same equation.

Theorem 9.4.19. *Let $\{x^*(k)\}$ be a positive solution of equation (9.4.94) and assume that*

$$\sum_{i=0}^{m} \sum_{k=0}^{\infty} \frac{x^*(k)}{x^*(k+1)} B_i(k)x^*(k-i) = \infty. \tag{9.4.95}$$

Then every positive solution of equation (9.4.95) oscillates about $\{x^*(k)\}$ if and only if every solution of the linear difference equation

$$y(k+1) - y(k) + \frac{x^*(k)}{x^*(k+1)} \sum_{i=0}^{m} B_i(k)x^*(k-1)y(k-i) = 0 \tag{9.4.96}$$

oscillates.

Proof. The change of variable $x(k) = x^*(k)e^{z(k)}$ reduces equation (9.4.94) to the equation

$$z(k+1) - z(k) - \ln\left(\frac{x^*(k)}{x^*(k+1)}\left[A(k) - \sum_{i=0}^{m} B_i(k)x^*(k-i)e^{z(k-i)}\right]\right) = 0. \tag{9.4.97}$$

Clearly, every positive solution of equation (9.4.94) oscillates about $\{x^*(k)\}$ if and only if every solution of equation (9.4.97) oscillates. Set

$$\begin{aligned} f(k,u_0,\dots,u_m) &= -\ln\left(\frac{x^*(k)}{x^*(k+1)}\left[A(k)-\sum_{i=0}^{m}B_i(k)x^*(k-i)e^{u_i}\right]\right),\\ g(k,u_0,\dots,u_m) &= f(k,u_0,\dots,u_m)-\frac{x^*(k)}{x^*(k+1)}\sum_{i=0}^{m}B_i(k)x^*(k-i)u_i. \end{aligned} \tag{9.4.98}$$

By noting that $\{x^*(k)\}$ satisfies equation (9.4.94), it follows that

$$\begin{aligned} \frac{\partial g}{\partial u_j} &= \frac{B_j(k)x^*(k-j)e^{u_j}}{A(k)-\sum_{i=0}^{m}B_i(k)x^*(k-i)e^{u_i}}-\frac{x^*(k)}{x^*(k+1)}B_j(k)x^*(k-j)\\ &= \frac{B_j(k)x^*(k-j)e^{u_j}}{A(k)-\sum_{i=0}^{m}B_i(k)x^*(k-i)e^{u_i}}-\frac{B_j(k)x^*(k-j)}{A(k)-\sum_{i=0}^{m}B_i(k)x^*(k-i)}. \end{aligned} \tag{9.4.99}$$

From this it is easy to see that

$$\frac{\partial g}{\partial u_j}<0 \quad \text{for } u_0,\dots,u_m<0, \tag{9.4.100}$$

and because $g(k,0,\dots,0)=0$, we see that

$$f(k,u_0,\dots,u_m)\geq \frac{x^*(k)}{x^*(k+1)}\sum_{i=0}^{m}B_i(k)x^*(k-i)u_i \quad \text{for } u_0,\dots,u_m<0. \tag{9.4.101}$$

In addition, for $j\in\{0,1,\dots,m\}$ and $u_0,\dots,u_m<0$,

$$\begin{aligned} \frac{\partial f(k,u_0,\dots,u_m)}{\partial u_j} &= \frac{B_j(k)x^*(k-j)e^{u_j}}{A(k)-\sum_{i=0}^{m}B_i(k)x^*(k-i)e^{u_i}}>0,\\ \frac{\partial f(k,0,\dots,0)}{\partial u_j} &= \frac{B_j(k)x^*(k-j)}{A(k)-\sum_{i=0}^{m}B_j(k)x^*(k-i)}=\frac{x^*(k)}{x^*(k+1)}B_j(k)x^*(k-j), \end{aligned} \tag{9.4.102}$$

and so $f(k,u_0,\dots,u_m)$ is increasing, $f(k,u_0,\dots,u_m)<0$ for $u_0,\dots,u_m<0$, and

$$\lim_{\substack{(u_0,\dots,u_m)\to(0,\dots,0)\\ u_iu_j>0,\ i,j\in\{0,1,\dots,m\}}}\left(\frac{f(k,u_0,\dots,u_m)}{(x^*(k)/x^*(k+1))\sum_{i=0}^{m}B_i(k)x^*(k-i)u_i}\right)\equiv 1. \tag{9.4.103}$$

Hence, by Theorem 9.3.11, every solution of equation (9.4.97) oscillates if and only if every solution of (9.4.96) oscillates. This completes the proof. □

By applying Theorem 9.4.18 to the difference equation

$$x(k+1) = Ax(k)\left[1 - \sum_{i=0}^{m} B_i x(k-i)\right], \tag{9.4.104}$$

where

$$m \in \mathbb{N}, \qquad A \in (1,\infty), \qquad B_i \in \mathbb{R}^+, \quad i \in \{0,1,\dots,m\} \tag{9.4.105}$$

and by taking the fixed solution of equation (9.4.104) to be the positive equilibrium

$$\overline{x} = \frac{A-1}{A\sum_{i=0}^{m} B_i}, \tag{9.4.106}$$

we obtain the following immediate consequence of Theorem 9.4.18.

Corollary 9.4.20. *Assume that (9.4.105) holds. Then every positive solution of equation (9.4.104) oscillates about the positive equilibrium $\overline{x}$ of equation (9.4.104) if and only if every solution of the linear equation*

$$y(k+1) - y(k) + A\overline{x}\sum_{i=0}^{m} B_i(k)y(k-i) = 0 \tag{9.4.107}$$

oscillates. Equivalently, the characteristic equation of equation (9.4.107), namely the equation

$$\lambda - 1 + A\overline{x}\sum_{i=0}^{m} B_i\lambda^{-i} = 0 \tag{9.4.108}$$

has no positive roots.

9.4.4. Oscillation of certain recursive sequences

Consider the difference equation

$$x(k+1) = \frac{\lambda x(k)}{(1+ax(k-\tau))^p + b\lambda x(k-\sigma)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.109}$$

where

$$a,b,p \in \mathbb{R}^+, \qquad \lambda \in (1,\infty), \qquad \tau,\sigma \in \mathbb{N}_0. \tag{9.4.110}$$

Let $r = \max\{\tau,\sigma\}$. We will assume that the initial conditions $x(-r),\dots,x(0)$ of equation (9.4.109) are arbitrary positive numbers. Equation (9.4.109) has a unique positive equilibrium $\overline{x}$. Furthermore, $\overline{x}$ is the unique positive solution of the equation

$$\lambda = (1+ax)^p + b\lambda x. \tag{9.4.111}$$

The following theorem gives necessary and sufficient conditions for the oscillation of all positive solutions of equation (9.4.109) about $\overline{x}$.

Theorem 9.4.21. *Assume that (9.4.110) holds and suppose that*

$$\frac{1}{\lambda}\left[a\overline{x}p(1+a\overline{x})^{p-1}\right] + b\overline{x} + \tau + \sigma \neq 1. \tag{9.4.112}$$

Then all positive solutions of equation (9.4.109) oscillate about $\overline{x}$ if and only if all solutions of the linear difference equation

$$z(k+1) - z(k) + \frac{a\overline{x}p(1+a\overline{x})^{p-1}}{\lambda} z(k-\tau) + b\overline{x}z(k-\sigma) = 0 \tag{9.4.113}$$

oscillate.

Proof. Set $x(k) = \overline{x}e^{y(k)}$. Then equation (9.4.109) becomes

$$y(k+1) - y(k) + \ln\left[\frac{1}{\lambda}\left(\left(1+a\overline{x}e^{y(k-\tau)}\right)^p + b\lambda\overline{x}e^{y(k-\sigma)}\right)\right] = 0. \tag{9.4.114}$$

Clearly, every positive solution of equation (9.4.109) oscillates about $\overline{x}$ if and only if every solution of equation (9.4.114) oscillates. Now, consider the function

$$f(x,y) = \ln\left[\frac{1}{\lambda}\left(\left(1+a\overline{x}e^{x}\right)^p + b\lambda\overline{x}e^{y}\right)\right]. \tag{9.4.115}$$

It is easy to see that

$$f(x,y)\begin{cases} \geq 0 & \text{for } x,y \in [0,\infty), \\ \leq 0 & \text{for } x,y \in (-\infty,0], \end{cases} \tag{9.4.116}$$

and $f(x,x) = 0$ only if $x = 0$. Since

$$D_x f(0,0) = \frac{(1+a\overline{x})^p}{\lambda}, \qquad D_y f(0,0) = b\overline{x}, \tag{9.4.117}$$

it follows that equation (9.4.113) is the linearized equation associated with equation (9.4.114). Therefore, by Theorem 9.3.3, it remains to show that there exists $\delta > 0$ such that

$$f(x,y) \geq \frac{1}{\lambda}(1+a\overline{x})^p x + b\overline{x}y \quad \text{for } x,y \in [-\delta,0]. \tag{9.4.118}$$

To this end, observe that the function

$$g(x,y) = f(x,y) - \frac{(1+a\overline{x})^p}{\lambda}x + b\overline{x}y \tag{9.4.119}$$

has a local minimum equal to 0 at the point $(0,0)$. Hence there exists $\delta > 0$ such that $g(x,y) \geq 0$ for $x, y \in [-\delta, 0]$. The proof is complete. □

Next we consider the difference equation

$$x(k+1) = \frac{x(k)}{a + bx^p(k-\tau) - cx^q(k-\tau)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.120}$$

where $a \in (0,1)$, $b, p, q \in \mathbb{R}^+$, $c \in \mathbb{R}$, $\tau \in \mathbb{N}$, and

$$p > q, \quad a + b\left(\frac{cq}{bp}\right)^{p/(p-q)} - c\left(\frac{cq}{bp}\right)^{q/(p-q)} > 0. \tag{9.4.121}$$

Equation (9.4.120) has a unique positive equilibrium $\overline{x}$. Furthermore, $\overline{x}$ is the unique positive solution of the equation

$$a + bx^p - cx^q = 1. \tag{9.4.122}$$

The following result gives necessary and sufficient conditions for the oscillation of all positive solutions of equation (9.4.120) about $\overline{x}$.

Theorem 9.4.22. *Assume that (9.4.121) holds. Then every solution of equation (9.4.120) oscillates about $\overline{x}$ if and only if*

$$pb(\overline{x})^p - qc(\overline{x})^q \begin{cases} \geq 1 & \text{if } \tau = 0, \\ > \dfrac{\tau^\tau}{(\tau+1)^{\tau+1}} & \text{if } \tau \geq 1. \end{cases} \tag{9.4.123}$$

Proof. The change of variable $x(k) = \overline{x}e^{y(k)}$ transforms equation (9.4.120) to the difference equation

$$y(k+1) - y(k) + \ln\left[a + b(\overline{x})^p e^{py(k-\tau)} - c(\overline{x})^q e^{qy(k-\tau)}\right] = 0. \tag{9.4.124}$$

Clearly, every solution of equation (9.4.120) oscillates about $\overline{x}$ if and only if every solution of equation (9.4.124) oscillates. Set

$$\begin{aligned} f(u) &= \ln\left[a + b(\overline{x}e^u)^p - c(\overline{x}e^u)^q\right], \\ g(u) &= f(u) - \left[pb(\overline{x})^p - qc(\overline{x})^q\right]u. \end{aligned} \tag{9.4.125}$$

If $c \leq 0$, then clearly

$$uf(u) > 0 \quad \text{for } u \neq 0. \tag{9.4.126}$$

Now assume that $c > 0$. As

$$p > q > 0, \quad b(\overline{x})^p - c(\overline{x})^q = 1 - a > 0, \tag{9.4.127}$$

it follows that

$$\begin{aligned} f(u) &\geq \ln\left[a + \left(b(\overline{x})^p - c(\overline{x})^q\right)e^{qu}\right] > 0 \quad \text{for } u > 0, \\ f(u) &\leq \ln\left[a + \left(b(\overline{x})^p - c(\overline{x})^q\right)e^{qu}\right] < 0 \quad \text{for } u < 0. \end{aligned} \tag{9.4.128}$$

Hence (9.4.126) holds for all $c \in \mathbb{R}$. Set $A = pb(\overline{x})^p - qc(\overline{x})^q$ and observe that

$$\begin{aligned} \frac{dg}{du} &= \frac{pb(\overline{x})^p e^{pu} - qc(\overline{x})^q e^{qu}}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}} - A \\ &= \frac{pb(\overline{x})^p e^{pu} - qc(\overline{x})^q e^{qu} - A\left(a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}\right)}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}} \\ &= \frac{pb(\overline{x})^p e^{pu} - qc(\overline{x})^q e^{qu} - Aa - Ab(\overline{x})^p e^{pu} + Ac(\overline{x})^q e^{qu}}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}} \\ &= \frac{\left(pb(\overline{x})^p - b(\overline{x})^p A\right)e^{pu} - \left(qc(\overline{x})^q - c(\overline{x})^q A\right)e^{qu} - aA}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}} \\ &\leq \frac{\left(pb(\overline{x})^p - Ab(\overline{x})^p - qc(\overline{x})^q + Ac(\overline{x})^q\right)e^{qu} - aA}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}} \\ &= \frac{A\left(1 - b(\overline{x})^p + c(\overline{x})^q\right)e^{qu} - aA}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}}, \\ A &= pb(\overline{x})^p - qc(\overline{x})^q \geq p\left[b(\overline{x})^p - c(\overline{x})^q\right] = p(1 - a) > 0. \end{aligned} \tag{9.4.129}$$

Hence

$$\frac{dg}{du} \leq \frac{A\left[1 - a - b(\overline{x})^p + c(\overline{x})^q\right]}{a + b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}} = 0 \quad \text{for } u < 0. \tag{9.4.130}$$

This together with $g(0) = 0$ implies that $g(u) > 0$ for $u < 0$, that is, $f(u) \geq Au$ for $u < 0$. Also, we have

$$\frac{d}{du} f(u)\Big|_{u=0} = A, \tag{9.4.131}$$

and so

$$\lim_{u \to 0} \frac{f(u)}{Au} = 1. \tag{9.4.132}$$

Hence, by Theorem 9.3.3, one can easily see that every solution of equation (9.4.124) oscillates if and only if (9.4.123) holds. This completes the proof. □

Finally, we investigate the oscillation of the delay difference equation

$$x(k+1) - x(k) = -ax(k) + bx(k-\tau)e^{-cx(k-\tau)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.4.133}$$

where $\tau \in \mathbb{N}_0$ and

$$a \in (0,1), \qquad b \in (a,\infty), \qquad c \in \mathbb{R}^+. \tag{9.4.134}$$

It is easy to see that when (9.4.134) holds and the initial conditions are such that $x(-\tau), x(-\tau+1), \dots, x(-1) \in [0,\infty)$ and $x(0) \in \mathbb{R}^+$, then the corresponding solution $\{x(k)\}$ is positive. Furthermore, the unique positive equilibrium of equation (9.4.134) is given by

$$\overline{x} = \frac{1}{c}\ln\left(\frac{b}{a}\right). \tag{9.4.135}$$

Theorem 9.4.23. *Assume that (9.4.134) holds and that*

$$a\left(\ln\frac{b}{a} - 1\right)\frac{(\tau+1)^{\tau+1}}{\tau^\tau} > (1-a)^{\tau+1}. \tag{9.4.136}$$

Then every positive solution of equation (9.4.133) oscillates about the positive equilibrium $\overline{x}$.

Proof. Assume for the sake of contradiction that equation (9.4.133) has a positive solution $\{x(k)\}$ which does not oscillate about $\overline{x}$. Set

$$x(k) = \overline{x} + \frac{1}{c}y(k) \quad \text{for } k \in \{-\tau, -\tau+1, \dots\}. \tag{9.4.137}$$

Then $\{y(k)\}$ is a nonoscillatory solution of the difference equation

$$y(k+1) - y(k) + ay(k) + ac\overline{x}\left(1 - e^{-y(k-\tau)}\right) - ay(k-\tau)e^{-y(k-\tau)} = 0. \tag{9.4.138}$$

We will assume that $\{y(k)\}$ is eventually positive. The case where $\{y(k)\}$ is eventually negative is similar and will be omitted. Let n_0 be an integer such that

$$y(k) > 0 \quad \text{for } k \geq n_0. \tag{9.4.139}$$

First, we claim that $\{y(k)\}$ is a bounded sequence. Otherwise, there exists a subsequence $\{y(k_i)\}$ such that for $i \in \mathbb{N}$,

$$k_i \geq n_0, \quad \lim_{i\to\infty} y(k_i) = \infty, \quad y(k_i+1) - y(k_i) \geq 0. \tag{9.4.140}$$

It follows from (9.4.138) that for i sufficiently large

$$y(k_i) + c\bar{x} \le [y(k_i - \tau) + c\bar{x}]e^{-y(k_i-\tau)} \tag{9.4.141}$$
$$\le y(k_i - \tau) + c\bar{x}. \tag{9.4.142}$$

From (9.4.141), we see that $\lim_{i\to\infty} y(k_i - \tau) = \infty$. But then (9.4.141) leads to a contradiction as $i \to \infty$.

Next, we claim that

$$\lim_{k\to\infty} y(k) = 0. \tag{9.4.143}$$

Otherwise, let

$$\mu = \limsup_{k\to\infty} y(k). \tag{9.4.144}$$

Then $\mu > 0$ and there exists a subsequence $\{y(k_i)\}$ such that for $i \in \mathbb{N}$,

$$k_i \ge n_0, \quad \lim_{i\to\infty} y(k_i) = \mu, \quad y(k_i + 1) - y(k_i) \ge 0. \tag{9.4.145}$$

Also, (9.4.141) and (9.4.142) hold. From (9.4.142) we see that

$$\mu \le \limsup_{i\to\infty} y(k_i - \tau) \tag{9.4.146}$$

and so, because of (9.4.144), $\limsup_{i\to\infty} x(k_i - \tau) = \mu$. But then (9.4.141) leads to

$$\mu + c\bar{x} \le (\mu + c\bar{x})e^{-\mu} < \mu + c\bar{x}, \tag{9.4.147}$$

which is impossible. Hence (9.4.143) holds.

Equation (9.4.138) can be rewritten in the form

$$y(k+1) - y(k) + ay(k) + q(k)y(k-\tau) = 0, \tag{9.4.148}$$

where

$$\begin{gathered} q(k) = ac\bar{x}\frac{1 - e^{-y(k-\tau)}}{y(k-\tau)} - ae^{-y(k-\tau)}, \\ \lim_{k\to\infty} q(k) = ac\bar{x} - a = a\left(\ln\frac{b}{a} - 1\right). \end{gathered} \tag{9.4.149}$$

One can easily see that the hypotheses of Lemma 9.3.1 are satisfied, and so the linear equation

$$z(k+1) - z(k) + az(k) + a\left(\ln\frac{b}{a} - 1\right)z(k-\tau) = 0 \tag{9.4.150}$$

has an eventually positive solution. Let $\{z(k)\}$ be an eventually positive solution of equation (9.4.150). Then $u(k) = (1-a)^{-k}z(k)$ is an eventually positive solution of

$$u(k+1) - u(k) + a(1-a)^{-\tau-1}\left(\ln\frac{b}{a} - 1\right)u(k-\tau) = 0. \qquad (9.4.151)$$

As before, condition (9.4.136) implies that equation (9.4.151) has no nonoscillatory solution, a contradiction which completes the proof. □

We note that equation (9.4.133) may be viewed as a model of the well-known *Nicholson blowflies*.

9.5. Global behavior of certain difference equations

Consider the more general nonlinear difference equation

$$x(k+1) = F\big(x(k), x(k-1), \dots, x(k-\tau)\big) \quad \text{for } k \in \mathbb{N}_0, \qquad (9.5.1)$$

where $F \in C(I^{\tau+1}, I)$, $I \subseteq \mathbb{R}$, and $\tau \in \mathbb{N}_0$.

First, we present some basic definitions and known results which will be useful in investigating equation (9.5.1).

Definition 9.5.1 (permanence). Equation (9.5.1) is said to be *permanent* if there exist numbers α and β with $0 < \alpha \le \beta < \infty$ such that for any initial conditions $x(-\tau), x(-\tau+1), \dots, x(0) \in \mathbb{R}^+$ there exists $N \in \mathbb{N}$ which depends on the initial conditions such that $\alpha \le x(k) \le \beta$ for $k \ge N$.

Again, an equilibrium point of equation (9.5.1) is a point $\overline{x} \in \mathbb{R}$ such that $\overline{x} = F(\overline{x}, \dots, \overline{x})$, that is, $\overline{x}$ is a fixed point of the function $F(x, \dots, x)$.

Definition 9.5.2 (stability). (i) The equilibrium point $\overline{x}$ of equation (9.5.1) is called *locally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(k) - \overline{x}| < \varepsilon$ for all $k \ge -\tau$ whenever $x(-\tau), x(-\tau+1), \dots, x(0) \in I$ with $\sum_{j=-\tau}^{0} |x(j) - \overline{x}| < \delta$.

(ii) The equilibrium point $\overline{x}$ of equation (9.5.1) is called *locally asymptotically stable* if $\overline{x}$ is a locally stable solution of equation (9.5.1) and there exists a constant $\gamma > 0$ such that $\lim_{k\to\infty} x(k) = \overline{x}$ whenever $x(-\tau), x(-\tau+1), \dots, x(0) \in I$ with $\sum_{j=-\tau}^{0} |x(j) - \overline{x}| < \gamma$.

(iii) The equilibrium point $\overline{x}$ of equation (9.5.1) is called a *global attractor* if $\lim_{k\to\infty} x(k) = \overline{x}$ whenever $x(-\tau), x(-\tau+1), \dots, x(0) \in I$.

(iv) The equilibrium point $\overline{x}$ of equation (9.5.1) is called *globally asymptotically stable* if $\overline{x}$ is locally stable and also a global attractor of equation (9.5.1).

(v) The equilibrium point $\overline{x}$ of equation (9.5.1) is called *unstable* if $\overline{x}$ is not locally stable.

(vi) The equilibrium point $\overline{x}$ of equation (9.5.1) is called a *source* if there exists $r > 0$ such that for all $x(-\tau), x(-\tau+1), \dots, x(0) \in I$ with $\sum_{j=-\tau}^{0} |x(j) - \overline{x}| < r$, there exists $N \in \mathbb{N}$ such that $|x(N) - \overline{x}| > r$.

Definition 9.5.3 (periodicity). A sequence $\{x(k)\}_{k=-\tau}^{\infty}$ is said to be *periodic* with period p if $x(k+p) = x(k)$ for all $k \geq -\tau$. A sequence $\{x(k)\}_{k=-\tau}^{\infty}$ is said to be periodic with *prime period* p if p is the smallest positive integer having this property.

Next, the *linearized equation* of (9.5.1) about the equilibrium $\overline{x}$ is the linear difference equation

$$y(k+1) = \sum_{i=0}^{\tau} \frac{\partial F}{\partial x(k-i)}(\overline{x},\dots,\overline{x})y(k-i) \quad \text{for } i \in \mathbb{N}_0. \tag{9.5.2}$$

The characteristic equation associated with this linearized equation is

$$P(\lambda) = \lambda^{\tau+1} - \sum_{i=0}^{\tau} \frac{\partial F}{\partial x(k-i)}(\overline{x},\dots,\overline{x})\lambda^{\tau-i} = 0. \tag{9.5.3}$$

For a linear homogeneous equation, the stability of the zero equilibrium is equivalent to boundedness of all solutions for $k \geq 0$. Also, the asymptotic stability of the zero equilibrium is equivalent to all solutions having limit zero as $k \to \infty$, which in turn is true if and only if every root of the characteristic equation lies in the open unit disk $|\lambda| < 1$. A linear equation will be called stable, asymptotically stable, or unstable provided that the zero equilibrium has that property.

The so-called Schur-Cohn criterion provides necessary and sufficient conditions for all roots of the equation

$$P(\lambda) = a_0\lambda^k + a_1\lambda^{k-1} + \cdots + a_{k-1}\lambda + a_k = 0 \tag{9.5.4}$$

with real coefficients to lie in the unit disk $|\lambda| < 1$.

Before we can discuss the Schur-Cohn criterion, we need the so-called Routh-Hurwitz criterion.

Theorem 9.5.4 (Routh-Hurwitz criterion). *Consider the polynomial equation (9.5.4) with real coefficients and $a_0 > 0$. Then a necessary and sufficient condition for all roots of equation (9.5.4) to have negative real part is $\Delta_n > 0$ for $n \in \{1,2,\dots,k\}$, where Δ_n is the principal minor of order n of the $k \times k$-matrix*

$$\begin{pmatrix} a_1 & a_3 & a_5 & \cdots & 0 \\ a_0 & a_2 & a_4 & \cdots & 0 \\ 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_k \end{pmatrix}. \tag{9.5.5}$$

Necessary and sufficient conditions for all roots of equation (9.5.4) to lie in the open disk $|\lambda| < 1$ are found from the Routh-Hurwitz criterion and the fact that the Möbius transformation $z = (\lambda+1)/(\lambda-1)$ transforms the open unit disk in the λ-plane onto the open left-half plane in the z-plane.

Theorem 9.5.5 (Schur-Cohn criterion). *Equation (9.5.4) has all its roots in the open unit disk* $|\lambda| < 1$ *if and only if the equation*

$$P\left(\frac{z+1}{z-1}\right) = 0 \tag{9.5.6}$$

has all its roots in the left-half plane $\operatorname{Re}(z) < 0$.

A special case of equation (9.5.1) is the following equation:

$$x(k+1) = f(x(k), x(k-1)) \quad \text{with } x(-1), x(0) \in I, \tag{9.5.7}$$

where $f : I^2 \to I$, $I \subseteq \mathbb{R}$, is continuous and has continuous first-order partial derivatives with respect to its variables.

The linearized equation of equation (9.5.7) about the equilibrium $\overline{x}$ is the linear difference equation

$$y(k+1) = py(k) + qy(k-1) \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.8}$$

where

$$p = \frac{\partial f}{\partial x(k)}(\overline{x}, \overline{x}), \qquad q = \frac{\partial f}{\partial x(k-1)}(\overline{x}, \overline{x}). \tag{9.5.9}$$

The characteristic equation of this linearized equation is the equation

$$\lambda^2 - p\lambda - q = 0 \tag{9.5.10}$$

with characteristic roots

$$\lambda_{\pm} = \frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 + 4q}. \tag{9.5.11}$$

The following well-known theorem, called the linearized stability theorem, is very important in determining the local stability character of the equilibrium $\overline{x}$ of equation (9.5.7).

Theorem 9.5.6 (linearized stability theorem). *The following statements are true.*

(I_1) *If both solutions of equation (9.5.10) have absolute value less than one, then the equilibrium point* $\overline{x}$ *of equation (9.5.7) is locally asymptotically stable.*

(I_2) *If at least one of the solutions of equation (9.5.10) has absolute value greater than one, then the equilibrium* $\overline{x}$ *of equation (9.5.7) is unstable.*

(I_3) *A necessary and sufficient condition for both roots of equation (9.5.10) to have absolute value less than one is* $|p| < 1 - q < 2$. *In this case* $\overline{x}$ *is called a sink.*

(I_4) *A necessary and sufficient condition for one root of equation (9.5.10) to have absolute value less than one and the other root of equation (9.5.10) to have absolute value greater than one is* $p^2 > -4q$ *and* $|p| > |1-q|$. *In this case* $\overline{x}$ *is called a saddle point.*

(I_5) *A necessary and sufficient condition for both roots of equation (9.5.10) to have absolute value greater than one is* $|q| > 1$ *and* $|p| < |1-q|$. *In this case* $\overline{x}$ *is called a source.*

The following theorem, called the stable manifold theorem in the plane, explains the significance of $\overline{x}$ being a saddle point.

Theorem 9.5.7 (stable manifold theorem in the plane). *Suppose that the mapping* $T : [0,\infty)\times[0,\infty) \to [0,\infty)\times[0,\infty)$ *is a continuously differentiable homeomorphism whose inverse is also continuously differentiable. Assume that* $p \in \mathbb{R}^+ \times \mathbb{R}^+$ *is a saddle point of* T, *that is,* $T(p) = p$ *and the Jacobian* $J_T(p)$ *has eigenvalues* s, u *with* $|s| < 1$ *and* $|u| > 1$. *Moreover, let* S *be the stable manifold of* p, *that is,* S *is the set of initial points* q *whose forward iterate is* $q, T(q), T^2(q), \dots$, *and let* U *be the unstable manifold of* p, *that is,* U *is the set of initial points* q *whose backward iterate under the inverse of* T *is* $q, T^{-1}(q), T^{-2}(q), \dots$ *and converges to* p. *Then* S *and* U *are each one-dimensional manifolds (curves) which contain* p. *Moreover, the eigenvectors* v_s *and* v_u *(corresponding to the eigenvalues* s *and* u*) are tangents, respectively, to* S *and* U *at* p.

The map T can be found as follows. For $k \in \mathbb{N}_0$, set $u(k) = x(k-1)$ and $v(k) = x(k)$. Then $u(k+1) = v(k)$ and $v(k+1) = f(v(k), u(k))$, and so

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ f(v,u) \end{pmatrix}. \tag{9.5.12}$$

Thus we see that the characteristic roots of the Jacobian $J_T\binom{\overline{x}}{\overline{x}}$ at the equilibrium $\overline{x}$ of equation (9.5.7) are the roots of equation (9.5.10).

Here we will present various results on global attractivity for the positive equilibrium of certain nonlinear difference equations. These results apply to certain known recursive sequences.

9.5.1. Global attractivity in a nonlinear second-order difference equation

In this subsection we obtain a global attractivity result for the positive equilibrium of equation (9.5.7). The result applies to the difference equation

$$x(k+1) = \frac{a + bx(k)}{c + x(k-1)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.13}$$

where $a, b, c \in \mathbb{R}^+$.

We will consider equation (9.5.7), where f satisfies the following hypotheses:

(i_1) $f \in C([0,\infty) \times [0,\infty), \mathbb{R}^+)$,
(i_2) $f(u,v)$ is nondecreasing in u and decreasing in v,
(i_3) the function $f(u,v)/u$ is nonincreasing in u,
(i_4) equation (9.5.7) has a unique positive equilibrium,
(i_5) if $\bar{x}$ denotes the unique positive equilibrium of equation (9.5.7), then the function $F(v) = f(f(\bar{x},v),v)$ has no periodic points of prime period 2.

If $a(-1) \in [0,\infty)$ and $a(0) \in \mathbb{R}^+$ are given, then equation (9.5.7) has a unique solution $\{x(k)\}$ satisfying the initial conditions $x(-1) = a(-1)$ and $x(0) = a(0)$. Clearly $x(k) > 0$ for $k \geq 0$. In the sequel we will only consider positive solutions of equation (9.5.7).

The following results are needed.

Lemma 9.5.8. *Let $F \in C([0,\infty), \mathbb{R}^+)$ be a nonincreasing function and let $\bar{x}$ denote the (unique) fixed point of F. Then the following statements are equivalent:*

(a_1) *$\bar{x}$ is the only fixed point of F^2 in $\mathbb{R}^+$,*
(a_2) *if $\lambda, \mu > 0$ such that $F(\mu) \leq \lambda \leq \bar{x} \leq \mu \leq F(\lambda)$, then $\lambda = \mu = \bar{x}$.*

Theorem 9.5.9. *Consider the difference equation*

$$x(k+1) = f(x(k)) \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.14}$$

where $f : I \to I \subseteq \mathbb{R}$ is a decreasing function. Assume that the unique equilibrium of equation (9.5.14) is locally asymptotically stable and that f has negative Schwarzian derivative

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 \tag{9.5.15}$$

everywhere on I, except for the points x^*, where $f(x^*) = 0$. Then the positive equilibrium of equation (9.5.13) is globally asymptotically stable.

The following lemma describes the semicycles of the strictly oscillatory solutions of equation (9.5.7).

Lemma 9.5.10. *Assume that the hypotheses (i_1)–(i_4) are satisfied and let $\{x(k)\}$ be a strictly oscillatory solution of equation (9.5.7). Then the following statements are true:*

(b_1) *if $x(-1)$ and $x(0)$ are not both equal to $\bar{x}$, then a positive semicycle cannot have two consecutive terms equal to $\bar{x}$,*
(b_2) *every semicycle of $\{x(k)\}$, except perhaps the first one, has at least two terms,*

(b$_3$) *the maximum in a positive semicycle and the minimum in a negative semicycle are equal to the first or to the second term of the semicycle. Furthermore, after the first term, the remaining terms in a positive semicycle are nonincreasing and the remaining terms in a negative semicycle are nondecreasing.*

Proof. The proof of (b$_1$) is simple and will be omitted. For the remaining statements we will only give the proof for positive semicycles of nontrivial solutions. The proof for negative semicycles is similar and will be omitted. The proof for the trivial semicycle where $x(-1) = x(0) = \overline{x}$ is obvious.

Now we show (b$_2$). If $x(k)$ is the first term in a positive semicycle (other than the first semicycle), then $x(k) \geq \overline{x} > x(k-1)$ and

$$x(k+1) = f\big(x(k), x(k-1)\big) \geq f\big(\overline{x}, x(k-1)\big) > f(\overline{x}, \overline{x}) = \overline{x}, \tag{9.5.16}$$

so $x(k+1)$ is also in the same semicycle.

If $x(k)$ and $x(k+1)$ are two consecutive terms in a positive semicycle, then

$$x(k+2) = x(k+1)\frac{f\big(x(k+1), x(k)\big)}{x(k+1)} \leq x(k+1)\frac{f(\overline{x}, \overline{x})}{\overline{x}} = x(k+1). \tag{9.5.17}$$

This proves (b$_3$). □

The following result is an immediate consequence of Lemma 9.5.10.

Corollary 9.5.11. *Let $\{x(k)\}$ be a solution of equation (9.5.7) such that for some $n_0 \in \mathbb{N}_0$, either $x(k) \geq \overline{x}$ for $k \geq n_0$ or $x(k) \leq \overline{x}$ for $k \geq n_0$. Then for $k \geq n_0 + 1$, the sequence $\{x(k)\}$ is monotonic and* $\lim_{k\to\infty} x(k) = \overline{x}$.

