INTRODUCTION TO THE THEORY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

METHODS AND APPLICATIONS

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Introduction to the Theory of Functional Differential Equations: Methods and Applications

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Preface

The aim of this book is to develop a unified approach to a wide class of equations. Previously, these equations were studied without any connection. We demonstrate how this general theory can be applied to specific classes of functional differential equations.

The equation

$$\dot{x} = Fx, \tag{1}$$

with an operator F defined on a set of absolutely continuous functions, is called the functional differential equation. Thus (1) is a far-reaching generalization of the differential equation

$$\dot{x}(t) = f(t, x(t)). \tag{2}$$

It covers also the integrodifferential equation

$$\dot{x}(t) = \int_a^b K(t, s, x(s))\,ds, \tag{3}$$

the "delay differential equation"

$$\begin{gathered} \dot{x}(t) = f(t, x[h(t)]), \quad t \in [a, b],\ h(t) \le t, \\ x(\xi) = \varphi(\xi) \quad \text{if } \xi < a, \end{gathered} \tag{4}$$

the "equation with distributed deviation of the argument"

$$\dot{x}(t) = f\left(t, \int_a^b x(s)\,d_sR(t, s)\right), \tag{5}$$

and so on.

Some distinctive properties of (2) used in investigations are defined by the specific character of the so-called "local operator." An operator $\Phi : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are functional spaces, is called local (see Shragin [209], Ponosov [176]) if the values of $y(t) = (\Phi x)(t)$ in any neighborhood of $t = t_0$ depend only on the values of $x(t)$ in the same neighborhood.

It is relevant to note that most hypotheses of classical physics assume that the rate $(d/dt)x$ of change of the state x of the process at the time t_0 depends only on the state of the process at the same time. Thus the mathematical description of such a process takes the form (2). The operator d/dt of differentiation as well as the Nemytskii operator

$$(Nx)(t) \stackrel{\text{def}}{=} f(t, x(t)) \tag{6}$$

are the representatives of the class of local operators. The property of being local of

$$(\Phi x)(t) \stackrel{\text{def}}{=} \dot{x}(t) - f(t, x(t)) \tag{7}$$

does not allow using equation (2) in description of processes where there is no way to ignore the past or future states of process. Thus some problems are in need of a generalization of (2), consisting in replacement of the local Nemytskii operator N by a more general F.

Here another principal generalization of (2) should be reminded of, where the finite-dimensional space $\mathbb{R}^n$ of the values of solutions x is replaced by an arbitrary Banach space $\mathbf{B}$. On the base of such generalization, there has arisen recently a new chapter of analysis: "the theory of ordinary differential equations in Banach spaces." This theory considers certain partial differential equations as the equation (2), where the values of $x(t)$ belong to an appropriate Banach space. But, under this generalization, the operator $(\Phi x)(t) = \dot{x}(t) - f(t, x(t))$ still remains to be local. Theory of equation (1) is thoroughly treated by Azbelev et al. in [32]. The following fact is of fundamental importance: the space $\mathbf{D}$ of absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ is isomorphic to the direct product of the space $\mathbf{L}$ of summable functions $z : [a, b] \to \mathbb{R}^n$ and the finite-dimensional space $\mathbb{R}^n$. Recall that the absolutely continuous function x is defined by

$$x(t) = \int_a^t z(s)\,ds + \alpha, \tag{8}$$

where $z \in \mathbf{L}$, $\alpha \in \mathbb{R}^n$. The space $\mathbf{D}$ is Banach under the norm

$$\|x\|_{\mathbf{D}} = \|\dot{x}\|_{\mathbf{L}} + \left\|x(a)\right\|_{\mathbb{R}^n}. \tag{9}$$

In the theory of (1), the specific character of the Lebesgue space $\mathbf{L}$ is used only in connection with the representation of operators in $\mathbf{L}$ and some of their properties. Only the fact that $\mathbf{L}$ is a Banach space is used, and most of the fundamentals of the general theory of (1) keep after replacement of the Lebesgue space $\mathbf{L}$ by an arbitrary Banach space $\mathbf{B}$. Thus there arises a new theory of the equations in the space $\mathbf{D}$ isomorphic to the direct product $\mathbf{B} \times \mathbb{R}^n$ ($\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$). The generalization consists here in replacement of the Lebesgue space $\mathbf{L}$ by an arbitrary Banach space $\mathbf{B}$ and in replacement of local operators by general ones acting from $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$

into **B**. The present book is devoted to the theory of such generalization and to some applications. The central idea of applications of the theory of abstract differential equation lies in the proper choice of the space **D** for each new problem. With the general theory, such a choice permits applying standard schemes and theorems of analysis to the problems which needed previously an individual approach and special constructions. The boundary value problem is the main point of consideration.

This theory was worked out during a quarter of century by a large group of mathematicians united by the so-called Perm Seminar. The results of the members of the seminar were published in journals as well as in the annuals "Boundary Value Problems" and "Functional Differential Equations" issued in 1976–1992 by the Perm Polytechnic Institute.

In the book, only the works closely related to the questions under consideration are cited.

It is assumed that the reader is acquainted with the foundations of functional analysis.

Let us give some remarks on the format. Each chapter is divided into numbered sections, some of which are divided into numbered subsections. Formulas and results, whether they are theorems, propositions, or lemmas, as well as remarks, are numbered consecutively within each chapter. For example, the fourth formula (theorem) of Chapter 2 is labeled (2.4) (Theorem 2.4). Formulas, propositions, and remarks of appendicies are numbered within each section. For example, the third formula of Section A is labeled (A.3).

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1 Linear abstract functional differential equation

1.1. Preliminary knowledge from the theory of linear equations in Banach spaces

The main assertions of the theory of linear abstract functional differential equations are based on the theorems about linear equations in Banach spaces. We give here without proofs certain results of the book of Krein [122] which we will need below. We formulate some of these assertions not in the most general form, but in the form satisfying our aims. The enumeration of the theorems in brackets means that the assertion either coincides with the corresponding result of the book of Krein [122] or is only an extraction from this result.

We will use the following notations.

$\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are Banach spaces; A, B are linear operators; $D(A)$ is a domain of definition of A; $R(A)$ is a range of values of A; and A^* is an operator adjoint to A. The set of solutions of the equation $Ax = 0$ is said to be *a null space* or *a kernel* of A and is denoted by $\ker A$. The dimension of a linear set M is denoted by $\dim M$.

Let A be acting from $\mathbf{X}$ into $\mathbf{Y}$. The equation

$$Ax = y \tag{1.1}$$

(the operator A) is said to be *normally solvable* if the set $R(A)$ is closed; (1.1) is said to be a *Noether equation* if it is a normally solvable one, and, besides, $\dim\ker A < \infty$ and $\dim\ker A^* < \infty$. The number $\operatorname{ind} A = \dim\ker A - \dim\ker A^*$ is said to be the *index of the operator A* (1.1). If A is a Noether operator and $\operatorname{ind} A = 0$, equation (1.1) (the operator A) is said to be a *Fredholm* one. The equation $A^*\varphi = g$ is said to be an *equation adjoint* to (1.1).

Theorem 1.1 (Krein [122, Theorem 3.2]). *An operator A is normally solvable if and only if* (1.1) *is solvable for such and only such right-hand side y which is orthogonal to all solutions of the homogenous adjoint equation* $A^*\varphi = 0$.

Theorem 1.2 (Krein [122, Theorem 16.4]). *The property of being Noether operator is stable in respect to completely continuous perturbations. By such a perturbation, the index of the operator does not change.*

Theorem 1.3 (Krein [122, Theorem 12.2]). *Let A be acting from* $\mathbf{X}$ *into* $\mathbf{Y}$ *and let* $D(B)$ *be dense in* $\mathbf{Y}$. *If A and B are Noether operators, BA is also a Noether one and* $\operatorname{ind}(BA) = \operatorname{ind} A + \operatorname{ind} B$.

Theorem 1.4 (Krein [122, Theorem 15.1]). *Let BA be a Noether operator and let* $D(B) \subset R(A)$. *Then B is a Noether operator.*

Theorem 1.5 (Krein [122, Theorem 2.4 and Lemma 8.1]). *Let A be defined on* $\mathbf{X}$ *and acting into* $\mathbf{Y}$. *A is normally solvable and* $\dim\ker A^* = n$ *if and only if the space* $\mathbf{Y}$ *is representable in the form of the direct sum* $\mathbf{Y} = R(A) \bigoplus M_n$, *where M_n is a finite-dimensional subspace of the dimension n.*

Theorem 1.6 (Krein [122, Theorem 12.2]). *Let* $D(A) \subset \mathbf{X}$, *let M_n be an n-dimensional subspace of* $\mathbf{X}$, *and let* $D(A) \cap M_n = \{0\}$. *If A is a Noether operator, then its linear extension $\widetilde{A}$ on $D(A) \oplus M_n$ is also a Noether operator. Besides* $\operatorname{ind} \widetilde{A} = \operatorname{ind} A + n$.

Theorem 1.7. *Let a Noether operator A be defined on* $\mathbf{X}$ *and acting into* $\mathbf{Y}$, *let* $D(B) = \mathbf{Y}$, *and let* $BA : \mathbf{X} \to \mathbf{Z}$ *be a Noether operator. Then B is also a Noether operator.*

Proof. By Theorem 1.4, we are in need only of the proof of the case $R(A) \neq \mathbf{Y}$. From Theorem 1.4, we obtain also that the restriction $\overline{B}$ of B on $R(A)$ is a Noether operator.

Let $\dim\ker A^* = n$. Then we have from Theorem 1.5 that

$$\mathbf{Y} = R(A) \oplus M_n = D(\overline{B}) \oplus M_n, \tag{1.2}$$

where $\dim M_n = n$.

From Theorem 1.6, we see that B is a Noether operator as a linear extension of $\overline{B}$ on $\mathbf{Y}$. □

A linear operator A acting from a direct product $\mathbf{X}_1 \times \mathbf{X}_2$ into $\mathbf{Y}$ is defined by a pair of operators $A_1 : \mathbf{X}_1 \to \mathbf{Y}$ and $A_2 : \mathbf{X}_2 \to \mathbf{Y}$ such that

$$A\{x_1, x_2\} = A_1x_1 + A_2x_2, \quad x_1 \in \mathbf{X}_1,\ x_2 \in \mathbf{X}_2, \tag{1.3}$$

where $A_1x_1 = A\{x_1, 0\}$ and $A_2x_2 = A\{0, x_2\}$. We will denote such an operator by $A = \{A_1, A_2\}$.

A linear operator A acting from $\mathbf{X}$ into a direct product $\mathbf{Y}_1 \times \mathbf{Y}_2$ is denoted by a pair of operators $A_1 : \mathbf{X} \to \mathbf{Y}_1$ and $A_2 : \mathbf{X} \to \mathbf{Y}_2$ so that $Ax = \{A_1x, A_2x\}$, $x \in \mathbf{X}$. We will denote such an operator by $A = [A_1, A_2]$.

The theory of linear abstract functional differential equation is using some operators defined on a product $\mathbf{B} \times \mathbb{R}^n$ or acting in such a product. We will formulate here certain assertions about such operators, preserving as far as possible the notation from Azbelev et al. [32, 33].

A linear operator acting from a direct product $\mathbf{B} \times \mathbb{R}^n$ of the Banach spaces $\mathbf{B}$ and $\mathbb{R}^n$ into a Banach space $\mathbf{D}$ is defined by a pair of linear operators $\Lambda : \mathbf{B} \to \mathbf{D}$

and $Y : \mathbb{R}^n \to \mathbf{D}$ in such a way that

$$\{\Lambda, Y\}\{z, \beta\} = \Lambda z + Y\beta, \quad z \in \mathbf{B},\ \beta \in \mathbb{R}^n. \tag{1.4}$$

A linear operator acting from a space $\mathbf{D}$ into a direct product $\mathbf{B}\times\mathbb{R}^n$ is defined by a pair of linear operators $\delta : \mathbf{D} \to \mathbf{B}$ and $r : \mathbf{D} \to \mathbb{R}^n$ so that

$$[\delta, r]x = \{\delta x, rx\}, \quad x \in \mathbf{D}. \tag{1.5}$$

If the norm in the space $\mathbf{B}\times\mathbb{R}^n$ is defined by a corresponding way, for instance, by

$$\big|\big|\{z, \beta\}\big|\big|_{\mathbf{B}\times\mathbb{R}^n} = \|z\|_{\mathbf{B}} + |\beta|, \tag{1.6}$$

the space $\mathbf{B} \times \mathbb{R}^n$ will be a Banach one (here and in what follows, $|\cdot|$ denotes a norm in $\mathbb{R}^n$).

If the bounded operator $\{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ is the inverse to the bounded operator $[\delta, r] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^n$, then

$$x = \Lambda\delta x + Yrx, \quad x \in \mathbf{D},\ \delta(\Lambda z + Y\beta) = z, r(\Lambda z + Y\beta) = \beta, \{z, \beta\} \in \mathbf{B} \times \mathbb{R}^n. \tag{1.7}$$

Hence

$$\Lambda\delta + Yr = I, \qquad \delta\Lambda = I, \qquad \delta Y = 0, \qquad r\Lambda = 0, \qquad rY = I. \tag{1.8}$$

We will identify the finite-dimensional operator $Y : \mathbb{R}^n \to \mathbf{D}$ with a vector $(y_1, \ldots, y_n)$, $y_i \in \mathbf{D}$, such that

$$Y\beta = \sum_{i=1}^{n} y_i\beta^i, \quad \beta = \operatorname{col}\{\beta^1, \ldots, \beta^n\}. \tag{1.9}$$

We denote the components of the vector functional r by $r^1, \ldots, r^n$.

If $l = [l^1, \ldots, l^m] : \mathbf{D} \to \mathbb{R}^m$ is a linear vector functional, $X = (x_1, \ldots, x_n)$ is a vector with components $x_i \in \mathbf{D}$, then lX denotes the $m\times n$ matrix, whose columns are the values of the vector functional l on the components of $X : lX = (l^i x_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Consider the form of the operators Λ, Y, δ, r for some actual spaces.

Let $\mathbf{D}$ be the space of absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ and let $\mathbf{L}$ be the space of summable $z : [a, b] \to \mathbb{R}^n$. The isomorphism between the space $\mathbf{D}$ and the product $\mathbf{B} \times \mathbb{R}^n$ may be defined, for instance, by

$$x(t) = \int_a^t z(s)ds + \beta, \quad x \in \mathbf{D},\ \{z, \beta\} \in \mathbf{L} \times \mathbb{R}^n. \tag{1.10}$$

In such a case,

$$(\Lambda z)(t) = \int_a^t z(s)ds, \qquad Y = E, \qquad \delta x = \dot{x}, \qquad rx = x(a), \tag{1.11}$$

where E is the identity $n \times n$ matrix.

In the case of the space $\mathbf{W}^n$ of functions $x : [a,b] \to \mathbb{R}^1$ with absolutely continuous derivative $x^{(n-1)}$, we obtain similarly that

$$\begin{gathered} (\Lambda z)(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)ds, \\ Y = \left(1, t-a, \ldots, \frac{(t-a)^{n-1}}{(n-1)!}\right), \\ \delta x = x^{(n)}, \qquad rx = \{x(a), \dot{x}(a), \ldots, x^{(n-1)}(a)\}, \end{gathered} \tag{1.12}$$

if the isomorphism between $\mathbf{W}^n$ and $\mathbf{L} \times \mathbb{R}^n$ is defined on the base of the representation

$$x(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} x^{(n)}(s)ds + \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} x^{(k)}(a) \tag{1.13}$$

of the element $x \in \mathbf{W}^n$.

Denote by $\mathbf{DS}[a, t_1, \ldots, t_m, b] = \mathbf{DS}(m)$ the space of functions $x : [a,b] \to \mathbb{R}^n$ permitting finite discontinuity at the points $t_1, \ldots, t_m \in (a,b)$ and being absolutely continuous on each $[a,t_1), [t_1,t_2), \ldots, [t_m, b]$. The element $x \in \mathbf{DS}(m)$ may be represented as

$$x(t) = \int_a^t \dot{x}(s)ds + x(a) + \sum_{i=1}^m \chi_{[t_i,b]}(t)\Delta x(t_i), \tag{1.14}$$

where $\Delta x(t_i) = x(t_i) - x(t_i - 0)$, $i = 1, \ldots, m$, $\chi_{[t_i,b]}$ is the characteristic function of the interval $[t_i, b]$. Thus the space $\mathbf{DS}(m)$ is isomorphic to the product $\mathbf{L} \times \mathbb{R}^{n(m+1)}$ and

$$\begin{gathered} (\Lambda z)(t) = \int_a^t z(s)ds, \\ Y = \left(E, E \cdot \chi_{[t_1,b]}, \ldots, E \cdot \chi_{[t_m,b]}\right), \\ \delta x = \dot{x}, \qquad rx = \{x(a), \Delta x(t_1), \ldots, \Delta x(t_m)\}. \end{gathered} \tag{1.15}$$

Theorem 1.8. *A linear bounded operator* $\{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ *has the bounded inverse if and only if the following conditions are satisfied.*

(a) *The operator* $\Lambda : \mathbf{B} \to \mathbf{D}$ *is Noether one and* $\operatorname{ind} \Lambda = -n$.

(b) $\dim \ker \Lambda = 0$.

(c) *If* $\lambda^1, \ldots, \lambda^n$ *is a basis for* $\ker \Lambda^*$ *and* $\lambda = \{\lambda^1, \ldots, \lambda^n\}$, *then* $\det \lambda Y \neq 0$.

Proof

Sufficiency. From (a) and (b), it follows that $\dim\ker\Lambda^* = n$. By virtue of Theorem 1.5, $\mathbf{D} = R(\Lambda) \oplus M_n$, where $\dim M_n = n$. It follows from (c) that any nontrivial linear combination of elements $y_1, \dots, y_n$ does not belong to $R(\Lambda)$, therefore $M_n = R(Y)$. Thus $\mathbf{D} = R(\Lambda) \oplus R(Y)$ and, consequently, the operator $\{\Lambda, Y\}$ has its inverse by virtue of Banach's theorem.

Necessity. From invertibility of $\{\Lambda, Y\}$, we have $\mathbf{D} = R(\Lambda) \oplus R(Y)$. Consequently, the operator Λ is normally solvable by virtue of Theorem 1.5 and $\dim\ker\Lambda^* = n$. Besides, $\dim\ker\Lambda = 0$. Therefore $\operatorname{ind}\Lambda = -n$. Assumption $\det\lambda Y = 0$ leads to the conclusion that a nontrivial combination of the elements $y_1, \dots, y_n$ belongs to $R(\Lambda)$. □

Theorem 1.9. *A linear bounded operator $[\delta, r] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^n$ has a bounded inverse if and only if the following conditions are satisfied.*

(a) *The operator $\delta : \mathbf{D} \to \mathbf{B}$ is a Noether one,* $\operatorname{ind}\delta = n$.

(b) $\dim\ker\delta = n$.

(c) *If $x_1, \dots, x_n$ is a basis of* $\ker\delta$ *and $X = (x_1, \dots, x_n)$, then* $\det rX \neq 0$.

Proof

Sufficiency. From (a) and (b), it follows that $\dim\ker\delta^* = 0$. Thus $R(\delta) = \mathbf{B}$. Each solution of the equation $\delta x = z$ has the form

$$x = \sum_{i=1}^{n} c_i x_i + v, \tag{1.16}$$

where $c_i = \text{const}$, $i = 1, \dots, n$, v is any solution of this equation. By virtue of (c), the system

$$\delta x = z, \qquad rx = \beta \tag{1.17}$$

has a unique solution for each pair $z \in \mathbf{B}$, $\beta \in \mathbb{R}^n$. Therefore, the operator $[\delta, r]$ has its bounded inverse.

Necessity. Let $[\delta, r]^{-1} = \{\Lambda, Y\}$. From the equality $\delta\Lambda = I$, by virtue of Theorem 1.7, it follows that δ is a Noether operator and, by virtue of Theorem 1.3, $\operatorname{ind}\delta = n$. As far as $R(\delta) = \mathbf{B}$, we have $\dim\ker\delta^* = 0$, and therefore $\dim\ker\delta = n$. If $\det rX = 0$, then the homogeneous system

$$\delta x = 0, \qquad rx = 0 \tag{1.18}$$

has a nontrivial solution. This gives a contradiction to the invertibility of the operator $[\delta, r]$. □

1.2. Linear equation and linear boundary value problem

The Cauchy problem

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) - P(t)x(t) = f(t), \quad x(a) = \alpha,\ t \in [a,b], \tag{1.19}$$

is uniquely solvable for any $\alpha \in \mathbb{R}^n$ and summable f if the elements of the $n \times n$ matrix P are summable. Thus, the representation of the solution

$$x(t) = X(t)\int_a^t X^{-1}(s)f(s)ds + X(t)\alpha \tag{1.20}$$

of the problem (the Cauchy formula), where X is a fundamental matrix such that $X(a)$ is the identity matrix, is also a representation of the general solution of the equation $\mathcal{L}x = f$. The Cauchy formula is the base for investigations on various problems in the theory of ordinary differential equations. The Cauchy problem for functional differential equations is not solvable generally speaking, but some boundary value problems may be solvable. Therefore the boundary value problem plays the same role in the theory of functional differential equations as the Cauchy problem does in the theory of ordinary differential equations.

We will call the equation

$$\mathcal{L}x = f \tag{1.21}$$

a *linear abstract functional differential equation* if $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is a linear operator, $\mathbf{D}$ and $\mathbf{B}$ are Banach spaces, and the space $\mathbf{D}$ is isomorphic to the direct product $\mathbf{B} \times \mathbb{R}^n (\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n)$.

Let $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ be a linear isomorphism and let $\mathcal{J}^{-1} = [\delta, r]$. Everywhere below, the norms in the spaces $\mathbf{B} \times \mathbb{R}^n$ and $\mathbf{D}$ are defined by

$$\big|\big|\{z,\beta\}\big|\big|_{\mathbf{B}\times\mathbb{R}^n} = \|z\|_{\mathbf{B}} + |\beta|, \qquad \|x\|_{\mathbf{D}} = \|\delta x\|_{\mathbf{B}} + |rx|. \tag{1.22}$$

By such a definition of the norms, the isomorphism $\mathcal{J}$ is an isometric one. Therefore,

$$\big|\big|\{\Lambda, Y\}\big|\big|_{\mathbf{B}\times\mathbb{R}^n\to\mathbf{D}} = 1, \qquad \big|\big|[\delta, r]\big|\big|_{\mathbf{D}\to\mathbf{B}\times\mathbb{R}^n} = 1. \tag{1.23}$$

Since

$$\|\Lambda z\|_{\mathbf{D}} = \big|\big|\{\Lambda, Y\}\{z,0\}\big|\big|_{\mathbf{D}} \le \big|\big|\{\Lambda, Y\}\big|\big|\,\big|\big|\{z,0\}\big|\big|_{\mathbf{B}\times\mathbb{R}^n} = \|z\|_{\mathbf{B}}, \tag{1.24}$$

$\|\Lambda\|_{\mathbf{B}\to\mathbf{D}} = 1$. Similarly it is stated that $\|Y\|_{\mathbb{R}^n\to\mathbf{D}} = 1$. Next, we have

$$\|\delta x\|_{\mathbf{B}} \le \|x\|_{\mathbf{D}}, \tag{1.25}$$

and if $rx = 0$,

$$\|\delta x\|_{\mathbf{B}} = \|x\|_{\mathbf{D}}. \tag{1.26}$$

Therefore $\|\delta\|_{\mathbf{D}\to\mathbf{B}} = 1$. Analogously $\|r\|_{\mathbf{D}\to\mathbb{R}^n} = 1$.

We will assume that the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is bounded. Applying $\mathcal{L}$ to both parts of (1.7), we get the decomposition

$$\mathcal{L}x = Q\delta x + Arx. \tag{1.27}$$

Here $Q = \mathcal{L}\Lambda : \mathbf{B} \to \mathbf{B}$ is the *principal part*, and $A = \mathcal{L}Y : \mathbb{R}^n \to \mathbf{B}$ is the *finite-dimensional part* of $\mathcal{L}$.

As examples of (1.21) in the case when $\mathbf{D}$ is a space $\mathbf{D}^n$ of absolutely continuous functions $x : [a,b] \to \mathbb{R}^n$ and $\mathbf{B}$ is a space $\mathbf{L}^n$ of summable functions $z : [a,b] \to \mathbb{R}^n$, we can take an ordinary differential equation

$$\dot{x}(t) - P(t)x(t) = f(t), \quad t \in [a,b], \tag{1.28}$$

where the columns of the matrix P belong to $\mathbf{L}^n$, or an integrodifferential equation

$$\dot{x}(t) - \int_a^b H_1(t,s)\dot{x}(s)ds - \int_a^b H(t,s)x(s)ds = f(t), \quad t \in [a,b]. \tag{1.29}$$

We will assume the elements $h_{ij}(t,s)$ of the matrix $H(t,s)$ to be measurable in $[a,b] \times [a,b]$, and the functions $\int_a^b h_{ij}(t,s)ds$ to be summable on $[a,b]$, and will assume the integral operator

$$(H_1 z)(t) = \int_a^b H_1(t,s)z(s)ds \tag{1.30}$$

on $\mathbf{L}^n$ into $\mathbf{L}^n$ to be completely continuous. The corresponding operators $\mathcal{L}$ for these equations in the form (1.27) have the representation

$$(\mathcal{L}x)(t) = \dot{x}(t) - P(t)\int_a^t \dot{x}(s)ds - P(t)x(a) \tag{1.31}$$

for (1.28) and

$$(\mathcal{L}x)(t) = \dot{x}(t) - \int_a^b \left\{H_1(t,s) + \int_s^b H(t,\tau)d\tau\right\}\dot{x}(s)ds - \int_a^b H(t,s)ds\, x(a) \tag{1.32}$$

for (1.29).

Theorem 1.10. *An operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is a Noether one if and only if the principal part $Q : \mathbf{B} \to \mathbf{B}$ of $\mathcal{L}$ is a Noether operator. In this case,* $\operatorname{ind}\mathcal{L} = \operatorname{ind} Q + n$.

Proof. If $\mathcal{L}$ is a Noether operator, $Q = \mathcal{L}\Lambda$ is also Noether as a product of Noether operators and $\operatorname{ind}\mathcal{L} = \operatorname{ind} Q + n$ (Theorems 1.8 and 1.3).

If Q is a Noether operator, $Q\delta$ is also Noether. Consequently, $\mathcal{L} = Q\delta + Ar$ is also Noether (Theorems 1.9 and 1.2). □

By Theorem 1.10, the equality $\operatorname{ind}\mathcal{L} = n$ is equivalent to the fact that Q is a Fredholm operator. The operator $Q : \mathbf{B} \to \mathbf{B}$ is a Fredholm one if and only if it is representable in the form $Q = P^{-1} + V(Q = P_1^{-1} + V_1)$, where P^{-1} is the inverse to a bounded operator P, and V is a compact operator (P_1^{-1} is the inverse to the bounded P_1, and V_1 is a finite-dimensional operator), see [108]. An operator $Q = (I + V) : \mathbf{B} \to \mathbf{B}$ is a Fredholm one, if a certain degree V^m of V is compact (see, e.g., [108]). If the operator V is compact, the operator $Q = I + V$ is said to be a *canonical Fredholm operator*.

In the examples given above, we have $Q = I - K$, where K is an integral operator. For (1.28),

$$(Kz)(t) = \int_a^t P(t)z(s)ds \tag{1.33}$$

and it is a compact operator. For (1.29),

$$(Kz)(t) = \int_a^b \left\{ H_1(t,s) + \int_s^b H(t,\tau)d\tau \right\} z(s)ds. \tag{1.34}$$

Here K^2 is a compact operator. The property of these operators being compact may be established by Maksimov's lemma [141, Lemma 1] (see also [32, Theorem 2.1]), which is given as Theorem B.1.

Theorem 1.11. *Let $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ be a Noether operator with $\operatorname{ind}\mathcal{L} = n$. Then $\dim\ker\mathcal{L} \geq n$ and also $\dim\ker\mathcal{L} = n$ if and only if the equation* (1.21) *is solvable for each $f \in \mathbf{B}$.*

Proof. Recall that $\dim\ker\mathcal{L} - \dim\ker\mathcal{L}^* = n$. Besides, the equation $\mathcal{L}x = f$ is solvable for each $f \in \mathbf{B}$ if and only if $\dim\ker\mathcal{L}^* = 0$ (Theorem 1.1). □

The vector $X = (x_1, \ldots, x_\nu)$ whose components constitute a basis for the kernel of $\mathcal{L}$ is called *the fundamental vector* of the equation $\mathcal{L}x = 0$ and the components $x_1, \ldots, x_\nu$ are called *the fundamental system* of solutions of this equation.

Let $l = [l^1, \ldots, l^m] : \mathbf{D} \to \mathbb{R}^m$ be a linear bounded vector functional, $\alpha = \operatorname{col}\{\alpha^1, \ldots, \alpha^m\} \in \mathbb{R}^m$. The system

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{1.35}$$

is called a *linear boundary value problem*.

If $R(\mathcal{L}) = \mathbf{B}$ and $\dim\ker\mathcal{L} = n$, the question about solvability of (1.35) is a one about solvability of a linear algebraic system with the matrix $lX = (l^i x_j)$, $i = 1,\dots,m$, $j = 1,\dots,n$. Really, since the general solution of the equation $\mathcal{L}x = f$ has the form

$$x = \sum_{j=1}^{n} c_j x_j + v, \tag{1.36}$$

where v is any solution of this equation, $c_1,\dots,c_n$ are arbitrary constants. Thus, problem (1.35) is solvable if and only if the algebraic system

$$\sum_{j=1}^{n} l^i x_j c_j = \alpha^i - l^i v, \quad i = 1,\dots,m, \tag{1.37}$$

is solvable with respect to $c_1,\dots,c_n$. So, problem (1.35) has a unique solution for each $f \in \mathbf{B}$, $\alpha \in \mathbb{R}^m$ if and only if $m = n$, and $\det lX \neq 0$. The determinant $\det lX$ is said to be the *determinant of the problem* (1.35).

By applying the operator l to the two parts of equality (1.7), we get the decomposition

$$lx = \Phi\delta x + \Psi r x, \tag{1.38}$$

where $\Phi : \mathbf{B} \to \mathbb{R}^m$ is a linear bounded vector functional. We will denote the matrix defined by the linear operator $\Psi : \mathbb{R}^n \to \mathbb{R}^m$ also by Ψ.

Using the representations (1.27) and (1.38), we can rewrite the problem (1.35) in the form of the equation

$$\begin{pmatrix} Q & A \\ \Phi & \Psi \end{pmatrix} \begin{pmatrix} \delta x \\ r x \end{pmatrix} = \begin{pmatrix} f \\ \alpha \end{pmatrix}. \tag{1.39}$$

The operator

$$\begin{pmatrix} Q^* & \Phi^* \\ A^* & \Psi^* \end{pmatrix} : \mathbf{B}^* \times (\mathbb{R}^m)^* \longrightarrow \mathbf{B}^* \times (\mathbb{R}^n)^* \tag{1.40}$$

is the adjoint one to the operator

$$\begin{pmatrix} Q & A \\ \Phi & \Psi \end{pmatrix} : \mathbf{B} \times \mathbb{R}^n \longrightarrow \mathbf{B} \times \mathbb{R}^m. \tag{1.41}$$

Taking into account the isomorphism between the spaces $\mathbf{B}^* \times (\mathbb{R}^n)^*$ and $\mathbf{D}^*$, we therefore call the equation

$$\begin{pmatrix} Q^* & \Phi^* \\ A^* & \Psi^* \end{pmatrix} \begin{pmatrix} \omega \\ \gamma \end{pmatrix} = \begin{pmatrix} g \\ \eta \end{pmatrix} \tag{1.42}$$

the equation adjoint to the problem (1.39).

Lemma 1.12. *The operator* $[\delta, l] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^m$ *is a Noether one,* $\operatorname{ind}[\delta, l] = n - m$.

Proof. We have $[\delta, l] = [\delta, 0] + [0, l]$, where the symbol "0" denotes a null operator on the corresponding space. The operator $[0, l] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^m$ is compact since the finite-dimensional operator $l : \mathbf{D} \to \mathbb{R}^m$ is a compact one. Compact perturbations do not change the index of the operator (Theorem 1.2). Therefore, it is sufficient to prove Lemma 1.12 only for the operator $[\delta, 0]$.

The direct product $\mathbf{B} \times \{0\}$ is the range of values of the operator $[\delta, 0]$. The homogeneous adjoint equation to the problem $[\delta, 0]x = \{f, 0\}$ is reducible to one equation $\omega = 0$ in the space $\mathbf{B}^* \times (\mathbb{R}^m)^*$. The solutions of this equation are the pairs $\{0, \gamma\}$. Therefore $\dim\ker[\delta, 0]^* = m$.

Thus $[\delta, 0] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^m$ is a Noether operator and $\operatorname{ind}[\delta, 0] = n - m$. □

Rewrite the problem (1.39) in the form of the equation

$$[\mathcal{L}, l]x = \{f, \alpha\}. \tag{1.43}$$

Theorem 1.13. *The problem* (1.43) *is a Noether one if and only if the principal part* $Q : \mathbf{B} \to \mathbf{B}$ *of* $\mathcal{L}$ *is a Noether operator and also* $\operatorname{ind}[\mathcal{L}, l] = \operatorname{ind} Q + n - m$.

Proof. The operator $[\mathcal{L}, l]$ has the representation

$$[\mathcal{L}, l] = \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l] + [Ar, 0], \tag{1.44}$$

where $I : \mathbb{R}^m \to \mathbb{R}^m$ is the identity operator, symbol "0" denotes the null operator in the corresponding space. Indeed

$$\begin{aligned} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l]x + [Ar, 0]x &= \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} \operatorname{col}\{\delta x, lx\} + \operatorname{col}\{Arx, 0\} \\ &= \operatorname{col}\{Q\delta x + Arx, lx\}. \end{aligned} \tag{1.45}$$

The operator $Q : \mathbf{B} \to \mathbf{B}$ is Noether if and only if the operator

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} : \mathbf{B} \times \mathbb{R}^m \longrightarrow \mathbf{B} \times \mathbb{R}^m \tag{1.46}$$

is a Noether one,

$$\operatorname{ind} \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} = \operatorname{ind} Q. \tag{1.47}$$

Therefore, the operator

$$\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} [\delta, l] : \mathbf{D} \longrightarrow \mathbf{B} \times \mathbb{R}^m \tag{1.48}$$

is a Noether one if and only if Q is a Noether operator and also

$$\operatorname{ind}\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}[\delta, l] = \operatorname{ind}\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix} + \operatorname{ind}[\delta, l] = \operatorname{ind} Q + n - m \tag{1.49}$$

(Theorems 1.3 and 1.7). The product $Ar : \mathbf{D} \to \mathbf{B}$ is compact. Hence the operator $[Ar, 0] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^m$ is also compact. Now we get the conclusion of Theorem 1.13 from the fact that compact perturbation does not violate the property of being a Noether operator and does not change the index. □

It should be noticed that the following corollaries are from Theorem 1.13 under the assumption that $\mathcal{L}$ is a Noether operator .

Corollary 1.14. *The problem* (1.43) *is a Fredholm one if and only if* $\operatorname{ind} Q = m - n$.

Corollary 1.15. *The problem* (1.43) *is solvable if and only if the right-hand side* $\{f, \alpha\}$ *is orthogonal to all the solutions* $\{\omega, \gamma\}$ *of the homogeneous adjoint equation*

$$\begin{aligned} Q^*\omega + \Phi^*\gamma &= 0, \\ A^*\omega + \Psi^*\gamma &= 0. \end{aligned} \tag{1.50}$$

The condition of being orthogonal has the form

$$\langle \omega, f \rangle + \langle \gamma, \alpha \rangle = 0. \tag{1.51}$$

Everywhere below, we assume that the operator $\mathcal{L}$ is a Noether one with $\operatorname{ind} \mathcal{L} = n$ which means that Q is a Fredholm operator. Under such an assumption, by virtue of Corollary 1.14, problem (1.43) is a Fredholm one if and only if $m = n$.

The functionals $l^1, \ldots, l^m$ are assumed to be linearly independent.

We will call the special case of (1.43) with $l = r$ *the principal boundary value problem*. The equation $[\delta, r]x = \{f, \alpha\}$ is just the problem which is the base of the isomorphism $\mathcal{J}^{-1} = [\delta, r]$ between $\mathbf{D}$ and $\mathbf{B} \times \mathbb{R}^n$.

Theorem 1.16. *The principal boundary value problem*

$$\mathcal{L}x = f, \qquad rx = \alpha \tag{1.52}$$

is uniquely solvable if and only if the principal part $Q : \mathbf{B} \to \mathbf{B}$ *of* $\mathcal{L}$ *has its bounded inverse* $Q^{-1} : \mathbf{B} \to \mathbf{B}$. *The solution* x *of* (1.52) *has the representation*

$$x = \Lambda Q^{-1} f + (Y - \Lambda Q^{-1} A)\alpha. \tag{1.53}$$

Proof. Using the decomposition (1.27), we can rewrite (1.52) in the form

$$Q\delta x + Arx = f, \quad rx = \alpha. \tag{1.54}$$

If Q is invertible, then

$$\delta x = Q^{-1}f - Q^{-1}A\alpha. \tag{1.55}$$

An application to this equality of the operator Λ yields (1.53) since $\Lambda\delta = I - Yr$.

If Q is not invertible and y is a nontrivial solution of the equation $Qy = 0$, the homogeneous problem

$$\mathcal{L}x = 0, \qquad rx = 0 \tag{1.56}$$

has a nontrivial solution x, for instance $x = \Lambda y$. □

From (1.53), one can see that the vector $X = Y - \Lambda Q^{-1}A$ is a fundamental one and also $rX = E$ (here A denotes the vector that defines the finite-dimensional operator $A : \mathbb{R}^n \to \mathbf{B}$).

Theorem 1.17. *The following assertions are equivalent.*

(a) $R(\mathcal{L}) = \mathbf{B}$.

(b) $\dim\ker\mathcal{L} = n$.

(c) *There exists a vector functional* $l : \mathbf{D} \to \mathbb{R}^n$ *such that problem* (1.43) *is uniquely solvable for each* $f \in \mathbf{B}$, $\alpha \in \mathbb{R}^n$.

Proof. The equivalence of the assertions (a) and (b) was established while proving Theorem 1.11.

Let $\dim\ker\mathcal{L} = n$ and $l = [l^1, \ldots, l^n]$, let the system $l^1, \ldots, l^n$ be biorthogonal to the bases $x_1, \ldots, x_n$ of the kernel of $\mathcal{L}$: $l^i x_j = \delta_{ij}$, $i, j = 1, \ldots, n$, where δ_{ij} is the Kronecker symbol. Then problem (1.43) with such an l has the unique solution

$$x = X(\alpha - lv) + v, \tag{1.57}$$

where $X = (x_1, \ldots, x_n)$ and v is any solution of $\mathcal{L}x = f$. This is seen by taking into account that $lX = E$. Conversely, if (1.43) is uniquely solvable for each f and α, then one can take the solutions of the problems

$$\mathcal{L}x = 0, \qquad lx = \alpha_i, \quad \alpha_i \in \mathbb{R}^n, \ i = 1, \ldots, n, \tag{1.58}$$

as the bases $x_1, \ldots, x_n$ if the matrix $(\alpha_1, \ldots, \alpha_n)$ is invertible. Thus the equivalence of the assertions (b) and (c) is proved. □

1.3. The Green operator

We will consider here the boundary value problem

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{1.59}$$

under the assumption that the dimension m of l (the number of the boundary conditions) is equal to n. By virtue of Corollary 1.15, such a condition is necessary for unique solvability of problem (1.59). Recall that we assume $\mathcal{L}$ to be a Noether operator, $\operatorname{ind}\mathcal{L} = n$ ($\operatorname{ind} Q = 0$). If $m = n$, then problem (1.59) is a Fredholm one ($[\mathcal{L}, l] : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^n$ is a Fredholm operator). Consequently, for this problem the assertions that "the problem has a unique solution for some kind of right-hand part $\{f, \alpha\}$ (the problem is uniquely solvable)," "the problem is solvable for each $\{f, \alpha\}$ (the problem is solvable everywhere)," and "the problem is everywhere and uniquely solvable" are equivalent.

Let (1.59) be uniquely solvable and let us denote $[\mathcal{L}, l]^{-1} = \{G, X\}$. Then the solution x of problem (1.59) has the representation

$$x = Gf + X\alpha. \tag{1.60}$$

The operator $G : \mathbf{B} \to \mathbf{D}$ is called *the Green operator of the problem* (1.59), the vector $X = (x_1, \dots, x_n)$ is a fundamental vector for the equation $\mathcal{L}x = 0$, and also $lX = E$.

It should be noted that Λ is the Green operator of the problem $\delta x = f$, $rx = \alpha$.

Theorem 1.18. *A linear bounded operator $G : \mathbf{B} \to \mathbf{D}$ is a Green operator of a boundary value problem* (1.59) *if and only if the following conditions are fulfilled.*

(a) *G is a Noether operator,* $\operatorname{ind} G = -n$.

(b) $\ker G = \{0\}$.

Proof. $\{G, X\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ is a one-to-one mapping if G is the Green operator of problem (1.59). So, (a) and (b) are fulfilled by virtue of Theorem 1.8. Conversely, let G be such that (a) and (b) are fulfilled. Then $\dim\ker G^* = n$. If $l^1, \dots, l^n$ constitute a basis of $\ker G^*$ and $l = [l^1, \dots, l^n]$, then $R(G) = \ker l$. G is the Green operator of problem (1.59), where

$$\mathcal{L}x = G^{-1}(x - Ulx) + Vlx, \tag{1.61}$$

G^{-1} is the inverse to $G : \mathbf{B} \to \ker l$; $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, is a vector such that $lU = E$; and $V = (v_1, \dots, v_n)$, $v_i \in \mathbf{B}$, is an arbitrary vector. □

Theorem 1.19. *Let the problem* (1.59) *be uniquely solvable and let G be the Green operator of this problem. Let further $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, $lU = E$. Then the vector*

$$X = U - G\mathcal{L}U \tag{1.62}$$

is a fundamental to the equation $\mathcal{L}x = 0$.

Proof. We have $\dim\ker\mathcal{L} = n$ by virtue of Theorem 2.8 and the unique solvability of (1.59). The components of X are linearly independent since $lX = E$. The equality $\mathcal{L}X = 0$ can be verified immediately. □

Theorem 1.20. *Let G and G_1 be Green operators of the problems*

$$\begin{aligned} \mathcal{L}x &= f, \qquad lx = \alpha, \\ \mathcal{L}x &= f, \qquad l_1x = \alpha. \end{aligned} \tag{1.63}$$

Let, further, X be the fundamental vector of $\mathcal{L}x = 0$. Then

$$G = G_1 - X(lX)^{-1}lG_1. \tag{1.64}$$

Proof. The general solution of $\mathcal{L}x = f$ has the representation

$$x = Xc + G_1f, \tag{1.65}$$

where $c \in \mathbb{R}^n$ is an arbitrary vector. Define c in such a way that $lx = 0$. We have

$$0 = lx = lXc + lG_1f. \tag{1.66}$$

Hence

$$c = -(lX)^{-1}lG_1f \tag{1.67}$$

and the solution x of the half-homogeneous problem $\mathcal{L}x = f$, $lx = 0$ has the form

$$x = \big(G_1 - X(lX)^{-1}lG_1\big)f = Gf. \tag{1.68}$$

□

At the investigation of particular boundary value problems and some properties of Green operator, it is useful to employ the "elementary Green operator" W_l that can be constructed for any boundary conditions $lx = \alpha$. Beforehand, we will prove the following lemma.

Lemma 1.21. *For any linear bounded vector functional $l = [l^1, \ldots, l^n] : \mathbf{D} \to \mathbb{R}^n$ with linearly independent components, there exists a vector $U = (u_1, \ldots, u_n)$, $u_i \in \mathbf{D}$, such that* $\det rU \neq 0$ *and* $\det lU \neq 0$.

Proof. Let U_1 and U_2 be n-dimensional vectors such that $\det rU_1 \neq 0$ and $lU_2 = E$. Let, further,

$$U = U_1 + \mu U_2, \tag{1.69}$$

where μ is a numerical parameter. The function $\psi(\mu) = \det rU$ is continuous and $\psi(0) \neq 0$. Hence $\psi(\mu) \neq 0$ on an interval $(-\mu_0, \mu_0)$. The polynomial $P(\mu) = \det lU = \det(lU_1 + \mu E)$ has no more than n roots. Consequently, there exists a $\mu_1 \in (-\mu_0, \mu_0)$ such that $P(\mu_1) \neq 0$. For $U = U_1 + \mu_1 U_2$, we have $\det rU \neq 0$ and $\det lU \neq 0$. □

Suppose $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, $\det rU \neq 0$, $lU = E$. Define the operator $W_l : \mathbf{B} \to \mathbf{D}$ as follows:

$$W_l = \Lambda - U\Phi, \tag{1.70}$$

where $U : \mathbb{R}^n \to \mathbf{D}$ is a finite-dimensional operator corresponding to the vector U, $\Phi : \mathbf{B} \to \mathbb{R}^n$ is the principal part of the vector functional l (see the equality (1.38)). Let, further, $\mathcal{L}_0 : \mathbf{D} \to \mathbf{B}$ be defined by

$$\mathcal{L}_0 x = \delta x - \delta U(rU)^{-1} rx. \tag{1.71}$$

Theorem 1.22. *W_l is the Green operator of the boundary value problem*

$$\mathcal{L}_0 x = f, \qquad lx = \alpha. \tag{1.72}$$

Proof. The principal boundary value problem for the equation $\mathcal{L}_0 x = f$ is uniquely solvable. Consequently, the dimension of the fundamental vector for $\mathcal{L}_0 x = 0$ equals n. By immediate substitution, we get $\mathcal{L}_0 U = 0$. Problem (1.72) is solvable since $lU = E$. We have

$$\begin{aligned} \mathcal{L}_0 W_l f &= \delta(\Lambda f - U\Phi f) - \delta U(rU)^{-1} r(\Lambda f - U\Phi f) \\ &= f - \delta U\Phi f + \delta U(rU)^{-1} rU\Phi f = f, \\ lW_l f &= \Phi\delta(\Lambda f - U\Phi f) + \Psi r(\Lambda f - U\Phi f) = \Phi f - lU\Phi f = 0. \end{aligned} \tag{1.73}$$

□

The collection of all Green operator corresponding to the given vector functional $l : \mathbf{D} \to \mathbb{R}^n$ is the set of operators of the form

$$G = W_l\Gamma, \tag{1.74}$$

where Γ is a linear homeomorphism of $\mathbf{B}$ into $\mathbf{B}$. Indeed, if $\Gamma : \mathbf{B} \to \mathbf{B}$ is a homeomorphism, then, by virtue of Theorem 1.18, $W_l\Gamma$ is a Green operator of a problem (1.59). Conversely, any Green operator $G : \mathbf{B} \to \ker l$ may be represented by (1.74), where $\Gamma = W_l^{-1}G$, $W_l^{-1} : \ker l \to \mathbf{B}$ is the inverse to $W_l : \mathbf{B} \to \ker l$.

Theorem 1.23. *The collection of all Green operators $G : \mathbf{B} \to \mathbf{D}$ is defined by*

$$G = (\Lambda - Uv)\Gamma, \tag{1.75}$$

where $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, $\det rU \neq 0$, $v : \mathbf{B} \to \mathbb{R}^n$ is a linear bounded vector functional, and Γ is a linear homeomorphism of the space $\mathbf{B}$ onto $\mathbf{B}$.

Proof. $W = \Lambda - Uv$ is the Green operator of problem (1.72), where $lx = v\delta x + [E - v\delta U](rU)^{-1}rx$. Indeed,

$$\begin{aligned}\mathcal{L}_0 Wf &= \delta(\Lambda - Uv)f - \delta U(rU)^{-1}r(\Lambda - Uv)f \\ &= f - \delta Uvf + \delta U(rU)^{-1}rUvf = f, \\ lWf &= v\delta(\Lambda - Uv)f + [E - v\delta U](rU)^{-1}r(\Lambda - Uv)f \\ &= vf - v\delta Uvf - [E - v\delta U]vf = 0.\end{aligned} \tag{1.76}$$

Now the assertion of Theorem 1.23 follows from the representation (1.74). □

Remark 1.24. The isomorphism $\{\Lambda, Y\} : \mathbf{B}\times\mathbb{R}^n \to \mathbf{D}$ may be constructed by using as Λ the Green operator of any uniquely solvable boundary value problem in the space $\mathbf{D}$. Thus, on the base of Theorem 1.23, we can assert the following.

If a Green operator $W : \mathbf{B} \to \mathbf{D}$ possesses a certain property and this property is invariant with respect to both finite-dimensional perturbations of this operator and multiplication by any linear bounded operator, then any other Green operator $G : \mathbf{B} \to \mathbf{D}$ possesses the same property.

In the investigation of boundary value problems, an important part belongs to the so-called *W-method* (Azbelev et al. [40]) which is based on an expedient choice of an auxiliary *model equation* $\mathcal{L}_1 x = f$. This method is based on the following assertion.

Theorem 1.25. *Let the model boundary value problem*

$$\begin{aligned}\mathcal{L}_1 x &= f, \\ lx &= 0\end{aligned} \tag{1.77}$$

be uniquely solvable and let $W : \mathbf{B} \to \mathbf{D}$ *be the Green operator of this problem. Problem* (1.59) *is uniquely solvable if and only if the operator* $\mathcal{L}W : \mathbf{B} \to \mathbf{B}$ *has the continuous inverse* $[\mathcal{L}W]^{-1}$. *In this event, the Green operator G of problem* (1.59) *has the representation*

$$G = W[\mathcal{L}W]^{-1}. \tag{1.78}$$

Proof. There is a one-to-one correspondence between the set of solutions $z \in \mathbf{B}$ of the equation $\mathcal{L}Wz = f$ and the set of solutions $x \in \mathbf{D}$ of problem (1.59) with homogeneous boundary conditions $lx = 0$. This correspondence is defined by $x = Wz$ and $z = \mathcal{L}_1 x$. Consequently, problem (1.59) is uniquely solvable and also the solution x of problem (1.59) for $\alpha = 0$ has the representation $x = W[\mathcal{L}W]^{-1}f$. Thus $G = W[\mathcal{L}W]^{-1}$. □

By the applications of Theorem 1.25, one may put $W = W_l$, where W_l is defined by (1.70). Let the operator $U : \mathbb{R}^n \to \mathbf{D}$ be defined as above by the vector

$U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}$, $\det rU \neq 0$, $lU = E$. Let, further, $\Phi : \mathbf{B} \to \mathbb{R}^n$ be the principal part of $l : \mathbf{D} \to \mathbb{R}^n$. Define the operator $F : \mathbf{B} \to \mathbf{B}$ by $F = \mathcal{L}U\Phi$.

Corollary 1.26. *The boundary value problem* (1.59) *is uniquely solvable if and only if the operator* $(Q - F) : \mathbf{B} \to \mathbf{B}$ *has its bounded inverse. The Green operator of this problem has the representation*

$$G = W_l(Q - F)^{-1}. \tag{1.79}$$

The proof follows from the fact that W_l is a Green operator of the model problem $\mathcal{L}_0 x = z$, $lx = 0$, where $\mathcal{L}_0$ is defined by (1.71) and

$$\mathcal{L}W_l = \mathcal{L}\Lambda - \mathcal{L}U\Phi = Q\delta\Lambda - Ar\Lambda - \mathcal{L}U\Phi = Q - \mathcal{L}U\Phi = Q - F. \tag{1.80}$$

The following assertions characterize some properties of the Green operator of problem (1.59) connected with the properties of the principal part Q of $\mathcal{L}$.

Theorem 1.27. *Assume that a boundary value problem* (1.59) *is uniquely solvable. Let* $P : \mathbf{B} \to \mathbf{B}$ *be a linear bounded operator with bounded inverse* P^{-1}. *The Green operator of this problem has the representation*

$$G = W_l(P + H), \tag{1.81}$$

where $H : \mathbf{B} \to \mathbf{B}$ *is a compact operator if and only if the principal part* Q *of* $\mathcal{L}$ *may be represented in the form* $Q = P^{-1} + V$, *where* $V : \mathbf{B} \to \mathbf{B}$ *is a compact operator.*

Proof. Let $G = W_l(Q - F)^{-1}$ (see (1.79)), $Q = P^{-1} + V$. Define $V_1 = V - F$. Then

$$(Q - F)^{-1} = \left(P^{-1} + V_1\right)^{-1} = \left(I + PV_1\right)^{-1}P = \left(I + H_1\right)P = P + H, \tag{1.82}$$

where $H : \mathbf{B} \to \mathbf{B}$ and $H_1 : \mathbf{B} \to \mathbf{B}$ are compact operators.

Conversely, if $(Q - F)^{-1} = P + H$, then

$$Q = F + (P + H)^{-1} = F + \left(I + P^{-1}H\right)^{-1}P^{-1} = F + \left(I + V_1\right)P^{-1} = P^{-1} + V, \tag{1.83}$$

where $V : \mathbf{B} \to \mathbf{B}$ and $V_1 : \mathbf{B} \to \mathbf{B}$ are compact operators. □

Theorem 1.28. *A linear bounded operator* $G : \mathbf{B} \to \mathbf{D}$ *is the Green operator of problem* (1.59), *where* $Q = P^{-1} + V$, *if and only if* $\ker G = \{0\}$ *and*

$$G = \Lambda P + T \tag{1.84}$$

with a compact operator $T : \mathbf{B} \to \mathbf{D}$.

Proof. If G is a Green operator and $Q = P^{-1} + V$, then (1.84) follows at once from (1.81) and (1.70).

Conversely, if G has the form (1.84), then G is a Noether operator, $\operatorname{ind} G = -n$. By virtue of Theorem 1.18, G is a Green operator of a problem (1.59). From $\mathcal{L}G = I$, it follows that $QP + \mathcal{L}T = I$. Hence $Q = P^{-1} + V$, where $V = -\mathcal{L}TP^{-1}$. □

We now state two corollaries of Theorem 1.28.

Corollary 1.29. *The representation $\delta G = P + H$, where $H : \mathbf{B} \to \mathbf{B}$ is a compact operator and $P : \mathbf{B} \to \mathbf{B}$ is a linear bounded operator with a bounded inverse P^{-1}, is possible if and only if G is the Green operator of a problem* (1.59) *with the principal part of $\mathcal{L}$ having the form $Q = P^{-1} + V$, where V is a compact operator.*

Proof. If $\delta G = P + H$,

$$G = \Lambda P + \Lambda H + YrG, \tag{1.85}$$

and by Theorem 1.28, $Q = P^{-1} + V$.

Conversely, if $Q = P^{-1} + V$, then $G = \Lambda P + T$, and consequently, $\delta G = P + \delta T$. □

Corollary 1.30. *The operator δG is a canonical Fredholm one if and only if the principal part Q of $\mathcal{L}$ is a canonical Fredholm one.*

The Green operator for ordinary differential equations and their generalizations is an integral one [32]. We consider below further generalizations of ordinary differential equations in various spaces $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$. The problem of the representation of the Green operator arises any time we use a new space $\mathbf{D}$ of functions for solutions. That is why we formulate the conditions under which the Green operator is representable with the Lebesgue integral in the most actual cases of the space $\mathbf{B}$.

Let $\mathbf{D}$ be a space of functions $x = \operatorname{col}\{x^1, \dots, x^N\} : [a,b] \to \mathbb{R}^N$ defined at any point and measurable on $[a,b]$. Suppose $\mathbf{D} \simeq \mathbf{L}_p \times \mathbb{R}^n$; $\mathbf{L}_p$, $1 \le p \le \infty$, is the Banach space of functions $z = \operatorname{col}\{z^1, \dots, z^N\} : [a,b] \to \mathbb{R}^N$ with components summable with power p for $1 \le p < \infty$, measurable and essentially bounded for $p = \infty$; $\|z\|_{L_p} = \{\int_a^b |z(t)|^p dt\}^{1/p}$ if $1 \le p < \infty$, $\|z\|_{\mathbf{L}_\infty} = \operatorname{ess\,sup}_{t\in[a,b]} |z(t)|$.

In the below assertions, all the boundary value problems are assumed to be uniquely solvable for any $f \in \mathbf{L}_p$ and $\alpha \in \mathbb{R}^n$.

First we consider the case of $1 \le p < \infty$. Let $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_p \times \mathbb{R}^n \to \mathbf{D}$ be an isomorphism and $\mathcal{J}^{-1} = [\delta, r]$. It is commonly known that for $1 \le p < \infty$, any linear bounded functional over the space $\mathbf{L}_p$ has the integral representation. Therefore the vector functional $\Phi : \mathbf{L}_p \to \mathbb{R}^n$ in decomposition (1.38) of the vector functional l is integral: $\Phi z = \int_a^b \Phi(s) z(s)\, ds$, where the columns of the $N \times n$ matrix $\Phi^\top$ belong to $\mathbf{L}_q$, $q = (p/(p-1))$($\cdot^\top$ is the symbol of transposition).

Theorem 1.31. *Let* $1 \leq p < \infty$. *If the Green operator of some problem* (1.59) *is integral, then the Green operator of any other problem* (1.59) *is also an integral one.*

Proof. The supposition that the Green operator to some problem from (1.59) is an integral one enables us to assume the operator $\Lambda : \mathbf{L}_p \to \mathbf{D}$ to be integral. Since in addition the operator $U\Phi : \mathbf{L}_p \to \mathbf{D}$ is integral and finite-dimensional, the operator W_l defined by (1.70) is integral too:

$$(W_l z)(t) = \int_a^b W(t,s)z(s)ds. \tag{1.86}$$

Then due to (1.79), we have

$$\begin{aligned}(Gf)(t) &= \int_a^b W(t,s)\left[(Q-F)^{-1}f\right](s)ds \\ &= \int_a^b \left[(Q^* - F^*)^{-1}W^\top(t,\cdot)\right](s)f(s)ds.\end{aligned} \tag{1.87}$$

□

Now let $p = \infty$. The linear bounded functional over the space $\mathbf{L}_\infty$ cannot in general be represented by means of the Lebesgue integral. Therefore the integral representation of the Green operator $G : \mathbf{L}_\infty \to \mathbf{D}$ can be ensured only by some restrictions on the operators $\mathcal{L}$ and l.

The Green operators $G_1\mathbf{L}_\infty \to \mathbf{D}$ and $G_2 : \mathbf{L}_\infty \to \mathbf{D}$ of the two boundary problems

$$\begin{aligned}\mathcal{L}x = f, \qquad l_1x = \alpha, \\ \mathcal{L}x = f, \qquad l_2x = \alpha\end{aligned} \tag{1.88}$$

for one and the same equation are linked by the equality

$$(G_2f)(t) = (G_1f)(t) - X(t)(l_2X)^{-1}l_2G_1f, \quad f \in \mathbf{L}_\infty, \tag{1.89}$$

due to Theorem 1.20. Here $X = (x_1, \dots, x_n)$ is the fundamental vector of the equation $\mathcal{L}x = 0$. This implies the following theorem.

Theorem 1.32. *Let* G_1 *be integral. The operator* G_2 *is integral if and only if the vector functional* $l_2G_1 : \mathbf{L}_\infty \to \mathbb{R}^n$ *has the integral representation.*

Let $\mathcal{J} : \mathbf{L}_\infty \times \mathbb{R}^n \to \mathbf{D}$ be an isomorphism and $[\delta, r] = \mathcal{J}^{-1}$.

The restrictions on the operators $\mathcal{L}$ and l in the next theorem are stipulated by the choice of the isomorphism $\mathcal{J}$.

We consider in what follows only vector functionals $l : \mathbf{D} \to \mathbb{R}^n$ such that the vector functional $\Phi \stackrel{\text{def}}{=} l\Lambda : \mathbf{L}_\infty \to \mathbb{R}^n$ in decomposition (1.38), as in the case of $1 \leq p < \infty$, is integral. Thus we restrict our attention to the case that l is of the form

$$lx = \int_a^b \Phi(s)(\delta x)(s)ds + \Psi rx, \tag{1.90}$$

where the columns of the $N \times n$ matrix $\Phi^\top$ belong to $\mathbf{L}_1$, Ψ is a constant $n \times n$ matrix.

Theorem 1.33. *Let the operator* $Q \stackrel{\text{def}}{=} \mathcal{L}\Lambda : \mathbf{L}_\infty \to \mathbf{L}_\infty$ *be adjoint to an operator* $Q_1 : \mathbf{L}_1 \to \mathbf{L}_1$, *and the vector functional* $l : \mathbf{D} \to \mathbb{R}^n$ *has representation* (1.90). *Then the Green operator of problem* (1.59) *is integral if and only if the Green operator* $\Lambda : \mathbf{L}_\infty \to \mathbf{D}$ *of the problem* $\delta x = z$, $rx = \alpha$ *is integral.*

Proof. If Λ is an integral operator, then, as in the proof of Theorem 1.32, the operator W_l is also integral:

$$(W_l)(t) = \int_a^b W(t,s)z(s)ds. \tag{1.91}$$

The finite-dimensional integral operator $F \stackrel{\text{def}}{=} \mathcal{L}U\Phi : \mathbf{L}_\infty \to \mathbf{L}_\infty$ is adjoint to the integral operator $F_1 : \mathbf{L}_1 \to \mathbf{L}_1$. Therefore by (1.79), we have

$$\begin{aligned} (Gf)(t) &= \int_a^b W(t,s)\Big[\big(Q_1^* - F_1^*\big)^{-1} f\Big](s)ds \\ &= \int_a^b \Big[\big(Q_1 - F_1\big)^{-1} W^\top(t,\cdot)\Big]^\top (s) f(s)ds. \end{aligned} \tag{1.92}$$

Now let the operator G be integral. It follows from (1.70) and (1.79) that

$$\Lambda = G(Q - F) + U\Phi. \tag{1.93}$$

Hence, as above, we get that the operator Λ is also integral. □

1.4. Problems lacking the everywhere and unique solvability

We assume, as above, that $\operatorname{ind} \mathcal{L} = n (\operatorname{ind} Q = 0)$ and in addition that the equation $\mathcal{L}x = 0$ has n-dimensional fundamental vector X. From Theorem 1.17, the equation $\mathcal{L}x = f$ is solvable for each $f \in \mathbf{B}$.

The boundary value problem

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{1.94}$$

will be considered without the assumption that the number m of boundary conditions equals n.

Denote $\rho = \operatorname{rank} lX$. In the case $\rho > 0$, we may assume without loss of generality that the determinant of the rank ρ composed from the elements in the left top of the matrix lX does not become zero. Let us choose the fundamental vector as follows. In the case that $\rho > 0$, the elements $x_1, \dots, x_\rho$ are selected in such a way that $l^i x_j = \delta_{ij}$, $i, j = 1, \dots, \rho$ (δ_{ij} is the Kronecker symbol). If $0 \leq \rho < n$, the homogeneous problem $\mathcal{L}x = 0$, $lx = 0$ has $n - \rho$ linearly independent solutions $u_1, \dots, u_{n-\rho}$. Everywhere below we will take as the fundamental vector the vector $X = (u_1, \dots, u_n)$ if $\rho = 0$, the vector $X = (x_1, \dots, x_\rho, u_1, \dots, u_{n-\rho})$ if $0 < \rho < n$, and the vector $X = (x_1, \dots, x_n)$ if $\rho = n$.

Recall that problem (1.94) cannot be a Fredholm one if $m \neq n$ (Corollary 1.14) and the question about solvability of problem (1.94) is the question about solvability of a linear algebraic system with the matrix lX.

Consider the cases corresponding to all possible relations between the numbers n, m, and ρ.

The case $n = m = \rho$ was investigated in the previous sections.

If $\rho = m < n$, the problem is solvable (but not uniquely) for any $f \in \mathbf{B}$, $\alpha = \{\alpha^1, \dots, \alpha^m\} \in \mathbb{R}^m$. To obtain the representation of the solution in this case, we can supplement the functionals $l^1, \dots, l^m$ by additional functionals $l^{m+1}, \dots, l^n$ such that

$$\det\left(l^{m+i} u_j\right)_{i,j=1}^{n-m} \neq 0. \tag{1.95}$$

The determinant of the problem

$$\mathcal{L}x = f, \qquad l^1 x = \alpha^1, \dots, \qquad l^n x = \alpha^n \tag{1.96}$$

does not become zero, and therefore this problem is uniquely solvable. Using the Green operator G of this problem, we can represent the solutions of problem (1.94) in the form

$$x = Gf + \sum_{i=1}^{m} \alpha^i x_i + \sum_{i=1}^{n-m} c_i u_i, \tag{1.97}$$

where $c_1, \dots, c_{n-m}$ are arbitrary constants.

In all the other cases, problem (1.94) is not everywhere solvable. The conditions of solvability can be obtained, using the Green operator of any uniquely solvable boundary value problem for the equation $\mathcal{L}x = f$. Such a problem exists by virtue of Theorem 1.17.

Let $\rho = n < m$. In this case, the homogeneous problem $\mathcal{L}x = 0$, $lx = 0$ has only the trivial solution. Thus, if problem (1.94) is solvable, the solution is unique and so is the solution of the problem

$$\mathcal{L}x = f, \qquad l^i x = \alpha^i, \quad i = 1,\dots,n, \tag{1.98}$$

(recall our convention that $(l^i x_j)_{i,j=1}^n = E$). If G is the Green operator of the latter problem, the solution of problem (1.94) in the event of its solvability has the representation

$$x = Gf + \sum_{i=1}^{n} \alpha^i x_i, \tag{1.99}$$

and the necessary and sufficient condition of solvability of problem (1.94) takes the form

$$\alpha^j = l^j Gf + \sum_{i=1}^{n} \alpha^i l^j x_i, \quad j = n+1,\dots,m. \tag{1.100}$$

If $\rho < n \le m$ or $\rho < m < n$, the solution of problem (1.94) cannot be a unique one. Let us choose functionals $\bar{l}^{\rho+1},\dots,\bar{l}^{n}$ such that $\det(\bar{l}^{\rho+i} u_j)_{i,j=1}^{n-\rho} \neq 0$. Then the problem

$$\mathcal{L}x = f, \qquad \bar{l}^i x = \alpha^i, \quad i = 1,\dots,n, \tag{1.101}$$

at $\rho = 0$ or the problem

$$\mathcal{L}x = f, \qquad l^i x = \alpha^i, \quad i = 1,\dots,\rho, \qquad \bar{l}^{\rho+j} x = \alpha^{\rho+j}, \quad j = 1,\dots,n-\rho, \tag{1.102}$$

at $\rho > 0$ is uniquely solvable. Using the Green operator G of this problem, we may write the solutions of problem (1.94) in the case of its solvability in the form

$$x = Gf + \sum_{i=1}^{n} c_i u_i \tag{1.103}$$

by $\rho = 0$ and in the form

$$x = Gf + \sum_{i=1}^{\rho} \alpha^i x_i + \sum_{i=1}^{n-\rho} c_i u_i \tag{1.104}$$

by $\rho > 0$. Here $c_1,\dots,c_{n-\rho}$ are arbitrary constants. The necessary and sufficient condition of solvability of (1.94) takes the form of the equalities

$$\alpha^j = l^j Gf, \quad j = 1,\dots,m, \tag{1.105}$$

by $\rho = 0$ and the equalities

$$\alpha^j = l^j G f + \sum_{i=1}^{\rho} \alpha^i l^j x_i, \quad j = \rho + 1, \dots, m, \tag{1.106}$$

by $\rho > 0$.

In the theory of ordinary differential equations, the so-called "generalized Green function" is widely used for representation of the solutions of the linear boundary value problem in the case when one has no unique solution. The construction of such a function (the kernel of the integral operator, the generalized Green operator) is based on the well-known construction of Schmidt (see, e.g., [219]). This one permits to construct, for a noninvertible operator H, a finite-dimensional operator F^0 such that there exists the bounded inverse $(H + F^0)^{-1}$. The classical scheme of the construction of generalized Green operators for differential equations is entirely extended for abstract functional differential equations. We will dwell here on this scheme.

By Corollary 1.26, the Fredholm operator $Q - F = \mathcal{L} W_l : \mathbf{B} \to \mathbf{B}$ is noninvertible if $\rho < m = n$. In this case, the half-homogeneous problem

$$\mathcal{L} x = f, \qquad lx = 0 \tag{1.107}$$

is solvable if and only if the function f is orthogonal to all the elements of the basis of $\ker(Q - F)^*$. Using the procedure which would be given below, we will construct an operator F^0 such that the operator $Q - F + F^0$ would have its inverse $\Gamma = (Q - F + F^0)^{-1}$. The product $G^0 = W_l \Gamma$ has the property that if problem (1.107) is solvable, then the solutions of this problem may be represented in the form

$$x = G^0 f + \sum_{i=1}^{n-\rho} c_i u_i, \tag{1.108}$$

where $u_i = W_l y_i$, $y_1, \dots, y_{n-\rho}$ is the basis of $\ker Q - F$, $c_1, \dots, c_{n-\rho}$ are arbitrary constants. This operator $G^0 : \mathbf{B} \to \ker l$ is said to be a *generalized Green operator* of problem (1.107). By virtue of (1.74), G^0 is the ordinary Green operator of a certain boundary value problem

$$\mathcal{L}^0 x = f, \qquad lx = \alpha. \tag{1.109}$$

To construct the operator F^0, let us choose any system $\varphi_1, \dots, \varphi_{n-\rho}$ of functionals from the space $\mathbf{B}^*$ that are biorthogonal to $y_1, \dots, y_{n-\rho}(\langle \varphi_i, y_j \rangle = \delta_{ij}, i, j = 1, \dots, n - \rho)$ and a system $z_1, \dots, z_{n-\rho}$, $z_i \in \mathbf{B}$, being biorthogonal to the bases $\omega_1, \dots, \omega_{n-\rho}$ of $\ker(Q - F)^*$. The Schmidt construction defines the operator $F^0 : \mathbf{B} \to \mathbf{B}$ by

$$F^0 y = \sum_{i=1}^{n-\rho} \langle \varphi_i, y \rangle z_i. \tag{1.110}$$

By virtue of the Schmidt lemma, Vainberg and Trenogin [219], there exists the bounded inverse $\Gamma = (Q - F + F^0)^{-1}$. And also, if y satisfies the equation $(Q - F + F^0)y = f$ and conditions of orthogonality $\langle \omega_i, f \rangle = 0$, $i = 1, \dots, n - \rho$, then $(Q - F)y = f$. Indeed, in this case we get from the equality $(Q - F)y = f - F^0 y$ that

$$\langle \omega_i, f - F^0 y \rangle = 0, \quad i = 1, \dots, n - \rho. \tag{1.111}$$

Hence,

$$\langle \omega_i, f \rangle - \langle \omega_i, F^0 y \rangle = -\left\langle \omega_i, \sum_{j=1}^{n-\rho} c_j z_j \right\rangle = 0, \quad i = 1, \dots, n - \rho, \tag{1.112}$$

where c_i are some arbitrary constants. But the latter equality is possible only if $c_1 = \cdots = c_{n-\rho} = 0$. Therefore $F^0 y = 0$ and, consequently, $(Q - F)y = f$, and $x = W_l \Gamma f$ is a solution of (1.107). Hence we get the representation (1.108).

Remark 1.34. To construct a generalized Green operator, one can use instead of W_l, defined by (1.79), the Green operator of any model problem $\mathcal{L}_1 x = f$, $lx = 0$ (see Theorem 1.25).

Not everywhere solvable problem (1.94) may become everywhere solvable by some generalization of the notion of the solution. For instance, the solution of (1.109) for the equation $\mathcal{L}^0 x = f$ constructed on the base of the Schmidt structure may be considered as a kind of such generalization. Below is proposed a notion of a generalized solution of problem (1.94) as an element of a finite-dimensional extension of the initial space. In this connection, the construction of the generalized (extended) everywhere solvable boundary value problem requires sometimes additional boundary conditions. So, the problem

$$\dot{x}(t) = f(t), \qquad x(a) - x(b) = 0 \tag{1.113}$$

has absolutely continuous solutions not for any summable f. If we declare the solution to be a function admitting a finite discontinuity at a fixed point $\tau \in (a, b)$, then the extended problem

$$\dot{y}(t) = f(t), \qquad y(a) - y(b) = \alpha, \qquad y(\xi) = \beta, \quad \xi \in (a, b), \tag{1.114}$$

has a unique solution for each f, α, and β. Indeed, in this case the fundamental system of solutions of the equation $\dot{y}(t) = 0$ consists of two functions $y_1 = 1$ and $y_2 = \chi_{[\tau,b]}(t)$ ($\chi_{[\tau,b]}(t)$ is the characteristic function of $[\tau, b]$). The determinant of the problem is not equal to zero:

$$\Delta = \begin{vmatrix} 0 & 1 \\ -1 & \chi_{[\tau,b]}(\xi) \end{vmatrix} \neq 0. \tag{1.115}$$

Next we will prove, under the assumption that the space **D** admits a finite-dimensional extension, that for any not everywhere solvable problem (1.94), it is possible that we construct an extended problem which is uniquely solvable.

Problem (1.94) is not everywhere solvable if $\rho = n < m$, $\rho < n \leq m$, or $\rho < m < n$. These cases are characterized by the inequality $m - \rho > 0$.

Let the space **D** be embedded into a Banach space $\widetilde{\mathbf{D}}$ so that $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^{\mu}$, where M^{μ} is a finite-dimensional subspace of the dimension μ. Any linear extension $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ of $\mathcal{L}$ is a Noether operator with $\operatorname{ind}\widetilde{\mathcal{L}} = \operatorname{ind}\mathcal{L} + \mu = n + \mu$. (Theorem 1.6). As far as $R(\mathcal{L}) = \mathbf{B}$, we have also $R(\widetilde{\mathcal{L}}) = \mathbf{B}$, therefore $\dim\ker\widetilde{\mathcal{L}} = n + \mu$.

Let $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ and let $\widetilde{l} : \widetilde{\mathbf{D}} \to \mathbb{R}^m$ be a linear extension of $\mathcal{L}$ and l.

Consider the boundary value problem

$$\widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}y = \alpha \tag{1.116}$$

in the space $\widetilde{\mathbf{D}}$. Since $\dim\ker\widetilde{\mathcal{L}} = n + \mu$, this problem may be uniquely and everywhere solvable only if $\mu = m - n$. If $\mu > m - n$, it is necessary to add to m boundary conditions some more $\mu + n - m$ conditions.

Problem (1.116) if $\mu + n - m = 0$, and problem

$$\widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}y = \alpha, \qquad \widetilde{l}_1 y = \alpha_1 \tag{1.117}$$

if $m + n - \mu > 0$ are called *extended boundary value problems*. Here $\widetilde{l}_1 : \widetilde{\mathbf{D}} \to \mathbb{R}^{\mu+n-m}$ is a linear bounded vector functional.

As it was noted above, the inequality $\mu \geq m - n$ is necessary for unique solvability of the extended problem.

Everywhere below, $y_1, \ldots, y_\mu$ are elements of fundamental system of the equation $\widetilde{\mathcal{L}}y = 0$, which do not belong to **D**.

For the beginning, consider an extended problem for a uniquely solvable problem (1.94).

Theorem 1.35. *Let $m = n$, let problem* (1.94) *be uniquely solvable, and let $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^{\mu}$. For any linear extensions $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$, $\widetilde{l} : \widetilde{\mathbf{D}} \to \mathbb{R}^n$ of $\mathcal{L} : \mathbf{D} \to \mathbf{B}$, and $l : \mathbf{D} \to \mathbb{R}^n$, there exists a vector functional $\widetilde{l}_1 : \widetilde{\mathbf{D}} \to \mathbb{R}^{\mu}$ such that problem* (1.117) *is uniquely solvable.*

Proof. For any linear extension $\tilde{l}$ of vector functional l, we have $\tilde{l}X = lX$. Therefore $\det \tilde{l}X \neq 0$. Let us choose $y_1, \dots, y_\mu$ in such a way that $\tilde{l}y_i = 0$, $i = 1, \dots, \mu$. It is possible since, letting

$$y_i = \overline{y}_i - \sum_{j=1}^{n} c_j x_j \tag{1.118}$$

for a fundamental system $x_1, \dots, x_n$, $\overline{y}_1, \dots, \overline{y}_\mu$ of the solutions of the equation $\tilde{\mathcal{L}}y = 0$, we get for constants $c_1, \dots, c_n$ the system

$$\sum_{j=1}^{n} c_j \tilde{l}^k x_j = \tilde{l}^k \overline{y}_i, \quad k = 1, \dots, n, \tag{1.119}$$

with a determinant that is not equal to zero. Let us take now a system of functionals $\tilde{l}^{n+i} : \tilde{\mathbf{D}} \to \mathbb{R}^1$, $i = 1, \dots, \mu$, such that

$$\Delta = \det \left(\tilde{l}^{n+i} y_j\right)_{i,j=1}^{\mu} \neq 0. \tag{1.120}$$

Then the determinant of problem (1.117) with $\tilde{l}_1 = [\tilde{l}^{n+1}, \dots, \tilde{l}^{n+\mu}]$ is equal to $\Delta \cdot \det lX \neq 0$. □

Any element $y \in \tilde{\mathbf{D}}$ has the representation

$$y = \pi y + \sum_{i=1}^{\mu} z_i \lambda^i y, \tag{1.121}$$

where $\pi : \tilde{\mathbf{D}} \to \mathbf{D}$ is a projector, $z_1, \dots, z_\mu$ constitute a basis of M^μ, $\lambda = [\lambda^1, \dots, \lambda^\mu] : \tilde{\mathbf{D}} \to \mathbb{R}^\mu$ is such a vector functional that $\lambda x = 0$ for each $x \in \mathbf{D}$ and $\lambda^i z_j = \delta_{ij}$, $i, j = 1, \dots, \mu$. From (1.121), it follows that any linear extension $\tilde{\mathcal{L}} : \tilde{\mathbf{D}} \to \mathbf{B}$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ has the representation

$$\tilde{\mathcal{L}} y = \mathcal{L}\pi y + \sum_{i=1}^{\mu} a_i \lambda^i y, \tag{1.122}$$

where $a_i = \tilde{\mathcal{L}} z_i$, and also for any $a_i \in \mathbf{B}$, $i = 1, \dots, \mu$, the latter equality defines a linear extension of $\mathcal{L}$ on the space $\tilde{\mathbf{D}}$. Similarly, the representation

$$\tilde{l} y = l\pi y + \Gamma \lambda y, \tag{1.123}$$

where $\Gamma = (\gamma_{ij})$ is a numerical $m \times n$ matrix, defines the general form of the linear extension $\tilde{l} : \tilde{\mathbf{D}} \to \mathbb{R}^m$ of the vector functional $l : \mathbf{D} \to \mathbb{R}^m$.

In what follows, $m - \rho > 0$. The next assertion recommends a more precise estimate of the number μ for uniquely solvable problem than the inequality $\mu \geq m - n$ given above.

Theorem 1.36. *Let* $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^{\mu}$. *If problem* (1.94) *has a uniquely solvable extended problem, then* $\mu \geq m - \rho$.

Proof. Let $\mu < m - \rho$. If $\rho = n$, then $\mu < m - n$. Therefore only the case $\rho < n$ needs the proof.

Let $\widetilde{\mathcal{L}}$ and $\widetilde{l}$ be any linear extensions on the space $\widetilde{\mathbf{D}}$ of $\mathcal{L}$ and l, respectively. If $\mu = m - n$, then the determinant of problem (1.116), the determinant of the order m, is equal to zero because it has nonzero elements only at the columns corresponding to $x_1, \dots, x_\rho, y_1, \dots, y_\mu$, if $\rho > 0$ or $y_1, \dots, y_\mu$, if $\rho = 0$. The number of such columns is equal to $\rho + \mu < m$.

Let $\mu > m - n$. Then the determinant of problem (1.117) is equal to zero. Really, the cofactors of the minors of the $(\mu + n - m)$-th order composed from the elements of the rows corresponding to the vector functional $\widetilde{l}_1$ are determinants of the mth order. These determinants are equal to zero. □

Theorem 1.37. *Let* $\widetilde{\mathbf{D}} = \mathbf{D} \oplus M^{m-\rho}$. *For any linear extension* $\widetilde{\mathcal{L}} : \widetilde{\mathbf{D}} \to \mathbf{B}$ *of the operator* $\mathcal{L} : \mathbf{D} \to \mathbf{B}$, *there exists a linear extension* $\widetilde{l} : \widetilde{\mathbf{D}} \to \mathbb{R}^m$ *of the vector functional* $l : \mathbf{D} \to \mathbb{R}^m$, *and in the case* $\rho < n$, *a vector functional* $\widetilde{l}_1 : \widetilde{\mathbf{D}} \to \mathbb{R}^{n-\rho}$ *such that the extended problem* (1.116) *if* $\rho = n$, *or the extended problem* (1.117) *if* $\rho < n$, *is uniquely solvable.*

Proof. The operator $\widetilde{\mathcal{L}}$ has the representation (1.122), where $\mu = m - \rho$. Denote by v_i any solution of the equation

$$\mathcal{L}x = -a_i \tag{1.124}$$

and let $y_i = v_i + z_i$, $i = 1, \dots, m - \rho$. Thus, $u_1, \dots, u_n, y_1, \dots, y_m$ is the fundamental system of solutions of the equation

$$\widetilde{\mathcal{L}}y = 0 \tag{1.125}$$

if $\rho = 0$, $x_1, \dots, x_\rho, u_1, \dots, u_{n-\rho}, y_1, \dots, y_{m-\rho}$ if $0 < \rho < n$, and $x_1, \dots, x_n, y_1, \dots, y_{m-n}$ if $\rho = n$.

Let $0 < \rho \leq n$. Denote $Y = (x_1, \dots, x_\rho, y_1, \dots, y_{m-\rho})$. We will show that it is possible to choose an $m \times (m - \rho)$ matrix Γ for the corresponding extension (1.123) of the vector functional l so that $\det \widetilde{l}Y \neq 0$. Due to special choice of $x_1, \dots, x_\rho$, we have $\widetilde{l}^i x_j = \delta_{ij}$, $i, j = 1, \dots, \rho$, for any extension $\widetilde{l}$. Further, $\pi Y = (x_1, \dots, x_\rho, v_1, \dots, v_{m-\rho})$; $\lambda x_i = 0$, $i = 1, \dots, \rho$; $\lambda^i y_j = \delta_{ij}$, $i, j = 1, \dots, m - \rho$.