Now we present the following result.

Theorem 9.5.12. *Assume that the hypotheses (i$_1$)–(i$_5$) are satisfied. Then $\overline{x}$ is a global attractor of all positive solutions of equation (9.5.7).*

Proof. We will show that

$$\lim_{k\to\infty} x(k) = \overline{x} \tag{9.5.18}$$

for all solutions $\{x(k)\}$ of (9.5.7). In view of Corollary 9.5.11, this is clearly true if $\{x(k)\}$ is not strictly oscillatory about $\overline{x}$. So assume that $\{x(k)\}$ is a strictly oscillatory solution. Let $\{x(q_i+1), x(q_i+2), \dots, x(p_i)\}$ be the ith negative semicycle of $\{x(k)\}$ followed by the ith positive semicycle $\{x(p_i+1), x(p_i+2), \dots, x(q_i)\}$. Let $x(m_i)$ and $x(M_i)$ be the minimum and the maximum values in these two semicycles, respectively, with the smallest possible indices m_i and M_i. Then from Lemma 9.5.10(b$_2$) it follows that

$$M_i - p_i \leq 2, \qquad m_i - q_i \leq 2. \tag{9.5.19}$$

First we consider the case where the maximum value in the positive semicycle is equal to the first term of the semicycle. Then $M_i - 1 = p_i$ and

$$\begin{aligned} x(m_i) &\le \min\{x(M_i-2), x(M_i-1)\} \\ &\le \max\{x(M_i-2), x(M_i-1)\} \\ &\le x(M_i), \end{aligned} \tag{9.5.20}$$

and so

$$x(M_i) = f(x(M_i-1), x(M_i-2)) \le f(\overline{x}, x(M_i-2)) < f(\overline{x}, x(m_i)). \tag{9.5.21}$$

From (i_2) and (i_3) it follows that

$$f(\overline{x}, x(m_i)) > \overline{x}, \qquad f(f(\overline{x}, x(m_i)), x(m_i)) > f(\overline{x}, x(m_i)). \tag{9.5.22}$$

Hence

$$x(M_i) < f(\overline{x}, x(m_i)) < f(f(\overline{x}, x(m_i)), x(m_i)) = F(x(m_i)). \tag{9.5.23}$$

Next we consider the case where the maximum value in the positive semicycle is equal to the second term of the semicycle. Then $M_i - 2 = p_i$ and

$$x(m_i) \le x(M_i-3), \qquad x(M_i-2) < \overline{x} \le x(M_i-1) < x(M_i). \tag{9.5.24}$$

Furthermore,

$$\begin{aligned} x(M_i) &= f(x(M_i-1), x(M_i-2)) \\ &= f(f(x(M_i-2), x(M_i-3)), x(m_i)) \\ &\le F(x(m_i)), \end{aligned} \tag{9.5.25}$$

that is, in all cases

$$x(M_i) \le F(x(m_i)). \tag{9.5.26}$$

From (9.5.26) it follows that

$$x(M_i) < f(f(\overline{x}, 0), 0) =: d. \tag{9.5.27}$$

By a parallel argument we obtain

$$x(m_i) > f(f(\overline{x}, d), d) =: c, \tag{9.5.28}$$

and so there exists an integer n_0 such that

$$c < x(k) < d \quad \text{for } k \geq n_0, \tag{9.5.29}$$

where the constants c and d are defined by (9.5.27) and (9.5.28), respectively. Let

$$\lambda = \liminf_{k\to\infty} x(k) = \liminf_{i\to\infty} x(m_i), \qquad \mu = \limsup_{k\to\infty} x(k) = \limsup_{i\to\infty} x(M_i), \tag{9.5.30}$$

which in view of (9.5.29) exist and are such that $0 < c \leq \lambda \leq \overline{x} \leq \mu \leq d$. To complete the proof it suffices to show that

$$\lambda = \mu = \overline{x}. \tag{9.5.31}$$

From (9.5.30) it follows that if $\eta \in \mathbb{R}^+$ and $\varepsilon \in (0, \lambda)$ are given, then there exists $n_0 \in \mathbb{N}$ such that

$$\lambda - \varepsilon \leq x(k) \leq \mu + \eta \quad \text{for } k \geq n_0 - 1. \tag{9.5.32}$$

From (9.5.26) we find $x(M_i) \leq F(\lambda - \varepsilon)$. Therefore, as $\varepsilon \in (0, \lambda)$ is arbitrary, it follows from (9.5.30) that

$$\mu \leq F(\lambda). \tag{9.5.33}$$

In a similar way we find

$$\lambda \geq F(\mu). \tag{9.5.34}$$

Clearly the function F is decreasing, and by applying Lemma 9.5.8, one can see that (9.5.31) is true. This completes the proof. □

From (9.5.29) and Corollary 9.5.11 we obtain the following result.

Corollary 9.5.13. *Assume that the hypotheses* (i_1)–(i_4) *are satisfied. Then equation (9.5.7) is permanent.*

Example 9.5.14. We apply Theorem 9.5.12 to the rational recursive sequence (9.5.13). If $\overline{x}$ denotes the unique positive equilibrium of equation (9.5.13), then clearly

$$\overline{x} = \frac{b-c}{2} + \frac{1}{2}\sqrt{(b-c)^2 + 4a}. \tag{9.5.35}$$

It is well known that $\overline{x}$ is locally asymptotically stable. Here we will show that when

$$(c-b)\overline{x} + c^2 > 0, \tag{9.5.36}$$

the equilibrium $\overline{x}$ is globally asymptotically stable. We will apply Theorem 9.5.12. In our case

$$\begin{aligned} f(u,v) &= \frac{a+bu}{c+v}, \\ f\big(f(\overline{x},v),v\big) &= \frac{a(c+v)+b(a+b\overline{x})}{(c+v)^2}. \end{aligned} \tag{9.5.37}$$

Clearly, the hypotheses (i_1)–(i_4) are satisfied. It remains to show that F has no periodic points of prime period 2. Since $F(\overline{x}) = \overline{x}$, in view of Lemma 9.5.8 it is sufficient to show that all solutions of the difference equation

$$y(k+1) = F\big(y(k)\big) \quad \text{for } k \in \mathbb{N}_0 \tag{9.5.38}$$

converge to $\overline{x}$. This will be accomplished by showing that the Schwarzian derivative of F is negative together with the observation that, because of (9.5.36), the equilibrium $\overline{x}$ of equation (9.5.38) is locally asymptotically stable. To this end, observe that

$$\begin{aligned} F(u) &= \frac{a(c+u)+2b(a+b\overline{x})}{(c+u)^2}, \\ \frac{F'''(u)}{F'(u)} &= \frac{a}{a(c+u)+2b(a+b\overline{x})} - \frac{3}{c+u} \\ &= -\frac{2a(c+u)+6b(a+b\overline{x})}{\big[a(c+u)+2b(a+b\overline{x})\big](c+u)}, \\ \left(\frac{F''(u)}{F(u)}\right)' &= \frac{F'''(u)}{F(u)} - \left(\frac{F''(u)}{F(u)}\right)^2 \\ &= \frac{3\big[a(c+u)+b(a+b\overline{x})\big]^2 - a^2(c+u)^2}{\big[a(c+u)+2b(a+b\overline{x})\big](c+u)}. \end{aligned} \tag{9.5.39}$$

Hence

$$\begin{aligned} SF(u) &= \left(\frac{F''(u)}{F(u)}\right)' - \frac{1}{2}\left(\frac{F''(u)}{F(u)}\right)^2 \\ &= -\frac{6ab(c+u)(a+b\overline{x}) + 15b^2(a+b\overline{x})^2}{\big[a(c+u)+2b(a+b\overline{x})\big]^2(c+u)^2} \\ &< 0, \end{aligned} \tag{9.5.40}$$

from which the result follows.

9.5.2. Global asymptotic stability of a second-order difference equation

We consider the second-order difference equation

$$x(k+1) = x(k)f\big(x(k), x(k-1)\big) \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.41}$$

where $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$. We will assume that for all $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$\frac{\partial}{\partial u} f(u,v) \le 0, \qquad \frac{\partial}{\partial v} f(u,v) < 0, \qquad \frac{d}{du}\big[uf(u,u)\big] > 0. \tag{9.5.42}$$

We will investigate the global asymptotic stability of the unique equilibrium $\overline{x}$ of equation (9.5.41).

The following technical lemma, which is a consequence of the condition (9.5.42) imposed on f, will be useful in the sequel.

Lemma 9.5.15. *Assume that* $0 < a < \overline{x} < b$. *Then the following inequalities are true:*

$$af(a,a) < \overline{x}, \qquad bf(b,b) > \overline{x}, \tag{9.5.43}$$

$$\overline{x}f\left(\frac{\overline{x}^2}{a}, \frac{\overline{x}^2}{a}\right) > a, \qquad \overline{x}f\left(\frac{\overline{x}^2}{b}, \frac{\overline{x}^2}{b}\right) < b, \tag{9.5.44}$$

$$\overline{x}f\big(\overline{x}f(a,a), \overline{x}f(a,a)\big) > a, \qquad \overline{x}f\big(\overline{x}f(b,b), \overline{x}f(b,b)\big) < b. \tag{9.5.45}$$

Proof. The inequalities (9.5.43) are a simple consequence of the increasing character of the function $uf(u,u)$ and the fact that $f(\overline{x},\overline{x}) = 1$. Since $(\overline{x})^2/a > \overline{x}$, the second inequality in (9.5.43) yields

$$\frac{\overline{x}^2}{a} f\left(\frac{\overline{x}^2}{a}, \frac{\overline{x}^2}{a}\right) > \overline{x}, \tag{9.5.46}$$

which proves the first inequality in (9.5.44). The second inequality in (9.5.44) is proved in a similar way. From (9.5.43), $\overline{x}f(a,a) < (\overline{x})^2/a$, and so, from (9.5.44) and the fact that $f(u,u)$ decreases in u, we have $\overline{x}f(\overline{x}f(a,a), \overline{x}f(a,a)) > a$. The second inequality in (9.5.45) is proved similarly. □

The next lemma presents a detailed description of the semicycles of $\{x(k)\}$.

Lemma 9.5.16. (i_1) *If* $x(-1)$ *and* $x(0)$ *are not both equal to* $\overline{x}$, *then a positive semicycle cannot have two consecutive terms equal to* $\overline{x}$.

(i_2) *Every semicycle of* $\{x(k)\}$, *except perhaps the first one, has at least two terms.*

(i_3) *The extreme in a semicycle is equal to the first or to the second term of the semicycle. More precisely, after the first term, the remaining terms in a positive semicycle decrease and the remaining terms in a negative semicycle increase.*

(i_4) *Except perhaps for the first semicycle of a solution, in a semicycle with finitely many terms, the extreme of the semicycle cannot be equal to the last term.*

(i_5) *The maxima in successive positive semicycles are decreasing and the minima in successive negative semicycles are increasing.*

Proof. The proof of (i_1) is simple and will be omitted. For the remaining statements we will only give the proof for positive semicycles whose terms are not equal to $\overline{x}$. The proof for negative semicycles is similar and will be omitted. The proof for the trivial semicycle where $x(-1) = x(0) = \overline{x}$ is obvious.

Now we address (i_2). If $x(k)$ is the first term in a positive semicycle (other than the first semicycle), then

$$x(k+1) = x(k)f(x(k), x(k-1)) > x(k)f(x(k), x(k)) \geq \overline{x}f(\overline{x}, \overline{x}) = \overline{x}, \quad (9.5.47)$$

so $x(k+1)$ is also in the same semicycle.

If $x(k)$ and $x(k+1)$ are two consecutive terms in a positive semicycle, then

$$x(k+2) = x(k+1)f(x(k+1), x(k)) < x(k+1)f(\overline{x}, \overline{x}) = x(k+1). \quad (9.5.48)$$

This shows (i_3).

Next, (i_4) follows from the observation that, unless $x(k) = x(k+1) = \overline{x}$, the inequality in (9.5.48) is strict. On the other hand, if $x(k) = x(k+1) = \overline{x}$, then by (i_1) the entire solution reduces to a positive semicycle which contradicts the hypothesis that the semicycle has finitely many terms.

Finally we prove (i_5). Consider four consecutive semicycles:

$$\begin{aligned} C_{r-1} &= \{x(k+1), x(k+2), \dots, x(l)\}\text{—negative semicycle,} \\ C_r &= \{x(l+1), x(l+2), \dots, x(m)\}\text{—positive semicycle,} \\ C_{r+1} &= \{x(m+1), x(m+2), \dots, x(n)\}\text{—negative semicycle,} \\ C_{r+2} &= \{x(n+1), x(n+2), \dots, x(p)\}\text{—positive semicycle.} \end{aligned} \quad (9.5.49)$$

If $b(r-1)$, $b(r)$, $b(r+1)$, $b(r+2)$ denote the extreme values in these four semicycles, respectively, we must prove that

$$b(r+2) < b(r), \qquad b(r-1) < b(r+1). \quad (9.5.50)$$

It follows from (i_3) that either $b(r) = x(l+1)$, or that $b(r) = x(l+2)$. In the first case

$$b(r) = x(l)f(x(l), x(l-1)) < \overline{x}f(b(r-1), b(r-1)). \quad (9.5.51)$$

In the second case

$$\begin{aligned} b(r) &= x(l+1)f(x(l+1), x(l)) \\ &= x(l)f(x(l), x(l-1))f(x(l+1), x(l)) \\ &\leq x(l)f(x(l), x(l))f(x(l), x(l-1)) \\ &< \overline{x}f(b(r-1), b(r-1)). \end{aligned} \quad (9.5.52)$$

Thus in either case

$$b(r) < \overline{x} f\big(b(r-1), b(r-1)\big). \tag{9.5.53}$$

In a similar way we obtain

$$b(r+1) > \overline{x} f\big(b(r), b(r)\big). \tag{9.5.54}$$

Hence

$$\begin{aligned} b(r+2) &< \overline{x} f\big(b(r+1), b(r+1)\big) \\ &< \overline{x} f\big(\overline{x} f\big(b(r), b(r)\big), \overline{x} f\big(b(r), b(r)\big)\big), \end{aligned} \tag{9.5.55}$$

$$\begin{aligned} b(r+1) &> \overline{x} f\big(b(r), b(r)\big) \\ &> \overline{x} f\big(\overline{x} f\big(b(r-1), b(r-1)\big), \overline{x} f\big(b(r-1), b(r-1)\big)\big). \end{aligned} \tag{9.5.56}$$

From these and (9.5.45) it follows that (9.5.50) holds, and the proof of Lemma 9.5.16 is complete. □

Now we present the following result.

Theorem 9.5.17. *Assume that (9.5.42) holds and equation (9.5.41) has a unique equilibrium $\overline{x}$. Then $\overline{x}$ is globally asymptotically stable.*

Proof. Let $\{x(k)\}$ be any solution of equation (9.5.41) with initial conditions $x(-1), x(0) \in \mathbb{R}^+$. We must prove that $\overline{x}$ is locally asymptotically stable and that

$$\lim_{k\to\infty} x(k) = \overline{x}. \tag{9.5.57}$$

The linearized equation associated with equation (9.5.41) about $\overline{x}$ is

$$y(k+1) + py(k) + qy(k-1) = 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.58}$$

where

$$p = -1 - \overline{x} f_u(\overline{x}, \overline{x}), \qquad q = -\overline{x} f_v(\overline{x}, \overline{x}). \tag{9.5.59}$$

By Theorem 9.5.6(I_3), the trivial solution of equation (9.5.58) is asymptotically stable if and only if $|p| < 1 + q < 2$. This is clearly satisfied because $f(\overline{x}, \overline{x}) = 1$ and from the hypotheses on f

$$f_u(\overline{x}, \overline{x}) \le 0, \qquad f_v(\overline{x}, \overline{x}) < 0, \qquad 1 + \overline{x} f_u(\overline{x}, \overline{x}) + \overline{x} f_v(\overline{x}, \overline{x}) > 0. \tag{9.5.60}$$

Next, we will establish (9.5.57). This is a simple consequence of Lemma 9.5.16(i_3), if the solution is nonoscillatory. So assume that $\{x(k)\}$ is oscillatory and set

$$\lambda = \liminf_{k\to\infty} x(k), \qquad \Lambda = \limsup_{k\to\infty} x(k), \tag{9.5.61}$$

which by Lemma 9.5.16(i_5) both exist and satisfy $0 < \lambda \le \bar{x} \le \Lambda$. It follows from (9.5.55) and (9.5.56) that

$$\Lambda \le \bar{x} f(\bar{x} f(\Lambda, \Lambda), \bar{x} f(\Lambda, \Lambda)), \qquad \lambda \ge \bar{x} f(\bar{x} f(\lambda, \lambda), \bar{x} f(\lambda, \lambda)). \tag{9.5.62}$$

In view of (9.5.45), these imply that $\lambda = \bar{x} = \Lambda$, and the proof of Theorem 9.5.17 is complete. □

Example 9.5.18. As an application of Theorem 9.5.17, one can easily show that the positive equilibrium of the nonlinear difference equation

$$x(k+1) = \frac{ax(k)}{1 + bx(k) + cx(k-1)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.63}$$

where $a \in (1, \infty)$ and $b, c \in \mathbb{R}^+$, is a global attractor of all positive solutions.

9.5.3. A rational recursive sequence

We consider the recursive sequence

$$x(k+1) = \frac{ax(k) + bx(k-1)}{cx(k) + dx(k-1)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.64}$$

where the parameters a, b, c, and d are positive numbers with $ad - bc \neq 0$, and the initial conditions are arbitrary positive numbers. Here we will investigate the global stability character and the periodic nature of solutions of equation (9.5.64). The only equilibrium of equation (9.5.64) is $\bar{x} = (a+b)/(c+d)$, and the linearized equation about $\bar{x}$ is

$$y(k+1) - \frac{ad - bc}{(a+b)(c+d)} y(k) + \frac{ad - bc}{(a+b)(c+d)} y(k-1) = 0. \tag{9.5.65}$$

By Theorem 9.5.6(I_3), the linearized equation is asymptotically stable if

$$\left| \frac{ad - bc}{(a+b)(c+d)} \right| < 1 + \frac{ad - bc}{(a+b)(c+d)} < 2. \tag{9.5.66}$$

This condition is satisfied if $ad - bc > 0$. On the other hand, when $ad - bc < 0$, (9.5.66) is satisfied if and only if

$$bc < ac + bd + 3ad. \tag{9.5.67}$$

We present the following result about the local stability of the equilibrium $\bar{x}$ of equation (9.5.64).

Theorem 9.5.19. (i_1) *Assume that $ad - bc > 0$. Then the positive equilibrium of equation (9.5.64) is locally asymptotically stable.*

(i_2) *Assume that $ad - bc < 0$. Then the positive equilibrium of equation (9.5.64) is locally asymptotically stable if (9.5.67) holds, and is unstable (and more precisely a saddle point equilibrium) if*

$$bc > ac + bd + 3ad. \tag{9.5.68}$$

9.5.3.1. Existence of a two-cycle

It is easy to see that when $ad - bc > 0$, equation (9.5.64) has no prime period-2 solutions. On the other hand when $ad - bc < 0$ and (9.5.68) holds, equation (9.5.64) possesses the (essentially unique) two-cycle:

$$\dots, p, q, p, q, \dots, \tag{9.5.69}$$

where p and q are the (positive and distinct) solutions of the quadratic equation

$$y^2 - \frac{b-a}{d}y + \frac{a(b-a)}{d(c-d)} = 0. \tag{9.5.70}$$

In order to investigate the stability nature of this two-cycle, we set $u(k) = x(k-1)$ and $v(k) = x(k)$ for $k \in \mathbb{N}_0$ and write equation (9.5.64) in the equivalent form ($k \in \mathbb{N}_0$)

$$\begin{aligned} u(k+1) &= v(k), \\ v(k+1) &= \frac{av(k) + bu(k)}{cv(k) + du(k)}. \end{aligned} \tag{9.5.71}$$

Let T be the function on $\mathbb{R}^+ \times \mathbb{R}^+$ defined by

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{av + bu}{cv + du} \end{pmatrix}. \tag{9.5.72}$$

Then $\binom{p}{q}$ is a fixed point of T^2, the second iterate of T. Now one can see that

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}, \tag{9.5.73}$$

where

$$g(u,v) = \frac{av + bu}{cv + du}, \qquad h(u,v) = \frac{a((av+bu)/(cv+du)) + bu}{c((av+bu)/(cv+du)) + du}. \tag{9.5.74}$$

The two-cycle is asymptotically stable if the eigenvalues of the Jacobian matrix J_{T^2}, evaluated at $\binom{p}{q}$, lie inside the unit disk. One can see that

$$J_{T^2}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -\dfrac{(ad-bc)q}{(cq+dp)^2} & \dfrac{(ad-bc)p}{(cq+dp)^2} \\ -\dfrac{(ad-bc)^2q^2}{(cq+dp)^2(cp+dq)^2} & \dfrac{(ad-bc)p}{(cp+dq)^2}\left(\dfrac{(ad-bc)q}{(cq+dp)^2}-1\right) \end{pmatrix}, \tag{9.5.75}$$

and its eigenvalues lie inside the unit disk $|\lambda| < 1$ if and only if

$$pq(ad-bc)^2 < (cq+dp)^2(cp+dq)^2. \tag{9.5.76}$$

This inequality is equivalent to $bc > ac + bd$, which is clearly satisfied.

We present the following result about the local stability of the two-cycle (9.5.69) of equation (9.5.64).

Theorem 9.5.20. *Assume that condition (9.5.68) holds. Then equation (9.5.64) possesses the (essentially unique) two-cycle (9.5.69), where p and q are the two positive and distinct roots of the equation (9.5.70). Furthermore, this two-cycle is locally asymptotically stable.*

Next, we offer a semicycle analysis of the solutions of (9.5.64).

Theorem 9.5.21. *Let $\{x(k)\}$ be a nontrivial positive solution of equation (9.5.64). Then the following statements are true.*

(i_1) *Assume $ad - bc > 0$. Then $\{x(k)\}$ oscillates about the equilibrium $\overline{x}$ with semicycles of length two or three, except possibly for the first semicycle which may have length one. The extreme in each semicycle occurs in the first term if the semicycle has two terms and in the second term if the semicycle has three terms.*

(i_2) *Assume $ad - bc < 0$. Then either $\{x(k)\}$ oscillates about the equilibrium $\overline{x}$ with semicycles of length one after the first semicycle, or $\{x(k)\}$ converges monotonically to $\overline{x}$.*

Proof. It follows from equation (9.5.64) that

$$x(k+1) - \overline{x} = \left(\frac{ad-bc}{c+d}\right)\left(\frac{x(k)-x(k-1)}{cx(k)+dx(k-1)}\right) \quad \text{for } k \in \mathbb{N}_0. \tag{9.5.77}$$

Concerning (i_1), we will first show that every positive semicycle has two or three terms, except possibly for the first semicycle which may contain only one term. The case of negative semicycles is similar and will be omitted. Let $x(N-1) > \overline{x}$ be the first term in a positive semicycle. First, assume that $x(N-2) < \overline{x}$. Then by equation (9.5.77), $x(N) > \overline{x}$. If $x(N) \geq x(N-1)$, then by equation (9.5.77),

$x(N+1) \geq \overline{x}$ and

$$x(N+1) = \frac{ax(N) + bx(N-1)}{cx(N) + dx(N-1)} < \frac{ax(N) + bx(N)}{c\overline{x} + d\overline{x}} = x(N). \tag{9.5.78}$$

Now, in view of equation (9.5.77), we have that $x(N+2) < \overline{x}$, which shows that the positive semicycle has three terms. If on the other hand $\overline{x} < x(N) < x(N-1)$, then by equation (9.5.77), $x(N+1) < \overline{x}$, and so the positive semicycle in this case has two terms. Finally, it remains to show that the very first semicycle of a nontrivial solution if it is a positive semicycle, contains at most three terms. More precisely, it is easy to show the following.

(i) If $\overline{x} \leq x(N-1) \leq x(N)$, then $x(N) \geq x(N+1) \geq \overline{x}$ and $x(N+2) < \overline{x}$.

(ii) If $\overline{x} \leq x(N) < x(N-1)$, then $x(N+1) < \overline{x}$.

Concerning (i_2), it is easy to see that every nontrivial nonoscillatory solution converges monotonically to the equilibrium. Next, assume that there are two consecutive terms $x(N-1)$ and $x(N)$ such that $x(N-1) > \overline{x} > x(N)$. Then by equation (9.5.77), $x(N+1) > \overline{x}$, $x(N+2) < \overline{x}$, and so forth. The case where $x(N-1) < \overline{x} < x(N)$ is similar. This completes the proof. □

9.5.3.2. Global stability analysis when $ad - bc < 0$

First, we will present a global stability result for the more general equation (9.5.7) with $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ that satisfies the following conditions:

(i) there exist $\alpha, \beta > 0$ such that $\alpha \leq f(x,y) \leq \beta$ for all $x, y > 0$,

(ii) $f(x,y)$ is decreasing in x for each fixed y, and $f(x,y)$ is increasing in y for each fixed x,

(iii) equation (9.5.7) has no two-cycle.

We present the following result.

Theorem 9.5.22. *Assume that conditions (i)–(iii) hold. Then equation (9.5.7) has a unique positive equilibrium $\overline{x}$, and every positive solution of equation (9.5.7) converges to $\overline{x}$.*

Proof. Set $m_0 = \alpha$ and $M_0 = \beta$. For $i \in \mathbb{N}$, set $m_i = f(M_{i-1}, m_{i-1})$ and $M_i = f(m_{i-1}, M_{i-1})$. Now observe that

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0, \tag{9.5.79}$$

and $m_i \leq x(k) \leq M_i$ for $k \geq 2i+1$. Set $m = \lim_{i\to\infty} m_i$ and $M = \lim_{i\to\infty} M_i$. Then clearly

$$M \geq \limsup_{i\to\infty} x(i) \geq \liminf_{i\to\infty} x(i) \geq m, \tag{9.5.80}$$

and by the continuity of f, $m = f(M,m)$ and $M = f(m,M)$. Hence $m = M$, for otherwise equation (9.5.7) would have the two-cycle $m, M, m, M, \ldots$ which would contradict the hypothesis. □

The following result is a corollary of Theorem 9.5.22.

Theorem 9.5.23. *Assume that* $ad - bc < 0$ *and condition (9.5.67) holds. Then the positive equilibrium* $\overline{x} = (a+b)/(c+d)$ *of equation (9.5.64) is globally asymptotically stable.*

Proof. Set $f(x, y) = (ax + by)/(cx + dy)$. Thus

$$\begin{aligned} f_x(x, y) &= (ad - bc)\frac{y}{(cx + dy)^2}, \\ f_y(x, y) &= -(ad - bc)\frac{x}{(cx + dy)^2}. \end{aligned} \tag{9.5.81}$$

Therefore $f(x, y)$ is decreasing in x for each fixed y, and increasing in y for each fixed x. Also, clearly

$$\frac{a}{c} \le f(x, y) \le \frac{b}{d} \quad \forall x, y > 0. \tag{9.5.82}$$

Finally, in view of condition (9.5.67), equation (9.5.64) has no two-cycles. Now the conclusion of Theorem 9.5.23 follows as a consequence of Theorem 9.5.22 and the fact that $\overline{x}$ is locally asymptotically stable. □

The method employed in the proof of Theorem 9.5.22 can also be used to establish that certain solutions of equation (9.5.64) converge to the two-cycle (9.5.69) when (9.5.68) holds instead of (9.5.67).

Theorem 9.5.24. *Assume that (9.5.68) holds. Let* $p, q, p, q, \ldots$ *with* $p < q$ *denote the two-cycle of equation (9.5.64). Assume that for some solution* $\{x(k)\}_{k=-1}^{\infty}$ *of equation (9.5.64) and for some index* $N \ge -1$,

$$x(N) \ge q, \qquad x(N + 1) \le p. \tag{9.5.83}$$

Then this solution of (9.5.64) converges to the two-cycle $p, q, p, q, \ldots$.

Proof. Assume that (9.5.83) holds. Set $f(x, y) = (ax + by)/(cx + dy)$. Then clearly

$$\begin{aligned} x(N + 2) &= f\big(x(N + 1), x(N)\big) \ge f(p, q) = q, \\ x(N + 3) &= f\big(x(N + 2), x(N + 1)\big) \le f(p, q) = p, \end{aligned} \tag{9.5.84}$$

and in general $x(N + 2n) \ge q$ and $x(N + 2n + 1) \le p$ for $n \in \mathbb{N}_0$. Now, as in the proof of Theorem 9.5.22, $\limsup_{k\to\infty} x(k) = q$ and $\liminf_{k\to\infty} x(k) = p$, from which we conclude that $\lim_{n\to\infty} x(N+2n) = q$ and $\lim_{n\to\infty} x(N+2n+1) = p$. This completes the proof. □

9.5.3.3. Global stability analysis when $ad - bc > 0$

We will present a global stability result for the more general equation (9.5.7) with $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ that satisfies the following conditions.

(i) There exist $\alpha, \beta > 0$ such that $\alpha \le f(x,y) \le \beta$ for $x, y > 0$.

(ii) $f(x,y)$ is increasing in x for each fixed y, and $f(x,y)$ is decreasing in y for each fixed x.

(iii) If $(m,M) \in \mathbb{R}^+ \times \mathbb{R}^+$ is a solution of the system $m = f(m,M)$ and $M = f(M,m)$, then $m = M$.

Now we present the following result.

Theorem 9.5.25. *Assume that conditions (i)–(iii) hold. Then equation (9.5.7) has a unique positive equilibrium $\overline{x}$, and every positive solution of equation (9.5.7) converges to $\overline{x}$.*

Proof. Set $m_0 = \alpha$ and $M_0 = \beta$. For $i \in \mathbb{N}$, set $M_i = f(M_{i-1}, m_{i-1})$ and $m_i = f(m_{i-1}, M_{i-1})$. Now observe that

$$m_0 \le m_1 \le \cdots \le m_i \le \cdots \le \cdots \le M_i \le \cdots \le M_1 \le M_0 \tag{9.5.85}$$

and $m_i \le x(k) \le M_i$ for $k \ge 2i+1$. Set $m = \lim_{i\to\infty} m_i$ and $M = \lim_{i\to\infty} M_i$. Then clearly (9.5.80) holds, and by the continuity of f, $m = f(m,M)$ and $M = f(M,m)$. Therefore, in view of (iii), $m = M$. This completes the proof. □

Now we obtain the following global stability result for equation (9.5.64) which is a special case of Theorem 9.5.25.

Theorem 9.5.26. *Assume that $ad - bc > 0$ and*

$$ad \le ac + bd + 3bc. \tag{9.5.86}$$

Then the positive equilibrium $\overline{x} = (a+b)/(c+d)$ of equation (9.5.64) is globally asymptotically stable.

Proof. In the case of equation (9.5.64) we have $f(x,y) = (ax+by)/(cx+dy)$, and when $ad - bc > 0$, f is increasing in x for each fixed y, and decreasing in y for each fixed x. Also

$$\frac{b}{d} \le f(x,y) \le \frac{a}{c} \quad \forall x, y > 0. \tag{9.5.87}$$

The system of M and m is

$$M = \frac{aM + bm}{cM + dm}, \qquad m = \frac{am + bM}{cm + dM}, \tag{9.5.88}$$

and we can easily show that when (9.5.86) holds, $m = M$. This completes the proof. □

The following result, which is a minor modification of Theorem 9.5.22, will be employed in the next subsection. We consider (9.5.7) with $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ that satisfies the following conditions:

(i) there exists $0 < \alpha < \beta$ such that $\alpha \le f(x, y) \le \beta$ for all $x, y \in [\alpha, \beta]$,

(ii) $f(x, y)$ is nonincreasing in $x \in [\alpha, \beta]$ for each $y \in [\alpha, \beta]$, and $f(x, y)$ is nondecreasing in $y \in [\alpha, \beta]$ for each $x \in [\alpha, \beta]$,

(iii) equation (9.5.7) has no solutions of prime period 2 in $[\alpha, \beta]$.

Also, we assume the initial conditions $x(-1), x(0) \in \mathbb{R}^+$.

Theorem 9.5.27. *Assume that conditions (i)–(iii) hold. Then there exists exactly one equilibrium $\overline{x}$ of equation (9.5.7) which lies in $[\alpha, \beta]$. Moreover, every solution of equation (9.5.7) which lies in $[\alpha, \beta]$ converges to $\overline{x}$.*

9.5.4. On the recursive sequence $x(k+1) = \alpha + (x(k-1)/x(k))$

We will study the global stability, the boundedness, and the periodic nature of the positive solutions of the difference equation

$$x(k+1) = \alpha + \frac{x(k-1)}{x(k)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.89}$$

where $\alpha \in [0, \infty)$ and where the initial conditions $x(-1)$ and $x(0)$ are arbitrary positive real numbers.

Clearly, the only equilibrium point of equation (9.5.89) is $\overline{x} = \alpha + 1$. The linearized equation of (9.5.89) about the equilibrium point $\overline{x} = \alpha + 1$ is

$$y(k+1) + \frac{1}{\alpha+1} y(k) - \frac{1}{\alpha+1} y(k-1) = 0 \quad \text{for } k \in \mathbb{N}_0. \tag{9.5.90}$$

The following lemma is a simple consequence of Theorem 9.5.6.

Lemma 9.5.28. *The following statements are true.*

(a_1) *The equilibrium point $\overline{x} = \alpha + 1$ of equation (9.5.89) is locally asymptotically stable if $\alpha > 1$.*

(a_2) *The equilibrium point $\overline{x} = \alpha + 1$ of equation (9.5.89) is unstable (and in fact is a saddle point) if $0 \le \alpha < 1$.*

The proofs of the following three lemmas follow from simple computation and will be omitted.

Lemma 9.5.29. *The following statements are true.*

(b_1) *Equation (9.5.89) has solutions of prime period 2 if and only if $\alpha = 1$.*

(b_2) *Suppose $\alpha = 1$. Let $\{x(k)\}_{k=-1}^{\infty}$ solve (9.5.89). Then $\{x(k)\}_{k=-1}^{\infty}$ is periodic with period 2 if and only if $x(-1) \neq 1$ and $x_0 = x(-1)/[x(-1) - 1]$.*

Lemma 9.5.30. *Let $\{x(k)\}_{k=-1}^{\infty}$ be a solution of equation (9.5.89) which is eventually constant. Then $\{x(k)\}_{k=-1}^{\infty}$ is the trivial solution $x(k) \equiv \alpha + 1$ for $k \in \mathbb{N}_0 \cup \{-1\}$.*

Lemma 9.5.31. *Let $\{x(k)\}_{k=-1}^{\infty}$ be a solution of equation (9.5.89), and let $L > \alpha$. Then the following statements are true:*

(c_1) $\lim_{k\to\infty} x(2k) = L$ *if and only if* $\lim_{k\to\infty} x(2k+1) = L/(L-\alpha)$,
(c_2) $\lim_{k\to\infty} x(2k+1) = L$ *if and only if* $\lim_{k\to\infty} x(2k) = L/(L-\alpha)$.

Next we present a semicycle analysis of the solutions of (9.5.89).

Lemma 9.5.32. *Let $\{x(k)\}_{k=-1}^{\infty}$ be a positive solution of equation (9.5.89) which consists of a single semicycle. Then $\{x(k)\}_{k=-1}^{\infty}$ converges monotonically to $\overline{x} = \alpha + 1$.*

Proof. Suppose $0 < x(k-1) < \alpha + 1$ for all $k \geq 0$. The case $x(k-1) > \alpha + 1$ for all $k \geq 0$ is similar and will be omitted. Note that for $k \in \mathbb{N}_0$,

$$0 < \alpha + \frac{x(k-1)}{x(k)} = x(k+1) < \alpha + 1, \tag{9.5.91}$$

and so $0 < x(k-1) < x(k) < \alpha + 1$ from which the result follows. □

Lemma 9.5.33. *Let $\{x(k)\}_{k=-1}^{\infty}$ be a positive solution of equation (9.5.89) which consists of at least two semicycles. Then $\{x(k)\}_{k=-1}^{\infty}$ is oscillatory. Moreover, with the possible exception of the first semicycle, every semicycle has length one and every term of $\{x(k)\}_{k=-1}^{\infty}$ is strictly greater than α, and with possible exception of the first two semicycles, no term of $\{x(k)\}_{k=-1}^{\infty}$ is ever equal to $\alpha + 1$.*

Proof. It suffices to consider the following two cases.

Case 1. Suppose $x(-1) < \alpha + 1 \leq x(0)$. Then

$$x(1) = \alpha + \frac{x(-1)}{x(0)} < \alpha + 1, \qquad x(2) = \alpha + \frac{x(0)}{x(1)} > \alpha + 1. \tag{9.5.92}$$

Case 2. Suppose $x(0) < \alpha + 1 \leq x(-1)$. Then

$$x(1) = \alpha + \frac{x(-1)}{x(0)} > \alpha + 1, \qquad x(2) = \alpha + \frac{x(0)}{x(1)} < \alpha + 1. \tag{9.5.93}$$

This completes the proof. □

The following lemma will be useful in the sequel when determining the limiting behavior of positive solutions of equation (9.5.89).

Lemma 9.5.34. *Let $\{x(k)\}_{k=-1}^{\infty}$ be a positive solution of equation (9.5.89) and let $N \in \mathbb{N}_0$. Then the following statements are true:*

(i_1) $x(N+1) > x(N-1)$ *if and only if* $x(N-1) + \alpha x(N) - x(N-1)x(N) > 0$,
(i_2) $x(N+1) = x(N-1)$ *if and only if* $x(N-1) + \alpha x(N) - x(N-1)x(N) = 0$,
(i_3) $x(N+1) < x(N-1)$ *if and only if* $x(N-1) + \alpha x(N) - x(N-1)x(N) < 0$.

Proof. The computation

$$\begin{aligned} x(N+1) - x(N-1) &= \left(\alpha + \frac{x(N-1)}{x(N)}\right) - x(N-1) \\ &= \frac{\alpha x(N) + x(N-1) - x(N-1)x(N)}{x(N)}. \end{aligned} \tag{9.5.94}$$

yields all claims. □

Next we consider equation (9.5.89) and the following three cases: $0 \le \alpha < 1$, $\alpha = 1$, and $\alpha > 1$.

9.5.4.1. The case $0 \le \alpha < 1$

In this case we will show that there exist positive solutions of equation (9.5.89) which are unbounded.

Theorem 9.5.35. *Let $0 \le \alpha < 1$ and let $\{x(k)\}_{k=-1}^{\infty}$ be a solution of equation (9.5.89) such that $0 < x(-1) < 1$ and $x(0) \ge 1/(1-\alpha)$. Then the following statements are true:*

(i) $\lim_{k\to\infty} x(2k) = \infty$,
(ii) $\lim_{k\to\infty} x(2k+1) = \alpha$.

Proof. Note that $1/(1-\alpha) > \alpha + 1$ and so $x(0) > \alpha + 1$. It suffices to show that $x(1) \in (\alpha, 1]$ and $x(2) \ge \alpha + x(0)$. Indeed

$$x(1) = \alpha + \frac{x(-1)}{x(0)} > \alpha. \tag{9.5.95}$$

Also

$$x(1) = \alpha + \frac{x(-1)}{x(0)} \le \alpha + \frac{1}{x(0)} \le \alpha + (1-\alpha) = 1, \tag{9.5.96}$$

and so $x(1) \in (\alpha, 1]$. Hence

$$x(2) = \alpha + \frac{x(0)}{x(1)} \ge \alpha + x(0), \tag{9.5.97}$$

which completes the proof. □

9.5.4.2. The case $\alpha = 1$

In this case we will show that every positive solution of equation (9.5.89) converges to a two-cycle. Clearly, if $\alpha = 1$, then the unique equilibrium point of equation (9.5.89) is $\bar{x} = 2$.

Theorem 9.5.36. *Let $\alpha = 1$ and let $\{x(k)\}_{k=-1}^{\infty}$ be a positive solution of equation (9.5.89). Then the following statements are true.*

(a_1) *Suppose $\{x(k)\}_{k=-1}^{\infty}$ consists of a single semicycle. Then $\{x(k)\}_{k=-1}^{\infty}$ converges monotonically to $\overline{x} = 2$.*

(a_2) *Suppose $\{x(k)\}_{k=-1}^{\infty}$ consists of at least two semicycles. Then $\{x(k)\}_{k=-1}^{\infty}$ converges to a prime period 2 solution of equation (9.5.89).*

Proof. From Lemma 9.5.32 it follows that if $\{x(k)\}_{k=-1}^{\infty}$ consists of a single semicycle, then $\{x(k)\}_{k=-1}^{\infty}$ converges monotonically to $\overline{x}$. Hence it suffices to consider the case where $\{x(k)\}_{k=-1}^{\infty}$ consists of at least two semicycles. So, assume that $\{x(k)\}_{k=-1}^{\infty}$ consists of at least two semicycles. By Lemma 9.5.33, $\{x(k)\}_{k=-1}^{\infty}$ is oscillatory and, except for possibly the first semicycle, every semicycle has length one and every term of $\{x(k)\}_{k=-1}^{\infty}$ is greater than $\alpha = 1$. Now observe that for $k \in \mathbb{N}_0$,

$$x(k) + x(k+1) - x(k)x(k+1) = \frac{x(k-1) + x(k) - x(k-1)x(k)}{x(k)} \tag{9.5.98}$$

and so, by Lemma 9.5.34, the following three statements are true.

(a) Suppose $x(-1) < x(1)$. Then we have $x(-1) < x(1) < x(3) < \cdots$ and $x(0) < x(2) < x(4) < \cdots$.

(b) Suppose $x(-1) = x(1)$. Then we have $x(-1) = x(1) = x(3) = \cdots$ and $x(0) = x(2) = x(4) = \cdots$.

(c) Suppose $x(-1) > x(1)$. Then we have $x(-1) > x(1) > x(3) > \cdots$ and $x(0) > x(2) > x(4) > \cdots$.

The proof of the theorem follows from Lemma 9.5.31 and the statements (a), (b), and (c) above. □

9.5.4.3. The case $\alpha > 1$

In this case we will show that the equilibrium point $\overline{x} = \alpha + 1$ of equation (9.5.89) is globally asymptotically stable.

First we give the following lemma which will be useful in the sequel.

Lemma 9.5.37. *Let $\alpha > 1$ and let $\{x(k)\}_{k=-1}^{\infty}$ be a positive solution of equation (9.5.89). Then*

$$\alpha + \frac{\alpha - 1}{\alpha} \le \liminf_{k\to\infty} x(k) \le \limsup_{k\to\infty} x(k) \le \frac{\alpha^2}{\alpha - 1}. \tag{9.5.99}$$

Proof. It follows by Lemmas 9.5.32 and 9.5.33 that we may assume that every semicycle of $\{x(k)\}_{k=-1}^{\infty}$ has length one, that $\alpha \le x(k)$ for all $k \ge -1$, and that $\alpha < x(0) < \alpha + 1 < x(-1)$. We will first show that

$$\limsup_{k\to\infty} x(k) \le \frac{\alpha^2}{\alpha - 1}. \tag{9.5.100}$$

Note that for $k \in \mathbb{N}_0$,

$$x(2k+1) < \alpha + \frac{x(2k-1)}{\alpha}. \tag{9.5.101}$$

So, as every solution of the difference equation

$$y(m+1) = \alpha + \frac{1}{\alpha} y(m) \quad \text{for } m \in \mathbb{N}_0 \tag{9.5.102}$$

converges to $\alpha^2/(\alpha-1)$, it follows that (9.5.100) holds. We will next show that

$$\alpha + \frac{\alpha-1}{\alpha} \leq \liminf_{k\to\infty} x(k). \tag{9.5.103}$$

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}_0$ such that for all $k \geq N$, $x(2k-1) < (\alpha^2+\varepsilon)/(\alpha-1)$. Let $k \geq N$. Then

$$\begin{aligned} x(2k) &= \alpha + \frac{x(2k-2)}{x(2k-1)} > \alpha + \alpha\left(\frac{\alpha-1}{\alpha^2+\varepsilon}\right) \\ &= \frac{\alpha^3 + \alpha\varepsilon + \alpha(\alpha-1)}{\alpha^2+\varepsilon}, \end{aligned} \tag{9.5.104}$$

and so

$$\liminf_{k\to\infty} x(k) \geq \frac{\alpha^3 + \alpha\varepsilon + \alpha(\alpha-1)}{\alpha^2+\varepsilon}. \tag{9.5.105}$$

Therefore, as ε is arbitrary, we have

$$\liminf_{k\to\infty} x(k) \geq \frac{\alpha^3 + \alpha(\alpha-1)}{\alpha^2} = \alpha + \frac{\alpha-1}{\alpha}. \tag{9.5.106}$$

This completes the proof. □

Finally we present the following result.

Theorem 9.5.38. *Let $\alpha > 1$. Then $\overline{x} = \alpha + 1$ is a globally asymptotically stable equilibrium point of equation (9.5.89).*

Proof. From Lemma 9.5.28 it follows that $\overline{x} = \alpha + 1$ is a locally asymptotically stable equilibrium point of equation (9.5.89). So, let $\{x(k)\}_{k=-1}^{\infty}$ be a positive solution of equation (9.5.89). It suffices to show that $\lim_{k\to\infty} x(k) = \alpha + 1$. For $x, y \in \mathbb{R}^+$, set $f(x,y) = \alpha + (y/x)$. Then $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that f is decreasing in $x \in \mathbb{R}^+$ for each $y \in \mathbb{R}^+$ and f is increasing in $y \in \mathbb{R}^+$

for each $x \in \mathbb{R}^+$. Recall that by Lemma 9.5.29 there exist no solutions of equation (9.5.89) with prime period 2. Let $\varepsilon > 0$, and set $a = \alpha$ and $b = (\alpha^2 + \varepsilon)/(\alpha - 1)$. Note that

$$\begin{gathered} f\left(\frac{\alpha^2+\varepsilon}{\alpha-1}, \alpha\right) = \alpha + \alpha\left(\frac{\alpha-1}{\alpha^2+\varepsilon}\right) > \alpha, \\ f\left(\alpha, \frac{\alpha^2+\varepsilon}{\alpha-1}\right) = \alpha + \frac{1}{\alpha}\left(\frac{\alpha^2+\varepsilon}{\alpha-1}\right) = \frac{\alpha^3+\varepsilon}{\alpha^2-\alpha} = \frac{\alpha^2+\varepsilon/\alpha}{\alpha-1} < \frac{\alpha^2+\varepsilon}{\alpha-1}. \end{gathered} \tag{9.5.107}$$

Hence

$$\alpha < f(x,y) < \frac{\alpha^2+\varepsilon}{\alpha-1} \quad \forall x, y \in \left[\alpha, \frac{\alpha^2+\varepsilon}{\alpha-1}\right]. \tag{9.5.108}$$

Finally, note that by Lemma 9.5.37,

$$\alpha < \alpha + \frac{\alpha-1}{\alpha} \le \liminf_{k\to\infty} x(k) \le \limsup_{k\to\infty} x(k) \le \frac{\alpha^2}{\alpha-1} < \frac{\alpha^2+\varepsilon}{\alpha-1}, \tag{9.5.109}$$

and so by Theorem 9.5.27,

$$\lim_{k\to\infty} x(k) = \alpha + 1. \tag{9.5.110}$$

This completes the proof. □

9.5.5. Global stability of a certain recursive sequence

We will investigate the global asymptotic stability of the positive equilibrium of equation (9.5.7), where $f(x,y)$ is strictly decreasing in both arguments. The result applies to the recursive sequence

$$x(k+1) = \frac{a}{x^p(k)} + \frac{1}{x^{1/p}(k-1)}, \tag{9.5.111}$$

where

$$x(-1), x(0), a \in \mathbb{R}^+, \qquad p > 1. \tag{9.5.112}$$

Theorem 9.5.39. *Assume that $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ is strictly decreasing in both arguments, and let $\overline{x}$ denote the unique positive equilibrium of equation (9.5.7). Set*

$$h(x) = \min\left\{f\left(f(x,x),\overline{x}\right), f\left(f(\overline{x},x), f(x,x)\right)\right\}. \tag{9.5.113}$$

Suppose that $h(x) > x$ for $x \in (0, \overline{x})$. Then $\overline{x}$ is globally asymptotically stable.

Proof. It is clear from the decreasing nature of f that the semicycles of every solution of (9.5.7) have length at most two. We will construct a strictly increasing sequence of lower bounds $\{L(k)\}$ for the minimum terms of consecutive negative semicycles. For any $N \in \mathbb{N}_0$, suppose that $x(N)$ is the last term of a negative semicycle. Choose $L(1) > 0$, so that $L(1) < \min\{x(N-1), x(N)\} < \overline{x}$. Let $x(M)$ be the last term of the next negative semicycle. Let $L(2) = h(L(1)) > L(1)$. We will show that $L(2) < \min\{x(M-1), x(M)\} < \overline{x}$. Set $g(x) = f(x,x)$. Now $x(N)$ is the last term of a negative semicycle, so it is true that $x(N+1) \geq \overline{x}$ and

$$x(N+1) = f\big(x(N), x(N-1)\big) < f\big(L(1), L(1)\big) = g\big(L(1)\big). \tag{9.5.114}$$

Thus we have

$$L(1) < \min\big\{x(N-1), x(N)\big\} < \overline{x} \leq x(N+1) < g\big(L(1)\big). \tag{9.5.115}$$

We consider the following two cases.

Case 1. $x(N+2) < \overline{x}$. Then

$$x(N+2) = f\big(x(N+1), x(N)\big) > f\big(g\big(L(1)\big), \overline{x}\big) \geq h\big(L(1)\big) = L(2). \tag{9.5.116}$$

Also

$$\begin{aligned} x(N+3) &= f\big(x(N+2), x(N+1)\big) > f\big(\overline{x}, g\big(L(1)\big)\big) \\ &> f\big(f\big(\overline{x}, L(1)\big), g\big(L(1)\big)\big) \\ &\geq h\big(L(1)\big) \\ &= L(2). \end{aligned} \tag{9.5.117}$$

This last statement follows from the fact that $f(\overline{x}, L(1)) > \overline{x}$. Thus the minimum in this semicycle is larger than $L(2)$, as desired.

Case 2. $x(N+2) \geq \overline{x}$. Then

$$x(N+2) = f\big(x(N+1), x(N)\big) < f\big(\overline{x}, L(1)\big) < g\big(L(1)\big). \tag{9.5.118}$$

Since any semicycle has a maximum length of two, it is clear that $x(N+2) < \overline{x}$. Also

$$\begin{aligned} x(N+3) &= f\big(x(N+2), x(N+1)\big) \\ &> f\big(f\big(\overline{x}, L(1)\big), g\big(L(1)\big)\big) \\ &\geq h\big(L(1)\big) \\ &= L(2). \end{aligned} \tag{9.5.119}$$

Finally,

$$\begin{aligned} x(N+4) &= f(x(N+3), x(N+2)) > f(\overline{x}, g(L(1))) \\ &> f(f(\overline{x}, L(1)), g(L(1))) \\ &\geq h(L(1)) \\ &= L(2). \end{aligned} \tag{9.5.120}$$

Thus the minimum of this semicycle is larger than $L(2)$ also.

So in either case $L(2) < \min\{x(M-1), x(M)\} < \overline{x}$. It is now clear that we can inductively construct a strictly increasing sequence of lower bounds

$$L(1) < L(2) < \cdots < L(k) < L(k+1) < \cdots < \overline{x} \tag{9.5.121}$$

for the minimum terms of consecutive negative semicycles, where we have that $L(k+1) = h(L(k))$. Thus $\lim_{k\to\infty} L(k) = L \leq \overline{x}$. However, since $x = \overline{x}$ is the only solution of $h(x) = \overline{x}$ for $x \leq \overline{x}$, it must be true that $L = \overline{x}$. This implies that the terms of the negative semicycles converge to $\overline{x}$. From here it can be seen that the terms of the positive semicycles also converge to $\overline{x}$, and so $\lim_{k\to\infty} x(k) = \overline{x}$. Hence $\overline{x}$ is a global attractor of all solutions of equation (9.5.7). Local asymptotic stability follows in a similar fashion. The proof is complete. □

Next, we find conditions for the positive equilibrium $\overline{x}$ of equation (9.5.111) to be globally asymptotically stable.

Theorem 9.5.40. *Suppose that (9.5.112) holds. Then a sufficient condition for the global asymptotic stability of the positive equilibrium $\overline{x}$ of equation (9.5.111) is that*

$$0 < a \leq \frac{2\left(1+\sqrt{4p^2-3}\right)^{2p-1}}{4^p\left(p^2-1\right)^p}. \tag{9.5.122}$$

Proof. It is clear that

$$f(x,y) = \frac{a}{x^p} + \frac{1}{y^{1/p}} \tag{9.5.123}$$

is strictly decreasing in both arguments. Thus by Theorem 9.5.39 it suffices to show that for each $x \in (0, \overline{x})$,

$$h(x) = \min\{f(g(x), \overline{x}), f(f(\overline{x}, x), g(x))\} > x, \tag{9.5.124}$$

where

$$g(x) = f(x,x) = \frac{a + x^{(p^2-1)/p}}{x^p}. \tag{9.5.125}$$

Set

$$c(x) = \frac{a(\overline{x})^{p^2}}{\left(ax^{1/p} + (\overline{x})^p\right)^p} + \frac{1}{\left(a + x^{(p^2-1)/p}\right)^{1/p}}. \tag{9.5.126}$$

Then

$$\begin{aligned} f\left(f(\overline{x},x),g(x)\right) &= \frac{a}{\left(a/(\overline{x})^p + 1/x^{1/p}\right)^p} + \frac{1}{\left(\left(a + x^{(p^2-1)/p}\right)/x^p\right)^{1/p}} \\ &= x\left[\frac{a(\overline{x})^{p^2}}{\left(ax^{1/p} + (\overline{x})^p\right)^p} + \frac{1}{\left(a + x^{(p^2-1)/p}\right)^{1/p}}\right] \\ &= xc(x). \end{aligned} \tag{9.5.127}$$

Thus

$$f\left(f(\overline{x},x),g(x)\right) > x \quad \text{for } 0 < x < \overline{x} \tag{9.5.128}$$

if and only if $c(x) > 1$ for $0 < x < \overline{x}$. Since $c(x)$ is decreasing for $a > 0$ and $p > 1$, condition (9.5.128) holds.

Now we consider $f(g(x),\overline{x})$. Set

$$b(x) = f\left(g(x),\overline{x}\right) = \frac{ax^{p^2}}{\left(a + x^{(p^2-1)/p}\right)^p} + \frac{1}{(\overline{x})^{1/p}} \quad \text{for } x > 0. \tag{9.5.129}$$

Then

$$b'(x) = \frac{ax^{p^2-1}\left(ap^2 + x^{(p^2-1)/p}\right)}{\left(a + x^{(p^2-1)/p}\right)^{p+1}} > 0 \quad \text{for } x > 0, \tag{9.5.130}$$

$$b'(\overline{x}) = \frac{a(\overline{x})^{p^2-1}\left(ap^2 + (\overline{x})^{(p^2-1)/p}\right)}{\left(a + (\overline{x})^{(p^2-1)/p}\right)^{p+1}}. \tag{9.5.131}$$

We will show that the following are equivalent:

(i) condition (9.5.122),

(ii) $b'(\overline{x}) \le 1$,

(iii) $b(x) > x$ for $x \in (0,\overline{x})$.

Now

$$\overline{x} = \frac{a}{(\overline{x})^p} + \frac{1}{(\overline{x})^{1/p}}, \tag{9.5.132}$$

and so

$$(\overline{x})^{(p^2-1)/p} = (\overline{x})^{p+1} - a. \tag{9.5.133}$$

Substituting this into (9.5.131) gives

$$b'(\overline{x}) = \frac{a^2p^2 - a^2 + a(\overline{x})^{p+1}}{(\overline{x})^{2(p+1)}}. \tag{9.5.134}$$

From here it is clear that $b'(\overline{x}) \le 1$ if and only if

$$(\overline{x})^{2(p+1)} - a(\overline{x})^{p+1} + a^2(1-p^2) \ge 0. \tag{9.5.135}$$

This is a quadratic inequality in $(\overline{x})^{p+1}$ with positive root

$$(\overline{x})^{p+1} = a\left(\frac{1+\sqrt{4p^2-3}}{2}\right). \tag{9.5.136}$$

Thus (9.5.135) holds if and only if

$$\overline{x} \ge a^{1/(p+1)}\left[\frac{1+\sqrt{4p^2-3}}{2}\right]^{1/(p+1)}. \tag{9.5.137}$$

Let $w(x) = x^{p+1} - x^{(p^2-1)/p} = a$ and let

$$y = a^{1/(p+1)}\left[\frac{1+\sqrt{4p^2-3}}{2}\right]^{1/(p+1)}. \tag{9.5.138}$$

It is easy to see that $x \le \overline{x}$ if and only if $w(x) \le 0$. Thus (9.5.137) holds if and only if $w(y) \le 0$. This condition is equivalent to (9.5.122) which establishes the equivalence of (i) and (ii).

Next,

$$b''(x) = \frac{x^{p^2-2}a^2(p-1)(p+1)\left(ap^3 + x^{(p^2-1)/p}\right)\left(-p^2+p+1\right)}{p\left(a + x^{(p^2-1)/p}\right)}. \tag{9.5.139}$$

Since $p > 1$, it is true that $b''(x) > 0$ for $x > 0$ if and only if $-p^2+p+1 \ge 0$, that is,

$$1 < p \le \frac{1+\sqrt{5}}{2}. \tag{9.5.140}$$

If (9.5.140) holds, then both $b'(x)$ and $b''(x)$ are positive for all $x > 0$. In this case, (ii) and (iii) are clearly equivalent. Now suppose that $p > (1+\sqrt{5})/2$. In this case $b(x)$ has exactly one point of inflection

$$\tilde{x} = \left(\frac{p^3a}{p^2-p-1}\right)^{p/(p^2-1)}, \tag{9.5.141}$$

where $b''(x) > 0$ for $0 < x < \tilde{x}$ and $b''(x) < 0$ for $x > \tilde{x}$. Consider the case if $\overline{x} > \tilde{x}$. Then $b''(x) > 0$ for $x < \overline{x}$. Just as before, (ii) holds if and only if (iii) holds. Finally suppose $\overline{x} \le \tilde{x}$. Then it can be seen that when $b'(\tilde{x}) \le 1$, it is true that $b'(\overline{x}) \le 1$. From here it follows that $b'(x) \le 1$ for all $x > 0$, and so $\tilde{x}$ is the only solution to $b(x) = x$. Thus, (iii) holds.

In any case, we see that condition (9.5.122) is equivalent to the hypotheses of Theorem 9.5.39 holding for equation (9.5.111). The proof is complete. □

9.5.6. Global attractivity in a nonlinear delay difference equation

Here we will investigate the global attractivity of the positive equilibrium $\overline{x}$ of equation (9.4.120) subject to condition (9.4.121).

Theorem 9.5.41. *Assume that condition (9.4.121) holds. Then every positive solution of equation (9.4.120) which is nonoscillatory about $\overline{x}$ tends to $\overline{x}$ as $k \to \infty$.*

Proof. Assume that $x(k) > \overline{x}$ for k sufficiently large. The proof if $x(k) < \overline{x}$ for k sufficiently large is similar and will be omitted. Set $x(k) = \overline{x}e^{y(k)}$. Then $y(k) > 0$ for k sufficiently large and

$$y(k+1) - y(k) + \ln\left[a + b(\overline{x})^p e^{py(k-\tau)} - c(\overline{x})^q e^{qy(k-\tau)}\right] = 0. \tag{9.5.142}$$

Thus

$$y(k+1) - y(k) \le -\ln\left[a + \left(b(\overline{x})^p - c(\overline{x})^q\right)e^{qy(k-\tau)}\right] \le 0, \tag{9.5.143}$$

and so $\lim_{k\to\infty} y(k) = \mu \in [0,\infty)$, say, exists.

We claim that $\mu = 0$. Otherwise, $\mu > 0$. Take

$$0 < \varepsilon < \frac{p-q}{p+q}\mu. \tag{9.5.144}$$

Then there exists $N_0 > 0$ such that for $k \ge N_0$,

$$\mu - \varepsilon < y(k-\tau) < \mu + \varepsilon. \tag{9.5.145}$$

First assume that $c \le 0$. From (9.5.142) and (9.5.145) it follows that

$$y(k+1) - y(k) + \ln\left[a + \left(b(\overline{x})^p - c(\overline{x})^q\right)e^{q(\mu-\varepsilon)}\right] \le 0 \quad \text{for } k \ge N_0, \tag{9.5.146}$$

and by summing (9.5.146) from N_0 to $m \to \infty$ we get a contradiction. Next assume that $c > 0$. Then (9.5.142) and (9.5.145) yield

$$y(k+1) - y(k) + \ln\left[a + b(\overline{x})^p e^{p(\mu-\varepsilon)} - c(\overline{x})^q e^{q(\mu+\varepsilon)}\right] \le 0. \tag{9.5.147}$$

In view of (9.5.144), we have

$$\ln\left[a + b(\overline{x})^p e^{p(\mu-\varepsilon)} - c(\overline{x})^q e^{q(\mu+\varepsilon)}\right] \geq \ln\left[a + \left(b(\overline{x})^p - c(\overline{x})^q\right)e^{q(\mu+\varepsilon)}\right], \qquad (9.5.148)$$

and so (9.5.147) yields

$$y(k+1) - y(k) + \ln\left[a + \left(b(\overline{x})^p - c(\overline{x})^q\right)e^{q(\mu+\varepsilon)}\right] \leq 0 \quad \text{for } k \geq N_0. \qquad (9.5.149)$$

By summing (9.5.149) from N_0 to $m \to \infty$, we obtain a contradiction. This completes the proof. □

We will need the following lemma.

Lemma 9.5.42. *Assume that condition (9.4.121) holds and set*

$$F(x) = a + bx^p - cx^q. \qquad (9.5.150)$$

Then there is a unique positive number $\overline{x}$ such that $F(\overline{x}) = 1$. Furthermore,

$$F(x)\begin{cases} < 1 & \text{for } 0 < x < \overline{x}, \\ > 1 & \text{for } \overline{x} < x < \infty. \end{cases} \qquad (9.5.151)$$

In addition, if $c \leq 0$, then $F(x)$ is increasing for $x > 0$, and if $c > 0$, then

$$F(x)\begin{cases} \text{is decreasing for} & 0 < x < \left(\dfrac{cq}{bp}\right)^{1/(p-q)}, \\ \text{is increasing for} & \left(\dfrac{cq}{bp}\right)^{1/(p-q)} < x < \infty. \end{cases} \qquad (9.5.152)$$

Theorem 9.5.43. *Assume that condition (9.4.121) holds. Set*

$$M_0 = \begin{cases} \left(\dfrac{1}{a}\right)^{\tau+1} & \text{if } c \leq 0, \\ \left[a + b\left(\dfrac{cq}{bp}\right)^p - c\left(\dfrac{cq}{bp}\right)^q\right]^{-\tau-1} & \text{if } c > 0. \end{cases} \qquad (9.5.153)$$

Suppose that

$$\frac{(\tau+1)}{\ln M_0}\ln\left[a + b\left(\overline{x}M_0\right)^p - c\left(\overline{x}M_0\right)^q\right] < 1. \qquad (9.5.154)$$

Then every positive solution of equation (9.4.120) which is oscillatory about $\overline{x}$ tends to $\overline{x}$ as $k \to \infty$.

Proof. Assume that $\{x(k)\}_{k\geq-\tau}$ is a solution of equation (9.4.120) which is oscillatory about $\overline{x}$. We will prove that $\lim_{k\to\infty} x(k) = \overline{x}$. Let $\{k_i\}$ be an increasing sequence of positive integers such that $k_i \to \infty$ as $i \to \infty$ satisfying $x(k_i) < \overline{x}$ and $x(k_i+1) \geq \overline{x}$ for $i \in \mathbb{N}$, and for each $i \in \mathbb{N}$, some of the terms $x(j)$ with $k_i < j \leq k_{i+1}$ are greater than $\overline{x}$ and some are less than $\overline{x}$. For $i \in \mathbb{N}$, let m_i and M_i be the integers in the interval $[k_i, k_{i+1}]$ such that $x(m_i + 1) = \min\{x(j) : k_i < j \leq k_{i+1}\}$ and $x(M_i + 1) = \max\{x(j) : k_i < j \leq k_{i+1}\}$. Then, for each $i \in \mathbb{N}$, $x(m_i + 1) < \overline{x}$ and $\Delta x(m_i) \leq 0$ while $x(M_i + 1) > \overline{x}$ and $\Delta x(M_i) \geq 0$.

By equation (9.4.120) we have

$$0 \geq \Delta x(m_i) = \frac{x(m_i)\left[1 - \left(a + bx^p(m_i - \tau) - cx^q(m_i - \tau)\right)\right]}{a + bx^p(m_i - \tau) - cx^q(m_i - \tau)}, \tag{9.5.155}$$

which indicates that $a + bx^p(m_i - \tau) - cx^q(m_i - \tau) \geq 1$, that is, $x(m_i - \tau) \geq \overline{x}$. Therefore there exists an integer $\overline{m}_i$ satisfying $\max\{k_i, m_i - \tau\} \leq \overline{m}_i < m_i + 1$ and

$$x(\overline{m}_i) \geq \overline{x}, \qquad x(j) < \overline{x} \quad \text{for } j \in \{\overline{m}_i + 1, \ldots, m_i + 1\}. \tag{9.5.156}$$

Similarly, there exists an integer $\overline{M}_i$ satisfying $\max\{k_i, M_i - \tau\} \leq \overline{M}_i < M_i + 1$ and

$$x(\overline{M}_i) \leq \overline{x}, \qquad x(j) > \overline{x} \quad \text{for } j \in \{\overline{M}_i + 1, \ldots, M_i + 1\}. \tag{9.5.157}$$

Now we show that $\{x(k)\}$ is bounded from above and below by positive constants. In fact, since $x(k) > 0$ for $k \geq 0$ it follows by (9.4.120) that

$$\frac{x(k+1)}{x(k)} = \frac{1}{a + bx^p(k - \tau) - cx^q(k - \tau)}. \tag{9.5.158}$$

First assume $c \leq 0$. Then we have

$$\frac{x(k+1)}{x(k)} \leq \frac{1}{a} \quad \text{for } k \in \mathbb{N}_0. \tag{9.5.159}$$

Hence, by multiplying (9.5.159) from $\overline{M}_i$ to M_i, we obtain

$$\frac{x(M_i + 1)}{x(\overline{M}_i)} \leq \left(\frac{1}{a}\right)^{M_i - \overline{M}_i + 1}, \tag{9.5.160}$$

and so $x(M_i + 1) < \overline{x}(1/a)^{\tau+1} = \overline{x}M_0$, which clearly implies that $x(k) \leq \overline{x}M_0$ for $k \in \mathbb{N}_0$. By using this fact in (9.5.158), we find that for $k \in \mathbb{N}_0$,

$$\frac{x(k+1)}{x(k)} \geq \frac{1}{a + b(\overline{x}M_0)^p - c(\overline{x}M_0)^q}, \tag{9.5.161}$$

and so

$$\frac{x(m_i+1)}{x(\overline{m}_i)} \geq \frac{1}{\left[a+b(\overline{x}M_0)^p - c(\overline{x}M_0)^q\right]^{\tau+1}} = M_1, \tag{9.5.162}$$

which implies that $x(k) \geq \overline{x}M_1$ for $k \geq 0$. Next assume that $c > 0$. Then, in view of Lemma 9.5.42, we see from (9.5.158) that for $k \in \mathbb{N}_0$,

$$\frac{x(k+1)}{x(k)} \leq \left[a+b\left(\frac{cq}{bp}\right)^p - c\left(\frac{cq}{bp}\right)^q\right]^{-1}. \tag{9.5.163}$$

Hence we have

$$\frac{x(M_i+1)}{x(\overline{M}_i)} \leq \left[a+b\left(\frac{cq}{bp}\right)^p - c\left(\frac{cq}{bp}\right)^q\right]^{-\tau-1} = M_0, \tag{9.5.164}$$

and so $x(M_i+1) \leq \overline{x}M_0$, which implies that $x(k) \leq \overline{x}M_0$ for $k \in \mathbb{N}_0$. Similarly, we have $x(k) \geq \overline{x}M_1$ for $k \in \mathbb{N}_0$. Therefore, we have $M_1\overline{x} \leq x(k) \leq \overline{x}M_0$ for $k \in \mathbb{N}_0$. Now set

$$g(u) = \begin{cases} \dfrac{1}{u}\ln\left[a+b(\overline{x})^p e^{pu} - c(\overline{x})^q e^{qu}\right] & \text{for } u \neq 0, \\ pb(\overline{x})^p - qc(\overline{x})^q & \text{for } u = 0. \end{cases} \tag{9.5.165}$$

Observe that the transformation $x(k) = \overline{x}e^{y(k)}$ transforms equation (9.4.120) into

$$y(k+1) - y(k) = -g\big(y(k-\tau)\big)y(k-\tau). \tag{9.5.166}$$

Clearly, to show that $\lim_{k\to\infty} x(k) = \overline{x}$, it suffices to show that

$$\lim_{k\to\infty} y(k) = 0. \tag{9.5.167}$$

To this end, observe that

$$\ln M_1 \leq y(k) \leq \ln M_0 \quad \text{for } k \in \mathbb{N}_0. \tag{9.5.168}$$

First we show that there exists $\delta > 0$ such that

$$\delta \le g(y(k)) \le g(\ln M_0) \quad \text{for } k \in \mathbb{N}_0. \tag{9.5.169}$$

Observe that

$$f(u) = \begin{cases} \dfrac{e^u - 1}{u} & \text{for } u \neq 0, \\ 1 & \text{for } u = 0 \end{cases} \tag{9.5.170}$$

is increasing, $f > 0$, $p > q$, and $pb(\overline{x})^p > qc(\overline{x})^q$. Thus, for $u < 0$,

$$\begin{aligned} g(u) &= \frac{1}{u} \ln\left[1 + b(\overline{x})^p(e^{pu} - 1) - c(\overline{x})^q(e^{qu} - 1)\right] \\ &\le pb(\overline{x})^p\left(\frac{e^{pu} - 1}{pu}\right) - qc(\overline{x})^q\left(\frac{e^{qu} - 1}{qu}\right) \\ &\le \left[pb(\overline{x})^p - qc(\overline{x})^q\right] f(qu) \\ &\le pb(\overline{x})^p - qc(\overline{x})^q \\ &= g(0), \\ g(u) &= \frac{1}{u}\left[a + b(\overline{x}e^u)^p - c(\overline{x}e^u)^q\right] > 0. \end{aligned} \tag{9.5.171}$$

Also, as g is increasing for $u > 0$, it follows that

$$g(0) \le g(u) \le g(\ln M_0) \quad \text{for } 0 \le u \le \ln M_0. \tag{9.5.172}$$

Therefore, by using (9.5.168), (9.5.171), and (9.5.172) and since g is continuous, we see that (9.5.169) holds.