Therefore

$$
\begin{aligned}
\widetilde{l}Y &= l\pi Y + \Gamma\lambda Y \\
&= \begin{pmatrix}
1 & 0 & \cdots & 0 & l^1 v_1 & \cdots & l^1 v_{m-\rho} \\
0 & 1 & \cdots & 0 & l^2 v_1 & \cdots & l^2 v_{m-\rho} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & l^\rho v_i & \cdots & l^\rho v_{m-\rho} \\
l^{\rho+1} x_1 & l^{\rho+1} x_2 & \cdots & l^{\rho+1} x_\rho & l^{\rho+1} v_1 & \cdots & l^{\rho+1} v_{m-\rho} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
l^m x_1 & l^m x_2 & \cdots & l^m x_\rho & l^m v_1 & \cdots & l^m v_{m-\rho}
\end{pmatrix} \\
&\quad + \begin{pmatrix}
0 & \cdots & 0 & \gamma_{11} & \cdots & \gamma_{1,m-\rho} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \gamma_{m1} & \cdots & \gamma_{m,m-\rho}
\end{pmatrix}.
\end{aligned}
\tag{1.126}
$$

The matrix Γ may be chosen, for instance, as follows. Let $\gamma_{ij} = -l^i v_j$ for $i = 1,\dots,\rho$, $j = 1,\dots,m-\rho$, and the numbers $\gamma_{\rho+i,j}$, $i, j = 1,\dots m-\rho$, are chosen so that

$$
\Delta = \det\left(l^{\rho+i} v_j + \gamma_{\rho+i,j}\right)_{i,j=1}^{m-\rho} \neq 0. \tag{1.127}
$$

Then $\det \widetilde{l}Y = \Delta \neq 0$.

If $\rho = n$, Theorem 1.37 is proved because problem (1.116) with the constructed extension $\widetilde{l}$ is uniquely solvable.

If $0 < \rho < n$, we choose in addition a vector functional

$$
\widetilde{l}_1 = \left[\widetilde{l}^{m+1},\dots,\widetilde{l}^{m+n-\rho}\right] : \widetilde{\mathbf{D}} \longrightarrow \mathbb{R}^{n-\rho} \tag{1.128}
$$

so that

$$
\Delta_1 = \det\left(\widetilde{l}^{m+i} u_j\right)_{i,j=1}^{n-\rho} \neq 0. \tag{1.129}
$$

The determinant of problem (1.117) with the extension $\widetilde{l}$ constructed above and the vector functional $\widetilde{l}_1$ is equal to $\Delta_1 \cdot \det \widetilde{l}Y \neq 0$.

If $\rho = 0$, let $Y = (y_1,\dots,y_m)$. In this case,

$$
\widetilde{l}Y = \left(l^i v_j + \gamma_{ij}\right)_{i,j=1}^{m}. \tag{1.130}
$$

Let us choose γ_{ij} such that $\det \widetilde{l}Y \neq 0$ and further, as above, take a vector functional

$$
\widetilde{l}_1 = \left[\widetilde{l}^{m+1},\dots,\widetilde{l}^{m+n}\right] : \widetilde{\mathbf{D}} \longrightarrow \mathbb{R}^{n} \tag{1.131}
$$

such that

$$
\Delta_1 = \det\left(\widetilde{l}^{m+i} u_j\right)_{i,j=1}^{n} \neq 0. \tag{1.132}
$$

Then the determinant of problem (1.117) will be equal to $\Delta_1 \cdot \det \widetilde{l}Y \neq 0$. □

Denote by $\widetilde{G}$ the Green operator of the extended problem (problem (1.116) if $\rho = n$ or (1.117) if $\rho < n$). Then the solution of the problem has the representation

$$y = \widetilde{G}f + Z(\widetilde{l}_\rho Z)^{-1}\alpha_\rho, \tag{1.133}$$

where Z is a fundamental vector of the equation $\widetilde{\mathcal{L}}y = 0$; $\widetilde{l}_\rho = \widetilde{l}$, $\alpha_\rho = \alpha$ if $\rho = n$ and $\widetilde{l}_\rho = [\widetilde{l}, \widetilde{l}_1]$, $\alpha_\rho = \{\alpha, \alpha_1\}$ if $\rho < n$.

Theorems 1.36 and 1.37 provide the minimal number $\mu = m - \rho$ for which there exists a uniquely solvable extended problem to problem (1.94). If $\mu > m - \rho$, the uniquely solvable extended problem also exists by virtue of Theorem 1.35. If the rank of the matrix lX is unknown, then we can take $\mu = m$ for the construction of uniquely solvable extended problem. It will demand n additional boundary conditions. The inequality $\mu \geq m - \rho$ could be used for the estimation of the rank of the matrix lX : if for a certain μ there exists a uniquely solvable extended problem, then rank $lX \geq m - \mu$.

1.5. Continuous dependence on parameters

One of the central places in the theory of differential equations is occupied by the question about conditions that guarantee continuous dependence of the solution of the Cauchy problem

$$\dot{x}(t) = f(t, x(t), \lambda), \qquad x(a) = \alpha \tag{1.134}$$

on parameters λ, α. Kurzweil [134] has approached this question in the following generalized formulation: under which conditions does the sequence $\{x_k\}$ of the solutions of the problems

$$\dot{x}(t) = f_k(t, x(t)), \quad x(a) = \alpha_k, \quad k = 1, 2, \ldots, \tag{1.135}$$

converge to the solution x_0 of the "limiting case"

$$\dot{x}(t) = f_0(t, x(t)), \qquad x(a) = \alpha_0 \tag{1.136}$$

of the problems?

Conditions for convergence of a sequence of solutions to linear boundary value problems in the space of absolutely continuous n-dimensional vector functions are given in [32, Theorem 4.1.1]. Let us formulate an abstract analog of the mentioned theorem.

Let $\mathcal{L}_k : \mathbf{D} \to \mathbf{B}$, ind $\mathcal{L}_k = n$, be linear bounded Noether operators, let $l_k : \mathbf{D} \to \mathbb{R}^n$ be linear bounded vector functionals, $f_k \in \mathbf{B}$, $\alpha_k \in \mathbb{R}^n$, $k = 1, 2, \ldots$. Assume further that

$$\begin{gathered} \lim_{k\to\infty} \|f_k - f_0\|_{\mathbf{B}} = 0, \qquad \lim_{k\to\infty} |\alpha_k - \alpha_0| = 0, \\ \lim_{k\to\infty} \|\mathcal{L}_k x - \mathcal{L}_0 x\|_{\mathbf{B}} = 0, \quad \lim_{k\to\infty} |l_k x - l_0 x| = 0 \quad \text{for each } x \in \mathbf{D}. \end{gathered} \tag{1.137}$$

Theorem 1.38. *Let $x_0 \in \mathbf{D}$ be the solution of the uniquely solvable problem*

$$\mathcal{L}_0 x = f_0, \qquad l_0 x = \alpha_0. \tag{1.138}$$

The problems

$$\mathcal{L}_k x = f_k, \qquad l_k x = \alpha_k \tag{1.139}$$

are uniquely solvable for all sufficiently large k and for their solutions $x_k \in \mathbf{D}$, the convergence

$$\lim_{k\to\infty} \|x_k - x_0\|_{\mathbf{D}} = 0 \tag{1.140}$$

holds if and only if there exists a vector functional $l : \mathbf{D} \to \mathbb{R}^n$ such that the problems

$$\mathcal{L}_k x = f, \qquad lx = \alpha \tag{1.141}$$

are uniquely solvable for $k = 0$ and all sufficiently large k and for each right-hand side $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$, the convergence of the solutions u_k of the problems

$$\lim_{k\to\infty} \|u_k - u_0\|_{\mathbf{D}} = 0 \tag{1.142}$$

holds.

A more general theorem will be proved below where each problem from the sequence of the boundary value problems is considered in its own space. This general assertion will contain Theorem 1.38.

We will formulate here the definitions and propositions of the paper by Vaĭnikko [220], which are required for the proof of the main theorem. We provide these results of Vaĭnikko in the form we are in need of. In the brackets, there are indicated general propositions of the paper by Vaĭnikko [220], on the base of which the theorems stated below are formulated.

Let $\mathbf{E}_0$ and $\mathbf{E}_k$, $k = 1, 2, \ldots$, be Banach spaces.

Definition 1.39. A system $\mathcal{P} = (\mathcal{P}_k)$, $k = 1, 2, \ldots$, of linear bounded operators $\mathcal{P}_k : \mathbf{E}_0 \to \mathbf{E}_k$ is said to be connecting for $\mathbf{E}_0$ and $\mathbf{E}_k$, $k = 1, 2, \ldots$, if

$$\lim_{k\to\infty} \|\mathcal{P}_k u\|_{\mathbf{E}_k} = \|u\|_{\mathbf{E}_0} \tag{1.143}$$

for any $u \in \mathbf{E}_0$.

Observe that the norms of the operators $\mathcal{P}_k$ are bounded in common ($\sup_k \|\mathcal{P}_k\| < \infty$) due to the principle of uniform boundedness.

Definition 1.40. The sequence $\{u_k\}$, $u_k \in \mathbf{E}_k$, is said to be $\mathcal{P}$-convergent to $u_0 \in \mathbf{E}_0$, which is denoted by $u_k \xrightarrow{\mathcal{P}} u_0$, if

$$\lim_{k\to\infty} \|u_k - \mathcal{P}_k u_0\|_{\mathbf{E}_k} = 0. \tag{1.144}$$

Observe that from the $\mathcal{P}$-convergence $u_k \xrightarrow{\mathcal{P}} u_0$, it follows in particular that $\lim_{k\to\infty} \|u_k\|_{\mathbf{E}_k} = \|u_0\|_{\mathbf{E}_0}$.

Definition 1.41. The sequence $\{u_k\}$, $u_k \in \mathbf{E}_k$, is said to be $\mathcal{P}$-compact if any of its subsequences includes a $\mathcal{P}$-convergent subsequence.

Let, further, $\mathbf{F}_0$ and $\mathbf{F}_k$, $k = 1,\dots,$ be Banach spaces; let $\mathcal{P} = (\mathcal{P}_k)$, $k = 1,2,\dots,$ be a connecting system for $\mathbf{E}_0$ and $\mathbf{E}_k$; let $\mathcal{Q} = (\mathcal{Q}_k)$, $k = 1,2,\dots,$ be a connecting system for $\mathbf{F}_0$ and $\mathbf{F}_k$; and let $A_k : \mathbf{E}_k \to \mathbf{F}_k$, $k = 0,1,\dots,$ be linear bounded operators.

Definition 1.42. A sequence $\{A_k\}$ is said to be $\mathcal{P}\mathcal{Q}$-convergent to A_0, which is denoted by $A_k \xrightarrow{\mathcal{P}\mathcal{Q}} A_0$, if the sequence $\{A_k u_k\}$ is $\mathcal{Q}$-convergent to $A_0 u_0$ for any sequence $\{u_k\}$, $u_k \in \mathbf{E}_k$, that is, $\mathcal{P}$-convergent to $u_0 \in \mathbf{E_0}$.

Theorem 1.43 (Vaĭnikko [220, Proposition 2.1]). *If* $A_k \xrightarrow{\mathcal{P}\mathcal{Q}} A_0$, *then* $\sup_k \|A_k\| < \infty$.

If a sequence $\{y_k\}$ of the elements of a Banach space converges to y_0 by the norm, we will denote this fact henceforth by $y_k \to y_0$.

Theorem 1.44 (Vaĭnikko [220, Proposition 3.5 and Theorem 4.1]). *Let the sequences* $\{B_k\}$ *and* $\{C_k\}$ *of linear bounded operators* $B_k : \mathbf{E}_k \to \mathbf{F}_k$, $C_k : \mathbf{E}_k \to \mathbf{F}_k$, $k = 1,2,\dots,$ *be* $\mathcal{P}\mathcal{Q}$*-convergent to* B_0 *and* C_0, *respectively. Let, further, the following conditions be fulfilled.*

(1) $R(B_0) = \mathbf{F}_0$, *there exist continuous inverses* B_k^{-1}, $k = 1,2,\dots,$ *and also* $\sup_k \|B_k^{-1}\| < \infty$.

(2) *The sequence* $\{C_k u_k\}$ *is* $\mathcal{Q}$*-compact for any bounded sequence* $\{u_k\}$, $u_k \in \mathbf{E}_k (\sup_k \|u_k\|_{\mathbf{E}_k} < \infty)$.

(3) *The operators* $A_k = B_k + C_k$, $k = 0,1,\dots,$ *are Fredholm ones,* $\ker A_0 = \{0\}$.

Then, for $k = 0$ *and all sufficiently large* k, *there exist bounded inverses* A_k^{-1} *and*

$$A_k^{-1} y_k \xrightarrow{\mathcal{P}} A_0^{-1} y_0 \quad \text{if } y_k \xrightarrow{\mathcal{Q}} y_0 \,(A_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} A_0^{-1}). \tag{1.145}$$

Remark 1.45. Condition (1) of Theorem 1.44 is equivalent to Condition 1*. There exist bounded inverses $B_k^{-1} : \mathbf{F}_k \to \mathbf{E}_k$, $k = 0,1,\dots,$ and also $B_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} B_0^{-1}$.

The implication $1^* \Rightarrow 1$ is obvious. Let us prove the implication $1 \Rightarrow 1^*$.

As it was shown by Vaĭnikko [220, Proposition 3.3], conditions imposed on the operators B_k guarantee the existence of a $\gamma > 0$ such that

$$||B_0 u||_{\mathbf{F}_0} \geq \gamma \|u\|_{\mathbf{E}_0} \tag{1.146}$$

for any $u \in \mathbf{E}_0$, and from $R(B_0) = \mathbf{F}_0$, there follows the existence of bounded inverse B_0^{-1}.

Let $y_k \xrightarrow{\mathcal{Q}} y_0$, $y_k \in \mathbf{F}_k$. We have

$$\begin{aligned}
&||B_k^{-1} y_k - \mathcal{P}_k B_0^{-1} y_0||_{\mathbf{E}_k} \\
&\qquad \leq ||B_k^{-1} y_k - B_k^{-1} \mathcal{Q}_k y_0||_{\mathbf{E}_k} + ||B_k^{-1} \mathcal{Q}_k y_0 - \mathcal{P}_k B_0^{-1} y_0||_{\mathbf{E}_k}, \\
&||B_k^{-1} y_k - B_k^{-1} \mathcal{Q}_k y_0||_{\mathbf{E}_k} \leq ||B_k^{-1}|| \, ||y_k - \mathcal{Q}_k y_0||_{\mathbf{F}_k} \longrightarrow 0.
\end{aligned} \tag{1.147}$$

Denote $B_0^{-1} y_0 = u_0$. Then

$$||B_k^{-1} \mathcal{Q}_k y_0 - \mathcal{P}_k B_0^{-1} y_0||_{\mathbf{E}_k} \leq ||B_k^{-1}|| \, ||\mathcal{Q}_k B_0 u_0 - B_k \mathcal{P}_k u_0||_{\mathbf{F}_k} \longrightarrow 0 \tag{1.148}$$

since $\mathcal{P}_k u_0 \xrightarrow{\mathcal{P}} u_0$ and $B_k \xrightarrow{\mathcal{P}\mathcal{Q}} B_0$.

Let $\mathbf{D}_k$ and $\mathbf{B}_k$ be Banach spaces, let $\mathbf{D}_k$ be isomorphic to the direct product $\mathbf{B}_k \times \mathbb{R}^n$, let

$$\{\Lambda_k, Y_k\} : \mathbf{B}_k \times \mathbb{R}^n \longrightarrow \mathbf{D}_k \Big([\delta_k, r_k] = \{\Lambda_k, Y_k\}^{-1} \Big) \tag{1.149}$$

be the isomorphisms, and

$$\|u\|_{\mathbf{D}_k} = ||\delta_k u||_{\mathbf{B}_k} + |r_k u|, \quad k = 0, 1, \ldots. \tag{1.150}$$

Let, further, $\mathcal{H} = (\mathcal{H}_k)$ be the connecting system for $\mathbf{B}_0$, $\mathbf{B}_k$ and let $\mathcal{P} = (\mathcal{P}_k)$ be the connecting system for $\mathbf{D}_0$, $\mathbf{D}_k$, $k = 1, 2, \ldots$. We denote by $\mathcal{H}_0$ and $\mathcal{P}_0$ the identical operators in the spaces $\mathbf{B}_0$ and $\mathbf{D}_0$, respectively.

Consider the sequences $\{\mathcal{L}_k\}$, $\{l_k\}$ of bounded linear Noether operators $\mathcal{L}_k : \mathbf{D}_k \to \mathbf{B}_k$, $\operatorname{ind} \mathcal{L}_k = n$, and bounded linear vector functionals $l_k : \mathbf{D}_k \to \mathbb{R}^n$ with linearly independent components, $k = 0, 1, \ldots$. We will assume that $\mathcal{L}_k \xrightarrow{\mathcal{P}\mathcal{H}} \mathcal{L}_0$ and that $l_k u_k \to l_0 u_0$ if $u_k \xrightarrow{\mathcal{P}} u_0$.

Let the boundary value problem

$$\mathcal{L}_0 x = f, \qquad l_0 x = \alpha \tag{1.151}$$

be uniquely solvable. Consider the question about conditions which provide the unique solvability of the problems

$$\mathcal{L}_k x = f, \qquad l_k x = \alpha \tag{1.152}$$

for all k large enough and also the convergence $x_k \xrightarrow{\mathcal{P}} x_0$ for any sequences $\{f_k\}$ and $\{\alpha_k\}$, $f_k \xrightarrow{\mathcal{H}} f_0$, $\alpha_k \to \alpha_0$. Here x_k is the solution of the problem

$$\mathcal{L}_k x = f_k, \qquad l_k x = \alpha_k \tag{1.153}$$

and x_0 is the solution of the problem

$$\mathcal{L}_0 x = f_0, \qquad l_0 x = \alpha_0. \tag{1.154}$$

We will assume the spaces $\mathbf{B}_k$, $k = 1, 2, \ldots$, to be isomorphic to $\mathbf{B}_0$ and also the operators $\mathcal{H}_k : \mathbf{B}_0 \to \mathbf{B}_k$ of the connecting system for $\mathbf{B}_0$ and $\mathbf{B}_k$ to be isomorphisms and $\sup_k \|\mathcal{H}_k^{-1}\| < \infty$.

Define the connecting system $\mathcal{Q} = (\mathcal{Q}_k)$ of the isomorphisms of the spaces $\mathbf{B}_0 \times \mathbb{R}^n$ and $\mathbf{B}_k \times \mathbb{R}^n$ by

$$\begin{aligned} \mathcal{Q}_k\{f, \alpha\} &= \{\mathcal{H}_k f, \alpha\}, \quad \{f, \alpha\} \in \mathbf{B}_0 \times \mathbb{R}^n, \\ \mathcal{Q}_k^{-1}\{f, \alpha\} &= \{\mathcal{H}_k^{-1} f, \alpha\}, \quad \{f, \alpha\} \in \mathbf{B}_k \times \mathbb{R}^n. \end{aligned} \tag{1.155}$$

Thus, if $f_k \xrightarrow{\mathcal{H}} f_0$ and $\alpha_k \to \alpha_0$, then $\{f_k, \alpha_k\} \xrightarrow{\mathcal{Q}} \{f_0, \alpha_0\}$. It is easy to see that

$$||\mathcal{Q}_k|| = \max\{||\mathcal{H}_k||, 1\}, \qquad ||\mathcal{Q}_k^{-1}|| = \max\{||\mathcal{H}_k^{-1}||, 1\}. \tag{1.156}$$

We choose the connecting system $\mathcal{P} = (\mathcal{P}_k)$ for the spaces $\mathbf{D}_0$ and $\mathbf{D}_k$ so that the operators $\mathcal{P}_k$ have bounded inverses and also $\sup_k \|\mathcal{P}_k^{-1}\| < \infty$. For instance,

$$\mathcal{P}_k = \Lambda_k \mathcal{H}_k \delta_0 + Y_k r_0 = \{\Lambda_k, Y_k\} \mathcal{Q}_k [\delta_0, r_0]. \tag{1.157}$$

Then

$$\begin{aligned} \mathcal{P}_k^{-1} &= \Lambda_0 \mathcal{H}_k^{-1} \delta_k + Y_0 r_k = \{\Lambda_0, Y_0\} \mathcal{Q}_k^{-1} [\delta_k, r_k], \\ ||\mathcal{P}_k|| &= ||\mathcal{Q}_k||, \qquad ||\mathcal{P}_k^{-1}|| = ||\mathcal{Q}_k^{-1}||. \end{aligned} \tag{1.158}$$

This system is a connecting one for $\mathbf{D}_0$ and $\mathbf{D}_k$. Really,

$$\delta_k \mathcal{P}_k u = \mathcal{H}_k \delta_0 u, \qquad r_k \mathcal{P}_k u = r_0 u \tag{1.159}$$

for any $u \in \mathbf{D}_0$. Therefore

$$||\mathcal{P}_k u||_{\mathbf{D}_k} = ||\mathcal{H}_k \delta_0 u||_{\mathbf{B}_k} + |r_0 u| \longrightarrow ||\delta_0 u||_{\mathbf{B}_0} + |r_o u| = \|u\|_{\mathbf{D}_0}. \tag{1.160}$$

(The possibility of choosing $\mathcal{P}_k$ will be considered more extensively at the end of this section.)

We will prove Theorem 1.46 under the assumptions as follows.

(a) There exists a connecting system $\mathcal{H} = (\mathcal{H}_k)$ of isomorphisms for the spaces $\mathbf{B}_0$ and $\mathbf{B}_k$ such that

$$\sup_k \|\mathcal{H}_k^{-1}\| < \infty. \tag{1.161}$$

(b) The connecting system $\mathcal{P} = (\mathcal{P}_k)$ for $\mathbf{D}_0$ and $\mathbf{D}_k$ is chosen in a way such that the operators $\mathcal{P}_k : \mathbf{B}_0 \to \mathbf{B}_k$ are isomorphisms and

$$\sup_k \|\mathcal{P}_k^{-1}\| < \infty. \tag{1.162}$$

(c) $\mathcal{L}_k \xrightarrow{\mathcal{P}\mathcal{H}} \mathcal{L}_0$ and $l_k u_k \to l_0 u_0$ if $u_k \xrightarrow{\mathcal{P}} u_0$.

Theorem 1.46. *Let problem* (1.151) *be uniquely solvable. Then problems* (1.152) *are uniquely solvable for all sufficiently large k; and for any sequences* $\{f_k\}$, $\{\alpha_k\}$, $f_k \xrightarrow{\mathcal{H}} f_0$, $\alpha_k \to \alpha_0$, *the solutions* x_k *of problems* (1.153) *are* $\mathcal{P}$*-convergent to the solution* x_0 *of problem* (1.154) *if and only if there exists a vector functional* $l : \mathbf{D}_0 \to \mathbb{R}^n$ *such that the problems*

$$\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k x = f, \qquad lx = \alpha \tag{1.163}$$

are uniquely solvable for $k = 0$ *and all sufficiently large k for any right-hand side* $\{f,\alpha\} \in \mathbf{B}_0 \times \mathbb{R}^n$ *and also the convergence* $v_k \to v_0$ *of the solutions* $v_k \in \mathbf{D}_0$ *of problems* (1.163) *holds.*

Let us rewrite problems (1.151)–(1.154) in the form

$$[\mathcal{L}_0, l_0]x = \{f,\alpha\}, \tag{1.164}$$

$$[\mathcal{L}_k, l_k]x = \{f,\alpha\}, \tag{1.165}$$

$$[\mathcal{L}_k, l_k]x = \{f_k,\alpha_k\}, \tag{1.166}$$

$$[\mathcal{L}_0, l_0]x = \{f_0,\alpha_0\}. \tag{1.167}$$

Then Theorem 1.46 may be stated as follows.

Let the operator $[\mathcal{L}_0, l_0] : \mathbf{D}_0 \to \mathbf{B}_0 \times \mathbb{R}^n$ be continuously invertible. Then theoperators $[\mathcal{L}_k, l_k] : \mathbf{D}_k \to \mathbf{B}_k \times \mathbb{R}^n$ are continuously invertible for all sufficiently large k and also

$$[\mathcal{L}_k, l_k]^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} [\mathcal{L}_0, l_0]^{-1} \tag{1.168}$$

if and only if there exists a vector functional $l : \mathbf{D}_0 \to \mathbb{R}^n$ such that the operators

$$[\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, l] : \mathbf{D}_0 \longrightarrow \mathbf{B}_0 \times \mathbb{R}^n \tag{1.169}$$

are continuously invertible for $k = 0$ and all sufficiently large k, and also

$$[\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, l]^{-1}\{f,\alpha\} \longrightarrow [\mathcal{L}_0, l]^{-1}\{f,\alpha\} \tag{1.170}$$

for any $\{f,\alpha\} \in \mathbf{B}_0 \times \mathbb{R}^n$.

Beforehand, we will prove two lemmas.

Denote $\mathcal{M}_k = \mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k$.

Lemma 1.47. *$\mathcal{M}_k u \to \mathcal{L}_0 u$ for any $u \in \mathbf{D}_0$ if and only if $\mathcal{L}_k \xrightarrow{\mathcal{P}\mathcal{H}} \mathcal{L}_0$.*

Proof. Let $\mathcal{L}_k \xrightarrow{\mathcal{P}\mathcal{H}} \mathcal{L}_0$. Since $\mathcal{P}_k u \xrightarrow{\mathcal{P}} u$ and $\sup_k \|\mathcal{H}_k^{-1}\| < \infty$, we have

$$\mathcal{M}_k u - \mathcal{L}_0 u = \mathcal{H}_k^{-1}(\mathcal{L}_k\mathcal{P}_k u - \mathcal{H}_k\mathcal{L}_0 u) \longrightarrow 0. \tag{1.171}$$

Conversely, let $\mathcal{M}_k u \to \mathcal{L}_0 u$ for any $u \in \mathbf{D}_0$ and $u_k \xrightarrow{\mathcal{P}} u_0$. We have

$$\begin{aligned}\mathcal{L}_k u_k - \mathcal{H}_k\mathcal{L}_0 u_0 &= \mathcal{H}_k\mathcal{M}_k\mathcal{P}_k^{-1}u_k - \mathcal{H}_k\mathcal{L}_0 u_0\\ &= \mathcal{H}_k\{\mathcal{M}_k(\mathcal{P}_k^{-1}u_k - u_0) + (\mathcal{M}_k u_0 - \mathcal{L}_0 u_0)\} \longrightarrow 0\end{aligned} \tag{1.172}$$

since $\mathcal{P}_k^{-1}u_k \to u_0$, $\mathcal{M}_k u_0 \to \mathcal{L}_0 u_0$, $\sup_k \|\mathcal{H}_k\| < \infty$, $\sup_k \|\mathcal{M}_k\| < \infty$. □

Denote

$$\begin{aligned}\Phi_k &= [\mathcal{L}_k, l\mathcal{P}_k^{-1}] : \mathbf{D}_k \longrightarrow \mathbf{B}_k \times \mathbb{R}^n,\\ F_k &= [\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, l] : \mathbf{D}_0 \longrightarrow \mathbf{B}_0 \times \mathbb{R}^n \quad (\Phi_0 = F_0).\end{aligned} \tag{1.173}$$

Lemma 1.48. *The operators Φ_k and F_k are continuously invertible (or not) simultaneously; $\Phi_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} \Phi_0^{-1}$ if and only if $F_k^{-1}y \to F_0^{-1}y$ for any $y \in \mathbf{B}_0 \times \mathbb{R}^n$.*

Proof. Simultaneous invertibility follows from the representation $\Phi_k = \mathcal{Q}_k F_k \mathcal{P}_k^{-1}$. Let $F_k^{-1}y \to F_0^{-1}y$ for any $y \in \mathbf{B}_0 \times \mathbb{R}^n$ and $y_k \xrightarrow{\mathcal{Q}} y_0$, $y_k \in \mathbf{B}_k \times \mathbb{R}^n$. We have

$$\Phi_k^{-1}y_k - \mathcal{P}_k\Phi_0^{-1}y_0 = \mathcal{P}_k F_k^{-1}\mathcal{Q}_k^{-1}(y_k - \mathcal{Q}_k y_0) + \mathcal{P}_k(F_k^{-1}y_0 - F_0^{-1}y_0). \tag{1.174}$$

From here, it follows that $\Phi_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} \Phi_0^{-1}$.

Conversely, let $\Phi_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} \Phi_0^{-1}$. We have

$$F_k^{-1}y - F_0^{-1}y = \mathcal{P}_k^{-1}(\Phi_k^{-1}\mathcal{Q}_k y - \mathcal{P}_k\Phi_0^{-1}y). \tag{1.175}$$

From here, $F_k^{-1}y \to F_0^{-1}y$. □

The proof of Theorem 1.46. Sufficiency. Let us represent the operator $[\mathcal{L}_k, l_k]$ in the form

$$[\mathcal{L}_k, l_k] = [\mathcal{L}_k, l\mathcal{P}_k^{-1}] + [0, l_k - l\mathcal{P}_k^{-1}]. \tag{1.176}$$

Since $\mathcal{L}_k \xrightarrow{\mathcal{P}\mathcal{H}} \mathcal{L}_0$ and $l\mathcal{P}_k^{-1}u_k \to lu_0$ if $u_k \xrightarrow{\mathcal{P}} u_0$, we have

$$\Phi_k = [\mathcal{L}_k, l\mathcal{P}_k^{-1}] \xrightarrow{\mathcal{P}\mathcal{Q}} [\mathcal{L}_0, l] = \Phi_0. \tag{1.177}$$

By virtue of Lemma 1.48, there exist, for all sufficiently large k, continuous inverses

$$\Phi_k^{-1} = [\mathcal{L}_k, l\mathcal{P}_k^{-1}]^{-1} : \mathbf{B}_k \times \mathbb{R}^n \longrightarrow \mathbf{D}_k \tag{1.178}$$

and $\Phi_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} \Phi_0^{-1}$. Thus, taking into account Theorem 1.43, condition (1) is fulfilled for the sequence $\{\Phi_k\}$.

Next consider the sequence of the operators

$$C_k = [0, l_k - l\mathcal{P}_k^{-1}] : \mathbf{D}_k \longrightarrow \mathbf{B}_k \times \mathbb{R}^n, \quad k = 1, 2, \ldots. \tag{1.179}$$

Let $u_k \xrightarrow{\mathcal{P}} u_0$ then $l_k u_k \to l_0 u_0$ due to the assumption (c) of the theorem and $l\mathcal{P}_k^{-1}u_k - lu_0 \to 0$ since $\mathcal{P}_k^{-1}u_k \to u_0$. Therefore

$$C_k \xrightarrow{\mathcal{P}\mathcal{Q}} C_0 = [0, l_0 - l]. \tag{1.180}$$

If the sequence $\{u_k\}$, $u_k \in \mathbf{D}_k$, is bounded, from the estimate

$$\left|(l_k - l\mathcal{P}_k^{-1})u_k\right| \leq ||l_k - l\mathcal{P}_k^{-1}||\,||u_k||_{\mathbf{D}_k} \tag{1.181}$$

and the boundedness in common of the norms $\|l_k - l\mathcal{P}_k^{-1}\|$, there follow boundedness in $\mathbb{R}^n$ and, consequently, compactness of the sequence $\{(l_k - l\mathcal{P}_k^{-1})u_k\}$. So, the sequence $\{C_k u_k\}$ is $\mathcal{Q}$-compact. Thus condition (2) of Theorem 1.44 is fulfilled for the operators C_k.

Further, $A_k = [\mathcal{L}_k, l_k] = \Phi_k + C_k$ are Fredholm operators, the equality $\ker A_0 = \{0\}$ follows from the unique solvability of problem (1.164). Thus, by virtue of Theorem 1.44, there exist continuous inverses $A_k^{-1} = [\mathcal{L}_k, l_k]^{-1}$ and also

$$[\mathcal{L}_k, l_k]^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} [\mathcal{L}_0, l_0]^{-1}. \tag{1.182}$$

Necessity. Let us show that we can take l_0 as the vector functional l. In other words, the operators

$$F_k = [\mathcal{H}_k^{-1}\mathcal{L}_k\mathcal{P}_k, l_0] : \mathbf{D}_0 \longrightarrow \mathbf{B}_0 \times \mathbb{R}^n \tag{1.183}$$

have, for all sufficiently large k, continuous inverses F_k^{-1} and $F_k^{-1}y \to F_0^{-1}y$ for any $y \in \mathbf{B}_0 \times \mathbb{R}^n$. By virtue of Lemma 1.48, it is sufficient to verify that for all sufficiently large k, the operators

$$\Phi_k = [\mathcal{L}_k, l_0\mathcal{P}_k^{-1}] : \mathbf{D}_k \longrightarrow \mathbf{B}_k \times \mathbb{R}^n \tag{1.184}$$

have continuous inverses with $\Phi_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} \Phi_0^{-1}$. We have

$$\Phi_k = [\mathcal{L}_k, l_k] + [0, l_0\mathcal{P}_k^{-1} - l_k]. \tag{1.185}$$

Under the condition

$$B_k = [\mathcal{L}_k, l_k] \xrightarrow{\mathcal{P}\mathcal{Q}} [\mathcal{L}_0, l_0] = B_0, \tag{1.186}$$

for $k = 0$ and all sufficiently large k, there exist continuous inverses B_k^{-1} and also $B_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} B_0^{-1}$.

Further we have

$$C_k = [0, l_0\mathcal{P}_k^{-1} - l_k] \xrightarrow{\mathcal{P}\mathcal{Q}} [0, 0] = C_0. \tag{1.187}$$

Really, if $u_k \xrightarrow{\mathcal{P}} u_0$, then

$$(l_0\mathcal{P}_k^{-1} - l_k)u_k = l_0\mathcal{P}_k^{-1}(u_k - \mathcal{P}_k u_0) - (l_k u_k - l_0 u_0) \longrightarrow 0. \tag{1.188}$$

$\mathcal{Q}$-compactness of the sequence $\{C_k u_k\}$ can be proved like it was done by the proof of sufficiency.

$\Phi_k = [\mathcal{L}_k, l_0\mathcal{P}_k^{-1}] = B_k + C_k$ are Fredholm operators and $\ker \Phi_0 = \{0\}$. Thus there exist, for all sufficiently large k, continuous inverses Φ_k^{-1} with $\Phi_k^{-1} \xrightarrow{\mathcal{Q}\mathcal{P}} \Phi_0^{-1}$. □

The condition $v_k \to v_0$ in the statement of Theorem 1.46 may be changed by another equivalent one due to Theorem 1.49.

Let $M_k : \mathbf{D}_0 \to \mathbf{B}_0$, $k = 0, 1, \ldots$, be linear bounded operators such that $M_k u \to M_0 u$ for any $u \in \mathbf{D}_0$ and let a linear bounded vector functional $l : \mathbf{D}_0 \to \mathbb{R}^n$ exist such that for each $k = 0, 1, \ldots$, the boundary value problem

$$M_k x = f, \qquad lx = \alpha \tag{1.189}$$

is uniquely and everywhere solvable. Denote by v_k the solution of this problem and denote by z_k the solution of the half-homogeneous problem

$$M_k x = f, \qquad lx = 0. \tag{1.190}$$

Let G_k be the Green operator of this problem.

Theorem 1.49. *The following assertions are equivalent.*

(a) $v_k \to v_0$ *for any* $\{f, \alpha\} \in \mathbf{B}_0 \times \mathbb{R}^n$.

(b) $\sup_k \|z_k\|_{\mathbf{D}_0} < \infty$ *for any* $f \in \mathbf{B}_0$.

(c) $G_k f \to G_0 f$ *for any* $f \in \mathbf{B}_0$.

Proof. The implication (a)⇒(b) is obvious.

The implication (b)⇒(c). The Green operator $G_k : \mathbf{B}_0 \to \ker l$ is an inverse to $M_k : \ker l \to \mathbf{B}_0$. From (b), it follows that $\sup_k \|G_k\| < \infty$. Thus, by virtue of Remark 1.45, we have (c).

Implication (c)⇒(a). The solution v_k has the representation

$$v_k = G_k f + X_k \alpha, \tag{1.191}$$

where X_k is the fundamental vector of the equation $M_k x = 0$ and also $lX_k = E$. By virtue of Theorem 1.19,

$$X_k = U - G_k M_k U, \tag{1.192}$$

where $U = (u_1, \dots, u_n)$, $u_i \in \mathbf{D}_0$, $lU = E$. Thus $X_k \alpha \to X_0 \alpha$ for any $\alpha \in \mathbb{R}^n$, and, consequently, $v_k \to v_0$. □

Next we dwell on the question of choosing the connecting systems of isomorphisms $\mathcal{H}_k : \mathbf{B}_0 \to \mathbf{B}_k$ and $\mathcal{P}_k : \mathbf{D}_0 \to \mathbf{D}_k$. It is natural to subordinate the operators $\mathcal{P}_k$ and $\mathcal{H}_k$ to the following requirement:

$$u_k \xrightarrow{\mathcal{P}} u_0 \Longleftrightarrow \delta_k u_k \xrightarrow{\mathcal{H}} \delta_0 u_0, \qquad r_k u_k \longrightarrow r_0 u_0. \tag{1.193}$$

Theorem 1.50. *Let*

$$\left\|(\mathcal{H}_k \delta_0 - \delta_k \mathcal{P}_k) u\right\|_{\mathbf{B}_k} \longrightarrow 0, \quad (r_0 - r_k \mathcal{P}_k) u \longrightarrow 0, \quad \forall u \in \mathbf{D}_0. \tag{1.194}$$

Then (1.193) *holds.*

Proof. The assertion follows from the inequalities

$$\begin{aligned}
\left\|\delta_k u_k - \mathcal{H}_k \delta_0 u_0\right\|_{\mathbf{B}_k} &\le \left\|\delta_k (u_k - \mathcal{P}_k u_0)\right\|_{\mathbf{B}_k} + \left\|(\delta_k \mathcal{P}_k - \mathcal{H}_k \delta_0) u_0\right\|_{\mathbf{B}_k}, \\
\left| r_k u_k - r_0 u_0 \right| &\le \left| r_k (u_k - \mathcal{P}_k u_0) \right| + \left| (r_k \mathcal{P}_k - r_0) u_0 \right|, \\
\left\|u_k - \mathcal{P}_k u_0\right\|_{\mathbf{D}_k} &= \left\|\delta_k (u_k - \mathcal{P}_k u_0)\right\|_{\mathbf{B}_k} + \left| r_k (u_k - \mathcal{P}_k u_0) \right| \\
&\le \left\|\delta_k u_k - \mathcal{H}_k \delta_0 u_0\right\|_{\mathbf{B}_k} + \left\|(\mathcal{H}_k \delta_0 - \delta_k \mathcal{P}_k) u_0\right\|_{\mathbf{B}_k} \\
&\quad + \left| r_k u_k - r_0 u_0 \right| + \left| (r_0 - r_k \mathcal{P}_k) u_0 \right|.
\end{aligned} \tag{1.195}$$

□

Conversely, if $\delta_k u_k \xrightarrow{\mathcal{H}} \delta_0 u_0$ and $r_k u_k \to r_0 u_0$, where $u_k \xrightarrow{\mathcal{P}} u_0$, the limiting relations (1.194) are fulfilled. This follows from $\mathcal{P}_k u \xrightarrow{\mathcal{P}} u$.

Thus, (1.193) are fulfilled if and only if the limiting relations (1.194) hold, in particular, if $\delta_k\mathcal{P}_k = \mathcal{H}_k\delta_0$ and $r_k\mathcal{P}_k = r_0$. Applying Λ_k to the first of these relations, we get

$$(I - Y_k r_k)\mathcal{P}_k = \Lambda_k\mathcal{H}_k\delta_0. \tag{1.196}$$

From here, taking into account the second equality, we obtain

$$\mathcal{P}_k = \Lambda_k\mathcal{H}_k\delta_0 + Y_k r_0 = \{\Lambda_k, Y_k\}\mathcal{Q}_k[\delta_0, r_0]. \tag{1.197}$$

The main statements of the theory of linear abstract differential equations were published by Anokhin [8, 10], Azbelev and Rakhmatullina [188], Anokhin [9], Azbelev et al. [32, 33], and by Anokhin and Rakhmatullina [11]. Applications of the assertions of Chapter 1 to some questions of the operator theory are considered by Islamov in [102, 103].

2 Equations in traditional spaces

2.1. Introduction

The first two sections of the chapter are devoted to the systems of linear functional differential equations and the scalar equations of the nth order. The theory of this generalization of the ordinary differential equations has been worked out by a large group of mathematicians united in 1975 by the so-called "Perm Seminar on Functional Differential Equations" at Perm Polytechnic Institute. The primary interest of the seminar arose while trying to clear out the numerous publications on the equations with deviated argument. Most parts of the publications were based on the conception accepted by Myshkis [163], Krasovskii [121], and Hale [98]. This conception was reasoned from a special definition of the solution as a continuous prolongation of the "initial function" by virtue of the equation. In the case of retarded equations, such a definition met no objection while the initial Cauchy problem was considered. The complications began to arise in studies of Cauchy problem with impulse impacts and particularly while studying the boundary value problems. In the case of general deviation of the argument, even simple linear equations have entirely no solution under such a definition. There is a considerable survey by Myshkis [165] of very extensive literature based on the conception above.

In [23, 34], a slight generalization of the notion of the solution was suggested. This generalization led to a more perfect conception which met no contradiction with the traditional one but simplified essentially some constructions. The new conception is natural and effective due to the description of the equation with deviated argument using the *composition operator* defined on the set of functions, $x : [a, b] \to \mathbb{R}^n$, by

$$(S_h x)(t) = \begin{cases} x[h(t)] & \text{if } h(t) \in [a, b], \\ 0 & \text{if } h(t) \notin [a, b]. \end{cases} \tag{2.1}$$

The new conception has led the seminar in a natural way to a richness in content general theory of the equation

$$\mathcal{L}x = f, \tag{2.2}$$

with the linear operator $\mathcal{L}$ defined on the Banach space of the absolutely continuous functions. This theory is treated below on the ground of the further generalization to which Chapter 1 was devoted. Such an approach shortens the presentation of the matter and allows us to consider wide classes of the problems from the unified point of view.

Much attention is given in Section 2.3 to the property of the fixed sign of Green function (to the problem of the validity of the functional differential analog to the Chaplygin theorem on differential inequality).

The third section is devoted to a new conception of the stability of solutions to the equations with aftereffect. It is emphasized that the new conception does not contradict to the classical one.