Next, for $k \ge N_0$ define the nonnegative function

$$V(y(k)) = \left[y(k) - \sum_{i=k-\tau}^{k} g(y(i))y(i)\right]^2 + \sum_{i=k-\tau}^{k}\left[g(y(i+\tau+1)) \sum_{j=i}^{k} g(y(j))y^2(j)\right]. \tag{9.5.173}$$

Calculating the difference of V along the solutions of (9.5.166) and using the fact that

$$2y(i)y(k+1) \le y^2(i) + y^2(k+1), \tag{9.5.174}$$

we see that

$$\begin{aligned}
\Delta V(y(k)) &= V(y(k+1)) - V(y(k)) \\
&= \left[y(k+1) - \sum_{i=k-\tau+1}^{k+1} g(y(i))y(i)\right]^2 - \left[y(k) - \sum_{i=k-\tau}^{k} g(y(i))y(i)\right]^2 \\
&\quad + \sum_{i=k-\tau+1}^{k+1}\left[g(y(i+\tau+1))\sum_{j=i}^{k+1} g(y(j))y^2(j)\right] \\
&\quad - \sum_{i=k-\tau}^{k}\left[g(y(i+\tau+1))\sum_{j=1}^{k} g(y(j))y^2(j)\right] \\
&= -g(y(k+1))y(k+1) \\
&\quad \times \left[2y(k+1) + g(y(k+1))y(k+1) - 2\sum_{i=k-\tau+1}^{k} g(y(i))y(i)\right] \\
&\quad + g(y(k+1))y^2(k+1)\sum_{i=k-\tau+1}^{k+1} g(y(i+\tau+1)) \\
&\quad - g(y(k+1))\sum_{i=k-\tau}^{k} g(y(i))y^2(i) \\
&= -2g(y(k+1))y^2(k+1) + 2g(y(k+1))y(k+1)\sum_{i=k-\tau+1}^{k+1} g(y(i))y(i) \\
&\quad - g(y(k+1))\sum_{i=k-\tau+1}^{k+1} g(y(i))y^2(i) - g^2(y(k+1))y^2(k+1) \\
&\quad + g(y(k+1))y^2(k+1)\sum_{i=k-\tau+1}^{k+1} g(y(i+\tau+1)) \\
&\quad + g^2(y(k+1))y^2(k+1) - g(y(k+1))g(y(k-\tau))y^2(k-\tau) \\
&\le -g(y(k+1))y^2(k+1)\left[2 - \sum_{i=k-\tau+1}^{k+1} g(y(i)) - \sum_{i=k-\tau+1}^{k+1} g(y(i+\tau+1))\right].
\end{aligned} \tag{9.5.175}$$

This, in view of (9.5.169), yields

$$V(y(k+1)) - V(y(k)) \le -2[1 - g(\ln M_0)(\tau+1)]g(y(k+1))y^2(k+1). \tag{9.5.176}$$

By summing both sides of (9.5.176), we see that for $k \ge N_0$,

$$V(y(k+1)) + 2[1 - g(\ln M_0)(\tau+1)]\sum_{i=N_0+1}^{k+1} g(y(i))y^2(i) \le V(y(N_0)). \tag{9.5.177}$$

Hence $\sum_{k=1}^{\infty} g(y(k))y^2(k) < \infty$, which in view of (9.5.169) implies $\sum_{k=1}^{\infty} y^2(k) < \infty$. Clearly, this fact implies that (9.5.167) holds. The proof is complete. □

9.5.7. On the recursive sequence $x(k+1) = ax(k) + (b + cx(k-1))e^{-x(k)}$

We will consider the so-called delay model of a *perennial grass*

$$x(k+1) = ax(k) + (b + cx(k-1))e^{-x(k)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.5.178}$$

where

$$a, c \in (0,1), \qquad b \in \mathbb{R}^+, \tag{9.5.179}$$

and where $x(-1)$ and $x(0)$ are arbitrary positive initial conditions.

First we need the following result.

Lemma 9.5.44. *Assume that (9.5.179) holds. Then equation (9.5.178) has a unique positive equilibrium $\overline{x}$.*

Proof. Set

$$h(u) = au + (b + cu)e^{-u} - u. \tag{9.5.180}$$

Then $h(0) = b > 0$ and $h(\infty) = -\infty$, and so there exists $\overline{x} > 0$ such that $h(\overline{x}) = 0$. Now

$$h'(u) = a - 1 + (c - b)e^{-u} - cue^{-u}. \tag{9.5.181}$$

Thus

$$h'(\overline{x}) = a - 1 + (c - b - c\overline{x})e^{-\overline{x}} = \frac{1-a}{b + c\overline{x}}\left[-b - b\overline{x} - c(\overline{x})^2\right] < 0, \tag{9.5.182}$$

and so $\overline{x}$ is unique. The proof is complete. □

In the following result we show that every solution of equation (9.5.178) is bounded and persists.

Theorem 9.5.45. *Assume that (9.5.179) holds. Then every solution of equation (9.5.178) is bounded and persists. Moreover, if $\{x(k)\}$ is a solution of equation (9.5.178) and if M is chosen in such a way that*

$$M \geq \max\left\{x(-1), x(0), \overline{x}, \frac{b}{1-c}\right\}, \tag{9.5.183}$$

then $be^{-M} < x(k) \leq M$ for $k \in \mathbb{N}$.

Proof. If h is the function defined by (9.5.180), then $h(M) < h(\overline{x}) = 0$ and so $aM + (b + cM)e^{-M} < M$. Set

$$p(x) = ax + (b + cM)e^{-x} \quad \text{for } x \in (0, M]. \tag{9.5.184}$$

Then $p(x)$ has a maximum at either $x = 0$ or at $x = M$. If the maximum occurs at $x = 0$, then

$$b + cM = p(0) \leq p(M) < M. \tag{9.5.185}$$

If the maximum occurs at $x = M$, then

$$p(M) = aM + (b + cM)e^{-M} < M. \tag{9.5.186}$$

Thus $p(x) \leq M$ for all $x \in (0, M]$. Note that $x(0) \in (0, M]$ and so

$$x(1) = ax(0) + [b + cx(-1)]e^{-x(0)} \leq ax(0) + [b + cM]e^{-x(0)} = p(x(0)) \leq M. \tag{9.5.187}$$

It follows by induction that $x(k) \leq M$ for $k \in \mathbb{N}$. Finally, note that for $k \in \mathbb{N}_0$, $x(k+1) > be^{-x(k)} \geq be^{-M}$. The proof is complete. □

9.5.7.1. Linearized stability analysis

The linearized equation for equation (9.5.178) about the positive equilibrium $\overline{x}$ is

$$y(k+1) + [(1-a)\overline{x} - a]y(k) - \frac{c(1-a)}{b + c\overline{x}}y(k-1) = 0. \tag{9.5.188}$$

Thus a sufficient condition for the local asymptotic stability of the equilibrium $\overline{x}$ is

$$|(1-a)\overline{x} - a| < 1 - \frac{c(1-a)\overline{x}}{b + c\overline{x}} < 2, \tag{9.5.189}$$

which is equivalent to

$$(1-a)c(\overline{x})^2 + [(1-a)b - 2ac]\overline{x} - b(1+a) < 0. \tag{9.5.190}$$

Let γ be the positive root of the quadratic polynomial in $\overline{x}$ in (9.5.190). Then

$$\gamma = \frac{a}{1-a} - \frac{b}{2c} + \frac{\left[(2ac - (1-a)b)^2 + 4(1-a^2)bc\right]^{1/2}}{2(1-a)c}, \tag{9.5.191}$$

and (9.5.190) is true if and only if

$$\overline{x} < \gamma. \tag{9.5.192}$$

This, for example, is true provided

$$\bar{x} \leq \frac{a}{1-a}. \tag{9.5.193}$$

An explicit condition in terms of a, b, and c for (9.5.193) to be true is that

$$h\left(\frac{a}{1-a}\right) \leq 0, \tag{9.5.194}$$

where h is the function defined in (9.5.180). One can see that (9.5.193) is equivalent to

$$b \leq a\left[e^{a/(1-a)} - \frac{c}{1-a}\right]. \tag{9.5.195}$$

9.5.7.2. Convergence of nonoscillatory solutions

Here, we will show that all nonoscillatory solutions of equation (9.5.178) converge to the positive equilibrium $\bar{x}$.

The following two lemmas are needed.

Lemma 9.5.46. *Let $\{x(k)\}$ be a solution of equation (9.5.178). Then the following statements are true.*

(a_1) *If there exists $N \in \mathbb{N}_0$ such that $x(N) > \bar{x}$ and $x(N-1) \leq x(N)$, then $x(N+1) < x(N)$.*

(a_2) *If there exists $N \in \mathbb{N}_0$ such that $x(N) < \bar{x}$ and $x(N-1) \geq x(N)$, then $x(N+1) > x(N)$.*

Proof. We will prove (a_1). The proof of (a_2) is similar and will be omitted. Note that

$$x(N+1) = ax(N) + \big(b + cx(N-1)\big)e^{-x(N)} \leq ax(N) + \big(b + cx(N)\big)e^{-x(N)} < x(N), \tag{9.5.196}$$

where the last inequality holds by the fact that the function $f(x) = ax + (b+cx)e^{-x}$ satisfies the negative feedback condition which is introduced below. □

Definition 9.5.47. A function $f \in C([0,\infty),[0,\infty))$ is said to satisfy the *negative feedback condition* if it has a unique positive fixed point $\bar{x}$ such that

$$[f(x) - x](x - \bar{x}) < 0 \quad \text{for } 0 < x \neq \bar{x}. \tag{9.5.197}$$

Lemma 9.5.48. *Let $\{x(k)\}$ be a solution of equation (9.5.178). Then the following statements are true.*

(b_1) *If for some $N \in \mathbb{N}_0$, $x(N-1) > x(N) \geq \bar{x}$, then $x(N+1) < x(N-1)$.*

(b_2) *If for some $N \in \mathbb{N}_0$, $x(N-1) < x(N) \leq \bar{x}$, then $x(N+1) > x(N-1)$.*

Proof. We will prove (b_1). The proof of (b_2) is similar and will be omitted. Note that

$$ax(N) + \big(b + cx(N)\big)e^{-x(N)} \leq x(N), \tag{9.5.198}$$

and so

$$\begin{aligned} x(N+1) &= ax(N) + \big(b + cx(N-1)\big)e^{-x(N)} \\ &= ax(N) + \big(b + cx(N)\big)e^{-x(N)} + c\big(x(N-1) - x(N)\big)e^{-x(N)} \\ &< x(N) + c\big(x(N-1) - x(N)\big) \\ &\leq x(N-1), \end{aligned} \tag{9.5.199}$$

which implies (b_1). □

Theorem 9.5.49. *Assume that (9.5.179) holds and let $\{x(k)\}$ be a nonoscillatory solution of equation (9.5.178). Then* $\lim_{k\to\infty} x(k) = \overline{x}$.

Proof. Let $\{x(k)\}$ be a nonoscillatory solution of equation (9.5.178). Then there exists $N \geq -1$ such that for $k \geq N$ exactly one of

(i_1) $x(k) < \overline{x}$,

(i_2) $x(k) > \overline{x}$

holds. We will present the proof when (i_1) holds. The proof when (i_2) holds is similar and will be omitted. Suppose $y(k) = \min\{x(k), x(k-1)\}$ for $k \geq N$. Then $y(k) < \overline{x}$, and by Lemmas 9.5.46 and 9.5.48, $x(k+1) > y(k)$. Therefore we have $y(k+1) = \min\{x(k+1), x(k)\} \geq y(k)$, and so $\lim_{k\to\infty} y(k) = y \leq \overline{x}$ exists. Now

$$\begin{aligned} x(k+1) &= ax(k) + \big(b + cx(k-1)\big)e^{-x(k)} \geq ay(k) + \big(b + cy(k)\big)e^{-\overline{x}}, \\ x(k+2) &\geq ay(k+1) + \big(b + cy(k+1)\big)e^{-\overline{x}} \geq ay(k) + \big(b + cy(k)\big)e^{-\overline{x}}. \end{aligned} \tag{9.5.200}$$

Thus

$$y(k+2) = \min\big\{x(k+2), x(k+1)\big\} \geq ay(k) + \big(b + cy(k)\big)e^{-\overline{x}}. \tag{9.5.201}$$

Taking limits as $k \to \infty$ gives

$$y \geq ay + (b + cy)e^{-\overline{x}} = ay + (b + cy)\frac{(1-a)\overline{x}}{b + c\overline{x}}, \tag{9.5.202}$$

from which it follows that $y \geq \overline{x}$ and so $y = \overline{x}$. The proof is complete. □

9.5.7.3. Global asymptotic stability when $\overline{x} \leq a/(1-a)$

The proof of the convergence of all solutions $\{x(k)\}$ of equation (9.5.178) to the positive equilibrium $\overline{x}$ involves looking at the function $f(u) = au + (b + c\overline{x})e^{-u}$.

This function has only one critical point and $f(u)$ achieves its minimum value there. The critical point is $x_c = \ln[(b + c\overline{x})/a]$. Note also that when $\overline{x} < a/(1-a)$ we have $x_c < \overline{x}$. Let $x^* < \overline{x}$ be the unique point in $(-\infty, \overline{x})$ such that $f(x^*) = 0$ and define $\tilde{x}$ as follows:

$$\tilde{x} = \begin{cases} x^* & \text{if } x^* > 0, \\ 0 & \text{if } x^* \le 0. \end{cases} \tag{9.5.203}$$

Note that $\tilde{x} = 0$ if and only if $b \le (1-c)\ln[1/(1-a)]$.

The following lemma is needed.

Lemma 9.5.50. *Assume that (9.5.179) holds and suppose that $\overline{x} < a/(1-a)$. Let $\{x(k)\}$ be a solution of equation (9.5.178). Then the following statements are true.*

(a_1) *If for some $N \ge -1$, $\tilde{x} < x(N) \le \overline{x}$ and $\tilde{x} < x(N+1) \le \overline{x}$ with one of the inequalities being strict, then $\tilde{x} < x(k) < \overline{x}$ for all $k > N+1$.*

(a_2) *If for some $N \ge -1$, $x(N) \ge \overline{x}$ and $x(N+1) \ge \overline{x}$ with one of the inequalities being strict, then $x(k) > \overline{x}$ for all $k > N+1$.*

Proof. We will prove (a_1). The proof of (a_2) is similar and will be omitted. It is easy to see that $f(x) < \overline{x}$ for $x \in (\tilde{x}, \overline{x})$ and $f(x) > \overline{x}$ for $x \in (\overline{x}, \infty)$. Now

$$\begin{aligned} x(N+2) &= ax(N+1) + \big(b + cx(N)\big)e^{-x(N+1)} \\ &\le ax(N+1) + (b + c\overline{x})e^{-x(N+1)} \\ &\le \overline{x}, \end{aligned} \tag{9.5.204}$$

with one of the inequalities being strict. Thus $x(N+2) < \overline{x}$. Also, by Lemmas 9.5.46 and 9.5.48, $x(N+2) > \tilde{x}$. It follows by induction that $x(k) \in (\tilde{x}, \overline{x})$ for all $k > N+1$, and the proof is complete. □

Corollary 9.5.51. *Assume that (9.5.179) holds and suppose that $\overline{x} < a/(1-a)$. Then the following statements are true.*

(b_1) *If $x(-1), x(0) \in (\tilde{x}, \overline{x})$, then $x(k) \in (\tilde{x}, \overline{x})$ for $k \in \mathbb{N}_0 \cup \{-1\}$.*

(b_2) *If $x(-1), x(0) \in (\overline{x}, \infty)$, then $\overline{x} < x(k)$ for $k \in \mathbb{N}_0 \cup \{-1\}$.*

Now we present the following result.

Theorem 9.5.52. *Assume that (9.5.178) holds and suppose that $\overline{x} \le a/(1-a)$. Then $\overline{x}$ is a global attractor of all positive solutions of equation (9.5.178).*

Proof. Suppose $\overline{x} < a/(1-a)$. The proof when $\overline{x} = a/(1-a)$ requires a slight modification and will be omitted. Let $\{x(k)\}$ be a solution of equation (9.5.178). If $\{x(k)\}$ is nonoscillatory, then the result follows from Theorem 9.5.49. So, assume that $\{x(k)\}$ is an oscillatory solution. We will show that

$$\lim_{k \to \infty} x(k) = \overline{x}. \tag{9.5.205}$$

There are two cases to consider.

Case 1. Suppose there exists $N \in \mathbb{N}$ such that $x(N), x(N+1) \in (\tilde{x}, \infty)$. Then, in view of Lemma 9.5.50, exactly one of

(i) $\tilde{x} < x(N) < \overline{x}$ and $\overline{x} < x(N+1)$,
(ii) $\overline{x} < x(N)$ and $\tilde{x} < x(N+1) < \overline{x}$

holds. Assume that (i) holds. The case where (ii) holds is similar and will be omitted. Then

$$\begin{aligned} x(N+2) &= ax(N+1) + \big(b + cx(N)\big)e^{-x(N+1)} \\ &= ax(N+1) + (b + c\overline{x})e^{-x(N+1)} - c\big(\overline{x} - x(N)\big)e^{-x(N+1)} \\ &> \overline{x} - c\big(\overline{x} - x(N)\big) \\ &> \overline{x} - \overline{x} + x(N) \\ &= x(N). \end{aligned} \tag{9.5.206}$$

Also, since $\{x(k)\}$ is an oscillatory solution, in view of Lemma 9.5.50, $x(N+2) < \overline{x}$. Similarly we can show that $\overline{x} < x(N+3) < x(N+1)$, and so we have

$$\tilde{x} < x(N) < x(N+2) < x(N+4) < \cdots < \overline{x} < \cdots < x(N+3) < x(N+1). \tag{9.5.207}$$

Since there cannot exist any solution of period 2 in the interval $(\overline{x}, \infty)$, it follows that (9.5.205) holds.

Case 2. Suppose no such N as in Case 1 exists. Then there exists $m > 0$ such that $x(m) < \tilde{x}$ and $\overline{x} \le x(m+1)$. Thus

$$\begin{aligned} x(m+2) &= ax(m+1) + \big(b + cx(m)\big)e^{-x(m+1)} \\ &= ax(m+1) + \big(b + cx(m) + c\overline{x} - c\overline{x}\big)e^{-x(m+1)} \\ &= ax(m+1) + (b + c\overline{x})e^{-x(m+1)} - c\big(\overline{x} - x(m)\big)e^{-x(m+1)}. \end{aligned} \tag{9.5.208}$$

But $x(m+1) > \overline{x}$, and so

$$x(m+2) > \overline{x} - c\big(\overline{x} - x(m)\big) > \overline{x} - \overline{x} + x(m) = x(m). \tag{9.5.209}$$

Also, since $x(m+1) \ge \overline{x}$, we must have that $x(m+2) \le \overline{x}$. Now

$$\begin{aligned} x(m+3) &= ax(m+2) + \big(b + cx(m+1)\big)e^{-x(m+2)} \\ &\ge ax(m+2) + (b + c\overline{x})e^{-x(m+2)} \\ &\ge \overline{x}, \end{aligned} \tag{9.5.210}$$

and so $x(m+3) \ge \overline{x}$. Thus, we have by induction that for $k \in \mathbb{N}_0$,

$$x(m) < x(m+2) < \cdots < x(m+2k) < \cdots < \tilde{x} < \overline{x} \le x(m+2k+1), \tag{9.5.211}$$

and so $\lim_{k\to\infty} x(m+2k) = L \le \tilde{x}$. Let $\varepsilon = (1-c)(\overline{x}-L)$. Then there exists $j > 0$ such that $x(j) \in (L-\varepsilon, L)$. Now

$$\begin{aligned} x(j+2) &= ax(j+1) + \big(b + cx(j)\big)e^{-x(j+1)} \\ &= ax(j+1) + \big(b + cx(j) + c\overline{x} - c\overline{x}\big)e^{-x(j+1)} \\ &= ax(j+1) + (b + c\overline{x})e^{-x(j+1)} - c\big(\overline{x} - x(j)\big)e^{-x(j+1)}. \end{aligned} \tag{9.5.212}$$

But, $x(j+1) > \overline{x}$, and so

$$x(j+2) > \overline{x} - c\big(\overline{x} - x(j)\big) = x(j) + (1-c)\big(\overline{x} - x(j)\big). \tag{9.5.213}$$

Also, $\overline{x} - x(j) > \overline{x} - L$, and so

$$x(j+2) > x(j) + (1-c)(\overline{x} - L) = x(j) + \varepsilon > L \tag{9.5.214}$$

which is a contradiction. Thus, such a solution cannot exist.

The proof is complete. □

9.5.7.4. Semicycle analysis

We will discuss the behavior of the semicycles of equation (9.5.178) for the case $\overline{x} > a/(1-a)$.

Theorem 9.5.53. *Suppose that (9.5.179) holds, $\overline{x} > a/(1-a)$, and $\{x(k)\}_{k=-1}^{\infty}$ is a nontrivial oscillatory solution of equation (9.5.178). Also, let $\tilde{x} > \overline{x}$ be the unique point such that $f(\tilde{x}) = \overline{x}$. Then the following statements are true.*

- (a_1) *Except for possibly the first semicycle, every negative semicycle has exactly one term.*
- (a_2) *Except for possibly the first semicycle, the maximum value in a positive semicycle occurs in the first term.*
- (a_3) *Except for possibly the first semicycle, if the first term of a positive semicycle is an element of $[\overline{x}, \tilde{x}]$, then the semicycle has length one. Thus, if $\{x(k)\}$ is a convergent oscillatory solution, then the length of every semicycle is eventually one.*
- (a_4) *If a positive semicycle has more than one term, then the last term in that semicycle is an element of $[\overline{x}, \tilde{x}]$.*
- (a_5) *If $\{x(k)\}$ is an oscillatory solution, then there exist terms of the solution in the interval $[\overline{x}, \tilde{x}]$ infinitely often, that is, for every $N > 0$ there exists $k > N$ such that $x(k) \in [\overline{x}, \tilde{x}]$.*

Proof. As before, we consider the function $f(x) = ax + (b + c\overline{x})e^{-x}$. When $\overline{x} > a/(1-a)$, it is easy to see that if either $x < \overline{x}$ or $x > \tilde{x}$, then $f(x) > \overline{x}$, and if $\overline{x} < x \le \tilde{x}$, then $f(x) \le \overline{x}$.

(a_1) Suppose $x(N-1) \ge \overline{x}$ and $x(N) < \overline{x}$. Then

$$\begin{aligned} x(N+1) &= ax(N) + \big(b + cx(N-1)\big)e^{-x(N)} \\ &\ge ax(N) + (b + c\overline{x})e^{-x(N)} \\ &= f\big(x(N)\big) \\ &> \overline{x}, \end{aligned} \tag{9.5.215}$$

and so $x(N+1) > \overline{x}$, and the negative semicycle has only one term.

(a_2) This is a direct consequence of Lemmas 9.5.46 and 9.5.48.

(a_3) Suppose $x(N-1) < \overline{x}$ and $x(N) \in [\overline{x}, \tilde{x}]$. Then

$$\begin{aligned} x(N+1) &= ax(N) + \big(b + cx(N-1)\big)e^{-x(N)} \\ &< ax(N) + (b + c\overline{x})e^{-x(N)} \\ &= f\big(x(N)\big) \\ &< \overline{x}. \end{aligned} \tag{9.5.216}$$

Hence $x(N+1) < \overline{x}$, and the positive semicycle has only one term.

(a_4) Suppose a positive semicycle has more than one term and $x(N)$ is the last term in that semicycle. Then $x(N-1) \ge \overline{x}$, and if $x(N) > \tilde{x}$, then

$$\begin{aligned} x(N+1) &= ax(N) + \big(b + cx(N-1)\big)e^{-x(N)} \\ &\ge ax(N) + (b + c\overline{x})e^{-x(N)} \\ &= f\big(x(N)\big) \\ &> \overline{x}. \end{aligned} \tag{9.5.217}$$

This contradicts the fact that $x(N)$ is the last term in the semicycle. Thus $x(N) \le \tilde{x}$.

(a_5) Suppose for the sake of contradiction that there exists $N \in \mathbb{N}$ such that $k > N$ implies $x(k) \notin [\overline{x}, \tilde{x}]$. Then by ($a_4$), every semicycle after N has length one. Assume without loss of generality that this is true for all semicycles and that $x(2k) < \overline{x}$ and $x(2k+1) > \tilde{x}$. Now

$$\begin{aligned} x(2k+2) &= ax(2k+1) + \big(b + cx(2k)\big)e^{-x(2k+1)} \\ &= ax(2k+1) + \big[b + cx(2k) + c\overline{x} - c\overline{x}\big]e^{-x(2k+1)} \\ &= ax(2k+1) + (b + c\overline{x})e^{-x(2k+1)} - c\big(\overline{x} - x(2k)\big)e^{-x(2k+1)}. \end{aligned} \tag{9.5.218}$$

But $x(2k+1) > \overline{x}$, and so

$$x(2k+2) > \overline{x} - c\big(\overline{x} - x(2k)\big) > \overline{x} - \overline{x} + x(2k) = x(2k). \tag{9.5.219}$$

Thus $x(2k+2) > x(2k)$ and the terms of the negative semicycles increase monotonically to $L \le \overline{x}$. Suppose $L < \overline{x}$ and set $\varepsilon = (1-c)(\overline{x}-L)$. Then there exists $j > 0$ such that $x(j) \in (L-\varepsilon, L)$, $x(j+1) > \tilde{x}$, and $x(j) < x(j+2) < L$. Now

$$\begin{aligned} x(j+2) &= ax(j+1) + \big(b + cx(j)\big)e^{-x(j+1)} \\ &= ax(j+1) + \big(b + cx(j) + c\overline{x} - c\overline{x}\big)e^{-x(j+1)} \\ &= ax(j+1) + (b + c\overline{x})e^{-x(j+1)} - c\big(\overline{x} - x(j)\big)e^{-x(j+1)}. \end{aligned} \tag{9.5.220}$$

But $x(j+1) > \overline{x}$, and so

$$x(j+2) > \overline{x} - c\big(\overline{x} - x(j)\big) = x(j) + (1-c)\big(\overline{x} - x(j)\big). \tag{9.5.221}$$

Also $\overline{x} - x(j) > \overline{x} - L$, and so

$$x(j+2) > x(j) + (1-c)(\overline{x} - L) = x(j) + \varepsilon > L, \tag{9.5.222}$$

which is a contradiction. Thus $x(2k)$ converges to $\overline{x}$. From this it can be seen that $x(2k+1)$ converges to $\tilde{x}$. Thus $\{\overline{x}, \tilde{x}, \overline{x}, \tilde{x}, \dots\}$ is a solution of equation (9.5.178) of period 2, which is impossible because $a\tilde{x} + (b + c\overline{x})e^{-\tilde{x}} < \overline{x}$.

The proof is complete. □

Next we present the following result in which we show that equation (9.5.178) is permanent.

Theorem 9.5.54. *Assume (9.5.179). Then equation (9.5.178) is permanent.*

Proof. If $\overline{x} \le a/(1-a)$, then $\overline{x}$ is globally asymptotically stable, and so equation (9.5.178) is permanent. Thus we need only to consider the case $\overline{x} > a/(1-a)$. If $\{x(k)\}$ is a nonoscillatory solution, then by Theorem 9.5.49 it converges to $\overline{x}$. Thus assume that $\{x(k)\}$ is an oscillatory solution of equation (9.5.178). By Theorem 9.5.53(a_5), there exists $N \in \mathbb{N}$ such that $x(N) \in [\overline{x}, \tilde{x}]$ and $x(N+1) < \overline{x}$. Let $M = \max\{\overline{x}, b/(1-c)\}$. We can now show as in the proof of Theorem 9.5.45 that $be^{-M} < x(k) \le M$ for $k \ge N$, and the proof is complete. □

9.5.7.5. Existence of a period 2 solution

Here we will show that when

$$(1-a)\overline{x} - a > 1 - ce^{-\overline{x}} \tag{9.5.223}$$

(a sufficient condition for the unique positive equilibrium $\overline{x}$ to be unstable), there exists a period 2 solution of equation (9.5.178).

Theorem 9.5.55. *Suppose that (9.5.179) and (9.5.223) hold. Then (9.5.178) has a period 2 solution $\{\alpha, \beta, \alpha, \beta, \dots\}$.*

Proof. Set

$$h(x) = \frac{ax + be^{-x}}{1 - ce^{-x}}, \qquad g(x) = h(h(x)) - x. \tag{9.5.224}$$

Observe that under the hypotheses of the theorem $g(0) > 0$, $g(\overline{x}) = 0$, $g'(\overline{x}) > 0$, and $g(\infty) < 0$. Hence there exist points α and β with $\alpha < \overline{x} < \beta$ such that $h(\alpha) = \beta$ and $h(\beta) = \alpha$. Now we claim that $\{\alpha, \beta, \alpha, \beta, \dots\}$ is a period 2 solution of equation (9.5.178). Indeed, if $x(-1) = \alpha$ and $x(0) = \beta$, then $x(1) = \alpha\beta + (b + c\alpha)e^{-\beta}$. But $h(\beta) = \alpha$, and so

$$\alpha = \frac{\alpha\beta + be^{-\beta}}{1 - ce^{-\beta}}. \tag{9.5.225}$$

Thus $\alpha = \alpha\beta + (b + c\alpha)e^{-\beta}$ and so $x(1) = \alpha$. Similarly $x(2) = \beta$, and by induction the proof is complete. □

9.5.8. Global attractivity in a differential equation with piecewise constant arguments

Here we consider the equation with piecewise constant arguments

$$x'(t) = rx(t)\left(1 - \sum_{j=0}^{m} \left(a_j x([t - \tau_j]) + b_j x^2([t - \tau_j])\right)\right), \tag{9.5.226}$$

where $[\cdot]$ denotes the greatest integer function, $r \in \mathbb{R}^+$, and for $j \in \{0, 1, \dots, m\}$, $\tau_j \in \mathbb{N}_0$ and $a_j, b_j \in [0, \infty)$ with $a_j + b_j > 0$.

We will provide sufficient conditions for the positive equilibrium of equation (9.5.226) to be a global attractor of all positive solutions. We will establish that every positive solution of equation (9.5.226) is bounded from above and from below.