Some characteristics of equations with aftereffect are connected with the situation, where the principal part $Q : \mathbf{L} \to \mathbf{L}$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ is Volterra by Tikhonov [215] and at the same time the inverse Q^{-1} is also Volterra. In Section 2.5, written by S. A. Gusarenko, the results are treated on preserving the mentioned characteristics when a more general conception of Volterra operators is accepted.

2.2. Equations in the space of absolutely continuous functions

2.2.1. Equations with deviated argument and their generalization

For any absolutely continuous function $x : [a,b] \to \mathbb{R}^n$, the identity

$$x(t) = \int_a^t \dot{x}(s)ds + x(a) \tag{2.3}$$

holds. Therefore, the space $\mathbf{D}$ of such functions is isomorphic to the direct product $\mathbf{L} \times \mathbb{R}^n$, where $\mathbf{L}$ is the Banach space of summable functions $z : [a,b] \to \mathbb{R}^n$ under the norm

$$\|z\|_{\mathbf{L}} = \int_a^b ||z(s)||_{\mathbb{R}^n} ds. \tag{2.4}$$

If

$$\|x\|_{\mathbf{D}} = \|\dot{x}\|_{\mathbf{L}} + ||x(a)||_{\mathbb{R}^n}, \tag{2.5}$$

the space $\mathbf{D}$ is Banach. The isomorphism $\mathcal{J} : \mathbf{L} \times \mathbb{R}^n \to \mathbf{D}$ in this case may be defined by

$$x(t) = \int_a^t z(s)ds + \beta, \quad \{z, \beta\} \in \mathbf{L} \times \mathbb{R}^n. \tag{2.6}$$

Therefore, the linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ as well as linear vector functional $l : \mathbf{D} \to \mathbb{R}^m$ may be represented in the form

$$\begin{aligned} \mathcal{L}x &= Q\dot{x} + Ax(a), \\ lx &= \int_a^b \Phi(s)\dot{x}(s)ds + \Psi\alpha. \end{aligned} \tag{2.7}$$

Here $Q : \mathbf{L} \to \mathbf{L}$ is the principal part of $\mathcal{L}$, which is defined by $Q = \mathcal{L}\Lambda$ $((\Lambda z)(t) = \int_a^t z(s)ds)$, the finite dimensional part $A : \mathbb{R}^n \to \mathbf{L}$ is defined by $(A\alpha)(t) = (\mathcal{L}E)(t)\alpha$ (here and below E is the identity $n\times n$ matrix), and $m\times n$ matrix Φ has measurable essentially bounded elements and may be constructed from the equality

$$l\left(\int_a^t z(s)ds\right) = \int_a^b \Phi(s)z(s)ds. \tag{2.8}$$

Any column of the $m \times n$ matrix Ψ is the result of application of the vector functional l to the corresponding column of the identity matrix E. Namely, $\Psi = lE$.

The general theory of Chapter 1 is applicable to the equation $\mathcal{L}x = f$ with linear $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ if $\mathcal{L}$ is bounded, is Noether with ind $\mathcal{L} = n$, or, what is the same, the principal part $Q = \mathcal{L}\Lambda : \mathbf{L} \to \mathbf{L}$ of $\mathcal{L}$ is Fredholm.

The differential equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + P(t)x(t) = f(t), \tag{2.9}$$

with the columns of the $n \times n$ matrix P from $\mathbf{L}$, as well as the generalization of the equation in the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + \int_a^b d_sR(t,s)x(s) = f(t); \tag{2.10}$$

under the assumption that the elements $r_{ij}(t,s)$ of the $n\times n$ matrix $R(t,s)$ are measurable in the square $[a,b]\times[a,b]$, the functions $r_{ij}(\cdot,s)$ for each $s \in [a,b]$ and the functions $\operatorname{var}_{s\in[a,b]} r_{ij}(\cdot,s)$ are summable on $[a,b]$, $R(t,b) \equiv 0$; are representatives of the equation $\mathcal{L}x = f$ with a Fredholm operator $\mathcal{L}\Lambda$.

Under the above assumptions, the operators $T : \mathbf{D} \to \mathbf{L}$ and $R : \mathbf{L} \to \mathbf{L}$, defined by

$$(Tx)(t) = \int_a^b d_sR(t,s)x(s), \tag{2.11}$$

$$(Rz)(t) = \int_a^b R(t,s)z(s)ds, \tag{2.12}$$

are compact. This follows from Theorem B.1.

The equation (2.10) takes the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) - \int_a^b R(t,s)\dot{x}(s)ds - R(t,a)x(a) = f(t) \tag{2.13}$$

after integration by parts of the Stiltjes integral. Thus the principal part of such an operator $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ has the form

$$Qz = z - Rz. \tag{2.14}$$

If the isomorphism $\mathcal{J} : \mathbf{L} \times \mathbb{R}^n \to \mathbf{D}$ is defined by (2.6), the principal boundary value problem for the equation $\mathcal{L}x = f$ is the Cauchy one:

$$\mathcal{L}x = f, \qquad lx \stackrel{\text{def}}{=} x(a) = \alpha. \tag{2.15}$$

By Theorem 1.16, this problem is uniquely solvable if and only if the principal part Q of $\mathcal{L}$ has the bounded inverse $Q^{-1} : \mathbf{L} \to \mathbf{L}$. Besides, the solution of the problem (the general solution of the equation) has the form

$$x(t) = \int_a^t (Q^{-1}f)(s)ds + \left[E - \int_a^t (Q^{-1}A)(s)ds\right]\alpha = (Gf)(t) + (X\alpha)(t). \tag{2.16}$$

Here $A = \mathcal{L}E$. The existence of the inverse Q^{-1} is equivalent to unique solvability of the equation $z = Rz + f$ in the space $\mathbf{L}$. Let it be unique solvable. Then

$$(Q^{-1}f)(t) = f(t) + \int_a^b H(t,s)f(s)ds. \tag{2.17}$$

Thus the solution of the Cauchy problem of (2.10) with $x(a) = 0$ is defined by

$$x(t) = (Gf)(t) = \int_a^t \left[f(s) + \int_a^b H(s,\tau)f(\tau)d\tau\right]ds. \tag{2.18}$$

By changing the integration order in the integral

$$\int_a^t \left\{\int_a^b H(s,\tau)f(\tau)d\tau\right\}ds, \tag{2.19}$$

we obtain the representation of the Green operator

$$(Gf)(t) = \int_a^b \left[\chi(t,s)E + \int_a^t H(\tau,s)\,d\tau\right]f(s)ds, \tag{2.20}$$

where $\chi(t,s)$ is the characteristic function of the triangle $a \le s \le t \le b$. Thus

$$G(t,s) = \chi(t,s)E + \int_a^t H(\tau,s)d\tau. \tag{2.21}$$

There is an extensive literature of the latter decades devoted to the equation with deviated argument,

$$\begin{gathered} \dot{x}(t) + P(t)x[h(t)] = v(t), \quad t \in [a,b], \\ x(\xi) = \varphi(\xi) \quad \text{if } \xi \notin [a,b], \end{gathered} \tag{2.22}$$

and some generalizations of this equation (see, for instance, [164, 165]). The second row in (2.22) is necessary in order to determine the value of $x[h(t)]$ when some values of h do not belong to $[a,b]$. The given function φ is called the initial one. In order to rewrite (2.22) in the form $\mathcal{L}x = f$ with linear $\mathcal{L} : \mathbf{D} \to \mathbf{L}$, we will introduce the notations

$$\begin{aligned} (S_h x)(t) &= \begin{cases} x[h(t)] & \text{if } h(t) \in [a,b], \\ 0 & \text{if } h(t) \notin [a,b], \end{cases} \\ \varphi^h(t) &= \begin{cases} 0 & \text{if } h(t) \in [a,b], \\ \varphi[h(t)] & \text{if } h(t) \notin [a,b]. \end{cases} \end{aligned} \tag{2.23}$$

Then (2.22) takes the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + P(t)(S_h x)(t) = f(t), \tag{2.24}$$

where $f(t) = v(t) - P(t)\varphi^h(t)$. Since

$$P(t)(S_h x)(t) = \int_a^b d_s R(t,s)x(s), \tag{2.25}$$

if $R(t,s) = -P(t)\chi_h(t,s)$, where $\chi_h(t,s)$ is the characteristic function of the set

$$\{(t,s) \in [a,b] \times [a,b] : a \leq s \leq h(t) < b\} \cup \{(t,s) \in [a,b] \times [a,b) : h(t) = b\}, \tag{2.26}$$

the equation (2.22) is of the form (2.10) if the elements of $n \times n$ matrix P are summable and $h : [a,b] \to \mathbb{R}^1$ is measurable. Sometimes we will designate the value of the composition operator S_h on the function x briefly as x_h and rewrite (2.22) in the form

$$\dot{x}(t) + P(t)x_h(t) = f(t). \tag{2.27}$$

The authors of numerous articles and monographs define the notion of the solution of (2.22) as a continuous prolongation onto $[a,b]$ of the initial function φ on the strength of the equation. More precisely, the mentioned authors define the solution of (2.22) as an absolutely continuous function $x : [a,b] \to \mathbb{R}^n$ that satisfies the equation and the boundary value conditions $x(a) = \varphi(a)$, $x(b) = \varphi(b)$. In this event, the number of the boundary value conditions $m = 2n > n$. The problem

$$\mathcal{L}x = f, \qquad x(a) = \varphi(a), \qquad x(b) = \varphi(b) \tag{2.28}$$

is not a Fredholm one because the index of such a problem is equal to $n - m = -n$. As it was shown in Section 1.4, problem (2.28) is solvable only for special f, $\varphi(a)$, $\varphi(b)$. Thus, (2.22) with the additional demand of continuous matching between function φ and solution x is, generally speaking, not solvable, even in the case $\dim\ker\mathcal{L} = n$. It should be noticed that, by Theorem 1.17, (2.22) under the condition $\dim\ker\mathcal{L} = n$ is solvable for any f without additional continuous matching conditions.

The requirement of continuous matching conditions $x(a) = \varphi(a)$, $x(b) = \varphi(b)$ had been involving numerous difficulties in attempts to outline a general theory of (2.22) even in the case $h(t) \leq t$ when (2.28) becomes a Cauchy problem, as well as by solving various applied problems connected with (2.22). Beginning with the works of Azbelev et al. [23], and Azbelev and Rakhmatullina [34], the participants of the Tambov Seminar did away with the requirement of the continuous matching, introduced the composition operator defined by (2.23), and began to use the form (2.10) for the equation (2.22). As a result, the fundamentals of the modern theory of the equations with deviating argument were accomplished to the middle of seventies. The boundary value problem has occupied the central point in this theory. A more detailed description of the development of the notion of the solution to the equation with deviated argument can be found in Rakhmatullina [187] and Azbelev et al. [32, 33].

2.2.2. The Green matrix

Consider the general linear boundary value problem

$$\mathcal{L}x = f, \qquad lx = \alpha, \tag{2.29}$$

where $l : \mathbf{D} \to \mathbb{R}^n$ is a linear bounded vector functional with linearly independent components.

If the problem (2.29) has a unique solution for each $\{f, \alpha\} \in \mathbf{L} \times \mathbb{R}^n$, the solution is defined by

$$x = Gf + X\alpha. \tag{2.30}$$

By Theorem 1.31, the Green operator $G : \mathbf{L} \to \mathbf{D}$ of the problem (2.29) is integral since the Green operator Λ in isomorphism (2.6) is integral. The kernel $G(t, s)$ of the Green operator

$$(Gf)(t) = \int_a^b G(t, s) f(s)\, ds \tag{2.31}$$

is called the *Green matrix* (see Azbelev et al. [32, 33]). The finite-dimensional operator $X : \mathbb{R}^n \to \mathbf{D}$ is defined by $n \times n$ matrix $X(t)$, the columns of which constitute a system of n linearly independent solutions of the homogeneous equation $\mathcal{L}x = 0$.

In order to investigate the Green operator in detail, it is convenient to introduce a special integral operator $W_l : \mathbf{L} \to \{x \in \mathbf{D} : lx = 0\}$ corresponding to the given vector functional l. Such an operator is defined by

$$(W_l z)(t) = \int_a^t z(s)ds - U(t)\int_a^b \Phi(s)z(s)ds, \tag{2.32}$$

where Φ is the $n \times n$ matrix from the representation (2.7) of the vector functional l, and U is an $n \times n$ matrix with the columns from $\mathbf{D}$ such that $lU = E$ and $\det U(a) \neq 0$.

Lemma 1.21 asserts the existence of such a matrix for any bounded vector functional l and, by virtue of Theorem 1.22, W_l is the Green operator of the "primary boundary value problem"

$$\mathcal{L}_0 x = z, \qquad lx = 0, \tag{2.33}$$

where

$$(\mathcal{L}_0 x)(t) = \dot{x}(t) - \dot{U}(t)U^{-1}(a)x(a). \tag{2.34}$$

The use of the "W-substitution" $x = W_l z$ to the equation $\mathcal{L}x = f$ leads to the equation

$$(\mathcal{L}W_l z)(t) \equiv (Qz)(t) - (\mathcal{L}U)(t)\int_a^b \Phi(s)z(s)ds = f(t) \tag{2.35}$$

with respect to z.

Define the degenerate operator $F : \mathbf{L} \to \mathbf{L}$ by

$$(Fz)(t) = (\mathcal{L}U)(t)\int_a^b \Phi(s)z(s)ds, \tag{2.36}$$

and rewrite (2.35) in the form

$$\mathcal{L}W_l z \equiv (Q - F)z = f. \tag{2.37}$$

The problem $\mathcal{L}x = f$, $lx = 0$ is equivalent to the equation in the following sense. Between the set of solutions $x \in \mathbf{D}$ of the problem and the set of solutions $z \in \mathbf{L}$ of (2.35), there is a one-to-one mapping defined by

$$x = W_l z, \quad z = \mathcal{L}_0 x. \tag{2.38}$$

The paraphrase of Theorem 1.25 and its Corollary 1.26 as applied to the concrete space $\mathbf{D}$ allows us to formulate the following assertion.

Theorem 2.1. *The boundary value problem* (2.29) *is uniquely solvable for each* $\{f, \alpha\} \in \mathbf{L} \times \mathbb{R}^n$ *if and only if there exists the bounded inverse* $(Q - F)^{-1} : \mathbf{L} \to \mathbf{L}$. *Therewith, the Green operator of the problem has the representation*

$$G = W_l(Q - F)^{-1}. \tag{2.39}$$

Let us dwell on the boundary value problem for the equation (2.10). In this case, $Qz = z - Rz$, where R is defined by (2.12). Let $K = R + F$. Then

$$(Kz)(t) = \int_a^b K(t,s)z(s)ds,$$
$$K(t,s) = R(t,s) + (\mathcal{L}U)(t)\Phi(s). \tag{2.40}$$

The equation (2.35) takes the form

$$z(t) = \int_a^b K(t,s)z(s)ds + f(t). \tag{2.41}$$

The sum $K = R + F$ of the compact R and the degenerated F is also compact. If $Q - F = I - K$ has the bounded inverse, $(I - K)^{-1} = I + H$, where

$$(Hf)(t) = \int_a^b H(t,s)f(s)ds \tag{2.42}$$

is compact. Thus

$$(Gf)(t) = \int_a^b W_l(t,s)\left\{f(s) + \int_a^b H(s,\tau)f(\tau)d\tau\right\}ds \tag{2.43}$$

and we obtain the following representation of the Green matrix:

$$\begin{aligned} G(t,s) &= W_l(t,s) + \int_a^b W_l(t,\tau)H(\tau,s)d\tau \\ &= \chi(t,s)E - U(t)\Phi(s) + \int_a^t H(\tau,s)d\tau - U(t)\int_a^b \Phi(\tau)H(\tau,s)d\tau, \end{aligned} \tag{2.44}$$

where $\chi(t,s)$ is the characteristic function of the set $\{(t,s) \in [a,b] \times [a,b] : a \le s \le t \le b\}$. On the base of this representation, we have the following assertion on the properties of the Green matrix.

Theorem 2.2. *The Green matrix $G(t,s)$ of the boundary value problem for the equation* (2.10) *has the following properties.*

(a) $G(\cdot,s)$ *is absolutely continuous on* $[a,s)$ *and* $(s,b]$ *for almost each* $s \in [a,b]$. *Besides*

$$G(s+0,s) - G(s-0,s) = E. \tag{2.45}$$

(b) *One has*

$$\frac{d}{dt}\int_a^b G(t,s)f(s)ds = f(t) + \int_a^b \frac{\partial}{\partial t}G(t,s)f(s)ds \tag{2.46}$$

for any $f \in \mathbf{L}$.

(c) $G(\cdot, s)$ *satisfies the equalities*

$$\frac{\partial}{\partial t}G(t,s) - \int_a^b R(t,\tau)\frac{\partial}{\partial \tau}G(\tau,s)d\tau - R(t,a)G(a,s) = R(t,s),$$
$$\int_a^b \Phi(\tau)\frac{\partial}{\partial \tau}G(\tau,s)d\tau + \Psi G(a,s) = -\Phi(s) \tag{2.47}$$

for almost each $s \in [a,b]$.

Proof. The assertions (a) and (b) follow at once from (2.44). The assertion (c) can be established by the substitution of $x(t) = \int_a^b G(t,s)f(s)ds$ into equation (2.10) and the boundary conditions. □

Remark 2.3. Theorem 2.2 is valid for the equation of the form

$$\dot{x}(t) - (M\dot{x})(t) - A(t)x(a) = f(t) \tag{2.48}$$

if the operator $M : \mathbf{L} \to \mathbf{L}$ is weakly compact [32, 33, Theorem 3.4.2].

Let us go a bit into the equation of more general form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} [(I-S)\dot{x}](t) + \int_a^b d_sR(t,s)x(s) = f(t), \tag{2.49}$$

where $(Sz)(t) = \sum_{i=1}^m B_i(t)(S_{g_i}z)(t)$, and the composition operators $S_{g_i} : \mathbf{L} \to \mathbf{L}$ are defined by (2.23). The operator $S : \mathbf{L} \to \mathbf{L}$ is bounded if the elements of the matrices B_i are measurable and essentially bounded and the functions g_i guarantee the action of the operators S_{g_i} in the space $\mathbf{L}$. By Theorem C.1, the operator S_g maps $\mathbf{L}$ into itself continuously if and only if

$$\mu \sup_{\substack{e\subset[a,b]\\ \operatorname{mes} e>0}} \frac{\operatorname{mes} g^{-1}(e)}{\operatorname{mes} e} < \infty \tag{2.50}$$

and therewith $\mu = \|S_g\|_{\mathbf{L}\to\mathbf{L}}$. By Theorem C.9, the operator $S : \mathbf{L} \to \mathbf{L}$ (if it differs from the null operator) cannot be compact. Under the above assumptions, the principal part $Q = I - S - R$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ is Fredholm if and only if there exists the bounded inverse $(I-S)^{-1} : \mathbf{L} \to \mathbf{L}$, see [18, 62, 64, 222]. The boundary value problem for the equation (2.49) is not reducible to the integral equation: here we have the functional equation $z = (K+S)z + f$ with compact K instead of an integral equation.

2.2.3. Equations with aftereffect

A special place in theory as well as in application is occupied by the *equations with aftereffect*, that is, by the equations with Volterra $\mathcal{L}$.

Let us call to mind that the linear operator $V : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are linear spaces of measurable on $[a,b]$ n-dimensional vector functions, is called *Volterra* operator if, for each $c \in (a,b)$ and any $x \in \mathbf{X}$ such that $x(t) \equiv 0$ on $[a,c]$, we have $(Vx)(t) = 0$ on $[a,c]$.

The equation (2.22) will be the one with Volterra $\mathcal{L}$ if $g_i(t) \le t$, $t \in [a,b]$, $i = 1,\dots,m$, and $R(t,s) = 0$ at $a \le t < s \le b$. If, besides, the isomorphism $\mathcal{J} : \mathbf{L} \times \mathbb{R}^n \to \mathbf{D}$ is defined by (2.6), the principal part $Q = I - (R+S)$ for (2.49) is Volterra. Under the assumption that the spectral radius of $(R+S) : \mathbf{L} \to \mathbf{L}$ is less than 1, there exists the Volterra inverse $Q^{-1} = I + (R+S) + (R+S)^2 + \cdots$. In this event, the Green operator of the Cauchy problem is said to be the *Cauchy operator* C and its kernel is called the *Cauchy matrix* which will be denoted by $C(t,s)$:

$$(Cf)(t) \stackrel{\text{def}}{=} \int_a^t (Q^{-1}f)(s)ds = \int_a^t C(t,s)f(s)ds. \tag{2.51}$$

In [24], it is shown that the spectral radius of Volterra $(R+S) : \mathbf{L} \to \mathbf{L}$ is equal to the spectral radius $\rho(S)$ of $S : \mathbf{L} \to \mathbf{L}$. In [24, 72], there are proposed some upper estimates of $\rho(S)$. We will cite one of the estimates from [32, Theorem 5.2.4].

For a fixed $\tau_i > 0$, define the set ω_i by

$$\omega_i = \{t \in [a,b] : t - g_i(t) \le \tau_i,\ g_i(t) \in [a,b]\}, \quad i = 1,\dots,m. \tag{2.52}$$

Then

$$\rho(S) \le \sum_{i=1}^m \mu_i \operatorname*{ess\,sup}_{t\in\omega_i} \|B_i(t)\|, \tag{2.53}$$

where $\mu = \|S_g\|_{\mathbf{L}\to\mathbf{L}}$, $\|B(t)\|$ is the norm of the matrix $B(t)$ agreed with the norm of $\mathbb{R}^n$. We have in mind that $\operatorname{ess\,sup}_{t\in\omega} \varphi(t) = 0$ if ω is empty.

From this it follows, in particular, that the existence of a constant $\tau > 0$ such that $t - g_i(t) \ge \tau$, $i = 1,\dots,m$, $t \in [a,b]$, provides the equality $\rho(S) = 0$.

Let us dwell on specific properties of the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + \int_a^t d_sR(t,s)x(s) = f(t) \tag{2.54}$$

with Volterra $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ under the assumption that $R(t,t) = 0$. It is a natural generalization of (2.22) with delay ($h(t) \le t$).

The spectral radius of the compact Volterra operator

$$(Rz)(t) = \int_a^t R(t,s)z(s)ds \tag{2.55}$$

is equal to zero, see, for instance, [229]. Therefore, the Cauchy problem for (2.54) is uniquely solvable and, besides,

$$Q^{-1}f = f + Rf + R^2f + \cdots = f + Hf, \tag{2.56}$$

where

$$(Hf)(t) = \int_a^t H(t,s)f(s)ds. \tag{2.57}$$

Thus the Cauchy operator for (2.54) is integral Volterra and the Cauchy matrix $C(t,s)$ defined by (2.21) has in this case the form

$$C(t,s) = E + \int_s^t H(\tau,s)d\tau, \quad a \le s \le t \le b. \tag{2.58}$$

The Cauchy matrix $C(t,s)$ is absolutely continuous with respect to $t \in [s,b]$ by virtue of the fact that $H(\tau,s)$ is summable for each $s \in [a,b)$. Therefore,

$$\frac{\partial}{\partial t}C(t,s) = H(t,s), \quad C(s,s) = E, \quad t \in [s,b], \tag{2.59}$$

holds at each $s \in [a,b]$.

For each fixed $s \in [a,b)$, we can write

$$\frac{\partial}{\partial t}C(t,s) = -\int_s^t d_\tau R(t,\tau)C(\tau,s), \quad t \in [s,b]. \tag{2.60}$$

Really, the kernels $R(t,s)$ and $H(t,s)$ are connected by the known equality

$$H(t,s) = \int_s^t R(t,\tau)H(\tau,s)d\tau + R(t,s). \tag{2.61}$$

Therefore, from (2.58) and (2.59), we have

$$\frac{\partial}{\partial t}C(t,s) = \int_s^t R(t,\tau)\frac{\partial}{\partial \tau}C(\tau,s)d\tau + R(t,s) = -\int_s^t d_\tau R(t,\tau)C(\tau,s). \tag{2.62}$$

For each fixed $s \in [a,b)$, the general solution of the equation

$$\dot{y}(t) + \int_s^t d_\tau R(t,\tau)y(\tau) = f(t), \quad t \in [s,b], \tag{2.63}$$

has the representation

$$y(t) = \int_s^t C(t,\tau)f(\tau)d\tau + C(t,s)y(s). \tag{2.64}$$

Really, the matrix $C(t,s)$ is the fundamental one for (2.63), besides, $C(s,s) = E$. Let us show that the function

$$v(t) = \int_s^t C(t,\tau)f(\tau)d\tau \tag{2.65}$$

satisfies (2.63). In fact, the equality (2.46) for (2.54) has the form

$$\frac{d}{dt}\int_a^t C(t,\tau)f(\tau)d\tau = \int_a^t \frac{\partial}{\partial t}C(t,\tau)f(\tau)d\tau + f(t), \quad t \in [a,b]. \tag{2.66}$$

Let $f(t)$ be defined on $[s,b]$ and prolonged on $[a,s)$ as zero. Then

$$\frac{d}{dt}\int_s^t C(t,\tau)f(\tau)d\tau = \int_s^t \frac{\partial}{\partial t}C(t,\tau)f(\tau)d\tau + f(t), \quad t \in [s,b]. \tag{2.67}$$

Using this equality and (2.60), we have

$$\begin{aligned}
\dot{v}(t) &+ \int_s^t d_\xi R(t,\xi)v(\xi) \\
&= f(t) + \int_s^t \frac{\partial}{\partial t}C(t,\tau)f(\tau)d\tau + \int_s^t d_\xi R(t,\xi)\left\{\int_s^\xi C(\xi,\tau)f(\tau)\,d\tau\right\} \\
&= f(t) - \int_s^t \left\{\int_\tau^t d_\xi R(t,\xi)C(\xi,\tau)\right\}f(\tau)d\tau \\
&\quad + \int_s^t d_\xi R(t,\xi)\left\{\int_s^\xi C(\xi,\tau)f(\tau)d\tau\right\} = f(t).
\end{aligned} \tag{2.68}$$

The latter equality is established here by immediate integration by parts of both Stiltjes integrals.

The representation (2.64) is called the Cauchy formula. In the case $s = a$, we obtain from (2.64) the representation

$$x(t) = \int_a^t C(t,s)f(s)ds + C(t,a)x(a) \tag{2.69}$$

of the general solution of (2.54).

It should be noticed that the Cauchy matrix $C(t,s)$ for the differential equation

$$\dot{x}(t) + P(t)x(t) = f(t) \tag{2.70}$$

(and only for such an equation) is connected with the fundamental matrix $X(t)$ by

$$C(t,s) = X(t)X^{-1}(s). \tag{2.71}$$

As for the properties of the Cauchy matrix and the Cauchy formula of the representation of the general solution in the general case of the equation with aftereffect, we will restrict ourselves to the following.

Let $Q : \mathbf{L} \to \mathbf{L}$ be a linear bounded Volterra operator. As it is known (see [109]), such an operator has the representation

$$(Qz)(t) = \frac{d}{dt}\int_a^t Q(t,s)z(s)ds. \tag{2.72}$$

Let, further, $A : \mathbb{R}^n \to \mathbf{L}$ be a linear bounded finite-dimensional operator. The equality

$$\mathcal{L}x = Q\dot{x} + Ax(a) \tag{2.73}$$

defines a linear bounded Volterra $\mathcal{L} : \mathbf{D} \to \mathbf{L}$.

We will assume below the existence of the bounded Volterra inverse $Q^{-1} : \mathbf{L} \to \mathbf{L}$. Thus

$$(Q^{-1}f)(t) = \frac{d}{dt}\int_a^t C(t,s)f(s)ds. \tag{2.74}$$

Now we can see that any $n \times n$ matrix $C(t,s)$, which defines the bounded operator (2.74) that acts in the space $\mathbf{L}$ and has the bounded Volterra inverse Q, is the Cauchy matrix for the equation $Q\dot{x} + Ax(a) = f$ for any A.

Let the columns of $n \times n$ matrix $X(t)$ belong to $\mathbf{D}$ and let $C(t,s)$ be a matrix such that the operator Q^{-1} defined by (2.74) is bounded and has bounded Volterra inverse $Q : \mathbf{L} \to \mathbf{L}$. Then the equality

$$x(t) = \int_a^t C(t,s)f(s)ds + X(t)x(a) \tag{2.75}$$

defines the general solution of the equation $Q\dot{x} + Ax(a) = f$. Here the operator $A : \mathbb{R}^n \to \mathbf{L}$ corresponds to the equality

$$X(t) = E - \int_a^t C(t,s)(AE)(s)ds. \tag{2.76}$$

It is relevant to remark that the matrices above, $Q(t,s)$ and $C(t,s)$, have similar properties and $Q(t,s)$ is also the matrix Cauchy for an equation $\mathcal{L}x = f$ such that $\mathcal{L}\Lambda = Q^{-1}$.

2.2.4. Control problems

Consider the Cauchy problem

$$\mathcal{L}x = f, \qquad x(a) = \alpha \tag{2.77}$$

under the assumption that the principal part Q of operator $\mathcal{L}$ has the bounded inverse $Q^{-1} : \mathbf{L} \to \mathbf{L}$ and the right-hand side f has the form $f = v + Bu$, where $v \in \mathbf{L}$ is a given function, and B is a given linear bounded operator mapping a Banach space $\mathbf{U}^r$ of functions $u : [a,b] \to \mathbb{R}^r$ into the space $\mathbf{L}$.

The equation

$$\mathcal{L}x = v + Bu \tag{2.78}$$

is called the control system with an eye to influence on the state x by the function $u \in \mathbf{U}^r$ called the control (the control action). As a rule, $r < n$, $B\mathbf{U}^r \neq \mathbf{L}$ in applied control problems.

In the classical control problem, one needs to find a control u taking system (2.78) from the given initial state $x(a) = \alpha$ to the desired terminal state $x(b) = \beta$, that is, to find $u \in \mathbf{U}^r$ such that the boundary value problem

$$\mathcal{L}x = v + Bu, \qquad x(a) = \alpha, \qquad x(b) = \beta \tag{2.79}$$

has the solution x_u. By any control u, the solution x_u is uniquely defined. In cases of ordinary differential equations and equations with delay, the control problem is the subject of wide literature (see, e.g., [7] and references therein).

Consider a more general control problem,

$$\mathcal{L}x = v + Bu, \qquad x(a) = \alpha, \qquad lx = \beta \tag{2.80}$$

where the aim of control is given by the general linear bounded vector functional $l : \mathbf{D} \to \mathbb{R}^n$. Such problems arise, in particular, in economic dynamics, where the aim of control can be formulated as the attainment of the given level, β, of certain characteristic of trajectory x. For example, in the case that the model (2.78) governs the production dynamics, the condition

$$lx = \int_a^b e^{-\lambda(t-a)} x(t)\,dt = \beta \tag{2.81}$$

gives the so-called integral discounted product with discount coefficient λ. The control problems in economic dynamics are studied in detail in [148].

Here we demonstrate that, for systems with the operator $\mathcal{L}$ being a Volterra one, conditions for the solvability of the control problem as well as the construction of corresponding control actions can be efficiently written due to the Cauchy matrix, $C(t,s)$ (the Green matrix of the Cauchy problem).

We will restrict our consideration to the problem (2.80) in the case

$$(\mathcal{L}x)(t) = \dot{x}(t) + \int_a^t d_sR(t,s)x(s). \tag{2.82}$$

In this case, the general solution of the equation $\mathcal{L}x = f$ has the representation (2.69)

$$x(t) = \int_a^t C(t,s)f(s)\,ds + C(t,a)x(a). \tag{2.83}$$

Thus the set of all possible trajectories to control system (2.78) is governed by the equality

$$x(t) = C(t,a)\alpha + \int_a^t C(t,s)v(s)\,ds + \int_a^t C(t,s)(Bu)(s)\,ds, \quad u \in \mathbf{U}^r. \tag{2.84}$$

Applying the vector functional

$$lx = \Psi x(a) + \int_a^b \Phi(\tau)\dot{x}(\tau)d\tau \tag{2.85}$$

to both of the sides of this equality and taking into account that

$$\frac{d}{dt}\int_a^t C(t,s)f(s)ds = \int_a^t C_t'(t,s)f(s)ds + f(t), \tag{2.86}$$

we obtain

$$\begin{aligned} lx = \Psi_1\alpha &+ \int_a^b \Phi(\tau)\int_a^\tau C_\tau'(\tau,s)v(s)ds\,d\tau + \int_a^b \Phi(\tau)v(\tau)d\tau \\ &+ \int_a^b \Phi(\tau)\int_a^\tau C_\tau'(\tau,s)(Bu)(s)ds\,d\tau + \int_a^b \Phi(\tau)(Bu)(\tau)d\tau = \beta, \end{aligned} \tag{2.87}$$

where

$$\Psi_1 = \Psi + \int_a^b \Phi(\tau)C_\tau'(\tau,a)d\tau. \tag{2.88}$$

After the interchange of the order of integration in the iterated integrals and the notation

$$\theta(s) = \Phi(s) + \int_s^b \Phi(\tau)C_\tau'(\tau,s)d\tau, \tag{2.89}$$

we come to the following equation concerning the control u:

$$\int_a^b \theta(s)(Bu)(s)ds = \gamma. \tag{2.90}$$

Here

$$\gamma = \beta - \Psi_1\alpha - \int_a^b \theta(s)v(s)ds. \tag{2.91}$$

The solvability of this equation is necessary and sufficient for the solvability of control problem (2.80).

The problem of constructing the control is more simple in case when the space $\mathbf{U}^r$ is Hilbert. First consider the most widespread case in the literature: $\mathbf{U}^r = \mathbf{L}_2^r$ is the space of square-summable functions $u : [a,b] \to \mathbb{R}^r$ with the inner product

$$(u_1,u_2)_{\mathbf{L}_2^r} = \int_a^b u_1^{\mathrm{T}}(s)u_2(s)ds \tag{2.92}$$

($\cdot^{\mathrm{T}}$ is the symbol of transposition).

Rewrite (2.90) in the form

$$\int_a^b [B^*\theta](s)u(s)ds = \gamma, \tag{2.93}$$

where $B^* : (\mathbf{L})^* \to (\mathbf{L}_2^r)^*$ is the adjoint operator to B, and try to find the control in the form

$$u = [B^*\theta]^{\mathrm{T}} \cdot \sigma + g, \tag{2.94}$$

where $\sigma \in \mathbb{R}^n$, and $g \in \mathbf{L}_2^r$ is the element of orthogonal complement to the linear manifold of elements of the form $[B^*\theta]^{\mathrm{T}} \cdot \sigma$:

$$\int_a^b [B^*\theta](s)g(s)ds = 0. \tag{2.95}$$

As is known, any element $u \in \mathbf{L}_2^r$ can be represented in the form (2.94). As for σ, we have the linear algebraic system

$$M \cdot \sigma = \gamma, \tag{2.96}$$

where $n \times n$ matrix M is defined by

$$M = \int_a^b [B^*\theta](s)[B^*\theta]^{\mathrm{T}}(s)ds. \tag{2.97}$$

It is the Gram matrix to the system of the rows of $B^*\theta$.

Thus the invertibility of M (i.e., the linear independence of the rows of $B^*\theta$) is the criterion of the solvability of control problem (2.80) for every $\alpha, \beta \in \mathbb{R}^n$, and $v \in \mathbf{L}$.

The control

$$\overline{u} = [B^*\theta]^{\mathrm{T}} M^{-1}\gamma \tag{2.98}$$

with the zero orthogonal complement g has the minimal norm among all controls that solve problem (2.80). It follows at once from

$$\|\overline{u} + g\|^2_{\mathbf{L}_2^r} = \|\overline{u}\|^2_{\mathbf{L}_2^r} + \|g\|^2_{\mathbf{L}_2^r}. \tag{2.99}$$

The application of the foregoing scheme assumes the construction (in the explicit form) of the Cauchy matrix $C(t,s)$ to the equation $\mathcal{L}x = f$ as well as the construction of the adjoint operator B^*.

Since the solvability of the control problem is a rough property (being conserved under small perturbations), establishing the solvability can be done using an approximation of $C(t,s)$ with enough high accuracy. Present-day computer-oriented methods and practices for efficiently constructing these approximations with guaranteed error bounds are presented in Chapter 6.

Representation (2.98) provides a way to reveal several properties of $\overline{u}$, in addition to belonging to $\mathbf{L}_2^r$ and the minimality of its norm. Explain the aforesaid by the case when $(Bu)(t) = B(t)u(t)$. In such a situation,

$$\overline{u}(t) = B^{\mathrm{T}}(t)\theta^{\mathrm{T}}(t)M^{-1}\gamma, \tag{2.100}$$

and the true smoothness of the $\overline{u}$ is defined by the smoothness of the functions $B(\cdot)$, $\Phi(\cdot)$, and $C'_\tau(\tau,\cdot)$. In applied control problems, the question on the solvability of the control problem within a class of functions of the given smoothness is of considerable importance. The properties of $C'_\tau(\tau,t)$ as the function of the arguments t and τ are studied in detail in [32, 33].

Another way of finding smooth controls is in connection with a special choice of the space $\mathbf{U}^r$. The question on the solvability of control problem (2.80) in the space $\mathbf{U}^r$ that is isomorphic to the direct product $\mathbf{L}_2^r \times \mathbb{R}^r \times \cdots \times \mathbb{R}^r$ is efficiently reduced to the question on the solvability of a linear algebraic system. For short, consider the case when $\mathbf{U}^r = \mathbf{D}_2^r \simeq \mathbf{L}_2^r \times \mathbb{R}^r$ is the Hilbert space of absolutely continuous functions $u : [a,b] \to \mathbb{R}^r$ with square-summable derivative and the inner product

$$(u_1,u_2)_{\mathbf{D}_2^r} = (u_1(a),u_2(a))_{\mathbb{R}^r} + (\dot{u}_1,\dot{u}_2)_{\mathbf{L}_2^r}. \tag{2.101}$$

Taking into account the representation

$$u(t) = u(a) + \int_a^t \dot{u}(s)ds, \tag{2.102}$$

we write (2.90) in the form

$$\begin{aligned}&\int_a^b \theta(s)(BE)(s)ds\cdot u(a) + \int_a^b \theta(s)\left[B\left(\int_a^{(\cdot)} \dot{u}(\tau)d\tau\right)\right](s)ds\\ &\quad= \int_a^b (B^*\theta)(s)ds\cdot u(a) + \int_a^b (\mathcal{B}^*\theta)(s)\dot{u}(s)ds,\end{aligned} \tag{2.103}$$

where $(\mathcal{B}z)(t) = [B(\int_a^{(\cdot)} z(\tau)d\tau)](t)$. Denoting

$$\mathcal{V} = \int_a^b (B^*\theta)(s)ds, \qquad \mathcal{W}(s) = (\mathcal{B}^*\theta)(s), \tag{2.104}$$

we come to a system

$$\mathcal{V}\cdot u(a) + \int_a^b \mathcal{W}(s)\dot{u}(s)ds = \gamma. \tag{2.105}$$

Any element $u \in \mathbf{D}_2^r$ can be represented in the form

$$u(t) = \mathcal{V}^{\mathrm{T}}\cdot\sigma_1 + g_1 + \int_a^t [\mathcal{W}^{\mathrm{T}}(s)\cdot\sigma_2 + g_2(s)]ds, \tag{2.106}$$

where $\sigma_1, \sigma_2 \in \mathbb{R}^n$,

$$(\mathcal{V}^{\mathrm{T}} \cdot \sigma_1, g_1)_{\mathbb{R}^n} = 0 \quad \forall \sigma_1 \in \mathbb{R}^n, \qquad (\mathcal{W}^{\mathrm{T}} \cdot \sigma_2, g_2)_{\mathbf{L}_2^r} = 0 \quad \forall \sigma_2 \in \mathbb{R}^n. \tag{2.107}$$

With (2.106) and (2.105), we get the following system with respect to vectors σ_1 and σ_2, that defines the control

$$\bar{u} \in \mathbf{D}_2^r, \qquad \bar{u}(t) = \mathcal{V}^{\mathrm{T}} \cdot \sigma_1 + \int_a^t \mathcal{W}^{\mathrm{T}}(s)ds \cdot \sigma_2 \tag{2.108}$$

with the minimal $\mathbf{D}_2^r$-norm:

$$M_1 \cdot \sigma_1 + M_2 \cdot \sigma_2 = \gamma. \tag{2.109}$$

Here $n \times n$ matrices M_1 and M_2 are defined by

$$M_1 = \mathcal{V} \cdot \mathcal{V}^{\mathrm{T}}, \qquad M_2 = \int_a^b \mathcal{W}(s)\mathcal{W}^{\mathrm{T}}(s)ds. \tag{2.110}$$

Consider the possibility of taking into account some additional linear restrictions concerning the control. Let $\lambda : \mathbf{D}_2^r \to \mathbb{R}^r$ be a given linear bounded vector functional with linearly independent components. The control problem with the restrictions can be written in the form of the system

$$\mathcal{L}x = v + Bu, \qquad x(a) = \alpha, \qquad lx = \beta, \qquad \lambda u = 0. \tag{2.111}$$

Obtain the criterion of the solvability of (2.111). Let $\mathcal{L}_\lambda : \mathbf{D}_2^r \to \mathbf{L}_2^r$ be a linear bounded operator such that the boundary value problem

$$\mathcal{L}_\lambda u = z, \qquad \lambda u = 0 \tag{2.112}$$

is uniquely solvable for every $z \in \mathbf{L}_2^r$. The set of all controls $u \in \mathbf{D}_2^r$ with the condition $\lambda u = 0$ is governed by the equality $u(t) = (Gz)(t)$, $z \in \mathbf{L}_2^r$, where $G : \mathbf{L}_2^r \to \ker \lambda$ is the Green operator of the problem (2.112). Using this representation as well as equation (2.90), we come to the following equation concerning an element $z \in \mathbf{L}_2^r$:

$$\int_a^b \theta(s)(BGz)(s)ds = \gamma. \tag{2.113}$$

Each solution of this equation, z, generates a control $u = Gz$ that solves problem (2.111). Denoting $\mathcal{B} = BG$, we get the equation

$$\int_a^b [\mathcal{B}^*\theta](s)z(s)ds = \gamma \tag{2.114}$$

and, next, doing again the consideration above, we come to the following criterion of the solvability of the problem (2.111):

$$\det \int_a^b [\mathcal{B}^*\theta](s)[\mathcal{B}^*\theta]^{\mathrm{T}}(s)ds \neq 0. \tag{2.115}$$

The steps of the development of the theory of linear equations in the space of absolutely continuous functions are reflected in the surveys [17, 31, 35].