Recall that equation (9.5.226) has a unique positive equilibrium. If we denote this equilibrium by $\overline{x}$, then

$$\sum_{j=0}^{m} \left(a_j \overline{x} + b_j (\overline{x})^2\right) = 1. \tag{9.5.227}$$

Theorem 9.5.56. (a_1) *Every positive solution of equation (9.5.226) is bounded from above and from below.*

(a_2) *Assume*

$$\overline{x}e^{r(\tau+1)} \sum_{j=0}^{m} \left(b_j \overline{x} e^{r(\tau+1)} + a_j\right) < 2, \tag{9.5.228}$$

where $\tau = \max\{\tau_0, \tau_1, \dots, \tau_m\}$. *Then the positive equilibrium* $\overline{x}$ *of equation (9.5.226) is a global attractor of all positive solutions.*

Proof. Without loss of generality, we may assume that we have $x(-j) > 0$ for each $j \in \{0, 1, \dots, \tau\}$. In view of the transformation $x(t) = \overline{x}e^{y(t)}$, to prove (a_1) it suffices to show that the solution $y(t)$ of the equation

$$y'(t) = -r\sum_{j=0}^{m} g_j\left(\overline{x}e^{y([t-\tau_j])}\right), \qquad (9.5.229)$$

where

$$g_j(u) = b_ju^2 + a_ju - \left[a_j\overline{x} + b_j(\overline{x})^2\right] \quad \text{for } j \in \{0, 1, \dots, m\}, \qquad (9.5.230)$$

which corresponds to the initial conditions

$$y(-j) = \ln\left(\frac{x(-j)}{\overline{x}}\right) \quad \text{for } j \in \{0, 1, \dots, \tau\}, \qquad (9.5.231)$$

is bounded, and to prove (a_2) it suffices to show that it satisfies

$$\lim_{t\to\infty} y(t) = 0. \qquad (9.5.232)$$

First we define

$$g(u) = \sum_{j=0}^{m} g_j\left(\overline{x}e^{u}\right) \quad \text{for } u \in \mathbb{R} \qquad (9.5.233)$$

and $M = r(\tau + 1)$. We first show that (9.5.232) is satisfied if y is an eventually nonnegative solution of equation (9.5.229). The proof when y is eventually non-positive is similar and will be omitted.

Note that for each $j \in \{0, 1, \dots, \tau\}$, $\overline{x}e^{y([t-\tau_j])} \geq \overline{x}$ eventually, so that

$$g_j\left(\overline{x}e^{y([t-\tau_j])}\right) \geq 0 \quad \text{eventually.} \qquad (9.5.234)$$

It follows from equation (9.5.229) that $y'(t) \leq 0$ for $k \leq t < k + 1$, where k is sufficiently large, say, $k \geq n_0$. Then $\lim_{t\to\infty} y(t) = \overline{L} \geq 0$. We claim that $L = 0$. Otherwise $\overline{L} > 0$, and by computing the limit as $t \to \infty$ on both sides of equation (9.5.229), we see that $\lim_{t\to\infty} y'(t) = -rg(\overline{L}) < 0$. Hence

$$y'(t) + rg(\overline{L}) \leq 0 \quad \text{for } n_0 \leq k \leq t < k + 1. \qquad (9.5.235)$$

Integrating both sides of (9.5.235) from k to t and then letting t approach $k + 1$ yields

$$y(k+1) - y(k) + rg(\overline{L}) \leq 0 \quad \text{for } k \geq n_0, \qquad (9.5.236)$$

which is clearly impossible for large values of k. Therefore every positive solution of equation (9.5.226) which is eventually in $[\bar{x}, \infty)$ or eventually in $(0, \bar{x}]$ is attracted to $\bar{x}$, and hence it is bounded. To complete the proof of (a_1) it remains to show that $y(t)$ is bounded when $y(t)$ is not eventually nonnegative or eventually nonpositive. In such a situation there exists a sequence of points $\{\xi(k)\}$ satisfying the following properties:

(i) $\lim_{k\to\infty} \xi(k) = \infty$,
(ii) $\tau < \xi(k) < \xi(k+1)$ and $y(\xi(k)) = 0$ for $k \in \mathbb{N}$,
(iii) $y(t)$ assumes both positive and negative values in each of the intervals $(\xi(k), \xi(k+1))$ for $k \in \mathbb{N}$.

Let t_k and s_k be points in $(\xi(k), \xi(k+1))$ such that for $k \in \mathbb{N}$, $y(t_k) = \max y(t)$ and $y(s_k) = \min y(t)$ for $t \in (\xi(k), \xi(k+1))$. Then, for $k \in \mathbb{N}$,

$$y(t_k) > 0, \qquad D^- y(t_k) \geq 0, \qquad y(s_k) < 0, \qquad D^- y(s_k) \leq 0, \tag{9.5.237}$$

where $D^- y$ denotes the left-sided derivative of y. We now claim that for $k \in \mathbb{N}$,

$$y(T_k) = 0 \quad \text{for some } T_k \in [t_k - \tau - 1, t_k), \tag{9.5.238}$$

$$y(S_k) = 0 \quad \text{for some } S_k \in [s_k - \tau - 1, s_k). \tag{9.5.239}$$

We prove (9.5.238). The proof of (9.5.239) is done similarly. Assume, for the sake of contradiction that (9.5.238) is wrong. Since $t_k \geq [t_k - \tau_j] \geq [t_k - \tau] \geq t_k - \tau - 1$ and $y(t)$ is positive in $[t_k - \tau - 1, t_k)$, we see that

$$g_i(\bar{x}e^{y([t_k - \tau_j])}) > 0 \quad \text{for } j \in \{0, 1, \dots, \tau\}. \tag{9.5.240}$$

Hence

$$D^- y(t_k) = -r \sum_{j=0}^{m} g_j(\bar{x}e^{y([t_k - \tau_j])}) < 0, \tag{9.5.241}$$

contradicting (9.5.237). By integrating equation (9.5.229) from T_k to t_k and by using that each g_j is an increasing continuous function on $[0, \infty)$ with the fact that $t_k - T_k \leq \tau + 1$, we find that for $k \in \mathbb{N}$,

$$\begin{aligned} x(t_k) &= -r \int_{T_k}^{t_k} \sum_{j=0}^{m} g_j(\bar{x}e^{y([t-\tau_j])})\, dt \\ &\leq -r \sum_{j=0}^{m} g_j(0)(t_k - T_k) \\ &\leq -r \sum_{j=0}^{m} g_j(0)(\tau + 1) \\ &= M, \end{aligned} \tag{9.5.242}$$

and so $y(t) \leq M$ for $t \geq \xi(1)$. We now use this upper bound to obtain the lower bound for $y(t)$. By integrating both sides of equation (9.5.229) from S_k to s_k and by using the facts that for each $j \in \{0, 1, \dots, m\}$, g_j is an increasing function on $[0, \infty)$ and that $0 < s_k - S_k \leq \tau + 1$, we obtain

$$\begin{aligned} y(s_k) &= -r \int_{S_k}^{s_k} \sum_{j=0}^{m} g_j\left(\overline{x} e^{y([t-\tau_j])}\right) dt \\ &\geq -r \int_{S_k}^{s_k} \sum_{j=0}^{m} g_j\left(\overline{x} e^{M}\right) dt \\ &= -rg(M)(s_k - S_k) \\ &\geq -Mg(M). \end{aligned} \tag{9.5.243}$$

We have established that $-Mg(M) \leq y(k) \leq M$ for $b \geq \xi(1) + \tau + 1$. Therefore (a_1) is proved.

We complete the proof of the theorem by showing that (9.5.232) holds when condition (9.5.228) is satisfied. Observe that condition (9.5.228) is equivalent to $g(M) < 1$, so that

$$-M \leq y(t) \leq M \quad \text{for } t \geq \xi(1) + \tau + 1. \tag{9.5.244}$$

Repeating the above argument with the bounds given in (9.5.244) and the fact that g satisfies $g(u) \geq -g(-u)$ for $u \in \mathbb{R}$, we obtain

$$-Mg(M) \leq y(t) \leq Mg(M) \quad \text{for } t \geq \xi(1) + 3(\tau + 1). \tag{9.5.245}$$

In fact, one can prove by induction that for $k \in \mathbb{N}_0$,

$$L(k) \leq x(t) \leq U(k) \quad \text{for } t \geq \xi(1) + (2k+1)(\tau + 1), \tag{9.5.246}$$

where $U(0) = M$ and

$$L(k) = -U(k), \qquad U(k+1) = Mg\big(U(k)\big). \tag{9.5.247}$$

Moreover,

$$L(k) \leq L(k+1) < 0 < U(k+1) \leq U(k) \quad \text{for } k \in \mathbb{N}_0. \tag{9.5.248}$$

Set $L = \lim_{k\to\infty} L(k)$ and $U = \lim_{k\to\infty} U(k)$. In view of (9.5.247) we observe that $U = -L$ and that U is a nonnegative zero of the function $\phi(y) = Mg(y) - y$. Since ϕ'' is a continuous nonnegative function on $\mathbb{R}$, $\phi(0) = 0$, $\phi(M) < 0$, and $0 \leq U \leq M$, we conclude that $L = U = 0$. In view of (9.5.246) we have that $\lim_{t\to\infty} y(t) = 0$. This completes the proof. $\square$

The following corollary is immediate.

Corollary 9.5.57. *Assume that* $b_j = 0$ *for* $j \in \{0, 1, \dots, m\}$ *and*

$$e^{r(\tau+1)} < 2. \tag{9.5.249}$$

Then the positive equilibrium $\overline{x} = 1/\sum_{j=0}^{m} a_j$ *of the equation*

$$x'(t) = rx(t)\left(1 - \sum_{j=0}^{m} a_j x([t - \tau_j])\right) \tag{9.5.250}$$

is a global attractor of all positive solutions.

9.6. Nonlinear difference equations with continuous variable

Consider the second-order nonlinear difference equation with continuous variable

$$\Delta_\tau^2 x(t) + f(t, x(t - \sigma)) = 0, \tag{9.6.1}$$

where $\Delta_\tau x(t) = x(t + \tau) - x(t)$, $\tau, \sigma > 0$, and $f \in C([t_0, \infty) \times \mathbb{R}, \mathbb{R})$. We assume that there exists a function $p \in C([t_0, \infty), \mathbb{R}^+)$ such that $f(t, u) \geq p(t)u$ for $t \geq t_0$ and $u \neq 0$. A function x is called a solution of equation (9.6.1) if $x(t) = \phi(t)$, $\phi \in C([t_0 - \max\{\tau, \sigma\}, \infty), \mathbb{R})$, and it satisfies equation (9.6.1) when $t \geq t_0$.

Here we will deal with the oscillatory behavior of equation (9.6.1).

Theorem 9.6.1. *Assume that* $q(t) = \min_{t \leq s \leq t+2\tau}\{p(s)\}$ *and for any* $t \geq t_0$ *there exists* $T \geq t$ *such that*

$$\sum_{i=0}^{\infty} q(T + i\tau) = \infty. \tag{9.6.2}$$

Then every solution of equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.6.1). Set

$$u(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} x(\eta)\, d\eta. \tag{9.6.3}$$

Then $u(t) > 0$, $u''(t) = \Delta_\tau^2 x(t) \leq 0$, and $u'(t) > 0$ eventually. Equation (9.6.1) takes the form

$$u''(t) + f(t, x(t - \sigma)) = 0, \tag{9.6.4}$$

and so

$$u''(t) + p(t)x(t - \sigma) \leq 0 \quad \text{eventually}. \tag{9.6.5}$$

Integrating (9.6.5) twice from s to $s+\tau$ and from t to $t+\tau$, we obtain

$$\Delta_\tau^2 u(t) + q(t)u(t-\sigma) \leq 0. \tag{9.6.6}$$

Define $w(t) = \Delta_\tau u(t)/u(t-\sigma)$. Since $u'(t) > 0$, we have $\Delta_\tau u(t) = u(t+\tau) - u(t) > 0$ and $w(t) > 0$ eventually. Then eventually

$$\begin{aligned}\Delta_\tau w(t) &= \frac{u(t+\tau-\sigma)\Delta_\tau^2 u(t) - \Delta_\tau u(t+\tau)\Delta_\tau u(t-\sigma)}{u(t+\tau-\sigma)u(t-\sigma)} \\ &\leq -q(t) - w^2(t+\sigma) \\ &\leq 0.\end{aligned} \tag{9.6.7}$$

Thus

$$\Delta_\tau w(t+i\tau) + q(t+i\tau) + w^2\big(t+(i+1)\tau\big) \leq 0 \quad \text{for } i \in \mathbb{N}_0. \tag{9.6.8}$$

Summing (9.6.8) from 0 to $k-1$, we obtain

$$w(t+k\tau) - w(t) + \sum_{i=0}^{k-1} q(t+i\tau) + \sum_{i=0}^{k-1} w^2\big(t+(i+1)\tau\big) \leq 0. \tag{9.6.9}$$

Hence we find

$$\sum_{i=0}^{k-1} q(t+i\tau) < w(t), \qquad \sum_{i=0}^{\infty} q(t+i\tau) \leq w(t) < \infty, \tag{9.6.10}$$

which contradicts condition (9.6.2). This completes the proof. □

Theorem 9.6.2. *Assume that there exists a function $\rho \in C([t_0,\infty),\mathbb{R}^+)$ with $\Delta_\tau\rho(t) \leq 0$ and for any $t \geq t_0$ there exists $T \geq t$ such that*

$$\sum_{i=0}^{\infty} \rho(T+i\tau)\left[q(T+i\tau) + \left(\frac{\Delta_\tau\rho(T+i\tau)}{2\rho(T+i\tau)}\right)^2 + \Delta_\tau\left(\frac{\Delta_\tau\rho(T+(i-1)\tau)}{2\rho(T+(i-1)\tau)}\right)\right] = \infty. \tag{9.6.11}$$

Then every solution of equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.6.1). As in the proof of Theorem 9.6.1, we define $u(t)$ by (9.6.3) and conclude that $u(t) > 0$, $\Delta_\tau u(t) > 0$, and inequality (9.6.6) hold. Define

$$w(t) = \rho(t)\left[\frac{\Delta_\tau u(t)}{u(t-\sigma)} - \frac{\Delta_\tau\rho(t-\tau)}{2\rho(t-\tau)}\right]. \tag{9.6.12}$$

Then $w(t) > 0$ and

$$\begin{aligned}
\Delta_\tau w(t) &= \frac{\Delta_\tau \rho(t)}{\rho(t+\tau)} w(t+\tau) \\
&\quad + \rho(t)\left[\frac{u(t+\tau-\sigma)\Delta_\tau^2 u(t) - \Delta_\tau u(t+\tau)\Delta_\tau u(t-\sigma)}{u(t+\tau-\sigma)u(t-\sigma)} - \Delta_\tau\left(\frac{\Delta_\tau \rho(t-\tau)}{2\rho(t-\tau)}\right)\right] \\
&\le \frac{\Delta_\tau \rho(t)}{\rho(t+\tau)} w(t+\tau) - \rho(t)\left[\left(\frac{\Delta_\tau u(t+\tau)}{u(t+\tau-\sigma)}\right)^2 + q(t) + \Delta_\tau\left(\frac{\Delta_\tau \rho(t-\tau)}{2\rho(t-\tau)}\right)\right] \\
&= -\rho(t)\left[q(t) + \left(\frac{\Delta_\tau \rho(t)}{2\rho(t)}\right)^2 + \Delta_\tau\left(\frac{\Delta_\tau \rho(t-\tau)}{2\rho(t-\tau)}\right)\right] - \frac{\rho(t) w^2(t+\tau)}{\rho^2(t+\tau)}.
\end{aligned} \tag{9.6.13}$$

Therefore

$$\begin{aligned}
&\Delta_\tau w(t+i\tau) + \rho(t+i\tau)\left[q(t+i\tau) + \left(\frac{\Delta_\tau \rho(t+i\tau)}{2\rho(t+i\tau)}\right)^2 + \Delta_\tau\left(\frac{\Delta_\tau \rho(t+(i-1)\tau)}{2\rho(t+(i-1)\tau)}\right)\right] \\
&\quad + \frac{\rho(t+i\tau) w^2(t+(i+1)\tau)}{\rho^2(t+(i+1)\tau)} \le 0.
\end{aligned} \tag{9.6.14}$$

Summing (9.6.14) from 0 to $k-1$, we find

$$\begin{aligned}
&w(t+k\tau) - w(t) \\
&\quad + \sum_{i=0}^{k-1} \rho(t+i\tau)\left[q(t+i\tau) + \left(\frac{\Delta_\tau \rho(t+i\tau)}{2\rho(t+i\tau)}\right)^2 + \Delta_\tau\left(\frac{\Delta_\tau \rho(t+(i-1)\tau)}{2\rho(t+(i-1)\tau)}\right)\right] \\
&\quad + \sum_{i=0}^{k-1} \frac{\rho(t+i\tau) w^2(t+(i+1)\tau)}{\rho^2(t+(i+1)\tau)} \le 0.
\end{aligned} \tag{9.6.15}$$

Thus we obtain

$$\begin{aligned}
&\sum_{i=0}^{\infty} \rho(t+i\tau)\left[q(t+i\tau) + \left(\frac{\Delta_\tau \rho(t+i\tau)}{2\rho(t+i\tau)}\right)^2 + \Delta_\tau\left(\frac{\Delta_\tau \rho(t+(i-1)\tau)}{2\rho(t+(i-1)\tau)}\right)\right] \\
&\quad \le w(t) < \infty,
\end{aligned} \tag{9.6.16}$$

which contradicts condition (9.6.11). This completes the proof. □

Theorem 9.6.3. *Assume that*

$$q(t) = \min_{t \le s \le t+2\tau} \left\{\frac{p(s)}{\tau^2}\right\}, \tag{9.6.17}$$

$$\int_t^\infty q(s)ds = \infty. \tag{9.6.18}$$

Then equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of (9.6.1). As in the proof of Theorem 9.6.1, we set $u(t)$ as in (9.6.3) and conclude that $u(t) > 0$, $u'(t) > 0$, and

$$\Delta_\tau^2 u(t) + \tau^2 q(t) u(t-\sigma) \leq 0. \tag{9.6.19}$$

Define

$$v(t) = \int_t^{t+\tau} ds \int_s^{s+\tau} u(\xi) d\xi > 0. \tag{9.6.20}$$

Since $u'(t) > 0$, we have $v(t) \leq \tau^2 u(t+2\tau)$, $v(t-\sigma-2\tau) \leq \tau^2 u(t-\sigma)$, and $v''(t) = \Delta_\tau^2 u(t) \leq 0$. Thus $v'(t) > 0$. Therefore (9.6.19) becomes

$$v''(t) + q(t) v(t-\sigma-2\tau) \leq 0. \tag{9.6.21}$$

Let $w(t) = v'(t)/v(t-\sigma-2\tau)$. Then

$$\begin{aligned} w'(t) &= \frac{v(t-2\tau-\sigma)v''(t) - v'(t)v'(t-\sigma-2\tau)}{v^2(t-2\tau-\sigma)} \\ &\leq -q(t) - \frac{v'(t)v'(t-\sigma-2\tau)}{v^2(t-2\tau-\sigma)}. \end{aligned} \tag{9.6.22}$$

Since $v''(t) \leq 0$ and $v'(t) > 0$, it follows that

$$w'(t) \leq -q(t) - \left(\frac{v'(t)}{v(t-2\tau-\sigma)}\right)^2 \leq -q(t) - w^2(t). \tag{9.6.23}$$

Integrating (9.6.23) from t to T provides

$$w(T) - w(t) \leq -\int_t^T q(s)ds - \int_t^T w^2(s)ds. \tag{9.6.24}$$

Thus we get $\int_t^T q(s)ds \leq w(t)$ or $\int_t^\infty q(s)ds \leq w(t) < \infty$, which contradicts condition (9.6.18). This completes the proof. □

The following theorem improves the previous result.

Theorem 9.6.4. *Let q be defined by (9.6.17). If*

$$\liminf_{t\to\infty} \left\{ t \int_t^\infty q(s)ds \right\} > \frac{1}{4}, \tag{9.6.25}$$

then equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.1.1). As in the proof of Theorem 9.6.3, we obtain (9.6.24) for $t_0 \leq t_1 \leq t \leq T < \infty$ and

$$w(t) \geq \int_t^\infty w^2(s)ds + \int_t^\infty q(s)ds. \tag{9.6.26}$$

Condition (9.6.25) implies that there exists a real number $t_2 \geq t_1$ such that $\int_t^\infty q(s)ds \geq \alpha_0/t$ for $t \geq t_2$, where $\alpha_0 > 1/4$, so

$$w(t) \geq \frac{\alpha_0}{t} + \int_t^\infty w^2(s)ds \quad \text{for } t \geq t_2. \tag{9.6.27}$$

It follows that $w(t) \geq \alpha_0/t$ for $t \geq t_2$, which together with (9.6.27) implies

$$w(t) \geq \frac{\alpha_0}{t} + \int_t^\infty w^2(s)ds \geq \frac{\alpha_0}{t} + \int_t^\infty \left(\frac{\alpha_0}{s}\right)^2 ds = \frac{\alpha_0 + \alpha_0^2}{t} \quad \text{for } t \geq t_2. \tag{9.6.28}$$

Set $\alpha_i = \alpha_{i-1}^2 + \alpha_0$ for $i \in \mathbb{N}$. Then $w(t) \geq \alpha_1/t$ for $t \geq t_2$. By induction, from (9.6.27) we can prove that $w(t) \geq \alpha_i/t$ for $i \in \mathbb{N}$ and $t \geq t_2$. Finally, we claim that $\alpha_i \to \infty$ as $i \to \infty$. To this end,

$$\alpha_i - \alpha_{i-1} = \alpha_{i-1}^2 - \alpha_{i-1} + \alpha_0 > \alpha_{i-1}^2 - \alpha_{i-1} + \frac{1}{4} = \left(\alpha_{i-1} - \frac{1}{2}\right)^2 \geq 0 \tag{9.6.29}$$

implies that $\{\alpha_i\}$ is a positive increasing sequence. Therefore there exists a constant $a \in [0, \infty]$ such that $\alpha_i \to a$ as $i \to \infty$. Now, if $0 \leq a < \infty$, then $a^2 - a + \alpha_0 = 0$. But this equation has no real roots. This contradiction yields that $a = \infty$ and the claim is proved. Thus we have $w(t) = \infty$, which is a contradiction. This completes the proof. □

Theorem 9.6.5. *Assume that there exists a function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that $\rho'(t) \leq 0$ and*

$$\int_t^\infty \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds = \infty, \tag{9.6.30}$$

where q is defined by (9.6.17). Then equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.6.1). As in the proof of Theorem 9.6.3, there exists a function v such that $v'(t) > 0$ and inequality (9.6.21) holds. Let

$$w(t) = \rho(t)\left[\frac{v'(t)}{v(t-2\tau-\sigma)} - \frac{\rho'(t)}{2\rho(t)}\right] > 0. \tag{9.6.31}$$

Then

$$\begin{aligned} w'(t) &= \rho'(t)\left[\frac{v'(t)}{v(t-2\tau-\sigma)} - \frac{\rho'(t)}{2\rho(t)}\right] \\ &\quad + \rho(t)\left[\frac{v''(t)v(t-2\tau-\sigma) - v'(t)v'(t-2\tau-\sigma)}{v^2(t-2\tau-\sigma)} - \left(\frac{\rho'}{2\rho}\right)'(t)\right] \\ &\le \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\left[q(t) + \left(\frac{\rho'}{2\rho}\right)'(t) + \left(\frac{v'(t)}{v(t-2\tau-\sigma)}\right)^2\right] \\ &= \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)\left[q(t) + \left(\frac{w(t)}{\rho(t)} + \frac{\rho'(t)}{2\rho(t)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(t)\right] \\ &= -\rho(t)\left[q(t) + \left(\frac{\rho'(t)}{2\rho(t)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(t)\right] - \frac{w^2(t)}{\rho(t)}. \end{aligned} \tag{9.6.32}$$

Thus

$$w(T) - w(t) \le -\int_t^T \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds - \int_t^T \frac{w^2(s)}{\rho(s)}ds. \tag{9.6.33}$$

So

$$\int_t^\infty \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds \le w(t) < \infty, \tag{9.6.34}$$

which contradicts condition (9.6.30). This completes the proof. □

The following result improves Theorem 9.6.5.

Theorem 9.6.6. *Assume that there exists a function $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that $\rho'(t) \le 0$ and*

$$\liminf_{t\to\infty} \frac{t}{\rho(t)} \int_t^\infty \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds > \frac{1}{4}, \tag{9.6.35}$$

where q is defined by (9.6.17). Then equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.6.1). As in the proof of Theorem 9.6.5, there exist a function $w \in C([t_0,\infty),\mathbb{R})$ and a real number $t_1 \geq t_0$ with $w(t) > 0$ for $t \geq t_1$ such that for $t_1 \leq t \leq T < \infty$, inequality (9.6.33) holds. So

$$w(t) \geq \int_t^\infty \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds + \int_t^\infty \frac{w^2(s)}{\rho(s)}ds. \tag{9.6.36}$$

By condition (9.6.35), there exists $t_2 \geq t_1$ such that

$$\int_t^\infty \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds \geq \alpha_0\frac{\rho(t)}{t} \quad \text{for } t \geq t_2, \tag{9.6.37}$$

where $\alpha_0 > 1/4$, so

$$w(t) \geq \int_t^\infty \frac{w^2(s)}{\rho(s)}ds + \alpha_0\frac{\rho(t)}{t} \quad \text{for } t \geq t_2. \tag{9.6.38}$$

It follows that

$$w(t) \geq \alpha_0\frac{\rho(t)}{t} \quad \text{for } t \geq t_2, \tag{9.6.39}$$

which together with (9.6.38) gives for $t \geq t_2$,

$$\begin{aligned} w(t) &\geq \int_t^\infty \frac{w^2(s)}{\rho(s)}ds + \alpha_0\frac{\rho(t)}{t} \\ &\geq \rho(t)\int_t^\infty \left(\frac{\alpha_0}{s}\right)^2 ds + \alpha_0\frac{\rho(t)}{t} \\ &= \left(\alpha_0^2 + \alpha_1\right)\frac{\rho(t)}{t}. \end{aligned} \tag{9.6.40}$$

Set $\alpha_i = \alpha_{i-1}^2 + \alpha_0$ for $i \in \mathbb{N}$. Then

$$w(t) \geq \alpha_1\frac{\rho(t)}{t} \quad \text{for } t \geq t_2. \tag{9.6.41}$$

By induction, from (9.6.38), we see that

$$w(t) \geq \alpha_i\frac{\rho(t)}{t} \quad \text{for } i \in \mathbb{N},\ t \geq t_2. \tag{9.6.42}$$

As in the proof of Theorem 9.6.5, we see that $\alpha_i \to \infty$ as $i \to \infty$, and hence we conclude that $w(t) = \infty$ for $t \geq t_2$. This contradiction completes the proof. □

Next, we construct the following sequence $\{Q_i\}$ for $i \in \{1, 2, \dots, m\}$, where $Q_i(t)$ are defined as follows:

$$
\begin{aligned}
Q_1(t) &= \int_t^\infty \rho(s)\left[q(s) + \left(\frac{\rho'(s)}{2\rho(s)}\right)^2 + \left(\frac{\rho'}{2\rho}\right)'(s)\right]ds,\\
Q_2(t) &= \int_t^\infty \frac{Q_1^2(s)}{\rho(s)} \exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)ds,\\
&\;\;\vdots\\
Q_m(t) &= \int_t^\infty \frac{Q_{m-1}^2(s)}{\rho(s)} \exp\left(2\int_t^s \frac{Q_{m-1}(\eta)}{\rho(\eta)}d\eta\right)ds,
\end{aligned}
\tag{9.6.43}
$$

where $\rho \in C([t_0, \infty), \mathbb{R}^+)$ and q is defined by (9.6.17).

Now we present the following result.

Theorem 9.6.7. *Assume that there exists a function $\rho \in C^1([t_0, \infty), \mathbb{R}^+)$ such that $\rho'(t) \le 0$ and*

$$0 < Q_1(t) < \infty, \tag{9.6.44}$$

but

$$Q_2(t) = \infty, \tag{9.6.45}$$

where Q_1 and Q_2 are as defined in (9.6.43). Then equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.6.1). As in the proof of Theorem 9.6.6, there exists a function $w(t) > 0$ that eventually satisfies inequality (9.6.36) which takes the form

$$w(t) \ge Q_1(t) + \int_t^\infty \frac{w^2(s)}{\rho(s)}ds. \tag{9.6.46}$$

Set

$$y(t) = \int_t^\infty \frac{w^2(s)}{\rho(s)}ds. \tag{9.6.47}$$

Then

$$
\begin{aligned}
y'(t) &= -\frac{w^2(t)}{\rho(t)} < 0,\\
y'(s)\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right) &= -\frac{w^2(s)}{\rho(s)}\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right).
\end{aligned}
\tag{9.6.48}
$$

Integrating this equation from t to $T \geq t$ provides

$$y(T)\exp\left(2\int_t^T \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right) - y(t) - \int_t^T \frac{2y(s)Q_1(s)}{\rho(s)}\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)ds$$
$$= -\int_t^T \frac{w^2(s)}{\rho(s)}\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)ds. \tag{9.6.49}$$

Therefore

$$y(t) \geq \int_t^T \exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)\left(\frac{w^2(s) - 2y(s)Q_1(s)}{\rho(s)}\right)ds. \tag{9.6.50}$$

Note that $w(t) \geq Q_1(t) + y(t)$ and $Q_1(t) > 0$, $y(t) > 0$. Hence

$$w^2(t) - 2Q_1(t)y(t) \geq Q_1^2(t) + y^2(t). \tag{9.6.51}$$

Thus we have

$$y(t) \geq \int_t^\infty \frac{Q_1^2(s)}{\rho(s)}\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)ds + \int_t^\infty \frac{y^2(s)}{\rho(s)}\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)ds, \tag{9.6.52}$$

which contradicts condition (9.6.45). This completes the proof. □

The following result extends Theorem 9.6.7.

Theorem 9.6.8. *If $Q_i(t)$ in (9.6.43) exist for $i \in \{1, 2, \ldots, m-1\}$ but $Q_m(t)$ does not exist, where $m \in \mathbb{N}$, then equation (9.6.1) is oscillatory.*

The following two results improve Theorems 9.6.7 and 9.6.8.

Theorem 9.6.9. *Assume that there exists a function $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that $\rho'(t) \leq 0$ and*

$$0 < \liminf_{t\to\infty} \frac{t}{\rho(t)}Q_1(t) \leq \frac{1}{4} \tag{9.6.53}$$

but

$$\liminf_{t\to\infty} \frac{t}{\rho(t)}\int_t^\infty \frac{Q_1^2(s)}{\rho(s)}\exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)}d\eta\right)ds > \frac{1}{4}, \tag{9.6.54}$$

where q and Q_1 are defined by (9.6.17) and (9.6.43), respectively. Then equation (9.6.1) is oscillatory.

Proof. Let x be an eventually positive solution of equation (9.6.1). As in the proof of Theorem 9.6.7, there exists $y \in C([t_0,\infty),\mathbb{R}^+)$ such that inequality (9.6.52) holds. By condition (9.6.54) there exists $t_1 \geq t_0$ such that

$$Q_2(t) \geq \alpha_0 \frac{\rho(t)}{t}, \quad Q_1(t) > 0 \quad \text{for } t \geq t_1, \tag{9.6.55}$$

where $\alpha_0 > 1/4$, so

$$y(t) \geq \int_t^\infty \frac{y^2(s)}{\rho(s)} \exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)} d\eta\right) ds + \alpha_0 \frac{\rho(t)}{t} \quad \text{for } t \geq t_1. \tag{9.6.56}$$

It follows that

$$y(t) \geq \alpha_0 \frac{\rho(t)}{t} \quad \text{for } t \geq t_1, \tag{9.6.57}$$

which together with (9.6.56) implies that for $t \geq t_1$,

$$\begin{aligned} y(t) &\geq \int_t^\infty \frac{y^2(s)}{\rho(s)} \exp\left(2\int_t^s \frac{Q_1(\eta)}{\rho(\eta)} d\eta\right) ds + \alpha_0 \frac{\rho(t)}{t} \\ &\geq \rho(t) \int_t^\infty \left(\frac{\alpha_0}{s}\right)^2 ds + \alpha_0 \frac{\rho(t)}{t} \\ &= \left(\alpha_0^2 + \alpha_0\right) \frac{\rho(t)}{t}. \end{aligned} \tag{9.6.58}$$

Set $\alpha_i = \alpha_{i-1}^2 + \alpha_0$ for $i \in \mathbb{N}$. Then

$$y(t) \geq \alpha_1 \frac{\rho(t)}{t} \quad \text{for } t \geq t_1. \tag{9.6.59}$$

By induction, from (9.6.56), we can prove that

$$y(t) \geq \alpha_i \frac{\rho(t)}{t} \quad \text{for } i \in \mathbb{N},\ t \geq t_2. \tag{9.6.60}$$

As before, it is easy to see that $y(t) = \infty$ for $t \geq t_1$, a contradiction which completes the proof. □

By using a similar proof as in Theorem 9.6.9, we have the following result which extends Theorem 9.6.9.

Theorem 9.6.10. *Assume that there exists a function $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ such that $\rho'(t) \le 0$. If*

$$\liminf_{t\to\infty} \frac{t}{\rho(t)} Q_i(t) \le \frac{1}{4} \quad \text{for } i \in \{1,2,\dots,m-1\}, \tag{9.6.61}$$

where Q_i is defined by (9.6.43), but

$$\liminf_{t\to\infty} \frac{t}{\rho(t)} Q_m(t) > \frac{1}{4}, \tag{9.6.62}$$

where $m \in \mathbb{N}$, then equation (9.6.1) is oscillatory.

The following example illustrates the methods presented above.

Example 9.6.11. Consider the difference equation

$$\Delta_\tau^2 x(t) + \frac{\alpha}{(t-2\tau)^2} x(t-\sigma) = 0 \quad \text{for } t > 2\tau, \tag{9.6.63}$$

where $\alpha,\tau,\sigma \in \mathbb{R}^+$, $q(t) = \alpha/(t^2\tau^2)$, and $\alpha \ge \tau^2/2$. Choose $\rho(t) \equiv 1$. Then

$$\begin{gathered} Q_1(t) = \int_t^\infty \frac{\alpha}{\tau^2 s^2} ds = \frac{\alpha}{\tau^2 t}, \\ Q_2(t) = \int_t^\infty \frac{\alpha^2}{\tau^4 s^2} \exp\left(2\int_t^s \frac{\alpha}{\tau^2 \eta} d\eta\right) ds = \frac{\alpha^2}{\tau^4 t^{2\alpha/\tau^2}} \int_t^\infty \frac{1}{s^{2(1-(\alpha/\tau^2))}} ds. \end{gathered} \tag{9.6.64}$$

Since $\alpha \ge \tau^2/2$, we get

$$\int_t^\infty s^{-2(1-(\alpha/\tau^2))} ds = \infty. \tag{9.6.65}$$

By Theorem 9.6.7 or 9.6.8, every solution of equation (9.6.63) is oscillatory.

9.7. Oscillation for systems of delay difference equations

Consider the system of delay difference equations

$$x_i(t) - x_i(t-\sigma) + \sum_{k=1}^{\ell} \sum_{j=1}^{n} q_{ijk} x_j(t-\tau_k) = 0 \quad \text{for } i \in \{1,2,\dots,n\}, \tag{9.7.1}$$

where

$$\sigma, \tau_k \in \mathbb{R}^+, \quad q_{ijk} \in \mathbb{R} \quad \text{for } i,j \in \{1,2,\dots,n\},\ k \in \{1,2,\dots,\ell\}. \tag{9.7.2}$$

Let $\gamma = \max\{\sigma,\tau_1,\dots,\tau_\ell\}$. By a solution $x(t) = (x_1(t),\dots,x_n(t))^T$ of (9.7.1), we mean a continuous function $x \in C([t_0-\gamma,\infty),\mathbb{R}^n)$ which satisfies (9.7.1) for all

$t \geq t_0$. A solution x is said to be oscillatory if at least one of its components x_i has arbitrarily large zeros. Otherwise, the solution is called nonoscillatory.

In this section we will study the oscillatory behavior of solutions of (9.7.1). First we will present some lemmas which will be utilized in the proofs of the main results of this section.

Lemma 9.7.1. *Assume that condition (9.7.2) holds. If (9.7.1) has a nonoscillatory solution x, then there are numbers $\delta_i \in \{-1, 1\}$, $i \in \{1, 2, \dots, n\}$, such that*

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{\ell} \sum_{j=1}^{n} q_{ijk}^* y_j(t - \tau_k) = 0 \quad \text{for } i \in \{1, 2, \dots, n\}, \tag{9.7.3}$$

where

$$q_{ijk}^* = \frac{\delta_i}{\delta_j} q_{ijk} \quad \text{for } i, j \in \{1, 2, \dots, n\},\ k \in \{1, 2, \dots, \ell\} \tag{9.7.4}$$

has a nonoscillatory solution $y(t) = (y_1(t), \dots, y_n(t))^T$ with eventually positive components y_i, $i \in \{1, 2, \dots, n\}$.

Proof. Suppose that x is a nonoscillatory solution of (9.7.1) with eventually positive or negative components. Then there exists $T \geq t_0$ such that $x_i(t) \neq 0$ for $t \geq T$ and $i \in \{1, 2, \dots, n\}$. Set $\delta_i = \operatorname{sgn}[x_i(t)]$ for $i \in \{1, 2, \dots, n\}$. It is easy to see that $y(t) = (\delta_1 x_1(t), \dots, \delta_n x_n(t))^T$ satisfies (9.7.3) and $y_i(t) = \delta_i x_i(t)$ for $t \geq T$ and $i \in \{1, 2, \dots, n\}$. This completes the proof. □

Remark 9.7.2. Clearly, from (9.7.4) for $i, j \in \{1, 2, \dots, n\}$ and $k \in \{1, 2, \dots, \ell\}$, $|q_{ijk}^*| = |q_{ijk}|$ and $q_{iik}^* = q_{iik}$.

Lemma 9.7.3. *Assume that $q_k, \tau_k \in \mathbb{R}^+$ for $k \in \{1, 2, \dots, n\}$, $\sigma \in \mathbb{R}^+$, and*

$$\sigma < \tau = \max_{1 \leq k \leq \ell} \{\tau_k\}. \tag{9.7.5}$$

If the difference inequality

$$u(t) - u(t-\sigma) + \sum_{k=1}^{\ell} q_k u(t - \tau_k) \leq 0 \tag{9.7.6}$$

has an eventually positive solution u, then the difference equation

$$v(t) - v(t-\sigma) + \sum_{k=1}^{\ell} q_k v(t - \tau_k) = 0 \tag{9.7.7}$$

also has an eventually positive solution v and $v(t) \leq z(t)$, where $z(t) = \int_{t-\sigma}^{t} u(s)ds$ and $\lim_{t \to \infty} z(t) = 0$.

Proof. Suppose that (9.7.6) has an eventually positive solution $u(t) > 0$ for $t \geq T \geq t_0 > 0$. By integrating both sides of inequality (9.7.6) on $[t-\sigma, t]$ for $t \geq T$, we obtain

$$\int_{t-\sigma}^{t} u(s)ds - \int_{t-\sigma}^{t} u(s-\sigma)ds + \sum_{k=1}^{\ell} q_k \int_{t-\sigma}^{t} u(s-\tau_k)ds \leq 0 \quad \forall t \geq T+\tau. \tag{9.7.8}$$

Set $z(t) = \int_{t-\sigma}^{t} u(s)ds$ for every $t \geq T+\tau$. Thus $z(t) > 0$ and

$$z'(t) = u(t) - u(t-\sigma) \leq -\sum_{k=1}^{\ell} q_k u(t-\tau_k) < 0. \tag{9.7.9}$$

Hence $z(t)$ is decreasing and

$$\lim_{t\to\infty} z(t) = L \in \mathbb{R}_0^+ = [0, \infty). \tag{9.7.10}$$

By using (9.7.8), it is easy to prove that $L = 0$ and

$$z(t) - z(t-\sigma) + \sum_{k=1}^{\ell} q_k z(t-\tau_k) \leq 0 \quad \text{for } t \geq T+\tau. \tag{9.7.11}$$

Further, we have

$$z(t+m\sigma) - z(t+(m-1)\sigma) + \sum_{k=1}^{\ell} q_k z(t+m\sigma-\tau_k) \leq 0 \tag{9.7.12}$$

for $t \geq T+\tau$ and $m \in \mathbb{N}$. Summing both sides of (9.7.12) from 1 to N, we have

$$z(t+N\sigma) + \sum_{m=1}^{N}\sum_{k=1}^{\ell} q_k z(t+m\sigma-\tau_k) \leq z(t) \quad \forall t \geq T+\tau. \tag{9.7.13}$$

Taking limits as $N \to \infty$ on the left-hand side of (9.7.13), in view of (9.7.10), we find

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\ell} q_k z(t+m\sigma-\tau_k) \leq z(t) \quad \forall t \geq T+\tau. \tag{9.7.14}$$

Now we consider the set W of all nonnegative continuous functions w defined by

$$W = \{w \in C([T+\tau, \infty), \mathbb{R}_0^+) : 0 \leq w(t) \leq z(t) \text{ for every } t \geq T+\tau\}, \tag{9.7.15}$$

and a mapping F on W defined by

$$(Fw)(t) = \begin{cases} \sum_{m=1}^{\infty} \sum_{k=1}^{\ell} q_k w(t + m\sigma - \tau_k) & \text{for } t \geq T + 2\tau - \sigma, \\ (Fw)(T + 2\tau - \sigma) + z(t) - z(t + 2\tau - \sigma) & \text{for } T + \tau \leq t \leq t + 2\tau - \sigma. \end{cases} \tag{9.7.16}$$

First we will prove that the mapping F is continuous. As $\lim_{t\to\infty} z(t) = 0$, for any $\varepsilon > 0$, there exists $T_1 \geq T + \tau$ such that $z(t) < \varepsilon$ for all $t \geq T_1$. We chose an integer $N \geq T_1/\sigma$. Then, from (9.7.14), for any $m_2 > m_1 \geq N$ and all $t \geq T + \tau$, we obtain

$$\begin{aligned} \sum_{m=m_1+1}^{m_2} \sum_{k=1}^{\ell} q_k w(t + m\sigma - \tau_k) &\leq \sum_{m=m_1+1}^{\infty} \sum_{k=1}^{\ell} q_k z(t + m\sigma - \tau_k) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^{\ell} q_k z(t + (m_1 + m)\sigma - \tau_k) \\ &< z(t + m_1\sigma) \\ &< \varepsilon, \end{aligned} \tag{9.7.17}$$

which implies that $\sum_{m=1}^{\infty} \sum_{k=1}^{\ell} q_k w(t+m\sigma-\tau_k)$ converges uniformly on $[T+\tau, \infty)$. Thus, from (9.7.16), $Fw : [T + \tau, \infty) \to \mathbb{R}_0^+$ is continuous. Next, (9.7.16) defines an increasing mapping $F : W \to W$. The increasing character of F is considered with respect to the usual pointwise ordering in W, that is, for any $w_1, w_2 \in W$ with $w_1(t) \leq w_2(t)$ we have $(Fw_1)(t) \leq (Fw_2)(t)$. Note also that by (9.7.14) we have $(Fw)(t) \leq z(t)$ for all $t \geq T + \tau$. Consider the decreasing sequence $\{v_n\}_{n=0}^{\infty}$ of functions in W defined by $v_0(t) = z(t)$ and $v_m(t) = (Fv_{m-1})(t)$ for $m \in \mathbb{N}$ and set

$$v(t) = \lim_{m\to\infty} v_m(t) \quad \text{pointwise on } [T + \tau, \infty). \tag{9.7.18}$$

Now, using (9.7.18) and (9.7.16), we can apply the convergence theorem to obtain $v(t) = (Fv)(t)$, that is,

$$v(t) = \sum_{m=1}^{\infty} \sum_{k=1}^{\ell} q_k v(t + m\sigma - \tau_k) \quad \text{for every } t \geq T + 2\tau - \sigma. \tag{9.7.19}$$

Since $\{v_m(t)\}$ converges uniformly on $[T + \tau, \infty)$, it follows from (9.7.19) that v is continuous on $[t + \tau, \infty)$ and

$$\begin{aligned} v(t) - v(t - \sigma) &= \sum_{m=1}^{\infty} \sum_{k=1}^{\ell} q_k v(t + m\sigma - \tau_k) - \sum_{m=1}^{\infty} \sum_{k=1}^{\ell} q_k v(t + (m-1)\sigma - \tau_k) \\ &= -\sum_{k=1}^{\ell} q_k v(t - \tau_k), \end{aligned} \tag{9.7.20}$$

which means that v is a solution on $[T+\tau,\infty)$ of (9.7.7) with

$$v(t) \le z(t) = \int_{t-\sigma}^{t} u(s)ds. \tag{9.7.21}$$

It remains to prove that v is positive on $[T+\tau,\infty)$. For $T+\tau \le t \le T+2\tau-\sigma$, from (9.7.16), we have $0 < z(t)-z(t-2\tau-\sigma) < v(t)$. Hence $v(t) > 0$ on $[T+\tau, T+2\tau-\sigma)$. Let $t^* = \inf\{t \ge T+2\tau-\sigma : v(t) = 0\}$. We will prove that $t^* = \infty$. Otherwise, $t^* \in [T+2\tau-\sigma,\infty)$. So, $v(t) > 0$ for $T+\tau < t < t^*$ and $v(t^*) = 0$. But, using (9.7.19) we have

$$v(t^*) = \sum_{m=1}^{\infty}\sum_{k=1}^{\ell} q_k v(t^*+m\sigma-\tau_k) \ge \sum_{k=1}^{\ell} q_k v(t^*+\sigma-\tau_k) > 0, \tag{9.7.22}$$

which is a contradiction. This contradiction implies $t^* = \infty$ and completes the proof. □

Next we will establish some sufficient conditions for the oscillation of system (9.7.1).

First we give the following comparison result.

Theorem 9.7.4. *Let*

$$q_k = \sum_{1\le i\le n}\left\{q_{iik} - \sum_{j=1,\, j\ne i}^{n} |q_{ijk}|\right\} > 0 \quad \text{for } k \in \{1,2,\dots,\ell\}. \tag{9.7.23}$$

Assume that (9.7.2) and (9.7.5) hold. If all solutions of the scalar difference equation

$$u(t) - u(t-\sigma) + \sum_{k=1}^{\ell} q_k u(t-\tau_k) = 0 \tag{9.7.24}$$

are oscillatory, then all solutions of the system (9.7.1) are also oscillatory.

Proof. Suppose that the delay system (9.7.1) has a nonoscillatory solution $x(t) = (x_1(t), x_2(t),\dots,x_n(t))^T$. Let $y_i(t) = \delta_i x_i(t)$ for $i \in \{1,2,\dots,n\}$. By Lemma 9.7.1 it follows from (9.7.1) that

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{\ell}\sum_{j=1}^{n} q_{ijk}^* y_j(t-\tau_k) = 0 \quad \text{for } i \in \{1,2,\dots,n\}, \tag{9.7.25}$$

where

$$q_{ijk}^* = \frac{\delta_i}{\delta_j} q_{ijk} \quad \text{for } i,j \in \{1,2,\dots,n\},\ k \in \{1,2,\dots,\ell\}, \tag{9.7.26}$$

and $y_i(t) > 0$ for $i \in \{1,2,\dots,n\}$. From (9.7.25), for $i \in \{1,2,\dots,n\}$ we have

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{\ell}\left(q_{iik}^{*}y_i(t-\tau_k) + \sum_{j=1,\, j\neq i}^{n} q_{ijk}^{*}y_j(t-\tau_k)\right) = 0. \tag{9.7.27}$$

By summing (vertically) both sides of (9.7.27), we find that

$$v(t) - v(t-\sigma) + \sum_{k=1}^{\ell}\sum_{i=1}^{n}\left(q_{iik}^{*}y_i(t-\tau_k) + \sum_{j=1,\, j\neq i}^{n} q_{ijk}^{*}y_j(t-\tau_k)\right) = 0, \tag{9.7.28}$$

where $v(t) = \sum_{i=1}^{n} y_i(t)$. As for $i \in \{1,2,\dots,n\}$ and $k \in \{1,2,\dots,\ell\}$ we have $|q_{iik}^{*}| = q_{iik}$ and $|q_{ijk}^{*}| = |q_{ijk}|$, it follows from (9.7.28) and (9.7.23) that

$$\begin{aligned} 0 &> v(t) - v(t-\sigma) + \sum_{k=1}^{\ell}\sum_{i=1}^{n}\left(q_{iik}y_i(t-\tau_k) - \sum_{j=1,\, j\neq i}^{n} |q_{ijk}|\, y_j(t-\tau_k)\right) \\ &\geq v(t) - v(t-\sigma) + \sum_{k=1}^{\ell} q_k v(t-\tau_k). \end{aligned} \tag{9.7.29}$$

From (9.7.29) (since it is easy to see that all the hypotheses of Lemma 9.7.3 are satisfied), we see that the corresponding scalar difference equation (9.7.24) has an eventually positive solution u, which contradicts the fact that equation (9.7.24) is oscillatory. This completes the proof. □

The following lemma is about the oscillation of equation (9.7.24).

Lemma 9.7.5. *Assume that $\sigma, \tau_k, q_k \in \mathbb{R}^+$ for $k \in \{1,2,\dots,\ell\}$ and*

$$\sigma < \tau_k \quad \text{for } k \in \{1,2,\dots,\ell\}. \tag{9.7.30}$$

If

$$\sum_{k=1}^{\ell} q_k \left[\frac{\tau_k^{\tau_k}}{\sigma^{\sigma}(\tau_k-\sigma)^{\tau_k-\sigma}}\right]^{1/\sigma} > 1, \tag{9.7.31}$$

then equation (9.7.24) is oscillatory.

By Theorem 9.7.4 and Lemma 9.7.5, we give the following explicit sufficient condition for the oscillation of system (9.7.1).

Corollary 9.7.6. *If conditions (9.7.2), (9.7.23), (9.7.30), and (9.7.31) hold, then system (9.7.1) is oscillatory.*

Theorem 9.7.4 can be extended to the nonautonomous system of delay difference equations

$$x_i(t) - x_i(t-\sigma) + \sum_{k=1}^{\ell} \sum_{j=1}^{n} q_{ijk}(t) x_j(t-\tau_k) = 0 \quad \text{for } i \in \{1,2,\dots,n\}, \tag{9.7.32}$$

where

$$\sigma, \tau_k \in \mathbb{R}^+, \quad q_{ijk} \in C([t_0,\infty),\mathbb{R}) \quad \text{for } i,j \in \{1,2,\dots,n\},\ k \in \{1,2,\dots,\ell\}. \tag{9.7.33}$$

Theorem 9.7.7. *Assume that conditions (9.7.5) and (9.7.33) hold and*

$$q_k = \inf_{t\in[t_0,\infty)} \min_{1\le i\le n} \left(q_{iik}(t) - \sum_{j=1,\, j\ne i}^{n} |q_{ijk}(t)| \right) > 0. \tag{9.7.34}$$

If equation (9.7.24) is oscillatory, then system (9.7.32) is also oscillatory.

Proof. Suppose that the delay system (9.7.32) has a nonoscillatory solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$. Let $y_i(t) = \delta_i x_i(t) > 0$ for $i \in \{1,2,\dots,n\}$. From system (9.7.32) we have

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{\ell} \sum_{i=1}^{n} q^*_{ijk}(t) y_j(t-\tau_k) = 0 \quad \text{for } i \in \{1,2,\dots,n\}, \tag{9.7.35}$$

where

$$q^*_{ijk}(t) = \frac{\delta_i}{\delta_j} q_{ijk}(t) \quad \text{for } i,j \in \{1,2,\dots,n\},\ k \in \{1,2,\dots,\ell\}. \tag{9.7.36}$$

It follows from (9.7.35) that for all $i \in \{1,2,\dots,n\}$,

$$y_i(t) - y_i(t-\sigma) + \sum_{k=1}^{\ell} \left(q^*_{iik}(t) y_i(t-\tau_k) + \sum_{j=1,\, j\ne i}^{n} q^*_{ijk}(t) y_j(t-\tau_k) \right) = 0. \tag{9.7.37}$$

Summing both sides of (9.7.37) and using (9.7.34), we find that

$$\begin{aligned} 0 &> v(t) - v(t-\sigma) + \sum_{k=1}^{\ell} \sum_{i=1}^{n} \left(q_{iik}(t) y_i(t-\tau_k) - \sum_{j=1,\, j\ne i}^{n} |q^*_{ijk}(t)| y_j(t-\tau_k) \right) \\ &\ge v(t) - v(t-\sigma) + \sum_{k=1}^{n} q_k v(t-\tau_k), \end{aligned} \tag{9.7.38}$$

where $v(t) = \sum_{i=1}^{n} y_i(t)$. By Lemma 9.7.3, equation (9.7.24) has an eventually positive solution, which is a contradiction. This completes the proof. □

Corollary 9.7.8. *Assume that conditions (9.7.5), (9.7.30), (9.7.31), (9.7.33), and (9.7.34) hold. Then system (9.7.32) is oscillatory.*

Example 9.7.9. As an application, we consider the system

$$\Delta x_i(t) + \sum_{k=1}^{\ell} \sum_{j=1}^{n} q_{ijk} x_j(t - \tau_k) = 0 \quad \text{for } i \in \{1, 2, \ldots, n\}, \tag{9.7.39}$$

where $q_{ijk} \in \mathbb{R}$ and $\tau_k \in \mathbb{R}^+$ for $i, j \in \{1, 2, \ldots, n\}$ and $k \in \{1, 2, \ldots, \ell\}$. System (9.7.39) is equivalent to system (9.7.1) with $\sigma = 1$. Therefore, system (9.7.39) is oscillatory if conditions (9.7.23), (9.7.30), and (9.7.31) are satisfied with $\sigma = 1$.

Next, when $\sigma \in \mathbb{N}$ is arbitrary, we see that the neutral difference systems

$$\Delta\left(x_i(t) + \sum_{m=1}^{\sigma-1} x_i(t-m)\right) + \sum_{k=1}^{\ell} \sum_{j=1}^{n} q_{ijk} x_j(t - \tau_k) = 0, \tag{9.7.40}$$

$$\Delta^2\left(x_i(t) + \sum_{m=1}^{\sigma-1} (m+1) x_i(t-m)\right) + \sum_{k=1}^{\ell} \sum_{j=1}^{n} q_{ijk} x_j(t - \tau_k) = 0 \tag{9.7.41}$$

are equivalent to the system (9.7.1). Now, as applications of Corollary 9.7.6, systems (9.7.40) and (9.7.41) are oscillatory provided that the conditions (9.7.2), (9.7.23), (9.7.30), and (9.7.31) are satisfied.

9.8. Oscillatory behavior of solutions of functional equations

We will consider functional equations of second order of the form

$$q_0(t)x(t) + q_1(t)x(g(t)) + q_2(t)x(g^2(t)) = 0, \tag{9.8.1}$$

where $q_i : I \to \mathbb{R}$, $i \in \{0, 1, 2\}$, and $g : I \to I$ are given functions, x is an unknown real-valued function, and I denotes an unbounded subset of $[0, \infty)$. By g^m we mean the mth iterate of the function g, that is,

$$g^0(t) = t, \quad g^{m+1}(t) = g(g^m(t)) \quad \text{for } t \in I,\ m \in \mathbb{N}_0. \tag{9.8.2}$$

By g^{-1} we mean the inverse function of g and $g^{-m-1}(t) = g^{-1}(g^{-m}(t))$.

Throughout this section upper indices at the sign of a function will denote iterates. In each instance we have the relation $g^1(t) = g(t)$. We will assume that

$$g(t) \neq t, \quad \lim_{t\to\infty} g(t) \quad \text{for } t \in I. \tag{9.8.3}$$

Moreover, we assume that g^{-1} exists.

By a solution of equation (9.8.1) we mean a function $x : I \to \mathbb{R}$ such that

$$\sup\{|x(s)| : s \in I_{t_0} = [t_0, \infty) \cap I\} > 0 \tag{9.8.4}$$

for any $t_0 \in \mathbb{R}$ and x satisfies equation (9.8.1) on I. A solution x of equation (9.8.1) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty} \subset I$ such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n)x(t_{n+1}) \leq 0$ for $n \in \mathbb{N}_0$. Otherwise, it is called nonoscillatory.

We note that functional equations, in particular, are recurrence equations which have numerous applications. Recurrence equations can be used to describe processes in many areas such as biology, meteorology, economics, and so forth (see [194]).

One can observe that existence of oscillatory solutions of equation (9.8.1) is connected with the sign of the functions $q_i(t)$ for $i \in \{0, 1, 2\}$ on I. For example, it is easy to prove that either $q_i(t) > 0$ or $q_i(t) < 0$ for $i \in \{0, 1, 2\}$ and $t \in I$, which implies that equation (9.8.1) possesses only oscillatory solutions. If one of the coefficients $q_i(t)$, $i \in \{0, 1, 2\}$, has a sign opposite to that of others, that is, $q_i(t) > 0$, $q_j(t) > 0$, and $q_k(t) < 0$ for $i \neq j \neq k$ on I, then equation (9.8.1) can possess both oscillatory and nonoscillatory solutions. For example, the functional equation

$$x(t + 2\pi) - (e^{\pi} + 1)x(t + \pi) + e^{\pi}x(t) = 0 \quad \text{for } t \in [0, \infty) \tag{9.8.5}$$

has a nonoscillatory solution $x(t) = e^t$ and an oscillatory solution $x(t) = \cos 2t$. Therefore a question arises: if the last case holds, under what additional conditions on the coefficients q_i will every solution of equation (9.8.1) be oscillatory? We will present some answers to this question in the case when

$$q_2(t) > 0, \quad q_1(t) < 0, \quad q_0(t) > 0 \quad \text{for } t \in I. \tag{9.8.6}$$

If we denote

$$p(t) = -\frac{q_0(t)}{q_1(t)} > 0, \quad q(t) = -\frac{q_2(t)}{q_1(t)} > 0 \quad \text{for } t \in I, \tag{9.8.7}$$

then equation (9.8.1) takes the form

$$x(g(t)) = p(t)x(t) + q(t)x(g^2(t)). \tag{9.8.8}$$

Now we present the following oscillation criterion for equation (9.8.8).

Lemma 9.8.1. *If*

$$\limsup_{I \ni t \to \infty} q(t)p(g(t)) > 1, \tag{9.8.9}$$

then equation (9.8.8) is oscillatory.

Proof. Suppose that x is a nonoscillatory solution of equation (9.8.8). Since $-x$ is also a solution of equation (9.8.8), without loss of generality we may assume that $x(t) > 0$ for $t \in I_{t_1}$ with $t_1 > 0$. Then, in view of (9.8.3), there exists a point $t_2 \in I_{t_1}$ such that $x(g^i(t)) > 0$ for $t \in I_{t_2}$ and $i \in \{1,2\}$. Therefore, from equation (9.8.8), we have $x(g(t)) \geq p(t)x(t)$ for $t \in I_{t_2}$, which gives

$$x(g^2(t)) \geq p(g(t))x(g(t)). \tag{9.8.10}$$

Using (9.8.10) in equation (9.8.8), we obtain

$$x(g(t)) = p(t)x(t) + q(t)x(g^2(t)) \geq q(t)p(g(t))x(g(t)), \tag{9.8.11}$$

which contradicts (9.8.9). This completes the proof. □

The following result is concerned with the case when condition (9.8.9) is violated.

Theorem 9.8.2. *If*

$$\liminf_{I \ni t \to \infty} q(t)p(g(t)) > \frac{1}{4}, \tag{9.8.12}$$

then equation (9.8.8) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of equation (9.8.8). Then, as in the proof of Lemma 9.8.1, x satisfies (9.8.10). Using (9.8.10) in equation (9.8.8), we obtain

$$x(g(t)) \geq p(t)x(t) + q(t)p(g(t))x(g(t)), \tag{9.8.13}$$

$$q(t)p(g(t)) \leq 1 - p(t)\frac{x(t)}{x(g(t))}. \tag{9.8.14}$$

From condition (9.8.12) it follows that there exists $\varepsilon > 0$ such that

$$q(t)p(g(t)) \geq \frac{1+\varepsilon}{4} > \frac{1}{4}. \tag{9.8.15}$$

Therefore, from (9.8.14) and (9.8.15), we have

$$\frac{1+\varepsilon}{4} \leq 1 - p(t)\frac{x(t)}{x(g(t))}, \tag{9.8.16}$$

which gives

$$p(t)\frac{x(t)}{x(g(t))} \leq 1 - \frac{1+\varepsilon}{4} \leq \frac{1}{1+\varepsilon}\max_{\varepsilon>0}(1+\varepsilon)\left(1 - \frac{1+\varepsilon}{4}\right) = \frac{1}{1+\varepsilon}. \tag{9.8.17}$$

Thus $(1+\varepsilon)p(t)x(t) \le x(g(t))$, and by iteration

$$(1+\varepsilon)p(g(t))x(g(t)) \le x(g^2(t)). \tag{9.8.18}$$

By using (9.8.18) in equation (9.8.8) and then by repeating the above arguments, we find that

$$(1+\varepsilon)^2 p(t)x(t) \le x(g(t)), \qquad (1+\varepsilon)^2 p(g(t))x(g(t)) \le x(g^2(t)). \tag{9.8.19}$$

Thus, by induction, we have for every $k \in \mathbb{N}$,

$$(1+\varepsilon)^k p(g(t))x(g(t)) \le x(g^2(t)). \tag{9.8.20}$$

Choose k such that

$$(1+\varepsilon)^{k+1} > 4. \tag{9.8.21}$$

Now, by using (9.8.18) in equation (9.8.8), we have

$$x(g(t)) = p(t)x(t) + q(t)x(g^2(t)) \ge (1+\varepsilon)^k q(t)p(g(t))x(g(t)). \tag{9.8.22}$$

Thus $1 \ge (1+\varepsilon)^k q(t)p(g(t))$, which by (9.8.15) gives $4 \ge (1+\varepsilon)^{k+1}$. The last inequality contradicts (9.8.20) and the proof is complete. □

Next we present the following result.

Theorem 9.8.3. *Suppose that for some $m \in \mathbb{N}_0$ the condition*

$$\limsup_{I\ni t\to\infty} \left[q(t)p(g(t)) + \sum_{i=0}^{m} \prod_{j=0}^{i} q(g^{j+1}(t))\, p(g^{j+2}(t)) \right] > 1. \tag{9.8.23}$$

is satisfied. Then equation (9.8.8) is oscillatory.

Proof. Assume that x is an eventually positive solution of equation (9.8.8). As in the proof of Lemma 9.8.1, we see that $x(g^i(t)) > 0$ for $i \ge 1$ and $t \in I_{t_2}$. Then from equation (9.8.8), we obtain

$$x(g(t)) \ge q(t)x(g^2(t)), \tag{9.8.24}$$

and $x(g(t)) \ge p(t)x(t)$. Thus, one can easily prove by induction the formula

$$x(g^i(t)) \ge x(t) \prod_{j=0}^{i-1} p(g^j(t)) \quad \text{for } i \in \mathbb{N}. \tag{9.8.25}$$

Replacing t by $g(t)$ in equation (9.8.8), we find

$$x(g^2(t)) = p(g(t))x(g(t)) + q(g(t))x(g^3(t)). \tag{9.8.26}$$

Now induction yields

$$x(g^{i+1}(t)) = p(g^i(t))x(g^i(t)) + q(g^i(t))x(g^{i+2}(t)) \quad \text{for } i \in \mathbb{N}. \tag{9.8.27}$$

Using (9.8.27) with $i = 2$ and next with $i = 3$ in (9.8.26), we obtain

$$\begin{aligned}
x(g^2(t)) &= p(g(t))x(g(t)) + q(g(t))p(g^2(t))x(g^2(t)) + q(g(t))q(g^2(t))x(g^4(t)) \\
&= p(g(t))x(g(t)) + q(g(t))p(g^2(t))x(g^2(t)) \\
&\quad + q(g(t))q(g^2(t))p(g^3(t))x(g^3(t)) + q(g(t))q(g^2(t))q(g^3(t))x(g^5(t)) \\
&= p(g(t))x(g(t)) + \sum_{i=0}^{1} p(g^{i+2}(t))x(g^{i+2}(t)) \prod_{j=0}^{i} q(g^{j+1}(t)) \\
&\quad + x(g^5(t)) \prod_{j=0}^{2} q(g^{j+1}(t)).
\end{aligned} \tag{9.8.28}$$

Then induction gives for $m > 1$,

$$\begin{aligned}
x(g^2(t)) = p(g(t))x(g(t)) &+ \sum_{i=0}^{m} p(g^{i+2}(t))x(g^{i+2}(t)) \prod_{j=0}^{i} q(g^{j+1}(t)) \\
&+ x(g^{m+4}(t)) \prod_{j=0}^{m+1} q(g^{j+1}(t)).
\end{aligned} \tag{9.8.29}$$

Now, in view of (9.8.24), (9.8.25), and the positivity of $x(g^i(t))$, we derive

$$x(g^2(t)) \geq q(t)p(g(t))x(g^2(t)) + \sum_{i=0}^{m} x(g^2(t)) \prod_{j=0}^{i} q(g^{j+1}(t))p(g^{j+2}(t)). \tag{9.8.30}$$

Dividing both sides of (9.8.30) by $x(g^2(t))$, we obtain a contradiction to (9.8.23). This completes the proof. □

Next we will apply the above results to difference equations of the form

$$\Delta_h x(t) = x(t+h) - x(t) = q(t)x(t+2h) \quad \text{with } h > 0, \tag{9.8.31}$$

$$b(k)x(k+1) = a(k)x(k) + c(k)x(k+2) \quad \text{for } k \in \mathbb{N}, \tag{9.8.32}$$

where $q : [0, \infty) \to [0, \infty)$ is a continuous function and $a, b, c : \mathbb{N} \to [0, \infty)$.

By applying the above results to equations (9.8.31) and (9.8.32), one can easily find the following results.

Theorem 9.8.4. *Equation (9.8.31) is oscillatory if one of the following conditions holds:*

$$\liminf_{t\to\infty} q(t) > \frac{1}{4} \tag{9.8.33}$$

or, for some $m \in \mathbb{N}_0$,

$$\limsup_{t\to\infty} \left[q(t) + \sum_{i=0}^{m} \prod_{j=0}^{i} q(t+(j+1)h) \right] > 1. \tag{9.8.34}$$

Theorem 9.8.5. *Equation (9.8.32) is oscillatory if one of the following conditions is satisfied:*

$$\liminf_{k\to\infty} \frac{a(k+1)c(k)}{b(k)b(k+1)} > \frac{1}{4} \tag{9.8.35}$$

or, for some $m \in \mathbb{N}_0$,

$$\limsup_{k\to\infty} \left[\frac{a(k+1)c(k)}{b(k)b(k+1)} + \sum_{i=0}^{m} \prod_{j=0}^{i} \frac{a(k+2+j)c(k+1+j)}{b(k+1+j)b(k+2+j)} \right] > 1. \tag{9.8.36}$$

The following examples illustrate the methods presented above.