The criticism of the conception of equations with deviated argument and the continuous matching between the solution and the initial function is presented in [187] (see also [32, 33]).

The composition operator in connection with equations with deviated argument was studied in [59, 62, 63, 76]. An extensive literature on the subject can be found in [12].

The reduction of the boundary value problem for equations with deviated argument to the integral equation of the second kind and the construction of the Green function on the base of the resolvent to the integral operator were proposed in [34].

First the decomposition of the linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ into the sum of two operators such that one of them is finite-dimensional was used in [185].

The representation of the general solution of the linear neutral equation was given in [16]. The Cauchy matrix of the equation resolved with respect to the derivative was thoroughly studied in [142, 144, 145].

The class of equivalent regularizators of the linear boundary value problem was described in [186, 188].

The first applications of the theory presented in Section 2.2 to the problem of controllability were given in [126, 127].

2.3. Equations of the nth order

2.3.1. The equation in the space of scalar functions with absolutely continuous derivative of the $(n-1)$th order

Denote by $\mathbf{W}^n$ the space of $(n-1)$-times differentiable functions $x : [a,b] \to \mathbb{R}^1$ with absolutely continuous derivative $x^{(n-1)}$. Let further $\mathbf{L}$ be the space of summable functions $z : [a,b] \to \mathbb{R}^1$. By virtue of the identity

$$x(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} x^{(n)}(s)ds + \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} x^{(k)}(a), \tag{2.116}$$

the element $x \in \mathbf{W}^n$ has the representation

$$\begin{gathered} x(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)ds + \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} \beta^{k+1}, \\ z \in \mathbf{L}, \quad \beta \overset{\text{def}}{=} \{\beta^1, \dots, \beta^n\} \in \mathbb{R}^n. \end{gathered} \tag{2.117}$$

The equality (2.117) defines the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbb{R}^n \to \mathbf{W}^n$, where

$$(\Lambda z)(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,ds, \qquad (Y\beta)(t) = \sum_{k=0}^{n-1} \frac{(t-a)^k}{k!} \beta^{k+1}. \tag{2.118}$$

Therewith $\mathcal{J}^{-1} = [\delta, r] : \mathbf{W}^n \to \mathbf{L} \times \mathbb{R}^n$, where

$$(\delta x)(t) = x^{(n)}(t), \qquad rx = \operatorname{col}\left\{x(a), \dots, x^{(n-1)}(a)\right\}. \tag{2.119}$$

The space $\mathbf{W}^n$ is Banach under the norm

$$\|x\|_{\mathbf{W}^n} = \|\dot{x}\|_{\mathbf{L}} + |x(a)| + \cdots + |x^{(n-1)}(a)|. \tag{2.120}$$

We will consider the equation with linear bounded $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ under the assumption that the principal part $Q = \mathcal{L}\Lambda : \mathbf{L} \to \mathbf{L}$ of the operator $\mathcal{L}$ is Fredholm. Such an equation is called the linear functional differential equation of the nth order.

Let the isomorphism $\mathcal{J} : \mathbf{L} \times \mathbb{R}^n \to \mathbf{W}^n$ be defined by (2.117). Then the operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ admits the decomposition

$$(\mathcal{L}x)(t) = \left(Qx^{(n)}\right)(t) + \sum_{i=0}^{n-1} p_i(t) x^{(i)}(a). \tag{2.121}$$

Here $Q = \mathcal{L}\Lambda$ and $p_i(t) = (\mathcal{L}y_i)(t)$, where $y_i(t) = (t-a)^i/i!$ are the components of the vector $Y = (y_0, \dots, y_{n-1})$.

The decomposition of the components $l^i : \mathbf{W}^n \to \mathbb{R}^1$ of the vector functional $l = [l^1, \dots, l^n]$ has the form

$$l^i x = \int_a^b \varphi^i(s) x^{(n)}(s)\,ds + \sum_{j=0}^{n-1} \psi^i_j x^{(j)}(a), \tag{2.122}$$

where φ^i are measurable and essentially bounded functions, $\psi^i_j = \text{const}$.

By Theorem 1.11, the fundamental system of the solutions of the homogeneous equation $\mathcal{L}x = 0$ is finite dimensional, besides, $\dim\ker\mathcal{L} \geq n$. Let $l^1, \dots, l^n$ be a linearly independent system of linear bounded functionals $l^i : \mathbf{W}^n \to \mathbb{R}^1$, and let the boundary value problem

$$\mathcal{L}x = f, \qquad l^i x = \alpha^i, \quad i = 1, \dots, n, \tag{2.123}$$

have a unique solution $x \in \mathbf{W}^n$ for each $\{f, \alpha\} \in \mathbf{L} \times \mathbb{R}^n$ ($\alpha = \operatorname{col}\{\alpha^1, \dots, \alpha^n\}$). Then $\dim\ker\mathcal{L} = n$, by Theorem 1.17, and the general solution of the equation has the representation

$$x(t) = (Gf)(t) + \sum_{i=1}^{n} c_i x_i(t), \quad c_i = \text{const}, \ i = 1, \dots, n. \tag{2.124}$$

Here $G : \mathbf{L} \to \{x \in \mathbf{W}^n : l^i x = 0,\ i = 1, \dots, n\}$ is the *Green operator* of the problem (2.123), and $x_1, \dots, x_n$ is the fundamental system of solutions of the homogeneous equation $\mathcal{L}x = 0$.

The Green operator of problem (2.123) is integral:

$$(Gf)(t) = \int_a^b G(t,s) f(s) ds, \tag{2.125}$$

by Theorem 1.31, since the Green operator Λ in isomorphism (2.117) is integral. The kernel $G(t,s)$ of the Green operator is said to be the *Green function*.

Let W be the Green operator of some boundary value problem with the functionals $l^1, \dots, l^n$. Then the problem (2.123) is uniquely solvable, by Theorem 1.25, if and only if the operator $\mathcal{L}W : \mathbf{L} \to \mathbf{L}$ has a bounded inverse. As it takes place, the Green operator of the problem (2.123) has the representation

$$G = W[\mathcal{L}W]^{-1}. \tag{2.126}$$

Thus, for the investigations of the problem (2.123), it is useful to get any operator W in the explicit form. Such an operator W can be constructed by the scheme provided by Lemma 1.21 and Theorem 1.22. For this purpose define by $u_1, \dots, u_n$ a linearly independent system from $\mathbf{W}^n$ such that

$$w(a) \stackrel{\text{def}}{=} \begin{vmatrix} u_1(a) & \cdots & u_n(a) \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)}(a) & \cdots & u_n^{(n-1)}(a) \end{vmatrix} \neq 0, \qquad \begin{pmatrix} l^1 u_1 & \cdots & l^1 u_n \\ \cdots & \cdots & \cdots \\ l^n u_1 & \cdots & l^n u_n \end{pmatrix} = E. \tag{2.127}$$

As such a "primary" operator W, the operator

$$(W_l z)(t) \stackrel{\text{def}}{=} \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s) ds - \int_a^b \sum_{i=1}^n u_i(t) \varphi^i(s) z(s) ds, \tag{2.128}$$

may be accepted, where the measurable and essentially bounded functions φ^i define the principal part of the linear functional l^i:

$$l^i \Lambda z = \int_a^b \varphi^i(s) z(s) ds. \tag{2.129}$$

By Theorem 1.22, W_l is the Green operator of the boundary value problem

$$\mathcal{L}_0 x = f, \qquad l^i x = \alpha^i, \quad i = 1, \dots, n, \tag{2.130}$$

where

$$(\mathcal{L}_0 x)(t) \stackrel{\text{def}}{=} \frac{1}{w(a)} \begin{vmatrix} u_1(a) & \cdots & u_n(a) & x(a) \\ \cdots & \cdots & \cdots & \cdots \\ u_1^{(n-1)}(a) & \cdots & u_n^{(n-1)}(a) & x^{(n-1)}(a) \\ u_1^{(n)}(t) & \cdots & u_n^{(n)}(t) & x^{(n)}(t) \end{vmatrix}. \tag{2.131}$$

One can make sure of it immediately:

$$\mathcal{L}_0 W_l f = f, \qquad l^i W_l f = l^i \Lambda z - \int_a^b \varphi^i(s) z(s)\,ds = 0, \quad i = 1, \dots, n. \tag{2.132}$$

Let us remark that the equation $\mathcal{L}_0 x = f$, in a sense, is a simplest equation with the given fundamental system $u_1, \dots, u_n$.

The "W-substitution," $x = W_l z$, establishes the one-to-one mapping between the set of solutions $x \in \mathbf{W}^n$ of the boundary value problem

$$\mathcal{L}x = f, \qquad l^k x = 0, \quad k = 1, \dots, n, \tag{2.133}$$

and the set of solutions $z \in \mathbf{L}$ of the equation

$$\mathcal{L} W_l z \equiv (Q - F)z = f, \tag{2.134}$$

where

$$(Fz)(t) = \int_a^b \sum_{k=1}^{n} (\mathcal{L}u_k)(t) \varphi^k(s) z(s)\,ds. \tag{2.135}$$

The equation of the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \sum_{k=0}^{n-1} \int_a^b x^{(k)}(s)\,d_s r_k(t,s) = f(t) \tag{2.136}$$

is a representative of the class of the equations of the nth order if the functions $r_k(t,s)$, $k = 0, \dots, n-1$, are measurable in the square $[a,b] \times [a,b]$, are summable for each $s \in [a,b]$, $\operatorname{var}_{s \in [a,b]} r_k(t,s)$, and are summable on $[a,b]$. We will also assume that $r_k(t,b) = 0$.

The principal part $Q = \mathcal{L}\Lambda$ of $\mathcal{L}$ for such an equation is defined by $Q = I - R$, where

$$\begin{gathered}
(Rz)(t) = \sum_{k=0}^{n-1} \int_a^b \frac{d^k}{ds^k} (\Lambda z)(s)\,d_s r_k(t,s) = \int_a^b R(t,s) z(s)\,ds, \\
R(t,s) = \sum_{k=0}^{n-2} \int_s^b \frac{(\tau - s)^{n-k-2}}{(n-k-2)!} r_k(t,\tau)\,d\tau + r_{n-1}(t,s) \quad \text{if } n \geq 2, \\
R(t,s) = r_0(t,s) \quad \text{if } n = 1.
\end{gathered} \tag{2.137}$$

The coefficients $p_i = \mathcal{L} y_i$ are defined by $p_0(t) = -r_0(t,a)$,

$$p_i(t) = -r_i(t,a) - \sum_{j=0}^{i-1} \int_a^b \frac{(\tau - a)^{i-j-1}}{(i-j-1)!} r_j(t,\tau)\,d\tau, \quad i = 1, \dots, n-1. \tag{2.138}$$

By Theorem B.1, the operator $R : \mathbf{L} \to \mathbf{L}$ is compact.

The Green function for (2.136) can be constructed by means of the resolvent kernel $H(t,s)$ of the integral operator $R+F$ as it has been done in the previous section. Namely, let

$$(I-R-F)^{-1}=I+H, \qquad (Hf)(t)=\int_a^b H(t,s)f(s)ds, \tag{2.139}$$

where $H:\mathbf{L}\to\mathbf{L}$ is compact. Then

$$(Gf)(t)=\left[W_l(I+H)f\right](t)=\int_a^b W_l(t,s)\left\{f(s)+\int_a^b H(s,\tau)f(\tau)d\tau\right\}ds. \tag{2.140}$$

Hence

$$\begin{aligned} G(t,s) &= W_l(t,s)+\int_a^b W_l(t,\tau)H(\tau,s)d\tau \\ &= \frac{(t-s)^{n-1}}{(n-1)!}\chi(t,s)-\sum_{i=1}^n u_i(t)\varphi^i(s) \\ &\quad +\int_a^t \frac{(t-\tau)^{n-1}}{(n-1)!}H(\tau,s)d\tau-\int_a^b\sum_{i=1}^n u_i(t)\varphi^i(\tau)H(\tau,s)d\tau. \end{aligned} \tag{2.141}$$

There holds the following assertion which is an analog to Theorem 2.2.

Theorem 2.4. *Let the problem* (2.123) *be uniquely solvable. The Green function $G(t,s)$ of the problem possesses the following properties.*

(a) *The function $G(\cdot,s)$ has at almost each $s\in[a,b]$ the absolutely continuous derivative of the $(n-1)$th order on $[a,s)$ and $(s,b]$ and, besides,*

$$\frac{\partial^{n-1}}{\partial t^{n-1}}G(t,s)\bigg|_{t=s+0}-\frac{\partial^{n-1}}{\partial t^{n-1}}G(t,s)\bigg|_{t=s-0}=1. \tag{2.142}$$

(b) *One has*

$$\frac{d^n}{dt^n}\int_a^b G(t,s)f(s)ds=f(t)+\int_a^b\frac{\partial^n}{\partial t^n}G(t,s)f(s)ds \tag{2.143}$$

for each $f\in\mathbf{L}$.

(c) *The function $G(\cdot,s)$ satisfies the equalities*

$$\begin{gathered} \frac{\partial^n}{\partial t^n}G(t,s)-\int_a^b R(t,\tau)\frac{\partial^n}{\partial\tau^n}G(\tau,s)d\tau+\sum_{i=0}^{n-1}p_i(t)\frac{\partial^i}{\partial t^i}G(t,s)\bigg|_{t=a}=R(t,s), \\ \int_a^b\varphi^i(\tau)\frac{\partial^n}{\partial\tau^n}G(\tau,s)\,d\tau+\sum_{j=0}^{n-1}\psi_j^i\frac{\partial^j}{\partial t^j}G(t,s)\bigg|_{t=a}=-\varphi^i(s), \quad i=1,\dots,n, \end{gathered} \tag{2.144}$$

at almost each $s\in[a,b]$.

Proof. The assertions (a) and (b) follow from the representation (2.141) of the Green function.

The assertion (c) may be gotten as a result of substitution of $x = Gf$ in the equation $\mathcal{L}x = f$ and the boundary conditions $l^k x = 0$, $k = 1, \dots, n$. $\square$

Let $\mathcal{L}$ be Volterra ($r_k(t,s) = 0$ for $a \le t < s \le b$) and let the principal part $Q = \mathcal{L}\Lambda$ be invertible and, besides, let Q^{-1} be bounded Volterra. Then the Green operator of the Cauchy problem is an integral Volterra operator. Denote such an operator by C:

$$(Cf)(t) = \int_a^t C(t,s)f(s)ds. \tag{2.145}$$

We will call it the *Cauchy operator* of (2.136) and the kernel $C(t,s)$ is said to be the *Cauchy function*. For the differential equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \sum_{k=1}^{n-1} p_k(t)x^{(k)}(t) = f(t), \tag{2.146}$$

the Cauchy function may be expressed by the fundamental system $x_1, \dots, x_n$:

$$C(t,s) = \frac{1}{w(t)} \begin{vmatrix} x_1(s) & \cdots & x_n(s) \\ \cdots & \cdots & \cdots \\ x_1^{(n-2)}(s) & \cdots & x_n^{(n-2)}(s) \\ x_1(t) & \cdots & x_n(t) \end{vmatrix}, \tag{2.147}$$

where

$$w(t) = \begin{vmatrix} x_1(t) & \cdots & x_n(t) \\ \cdots & \cdots & \cdots \\ x_1^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix}. \tag{2.148}$$

Rather a general representative of the equations of the nth order is the equation of the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) - \sum_{i=1}^{m} b_i(t)(S_{g_i}x^{(n)})(t) + \sum_{k=0}^{n-1} \int_a^b x^{(k)}(s)d_s r_k(t,s) = f(t), \tag{2.149}$$

where the operator $S : \mathbf{L} \to \mathbf{L}$ defined by

$$(Sz)(t) = \sum_{i=1}^{m} b_i(t)(S_{g_i}z)(t) \tag{2.150}$$

is bounded and the operator $I - S$ has the bounded inverse. The conditions which guarantee the boundedness of S and the invertibility of $I - S$ were discussed in the previous section.

The principal part $Q = \mathcal{L}\Lambda$ of $\mathcal{L}$ in the case of such an equation has the form $Q = I - S - R$, where R is defined by (2.137). Therefore, in this case, (2.134) takes the form

$$\mathcal{L}W_l z \equiv z - (S + R + F)z = f \tag{2.151}$$

which, by this time, is not an integral one.

2.3.2. The monotonicity conditions for the Green operator

For the differential equation of the first order $\dot{x}(t) + p(t)x(t) = f(t)$, the differential inequality

$$\dot{z}(t) + p(t)z(t) - f(t) \overset{\text{def}}{=} \varphi(t) \geq 0, \quad t \in [a,b], \quad z(a) = x(a), \tag{2.152}$$

guarantees the estimate $z(t) \geq x(t)$, $t \in [a,b]$, for the solution x of the equation. Indeed, the difference $y = z - x$ satisfies the Cauchy problem

$$\dot{y}(t) + p(t)y(t) = \varphi(t), \qquad y(a) = 0. \tag{2.153}$$

Such a problem may be solved in quadrature:

$$y(t) = \int_a^t \exp\left\{-\int_s^t p(\tau)d\tau\right\}\varphi(s)ds = \int_a^t C(t,s)\varphi(s)ds \geq 0 \quad \text{if } \varphi \geq 0. \tag{2.154}$$

For the equation $\mathcal{L}x = f$ of the nth order, the inequality

$$(\mathcal{L}z)(t) - f(t) \overset{\text{def}}{=} \varphi(t) \geq 0, \qquad z^{(k)}(a) = x^{(k)}(a), \quad k = 0, \dots, n-1, \tag{2.155}$$

yields the estimate $z(t) \geq x(t)$ only under special conditions. The question about such conditions is called the problem of applicability of the Chaplygin theorem on differential inequality, the Chaplygin problem in short, see [44, 140]. Without dwelling on the long and interesting history of the question, we will mark that the estimate $z(t) \geq x(t)$ under the condition $\varphi(t) \geq 0$ is provided by the isotonic property of the Cauchy operator C since $z(t) - x(t) = (C\varphi)(t) \geq 0$. We will dwell here on the following natural generalization of the Chaplygin problem.

What conditions do guarantee the isotonic (antitonic) property of the Green operator to (2.123)?

One of the schemes of solving this problem is provided by the following criterion.

Theorem 2.5. *The problem* (2.123) *is uniquely solvable and, besides, the Green operator G of this problem is isotonic (antitonic) if the Green operator W of some model problem*

$$\mathcal{L}^0 x = z, \qquad l^i x = 0, \quad i = 1, \dots, n, \tag{2.156}$$

is isotonic (antitonic), the operator $\Omega \stackrel{\text{def}}{=} I - \mathcal{L}W$ is isotonic, and its spectral radius $\rho(\Omega) < 1$.

Proof. There exists the one-to-one mapping $x = Wz$, $z = \mathcal{L}^0 x$, between the set of solutions $z \in \mathbf{L}$ of the equation $\mathcal{L}Wz = f$ and the set of solutions $x \in \mathbf{W}^n$ of the problem (2.123) under $l^i x = 0$, $i = 1, \dots, n$. The operator

$$[\mathcal{L}W]^{-1} = I + \Omega + \Omega^2 + \cdots \tag{2.157}$$

is isotonic. Thus the solution $x = Wz = Gf$ of the problem (2.123) is positive (negative) for each $f(t) \geq 0$. □

To illustrate Theorem 2.5, consider the two-point problem

$$\begin{gathered} (\mathcal{L}x)(t) \stackrel{\text{def}}{=} (\mathcal{L}^0 x)(t) - (Tx)(t) = f(t), \\ n = 2, \qquad x(a) = 0, \qquad x(b) = 0 \end{gathered} \tag{2.158}$$

under the assumption that the problem

$$\mathcal{L}^0 x = z, \qquad x(a) = 0, \qquad x(b) = 0 \tag{2.159}$$

is uniquely solvable and its Green operator W is antitonic. Let, further, T be isotonic. By Theorem 2.5, the estimate $\|TW\|_{\mathbf{L}\to\mathbf{L}} < 1$ guarantees for the given problem the unique solvability and the isotonic property of the Green operator.

If

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + p(t)(S_h x)(t), \quad p(t) \geq 0, \tag{2.160}$$

we may assume that $(\mathcal{L}^0 x)(t) = \ddot{x}(t)$. Then the model equation is integrable and the Green function $W(t,s)$ can be written in the explicit form

$$W(t,s) = \begin{cases} -\dfrac{(s-a)(b-t)}{b-a} & \text{for } a \leq s \leq t \leq b, \\ -\dfrac{(t-a)(b-s)}{b-a} & \text{for } a \leq t < s \leq b. \end{cases} \tag{2.161}$$

Thus $W(t,s) \leq 0$. Let $W(t,s) = 0$ outside the square $[a,b] \times [a,b]$. Then

$$(\Omega z)(t) = -\int_a^b p(t)W[h(t),s]z(s)ds,$$
$$\|\Omega\|_{\mathbf{L}\to\mathbf{L}} \leq \int_a^b p(t) \max_{s\in[a,b]} |W(h(t),s)|dt. \tag{2.162}$$

Since $|W(t,s)| \leq (t-a)(b-t)/(b-a)$, the estimate

$$\|\Omega\|_{\mathbf{L}\to\mathbf{L}} \leq \int_a^b p(t)\sigma_h(t)\frac{[h(t)-a][b-h(t)]}{b-a}dt \tag{2.163}$$

holds, where

$$\sigma_h(t) = \begin{cases} 1 & \text{if } h(t) \in [a,b], \\ 0 & \text{if } h(t) \notin [a,b]. \end{cases} \tag{2.164}$$

In such a way, the inequality

$$\int_a^b p(t)\sigma_h[h(t)-a][b-h(t)]dt < b-a \tag{2.165}$$

guarantees, by Theorem 2.5, the unique solvability of the problem

$$\ddot{x}(t) + p(t)(S_h x)(t) = f(t), \quad x(a) = 0, \quad x(b) = 0, \quad p(t) \geq 0, \tag{2.166}$$

and the antitonicity of the Green operator. This inequality holds if

$$\int_a^b p(t)\sigma_h(t)dt \leq \frac{4}{b-a}. \tag{2.167}$$

The latter inequality is well known in the case of differential equation ($h(t) \equiv t$) as the Lyapunov-Zhukovskii inequality.

The explicit form of the operator $\Omega = I - \mathcal{L}W$ is not always known and in such a case the application of Theorem 2.5 meets difficulty. The following theorem offers some other schemes for investigation of boundary value problems.

Let us denote by $\mathbf{C}$ the space of continuous functions $x : [a,b] \to \mathbb{R}^1$ with $\|x\|_{\mathbf{C}} = \max_{t\in[a,b]} |x(t)|$.

We will assume that there exists the decomposition $\mathcal{L} = \mathcal{L}_0 - T$ where $T : \mathbf{C} \to \mathbf{L}$ is bounded isotonic (antitonic) operator and $\mathcal{L}_0 : \mathbf{W}^n \to \mathbf{L}$ is in possession of the following properties.

(1) The problem

$$\mathcal{L}_0 x = f, \qquad l^i x = \alpha^i, \quad i = 1,\dots,n, \tag{2.168}$$

has the unique solution $x \in \mathbf{W}^n$ for each $\{f,\alpha\} \in \mathbf{L} \times \mathbb{R}^n$.

(2) The Green function $W(t,s)$ of the problem satisfies the estimate $W(t,s) \geq 0$ ($W(t,s) \leq 0$) in the square $[a,b] \times [a,b]$.

(3) There exists a solution u_0 of the homogeneous equation $\mathcal{L}_0 x = 0$ such that $u_0(t) > 0$, $t \in [a,b] \setminus \{v\}$, where the set $\{v\}$ is defined as follows. If among the functionals l^i there are functionals such that $l^i x \stackrel{\text{def}}{=} x(v_i)$, $v_i \in [a,b]$, the set $\{v\}$ is the set of all such points v_i; otherwise $\{v\}$ denotes the empty set.

Theorem 2.6. *The following assertions are equivalent.*

(a) *There exists $v \in \mathbf{W}^n$ such that*

$$v(t) > 0, \quad r(t) \stackrel{\text{def}}{=} (W\varphi)(t) + g(t) > 0, \quad t \in [a,b] \setminus \{v\}, \tag{2.169}$$

where $\varphi = \mathcal{L}v$, g is the solution of the problem

$$\mathcal{L}_0 x = 0, \qquad l^i x = l^i v, \quad i = 1, \dots, n. \tag{2.170}$$

(b) *The spectral radius of the operator $WT : \mathbf{C} \to \mathbf{C}$ is less than* 1.

(c) *The problem* (2.123) *is uniquely solvable and, besides, the Green operator G of the problem is isotonic (antitonic).*

(d) *The homogeneous equation $\mathcal{L}x = 0$ has a positive solution $u(u(t) > 0$, $t \in [a,b] \setminus \{v\})$ satisfying the boundary conditions $l^i x = l^i v$, $i = 1, \dots, n$.*

(e) *The problem* (2.123) *is uniquely solvable and, besides, the inequality $G(t,s) \geq W(t,s)$ ($G(t,s) \leq W(t,s)$), $(t,s) \in [a,b] \times [a,b]$, for the Green functions $G(t,s)$ and $W(t,s)$ holds.*

Theorem 2.6 is a concrete realization of Theorem C.11 and Remark C.12.

Let us apply Theorem 2.6 to the problem

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + p(t)(S_h x)(t) = f(t), \qquad x(a) = 0, \qquad x(b) = 0, \tag{2.171}$$

considered above, where $p(t) \geq 0$. Letting $\mathcal{L}_0 x = \ddot{x}$, $Tx = -pS_h x$, $v(t) = (t-a)(b-t)$, we get, by Theorem 2.6, that the inequality

$$(\mathcal{L}v)(t) = -2 + p(t)\sigma_h(t)[h(t) - a][b - h(t)] < 0 \tag{2.172}$$

guarantees the unique solvability of the problem and the strict negativity of the Green function in the open square $(a,b) \times (a,b)$. The latter inequality holds if

$$\operatorname*{ess\,sup}_{t \in [a,b]} p(t)\sigma_h(t) < \frac{8}{(b-a)^2}. \tag{2.173}$$

As another example, consider the two-point boundary value problem

$$\begin{gathered} (\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) - q(t)(S_g \ddot{x})(t) + p(t)(S_h x)(t) = f(t), \\ x(a) = x(b) = 0 \end{gathered} \tag{2.174}$$

for the so-called neutral equation. Assume that the function $q : [a,b] \to \mathbb{R}^1$ is measurable and essentially bounded, $q(t) \geq 0$; $g(t) = t - \tau$, $\tau = \text{const} > 0$; $p \in \mathbf{L}$, $p(t) \geq 0$; and the function $h : [a,b] \to \mathbb{R}^1$ is measurable.

Let

$$\mathcal{L}_0 x = \ddot{x} - qS_g\ddot{x}, \qquad (\Lambda z)(t) = \int_a^t (t-s)z(s)ds. \tag{2.175}$$

The principal part Q_0 of $\mathcal{L}_0 : \mathbf{W}^2 \to \mathbf{L}$ has the form

$$Q_0 z = z - Sz, \tag{2.176}$$

where

$$Sz = qS_g z. \tag{2.177}$$

S is a nilpotent isotonic operator. Therefore

$$Q_0^{-1} = I + S + S^2 + \cdots + S^m, \quad m = \left[\frac{b-a}{\tau}\right] \tag{2.178}$$

(m is the integer part of the fraction $(b-a)/\tau$), and so the inverse Q_0^{-1} is also isotonic. The model equation $\mathcal{L}_0 x = f$ is equivalent to the equation of the form $\ddot{x} = Q_0^{-1} f$. Therefore the Green operator W of the model problem

$$\mathcal{L}_0 x = f, \qquad x(a) = x(b) = 0 \tag{2.179}$$

is antitonic ($W = W^0 Q_0^{-1}$, where W^0 is defined by (2.161)). The homogeneous equation $\mathcal{L}_0 x = 0$ has the solution $u_0(t) = t - a$.

Thus all the conditions of Theorem 2.6, as applied to the problem (2.174), are fulfilled. So the problem (2.174) is uniquely solvable and its Green operator G is antitonic if and only if the problem

$$\mathcal{L}x = 0, \qquad x(a) = 0, \qquad x(b) = b - a \tag{2.180}$$

has a solution $u(t)$ positive for $t \in (a,b]$.

It should be remarked that, in the examples above, the condition that the coefficients hold their fixed signs is essential. We will consider below the assertions that permit considering the equations with alternating coefficients.

2.3.3. The P-property

The system $u_1, \ldots, u_n \in \mathbf{W}^n$ is called nonoscillatory if any nontrivial linear combination $u = c_1 i_1 + \cdots + c_n u_n$ has no more than $n-1$ zeros, counting each multiple zero according to its multiplicity. Thus the Wronskian

$$w(t) = \begin{vmatrix} u_1(t) & \cdots & u_n(t) \\ \cdots & \cdots & \cdots \\ u_1^{(n-1)}(t) & \cdots & u_n^{(n-1)}(t) \end{vmatrix} \tag{2.181}$$

of the nonoscillatory system has no zeros since otherwise the system has an n-multiple zero.

The fundamental system of a second-order differential equation is nonoscillatory on the interval $[a,b]$ if and only if the homogeneous equation has a positive solution on $[a,b]$. It follows from the Sturm theorem on separation of zeros.

For the whole class of boundary value problems, one can reveal a connection between the invariance of the sign of Green functions and the nonoscillatory property of the fundamental system. For instance, the Green function $G(t,s)$ of any uniquely solvable boundary value problem for the differential equation of the second order may be strictly positive ($G(t,s) > 0$) or strictly negative ($G(t,s) < 0$) in the square $(a,b) \times (a,b)$ only under the condition that the interval $[a, b - \varepsilon]$ is the interval of nonoscillatory of the fundamental system for any $\varepsilon > 0$ being as small as we wish. It follows from the properties of the section $g(t) = G(t,s)$ of the Green function at the fixed $s \in (a,b)$ (Theorem 2.4) and from the Sturm theorem. An analogous phenomenon may be observed for some functional differential equations.

There is particularly interesting connections between the nonoscillatory property of the fundamental system and the properties of Green functions of the Vallee-Poussin boundary value problem of the nth order

$$\mathcal{L}x = f, \qquad x^{(j)}(t_i) = 0, \qquad a = t_1 < t_2 < \cdots < t_m = b, \tag{2.182}$$

$j = 0,\dots,k_i - 1$, $i = 1,\dots,m$, $k_1 + \cdots + k_m = n$, $\dim\ker\mathcal{L} = n$. The problem is uniquely solvable if $[a,b]$ is the interval of nonoscillation of the fundamental system, and in the case of the differential equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \sum_{k=0}^{n-1} p_k(t)x^{(k)}(t) = f(t), \tag{2.183}$$

the nonoscillatory property of the fundamental system guarantees the "regular behavior" of the Green function $W(t,s)$ of any Vallee-Poussin problem. Namely,

$$W(t,s) \cdot \prod_{i=1}^{m} (t - t_i)^{k_i} > 0, \quad t \in [a,b] \setminus \{t_1,\dots,t_m\}, \tag{2.184}$$

at any fixed $s \in (a,b)$. It was shown by Chichkin [61].

There is an extensive literature on the tests of the nonoscillatory property of fundamental systems.

If the Wronskian w of the fundamental system $x_1, \dots, x_n$ of an equation $\mathcal{L}x = f$ has no zeros, the homogeneous $\mathcal{L}x = 0$ is equivalent to the homogeneous differential equation

$$(Mx)(t) \overset{\text{def}}{=} \frac{1}{w(t)} \begin{vmatrix} x_1(t) & \cdots & x_n(t) & x(t) \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(n)}(t) & \cdots & x_n^{(n)}(t) & x^{(n)}(t) \end{vmatrix} = 0. \tag{2.185}$$

Besides, there exists an invertible $P : \mathbf{L} \to \mathbf{L}$ such that the sets of solutions of $\mathcal{L}x = f$ and $Mx = Pf$ coincide. The property of P being isotonic is called the *P-property* of the equation $\mathcal{L}x = f$.

Thus the P-property is defined by the following: the equation has an n-dimensional fundamental system, the Wronskian of the system has no zeros on $[a, b]$, and the operator P is isotonic.

The P-property and the nonoscillatory property of the fundamental system guarantees for the Green function $G(t, s)$ of the Vallee-Poussin problem the inequality

$$G(t, s) \cdot \prod_{i=1}^{m} (t - t_i)^{k_i} > 0, \quad t \in [a, b] \setminus \{t_1, \dots, t_m\}, \tag{2.186}$$

for almost all $s \in [a, b]$.

We will give two effective tests of the P-property of the equation

$$(\mathcal{L}x)(t) \overset{\text{def}}{=} x^{(n)}(t) + \int_a^b x(s) d_s r(t, s) = f(t). \tag{2.187}$$

Theorem 2.7. *Let n be even, let the function $r(t, s)$ do not increase with respect to the second argument for almost all $t \in [a, b]$, and let at least one of the inequalities*

$$\int_a^b [r(t, a) - r(t, b)] dt \leq \frac{(n-1)!}{(b-a)^{n-1}} \tag{2.188}$$

or

$$\operatorname*{ess\,sup}_{t \in [a,b]} [r(t, a) - r(t, b)] \leq \frac{n!}{(b-a)^n} \tag{2.189}$$

hold.

Then (2.187) *possesses the P-property.*

Proof. Let

$$(W^\tau z)(t) = \int_a^b W^\tau(t, s) z(s) ds \tag{2.190}$$

be the Green operator of the problem

$$x^{(n)} = z, \qquad x^{(k)}(\tau) = 0, \quad k = 0, \dots, n-1. \tag{2.191}$$

Thus

$$W^{\tau}(t,s) = \begin{cases} \dfrac{(t-s)^{n-1}}{(n-1)!} & \text{if } \tau \le s \le t \le b, \\ -\dfrac{(t-s)^{n-1}}{(n-1)!} & \text{if } a \le t \le s \le \tau, \\ 0 & \text{in other points of the square } [a,b] \times [a,b]. \end{cases} \tag{2.192}$$

Denote

$$(A_{\tau}x)(t) = -\int_a^b W^{\tau}(t,s)\left\{\int_a^b x(\xi)d_{\xi}r(s,\xi)\right\}ds. \tag{2.193}$$

The operator A_{τ} acts continuously in the space $\mathbf{C}$. The condition (2.188) guarantees the estimate $\|A_{\tau}\|_{\mathbf{C}\to\mathbf{C}} < 1$ (and, therefore, $\rho(A_{\tau}) < 1$) for each $\tau \in [a,b]$. From here, by Theorem 2.6, the problem

$$\mathcal{L}x = f, \qquad x^{(k)}(\tau) = 0, \quad k = 0, \dots, n-1, \tag{2.194}$$

is uniquely solvable for each $\tau \in [a,b]$ (the Wronskian w has no zero on $[a,b]$), besides, the Green operator G_{τ} of the problem is isotonic.

The latter assertion is valid by (2.189) also. Indeed, taking $v(t) = (\tau - t)^n$, we obtain, by virtue of Theorem 2.6, that the inequality

$$v^{(n)}(t) + \int_a^b (\tau - s)^n d_s r(t,s) > 0 \tag{2.195}$$

guarantees the unique solvability of the problem (2.194) and the isotonicity of the Green operator. The latter inequality is valid under the condition (2.189) for each $\tau \in [a,b]$.

Let, further, $f(t) \ge 0$ for each $t \in [a,b]$, $f(t) \not\equiv 0$, let u be a solution of the equation $\mathcal{L}x = f$, and let m be a set of zero measure such that any solution

$$y(t) = \sum_{k=1}^{n} c_k x_k(t) + u(t) \tag{2.196}$$

of the equation satisfies the equation at each point of the set $E = [a,b] \setminus m$. If $\tau \in E$, $y = G_\tau f$,

$$(My)(\tau) = \frac{1}{w(\tau)} \begin{vmatrix} x_1(\tau) & \cdots & x_n(\tau) & 0 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(n-1)}(\tau) & \cdots & x_n^{(n-1)}(\tau) & 0 \\ x_1^{(n)}(\tau) & \cdots & x_n^{(n)}(\tau) & y^{(n)}(\tau) \end{vmatrix} = -\int_a^b (G_\tau f)(s) d_s r(\tau,s) + f(\tau). \tag{2.197}$$

Thus for each $\tau \in E$ (consequently, a.e. on $[a,b]$), the inequality $(My)(t) \geq f(t) \geq 0$ holds for the solution of the equation $\mathcal{L}x = f$, where $f(t) \geq 0$. It means the isotonicity of P. □

Let us come back to the problem

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + p(t)(S_h x)(t) = f(t), \qquad x(a) = x(b) = 0, \tag{2.198}$$

without any assumption of the sign of the coefficient p. Let $p = p^+ - p^-$, $p^+(t) \geq 0$, $p^-(t) \geq 0$, and let at least one of the inequalities

$$\int_a^b p^-(s)\sigma_h(s)ds < \frac{1}{b-a}, \qquad \operatorname*{ess\,sup}_{t\in[a,b]} p^-(t)\sigma_h(t) < \frac{2}{(b-a)^2} \tag{2.199}$$

holds. By Theorem 2.7, the Green operator W of the auxiliary problem

$$(\mathcal{L}_0 x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) - p^-(t)(S_h x)(t) = z(t), \qquad x(a) = x(b) = 0 \tag{2.200}$$

is antitonic, since, for the case $n = 2$, the inequalities (2.188) and (2.189) are the conditions (2.199), $r(t,s) = p^-(t)\sigma(t,s)$, where $\sigma(t,s)$ is the characteristic function of the set

$$\{(t,s) \in [a,b] \times [a,b] : a \leq s \leq h(t) < b\} \cup \{(t,s) \in [a,b] \times [a,b) : h(t) = b\}. \tag{2.201}$$

Taking $v(t) = (t-a)(b-t)$, we obtain

$$(\mathcal{L}v)(t) = -2 - p^-(t)\sigma_h(t)[h(t)-a][b-h(t)] + p^+(t)\sigma_h(t)[h(t)-a][b-h(t)]. \tag{2.202}$$

Thus the inequality

$$-2 + p^+(t)\sigma_h(t)[h(t)-a][b-h(t)] < 0, \quad t \in [a,b], \tag{2.203}$$

guarantees the solvability of the problem (2.198) and the antitonicity of the Green operator of this problem. The latter inequality holds if

$$\operatorname*{ess\,sup}_{t\in[a,b]} \sigma_h(t)p^+(t) < \frac{8}{(b-a)^2}. \tag{2.204}$$

Using the Volterra property of the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \int_a^t x(s)d_s r(t,s) = f(t), \tag{2.205}$$

we are in position to state and prove the following test.

Theorem 2.8. *Let n be odd, let the function $r(t,s)$ do not decrease with respect to the second argument, and let at least one of the inequalities*

$$\int_a^b [r(t,b) - r(t,a)]dt < \frac{(n-1)!}{(b-a)^{n-1}}, \tag{2.206}$$

$$\operatorname*{ess\,sup}_{t\in[a,b]} [r(t,b) - r(t,a)] < \frac{n!}{(b-a)^n} \tag{2.207}$$

hold.

Then (2.205) *possesses P-property.*

Proof. Let $\tau \in (a,b]$ be fixed. Consider the equation

$$(\mathcal{L}^\tau x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \int_a^t x(s)d_s r^\tau(t,s) = f(t), \quad t \in [a,\tau]. \tag{2.208}$$

Emphasize that the operator $\mathcal{L}^\tau$ is defined on the space of the functions $x : [a,\tau] \to \mathbb{R}^1$. The boundary value problem

$$(\mathcal{L}^\tau x)(t) = f(t), \quad x^{(k)}(\tau) = 0, \quad k = 0,\dots,n-1,\ t \in [a,\tau], \tag{2.209}$$

is equivalent to the equation $x = A^\tau x + g$, where the operator $A^\tau : \mathbf{C}[a,\tau] \to \mathbf{C}[a,\tau]$ is defined by

$$\begin{gathered}(A^\tau x)(t) = -\int_a^t \frac{(t-s)^{n-1}}{(n-1)!}\left\{\int_a^s x(\xi)d_\xi r(s,\xi)\right\}ds, \quad t \in [a,\tau], \\ g(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds.\end{gathered} \tag{2.210}$$

The condition (2.206) guarantees the estimate $\rho(A^\tau) < 1$. This implies, by Theorem 2.6, that the problem (2.209) is uniquely solvable for each $\tau \in (a,b]$ and, besides, the Green operator G^τ of the problem is antitonic. The same assertion holds under (2.207). It follows from Theorem 2.6 if $v(t) = (\tau - t)^n$. Then

$(\mathcal{L}^\tau v)(t) < 0$, $t \in [a, \tau]$. The operators $\mathcal{L}$ and $\mathcal{L}^\tau$ are Volterra. Consequently, any solution x of the equation $\mathcal{L}x = f$ is the extension on $(\tau, b]$ of a solution x^τ of the equation $\mathcal{L}^\tau x = f$. From this it follows, in particular, that the conditions of Theorem 2.8 guarantee that the Wronskian of the fundamental system $x_1, \dots, x_n$ of the solutions of $\mathcal{L}x = 0$ has no zeros.

Following the scheme of the proof of Theorem 2.7, we have, at each $\tau \in E = [a, b] \setminus m$, where $m \subset [a, b]$ is a set of zero measure, that the solution y of the problem

$$\mathcal{L}x = f, \qquad x^{(i)}(\tau) = 0, \quad i = 0, \dots, n-1, \tag{2.211}$$

satisfies the equality

$$\begin{aligned}(My)(\tau) &= \frac{1}{w(\tau)} \begin{vmatrix} x_1(\tau) & \cdots & x_n(\tau) & 0 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(n-1)}(\tau) & \cdots & x_n^{(n-1)}(\tau) & 0 \\ x_1^{(n)}(\tau) & \cdots & x_n^{(n)}(\tau) & y^{(n)}(\tau) \end{vmatrix} \\ &= -\int_a^t (G_\tau f)(s) d_s r(\tau, s) + f(\tau). \end{aligned} \tag{2.212}$$

Thus, if $f(t) \geq 0$ a.e. on $[a, b]$, any solution x of the equation $\mathcal{L}x = f$ satisfies the inequality $(Mx)(t) \geq f(t) \geq 0$. Thus the operator P is isotonic. □

Remark 2.9. Under the assumptions of Theorems 2.7 and 2.8 the fundamental system of solutions of the equation $\mathcal{L}x = 0$ is nonoscillatory on $[a, b]$. It follows from the estimates of the Green functions of the Vallee-Poussin problems for the equation $x^{(n)} = f$ given by Beesack [49].

To illustrate Theorems 2.6 and 2.8, consider the problem

$$\begin{gathered}(\mathcal{L}x)(t) \overset{\text{def}}{=} \dddot{x}(t) + p(t)(S_h x)(t) = f(t), \qquad h(t) \leq t,\ t \in [a, b], \\ x(a) = x(b) = \dot{x}(b) = 0. \end{gathered} \tag{2.213}$$

Let $p = p^+ - p^-$, $p^+(t) \geq 0$, $p^-(t) \geq 0$, let $\mathcal{L}_0 x = \dddot{x} + p^+ S_h x$, and let the inequality

$$\operatorname*{ess\,sup}_{t \in [a,b]} p^+(t)\sigma_h(t) < \frac{6}{(b-a)^3} \tag{2.214}$$

holds. By Theorem 2.8 and Remark 2.9, the equation $\mathcal{L}_0 x = f$ possesses P-property and the fundamental system of the equation is nonoscillatory. Therefore, by virtue of (2.186) the Green function of the problem

$$\mathcal{L}_0 x = f, \qquad x(a) = x(b) = \dot{x}(b) = 0 \tag{2.215}$$

is strictly positive in the square $(a,b)\times(a,b)$. Thus Theorem 2.6 is applicable to the equation $\mathcal{L}x \equiv \mathcal{L}_0x - p^-S_hx = f$.

Let $v(t) = (t-a)(b-t)^2$. Then

$$(\mathcal{L}v)(t) = 6 + p^+(t)\sigma_h(t)\big[h(t)-a\big]\big[b-h(t)\big]^2 - p^-(t)\sigma_h(t)\big[h(t)-a\big]\big[b-h(t)\big]^2. \tag{2.216}$$

Therefore, by Theorem 2.6, the inequalities (2.214) and

$$6 - p^-(t)\sigma_h(t)\big[h(t)-a\big]\big[b-h(t)\big]^2 > 0, \quad t\in[a,b], \tag{2.217}$$

guarantee the unique solvability of the problem (2.213) and the strict positiveness of the Green function of the problem in the square $(a,b)\times(a,b)$. Since $\max_{t\in[a,b]} v(t) = 4(b-a)^3/27$, the inequality (2.217) holds if

$$\operatorname*{ess\,sup}_{t\in[a,b]} p^-(t)\sigma_h(t) < \frac{81}{2(b-a)^3}. \tag{2.218}$$

There is an extensive literature on the problem of fixed sign of the Green function and on the connection of the problem with the rules of distribution of zeros of solutions to homogeneous equation. This literature begins with [61, 168, 214] (see also the survey by Azbelev and Tsalyuk [44]).

The results of researches on the similar problems for the equations with deviated argument were published in Differential Equations and Russian Mathematics (IzVUZ) as well as in the yearly Boundary Value Problems and Functional-Differential Equations issued by the Perm Politechnic Institute.

2.4. Equations in spaces of functions defined on the semiaxis

The stability theory of differential equations arose in connection with some problems in mechanics a century ago. It was being developed until recently in the direction given by Lyapunov. The methods of Lyapunov, like all techniques of the qualitative theory of differential equations, are closely connected to the properties of the local operator $(\Phi x)(t) \overset{\text{def}}{=} \dot{x}(t) - f(t,x(t))$. Thus, the extension of the classical qualitative theory, from the first steps, came across many unexpected difficulties. Indeed, the techniques connected with the field of directions are useless for the equations differing from the ordinary differential ones. The method of Lyapunov functions is based on the Chaplygin theorem on differential inequalities which is not applicable, generally speaking, to delay differential equations.

The classical theory makes use of the so-called "semigroup equality"

$$X(t)X^{-1}(s) \overset{\text{def}}{=} C(t,s) = C(t,\tau)C(\tau,s) \tag{2.219}$$

for the fundamental matrix $X(t)$. But this equality holds only for ordinary differential equations. Thus the creation of a general theory of stability demands new ideas. One of such ideas was being developed in the monographs of Barbashin [46] and Massera and Schaffer [152], where the notion of stability was associated with solvability of equations in specific spaces. But the famous authors considered the problem in the terms of the theory of "ordinary differential equations in a Banach space." Therefore, they utilized the properties of local operators.

Some ideas of the monograph [46] as well as the results of the Perm Seminar on the delay differential equations were laid to the base of the works of Tyshkevich [218] on stability of solutions of the equations with aftereffect, where in particular the semigroup equality was replaced by its generalization.

The development of the theory of abstract functional differential equation had been leading to a new conception of stability. This conception does not contradict the classical one. It gives in addition efficient ways to investigate some forms of asymptotic behavior of solutions for a wide class of equations.

2.4.1. Linear manifold of solutions

Denote by $\mathbf{Y}$ a linear manifold of functions $x : [0,\infty) \to \mathbb{R}^n$ that are absolutely continuous on any finite $[0,b]$, by $\mathbf{Z}$ we denote a linear manifold of functions $z : [0,\infty) \to \mathbb{R}^n$ that are summable on any finite $[0,b]$. Let $\mathcal{L}_0 : \mathbf{Y} \to \mathbf{Z}$ be linear Volterra and suppose that the Cauchy problem

$$\mathcal{L}_0 x = z, \qquad x(0) = \alpha \tag{2.220}$$

has a unique solution $x \in \mathbf{Y}$ for each $\{z,\alpha\} \in \mathbf{Z} \times \mathbb{R}^n$ and the solution has the representation by the Cauchy formula

$$x(t) = \int_0^t W(t,s)z(s)ds + U(t)\alpha \overset{\text{def}}{=} (Wz)(t) + \big(Ux(0)\big)(t) \tag{2.221}$$

in the explicit form. We will call $\mathcal{L}_0 x = z$ a "model" equation. Let, further, $\mathbf{B} \subset \mathbf{Z}$ be a linear manifold of elements $z \in \mathbf{Z}$. Then (2.221) defines for each $\{z,\alpha\} \in \mathbf{B} \times \mathbb{R}^n$ the element $x \in \mathbf{Y}$ of the linear manifold $W\mathbf{B} + U\mathbb{R}^n \overset{\text{def}}{=} \mathbf{D}(\mathcal{L}_0, \mathbf{B})$. The manifold $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ consists of all solutions of the model equation $\mathcal{L}_0 x = z$ at all $z \in \mathbf{B}$.

Together with $\mathcal{L}_0 x = z$ consider an equation $\mathcal{L}x = f$ with a linear Volterra operator $\mathcal{L} : \mathbf{Y} \to \mathbf{Z}$ supposing that the Cauchy problem $\mathcal{L}x = f$, $x(0) = \alpha$ has a unique solution $x \in \mathbf{Y}$, and for this solution, the Cauchy formula

$$x(t) = \int_0^t C(t,s)f(s)ds + X(t)x(0) \overset{\text{def}}{=} (Cf)(t) + \big(Xx(0)\big)(t) \tag{2.222}$$

holds. The explicit form of the operators $C : \mathbf{Z} \to \mathbf{Y}$ and $X : \mathbb{R}^n \to \mathbf{Y}$ may be unknown. All solutions of $\mathcal{L}x = f$ at all $f \in \mathbf{B}$ form the linear manifold

$$\mathbf{D}(\mathcal{L}, \mathbf{B}) = C\mathbf{B} + X\mathbb{R}^n. \tag{2.223}$$

We will say that the equation $\mathcal{L}x = f$ possesses $\mathbf{D}_0$-property (the equation is $\mathbf{D}_0$-stable) if the manifolds $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ and $\mathbf{D}(\mathcal{L}, \mathbf{B})$ coincide.

Some properties of the elements $x \in \mathbf{D}(\mathcal{L}_0, \mathbf{B})$ (the properties of solutions of the equation which is solvable in the explicit form) are quite definite. The equality $\mathbf{D}(\mathcal{L}, \mathbf{B}) = \mathbf{D}(\mathcal{L}_0, \mathbf{B})$ provides the existence of the same properties of solutions of $\mathcal{L}x = f$.

Let us clarify the said by examples.

Example 2.10. Let $\mathcal{L}_0 x \overset{\text{def}}{=} \dot{x} + x$. Then the element $x \in \mathbf{D}(\mathcal{L}_0, \mathbf{B})$ has the form

$$x(t) = e^{-t} \int_0^t e^s z(s)\,ds + e^{-t}\alpha. \tag{2.224}$$

Let $\mathbf{B}^0$ be a manifold of elements $z \in \mathbf{Z}$ such that $\sup_{t \geq 0} \|z(t)\|_{\mathbb{R}^n} < \infty$, and let $\mathbf{B}^\gamma$ be the manifold of functions of the form $z(t) = e^{-\gamma t} y(t)$, where $y \in \mathbf{B}^0$, $0 < \gamma < 1$. Then the $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$-property of $\mathcal{L}x = f$ yields the boundedness of any solution $x(\sup_{t \geq 0} \|x(t)\|_{\mathbb{R}^n} < \infty)$ if $\mathbf{B} = \mathbf{B}^0$ and does the existence of exponential estimate $\|x(t)\|_{\mathbb{R}^n} \leq M_x e^{-\gamma t}$ if $\mathbf{B} = \mathbf{B}^\gamma$. Thus $\mathbf{D}(\mathcal{L}_0, \mathbf{B}^0)$-stability provides Lyapunov's stability of solutions of $\mathcal{L}x = f$ and $\mathbf{D}(\mathcal{L}_0, \mathbf{B}^\gamma)$-stability gives the exponential stability.

Example 2.11. Let the model equation be

$$(\mathcal{L}_0 x)(t) \overset{\text{def}}{=} \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} z_1(t) \\ z_2(t) \end{Bmatrix}. \tag{2.225}$$

Then the components x_1 and x_2 of the element $x = \operatorname{col}\{x_1, x_2\}$ of $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ are defined by

$$x_1(t) = e^{-t} \int_0^t e^s z_1(s)\,ds + e^{-t}\alpha_1, \qquad x_2(t) = e^t \int_0^t e^{-s} z_2(s)\,ds + e^t \alpha_2. \tag{2.226}$$

In this case, $\mathbf{D}(\mathcal{L}_0, \mathbf{B}^\gamma)$-stability with $0 < \gamma < 1$ guarantees the exponential stability of solutions of $\mathcal{L}x = f$ with respect to the first component.

Theorem 2.12. *Let the operator $\mathcal{L}$ be acting from $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ to $\mathbf{B}$. Then the following assertions are equivalent.*

(a) *The manifolds $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ and $\mathbf{D}(\mathcal{L}, \mathbf{B})$ coincide.*

(b) $\mathcal{L}W\mathbf{B} = \mathbf{B}$ *(there exists $[\mathcal{L}W]^{-1} : \mathbf{B} \to \mathbf{B}$).*

(c) *For each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$, the solution of the Cauchy problem*

$$\mathcal{L}x = f, \qquad x(0) = \alpha \tag{2.227}$$

belongs to $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$.

Proof. Between the set of solutions $x \in \mathbf{D}(\mathcal{L}_0, \mathbf{B})$ of the problem (2.227) and the set of solutions $z \in \mathbf{B}$ of the equation

$$\mathcal{L}Wz = f - \mathcal{L}U\alpha, \tag{2.228}$$

there exists the one-to-one mapping

$$x = Wz + U\alpha, \quad z = \mathcal{L}_0 x, \ \alpha = x(0). \tag{2.229}$$

Really, if (b) holds, a solution $z \in \mathbf{B}$ of (2.228) corresponds to each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$. Consequently, a solution $x = Wz + U\alpha$ of the problem (2.227) corresponds to each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$.

Now let a solution x of the problem (2.227) belong to $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$. Since each $\xi \in \mathbf{B}$ may be represented in the form $\xi = f - \mathcal{L}U\alpha$ for an element $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$, it follows from (2.229) that the equation $\mathcal{L}Wz = \xi$ has the solution $z \in \mathbf{B}$ for each $\xi \in \mathbf{B}$. Thus $\mathcal{L}W\mathbf{B} = \mathbf{B}$.

The equivalence (b)$\Leftrightarrow$(c) is proved.

If (c) is valid, $C\mathbf{B} + X\mathbb{R}^n \subset W\mathbf{B} + U\mathbb{R}^n$. Since $\mathcal{L}[\mathbf{D}(\mathcal{L}_0, \mathbf{B})] \subset \mathbf{B}$ due to the condition, $W\mathbf{B} + U\mathbb{R}^n \subset C\mathbf{B} + X\mathbb{R}^n$.

The implication (c)$\Rightarrow$(a) is proved.

The implication (a)$\Rightarrow$(c) is obvious because the equality $\mathbf{D}(\mathcal{L}_0, \mathbf{B}) = \mathbf{D}(\mathcal{L}, \mathbf{B})$ means that the solution of the problem (2.227) belongs to $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$. □

Remark 2.13. Under the assumption of $\mathcal{L}$, the problem (2.227) has the unique solution $x = Cf + X\alpha \in \mathbf{Y}$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$. Therefore, the assertion (c) of Theorem 2.12 is equivalent to the assertion on the existence of unique solution $x \in \mathbf{D}(\mathcal{L}_0, \mathbf{B})$ of the Cauchy problem for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$, as well as the assertion (b) that is equivalent to the assertion on the solvability of the equation $\mathcal{L}Wz = \xi$ in the space $\mathbf{B}$ for each $\xi \in \mathbf{B}$.

The asymptotic behavior of solutions of differential equation does not depend on the behavior on any finite $[0, b]$. It is obvious due to local property of $\mathcal{L}$. We will state the conditions that provide an analogous property of solutions of equations $\mathcal{L}x = f$ with Volterra $\mathcal{L}$.

Denote by $\chi_\omega(t)$ the characteristic function of the set $\omega \subset [0, \infty)$. Let $b > 0$. As for $\mathbf{B}$, we assume that $z_b \overset{\text{def}}{=} \chi_{[0,b)} z \in \mathbf{B}$ for each $z \in \mathbf{B}$. Define linear manifolds $\mathbf{B}^b$ and $\mathbf{B}_b$ by

$$\begin{aligned} \mathbf{B}^b &= \{z \in \mathbf{B} : z(t) = 0 \text{ a.e. on } [0, b)\}, \\ \mathbf{B}_b &= \{z \in \mathbf{B} : z(t) = 0 \text{ a.e. on } [b, \infty)\}. \end{aligned} \tag{2.230}$$

Let, further, $K \overset{\text{def}}{=} (\mathcal{L}_0 - \mathcal{L})W$, $K^b : \mathbf{B}^b \to \mathbf{B}^b$, be the restriction of Volterra K on $\mathbf{B}^b$.

Theorem 2.14. *Let $\mathcal{L}$ be acting from the manifold* $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ *into* $\mathbf{B}$. *Then* $\mathbf{D}(\mathcal{L}_0, \mathbf{B}) = \mathbf{D}(\mathcal{L}, \mathbf{B})$ *if the equation*

$$\varphi - K^b \varphi = \xi \tag{2.231}$$

has the solution $\varphi \in \mathbf{B}^b$ *for each* $\xi \in \mathbf{B}^b$.

Proof. By Theorem 2.12, it is sufficient to establish the solvability in $\mathbf{B}$ of the equation

$$\mathcal{L} W z \equiv z - Kz = f. \tag{2.232}$$

Define $K_b : \mathbf{B}_b \to \mathbf{B}_b$ and $K_b^b : \mathbf{B}_b \to \mathbf{B}^b$ by

$$(K_b z)(t) = \chi_{[0,b)}(t)(Kz)(t), \qquad (K_b^b z)(t) = \chi_{[b,\infty)}(t)(Kz)(t). \tag{2.233}$$

For each $z \in \mathbf{B}_b$, we have

$$Kz = K_b z + K_b^b z. \tag{2.234}$$

Therefore,

$$Kz = Kz_b + Kz^b = K_b z_b + K^b z^b + K_b^b z_b \tag{2.235}$$

for each $z \in \mathbf{B}$, where $z_b = \chi_{[0,b)} z$, $z^b = \chi_{[b,\infty)} z$. Using such notation, we may rewrite (2.232) in the form of two equations

$$\begin{aligned} z_b - K_b z_b &= f_b, \\ z^b - K^b z^b &= f^b + K_b^b z_b. \end{aligned} \tag{2.236}$$

If (2.236) has a solution $z_b \in \mathbf{B}_b$, the whole system has a solution $\{z_b, z^b\}$ for each $\{f_b, f^b\}$. Consequently, (2.232) has the solution $z = z^b + z_b$ for each $f \in \mathbf{B}$. Let us establish the solvability of (2.232).

There exists the one-to-one mapping

$$x = Wz, \qquad z = \mathcal{L}_0 x \tag{2.237}$$

between the set of solutions $x = C f_b \in \mathbf{Y}$ of the Cauchy problem

$$\mathcal{L}x = f_b, \quad x(0) = 0, \tag{2.238}$$

and the set of solutions $z \in \mathbf{B}$ of the equation

$$z - Kz = f_b. \tag{2.239}$$

Therefore, $z_b = \chi_{[0,b)}z = \chi_{[0,b)}\mathcal{L}_0 C f_b \in \mathbf{B}_b$ and it satisfies (2.236). It is clear after multiplication of (2.239) by $\chi_{[0,b)}$:

$$\chi_{[0,b)}z - \chi_{[0,b)}Kz = f_b. \tag{2.240}$$

Since K is Volterra,

$$\chi_{[0,b)}Kz = \chi_{[0,b)}K\big[\chi_{[0,b)}z\big] = K_b z_b. \tag{2.241}$$

Thus (2.236) has the solution $z_b \in \mathbf{B}_b$ for each $f_b \in \mathbf{B}$. □

2.4.2. Banach space of solutions

Assuming $\mathbf{B}$ to be a Banach space and making a proper choice, we are able to establish the connection between classical notions of stability and the notion of $\mathbf{D}_0$-stability and obtain various tests of $\mathbf{D}_0$-stability. In the case $\mathbf{B}$ is a Banach space, the space $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$ is also Banach under the norm

$$\|x\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})} = \big\|\mathcal{L}_0 x\big\|_{\mathbf{B}} + \big\|x(0)\big\|_{\mathbb{R}^n}. \tag{2.242}$$

We assume everywhere below that $\mathbf{B}$ and $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$ are Banach spaces.

Lemma 2.15. *Let the operator $\mathcal{L}$ as $\mathcal{L} : \mathbf{D}(\mathcal{L}_0,\mathbf{B}) \to \mathbf{B}$ be bounded and let the linear manifolds $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$ and $\mathbf{D}(\mathcal{L},\mathbf{B})$ coincide. Then the norms*

$$\|x\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})} = \big\|\mathcal{L}_0 x\big\|_{\mathbf{B}} + \big\|x(0)\big\|_{\mathbb{R}^n}, \qquad \|x\|_{\mathbf{D}(\mathcal{L},\mathbf{B})} = \|\mathcal{L}x\|_{\mathbf{B}} + \big\|x(0)\big\|_{\mathbb{R}^n} \tag{2.243}$$

are equivalent.

Proof. For each $x \in \mathbf{D}(\mathcal{L}_0,\mathbf{B})$, we have

$$x = Cf + X\alpha = Wz + U\alpha, \quad \text{where } f = \mathcal{L}x,\ z = \mathcal{L}_0 x,\ \alpha = x(0). \tag{2.244}$$

Therefore,

$$\begin{aligned} \|x\|_{\mathbf{D}(\mathcal{L},\mathbf{B})} &= \big\|\mathcal{L}(Wz + U\alpha)\big\|_{\mathbf{B}} + \|\alpha\|_{\mathbb{R}^n} \\ &= \big\|\mathcal{L}W(\mathcal{L}_0 x) + \mathcal{L}U\alpha\big\|_{\mathbf{B}} + \|\alpha\|_{\mathbb{R}^n} \le M\|x\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})}, \end{aligned} \tag{2.245}$$

where

$$M = \max\big\{\|\mathcal{L}W\|_{\mathbf{B}\to\mathbf{B}}, \|\mathcal{L}U\|_{\mathbb{R}^n\to\mathbf{B}} + 1\big\}. \tag{2.246}$$

In much the same way,

$$\|x\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})} = \big\|\mathcal{L}_0(Cf + X\alpha)\big\|_{\mathbf{B}} + \|\alpha\|_{\mathbb{R}^n} \le N\|x\|_{\mathbf{D}(\mathcal{L},\mathbf{B})}, \tag{2.247}$$

where

$$N = \max\{||\mathcal{L}_0 C||_{\mathbf{B}\to\mathbf{B}}, ||\mathcal{L}_0 X||_{\mathbb{R}^n\to\mathbf{B}} + 1\}. \tag{2.248}$$

□

The following rephrasing of Theorem 2.12 is useful in getting efficient tests of $\mathbf{D}_0$-stability.

Theorem 2.16 (Theorem 2.12 bis). *Let $\mathcal{L}$ be acting from $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ into $\mathbf{B}$ and let it be bounded. Then the following assertions are equivalent.*

(a) *The manifolds $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ and $\mathbf{D}(\mathcal{L}, \mathbf{B})$ coincide and, besides, the norms*

$$\begin{aligned} \|x\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})} &= ||\mathcal{L}_0 x||_{\mathbf{B}} + ||x(0)||_{\mathbb{R}^n}, \\ \|x\|_{\mathbf{D}(\mathcal{L},\mathbf{B})} &= \|\mathcal{L}x\|_{\mathbf{B}} + ||x(0)||_{\mathbb{R}^n} \end{aligned} \tag{2.249}$$

are equivalent.

(b) *There exists the bounded inverse $[\mathcal{L}W]^{-1} : \mathbf{B} \to \mathbf{B}$.*

(c) *The solution of the Cauchy problem*

$$\mathcal{L}x = f, \qquad x(0) = \alpha \tag{2.250}$$

belongs to $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$.

The space $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ is isomorphic to the product $\mathbf{B} \times \mathbb{R}^n$ with the isomorphism defined by (2.221). $\mathbf{D}_0$-property guarantees that the principal part $\mathcal{L}W$ of the operator $\mathcal{L} : \mathbf{D}(\mathcal{L}_0, \mathbf{B}) \to \mathbf{B}$ is Fredholm (in our case $\mathcal{L}W$ is invertible). Thus the assertions of the general theory are applicable to the equation $\mathcal{L}x = f$ which is $\mathbf{D}_0$-stable. In particular, the solution of the Cauchy problem depends continuously on f and α in the metric of the space $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$. The metric differs from Chebyshev's one which is the base of classical notions of stability. Nevertheless, as it will be seen from Lemma 2.17 given below, the main problems of the classical theory of stability will find their solutions if the $\mathbf{D}_0$-property is established at the proper choice of a model equation and a space $\mathbf{B}$.

The theory of stability of differential equations considers the problem of stability with respect to the right-hand side of the equation, f. We formulate the problem in the following form that is convenient for our purposes.

Let $\mathbf{V}$ be a Banach space of functions $x : [0, \infty) \to \mathbb{R}^n$ under the norm $\|\cdot\|_{\mathbf{V}}$. We will say that the equation $\mathcal{L}x = f$ is $\mathbf{V}$-stable if the solution x of the Cauchy problem $\mathcal{L}x = f$, $x(0) = \alpha$, belongs to $\mathbf{V}$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$ and this solution depends continuously on f and α: for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x - x_1\|_{\mathbf{V}} < \varepsilon$ if $\|f - f_1\|_{\mathbf{B}} < \delta$, $\|\alpha - \alpha_1\|_{\mathbb{R}^n} < \delta$, where x_1 is the solution of the Cauchy problem $\mathcal{L}x = f_1$, $x(0) = \alpha_1$.

Thus the $\mathbf{V}$-stability means that $\mathbf{D}(\mathcal{L}, \mathbf{B}) \subset \mathbf{V}$ and the operators $C : \mathbf{B} \to \mathbf{V}$ and $X : \mathbb{R}^n \to \mathbf{V}$ are bounded. Besides, the imbedding $\mathbf{D}(\mathcal{L}, \mathbf{B}) \subset \mathbf{V}$ is continuous:

there exists a constant $k > 0$ such that $\|x\|_{\mathbf{V}} \le k\|x\|_{\mathbf{D}(\mathcal{L},\mathbf{B})}$. Really, let

$$k = \max\left\{\|C\|_{\mathbf{B}\to\mathbf{V}}, \|X\|_{\mathbb{R}^n\to\mathbf{V}}\right\}. \tag{2.251}$$

If $x \in \mathbf{D}(\mathcal{L},\mathbf{B})$, we have $x = Cf + X\alpha$,

$$\|x\|_{\mathbf{V}} \le k\left(\|f\|_{\mathbf{B}} + \|\alpha\|_{\mathbb{R}^n}\right) = k\left(\|\mathcal{L}x\|_{\mathbf{B}} + \|\alpha\|_{\mathbb{R}^n}\right) = k\|x\|_{\mathbf{D}(\mathcal{L},\mathbf{B})}. \tag{2.252}$$

Lemma 2.17. *Let $\mathcal{L} : \mathbf{D}(\mathcal{L}_0,\mathbf{B}) \to \mathbf{B}$ be bounded, let the equation $\mathcal{L}x = f$ be $\mathbf{D}_0$-stable, and let the imbedding $\mathbf{D}(\mathcal{L}_0,\mathbf{B}) \subset \mathbf{V}$ be continuous. Then the equation is $\mathbf{V}$-stable.*

Proof. It is sufficient to prove the boundedness of $C : \mathbf{B} \to \mathbf{V}$ and $X : \mathbb{R}^n \to \mathbf{V}$. Since the equation is $\mathbf{D}_0$-stable, the operators $C : \mathbf{B} \to \mathbf{D}(\mathcal{L}_0,\mathbf{B})$ and $X : \mathbb{R}^n \to \mathbf{D}(\mathcal{L}_0,\mathbf{B})$ are bounded. The imbedding $\mathbf{D}(\mathcal{L}_0,\mathbf{B}) \subset \mathbf{V}$ is continuous due to the conditions. Therefore,

$$\|Cf\|_{\mathbf{V}} \le k\|Cf\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})} \le k\|C\|_{\mathbf{B}\to\mathbf{D}(\mathcal{L}_0,\mathbf{B})}\|f\|_{\mathbf{B}}. \tag{2.253}$$

Hence

$$\|C\|_{\mathbf{B}\to\mathbf{V}} \le k\|C\|_{\mathbf{B}\to\mathbf{D}(\mathcal{L}_0,\mathbf{B})}. \tag{2.254}$$

The boundedness of $X : \mathbb{R}^n \to \mathbf{V}$ is obtained similarly. □

Lemma 2.18. *Let $\mathcal{L}$ be acting from $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$ into $\mathbf{B}$ and let it be bounded. Let, further, the equation $\mathcal{L}x = f$ be $\mathbf{V}$-stable. If the operator $\mathcal{L}_0 - \mathcal{L}$ is defined on the whole $\mathbf{V}$ and $(\mathcal{L}_0 - \mathcal{L})\mathbf{V} \subset \mathbf{B}$, then the equation $\mathcal{L}x = f$ is $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$-stable.*

Proof. Since $\mathbf{D}(\mathcal{L}_0,\mathbf{B}) \subset \mathbf{D}(\mathcal{L},\mathbf{B})$, it is sufficient to prove that the solution $x \in \mathbf{V}$ of the Cauchy problem belongs to $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$. Rewriting the problem in the form

$$\mathcal{L}_0 x = (\mathcal{L}_0 - \mathcal{L})x + f, \qquad x(0) = \alpha \tag{2.255}$$

we observe that any solution of the problem satisfies the equation

$$x = W(\mathcal{L}_0 - \mathcal{L})x + Wf + U\alpha. \tag{2.256}$$

Since W is acting from $\mathbf{B}$ into $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$, any solution $x \in \mathbf{V}$ of the latter equation and, consequently, of the Cauchy problem, belongs to $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$. □

On the base of Lemmas 2.17 and 2.18, we are in position to establish the following assertion.

Theorem 2.19. *Let the imbedding $\mathbf{D}(\mathcal{L}_0,\mathbf{B}) \subset \mathbf{V}$ be continuous, let $\mathcal{L}$ be acting from $\mathbf{D}(\mathcal{L}_0,\mathbf{B})$ into $\mathbf{B}$ and bounded. If, besides, the difference $\mathcal{L}_0 - \mathcal{L}$ is defined on the whole $\mathbf{V}$ and $(\mathcal{L}_0 - \mathcal{L})\mathbf{V} \subset \mathbf{B}$, then the $\mathbf{V}$-stability and $\mathbf{D}_0$-stability are equivalent.*

Further in examples we will use the following Banach spaces.

(i) The space $\mathbf{L}_\infty$ of measurable and essentially bounded functions $z : [0,\infty) \to \mathbb{R}^n$, $\|z\|_{\mathbf{L}_\infty} = \operatorname{ess\,sup}_{t\geq 0} \|z(t)\|_{\mathbb{R}^n}$.

(ii) The space $\mathbf{L}_\infty^\gamma$ of all functions of the form $z(t) = e^{-\gamma t}y(t)$, where $y \in \mathbf{L}_\infty$, $\gamma > 0$, $\|z\|_{\mathbf{L}_\infty^\gamma} = \|y\|_{\mathbf{L}_\infty}$.

(iii) The space $\mathbf{C}$ of continuous bounded functions $x : [0,\infty) \to \mathbb{R}^n$, $\|x\|_{\mathbf{C}} = \sup_{t\geq 0} \|x(t)\|_{\mathbb{R}^n}$.

(iv) The space $\mathbf{C}^\gamma$ of all functions of the form $x(t) = e^{-\gamma t}y(t)$, where $y \in \mathbf{C}$, $\gamma > 0$, $\|x\|_{\mathbf{C}^\gamma} = \|y\|_{\mathbf{C}}$.

Lemma 2.20. *Let $\mathcal{L}_0 x = \dot{x} + \beta x$, $\beta = \text{const} > 0$, $\mathbf{B} = \mathbf{L}_\infty^\gamma$, $0 < \gamma < \beta$. Then the imbedding $\mathbf{D}(\mathcal{L}_0, \mathbf{B}) \subset \mathbf{C}^\gamma$ is continuous.*

Proof. If $x \in \mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty^\gamma)$, we have

$$x(t) = \int_0^t e^{-\beta(t-s)}e^{-\gamma s}y(s)ds + e^{-\beta t}x(0), \quad y \in \mathbf{L}_\infty;$$

$$\begin{aligned} \|x\|_{\mathbf{C}^\gamma} &\leq \sup_{t\geq 0} e^{-(\beta-\gamma)t} \int_0^t e^{(\beta-\gamma)s}\|y(s)\|_{\mathbb{R}^n} ds + \|x(0)\|_{\mathbb{R}^n} \\ &= \frac{1}{\beta-\gamma}\|\mathcal{L}_0 x\|_{\mathbf{L}_\infty^\gamma} + \|x(0)\|_{\mathbb{R}^n} \leq k\|x\|_{\mathbf{D}(\mathcal{L}_0,\mathbf{L}_\infty^\gamma)}, \end{aligned} \tag{2.257}$$

where $k = \max\{1, 1/(\beta-\gamma)\}$. □

Remark 2.21. If $\mathcal{L}_0 x = \dot{x} + \beta x$, $\mathbf{B} = \mathbf{L}_\infty^\gamma$, $0 < \gamma < \beta$, then $\mathbf{D}_0$-stability of the equation $\mathcal{L}x = f$ provides, by Lemmas 2.17 and 2.20, $\mathbf{C}^\gamma$-stability of this equation. Consequently, in particular, such a $\mathbf{D}_0$-stability of the equation $\mathcal{L}x = f$ gives the exponential stability. If, besides, the difference $\mathcal{L}_0 - \mathcal{L}$ is defined on the whole $\mathbf{C}^\gamma$ and $(\mathcal{L}_0 - \mathcal{L})\mathbf{C}^\gamma \subset \mathbf{L}_\infty^\gamma$, then, by Lemma 2.18, $\mathbf{D}_0$-stability follows from $\mathbf{C}^\gamma$-stability.

$\mathbf{D}_0$-stability at the given $\mathcal{L}_0$ and $\mathbf{B}$ defines a wide class of equations $\mathcal{L}x = f$, for which the manifolds $\mathbf{D}(\mathcal{L}, \mathbf{B})$ coincide with each other and with the manifold $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$. For instance, all $\mathbf{D}(\mathcal{L}^\beta, \mathbf{B})$ with $\mathcal{L}^\beta x = \dot{x} + \beta x$, $\beta > 0$, $\mathbf{B} = \mathbf{L}_\infty$, coincide with each other and with the linear manifold of $x \in \mathbf{Y}$ with the property that $\sup_{t\geq 0}\|x(t)\|_{\mathbb{R}^n} < \infty$, $\operatorname{ess\,sup}_{t\geq 0}\|\dot{x}(t)\|_{\mathbb{R}^n} < \infty$. Thus any equation of the class under consideration may be taken as a model equation and, consequently, the selection of the model equation is sufficiently wide.

2.4.3. Application of W-method

Theorem 2.16 offers a scheme of deciding whether the elements $x \in \mathbf{D}(\mathcal{L}, \mathbf{B})$ (the solutions x of the equation $\mathcal{L}x = f$ by $f \in \mathbf{B}$) possess the given property (for instance, the property that $x \in \mathbf{L}_\infty$ if $f \in \mathbf{L}_\infty$). This scheme is called *W-method* and is reducible to selecting the model equation $\mathcal{L}_0 x = z$ (or an operator W) so that the following conditions are fulfilled.

(a) The manifold $\mathbf{Y}_0 \subset \mathbf{Y}$ with the given asymptotical properties coincides with $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$.

(b) The operator $\mathcal{L}$ is acting from $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ into $\mathbf{B}$.

(c) The operator $\mathcal{L}W : \mathbf{B} \to \mathbf{B}$ is invertible.

The invertibility of $\mathcal{L}W = I - (\mathcal{L}_0 - \mathcal{L})W$ (the existence of the fix point of the operator $(\mathcal{L}_0 - \mathcal{L})W : \mathbf{B} \to \mathbf{B}$) is guaranteed by the Banach principle if the estimate

$$||(\mathcal{L}_0 - \mathcal{L})W||_{\mathbf{B}\to\mathbf{B}} < 1 \tag{2.258}$$

holds.

By virtue of the equivalence of the assertions (b) and (c) of Theorem 2.16 the invertibility of $\mathcal{L}W : \mathbf{B} \to \mathbf{B}$ is equivalent to the invertibility of $W\mathcal{L} : \mathbf{D}(\mathcal{L}_0, \mathbf{B}) \to \mathbf{D}(\mathcal{L}_0, \mathbf{B})$. Moreover the assertion (c) on the belonging of the solutions of the Cauchy problem to $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$ is equivalent to the solvability of the equation

$$W\mathcal{L}x \equiv x - W(\mathcal{L}_0 - \mathcal{L})x = Wf + U\alpha \tag{2.259}$$

in the space $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$. Thus the estimate

$$||W(\mathcal{L}_0 - \mathcal{L})||_{\mathbf{D}(\mathcal{L}_0,\mathbf{B})\to\mathbf{D}(\mathcal{L}_0,\mathbf{B})} < 1 \tag{2.260}$$

guarantees $\mathbf{D}_0$-stability.

It might be useful to observe that in the case of continuous acting of $\mathcal{L}_0 - \mathcal{L}$ from $\mathbf{V}$ into $\mathbf{B}$ and under the assumption that $\mathbf{D}(\mathcal{L}_0, \mathbf{B}) \subset \mathbf{V}$ the estimate (2.260) may be replaced by

$$||W(\mathcal{L}_0 - \mathcal{L})||_{\mathbf{V}\to\mathbf{V}} < 1. \tag{2.261}$$

It follows from the fact that any solution $x \in \mathbf{V}$ of (2.259) belongs to $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$.

Let us observe also that at the establishment of any of estimates (2.258), (2.260), or (2.261) it may be assumed, by Theorem 2.14, that $(\mathcal{L}_0x - \mathcal{L}x)(t) = 0$, $t \in [0, b]$.

For the purposes of illustration, let us consider as a typical example the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + P(t)x(t) = f(t) \tag{2.262}$$

under the assumption that the columns of the $n \times n$ matrix P belong to $\mathbf{L}_\infty$.

Let $\mathcal{L}_0x = \dot{x} + \beta x$, $\beta > 0$, $\mathbf{B} = \mathbf{L}_\infty^\gamma$, $0 < \gamma < \beta$. We have

$$\mathcal{L}Wz = z - (\mathcal{L}_0 - \mathcal{L})Wz, \tag{2.263}$$

where

$$[(\mathcal{L}_0 - \mathcal{L})Wz](t) = [E\beta - P(t)]\int_0^t e^{-\beta(t-s)}z(s)ds;$$

$$\begin{aligned}\|(\mathcal{L}_0 - \mathcal{L})Wz\|_{\mathbf{L}_\infty^\gamma \to \mathbf{L}_\infty^\gamma} &\le \operatorname*{ess\,sup}_{t\ge 0}\left\{\|E\beta - P(t)\|e^{\gamma t}\int_0^t e^{-\beta(t-s)}e^{-\gamma s}e^{\gamma s}\|z(s)\|_{\mathbb{R}^n}ds\right\}\\ &\le \operatorname*{ess\,sup}_{t\ge 0}\|E\beta - P(t)\|\frac{1}{\beta-\gamma}\|z\|_{\mathbf{L}_\infty^\gamma}.\end{aligned} \tag{2.264}$$

Here and below, $\|A\|$ is the norm of the matrix A agreed with the norm in $\mathbb{R}^n$.

By Theorem 2.16 and Theorem 2.14, we obtain that if there exists $b > 0$ such that the inequality

$$\operatorname*{ess\,sup}_{t\ge b}\|E\beta - P(t)\| < \beta - \gamma \tag{2.265}$$

holds, then we have $\mathbf{D}_0$-stability of (2.262).

Using Lemma 2.17, we establish the following assertion.

Theorem 2.22. *Let there exist a number $\beta > 0$ such that*

$$\operatorname*{ess\,lim}_{t\to\infty}\|E\beta - P(t)\| < \beta. \tag{2.266}$$

Then (2.262) *is $\mathbf{C}^\gamma$-stable for a sufficiently small $\gamma > 0$.*

If $\|\alpha\|_{\mathbb{R}^n}$ is defined by $\|\alpha\| = \max\{|\alpha^1|,\dots,|\alpha^n|\}$, the corresponding norm of the $n\times n$ matrix $P = (p_{ij})$ is defined by $\|P\| = \max_i\sum_{j=1}^n |p_{ij}|$. Under such a norm the estimate (2.266) holds if

$$\operatorname*{ess\,lim}_{t\to\infty}\left\{|\beta - p_{ii}(t)| + \sum_{j\ne i}|p_{ij}(t)|\right\} < \beta, \quad i = 1,\dots,n. \tag{2.267}$$

In the case $\beta > \max_i\{\operatorname{ess\,lim}_{t\to\infty} p_{ii}(t)\}$, we obtain the following.

Corollary 2.23. *Let*

$$\operatorname*{ess\,lim}_{t\to\infty}\left\{p_{ii}(t) - \sum_{j\ne i}|p_{ij}(t)|\right\} > 0, \quad i = 1,\dots,n. \tag{2.268}$$

Then there exists $\gamma > 0$ such that (2.262) *is $\mathbf{C}^\gamma$-stable.*

The assertions above may be sharpened by using more complicated model equations. Thus, taking $(\mathcal{L}_0x)(t) = \dot{x}(t) + P_0x(t)$ with diagonal matrix $P_0 = \operatorname{diag}(\beta^1,\dots,\beta^n)$, we obtain the following test of stability.