Example 9.8.6. Consider the damped second-order difference equation

$$\Delta^2 x(k) + p(k)\Delta x(k+1) + q(k)x(k+1) = 0, \tag{9.8.37}$$

where $\{p(k)\}$ and $\{q(k)\}$ are sequences of real numbers satisfying $p(k) > -1$ and $q(k) < 2 + p(k)$ for $k \in \mathbb{N}$. We rewrite equation (9.8.37) in the form

$$x(k+1) = \frac{1}{2+p(k)-q(k)} x(k) + \frac{1+p(k)}{2+p(k)-q(k)} x(k+2). \tag{9.8.38}$$

By applying Theorem 9.8.5 we see that equation (9.8.37) is oscillatory if

$$\liminf_{k\to\infty} \frac{1+p(k)}{(2+p(k)-q(k))(2+p(k+1)-q(k+1))} > \frac{1}{4} \tag{9.8.39}$$

or, for some $m \in \mathbb{N}_0$,

$$\limsup_{k\to\infty} \left[\frac{1+p(k)}{(2+p(k)-q(k))(2+p(k+1)-q(k+1))} + \sum_{i=0}^{m} \prod_{j=0}^{i} \frac{1+p(k+1+j)}{(2+p(k+1+j)-q(k+1+j))(2+p(k+2+j)-q(k+2+j))} \right] > 1. \tag{9.8.40}$$

Example 9.8.7. Consider the neutral difference equation

$$\Delta\big(x(k) + px(k+1)\big) + q(k)x(k) = 0, \tag{9.8.41}$$

where $\{q(k)\}$ is a sequence of real numbers with $q(k) > 1$ for $k \in \mathbb{N}$ and p is a real number with $p > 1$. Clearly equation (9.8.41) can be written as

$$x(k+1) = \frac{q(k)-1}{p-1}x(k) + \frac{p}{p-1}x(k+2). \tag{9.8.42}$$

This equation is oscillatory by Theorem 9.8.5 provided

$$\liminf_{k\to\infty}\big(q(k+1)-1\big) > \frac{1}{4}\frac{(p-1)^2}{p} \tag{9.8.43}$$

or, for some $m \in \mathbb{N}_0$,

$$\limsup_{k\to\infty}\left[q(k+1)-1+\sum_{i=0}^{m}\prod_{j=0}^{i}\big(q(k+2+j)-1\big)\right] > \frac{(p-1)^2}{p}. \tag{9.8.44}$$

Example 9.8.8. Consider the perturbed neutral difference equation

$$\Delta^2\big(x(k)+x(k-1)\big) + p(k)x(k) + q(k)x(k+2) = x(k-1), \tag{9.8.45}$$

where $\{p(k)\}$ and $\{q(k)\}$ are sequences of real numbers satisfying $p(k) > 1$ and $q(k) > -1$ for $k \in \mathbb{N}$. Equation (9.8.45) can be rewritten as

$$x(k+1) = \big(p(k)-1\big)x(k) + \big(q(k)+1\big)x(k+2). \tag{9.8.46}$$

If

$$\liminf_{k\to\infty}\big(p(k+1)-1\big)\big(q(k)+1\big) > \frac{1}{4} \tag{9.8.47}$$

or, for some $m \in \mathbb{N}_0$,

$$\limsup_{k\to\infty}\left[\big(p(k+1)-1\big)\big(q(k)+1\big)+\sum_{i=0}^{m}\prod_{j=0}^{i}\big(p(k+2+j)-1\big)\big(q(k+1+j)+1\big)\right] > 1, \tag{9.8.48}$$

then equation (9.8.45) is oscillatory by Theorem 9.8.5.

9.9. Notes and general discussions

(1) The results of Section 9.1 are taken from Qian and Yan [239].
(2) The results of Section 9.2.1 are due to Kordonis et al. [180]. We note that in the special case where $a(k) \equiv 1$ for $k \in \mathbb{N}_0$, equation (9.2.1) becomes

$$x(k+1) - x(k) + px(k-\tau) + qx(k-\sigma) = 0. \tag{9.9.1}$$

The following "if and only if" criterion is known (cf. [190] or [150, Chapter 7]) for the oscillation of equation (9.9.1).

Theorem 9.9.1. *All solutions of equation (9.9.1) are oscillatory if and only if its characteristic equation*

$$\lambda - 1 + p\lambda^{-\tau} + q\lambda^{-\sigma} = 0 \tag{9.9.2}$$

has no positive roots.

The "only if" part can be obtained from Theorem 9.2.4, while the "if" part can be obtained from Theorem 9.2.4 only in the case where the following assumptions fail to hold: $\tau > \sigma > 0$, $p + q > 0$, $p > 0$, and $p(\tau - \sigma) > 1$.

It remains an open problem to establish result Theorem 9.2.4(i_2) without the restriction that (9.2.51) fails to hold. Note that, in the case of linear delay differential equations with periodic coefficients, an analogous restriction is not imposed.

The results of Section 9.2.2 are taken from Kordonis et al. [180]. Note that by using the same arguments with those applied in proving result Theorem 9.2.4(i_1), one can establish an analogous result for equation (9.2.105). Indeed, the following proposition can be proved.

Proposition 9.9.2. *A necessary condition for the oscillation of equation (9.2.105) is that there is no positive root λ_0 of the equation*

$$\lambda^m - \prod_{r=0}^{m-1} \left(1 - \left[p(r)\lambda^{-\tau} - q(r)\lambda^{-\sigma}\right]\right) = 0 \tag{9.9.3}$$

with the following property: if $m > 1$, then

$$p(r)\lambda_0^{-\tau} - q(r)\lambda_0^{-\sigma} < 1 \quad \textit{for } r \in \{1, 2, \dots, m-1\}. \tag{9.9.4}$$

By applying this proposition one can obtain explicit necessary conditions (in terms of m, τ, σ, and the coefficient sequences $\{p(k)\}_{k\geq 0}$ and $\{q(k)\}_{k\geq 0}$) for the oscillation of all solutions of equation (9.2.105). It suffices to find necessary conditions for equation (9.9.3) that have no positive roots λ_0 with the following property: if $m > 0$, then (9.9.4) holds.

The results in Sections 9.2.3 and 9.2.4 are taken from Kordonis et al. [180]. By applying Theorem 9.2.8, in the special case where $c = 0$, we obtain the main result in [233], while Theorem 9.2.10 when $c = 0$ leads to the main result in [231].

Now, consider the special case of neutral linear difference equations with constant coefficients, namely, the equation

$$\Delta(x(k) + cx(k-\tau)) + \sum_{j=0}^{N} p_j x(k-\sigma_j) = 0, \tag{9.9.5}$$

where $c \in \mathbb{R}$, $\tau \in \mathbb{N}$, $p_0 \geq 0$, and $p_j > 0$ for $j \in \{1, 2, \ldots, N\}$ with $N \in \mathbb{N}$ are real numbers, and $\sigma_j \in \mathbb{N}_0$ for $j \in \{0, 1, \ldots, N\}$ are such that $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_N$. In this case, the period is $m = 1$ and the characteristic equation of (9.9.5) is

$$\lambda - 1 + \frac{1}{1 + c\lambda^{-\tau}} \sum_{j=0}^{N} p_j \lambda^{-\sigma_j} = 0. \tag{9.9.6}$$

By applying Theorem 9.2.8 to equation (9.9.5), we obtain the following "if and only if" criterion for the oscillation of equation (9.9.5).

Theorem 9.9.3. *Assume that* $-1 < c \leq 0$. *Then a necessary and sufficient condition for the oscillation of equation (9.9.5) is that equation (9.9.6) has no roots in the interval* $((-c)^{1/\tau}, 1)$.

Next, consider the difference equation

$$\Delta(x(k) + cx(k+\tau)) - q_0 x(k) - \sum_{j \in J} q_j x(k + \sigma_j^*) = 0, \tag{9.9.7}$$

where $c \in \mathbb{R}$, $\tau \in \mathbb{N}$, J is a nonempty (which may be infinite) subset of $\mathbb{N}$, $q_0 \geq 0$, $q_j > 0$ for $j \in J$ are real numbers, and $\sigma_j^* \in \mathbb{N}$ for $j \in J$ are such that $\sigma_{j_1}^* \neq \sigma_{j_2}^*$ if $j_1, j_2 \in J$ with $j_1 \neq j_2$. Here $m = 1$ and the characteristic equation (9.9.6) appears as

$$\lambda - 1 - \frac{1}{1 + c\lambda^{\tau}} \left(q_0 + \sum_{j \in J} q_j \lambda^{\sigma_j^*} \right) = 0. \tag{9.9.8}$$

An application of Theorem 9.2.10 to equation (9.9.7) leads to the following "if and only if" oscillation criterion.

Theorem 9.9.4. *Assume that* $-1 < c \leq 0$. *Then a necessary and sufficient condition for the oscillation of equation (9.9.7) is that equation (9.9.8) has no roots in the interval* $(1, 1/(-c)^{1/\tau})$.

(3) Lemmas 9.3.1 and 9.3.2 are taken from Győri and Ladas [149] while Theorem 9.3.3 is due to Kocić and Ladas [175]. The results in Section 9.3.2 are taken from Ladas and Qian [193].
(4) The results of Section 9.4.1 are due to Jaroma et al. [167] while the results in Section 9.4.2 are extracted from Kocić et al. [178]. The results

in Section 9.4.3 are due to Kocić and Ladas [175] and Ladas and Qian [193]. The results of Section 9.4.4 are taken from Kocić and Ladas [176] while Theorem 9.4.22 is due to Cheng and Yan [85].

(5) The results of Section 9.5.1 are due to Kocić and Ladas [177]. The results of Section 9.5.2 are taken from Jaroma et al. [166] while the results in Section 9.5.3 are extracted from Kulenović et al. [187]. The results of Section 9.5.4 are due to Amleh et al. [41], and the results in Section 9.5.5 are taken from DeVault et al. [103]. The results in Section 9.5.6 are taken from Cheng and Yan [85] while the results in Section 9.5.7 are due to DeVault et al. [102]. Finally, the result in Section 9.5.8 is due to Rodrigues [252].

It is interesting to note that computer analysis indicates that equation (9.5.178) exhibits some sort of period doubling behavior. If a and c are fixed and b is small, then all solutions tend to the unique positive equilibrium $\overline{x}$. When b is increased, $\overline{x}$ becomes unstable and a period-2 cycle is born, which seems to attract all solutions. As b is increased further, a four-cycle comes into being which seems to attract all solutions, and then an eight-cycle and so forth. The behavior is very similar to that of first-order difference equations that exhibit the period doubling route to chaos. In fact, when $c = 0$, we get the first-order equation

$$x(k+1) = ax(k) + be^{-x(k)}, \tag{9.9.9}$$

which exhibits chaotic dynamics.

For further discussions on the oscillatory and global asymptotic stability of the well-known Riccati difference equation

$$x(k+1) = \frac{a(k)x(k) + b(k)}{c(k)x(k) + d(k)} \quad \text{for } k \in \mathbb{N}_0, \tag{9.9.10}$$

where $\{a(k)\}$, $\{b(k)\}$, $\{c(k)\}$, and $\{d(k)\}$ are given sequences of real numbers such that

$$a(k)d(k) - b(k)c(k) \neq 0, \quad c(k) \neq 0 \quad \text{for } k \in \mathbb{N}_0, \tag{9.9.11}$$

as well as some interesting recursive sequences, we refer the reader to the monographs due to Győri and Ladas [150], Kocić and Ladas [176], and others appeared recently.

(6) Theorems 9.6.1–9.6.3, 9.6.5, 9.6.7, and 9.6.8 are taken from Zhang et al. [291] while Theorems 9.6.4, 9.6.6, 9.6.9, and 9.6.10 improve results due to Deng [100].

(7) The results of Section 9.7 are taken from Yan and Zhang [287]. The obtained results can be generalized to nonlinear systems of difference equations as well as higher-order systems of difference equations.

For further discussion on this topic, we refer the reader to the monographs of Agarwal [4], Agarwal et al. [19], Gopalsamy [131], Győri and Ladas [150], and Kocić and Ladas [176].

(8) The results of Section 9.8 are taken from Golda and Werbowski [130]. We note that the results of this section can be extended to more general functional equations of the form

$$q_0(t)x(t) + q_1(t)x\big(g(t)\big) + q_2(t)x\big(g^2(t)\big) + \cdots + q_{m+1}(t)x\big(g^{m+1}(t)\big) = 0, \quad (9.9.12)$$

where $q_i : I \to \mathbb{R}$ for $i \in \{0, 1, \ldots, m+1\}$ and $g : I \to I$, where I denotes an unbounded subset of $[0, \infty)$.

For further studies of this equation we refer the reader to the papers of Nowakowska and Werbowski [208, 209].

Bibliography

[1] A. R. Aftabizadeh and J. Wiener, *Oscillatory properties of first order linear functional-differential equations*, Appl. Anal. **20** (1985), no. 3-4, 165–187.

[2] ______, *Oscillatory and periodic solutions for systems of two first order linear differential equations with piecewise constant argument*, Appl. Anal. **26** (1988), no. 4, 327–333.

[3] A. R. Aftabizadeh, J. Wiener, and J.-M. Xu, *Oscillatory and periodic solutions of delay differential equations with piecewise constant argument*, Proc. Amer. Math. Soc. **99** (1987), no. 4, 673–679.

[4] R. P. Agarwal, *Difference Equations and Inequalities. Theory, Methods, and Applications*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 155, Marcel Dekker, New York, 1992.

[5] R. P. Agarwal, C. D. Ahlbrandt, M. Bohner, and A. C. Peterson, *Discrete linear Hamiltonian systems: a survey*, Dynam. Systems Appl. **8** (1999), no. 3-4, 307–333, Special issue on "discrete and continuous Hamiltonian systems," edited by R. P. Agarwal and M. Bohner.

[6] R. P. Agarwal and M. Bohner, *Quadratic functionals for second order matrix equations on time scales*, Nonlinear Anal. **33** (1998), no. 7, 675–692.

[7] ______, *Basic calculus on time scales and some of its applications*, Results Math. **35** (1999), no. 1-2, 3–22.

[8] R. P. Agarwal, M. Bohner, and W.-T. Li, *Nonoscillation and Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 2004.

[9] R. P. Agarwal, M. Bohner, D. O'Regan, and A. C. Peterson, *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. **141** (2002), no. 1-2, 1–26, Special issue on "dynamic equations on time scales," edited by R. P. Agarwal, M. Bohner, and D. O'Regan.

[10] R. P. Agarwal, M. Bohner, and A. C. Peterson, *Inequalities on time scales: a survey*, Math. Inequal. Appl. **4** (2001), no. 4, 535–557.

[11] R. P. Agarwal, M. Bohner, and P. Řehák, *Half-linear dynamic equations*, Nonlinear Analysis and Applications: to V. Lakshmikantham on His 80th Birthday. Vol. 1, 2, Kluwer Academic Publishers Group, Dordrecht, 2003, pp. 1–57.

[12] R. P. Agarwal, M. Bohner, and P. J. Y. Wong, *Sturm-Liouville eigenvalue problems on time scales*, Appl. Math. Comput. **99** (1999), no. 2-3, 153–166.

[13] R. P. Agarwal and S. R. Grace, *The oscillation of certain difference equations*, Math. Comput. Modelling **30** (1999), no. 1-2, 53–66.

[14] ______, *Oscillation of higher-order nonlinear difference equations of neutral type*, Appl. Math. Lett. **12** (1999), no. 8, 77–83.

[15] ______, *Oscillation theorems for certain neutral functional-differential equations*, Comput. Math. Appl. **38** (1999), no. 11-12, 1–11.

[16] ______, *On the oscillation of certain second order differential equations*, Georgian Math. J. **7** (2000), no. 2, 201–213.

[17] ______, *Oscillation theorems for certain difference equations*, Dynam. Systems Appl. **9** (2000), no. 2, 299–308.

[18] ______, *Oscillation criteria for certain higher order difference equations*, Math. Sci. Res. J. **6** (2002), no. 1, 60–64.

[19] R. P. Agarwal, S. R. Grace, and D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer Academic Publishers Group, Dordrecht, 2000.

[20] ______, *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Kluwer Academic Publishers Group, Dordrecht, 2002.

[21] ______, *Nonoscillatory solutions for discrete equations*, Comput. Math. Appl. **45** (2003), no. 6-9, 1297–1302.

[22] ______, *On the oscillation of certain second order difference equations*, J. Difference Equ. Appl. **9** (2003), no. 1, 109–119, In honour of Professor Allan Peterson on the occasion of his 60th birthday, Part II.

[23] ______, *Oscillation Theory for Second Order Dynamic Equations*, Series in Mathematical Analysis and Applications, vol. 5, Taylor & Francis Ltd., London, 2003.

[24] R. P. Agarwal and D. O'Regan, *Existence principles for continuous and discrete equations on infinite intervals in Banach spaces*, Math. Nachr. **207** (1999), 5–19.

[25] R. P. Agarwal and P. J. Y. Wong, *Advanced Topics in Difference Equations*, Mathematics and Its Applications, vol. 404, Kluwer Academic Publishers Group, Dordrecht, 1997.

[26] C. D. Ahlbrandt, *Discrete variational inequalities*, General Inequalities 6 (Oberwolfach, 1990), Internat. Ser. Numer. Math., vol. 103, Birkhäuser, Basel, 1992, pp. 93–107.

[27] ______, *Continued fraction representations of maximal and minimal solutions of a discrete matrix Riccati equation*, SIAM J. Math. Anal. **24** (1993), no. 6, 1597–1621.

[28] ______, *Equivalence of discrete Euler equations and discrete Hamiltonian systems*, J. Math. Anal. Appl. **180** (1993), no. 2, 498–517.

[29] C. D. Ahlbrandt, M. Bohner, and J. Ridenhour, *Hamiltonian systems on time scales*, J. Math. Anal. Appl. **250** (2000), no. 2, 561–578.

[30] C. D. Ahlbrandt, M. Bohner, and T. Voepel, *Variable change for Sturm-Liouville differential expressions on time scales*, J. Difference Equ. Appl. **9** (2003), no. 1, 93–107, In honour of Professor Allan Peterson on the occasion of his 60th birthday, Part II.

[31] C. D. Ahlbrandt, S. L. Clark, J. W. Hooker, and W. T. Patula, *A discrete interpretation of Reid's roundabout theorem for generalized differential systems*, Comput. Math. Appl. **28** (1994), no. 1-3, 11–21.

[32] C. D. Ahlbrandt and M. Heifetz, *Discrete Riccati equations of filtering and control*, Proceedings of the First International Conference on Difference Equations (San Antonio, Tex, 1994) (S. N. Elaydi, J. R. Graef, G. Ladas, and A. C. Peterson, eds.), Gordon and Breach, Luxembourg, 1995, pp. 1–16.

[33] C. D. Ahlbrandt, M. Heifetz, J. W. Hooker, and W. T. Patula, *Asymptotics of discrete time Riccati equations, robust control, and discrete linear Hamiltonian systems*, Panamer. Math. J. **5** (1995), no. 2, 1–39.

[34] C. D. Ahlbrandt and J. W. Hooker, *Riccati matrix difference equations and disconjugacy of discrete linear systems*, SIAM J. Math. Anal. **19** (1988), no. 5, 1183–1197.

[35] C. D. Ahlbrandt and W. T. Patula, *Recessive solutions of block tridiagonal nonhomogeneous systems*, J. Differ. Equations Appl. **1** (1995), no. 1, 1–15.

[36] C. D. Ahlbrandt and A. C. Peterson, *The (n,n)-disconjugacy of a $2n$th order linear difference equation*, Comput. Math. Appl. **28** (1994), no. 1-3, 1–9.

[37] ______, *Discrete Hamiltonian Systems. Difference Equations, Continued Fractions, and Riccati Equations*, Kluwer Texts in the Mathematical Sciences, vol. 16, Kluwer Academic Publishers Group, Dordrecht, 1996.

[38] E. Akin, M. Bohner, L. H. Erbe, and A. C. Peterson, *Existence of bounded solutions for second order dynamic equations*, J. Difference Equ. Appl. **8** (2002), no. 4, 389–401, In honor of Professor Lynn Erbe.

[39] E. Akin-Bohner and M. Bohner, *Miscellaneous dynamic equations*, Methods Appl. Anal. **10** (2003), no. 1, 11–30.

[40] E. Akin-Bohner and J. Hoffacker, *Oscillation properties of an Emden-Fowler type equation on discrete time scales*, J. Difference Equ. Appl. **9** (2003), no. 6, 603–612.

[41] A. M. Amleh, E. A. Grove, G. Ladas, and D. A. Georgiou, *On the recursive sequence $x_{n+1} = \alpha + x_{n-1}/x_n$*, J. Math. Anal. Appl. **233** (1999), no. 2, 790–798.

[42] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Mathematics in Science and Engineering, vol. 8, Academic Press, New York, 1964.

[43] G. Baur and W. Kratz, *A general oscillation theorem for selfadjoint differential systems with applications to Sturm-Liouville eigenvalue problems and quadratic functionals*, Rend. Circ. Mat. Palermo (2) **38** (1989), no. 3, 329–370.

[44] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1974.

[45] M. Bohner, *Controllability and disconjugacy for linear Hamiltonian difference systems*, Proceedings of the First International Conference on Difference Equations (San Antonio, Tex, 1994) (S. N. Elaydi, J. R. Graef, G. Ladas, and A. C. Peterson, eds.), Gordon and Breach, Luxembourg, 1995, pp. 65–77.

[46] ———, *Zur Positivität diskreter quadratischer Funktionale* [On positivity of discrete quadratic functionals], Ph.D. thesis, Universität Ulm, Ulm, 1995.

[47] ———, *Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions*, J. Math. Anal. Appl. **199** (1996), no. 3, 804–826.

[48] ———, *On disconjugacy for Sturm-Liouville difference equations*, J. Differ. Equations Appl. **2** (1996), no. 2, 227–237.

[49] ———, *An oscillation theorem for a Sturm-Liouville eigenvalue problem*, Math. Nachr. **182** (1996), 67–72.

[50] ———, *Riccati matrix difference equations and linear Hamiltonian difference systems*, Dynam. Contin. Discrete Impuls. Systems **2** (1996), no. 2, 147–159.

[51] ———, *Inhomogeneous discrete variational problems*, Proceedings of the 2nd International Conference on Difference Equations (Veszprém, 1995) (S. N. Elaydi, I. Győri, and G. Ladas, eds.), Gordon and Breach, Amsterdam, 1997, pp. 89–97.

[52] ———, *Positive definiteness of discrete quadratic functionals*, General Inequalities 7 (Oberwolfach, 1995) (C. Bandle, W. N. Everitt, L. Losonczi, and W. Walter, eds.), Internat. Ser. Numer. Math., vol. 123, Birkhäuser, Basel, 1997, pp. 55–60.

[53] ———, *Symplectic systems and related discrete quadratic functionals*, Facta Univ. Ser. Math. Inform. **12** (1997), 143–156.

[54] ———, *Asymptotic behavior of discretized Sturm-Liouville eigenvalue problems*, J. Differ. Equations Appl. **3** (1998), no. 3-4, 289–295.

[55] ———, *Discrete linear Hamiltonian eigenvalue problems*, Comput. Math. Appl. **36** (1998), no. 10-12, 179–192.

[56] ———, *Discrete Sturmian theory*, Math. Inequal. Appl. **1** (1998), no. 3, 375–383.

[57] M. Bohner, S. Clark, and J. Ridenhour, *Lyapunov inequalities for time scales*, J. Inequal. Appl. **7** (2002), no. 1, 61–77.

[58] M. Bohner and O. Došlý, *Disconjugacy and transformations for symplectic systems*, Rocky Mountain J. Math. **27** (1997), no. 3, 707–743.

[59] ———, *Positivity of block tridiagonal matrices*, SIAM J. Matrix Anal. Appl. **20** (1998), no. 1, 182–195.

[60] ———, *Trigonometric transformations of symplectic difference systems*, J. Differential Equations **163** (2000), no. 1, 113–129.

[61] ———, *The discrete Prüfer transformation*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2715–2726.

[62] ———, *Trigonometric systems in oscillation theory of difference equations*, Dynamic Systems and Applications, Vol. 3 (Atlanta, Ga, 1999) (G. S. Ladde, N. G. Medhin, and M. Sambandham, eds.), Dynamic Publishers, Georgia, 2001, pp. 99–104.

[63] ———, *Oscillation of symplectic dynamic systems*, ANZIAM J. **46** (2004), no. 1, 17–32.

[64] M. Bohner, O. Došlý, and R. Hilscher, *Linear Hamiltonian dynamic systems on time scales: Sturmian property of the principal solution*, Nonlinear Anal. **47** (2001), no. 2, 849–860.

[65] M. Bohner, O. Došlý, R. Hilscher, and W. Kratz, *Diagonalization approach to discrete quadratic functionals*, Arch. Inequal. Appl. **1** (2003), no. 2, 261–274.

[66] M. Bohner, O. Došlý, and W. Kratz, *Inequalities and asymptotics for Riccati matrix difference operators*, J. Math. Anal. Appl. **221** (1998), no. 1, 262–286.

[67] ———, *A Sturmian theorem for recessive solutions of linear Hamiltonian difference systems*, Appl. Math. Lett. **12** (1999), no. 2, 101–106.

[68] ———, *An oscillation theorem for discrete eigenvalue problems*, Rocky Mountain J. Math. **33** (2003), no. 4, 1233–1260.

[69] ———, *Positive semidefiniteness of discrete quadratic functionals*, Proc. Edinb. Math. Soc. (2) **46** (2003), no. 3, 627–636.

[70] M. Bohner and P. W. Eloe, *Higher order dynamic equations on measure chains: Wronskians, disconjugacy, and interpolating families of functions*, J. Math. Anal. Appl. **246** (2000), no. 2, 639–656.

[71] M. Bohner and R. Hilscher, *An eigenvalue problem for linear Hamiltonian systems on time scales*, to appear in Fasc. Math.

[72] M. Bohner and V. Lakshmikantham, *Formulas of Bendixson and Alekseev for difference equations*, Bull. London Math. Soc. **36** (2004), no. 1, 65–71.

[73] M. Bohner and D. A. Lutz, *Asymptotic behavior of dynamic equations on time scales*, J. Differ. Equations Appl. **7** (2001), no. 1, 21–50, Special issue in memory of W. A. Harris Jr.

[74] M. Bohner and A. C. Peterson, *Dynamic Equations on Time Scales. An Introduction with Applications*, Birkhäuser Boston, Massachusetts, 2001.

[75] ———, *First and second order linear dynamic equations on time scales*, J. Differ. Equations Appl. **7** (2001), no. 6, 767–792, On the occasion of the 60th birthday of Calvin D. Ahlbrandt.

[76] ———, *Laplace transform and Z-transform: unification and extension*, Methods Appl. Anal. **9** (2002), no. 1, 151–157.

[77] M. Bohner and A. C. Peterson (eds.), *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Massachusetts, 2003.

[78] M. Bohner and S. H. Saker, *Oscillation of second order nonlinear dynamic equations on time scales*, Rocky Mountain J. Math. **34** (2004), no. 4, 1239–1254.

[79] S. Busenberg and K. Cooke, *Vertically Transmitted Diseases*, Biomathematics, vol. 23, Springer-Verlag, Berlin, 1993.

[80] M. Cecchi, Z. Došlá, and M. Marini, *Positive decreasing solutions of quasi-linear difference equations*, Comput. Math. Appl. **42** (2001), no. 10-11, 1401–1410.

[81] ———, *Unbounded solutions of quasi-linear difference equations*, Comput. Math. Appl. **45** (2003), no. 6-9, 1113–1123.

[82] S. Z. Chen, *Disconjugacy, disfocality, and oscillation of second order difference equations*, J. Differential Equations **107** (1994), no. 2, 383–394.

[83] S. Z. Chen and L. H. Erbe, *Oscillation and nonoscillation for systems of selfadjoint second-order difference equations*, SIAM J. Math. Anal. **20** (1989), no. 4, 939–949.

[84] ———, *Riccati techniques and discrete oscillations*, J. Math. Anal. Appl. **142** (1989), no. 2, 468–487.

[85] D. Cheng and J. Yan, *Oscillation and global attractivity in a nonlinear delay difference equation*, Math. Slovaca **50** (2000), no. 3, 335–344.

[86] S. S. Cheng, *Monotone solutions of* $\Delta^2 x(k) = Q(k)x(k+1)$, Chinese J. Math. **10** (1982), no. 1, 71–75.

[87] ———, *Sturmian comparison theorems for three-term recurrence equations*, J. Math. Anal. Appl. **111** (1985), no. 2, 465–474.

[88] ———, *Discrete quadratic Wirtinger's inequalities*, Linear Algebra Appl. **85** (1987), 57–73.

[89] ———, *Hille-Wintner type comparison theorems for nonlinear difference equations*, Funkcial. Ekvac. **37** (1994), no. 3, 531–535.

[90] S. S. Cheng and H. J. Li, *Oscillatory behavior of a class of nonlinear three term recurrence equations*, Funkcial. Ekvac. **31** (1988), no. 1, 75–87.

[91] S. S. Cheng, H. J. Li, and W. T. Patula, *Bounded and zero convergent solutions of second-order difference equations*, J. Math. Anal. Appl. **141** (1989), no. 2, 463–483.

[92] S. S. Cheng and R. F. Lu, *A generalization of the discrete Hardy's inequality*, Tamkang J. Math. **24** (1993), no. 4, 469–475.

[93] S. S. Cheng and W. T. Patula, *An existence theorem for a nonlinear difference equation*, Nonlinear Anal. **20** (1993), no. 3, 193–203.

[94] S. S. Cheng, T. C. Yan, and H. J. Li, *Oscillation criteria for second order difference equation*, Funkcial. Ekvac. **34** (1991), no. 2, 223–239.

[95] S. S. Cheng and B. G. Zhang, *Monotone solutions of a class of nonlinear difference equations*, Comput. Math. Appl. **28** (1994), no. 1-3, 71–79.

[96] K. L. Cooke and J. Wiener, *Retarded differential equations with piecewise constant delays*, J. Math. Anal. Appl. **99** (1984), no. 1, 265–297.

[97] ———, *An equation alternately of retarded and advanced type*, Proc. Amer. Math. Soc. **99** (1987), no. 4, 726–732.

[98] ———, *A survey of differential equations with piecewise continuous arguments*, Delay Differential Equations and Dynamical Systems (Claremont, Calif, 1990), Lecture Notes in Math., vol. 1475, Springer-Verlag, Berlin, 1991, pp. 1–15.

[99] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, vol. 220, Springer-Verlag, Berlin, 1971.

[100] J. Deng, *A note on oscillation of second-order nonlinear difference equation with continuous variable*, J. Math. Anal. Appl. **280** (2003), no. 1, 188–194.

[101] R. DeVault and L. Galminas, *Global stability of* $x_{n+1} = A/x_n^p + 1/x_{n-1}^{1/p}$, J. Math. Anal. Appl. **231** (1999), no. 2, 459–466.

[102] R. DeVault, E. A. Grove, G. Ladas, R. Levins, and C. Puccia, *Oscillation and stability in a delay model of a perennial grass*, J. Differ. Equations Appl. **1** (1995), no. 2, 173–185.

[103] R. DeVault, G. Ladas, and S. W. Schultz, *Necessary and sufficient conditions for the boundedness of* $x_{n+1} = A/x_n^p + B/x_{n-1}^q$, J. Differ. Equations Appl. **3** (1998), no. 3-4, 259–266.

[104] ———, *On the recursive sequence* $x_{n+1} = A/x_n + 1/x_{n-2}$, Proc. Amer. Math. Soc. **126** (1998), no. 11, 3257–3261.

[105] O. Došlý, *Conjugacy criteria for second order differential equations*, Rocky Mountain J. Math. **23** (1993), no. 3, 849–861.

[106] ———, *Transformations of linear Hamiltonian difference systems and some of their applications*, J. Math. Anal. Appl. **191** (1995), no. 2, 250–265.

[107] ———, *Oscillation criteria for higher order Sturm-Liouville difference equations*, J. Differ. Equations Appl. **4** (1998), no. 5, 425–450.

[108] ———, *Methods of oscillation theory of half-linear second order differential equations*, Czechoslovak Math. J. **50(125)** (2000), no. 3, 657–671.

[109] ———, *A remark on the half-linear extension of the Hartman-Wintner theorem*, Proceedings of the Fourth Mississippi State Conference on Difference Equations and Computational Simulations (Miss, 1999), Electron. J. Differ. Equ., conf. 03, Southwest Texas State University, Texas, 2000, pp. 29–37.

[110] O. Došlý, J. R. Graef, and J. Jaroš, *Forced oscillation of second order linear and half-linear difference equations*, Proc. Amer. Math. Soc. **131** (2003), no. 9, 2859–2867.

[111] O. Došlý and P. Řehák, *Conjugacy criteria for second-order linear difference equations*, Arch. Math. (Brno) **34** (1998), no. 2, 301–310.

[112] ———, *Nonoscillation criteria for half-linear second-order difference equations*, Comput. Math. Appl. **42** (2001), no. 3-5, 453–464.

[113] J. Dugundji and A. Granas, *Fixed Point Theory. I*, Monografie Matematyczne, vol. 61, Państwowe Wydawnictwo Naukowe (PWN), Warsaw, 1982.

[114] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure and Applied Mathematics, vol. 7, Interscience Publishers, New York, 1958, with the assistance of W. G. Bade and R. G. Bartle.

[115] O. Dunkel, *The alternation of nodes of linearly independent solutions of second order difference equations*, Bull. Amer. Math. Soc. **32** (1926), 333–334.

[116] Á. Elbert, *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations*, Ordinary and Partial Differential Equations (Dundee, 1982), Lecture Notes in Math., vol. 964, Springer-Verlag, Berlin, 1982, pp. 187–212.

[117] P. W. Eloe, *Criteria for right disfocality of linear difference equations*, J. Math. Anal. Appl. **120** (1986), no. 2, 610–621.

[118] P. W. Eloe and J. Henderson, *Analogues of Fekete and Descartes systems of solutions for difference equations*, J. Approx. Theory **59** (1989), no. 1, 38–52.

[119] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 190, Marcel Dekker, New York, 1995.

[120] L. H. Erbe and P. X. Yan, *Disconjugacy for linear Hamiltonian difference systems*, J. Math. Anal. Appl. **167** (1992), no. 2, 355–367.

[121] ———, *Qualitative properties of Hamiltonian difference systems*, J. Math. Anal. Appl. **171** (1992), no. 2, 334–345.

[122] ———, *Weighted averaging techniques in oscillation theory for second order difference equations*, Canad. Math. Bull. **35** (1992), no. 1, 61–69.

[123] ———, *Oscillation criteria for Hamiltonian matrix difference systems*, Proc. Amer. Math. Soc. **119** (1993), no. 2, 525–533.

[124] ———, *On the discrete Riccati equation and its applications to discrete Hamiltonian systems*, Rocky Mountain J. Math. **25** (1995), no. 1, 167–178.

[125] L. H. Erbe and B. G. Zhang, *Oscillation of second order linear difference equations*, Chinese J. Math. **16** (1988), no. 4, 239–252.

[126] T. Fort, *Finite Differences and Difference Equations in the Real Domain*, Oxford University Press, London, 1948.

[127] W. Gautschi, *Computational aspects of three-term recurrence relations*, SIAM Rev. **9** (1967), 24–82.

[128] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, New Jersey, 1963, revised English edition translated and edited by Richard A. Silverman.

[129] A. Gleska and J. Werbowski, *Comparison theorems for the asymptotic behavior of solutions of nonlinear difference equations*, J. Math. Anal. Appl. **226** (1998), no. 2, 456–465.