Theorem 2.24 (Theorem 2.22 bis). *Assume that there exist positive constants $\beta^1, \ldots, \beta^n$ such that*

$$\operatorname*{ess\,lim}_{t\to\infty}\left\{\beta^i p_{ii}(t) - \sum_{j\neq i}\beta^i \,|\, p_{ij}(t)\,|\right\} > 0, \quad i = 1, \ldots, n. \tag{2.269}$$

Then there exists a $\gamma > 0$ such that (2.262) *is* $\mathbf{C}^\gamma$*-stable.*

The "diagonal prevalence" in the tests of $\mathbf{C}^\gamma$-stability formulated above (and well known) is a result of the fact that the matrix P of (2.262) was compared with a diagonal matrix P_0 of the model equation. Such a rough comparison may be explained by the necessity to integrate the model equation in the explicit form. There are some other possible variants of model equations where we can construct the Cauchy matrix in the explicit form. For instance, the differential equation with triangular matrix P_0, with constant matrix P_0, or the equation

$$(\mathcal{L}_0 x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + P_0(t)x(t) = z(t), \quad \text{where } P_0(t) = -\dot{U}(t)U^{-1}(t), \tag{2.270}$$

and the matrix U with the columns from $\mathbf{Y}$ guarantees the proper estimate of the matrix $W(t,s) = U(t)U^{-1}(s)$. But we do not know any detailed investigation in such a direction yet.

As another simple example, consider the scalar ($n = 1$) equation

$$\mathcal{L}x \equiv \mathcal{L}_0 x - Tx = f \tag{2.271}$$

under the following assumptions:

$$\nu \stackrel{\text{def}}{=} \sup_{t\geq 0} |\, U(t)\,| < \infty, \qquad \sigma \stackrel{\text{def}}{=} \sup_{t\geq 0} \int_0^t |\, W(t,s)\,|\, ds < \infty, \tag{2.272}$$

the operator $T : \mathbf{C} \to \mathbf{L}_\infty$ is continuous and monotone (isotonic or antitonic).

The equation (2.259) that is equivalent to the Cauchy problem $\mathcal{L}x = f$, $x(0) = \alpha$ has the form

$$(W\mathcal{L}x)(t) \equiv x(t) - \int_0^t W(t,s)(Tx)(s)ds = (Wf)(t) + (U\alpha)(t). \tag{2.273}$$

The solvability of the equation in the space $\mathbf{C}$ guarantees $\mathbf{D}_0$-stability which means in this case that all the solutions of (2.271) belong to $\mathbf{L}_\infty$ if $f \in \mathbf{L}_\infty$.

Denoting

$$\tau_b = \operatorname*{ess\,sup}_{t\geq b} |\,[T(1)](t)\,|, \qquad \sigma_b = \sup_{t\geq b}\int_b^t |\, W(t,s)\,|\, ds, \tag{2.274}$$

and using the scheme above, we obtain the following.

Theorem 2.25. *The equation* (2.271) *is* **C**-*stable if there exists* $b > 0$ *such that*

$$\sigma_b \cdot \tau_b < 1. \tag{2.275}$$

Proof. The equation (2.271) is $\mathbf{D}_0$-stable since the inequality (2.261) holds if $\mathbf{V} = \mathbf{C}$ and $(Tx)(t) = 0$ at $t \in [0, b)$. The imbedding $\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty) \subset \mathbf{C}$ is continuous. Indeed,

$$\begin{aligned} \|x\|_{\mathbf{C}} &= \sup_{t\geq 0} |x(t)| = \sup_{t \geq 0} \left|(Wz)(t) + \big(Ux(0)\big)(t)\right| \\ &\leq \sigma \|z\|_{\mathbf{L}_\infty} + \nu |x(0)| = \sigma \left\|\mathcal{L}_0 x\right\|_{\mathbf{L}_\infty} + \nu |x(0)| \leq k \|x\|_{\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty)}, \end{aligned} \tag{2.276}$$

where $k = \max\{\delta, \nu\}$.

The reference to Lemma 2.17 completes the proof. □

Applying Theorem 2.25 to the scalar equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + p x_h(t) = f(t) \quad (p = \text{const} > 0), \tag{2.277}$$

we obtain the following.

Corollary 2.26. *The equation* (2.277) *is* **C**-*stable if there exists* $b > 0$ *such that*

$$t - h(t) < \frac{1}{p} \quad \textit{for a.a. } t > b. \tag{2.278}$$

Proof. Putting $\mathcal{L}_0 x = \dot{x} + px$, we have

$$W(t,s) = e^{-p(t-s)}, \qquad \sigma = \sup_{t \geq 0} \int_0^t W(t,s)\,ds = \frac{1}{p}. \tag{2.279}$$

The difference $t - h(t)$ is bounded, so we will assume that $h(t) > 0$ for a sufficiently large b. Under such an assumption,

$$(\mathcal{L}x)(t) = \big(\mathcal{L}_0 x\big)(t) - p\big(x(t) - x_h(t)\big) = \dot{x}(t) + px(t) - p\int_{h(t)}^t \dot{x}(s)\,ds. \tag{2.280}$$

Since $\dot{x} = f - px_h$ for the solution of (2.277), such a solution satisfies the equation

$$\big(\mathcal{L}_1 x\big)(t) \stackrel{\text{def}}{=} \dot{x}(t) + px(t) + p^2 \int_{h(t)}^t x_h(s)\,ds = f(t) + p\int_{h(t)}^t f(s)\,ds. \tag{2.281}$$

Applying Theorem 2.25 to this equation, we complete the proof. □

As it was shown in [92], the other choice of the model equation and some more sophistical reasoning guarantee **C**-stability of the equation

$$\dot{x}(t) + p(t)x_h(t) = f(t) \tag{2.282}$$

under the conditions

$$p(t) \geq \text{const} > 0, \qquad \operatorname*{ess\,lim}_{t\to\infty} \int_{h(t)}^{t} p(s)ds < 1 + \frac{1}{e}. \tag{2.283}$$

There is an extensive literature on the so-called Bohl-Perron-like theorems [171] (see, for instance, [41, 47, 96, 98, 218]). Under the conditions of such theorems, one may state that a $\mathbf{D}(\mathcal{L}_0, \mathbf{B})$-property involves a more refined $\mathbf{D}(\mathcal{L}_0, \mathbf{B}_0)$-property where $\mathbf{B}_0 \subset \mathbf{B}$. We give below without proof one of the simplest Bohl-Perron-like assertions.

Theorem 2.27 (see [26, 41]). *Let $\mathcal{L}_0 x = \dot{x} + \beta x$, $\beta > 0$, let $\mathcal{L} : \mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty) \to \mathbf{L}_\infty$ be bounded, and let it satisfy the "Δ-condition." There exist positive numbers N and β such that*

$$||(\mathcal{L}x)(t)||_{\mathbb{R}^n} < Ne^{-\beta t} \tag{2.284}$$

for each $x \in \mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty)$ such that $\|\dot{x}(t)\|_{\mathbb{R}^n} + \|x(t)\|_{\mathbb{R}^n} < e^{-\beta t}$.

Then there exists $\gamma > 0$ at which the equation $\mathcal{L}x = f$ is $\mathbf{D}(\mathcal{L}_0, \mathbf{L}_\infty^\gamma)$-stable.

The Δ-condition is fulfilled, for example, if

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + \int_0^t d_s R(t,s)x(s), \tag{2.285}$$

where the operator T, $(Tx)(t) = \int_0^t d_s R(t,s)x(s)$, acts continuously from $\mathbf{C}$ into $\mathbf{L}_\infty$, and there exists $\delta > 0$ such that $R(t,s) = 0$ in the triangle $0 \leq s \leq t \leq \delta$. Thus Δ-condition is fulfilled for the ordinary differential equation and the equation of the form

$$\dot{x}(t) + P(t)x_h(t) = f(t) \tag{2.286}$$

with essentially bounded elements of a matrix $P(\cdot)$ and the "bounded delay": $t - h(t) <$ const.

The equation (2.271) under the Δ-condition and the inequality $\sigma_b \cdot \tau_b < 1$ is $\mathbf{C}^\gamma$-stable for a sufficiently large b at a $\gamma > 0$ by virtue of Theorem 2.27. Also (2.277) is $\mathbf{C}^\gamma$-stable if $[t - h(t)] < 1/p$.

The book [41] provided with an extensive bibliography is devoted to researches on asymptotic behavior and stability of solutions of equations with aftereffect. It should be remarked that the first results on stability, obtained on the ground of the representation (2.69) of the general solution were published in [42, 218].

2.5. Equations with generalized Volterra operators

Let[1] $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$, let an isomorphism $J = \{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ be defined by equality $x = \Lambda z + Y\beta$, where $\{z, \beta\} \in \mathbf{B} \times \mathbb{R}^n$, and let $J^{-1} = [\delta, r]$. Consider the principal boundary value problem

$$\mathcal{L}x = f, \qquad rx = \alpha. \tag{2.287}$$

By Theorem 1.16, this problem is uniquely solvable if and only if the operator $Q = \mathcal{L}\Lambda : \mathbf{B} \to \mathbf{B}$ has the bounded inverse Q^{-1}. Some special features of equations with aftereffect are connected with such a situation when operators Q and Q^{-1} are Volterra ones in sense of the definition by Tikhonov [215] (see, for instance, Azbelev et al. [32, Chapter 5]). In this situation, one can sometimes use some specific techniques for the study of an equation under consideration. The special features and techniques above are still retained in the case $Q : \mathbf{B} \to \mathbf{B}$ and Q^{-1} are Volterra operators in a generalized sense. Certain notions of the generalized Volterra property had been introduced and applied in the works [57, 86, 91, 213, 227, 228, 232] and others.

We define a generalization of the Volterra property as follows.

Let us assume that, for every $\tau \in [0,1]$, a linear projector $P^\tau : \mathbf{B} \to \mathbf{B}$ is given such that $P^0 = 0$, $P^1 = I$, and $P^\tau P^\sigma = P^{\min(\tau,\sigma)}$.

An operator $Q : \mathbf{B} \to \mathbf{B}$ is called *Volterra operator (in the generalized sense)* if, for all $\tau \in [0,1]$, $P^\tau y = 0$ implies $P^\tau Q y = 0$. This condition is equivalent to the following: $P^\tau Q P_\tau = 0$ for all $\tau \in [0,1]$, where the projector $P_\tau : \mathbf{B} \to \mathbf{B}$ is defined by $P_\tau = I - P^\tau$.

Note that the classical Volterra property by Tikhonov in the space $\mathbf{B} = \mathbf{L}$ is a specific case of the general one, it is defined by the projectors

$$(P^\tau z)(t) = \begin{cases} z(t) & \text{if } t \in [a, a + (b-a)\tau], \\ 0 & \text{if } t \in (a + (b-a)\tau, b]. \end{cases} \tag{2.288}$$

It is easy to see that the sum, the product, as well as the limit of the sequence of Volterra operators that converges at any point of $\mathbf{B}$ are Volterra operators. Therefore, if $Q = I - A$ and the spectral radius of A is less than 1, the inverse $Q^{-1} : \mathbf{B} \to \mathbf{B}$ exists and is Volterra too.

To estimate the spectral radius of Volterra operator A, one can apply the following technique. If z is a solution of the equation

$$z = Az + f, \tag{2.289}$$

then the element $z^\tau = P^\tau z$ is a solution of the equations $z^\tau = P^\tau A z^\tau + P^\tau f$ with parameter τ. We will call this z^τ *local solution* of (2.289). The Volterra property of A allows us to construct the solution of (2.289) by the prolongations of the local

[1]written by S. A. Gusarenko

solutions in the parameter. In particular, if $\|P^\sigma AP_\tau\| < 1$ for some $\sigma \in (\tau, 1)$, then due to the Banach principle, there exists the local solution z^σ of (2.289), as $z^\sigma = P^\sigma AP_\tau z^\sigma + P^\sigma Az^\tau + P^\sigma f$. In connection with this consideration, we give the following definition.

An operator $A : \mathbf{B} \to \mathbf{B}$ is called *q-bettering* if there exists a sequence $\{\tau_i\}$, $0 = \tau_0 < \tau_1 < \cdots < \tau_i < \cdots < \tau_n = 1$, such that $\|P^{\tau_{i+1}}AP_{\tau_i}\| \leq q$ for any $i = 0, \dots, n-1$.

It is clear that in case that $\|A\| \leq q$, the operator A is q-bettering. The reverse is not true, as the following example shows. Let $\mathbf{B}$ be the space of continuous functions $z : [0,1] \to \mathbb{R}$ such that $z(0) = 0$ ($\|z\|_{\mathbf{B}} = \sup_{t\in[0,1]} \|z(t)\|$), and

$$(P^\tau z)(t) = \begin{cases} z(t) & \text{if } t \in [0,\tau], \\ z(t) - z(\tau) & \text{if } t \in (\tau, 1]. \end{cases} \tag{2.290}$$

Define the operator $A : \mathbf{B} \to \mathbf{B}$ by the equality $(Az)(t) = (\alpha + \beta t)z(\lambda t)$, where $\alpha \geq 0$, $\beta \geq 0$, $0 \leq \lambda < 1$. Then $\|A\| = \alpha + \beta$ and A is q-bettering if $q > \alpha$.

Theorem 2.28. *Let a Volterra operator A be q-bettering. Then its spectral radius is not greater than q.*

Proof. Show that in the case $|\lambda| < q^{-1}$ the equation $z = \lambda Az + f$ is uniquely solvable for any $f \in \mathbf{B}$. The set of local solutions $z_i = \lambda P^{\tau_i}Az_i + P^{\tau_i}f$ of this equation can be obtained with recurrent formula $z_{i+1} = (I - \lambda P^{\tau_{i+1}}AP_{\tau_i})^{-1}P^{\tau_{i+1}}(Az_i + f)$. Hence the equation $z = \lambda Az + f$ has the unique solution. □

To compute q, one can apply the following.

Theorem 2.29. *Let Volterra operators A_1, A_2 be q_1- and q_2-bettering, respectively. Then $A_1 + A_2$ and A_1A_2 are $(q_1 + q_2)$- and q_1q_2-bettering, respectively.*

Proof. Let $\|P^{\theta_{i+1}}A_1P_{\theta_i}\| \leq q_1$ and $\|P^{\sigma_{i+1}}A_2P_{\sigma_i}\| \leq q_2$ hold for the partitions $0 = \theta_0 < \theta_1 < \cdots < \theta_n = 1$, $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_m = 1$. Denote $\theta = \{\theta_i\}_{i=1}^n$, $\sigma = \{\sigma_i\}_{i=1}^m$. For the points τ_i belonging to the partition $\tau = \theta \cup \sigma$, we have $\|P^{\tau_{i+1}}(A_1 + A_2)P_{\tau_i}\| \leq \|P^{\tau_{i+1}}A_1P_{\tau_i}\| + \|P^{\tau_{i+1}}A_2P_{\tau_i}\| \leq q_1 + q_2$ and $\|P^{\tau_{i+1}}A_1A_2P_{\tau_i}\| = \|P^{\tau_{i+1}}A_1P_{\tau_i}P^{\tau_{i+1}}A_2P_{\tau_i}\| \leq \|P^{\tau_{i+1}}A_1P_{\tau_i}\|\|P^{\tau_{i+1}}A_2P_{\tau_i}\| \leq q_1q_2$. □

We will call a Volterra operator $A : \mathbf{B} \to \mathbf{B}$ *strongly bettering* one if it is q-bettering for any $q > 0$. The spectral radius of the strongly bettering operator is equal to zero. Theorem 2.2 implies that the sum of the q-bettering operator and the strongly bettering one is q-bettering operator, and the product of the q-bettering operator and the strongly bettering one is strongly bettering.

Since the mapping, being continuous on a compact set, is uniformly continuous on that set and $\|P^{\tau_{i+1}}AP_{\tau_i}\| = \|(P^{\tau_{i+1}} - P^{\tau_i})AP_{\tau_i}\| \leq \|P^{\tau_{i+1}}A - P^{\tau_i}A\|$, the continuity of the mapping $A(\tau) \overset{\text{def}}{=} P^\tau A$ of the segment $[0,1]$ into the space of linear bounded operators implies the property of A of being strongly bettering.

Theorem 2.30. *Let* $\lim_{\sigma\to\tau} P^\sigma z = P^\tau z$ *for each* $\tau \in [0,1]$ *and let* $z \in \mathbf{B}$. *If operator A is Volterra and compact, then A is strongly bettering.*

Proof. Let us show that the mapping $A(\tau)$ is continuous. Actually, let $A(\tau)$ be not continuous. Then there exist numbers $\tau \in [0,1]$, $\varepsilon > 0$, and sequences $\{\tau_i\} \in [0,1]$, $\{z_i\} \in S$, $S = \{z \in \mathbf{B} : \|z\| \le 1\}$, such that $\tau_i \to \tau$, $\|(P^{\tau_i} - P^\tau)Az_i\| \ge \varepsilon$. Take from the sequence $\{Az_i\}$ a subsequence $\{y_i\}$ converging to a point $y \in S$. For a sufficiently large i, we will have $\|y_i - y\| < \varepsilon/4$, $\|(P^{\tau_i} - P^\tau)y\| < \varepsilon/4$. Therefore,

$$\|(P^{\tau_i} - P^\tau)y_i\| \le \|P^{\tau_i}(y_i - y)\| + \|(P^{\tau_i} - P^\tau)y\| + \|P^\tau(y - y_i)\| \le \frac{3}{4}\varepsilon < \varepsilon. \tag{2.291}$$

The contradiction completes the proof. □

Corollary 2.31. *Let* $\lim_{\sigma\to\tau} P^\sigma z = P^\tau z$ *for each* $\tau \in [0,1]$ *and let* $z \in \mathbf{B}$. *The sum of the Volterra q-bettering operator and the Volterra compact one is q-bettering.*

Corollary 2.32. *Let* $\lim_{\sigma\to\tau} P^\sigma z = P^\tau z$ *for each* $\tau \in [0,1]$ *and let* $z \in \mathbf{B}$. *Then the spectral radius of Volterra compact operator equals zero.*

Note that similar propositions on being equal to zero of spectral radius for certain classes of Volterra operators are obtained by a number of authors (see, for instance, [6, 227]). It should be remarked especially that there are necessary and sufficient conditions for the above property within some classes of linear bounded operators. See [86, 190, 191, 232].

The class of strongly bettering operators contains not only Volterra compact operators. For example, the operator

$$(Kz)(t) = \begin{cases} p(t)z(t-\omega) & \text{if } t \in [a+\omega, b), \\ 0 & \text{if } t \in [a, a+\omega], \end{cases} \tag{2.292}$$

with $p \in \mathbf{L}$ is not compact, it is Volterra in Tikhonov's sense (the projectors are defined by equalities (2.288)), but it is strongly bettering.

An example of a Volterra property (in the space $\mathbf{L}$) different from classical Tikhonov's one is given by the projectors

$$(P^\tau z)(t) = \begin{cases} z(t) & \text{if } t \in [u(\tau), v(\tau)], \\ 0 & \text{if } t \in [a, u(\tau)) \cup (v(\tau), b], \end{cases} \tag{2.293}$$

where continuous function $u : [0,1] \to [a,c]$ is strictly decreasing, $u(0) = c$, $u(1) = a$, and continuous function $v : [0,1] \to [c,b]$ is strictly increasing, $v(0) = c$, $v(1) = b$, $c \in (a,b)$. For this case, we formulate below the conditions of being Volterra for some linear operators.

Integral operator K acting in space $\mathbf{L}$,

$$(Kz)(t) = \int_a^b \mathcal{K}(t,s)z(s)ds, \tag{2.294}$$

is Volterra if $\mathcal{K}(t,s) = 0$ as $s \le t \le v(u^{-1}(s))$ and as $u(v^{-1}(s)) \le t \le s$. The operator K is strongly bettering if $\lim_{\delta\to 0}\int_{\theta-\delta}^{\theta+\delta} \operatorname{ess\,sup}_{s\in[a,b]} |\mathcal{K}(t,s)|dt = 0$ for any $\theta \in [a,b]$.

The inner superposition operator $S : \mathbf{L} \to \mathbf{L}$ defined by the equality

$$(Sz)(t) = \begin{cases} b(t)z[g(t)] & \text{if } g(t) \in [a,b], \\ 0 & \text{if } g(t) \notin [a,b], \end{cases} \tag{2.295}$$

is Volterra if $t \le g(t) \le v(u^{-1}(t))$ as $a \le t \le c$ and $u(v^{-1}(t)) \le g(t) \le t$ as $c \le t \le b$.

Operator S is strongly bettering if there exists $\delta > 0$ such that $t+\delta \le g(t) \le v(u^{-1}(t)) - \delta$ as $a \le t \le c$ and $u(v^{-1}(t)) + \delta \le g(t) \le t - \delta$ as $c \le t \le b$. These conditions mean that function g has no singular point (a point $\theta \in [a,b]$ is called *singular* if, for any $\delta > 0$, there exists a set $e \subset [a,b]$, $\operatorname{mes}(e) < \delta$ such that $\operatorname{mes}(e \cap g^{-1}(e)) > 0$). Operator S is q-bettering if there exists $\delta > 0$ such that

$$\operatorname*{ess\,sup}_{\substack{t\in\{e\cap g^{-1}(e)\} \\ \operatorname{mes}(e)<\delta}} |B(t)| \frac{\operatorname{mes}\left(g^{-1}(e) \cap [a,b]\right)}{\operatorname{mes}(e)} < q. \tag{2.296}$$

Consider the case, when operator A is representable in the form of the sum of an integral Volterra operator and an operator which, generally, is not Volterra.

Theorem 2.33. *Let $A = K + U$, where the kernel of integral operator $K : \mathbf{L} \to \mathbf{L}$ satisfies the estimate $|\mathcal{K}(t,s)| \le k(t)$, $k \in \mathbf{L}$, and operator $U : \mathbf{L} \to \mathbf{L}$ is bounded. If*

$$\|U\| < e^{-\int_a^b k(s)ds}, \tag{2.297}$$

then operator $Q = I - A$ has the bounded inverse $Q^{-1} : \mathbf{L} \to \mathbf{L}$.

Proof. Since $\|U\| < 1$ and operator K is weakly compact, the operator $I - U - K$ is Fredholm. Let us demonstrate that the equation $z = Uz + Kz$ has only zero solution. Indeed, denote $m(\tau) = \int_{u(\tau)}^{v(\tau)} |z(s)|ds$. Then $m(\tau) \le \|U\|m(1) + \int_{u(\tau)}^{c} k(t)\int_t^{v(u^{-1}(t))}|z(s)|ds\,dt + \int_c^{v(\tau)} k(t)\int_{u(v^{-1}(t))}^{t}|z(s)|ds\,dt \le \|U\|m(1) + \int_0^\tau \varphi(\eta)m(\eta)d\eta$, where $\int_0^1\varphi(\eta)d\eta = \int_a^b k(s)ds$. Therefore, $m(\tau) \le \|U\|m(1)e^{\int_0^\tau \varphi(\eta)d\eta}$, and $m(1) \le \|U\|m(1)e^{\int_a^b k(s)ds}$. Clearly, $m(1) = 0$, $z = 0$, and the equation $z = Uz + Kz + f$ is solvable for any $f \in \mathbf{L}$. □

Note that standard application of the Banach principle allows one to obtain no more than the condition $\|U\| < 1 - \int_a^b k(s)ds$.

Example 2.34. Consider the boundary value problem

$$\begin{gathered}
\dot{x}(t) = (S\dot{x})(t) + \int_a^b x(s)d_s r(t,s) + f(t), \quad t \in [a,b], \\
\dot{x}(\xi) = 0 \quad \text{if } \xi \notin [a,b], \\
x(c) + \int_a^b \phi(s)\dot{x}(s)ds = \alpha,
\end{gathered} \tag{2.298}$$

under the conditions (see Section 2.2) yielding the continuity of the operator S_g in the space $\mathbf{L}$ and the continuity of R,

$$(Rx)(t) = \int_a^b x(s)d_s r(t,s), \tag{2.299}$$

as an operator from $\mathbf{W}^1$ in $\mathbf{L}$; $\psi \in \mathbf{L}_\infty$, $f \in \mathbf{L}$. Define the isomorphism between spaces $\mathbf{W}^1$ and $\mathbf{L} \times \mathbb{R}^1$ by the equality $x(t) = \beta - \int_a^b \phi(s)z(s)ds + \int_c^t z(s)ds$. Then $Q = I - K - K_0 - S$, where $\mathcal{K}$ is integral operator with the kernel

$$\mathcal{K}(t,s) = \begin{cases} r(t,a) - r(t,s) & \text{if } a \le s \le c, \\ -r(t,s) & \text{if } c < s \le b, \end{cases} \tag{2.300}$$

and K_0 is integral operator with the kernel $\mathcal{K}_0(t,s) = r(t,a)\phi(s)$.

Theorem 2.35. (a) *Suppose $r(t,s) = 0$ as $u(v^{-1}(s)) \le t \le s$ and $r(t,s) = r(t,a)$ as $s \le t \le v(u^{-1}(s))$; $k(t) = \max(\operatorname{ess\,sup}_{s\in[a,c]} |r(t,a) - r(t,s)|, \operatorname{ess\,sup}_{s\in[c,b]} |r(t,s)|)$; $k \in \mathbf{L}$; and*

$$\|S\| + \|\phi\| \int_a^b |r(t,a)|\,dt < e^{-\int_a^b k(s)ds}. \tag{2.301}$$

Then problem (2.137) *is uniquely solvable for any $f \in \mathbf{L}$ and $\alpha \in \mathbb{R}$.*

(b) *Suppose $\phi(s)r(t,a) = r(t,s)$ as $u(v^{-1}(s)) \le t \le s$ and $\phi(s)r(t,a) + r(t,a) = r(t,s)$ as $s \le t \le v(u^{-1}(s))$; $\operatorname{ess\,sup}_{s\in[a,b]} |(\phi(s)+1)r(\cdot,a) - r(\cdot,s)| \in \mathbf{L}$; $t \le g(t) \le v(u^{-1}(t))$ as $a \le t \le c$; $u(v^{-1}(t)) \le g(t) \le t$ as $c \le t \le b$. Let, further, there exist $\delta > 0$ such that*

$$\operatorname*{ess\,sup}_{\substack{t\in\{e\cap g^{-1}(e)\}\\ \text{mes}(e)<\delta}} |b(t)| \frac{\text{mes}\,(g^{-1}(e) \cap [a,b])}{\text{mes}(e)} < 1. \tag{2.302}$$

Then the operator Q is invertible, the operator Q^{-1} is Volterra with respect to the projectors system (2.293), *and problem* (2.298) *is uniquely solvable for any $f \in \mathbf{L}$ and $\alpha \in \mathbb{R}^1$.*

Proof. The statement (a) follows from Theorem 2.33, the statement (b) follows from the property of Volterra operator $K + K_0$ of being strongly bettering, the property of Volterra operator S of being q-bettering with $q < 1$, and Theorem 2.28. □

Remark 2.36. Notice that a standard application of the Banach principle leads to the following condition of the unique solvability of problem (2.298):

$$\|S\| + \|\phi\| \int_a^b |r(t,a)|\,dt < 1 - \int_a^b k(s)ds. \tag{2.303}$$

Let systems of Volterra projectors $P_i^\tau : \mathbf{B}_i \to \mathbf{B}_i$ be given on Banach spaces $\mathbf{B}_i$, $i = 1,\dots,n$. Define the Volterra projector $P^\tau = (P_1^\tau,\dots,P_n^\tau) : \mathbf{B} \to \mathbf{B}$ on the direct product $\mathbf{B} = \mathbf{B}_1 \times \cdots \times \mathbf{B}_n$ by the equality $P^\tau(x_1,\dots,x_n) = (P_1^\tau x_1,\dots,P_n^\tau x_n)$.

Theorem 2.37. *Operator $A : \mathbf{B} \to \mathbf{B}$ is Volterra if and only if $P_i^\tau A_{ij} P_{j\tau} = 0$ for any $\tau \in [0,1]$, $i, j = 1,\dots,n$, where $A_{ij} : \mathbf{B}_i \to \mathbf{B}_j$ are corresponding components of A.*

Example 2.38. Consider the problem

$$\begin{gathered}\dot{x}_i(t) = p_{i1}(t)x_1[h_{i1}(t)] + \cdots + p_{in}(t)x_n[h_{in}(t)] + f_i(t), \quad t \in [a,b],\\ x_i(\xi) = 0 \quad \text{if } \xi \notin [a,b],\\ x_i(c_i) = \alpha_i, \quad i = 1,\dots,n,\ c_i \in (a,b),\end{gathered} \tag{2.304}$$

where $p_{ij} \in \mathbf{L}$, $f_i \in \mathbf{L}$, functions $h_{ij} : [a,b] \to \mathbb{R}^1$ are measurable.

The substitution $x_i(t) = \alpha_i + \int_{c_i}^t z_i(s)ds$ reduces this problem to the equation $z = Kz + f$, $f = (f_1,\dots,f_n)$, where an operator $K : \mathbf{L}^n \to \mathbf{L}^n$ ($\mathbf{L}^n = \underbrace{\mathbf{L} \times \cdots \times \mathbf{L}}_{n \text{ times}}$) is compact. The conditions

$$\begin{gathered}A + \frac{c_j - a}{c_i - a}(t-a) \le h_{ij}(t) \le b + \frac{b - c_j}{a - c_l}(t-a) \quad \text{if } a \le t \le c_i,\\ A + \frac{c_j - a}{c_i - b}(t-b) \le h_{ij}(t) \le b + \frac{b - c_j}{b - c_i}(t-b) \quad \text{if } c_i \le t \le b,\end{gathered} \tag{2.305}$$

ensure that operator K is Volterra with respect to the system

$$(P_i^\tau z)(t) = \begin{cases} z(t) & \text{if } t \in [c_i - \tau(c_i - a),\ c_i + \tau(b - c_i)],\\ 0 & \text{if } t \in [a, c_i - \tau(c_i - a)) \cup (c_i + \tau(b - c_i), b],\end{cases} \tag{2.306}$$

and, therefore, problem (2.304) is uniquely solvable.

Corollary 2.39. *The problem*

$$\begin{gathered}\dot{x}(t)=p_1(t)x_{h_1}(t)+q_1(t)y_{g_1}(t)+f_1(t),\\ \dot{y}(t)=p_2(t)x_{h_2}(t)+q_2(t)y_{g_2}(t)+f_2(t),\quad t\in[a,b],\\ x(A)=\alpha,\qquad y(b)=\beta,\end{gathered}\tag{2.307}$$

with p_i, $q_i \in \mathbf{L}$, is uniquely solvable if $h_1(t) \le t \le g_2(t)$, $h_2(t) \le a+b-t \le g_1(t)$.

Conclusively it is pertinent to give some remarks and discussion.

The notion of Volterra operator arose simultaneously in some different fields of mathematics (for instance, theory of integral operators, spectral theory, general theory of systems, theory of functional differential equations) and was being studied separately in the context of these theories. Thus in a number of papers, the results obtained by different authors are repeated many times, there are no uniform definitions and terms. Even the title "Volterra operator" is not commonly used, there are many titles such as operators of Volterra type, delaying operators, causal operator, retarding ones, nonantissipative ones, and so on, which are used to give the description of some operator classes with close properties. As a rule, the definitions have been based on the most important properties of the integral Volterra operator $(Kx)(t) = \int_a^t \mathcal{K}(t,s)x(s)ds$. Therewith part of the authors used the property of compactness with the property of being quasinilpotent, another one did this with the evolution property.

As it seems, Tonelli [216] was the first who entered a class of operators of Volterra type, namely, operators F such that the equality $x(s) = y(s)$, $s \le t$, implies the equality $(Fx)(s) = (Fy)(s)$, $s \le t$. Then Graffi [87] and Cinquini [65] obtained first results in the theory of Volterra operators. In 1938, a definition of functional Volterra operator as an operator $(F\varphi)(t)$ whose value is defined by the values of function $\varphi(\tau)$ on the interval $[0,t)$ appeared in the work of Tikhonov [215] devoted to applications of such operators to the problems in mathematical physics. This work has been world known, and under the influence of this work, the theory of Volterra operators in functional spaces has got the further development, and the operators satisfying the Tikhonov definition were called Volterra ones in Tikhonov's sense. Just this definition is used in works on functional differential equations with Volterra-Tikhonov operators, see [32, 33, 67].

A period of time ago, works on generalization of the Volterra-Tikhonov operator appeared. An immediate generalization of the Tikhonov definition for spaces of summable functions is proposed by V. I. Sumin. In the work by Sumin [213], an operator $F : \mathbf{L}_p^m(T) \to S^l(T)$ is called Volterra on the set system Θ, where Θ is a part of the σ-algebra T of Lebesgue measurable subsets if the equality $x = y$ on $M \in \Theta$ implies the equality $Fx = Fy$ on M. A similar definition of generalized Volterra operators acting in $\mathbf{L}_p[a,b]$ with all systems of subsets $[a,b]$ ordered by inclusion, whose measure varies continuously from 0 up to $b-a$, has been introduced and studied by Zhukovskii [232, 233]. In the works by Gusarenko [91] and

Vâth [221], the definitions of generalized Volterra operators are based on chains of ordered projectors.

Zabreiko [227, 228] proposed a generalization of the notion of integral Volterra operator that is based on the properties of its kernel such that it is guaranteed that the integral operator has a chain of invariant subspaces. P. P. Zabreiko obtained a formula for the spectral radius and proved that the property of being equal to zero for the spectral radius follows from the Ando's property (Ando [6]). Gokhberg and Krein [86] designated a linear operator as abstract Volterra operator in Hilbert space if it is compact and its spectral radius equals zero. Bukhgeim [57] extended the theory of such operators onto operators in Banach spaces. His definition is based on a specific chain of projectors. A similar construction, which is based on chains of subspaces embedded in each other, is considered in Kurbatov [129–131].

The theory of Volterra operators in Hilbert spaces was intensively developed by American mathematicians. Here the role of a start point was played by the work by Youla et al. [226]. It was the first work where the valuable role of the Volterra property in the general theory of systems was outlined. The fundamental results of the theory of Volterra operators were formulated at first for the space $\mathbf{L}_2$ and then for arbitrary Hilbert space. By Feintuch and Saeks [82], a linearly ordered closed set of orthogonal projectors P in $\mathbf{H}$ is called a decomposition of unit if $P^\tau \mathbf{H} \subset P^\theta \mathbf{H}$ for $0 \le \tau \le \theta \le 1$ and $P^0 = 0$, $P^1 = I$. An operator F is said to be causal if it possesses the property that $P^\tau F x = P^\tau F y$. Also the notions of anticausal and memoryless operators are introduced. The questions of decomposition, factorization, and invertibility of linear operators are mainly studied in the mentioned works.

3 Equations in finite-dimensional extensions of traditional spaces

3.1. Introduction

Sections 3.2 and 3.3 of this chapter are concerned with equations known as impulsive equations (equations with impulses). Steady interest in such equations arose in the mid-twentieth century. These equations work as the models for systems characterized by the fact that the state of the system may vary step-wisely at discrete times, whereas the state on the intervals between mentioned times is defined by a differentiable function being the solution of a differential equation in the ordinary sense. The systematic study of impulsive differential equations and their generalizations, differential equations in distributions, is related to many well-known scientists (see, e.g., [97, 154, 199, 200, 212, 217, 230]). The contemporary theory of impulsive systems is based on the theory of generalized functions (distributions) whose heart was created by Sobolev [210] and Schwartz [204, 205]. A somewhat different approach to the study of differential equations with discontinuous solutions is associated with the so-called "generalized ordinary differential equations" whose theory was initiated by Kurzweil [133–135]. Nowadays this theory is highly developed (see, e.g., [13–15, 202]). According to the accepted approaches impulsive equations are considered within the class of functions of bounded variation. In this case the solution is understood as a function of bounded variation satisfying an integral equation with the Lebesgue-Stiltjes integral or Perron-Stiltjes one. Integral equations in the space of functions of boundary variation became the subject of its own interest and are studied in detail in [203]. Recall that the function of bounded variation is representable in the form of the sum of an absolutely continuous function, a break function and a singular component (a continuous function with the derivative being equal zero almost everywhere). The solutions of the equations with impulse impact considered below do not contain the singular component and may have discontinuity only at finite number of fixed points. We consider these equations on a finite-dimensional extension of the traditional space of absolutely continuous functions. Thus the theorems of Chapter 1 are applicable to these equations. This approach to the equations with impulse impact does not use the complicated theory of generalized functions, turned out to be rich in content, and finds many applications in the cases where the question about the singular component does not arise. In particular, such is the case of certain problems in economic dynamics, see [148].

The approach below was offered in [8].

Section 3.4 is devoted to the multipoint boundary value problem for the Poisson equation and some linear perturbations thereof. The results of Bondareva [50, 51] are presented. The problem is considered in a space $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ where the finite-dimensional component is constructed according to a fixed system of points $t_1, \ldots, t_n$ belonging to a closed bounded set $\Omega \subset \mathbb{R}^3$, and $\mathbf{B}$ is the Banach space of Hölder functions $z : \Omega \to \mathbb{R}^1$. It is shown that the problem with conditions at the points t_i, $i = 1, \ldots, n$, is Fredholm. An effective way to regularize the problem is proposed; some conditions of the correct solvability as well as a presentation of the solutions are obtained.

3.2. Equations in the space of piecewise absolutely continuous functions

The space of piecewise absolutely continuous functions $y : [a, b] \to \mathbb{R}^n$ with fixed points $t_i \in (a, b)$ of discontinuity and representable in the form

$$y(t) = \int_a^t \dot{y}(s)ds + y(a) + \sum_{i=1}^{m} \chi_{[t_i,b]}(t)\Delta y(t_i), \tag{3.1}$$

is denoted by $\mathbf{DS}(m) = \mathbf{DS}[a, t_1, \ldots, t_m, b]$.

Here $a < t_1 < \cdots < t_m < b$, $\Delta y(t_i) = y(t_i) - y(t_i - 0)$, $\chi_{[t_i,b]}(t)$ is the characteristic function of $[t_i, b]$. Thus the elements of $\mathbf{DS}(m)$ are the functions which are absolutely continuous on each $[a, t_1)$, $[t_i, t_{i+1})$, $i = 1, \ldots, m-1$, and $[t_m, b]$; and continuous from the right at the points $t_1, \ldots, t_m$. This space is isomorphic to the product $\mathbf{L} \times \mathbb{R}^{n+nm}$, an isomorphism $\mathcal{J} : \{\Lambda, Y\} : \mathbf{L} \times \mathbb{R}^{n+nm} \to \mathbf{DS}(m)$ may be defined by

$$(\Lambda z)(t) = \int_a^t z(s)ds, \qquad (Y\beta)(t) = Y(t)\beta, \tag{3.2}$$

where $\beta = \operatorname{col}\{\beta^1, \ldots, \beta^{n+nm}\}$, $Y(t) = (E, \chi_{[t_1,b]}(t)E, \ldots, \chi_{[t_m,b]}(t)E)$. Here, as above, E is the identity $n \times n$ matrix. The inverse $\mathcal{J}^{-1} = [\delta, r] : \mathbf{DS}(m) \to \mathbf{L} \times \mathbb{R}^{n+nm}$ is defined by

$$\delta y = \dot{y}, \qquad ry = (y(a), \Delta y(t_1), \ldots, \Delta y(t_m)). \tag{3.3}$$

Under the norm

$$\|y\|_{\mathbf{DS}(m)} = \|\dot{y}\|_{\mathbf{L}} + \|ry\|_{\mathbb{R}^{n+nm}}, \tag{3.4}$$

the space $\mathbf{DS}(m)$ is Banach.

The space **D** of absolutely continuous functions $x : [a,b] \to \mathbb{R}^n$ which was introduced in the previous chapter is continuously imbedded into $\mathbf{DS}(m)$, besides $\mathbf{DS}(m) = \mathbf{D} \oplus \mathbf{M}^{nm}$, where $\mathbf{M}^{nm}$ is the finite-dimensional space of the nm-dimension. Therefore, any linear operator on $\mathbf{DS}(m)$ is a linear extension on this space of a linear operator $\mathcal{L}$ defined on **D**. Making stress on this circumstance, we will denote linear operators defined on $\mathbf{DS}(m)$ by $\widetilde{\mathcal{L}}$.

All assertions of Chapter 1 are valid for $\widetilde{\mathcal{L}}y = f$ with linear bounded Noether operator $\widetilde{\mathcal{L}} : \mathbf{DS}(m) \to \mathbf{L}$ and $\operatorname{ind}\widetilde{\mathcal{L}} = n + nm$.

In the theory of differential equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + P(t)x(t) = f(t) \tag{3.5}$$

the solutions of this equation with discontinuity at the points $t_i \in (a,b)$ are treated as solutions of

$$\dot{y}(t) + P(t)y(t) = f(t) + \sum_{i=1}^{m} \gamma_i \delta(t - t_i), \tag{3.6}$$

where δ is the Dirac function. The latter equation is understood as the equality between the functionals defined on the space **C** of continuous n-dimensional vector functions under the assumption that $f \in \mathbf{C}^*$, where $\mathbf{C}^*$ is the dual space to **C**. The solution of (3.6) is identified with the element of the space **BV** of n-dimensional vector functions with components of bounded variation on $[a,b]$. Let the right-hand side f of the equation be not arbitrary $f \in \mathbf{C}^*$, but only functionals generated by absolutely continuous functions. Then (3.6) may be considered as (3.5) with the special linear extension $\widetilde{\mathcal{L}}$ of $\mathcal{L}$:

$$(\widetilde{\mathcal{L}}y)(t) \stackrel{\text{def}}{=} \dot{y}(t) + P(t)y(t) = \dot{x}(t) + P(t)x(t) + P(t)\sum_{i=1}^{m} \chi_{[t_i,b]}(t)\Delta y(t_i) \tag{3.7}$$

and additional boundary conditions $\Delta y(t_i) = \gamma_i$. Here $x(t) = \int_a^t \dot{y}(s)ds + y(a)$ is the absolutely continuous summand in the representation (3.1). In other words, (3.6) is the boundary value problem

$$\dot{y}(t) + P(t)y(t) = f(t), \qquad \Delta y(t_i) = \gamma_i, \quad i = 1,\dots,m, \tag{3.8}$$

in the space $\mathbf{DS}(m)$.