[130] W. Golda and J. Werbowski, *Oscillation of linear functional equations of the second order*, Funkcial. Ekvac. **37** (1994), no. 2, 221–227.

[131] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Mathematics and Its Applications, vol. 74, Kluwer Academic Publishers Group, Dordrecht, 1992.

[132] K. Gopalsamy, I. Győri, and G. Ladas, *Oscillations of a class of delay equations with continuous and piecewise constant arguments*, Funkcial. Ekvac. **32** (1989), no. 3, 395–406.

[133] K. Gopalsamy, M. R. S. Kulenović, and G. Ladas, *On a logistic equation with piecewise constant arguments*, Differential Integral Equations **4** (1991), no. 1, 215–223.

[134] S. R. Grace, *Oscillation criteria for nth order neutral functional-differential equations*, J. Math. Anal. Appl. **184** (1994), no. 1, 44–55.

[135] ———, *On the oscillations of mixed neutral equations*, J. Math. Anal. Appl. **194** (1995), no. 2, 377–388.

[136] ———, *Oscillations of mixed neutral functional-differential equations*, Appl. Math. Comput. **68** (1995), no. 1, 1–13.

[137] ———, *Oscillation of certain neutral difference equations of mixed type*, J. Math. Anal. Appl. **224** (1998), no. 2, 241–254.

[138] ———, *Oscillation of certain difference equations*, Czechoslovak Math. J. **50(125)** (2000), no. 2, 347–358.

[139] ———, *Oscillatory behaviour of certain difference equations*, Math. Slovaca **50** (2000), no. 3, 345–355.

[140] S. R. Grace, A. A. Abadeer, and H. A. El-Morshedy, *On the oscillation of certain second order difference equations*, Commun. Appl. Anal. **2** (1998), no. 4, 447–455.

[141] ———, *Summation averaging technique for the oscillation of second order linear difference equations*, Publ. Math. Debrecen **55** (1999), no. 3-4, 333–347.

[142] S. R. Grace and H. A. El-Morshedy, *Oscillation and nonoscillation theorems for certain second-order difference equations with forcing term*, J. Math. Anal. Appl. **216** (1997), no. 2, 614–625.

[143] ———, *Oscillation criteria for certain second order nonlinear difference equations*, Bull. Austral. Math. Soc. **60** (1999), no. 1, 95–108.

[144] ———, *Oscillation criteria of comparison type for second order difference equations*, J. Appl. Anal. **6** (2000), no. 1, 87–103.

[145] S. R. Grace and G. G. Hamedani, *On the oscillation of certain neutral difference equations*, Math. Bohem. **125** (2000), no. 3, 307–321.

[146] S. R. Grace and B. S. Lalli, *Oscillation theorems for forced neutral difference equations*, J. Math. Anal. Appl. **187** (1994), no. 1, 91–106.

[147] ———, *Oscillation theorems for second order delay and neutral difference equations*, Utilitas Math. **45** (1994), 197–211.

[148] E. A. Grove, I. Győri, and G. Ladas, *On the characteristic equation for equations with continuous and piecewise constant arguments*, Rad. Mat. **5** (1989), no. 2, 271–281.

[149] I. Győri and G. Ladas, *Linearized oscillations for equations with piecewise constant arguments*, Differential Integral Equations **2** (1989), no. 2, 123–131.

[150] ———, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991.

[151] A. Halanay and V. Ionescu, *Properties of some global solutions to the discrete-time Riccati equation associated to a contracting input-output operator*, J. Differ. Equations Appl. **1** (1995), no. 1, 61–71.

[152] D. Hankerson, *Right and left disconjugacy in difference equations*, Rocky Mountain J. Math. **20** (1990), no. 4, 987–995.

[153] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1959.

[154] P. Hartman, *On non-oscillatory linear differential equations of second order*, Amer. J. Math. **74** (1952), 389–400.

[155] ———, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[156] ———, *Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity*, Trans. Amer. Math. Soc. **246** (1978), 1–30.

[157] P. Hartman and A. Wintner, *On linear difference equations of second order*, Amer. J. Math. **72** (1950), 124–128.

[158] X. Z. He, *Oscillatory property of solutions of a second order nonlinear difference equation*, Ann. Differential Equations **7** (1991), no. 4, 415–422.

[159] E. Hille, *Non-oscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), 234–252.

[160] D. B. Hinton and R. T. Lewis, *Spectral analysis of second order difference equations*, J. Math. Anal. Appl. **63** (1978), no. 2, 421–438.

[161] J. W. Hooker, M. K. Kwong, and W. T. Patula, *Oscillatory second order linear difference equations and Riccati equations*, SIAM J. Math. Anal. **18** (1987), no. 1, 54–63.

[162] J. W. Hooker and W. T. Patula, *Riccati type transformations for second-order linear difference equations*, J. Math. Anal. Appl. **82** (1981), no. 2, 451–462.

[163] ———, *A second-order nonlinear difference equation: oscillation and asymptotic behavior*, J. Math. Anal. Appl. **91** (1983), no. 1, 9–29.

[164] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[165] V. Ionescu and M. Weiss, *Two-Riccati formulae for the discrete-time H^∞-control problem*, Internat. J. Control **57** (1993), no. 1, 141–195.

[166] J. H. Jaroma, V. L. Kocić, and G. Ladas, *Global asymptotic stability of a second order difference equation*, preprint, 2003.

[167] J. H. Jaroma, S. A. Kuruklis, and G. Ladas, *Oscillations and stability of a discrete delay logistic model*, Ukrain. Mat. Zh. **43** (1991), no. 6, 734–744.

[168] J. Jaroš and T. Kusano, *A Picone type identity for second order half-linear differential equations*, Acta Math. Univ. Comenian. (N.S.) **68** (1999), no. 1, 137–151.

[169] N. Kandelaki, A. Lomtatidze, and D. Ugulava, *On oscillation and nonoscillation of a second order half-linear equation*, Georgian Math. J. **7** (2000), no. 2, 329–346.

[170] A. G. Kartsatos, *On the nth-order differential inequalities*, J. Math. Anal. Appl. **52** (1975), no. 1, 1–9.

[171] A. G. Kartsatos and M. N. Manougian, *Perturbations causing oscillations of functional-differential equations*, Proc. Amer. Math. Soc. **43** (1974), 111–117.

[172] A. G. Kartsatos and J. Toro, *Comparison and oscillation theorems for equations with middle terms of order $n-1$*, J. Math. Anal. Appl. **66** (1978), no. 2, 297–312.

[173] W. G. Kelley and A. C. Peterson, *Difference Equations. An Introduction with Applications*, 2nd ed., Harcourt/Academic Press, California, 2001.
[174] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Mathematics and Its Applications (Soviet Series), vol. 89, Kluwer Academic Publishers Group, Dordrecht, 1993, translated from the 1985 Russian original.
[175] V. L. Kocić and G. Ladas, *Linearized oscillations for difference equations*, Hiroshima Math. J. **22** (1992), no. 1, 95–102.
[176] ———, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Mathematics and Its Applications, vol. 256, Kluwer Academic Publishers Group, Dordrecht, 1993.
[177] ———, *Global attractivity in a nonlinear second-order difference equation*, Comm. Pure Appl. Math. **48** (1995), no. 9-10, 1115–1122, A special double issue dedicated to Jack Schwartz.
[178] V. L. Kocić, G. Ladas, and I. W. Rodrigues, *On rational recursive sequences*, J. Math. Anal. Appl. **173** (1993), no. 1, 127–157.
[179] I.-G. E. Kordonis and Ch. G. Philos, *Oscillation of neutral difference equations with periodic coefficients*, Comput. Math. Appl. **33** (1997), no. 7, 11–27.
[180] I.-G. E. Kordonis, Ch. G. Philos, and I. K. Purnaras, *On the oscillation of some linear difference equations with periodic coefficients*, J. Comput. Appl. Math. **84** (1997), no. 2, 219–241.
[181] W. A. Kosmala, *Oscillation and asymptotic behaviour of nonlinear equations with middle terms of order $N-1$ and forcings*, Nonlinear Anal. **6** (1982), no. 10, 1115–1133.
[182] W. Kratz, *Quadratic Functionals in Variational Analysis and Control Theory*, Mathematical Topics, vol. 6, Akademie Verlag, Berlin, 1995.
[183] W. Kratz and A. Peyerimhoff, *A treatment of Sturm-Liouville eigenvalue problems via Picone's identity*, Analysis **5** (1985), no. 1-2, 97–152.
[184] M. R. S. Kulenović and M. Budinčević, *Asymptotic analysis of nonlinear second order difference equation*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **30** (1984), no. 3, 39–52.
[185] M. R. S. Kulenović and S. Hadžiomerspahić, *Existence of nonoscillatory solution of second order linear neutral delay equation*, J. Math. Anal. Appl. **228** (1998), no. 2, 436–448.
[186] M. R. S. Kulenović, G. Ladas, and N. R. Prokup, *On the recursive sequence* $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(A + x_n)$, J. Differ. Equations Appl. **6** (2000), no. 5, 563–576.
[187] M. R. S. Kulenović, G. Ladas, and W. S. Sizer, *On the recursive sequence* $x_{n+1} = (\alpha x_n + \beta x_{n-1})/(\gamma x_n + \delta x_{n-1})$, Math. Sci. Res. Hot-Line **2** (1998), no. 5, 1–16.
[188] T. Kusano, Y. Naito, and A. Ogata, *Strong oscillation and nonoscillation of quasilinear differential equations of second order*, Differential Equations Dynam. Systems **2** (1994), no. 1, 1–10.
[189] M. K. Kwong, J. W. Hooker, and W. T. Patula, *Riccati type transformations for second-order linear difference equations. II*, J. Math. Anal. Appl. **107** (1985), no. 1, 182–196.
[190] G. Ladas, *Oscillations of difference equations with positive and negative coefficients*, Rocky Mountain J. Math. **20** (1990), no. 4, 1051–1061.
[191] G. Ladas, E. C. Partheniadis, and J. Schinas, *Oscillation and stability of second order differential equations with piecewise constant argument*, Rad. Mat. **5** (1989), no. 1, 171–178.
[192] G. Ladas and C. Qian, *Comparison results and linearized oscillations for higher-order difference equations*, Int. J. Math. Math. Sci. **15** (1992), no. 1, 129–142.
[193] ———, *Linearized oscillations for nonautonomous delay difference equations*, Oscillation and Dynamics in Delay Equations (San Francisco, Calif, 1991), Contemp. Math., vol. 129, American Mathematical Society, Rhode Island, 1992, pp. 115–125.
[194] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations. Numerical Methods and Applications*, Mathematics in Science and Engineering, vol. 181, Academic Press, Massachusetts, 1988.
[195] B. S. Lalli and S. R. Grace, *Oscillation theorems for second order neutral difference equations*, Appl. Math. Comput. **62** (1994), no. 1, 47–60.
[196] H. J. Li and S. S. Cheng, *An oscillation theorem for a second order nonlinear difference equation*, Utilitas Math. **44** (1993), 177–181.
[197] H. J. Li and C. C. Yeh, *Existence of positive nondecreasing solutions of nonlinear difference equations*, Nonlinear Anal. **22** (1994), no. 10, 1271–1284.

[198] W.-T. Li, *Oscillation theorems for second-order nonlinear difference equations*, Math. Comput. Modelling **31** (2000), no. 6-7, 71–79.

[199] B. Liu and S. S. Cheng, *Positive solutions of second order nonlinear difference equations*, J. Math. Anal. Appl. **204** (1996), no. 2, 482–493.

[200] W. E. Mahfoud, *Oscillation and asymptotic behavior of solutions of Nth order nonlinear delay differential equations*, J. Differential Equations **24** (1977), no. 1, 75–98.

[201] ———, *Remarks on some oscillation theorems for nth-order differential equations with a retarded argument*, J. Math. Anal. Appl. **62** (1978), no. 1, 68–80.

[202] ———, *Comparison theorems for delay differential equations*, Pacific J. Math. **83** (1979), no. 1, 187–197.

[203] M. Marini and P. Zezza, *On the asymptotic behavior of the solutions of a class of second-order linear differential equations*, J. Differential Equations **28** (1978), no. 1, 1–17.

[204] P. J. McCarthy, *Note on the oscillation of solutions of second order linear difference equations*, Portugal. Math. **18** (1959), 203–205.

[205] A. B. Mingarelli, *Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions*, Lecture Notes in Mathematics, vol. 989, Springer-Verlag, Berlin, 1983.

[206] M. Morse, *Variational Analysis: Critical Extremals and Sturmian Extensions*, Pure and Applied Mathematics, John Wiley & Sons, New York, 1973.

[207] Z. Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc. **85** (1957), 428–445.

[208] W. Nowakowska and J. Werbowski, *Oscillation of linear functional equations of higher order*, Arch. Math. (Brno) **31** (1995), no. 4, 251–258.

[209] ———, *Oscillatory behavior of solutions of functional equations*, Nonlinear Anal. Ser. A: Theory Methods **44** (2001), no. 6, 767–775.

[210] F. W. J. Olver, *Bounds for the solutions of second-order linear difference equations*, J. Res. Nat. Bur. Standards Sect. B **71B** (1967), 161–166.

[211] ———, *Numerical solution of second-order linear difference equations*, J. Res. Nat. Bur. Standards Sect. B **71B** (1967), 111–129.

[212] F. W. J. Olver and D. J. Sookne, *Note on backward recurrence algorithms*, Math. Comp. **26** (1972), 941–947.

[213] G. Papaschinopoulos and J. Schinas, *Some results concerning second and third order neutral delay differential equations with piecewise constant argument*, Czechoslovak Math. J. **44(119)** (1994), no. 3, 501–512.

[214] T. Pappas, A. J. Laub, and N. R. Sandell Jr., *On the numerical solution of the discrete-time algebraic Riccati equation*, IEEE Trans. Automat. Control **25** (1980), no. 4, 631–641.

[215] E. C. Partheniadis, *Stability and oscillation of neutral delay differential equations with piecewise constant argument*, Differential Integral Equations **1** (1988), no. 4, 459–472.

[216] W. T. Patula, *Growth and oscillation properties of second order linear difference equations*, SIAM J. Math. Anal. **10** (1979), no. 1, 55–61.

[217] ———, *Growth, oscillation and comparison theorems for second-order linear difference equations*, SIAM J. Math. Anal. **10** (1979), no. 6, 1272–1279.

[218] T. Peil and A. C. Peterson, *Criteria for C-disfocality of a selfadjoint vector difference equation*, J. Math. Anal. Appl. **179** (1993), no. 2, 512–524.

[219] ———, *Asymptotic behavior of solutions of a two term difference equation*, Rocky Mountain J. Math. **24** (1994), no. 1, 233–252.

[220] M. Peng, W. Ge, and Q. Xu, *New criteria for the oscillation and existence of monotone solutions of second-order nonlinear difference equations*, Appl. Math. Comput. **114** (2000), no. 1, 103–114.

[221] A. C. Peterson, *Boundary value problems for an nth order linear difference equation*, SIAM J. Math. Anal. **15** (1984), no. 1, 124–132.

[222] ———, *C-disfocality for linear Hamiltonian difference systems*, J. Differential Equations **110** (1994), no. 1, 53–66.

[223] A. C. Peterson and J. Ridenhour, *Atkinson's superlinear oscillation theorem for matrix difference equations*, SIAM J. Math. Anal. **22** (1991), no. 3, 774–784.

[224] ______, *Oscillation of second order linear matrix difference equations*, J. Differential Equations **89** (1991), no. 1, 69–88.

[225] ______, *A disconjugacy criterion of W. T. Reid for difference equations*, Proc. Amer. Math. Soc. **114** (1992), no. 2, 459–468.

[226] ______, *The* $(2,2)$*-disconjugacy of a fourth order difference equation*, J. Differ. Equations Appl. **1** (1995), no. 1, 87–93.

[227] ______, *A disfocality criterion for an nth order difference equation*, Proceedings of the First International Conference on Difference Equations (San Antonio, Tex, 1994) (S. N. Elaydi, J. R. Graef, G. Ladas, and A. C. Peterson, eds.), Gordon and Breach, Luxembourg, 1995, pp. 411–418.

[228] Ch. G. Philos, *On oscillations of some difference equations*, Funkcial. Ekvac. **34** (1991), no. 1, 157–172.

[229] ______, *Oscillations of some delay differential equations with periodic coefficients*, J. Math. Anal. Appl. **162** (1991), no. 2, 452–475.

[230] ______, *Oscillations in a class of difference equations*, Appl. Math. Comput. **48** (1992), no. 1, 45–57.

[231] ______, *Oscillations in difference equations with periodic coefficients*, Arch. Math. (Basel) **58** (1992), no. 5, 453–461.

[232] Ch. G. Philos and I. K. Purnaras, *Oscillations in linear difference equations with variable coefficients*, J. Appl. Math. Stochastic Anal. **4** (1991), no. 3, 241–257.

[233] ______, *Oscillations in a class of linear difference equations with periodic coefficients*, Appl. Anal. **53** (1994), no. 3-4, 175–183.

[234] Ch. G. Philos and Y. G. Sficas, *Positive solutions of difference equations*, Proc. Amer. Math. Soc. **108** (1990), no. 1, 107–115.

[235] M. Picone, *Sulle autosoluzioni e sulle formule di maggiorazione per gli integrali delle equazioni differenziali lineari ordinarie autoaggiunte*, Math. Z. **28** (1928), 519–555 (Italian).

[236] J. Popenda, *Oscillation and nonoscillation theorems for second-order difference equations*, J. Math. Anal. Appl. **123** (1987), no. 1, 34–38.

[237] J. Popenda and J. Werbowski, *On the asymptotic behaviour of the solutions of difference equations of second order*, Comment. Math. Prace Mat. **22** (1980), no. 1, 135–142.

[238] B. Przeradzki, *The existence of bounded solutions for differential equations in Hilbert spaces*, Ann. Polon. Math. **56** (1992), no. 2, 103–121.

[239] C. X. Qian and J. R. Yan, *Existence and comparison results of positive solutions of difference equations*, Funkcial. Ekvac. **35** (1992), no. 3, 581–594.

[240] T. T. Read, *Bounds and quantitative comparison theorems for nonoscillatory second order differential equations*, Pacific J. Math. **63** (1976), no. 1, 231–245.

[241] P. Řehák, *Half-linear discrete oscillation theory*, Ph.D. thesis, Masaryk University, Brno, 2000.

[242] P. Řehák, *Half-linear discrete oscillation theory*, Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations (Szeged, 1999), Electron. J. Qual. Theory Differ. Equ., no. 24, University of Szeged, Szeged, 2000, p. 14.

[243] ______, *Hartman-Wintner type lemma, oscillation, and conjugacy criteria for half-linear difference equations*, J. Math. Anal. Appl. **252** (2000), no. 2, 813–827.

[244] ______, *Generalized discrete Riccati equation and oscillation of half-linear difference equations*, Math. Comput. Modelling **34** (2001), no. 3-4, 257–269.

[245] ______, *Oscillation and nonoscillation criteria for second order linear difference equations*, Fasc. Math. (2001), no. 31, 71–89.

[246] ______, *Oscillation criteria for second order half-linear difference equations*, J. Differ. Equations Appl. **7** (2001), no. 4, 483–505.

[247] ______, *Oscillatory properties of second order half-linear difference equations*, Czechoslovak Math. J. **51(126)** (2001), no. 2, 303–321.

[248] ______, *Comparison theorems and strong oscillation in the half-linear discrete oscillation theory*, Rocky Mountain J. Math. **33** (2003), no. 1, 333–352.

[249] W. T. Reid, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1971.

[250] ______, *Riccati Differential Equations*, Mathematics in Science and Engineering, vol. 86, Academic Press, New York, 1972.

[251] ———, *Sturmian Theory for Ordinary Differential Equations*, Applied Mathematical Sciences, vol. 31, Springer-Verlag, New York, 1980.

[252] I. W. Rodrigues, *Oscillation and attractivity in a differential equation with piecewise constant arguments*, Rocky Mountain J. Math. **24** (1994), no. 1, 261–271.

[253] ———, *Systems of differential equations of alternately retarded and advanced type*, J. Math. Anal. Appl. **209** (1997), no. 1, 180–190.

[254] D. F. Saint Mary, *Riccati integral equations and nonoscillation of selfadjoint linear systems*, J. Math. Anal. Appl. **121** (1987), no. 1, 109–118.

[255] S. M. Shah and J. Wiener, *Advanced differential equations with piecewise constant argument deviations*, Int. J. Math. Math. Sci. **6** (1983), no. 4, 671–703.

[256] J. H. Shen and I. P. Stavroulakis, *Oscillatory and nonoscillatory delay equations with piecewise constant argument*, J. Math. Anal. Appl. **248** (2000), no. 2, 385–401.

[257] M. Slemrod and E. F. Infante, *Asymptotic stability criteria for linear systems of difference-differential equations of neutral type and their discrete analogues*, J. Math. Anal. Appl. **38** (1972), 399–415.

[258] C. Sturm, *Mémoire sur les équations différentielles linéaires du second ordre*, Journal de Liouville **1** (1836), 106–186 (French).

[259] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Mathematics in Science and Engineering, vol. 48, Academic Press, New York, 1968.

[260] Z. Szafrański and B. Szmanda, *Oscillation theorems for some nonlinear difference equations*, Appl. Math. Comput. **83** (1997), no. 1, 43–52.

[261] B. Szmanda, *Oscillation criteria for second order nonlinear difference equations*, Ann. Polon. Math. **43** (1983), no. 3, 225–235.

[262] ———, *Nonoscillation, oscillation and growth of solutions of nonlinear difference equations of second order*, J. Math. Anal. Appl. **109** (1985), no. 1, 22–30.

[263] ———, *Characterization of oscillation of second order nonlinear differences equations*, Bull. Polish Acad. Sci. Math. **34** (1986), no. 3-4, 133–141.

[264] E. Thandapani, *Oscillation theorems for perturbed nonlinear second order difference equations*, Comput. Math. Appl. **28** (1994), no. 1-3, 309–316.

[265] ———, *Note on: "Oscillation theorems for certain second order nonlinear difference equations"*, J. Math. Anal. Appl. **224** (1998), no. 2, 349–355.

[266] E. Thandapani, J. R. Graef, and P. W. Spikes, *Monotonicity and summability of solutions of a second order nonlinear difference equation*, Bull. Inst. Math. Acad. Sinica **23** (1995), no. 4, 343–356.

[267] E. Thandapani, I. Győri, and B. S. Lalli, *An application of discrete inequality to second order nonlinear oscillation*, J. Math. Anal. Appl. **186** (1994), no. 1, 200–208.

[268] E. Thandapani, M. M. S. Manuel, J. R. Graef, and P. W. Spikes, *Monotone properties of certain classes of solutions of second-order difference equations*, Comput. Math. Appl. **36** (1998), no. 10-12, 291–297.

[269] E. Thandapani and S. Pandian, *Oscillation theorems for nonlinear second order difference equations with a nonlinear damping term*, Tamkang J. Math. **26** (1995), no. 1, 49–58.

[270] E. Thandapani and K. Ravi, *Bounded and monotone properties of solutions of second-order quasilinear forced difference equations*, Comput. Math. Appl. **38** (1999), no. 2, 113–121.

[271] E. Thandapani, K. Ravi, and J. R. Graef, *Oscillation and comparison theorems for half-linear second-order difference equations*, Comput. Math. Appl. **42** (2001), no. 6-7, 953–960.

[272] E. A. van Doorn, *A note on orthogonal polynomials and oscillation criteria for second order linear difference equations*, J. Math. Anal. Appl. **120** (1986), no. 1, 354–359.

[273] Z. Wang and J. S. Yu, *Oscillation criteria for second order nonlinear difference equations*, Funkcial. Ekvac. **34** (1991), no. 2, 313–319.

[274] J. Wiener, *Generalized Solutions of Functional-Differential Equations*, World Scientific Publishing, New Jersey, 1993.

[275] ———, *A second-order delay differential equation with multiple periodic solutions*, J. Math. Anal. Appl. **229** (1999), no. 2, 659–676.

[276] J. Wiener and K. L. Cooke, *Oscillations in systems of differential equations with piecewise constant argument*, J. Math. Anal. Appl. **137** (1989), no. 1, 221–239.

[277] J. Wiener and L. Debnath, *A survey of partial differential equations with piecewise continuous arguments*, Int. J. Math. Math. Sci. **18** (1995), no. 2, 209–228.

[278] ______, *Boundary value problems for the diffusion equation with piecewise continuous time delay*, Int. J. Math. Math. Sci. **20** (1997), no. 1, 187–195.

[279] D. Willett, *Classification of second order linear differential equations with respect to oscillation*, Adv. Math. **3** (1969), 594–623.

[280] J. S. W. Wong, *Oscillation and nonoscillation of solutions of second order linear differential equations with integrable coefficients*, Trans. Amer. Math. Soc. **144** (1969), 197–215.

[281] ______, *Second order nonlinear forced oscillations*, SIAM J. Math. Anal. **19** (1988), no. 3, 667–675.

[282] P. J. Y. Wong and R. P. Agarwal, *Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations*, Math. Comput. Modelling **21** (1995), no. 3, 63–84.

[283] ______, *Oscillation and monotone solutions of second order quasilinear difference equations*, Funkcial. Ekvac. **39** (1996), no. 3, 491–517.

[284] ______, *Oscillation theorems for certain second order nonlinear difference equations*, J. Math. Anal. Appl. **204** (1996), no. 3, 813–829.

[285] ______, *Summation averages and the oscillation of second-order nonlinear difference equations*, Math. Comput. Modelling **24** (1996), no. 9, 21–35.

[286] ______, *Oscillations and nonoscillations of half-linear difference equations generated by deviating arguments*, Comput. Math. Appl. **36** (1998), no. 10-12, 11–26.

[287] J. Yan and F. Zhang, *Oscillation for system of delay difference equations*, J. Math. Anal. Appl. **230** (1999), no. 1, 223–231.

[288] B. Zhang and S. S. Cheng, *Comparison and oscillation theorems for an advanced type difference equation*, Ann. Differential Equations **11** (1995), no. 4, 485–494.

[289] B. G. Zhang, *Oscillation and asymptotic behavior of second order difference equations*, J. Math. Anal. Appl. **173** (1993), no. 1, 58–68.

[290] B. G. Zhang and G. D. Chen, *Oscillation of certain second order nonlinear difference equations*, J. Math. Anal. Appl. **199** (1996), no. 3, 827–841.

[291] Z. Zhang, P. Bi, and J. Chen, *Oscillation of second order nonlinear difference equation with continuous variable*, J. Math. Anal. Appl. **255** (2001), no. 1, 349–357.

[292] Z. Zhang and Q. Li, *Oscillation theorems for second-order advanced functional difference equations*, Comput. Math. Appl. **36** (1998), no. 6, 11–18.

[293] Z. Zhang and J. Zhang, *Oscillation criteria for second-order functional difference equations with "summation small" coefficient*, Comput. Math. Appl. **38** (1999), no. 1, 25–31.

Index

Abel's formula, 131
admissible, 187, 204, 247
advanced difference
  equation, 519, 522
  inequality, 694, 707
almost oscillatory, 600, 603, 604, 606, 607, 609, 611, 613
α-degree functional, 268
alternately retarded and advanced equation, 766, 776
arithmetic-geometric mean inequality, 109, 122, 798
Arzelá-Ascoli theorem, 343
associated solution, 195
asymptotically stable, 769, 775, 856
averaging technique, 87, 102

backward difference operator, 36
Banach's contraction principle, 420, 793
big Hamiltonian system, 197, 210
boundary condition, 188, 204
boundary value problem, 18, 188, 754–756
bounded solution, 180

Casoratian, 5, 22, 131
Cauchy function, 7, 9
characteristic equation, 585, 599, 693, 717, 750, 777, 792, 796, 804, 805, 812, 841, 853, 872, 942
comparison theorem, 17, 69, 162, 343, 347, 541, 543, 581, 595, 643, 932
conditionally oscillatory, 361
conjoined basis, 195
conjugate, 34, 37, 39, 40, 268, 299, 303
continued fraction, 28, 29, 71
contraction mapping, 420
controllability
  index, 220, 223
  matrix, 193, 222
controllable, 220

damped equation, 117, 365, 426, 486, 594, 595, 615, 940
damped spring-mass system, 782
delay difference
  equation, 521
  inequality, 694, 707
delay logistic equation, 863
Descartes rule of signs, 757
difference operator
  backward, 36
  forward, 2
disconjugate, 9, 16, 18–21, 217, 251, 268
discrete quadratic functional, 10, 16, 203
discrete variational problem, 189
disfocal, 31, 238
dominant solution, 23, 136, 141, 142, 144, 149
dual equation, 148, 342
dynamic equation, 264

equation of motion, 188, 204
equilibrium point, 786, 849, 871, 886, 888, 889
Euler
  difference equation, 364
  differential equation, 47
Euler-Lagrange equation, 188, 269
existence of monotone solutions, 74, 491, 764

Fibonacci difference equation, 16
first order difference equation, 438, 875
first variation, 187, 188
focal point, 213, 234
forced equation, 131, 135, 372, 598, 615, 617, 643, 647
forward difference operator, 2

generalized zero, 9, 16, 217, 250, 251, 268
global attractor, 871, 876, 909, 914
globally asymptotically stable, 779, 782, 871, 882, 887, 888, 893, 894, 896
Green's theorem, 6
Gronwall inequality, 173, 174

half-linear
  difference equation, 267, 347, 365
  difference inequality, 367
  differential equation, 266, 321
Hardy inequality, 348
Hartman criterion, 319
Hartman-Wintner criterion, 91

higher order linear difference equation, 223, 244
Hinton-Lewis criterion, 62, 340
Hölder's inequality, 49, 172, 352

inhomogeneous variational problem, 185

Jacobi condition, 251
Jacobian matrix, 885

Knaster-Tarski fixed point theorem, 488
Krasnoselskii fixed point theorem, 648

Lagrange identity, 6
Lebesgue dominated convergence theorem, 496, 672
Leighton-Wintner criterion, 60, 337
Leray-Schauder alternative, 648
Liapunov function, 443, 447
limit circle, 169
limit point, 169, 170
linear Hamiltonian difference system, 191, 195, 245
linearized equation, 635, 840, 842, 848, 862, 863, 866, 872, 873, 906
linearized stability theorem, 873
Liouville's formula, 6, 22
Lipschitz condition, 172
locally asymptotically stable, 871, 884, 885
locally stable, 871
logistic equation, 784, 863

mean value theorem, 39
mixed difference equation, 176
mixed neutral difference equation, 692
Möbius transformation, 872
Moore-Penrose inverse, 257, 261
mutually conjugate, 270, 342

natural boundary conditions, 188
negative feedback condition, 907
negative semicycle, 850
neutral difference equation, 627
Nicholson blowfly, 871
nonoscillation criterion, 14, 47, 51, 59, 96, 119, 137, 167, 285, 288, 289, 292, 296, 299, 413, 541, 559, 561, 655, 656, 670, 763
nonoscillatory, 10, 268, 849
normalized conjoined bases, 195

oscillation constant, 361–363
oscillation criterion, 44–46, 48, 52, 55, 57, 59, 62, 66, 69, 93, 95, 111–113, 306, 308, 314, 315, 321, 332, 362, 373, 378, 380, 382, 390, 391, 396, 403, 404, 420, 424, 454, 455, 462, 463, 465, 472, 474, 524, 526, 528, 529, 531, 554, 616, 628, 630, 632, 682, 689, 695, 699, 702, 703, 713, 717, 719, 728, 729, 745, 823, 834, 918, 919, 921, 922, 928
oscillatory, 10, 268, 849
oscillatory about a point, 849, 850, 853, 855, 857, 859, 867, 869, 900
oscillatory about a sequence, 849

perennial grass, 905
periodic, 755, 759, 765, 766, 858, 872
periodic coefficients, 707
permanent, 871, 878, 913
Picone identity, 15, 208, 211, 273
piecewise constant argument, 727
Pólya factorization, 7, 19
positive definite, 10, 16, 204
positive semicycle, 849
positive semidefinite, 191
prime period, 872, 892
principal solution, 195
Property A, 479, 485
Property B, 479
Property C, 484

recessive solution, 23, 135, 141, 142, 144, 148, 149, 152
reciprocal equation, 148, 342
reciprocity principle, 148, 341, 342, 508
Reid roundabout theorem, 16, 226, 231, 242, 255, 278
relatively compact, 343, 648
retarded difference equation, 584
Riccati equation, 11, 13, 16, 40, 75, 198, 202, 239, 319
right disfocal, 31
Rolle's theorem, 31
Routh-Hurwitz criterion, 872

saddle point, 874
Schauder fixed point theorem, 343, 345, 414, 557, 574
Schur-Cohn criterion, 873
Schwarz inequality, 164, 170, 172, 427
Schwarzian derivative, 875
second mean value theorem, 336
second order linear
  difference equation, 1, 3, 213, 319
  differential equation, 1, 3, 319
second order nonlinear difference equation, 377, 453, 516

second variation, 187, 188
semicycle, 849, 860, 875, 885, 892, 911
separated boundary conditions, 235, 238
sink, 873
source, 871, 874
special normalized conjoined bases, 195
stable manifold theorem in the plane, 874
strictly oscillatory, 849
strictly oscillatory about a point, 849, 856, 860
string, 849
strongly nonoscillatory, 361, 364
strongly oscillatory, 361, 364
strongly sublinear equation, 412, 419
strongly superlinear equation, 412, 416
Sturm comparison theorem, 16, 70, 229, 282
Sturm separation theorem, 16, 227, 282
sublinear equation, 378, 399, 412, 417, 424, 474, 534
superlinear equation, 378, 391, 412, 417, 463
symplectic
  matrix, 192, 196, 200, 227
  system, 227, 263

three-term recurrence, 2, 26, 73, 127
transition matrix, 193
two-cycle, 885, 887, 891

uniformly Cauchy, 343
unstable, 871

variation of constants, 8
variational principle, 40, 336

Wintner criterion, 319
Wirtinger inequality, 295
Wronskian, 5, 22, 131

Young inequality, 312