Applying $\widetilde{\mathcal{L}} : \mathbf{DS}(m) \to \mathbf{L}$ to both sides of (3.1), we get the decomposition

$$(\widetilde{\mathcal{L}}y)(t) = (Q\dot{y})(t) + A_0(t)y(a) + \sum_{i=1}^{m} A_i(t)\Delta y(t_i), \tag{3.9}$$

where $Q = \widetilde{\mathcal{L}}\Lambda$, $A_0 = \widetilde{\mathcal{L}}E$, $A_i = \widetilde{\mathcal{L}}(\chi_{[t_i,b]}E)$, $i = 1,\dots,m$. For any matrices $A_1,\dots,A_m$ with the columns from **L**, the operator $\widetilde{\mathcal{L}}$ defined by (3.9) is a linear extension on $\mathbf{DS}(m)$ of $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ such that

$$(\mathcal{L}x)(t) = (Q\dot{x})(t) + A_0(t)x(a). \tag{3.10}$$

In this case the principal parts $\mathcal{L}\Lambda$ and $\widetilde{\mathcal{L}}\Lambda$ of $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ and $\widetilde{\mathcal{L}} : \mathbf{DS}(m) \to \mathbf{L}$ coincide: $\mathcal{L}\Lambda = \widetilde{\mathcal{L}}\Lambda = Q$.

The linear bounded functional $\tilde{l} : \mathbf{DS}(m) \to \mathbb{R}^N$ has the representation

$$\tilde{l}y = \int_a^b \widetilde{\Phi}(s)\dot{y}(s)ds + \widetilde{\Psi}ry, \tag{3.11}$$

where $N \times n$ matrix $\widetilde{\Phi}$ has measurable essentially bounded elements, $\widetilde{\Psi}$ is a constant $N \times (n + nm)$ matrix.

Let us rewrite the general boundary value problem

$$\widetilde{\mathcal{L}}x = f, \qquad \tilde{l}x = \alpha \tag{3.12}$$

in the form

$$\begin{gathered}(Q\dot{y})(t) + A(t)ry = f(t),\\ \int_a^b \widetilde{\Phi}(s)\dot{y}(s)ds + \widetilde{\Psi}ry = \alpha,\end{gathered} \tag{3.13}$$

where $A = (A_0, A_1, \dots, A_m)$, $ry = \operatorname{col}(y(a), \Delta y(t_1), \dots, \Delta y(t_m))$. Then the adjoint problem to (3.12) takes the form

$$\begin{gathered}(Q^*\omega)(t) + \gamma\widetilde{\Phi}(t) = g(t),\\ \int_a^b \omega(s)A(s)ds + \gamma\widetilde{\Psi} = \eta.\end{gathered} \tag{3.14}$$

Here $\omega, g \in \mathbf{L}_\infty$, $\gamma \in (\mathbb{R}^N)^*$, $\eta \in (\mathbb{R}^{n+nm})^*$, $\mathbf{L}_\infty$ is the Banach space of Lebesgue's measurable essentially bounded functions $\omega : [a,b] \to \mathbb{R}^n$, $\|\omega\|_{\mathbf{L}_\infty} = \operatorname{ess\,sup}_{t\in[a,b]} \|\omega(t)\|_{\mathbb{R}^n}$. By virtue of Corollary 1.15 the problem (3.12) is solvable if and only if the right-hand side $\{f, \alpha\}$ of the problem (3.12) is orthogonal to the solutions $\{\omega, \gamma\}$ of the adjoint homogeneous problem

$$\begin{gathered}(Q^*\omega)(t) + \gamma\widetilde{\Phi}(t) = 0,\\ \int_a^b \omega(s)A(s)ds + \gamma\widetilde{\Psi} = 0.\end{gathered} \tag{3.15}$$

The necessary condition for the unique solvability of (3.12) is the equality $N = n + nm$ (see Corollary 1.14). In the case of unique solvability of the problem

(3.12) the solution has the representation

$$y(t) = (\widetilde{G}f)(t) + X(t)\alpha. \tag{3.16}$$

By Theorem 1.31, the Green operator $\widetilde{G} : \mathbf{L} \to \mathbf{DS}(m)$ is an integral one since the operator $\Lambda : \mathbf{L} \to \mathbf{DS}(m)$ is integral (see (3.2)).

Theorem 3.1. *Let $\widetilde{\mathcal{L}} : \mathbf{DS}(m) \to \mathbf{L}$ and $\widetilde{l} : \mathbf{DS}(m) \to \mathbb{R}^n$ be linear extensions of $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ and $l : \mathbf{D} \to \mathbb{R}^n$. Then the unique solvability of one of the problems*

$$\mathcal{L}x = f, \qquad lx = \alpha, \tag{3.17}$$

$$\widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}y = \alpha, \qquad \Delta y(t_i) = \gamma_i, \quad i = 1,\dots,m, \tag{3.18}$$

ensures the unique solvability of the other. If the problems are uniquely solvable, the Green operator of (3.17) *is also the Green operator of* (3.18).

Proof. The problem (3.17) and the problem

$$\widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}y = \alpha, \qquad \Delta y(t_i) = 0, \quad i = 1,\dots,m \tag{3.19}$$

are equivalent. Therefore, the unique solvability of one of the problems (3.17), (3.18) with a right-hand side implies the unique solvability of the other problem with any right-hand side. If $x = Gf$ is the unique solution of (3.17) with $\alpha = 0$, this x is also the unique solution of (3.18) with $\alpha = 0$, $\gamma_i = 0$, $i = 1,\dots,m$. It means that G is the Green operator of the problem (3.18). □

It should be noticed that Theorems 3.1 and 1.20 imply the following corollary.

Corollary 3.2. *Let $\widetilde{\mathcal{L}} : \mathbf{DS}(m) \to \mathbf{L}$ be a linear extension of $\mathcal{L} : \mathbf{D} \to \mathbf{L}$. Let, further, $G : \mathbf{L} \to \mathbf{D}$ be the Green operator of some boundary value problem for $\mathcal{L}x = f$. Then the Green operator $\widetilde{G}$ of any uniquely solvable boundary value problem for $\widetilde{\mathcal{L}}x = f$ has the form*

$$\widetilde{G} = G + H, \tag{3.20}$$

where $H : \mathbf{L} \to \mathbf{DS}(m)$ is a degenerated operator.

Let $Q = I - R$, where $R : \mathbf{L} \to \mathbf{L}$ is an integral compact operator

$$(Rz)(t) = \int_a^b R(t,s)z(s)\,ds. \tag{3.21}$$

Then (see Theorem 2.2) the Green matrix $G(\cdot,s)$ of the problem (3.17) satisfies, at almost each $s \in (a,b)$, the following matrix equations:

$$\begin{gathered}(\widetilde{\mathcal{L}}Z)(t) \stackrel{\text{dim}}{=} \dot{Z}(t) - \int_a^b R(t,\tau)\dot{Z}(\tau)d\tau + A_0(t)Z(a) - R(t,s)\Delta Z(s) = 0,\\ \widetilde{l}Z \stackrel{\text{dim}}{=} \int_a^b \Phi(\tau)\dot{Z}(\tau)d\tau + \Psi Z(a) + \Phi(s)\Delta Z(s) = 0\\ (\Delta Z(s) = Z(s) - Z(s-0)),\end{gathered} \tag{3.22}$$

and besides

$$G(s,s) - G(s-0,s) = E \tag{3.23}$$

(we may presume that $G(\cdot,s)$ in the point s is continuous from the right). Thus, if the linear extensions of $\mathcal{L}$ and l on $\mathbf{DS}[a,s,b]$ are constructed as follows:

$$(\widetilde{\mathcal{L}}y)(t) = \dot{y}(t) - \int_a^b R(t,\tau)\dot{y}(\tau)d\tau + A_0(t)y(a) - R(t,s)\Delta y(s), \tag{3.24}$$

$$\widetilde{l}y = \int_a^b \Phi(\tau)\dot{y}(\tau)d\tau + \Psi y(a) + \Phi(s)\Delta y(s), \tag{3.25}$$

then the matrix $G(\cdot,s)$, at almost each $s \in (a,b)$, satisfies the matrix boundary value problem

$$\widetilde{\mathcal{L}}Z = 0, \qquad \widetilde{l}Z = 0, \qquad \Delta Z(s) = E. \tag{3.26}$$

As for extensions (3.24), (3.25) of $\mathcal{L}$, l, the following should be noticed. Let

$$\begin{aligned}(\mathcal{L}x)(t) &\stackrel{\text{dim}}{=} \dot{x}(t) + P(t)(S_hx)(t)\\ &= \dot{x}(t) + \int_a^b P(t)\chi_h(t,\tau)\dot{x}(\tau)d\tau + P(t)\chi_h(t,a)x(a),\end{aligned} \tag{3.27}$$

where $\chi_h(t,\tau)$ is the characteristic function of the set $\{(t,\tau) \in [a,b]\times[a,b] : \tau \le h(t) \le b\}$. Then the extension (3.24) preserves the initial form

$$(\widetilde{\mathcal{L}}y)(t) = \dot{y}(t) + P(t)(S_hy)(t). \tag{3.28}$$

It follows from the representation that

$$(S_hy)(t) = \int_a^b \chi_h(t,\tau)\dot{y}(\tau)d\tau + y(a) + \chi_h(t,s)\Delta y(s). \tag{3.29}$$

Similarly, for the vector functional

$$lx \stackrel{\text{def}}{=} x(\xi) = \int_a^b \chi_{[a,\xi]}(\tau)\dot{x}(\tau)d\tau + x(a) \quad (\xi \in [a,b]), \tag{3.30}$$

the initial form is preserved by the extension (3.25),

$$\widetilde{l}y = \int_a^b \chi_{[a,\xi]}(\tau)\dot{y}(\tau)d\tau + y(a) + \chi_{[a,\xi]}(s)\Delta y(s) = y(\xi). \tag{3.31}$$

In a more general case, the form of $\mathcal{L}$ and l may be changed by extension. Let, for instance,

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + \int_a^b d_\tau R(t,\tau)x(\tau) = \dot{x}(t) - \int_a^b R(t,\tau)\dot{x}(\tau)d\tau - R(t,a)x(a). \tag{3.32}$$

Without loss of generality we may presume that $R(t,\cdot)$ is continuous from the left in each point $s \in (a,b)$. Then the extension (3.24) may be written in the form

$$(\widetilde{\mathcal{L}}y)(t) = \dot{y}(t) + \int_a^s d_\tau R(t,\tau)y(\tau) + \int_s^b d_\tau R(t,\tau)y(\tau). \tag{3.33}$$

Indeed,

$$\begin{gathered}\int_a^s d_\tau R(t,\tau)y(\tau) = R(t,s)y(s-0) - R(t,a)y(a) - \int_a^s R(t,\tau)\dot{y}(\tau)d\tau, \\ \int_s^b d_\tau R(t,\tau)y(\tau) = -R(t,s)y(s) - \int_s^b R(t,\tau)\dot{y}(\tau)d\tau.\end{gathered} \tag{3.34}$$

Hence

$$\begin{aligned}&\int_a^s d_\tau R(t,\tau)y(\tau) + \int_s^b d_\tau R(t,\tau)y(\tau) \\ &\qquad = -\int_a^b R(t,\tau)\dot{y}(\tau)d\tau - R(t,a)y(a) - R(t,s)\Delta y(s).\end{aligned} \tag{3.35}$$

Consider on the base of Theorem 1.44 the conditions which guarantee the continuous dependence of the solution of the problem (3.12) on parameters, in particular, on the position of the points $a, t_1, \ldots, t_m, b$.

For each $k = 0, 1, \ldots$, let us determine the system of the points $a^k = t_0^k < t_1^k < \cdots < t_{m+1}^k = b^k$ such that $\lim_{k\to\infty} t_i^k = t_i^0$, $i = 0, 1, \ldots, m+1$. Let, further,

$$\mathbf{D}_k = \mathbf{DS}[a^k, t_1^k, \ldots, t_m^k, b^k], \qquad \mathbf{B}_k = \mathbf{L}[a^k, b^k]. \tag{3.36}$$

The element $y \in \mathbf{D}_k$ has the representation

$$y(t) = \int_{a^k}^{t} \dot{y}(s)ds + y(a^k) + \sum_{i=1}^{m} \chi_{[t_i^k, b^k]}(t)\Delta y(t_i^k). \tag{3.37}$$

The space $\mathbf{D}_k$ is isomorphic to the direct product $\mathbf{B}_k \times \mathbb{R}^{n+nm}$, the isomorphism $\mathcal{J}_k = \{\Lambda_k, Y_k\} : \mathbf{B}_k \times \mathbb{R}^{n+nm} \to \mathbf{D}_k$ is defined by the equalities

$$\begin{gathered}
(\Lambda_k z)(t) = \int_{a^k}^{t} z(s)ds, \qquad Y_k(t) = \big(E, \chi_{[t_1^k, b^k]}(t)E, \dots, \chi_{[t_m^k, b^k]}(t)E\big), \\
\mathcal{J}_k^{-1} = [\delta_k, r_k], \quad \text{where } \delta_k y = \dot{y},\ r_k y = \big(y(a^k), \Delta y(t_1^k), \dots, \Delta y(t_m^k)\big), \\
\|y\|_{\mathbf{D}_k} = \|\dot{y}\|_{\mathbf{B}_k} + \|r_k y\|_{\mathbb{R}^{n+nm}}.
\end{gathered} \tag{3.38}$$

Define the functional $\omega_k : [a^k, b^k] \to [a^0, b^0]$ by

$$\begin{gathered}
\omega_0(t) = t, \\
\omega_k(t) = \sum_{i=0}^{m} \left[\frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}(t - t_i^k) + t_i^0\right]\chi_{[t_i^k, t_{i+1}^k]}(t), \quad k = 1, 2, \dots.
\end{gathered} \tag{3.39}$$

Note that ω_k has the inverse

$$\omega_k^{-1}(t) = \sum_{i=0}^{m} \left[\frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0}(t - t_i^0) + t_i^k\right]\chi_{[t_i^0, t_{i+1}^0]}(t), \quad t \in [a^0, b^0]. \tag{3.40}$$

Define $\mathcal{H}_k : \mathbf{B}_0 \to \mathbf{B}_k$ by $(\mathcal{H}_k z)(t) = z[\omega_k(t)]$. Then $(\mathcal{H}_k^{-1} z)(t) = z[\omega_k^{-1}(t)]$. Thus

$$\|\mathcal{H}_k z\|_{\mathbf{B}_k} = \sum_{i=0}^{m} \int_{t_i^k}^{t_{i+1}^k} \left| z\left[\frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}(t - t_i^k) + t_i^0\right]\right| dt = \sum_{i=0}^{m} \frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0} \int_{t_i^0}^{t_{i+1}^0} |z(\tau)|\, d\tau. \tag{3.41}$$

From here

$$\begin{gathered}
\|\mathcal{H}_k z\|_{\mathbf{B}_k} \le \max_i \frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0} \|z\|_{\mathbf{B}_0}, \\
\|\mathcal{H}_k\| = \max_i \frac{t_{i+1}^k - t_i^k}{t_{i+1}^0 - t_i^0}, \\
\lim_{k\to\infty} \|\mathcal{H}_k z\|_{\mathbf{B}_k} = \|z\|_{\mathbf{B}_0}
\end{gathered} \tag{3.42}$$

for all $z \in \mathbf{B}_0$.

In the same way,

$$||\mathcal{H}_k^{-1}|| = \max_i \frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}. \tag{3.43}$$

Let $(\mathcal{P}_k y)(t) = y[\omega_k(t)]$, $y \in \mathbf{D}_0$. Then

$$\begin{aligned}(\mathcal{P}_k y)(t) &= \int_{a^0}^{\omega_k(t)} \dot{y}(s)ds + y(a^0) + \sum_{i=1}^{m} \chi_{[t_i^0, b^0]}[\omega_k(t)]\Delta y(t_i^0)\\ &= \int_{a^k}^{t} \frac{d}{ds}(y[\omega_k(s)])ds + y(a^0) + \sum_{i=1}^{m} \chi_{[t_i^k, b^k]}(t)\Delta y(t_i^0).\end{aligned} \tag{3.44}$$

Thus, $\mathcal{P}_k y \in \mathbf{D}_k$, $r_k \mathcal{P}_k y = r_0 y$, $(\mathcal{P}_k^{-1} y)(t) = y[\omega_k^{-1}(t)]$. Further,

$$\begin{aligned}||\mathcal{P}_k y||_{\mathbf{D}_k} &= \left|\left|\frac{d}{dt}\mathcal{P}_k y\right|\right|_{\mathbf{B}_k} + ||r_k \mathcal{P}_k y||_{\mathbb{R}^{n+nm}}\\ &= \sum_{i=0}^{m} \int_{t_i^k}^{t_{i+1}^k} \left|\dot{y}\left[\frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k}(t - t_i^k) + t_i^0\right]\right| \frac{t_{i+1}^0 - t_i^0}{t_{i+1}^k - t_i^k} dt + ||r_0 y||_{\mathbb{R}^{n+nm}}\\ &= \sum_{i=0}^{m} \int_{t_i^0}^{t_{i+1}^0} |\dot{y}(\tau)|\, d\tau + ||r_0 y||_{\mathbb{R}^{n+nm}} = \|y\|_{\mathbf{D}_0}.\end{aligned} \tag{3.45}$$

Thus the systems $\{\mathcal{H}_k\}$ and $\{\mathcal{P}_k\}$ are the connected ones for $\mathbf{B}_0$, $\mathbf{B}_k$ and $\mathbf{D}_0$, $\mathbf{D}_k$, respectively, such that the conditions of Theorem 1.44 are fulfilled.

Let $\widetilde{\mathcal{L}}_k : \mathbf{DS}[a^k, t_1^k, \dots, t_m^k, b^k] \to \mathbf{L}[a^k, b^k]$ be a linear bounded Noether operator with $\operatorname{ind}\widetilde{\mathcal{L}} = n + nm$ and let $\widetilde{l}_k : \mathbf{DS}[a^k, t_1^k, \dots, t_m^k, b^k] \to \mathbb{R}^{n+nm}$ be a linear bounded vector functional, $k = 0, 1, \dots$. Under the assumption $\widetilde{\mathcal{L}}_k \xrightarrow{\mathcal{P}\mathcal{H}} \widetilde{\mathcal{L}}_0$, $\widetilde{l}_k u_k \to \widetilde{l}_0 u_0$, as $u_k \xrightarrow{\mathcal{P}} u_0$, we establish the following assertion.

Theorem 3.3. *Let y_0 be the solution of the uniquely solvable boundary value problem*

$$\widetilde{\mathcal{L}}_0 y = f, \qquad \widetilde{l}_0 y = \alpha. \tag{3.46}$$

The problems

$$\widetilde{\mathcal{L}}_k y = f, \qquad \widetilde{l}_k y = \alpha \tag{3.47}$$

are uniquely solvable for a sufficiently large k, and, for solutions y_k of the problems

$$\widetilde{\mathcal{L}}_k y = f_k, \qquad \widetilde{l}_k y = \alpha_k, \tag{3.48}$$

the convergence $y_k \xrightarrow{\mathcal{P}} y_0$ *holds for any* $f_k \xrightarrow{\mathcal{H}} f_0$ *and* $\alpha_k \to \alpha_0$ *if and only if there exists the vector functional*

$$l : \mathbf{DS}[a^0, t_1^0, \dots, t_m^0, b^0] \longrightarrow \mathbb{R}^{n+nm} \tag{3.49}$$

such that the problems

$$\mathcal{H}_k^{-1} \widetilde{\mathcal{L}}_k \mathcal{P}_k y = f, \qquad ly = \alpha \tag{3.50}$$

are uniquely solvable for $k = 0$ *and a sufficiently large k; and for any* $\{f, \alpha\} \in \mathbf{L}[a^0, b^0] \times \mathbb{R}^{n+nm}$, *the convergence* $v_k \to v_0$ *under the norm of the space* $\mathbf{DS}[a^0, t_1^0, \dots, t_m^0, b^0]$ *holds for the solutions* $v_k \in \mathbf{DS}[a^0, t_1^0, \dots, t_m^0, b^0]$ *of the problem* (3.50).

3.3. Equations of the nth order with impulse effect

The scheme of the investigation of the equation of the nth order in the case when the discontinuity of solutions and their derivatives of various order are admissible in the finite number of the points was developed by Plaksina [172]. Here we restrict our consideration to the specific case when the discontinuity is admissible only for the derivative of the $(n-1)$th order of solution.

Let $t_1, \dots, t_m$ be a fixed ordered system of the points of (a, b). Denote by $\mathbf{W}^n\mathbf{S}(m)$ the space of functions $y : [a, b] \to \mathbb{R}^1$ representable in the form

$$\begin{aligned} y(t) = {} & \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} y^{(n)}(s)\,ds + \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} y^{(i)}(a) \\ & + \sum_{k=1}^{m} \frac{(t-t_k)^{n-1}}{(n-1)!} \chi_{[t_k, b]} \Delta y^{(n-1)}(t_k). \end{aligned} \tag{3.51}$$

Here $\chi_{[t_k, b]}$ is the characteristic function of the segment $[t_k, b]$,

$$\Delta y^{(n-1)}(t_k) = y^{(n-1)}(t_k) - y^{(n-1)}(t_k - 0). \tag{3.52}$$

Such a space is Banach under the norm

$$\begin{aligned} & \|y\|_{\mathbf{W}^n\mathbf{S}(m)} \\ & \quad = \|y^{(n)}\|_{\mathbf{L}} + \|\operatorname{col}\{y(a), \dots, y^{(n-1)}(a), \Delta y^{(n-1)}(t_1), \dots, \Delta y^{(n-1)}(t_m)\}\|_{\mathbb{R}^{n+m}}. \end{aligned} \tag{3.53}$$

Let $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ be a linear bounded Noether operator with $\operatorname{ind}\mathcal{L} = n$. Any linear extension $\widetilde{\mathcal{L}} : \mathbf{W}^n\mathbf{S}(m) \to \mathbf{L}$ of such an operator may be represented in the form

$$(\widetilde{\mathcal{L}}y)(t) = (\mathcal{L}x)(t) + \sum_{i=1}^{m} a_i(t)\Delta y^{(n-1)}(t_i). \tag{3.54}$$

Here $a_i \in \mathbf{L}$,

$$x(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} y^{(n)}(s)ds + \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} y^{(i)}(a). \tag{3.55}$$

The linear extension $\widetilde{\mathcal{L}} : \mathbf{W}^n\mathbf{S}(m) \to \mathbf{L}$ of a linear bounded Noether operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ of $\operatorname{ind}\mathcal{L} = n$ is the bounded Noether operator with $\operatorname{ind}\widetilde{\mathcal{L}} = n + m$. Therefore, just as in the section above the assertions of Chapter 1 are valid for $\widetilde{\mathcal{L}}y = f$. Here we restrict ourselves to application of the general theory to investigation of the Green function for the boundary value problem for (2.136),

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \sum_{k=0}^{n-1} \int_a^b x^{(k)}(s)d_s r_k(t,s) = f(t). \tag{3.56}$$

This application is based on the fact that any section $G(\cdot, s)$ of the Green function may be treated as the solution of the corresponding boundary value problem in the space of functions y such that the derivative $y^{(n-1)}$ may have discontinuity at the point $s \in (a,b)$.

So, consider the space $\mathbf{W}^n\mathbf{S}(m)$ in the case $m = 1$. Denote $t_1 = s$, $\mathbf{W}^n\mathbf{S}(1) = \mathbf{W}^n\mathbf{S}[a,s,b]$ and define the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbb{R}^{n+1} \to \mathbf{W}^n\mathbf{S}[a,s,b]$ by

$$(\Lambda z)(t) = \int_a^t \frac{(t-\tau)^{n-1}}{(n-1)!} z(\tau)d\tau,$$

$$(Y\beta)(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!}\beta^{i+1} + \frac{(t-s)^{n-1}}{(n-1)!}\chi_{[s,b]}(t)\beta^{n+1}, \quad \beta = \{\beta^1, \dots, \beta^{n+1}\}. \tag{3.57}$$

Let $\widetilde{\mathcal{L}} : \mathbf{W}^n\mathbf{S}[a,s,b] \to \mathbf{L}$ and $\widetilde{l}^i : \mathbf{W}^n\mathbf{S}[a,s,b] \to \mathbb{R}^1$ be linear extensions of the operator $\mathcal{L} : \mathbf{W}^n \to \mathbf{L}$ and the functional $l^i : \mathbf{W}^n \to \mathbb{R}^1$. Similarly, by the proof of Theorem 3.1, we make sure that the boundary value problems

$$\begin{gathered} \mathcal{L}x = f, \qquad l^i x = \alpha^i, \quad i = 1, \dots, n, \\ \widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}^i y = \alpha^i, \quad i = 1, \dots, n, \qquad \Delta y^{(n-1)}(s) = \alpha^{n+1} \end{gathered} \tag{3.58}$$

are uniquely solvable (or not) simultaneously one with the other. In the case of their unique solvability, the Green functions of the problems coincide.

Define the linear extensions $\widetilde{\mathcal{L}}$ and $\widetilde{l}^i$ as follows:

$$\begin{aligned}(\widetilde{\mathcal{L}}y)(t) = y^{(n)}(t) - \int_a^b R(t,\tau)y^{(n)}(\tau)d\tau \\ + \sum_{k=0}^{n-1} p_k(t)y^{(k)}(a) - R(t,s)\Delta y^{(n-1)}(s),\end{aligned} \tag{3.59}$$

$$\widetilde{l}^i y = \int_a^b \varphi^i(\tau)y^{(n)}(\tau)d\tau + \sum_{k=0}^{n-1} \psi_k^i y^{(k)}(a) + \varphi^i(s)\Delta y^{(n-1)}(s). \tag{3.60}$$

Then by virtue of Theorem 2.4 the section $G(\cdot,s)$ of the Green function of the problem (3.58) at almost each $s \in (a,b)$ is the solution of the extended problem

$$(\widetilde{\mathcal{L}}y)(t) = 0, \qquad \widetilde{l}^i y = 0, \quad i = 1,\dots,n, \qquad \Delta y^{(n-1)}(s) = 1. \tag{3.61}$$

Being the kernel of the integral operator $G : \mathbf{D} \to \mathbf{L}$, the Green function can change significantly on any set of zero measure at each $t \in [a,b]$. Under the notion of Green function we will understand the function $G(t,s)$ which is the solution of the problem (3.61) for each $s \in (a,b)$. Thus the question of the unique solvability and the property of having fixed sign by the Green function of the problem (3.58) may be reduced to the question of unique solvability and a fixed sign of the solution of the problem (3.61) for each $s \in (a,b)$.

Let us illustrate the said by the example of the boundary value problem

$$\begin{gathered}(\mathcal{L}x)(t) \stackrel{\text{def}}{=} x^{(n)}(t) + \int_a^b x(s)d_s r(t,s) = f(t) \quad (n \geq 2), \\ l^i x = 0, \quad i = 1,\dots,n,\end{gathered} \tag{3.62}$$

for a special equation with a function $r(t,s)$ nonincreasing (nondecreasing) with respect to the second argument for almost all $t \in [a,b]$.

Denote by $\{v\}$ the set of $v_1,\dots,v_m \in [a,b]$ such that $l^i x \stackrel{\text{def}}{=} x(v_i)$. If there is no such a point, the symbol $\{v\}$ denotes the empty set. Let the auxiliary problem

$$\mathcal{L}_0 x \stackrel{\text{def}}{=} x^{(n)} = f, \qquad l^i x = 0, \quad i = 1,\dots,n, \tag{3.63}$$

be uniquely solvable and let its Green function $W(t,s)$ be strictly negative (positive) at each fixed $s \in (a,b)$ for $t \in [a,b] \setminus \{v\}$.

Theorem 3.4. *Let the inequality*

$$\varphi_s(t) \stackrel{\text{def}}{=} W(t,s) - \int_a^b W(t,\tau)\left\{\int_a^b W(\xi,s)d_\xi r(\tau,\xi)\right\}d\tau < 0, \quad t \in [a,b] \setminus \{v\}, \tag{3.64}$$

hold for each fixed $s \in (a,b)$. Then the problem (3.62) *is uniquely solvable, and for the Green function $G(t,s)$ of the problem, the estimates*

$$\varphi_s(t) \le G(t,s) \le W(t,s) \quad (\varphi_s(t) \ge G(t,s) \ge W(t,s)), \quad (t,s) \in [a,b] \times (a,b), \tag{3.65}$$

are valid.

Proof. Let, for definiteness, $W(t,s) \le 0$. As it was said above, $W(t,s)$ is also the Green function of the extended problem

$$\widetilde{\mathcal{L}}_0 y \overset{\text{def}}{=} y^{(n)} = f, \qquad \widetilde{l}^i y = 0, \quad i = 1,\dots,n, \qquad \Delta y^{(n-1)}(s) = 0, \tag{3.66}$$

and the section $w_s(t) = W(t,s)$ satisfies the problem

$$\widetilde{\mathcal{L}}_0 y \overset{\text{def}}{=} y^{(n)} = 0, \qquad \widetilde{l}^i y = 0, \quad i = 1,\dots,n, \qquad \Delta y^{(n-1)}(s) = 1, \tag{3.67}$$

where the extension $\widetilde{l}^i$ is defined by (3.60). Therefore, the extended problem

$$\begin{gathered}(\widetilde{\mathcal{L}} y)(t) \overset{\text{def}}{=} y^{(n)}(t) + \int_a^b y(\tau) d_\tau r(t,\tau) = 0, \\ \widetilde{l}^i y = 0, \quad i = 1,\dots,n, \qquad \Delta y^{(n-1)}(s) = 1\end{gathered} \tag{3.68}$$

is equivalent to

$$y(t) = -\int_a^b W(t,\tau)\left\{\int_a^b y(\xi) d_\xi r(\tau,\xi)\right\} d\tau + w_s(t) \tag{3.69}$$

in the space $\mathbf{W}^n\mathbf{S}[a,s,b]$.

Denote

$$(Ay)(t) = \int_a^b W(t,\tau)\left\{\int_a^b y(\xi) d_\xi r(t,\xi)\right\} d\tau \tag{3.70}$$

and rewrite the latter equation in the form

$$y + Ay = w_s. \tag{3.71}$$

Consider this equation in the space $\mathbf{C}$ of all continuous functions on $[a,b]$. It is possible because any continuous solution of (3.71) belongs to $\mathbf{W}^n\mathbf{S}[a,s,b]$. Theorem A.5 may be applied to (3.71). Indeed, the condition (a) of this theorem is fulfilled if $\{v\}$ is empty. If not, the condition (c) is fulfilled. Thus the solution y_s of (3.71) satisfies the inequalities $\varphi_s(t) \le y_s(t) \le w_s(t)$, $t \in [a,b] \setminus \{v\}$. Since $y_s(t) = G(t,s)$, the proof is completed. □

Being applied to the two-point boundary value problem

$$\ddot{x}(t) + \int_a^b x(s)d_s r(t,s) = f(t), \qquad x(a) = 0, \quad \dot{x}(b) = 0, \tag{3.72}$$

Theorem 3.4 permits asserting that under the assumption of nonincreasing $r(t,s)$ with respect to the second argument, the inequality

$$\int_a^b \operatorname*{var}_{\xi\in[a,b]} r(t,\xi)dt < \frac{1}{b-a} \tag{3.73}$$

guarantees the unique solvability of the problem and negativity of its Green function.

In the case under consideration, we have

$$W(t,s) = \begin{cases} -\dfrac{(s-a)(b-t)}{b-a} & \text{if } a \le s \le t \le b, \\ -\dfrac{(t-a)(b-s)}{b-a} & \text{if } a \le t < s \le b. \end{cases} \tag{3.74}$$

By virtue of (3.73) and the estimates

$$|W(t,s)| \le s-a, \quad t \in [a,b]; \qquad |W(t,s)| \le t-a, \quad s \in [a,b], \tag{3.75}$$

we obtain the inequality (3.64) required by Theorem 3.4. Indeed, if $t \in (a,s)$,

$$\begin{aligned} \varphi_s(t) &\le -(t-a) + (t-a)(b-a)\int_a^b \operatorname*{var}_{\xi\in[a,b]} r(\tau,\xi)d\tau \\ &= -(t-a)\left[1-(b-a)\int_a^b \operatorname*{var}_{\xi\in[a,b]} r(\tau,\xi)d\tau\right] < 0; \end{aligned} \tag{3.76}$$

if $t \in [s,b]$,

$$\begin{aligned} \varphi_s(t) &\le -(s-a) + (b-a)(s-a)\int_a^b \operatorname*{var}_{\xi\in[a,b]} r(\tau,\xi)d\tau \\ &= -(s-a)\left[1-(b-a)\int_a^b \operatorname*{var}_{\xi\in[a,b]} r(\tau,\xi)d\tau\right] < 0. \end{aligned} \tag{3.77}$$

The fact of a fixed sign of the Green function to the problem (3.58) may be established on the base of the theorem below.

Let us fix a point $\theta \in [a,b]$ such that the functionals $l^1,\dots,l^n,l^{n+1}$, where $l^{n+1}x = x(\theta)$, are linearly independent. Define the linear extensions of $\mathcal{L}$ and l^i by (3.59) and (3.60).

Theorem 3.5. *Let the problem* (3.58) *be uniquely solvable. The Green function* $G(t,s)$ *of the problem possesses the property* $G(\theta,s)\neq 0$ *if and only if the problem*

$$\widetilde{\mathcal{L}}y=0,\qquad \widetilde{l}^{i}y=0,\quad i=1,\dots,n,\qquad y(\theta)=0 \tag{3.78}$$

has only the trivial solution.

Proof. Let $x_1,\dots,x_n$ be the fundamental system of the homogeneous equation $\mathcal{L}x=0$ and let

$$\Delta=\begin{vmatrix} l^1x_1 & \cdots & l^1x_n \\ \cdots & \cdots & \cdots \\ l^nx_1 & \cdots & l^nx_n \end{vmatrix} \tag{3.79}$$

be the determinant of the problem (3.58). Denote $g_s(t)=G(t,s)$. The functions $x_1,\dots,x_n$, g_s constitute the fundamental system for $\widetilde{\mathcal{L}}y=0$. The determinant of the problem (3.78) has the form

$$\widetilde{\Delta}=\begin{vmatrix} l^1x_1 & \cdots & l^1x_n^0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l^nx_1 & \cdots & l^nx_n & 0 \\ x_1(\theta) & \cdots & x_n(\theta) & g_s(\theta) \end{vmatrix}=G(\theta,s)\Delta. \tag{3.80}$$

Since $\Delta\neq 0$, the theorem is proved. □

Consider as an example of application of Theorem 3.5 the following periodic boundary value problem:

$$\begin{gathered}(\mathcal{L}x)(t)\stackrel{\text{def}}{=}\ddot{x}(t)+\int_a^b x(\tau)d_\tau r(t,\tau)=f(t),\\ x(b)-x(a)=0,\qquad \dot{x}(b)-\dot{x}(a)=0,\end{gathered} \tag{3.81}$$

observing that in [167] Theorem 3.5 is applied to the periodic problem for a more general equation.

We will demonstrate that under the assumption of the unique solvability of the problem (3.81) the inequality

$$\int_a^b \operatorname*{var}_{\tau\in[a,b]} r(t,\tau)dt<\frac{1}{b-a} \tag{3.82}$$

guarantees a fixed sign of the Green function.

Indeed, by virtue of Theorem 3.5 the Green function $G(t,s)$ has no zero on the square $[a,b] \times (a,b)$ if the boundary value problem

$$\begin{gathered}(\widetilde{\mathcal{L}}y)(t) \overset{\text{dim}}{=} \ddot{y}(t) + \int_a^b y(\tau)d_\tau r(t,\tau) = f(t),\\ y(b)-y(a)=0, \qquad \dot{y}(b)-\dot{y}(a)=0, \qquad y(\theta)=0\end{gathered} \tag{3.83}$$

is uniquely solvable for each $s \in (a,b)$ and every $\theta \in [a,b]$.

Using Theorem 1.25, let us establish the unique solvability of the problem (3.83). As the model problem, we take

$$\begin{gathered}\ddot{y}=z, \qquad y(b)-y(a)=0,\\ \dot{y}(b)-\dot{y}(a)=0, \qquad y(\theta)=0.\end{gathered} \tag{3.84}$$

The functions 1, t, and $(t-s)\chi_{[s,b]}(t)$ constitute the fundamental system of $\ddot{y}=0$ in the space $\mathbf{WS}^2[a,s,b]$. The determinant of the problem (3.84) is not equal to zero:

$$\begin{vmatrix} 0 & b-a & b-s \\ 0 & 0 & 1 \\ 1 & \theta & (\theta-s)\chi_{[s,b]}(\theta) \end{vmatrix} = -(b-a) \neq 0. \tag{3.85}$$

Consequently, the problem (3.84) is uniquely solvable. The Green function of this problem was constructed in [167]:

$$\begin{aligned}W_{\theta,s}(t,\tau) &= \chi_{[a,t]}(\tau)(t-\tau) - \chi_{[a,\theta]}(\tau)(\theta-\tau)\\ &\quad - \chi_{[s,b]}(t)(t-s) + \chi_{[s,b]}(\theta)(\theta-s) + \frac{\tau-s}{b-a}(t-\theta)\end{aligned} \tag{3.86}$$

and has the estimate

$$\left| W_{\theta,s}(t,\tau) \right| \leq b-a, \quad (t,\tau) \in [a,b]\times[a,b],\ s\in(a,b),\ \theta\in[a,b]. \tag{3.87}$$

We have $\widetilde{\mathcal{L}}W_{\theta,s} = I - \Omega$, where $W_{\theta,s}$ is the Green operator of the problem (3.84),

$$\begin{aligned}(\Omega z)(t) &= \int_a^b \left\{ \int_a^b W_{\theta,s}(\xi,\tau)z(\tau)d\tau \right\} d_\xi r(t,\xi)\\ &= \int_a^b \left\{ \int_a^b W_{\theta,s}(\xi,\tau)d_\xi r(t,\xi) \right\} z(\tau)d\tau.\end{aligned} \tag{3.88}$$

Since

$$\|\Omega\|_{\mathbf{L}\to\mathbf{L}} \leq (b-a)\int_a^b \operatorname*{var}_{\xi\in[a,b]} r(t,\xi)dt < 1, \tag{3.89}$$

the problem (3.83) is uniquely solvable.

Equations in finite-dimensional extensions of traditional spaces were studied from the point of view of the theory of abstract functional differential equation in [8, 10]. Theorem 3.5 was used in [138, 167, 172] to establish the property of fixed sign of the Green function.

3.4. Multipoint boundary value problem for the Poisson equation

In this section, we follow the works of Bondareva (see [50, 51]).

3.4.1. On setting up the problem

Boundary conditions of the classical Dirichlet and Neumann problems are defined at the domain boundary by operators (and not by functionals) in a corresponding space $\mathbf{B}_0$ of functions. Thus these problems may be considered in a space $\mathbf{D}$ isomorphic to the direct product $\mathbf{B}\times\mathbf{B}_0$, where $\mathbf{B}$ and $\mathbf{B}_0$ are both infinite-dimensional spaces. Such an approach with the use of some ideas from Chapter 1 has been employed by Gusarenko (see [93]). In this section, the Poisson equation and its perturbations by linear operators are considered with multipoint conditions. In such a case the values of the solution are given at n points of the domain and its boundary. This approach makes it possible to employ immediately the results from Chapter 1 to some topical problems. Let us consider two examples.

Example 3.6. In [66] the problem on the twist of the beam with transversal section Ω:

$$\begin{gathered}\Delta u \overset{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -1, \quad (x, y) \in \Omega; \\ u(x, y) = 0 \quad \text{on the boundary } \partial\Omega,\end{gathered} \tag{3.90}$$

is considered. An approximate solution of the problem is constructed in the following way. A system of points $t_i = (x_i, y_i) \in \partial\Omega$, $i = 1, \dots, n$, is fixed and the problem

$$\Delta v = -1 \quad \text{in } \Omega; \qquad v(t_i) = 0, \quad i = 1, \dots, n, \tag{3.91}$$

is considered. The approximate solution is sought in the form

$$\widetilde{v} = v_0 + \sum_{j=1}^{q} a_j v_j, \tag{3.92}$$

where v_0 is a solution of $\Delta v = -1$ and functions v_j, $j = 1, \dots, q$, are harmonic on Ω. Coefficients a_j are defined in such a way as to minimize $\max_{i=1,\dots,n} |\widetilde{v}(t_i)|$.

Example 3.7. Let us quote a text from the book of Sologub [211]. "Kirchgoff considered a system of nonlinear contacting conductors penetrated by electric current. At first, he deduces the Laplace equation describing the voltage distribution $v(x, y, z)$ in every conductor of the system $\Delta v = 0$. Next, Kirchgoff finds boundary conditions for $v(x, y, z)$. On the part of the surface S of the conductor contacting with dielectric, say, dry air, where there is no leak of electricity, we have

$$\frac{\partial v}{\partial \nu} = 0; \tag{3.93}$$

on the part of the surface S contacting with other conductors, the equalities

$$\begin{gathered} k\frac{\partial v}{\partial \nu} + k'\frac{\partial v'}{\partial \nu'} = 0, \\ v - v' = c = \text{const} \end{gathered} \tag{3.94}$$

hold, where $v'(x, y, z)$ is the voltage of the neighbouring conductor; k, k' are the heat conduction coefficients; ν, ν' are the corresponding inward-directed normals to S; c is a given constant defining the electromotive force on the contact surface of the conductors."

Thus in the case of pointwise contacting of conductors the corresponding boundary value problem becomes natural.

3.4.2. Construction of the space D

Consider the multipoint boundary value problem for the Poisson equation and its perturbations on a set in $\mathbb{R}^3$.

Let $\Omega \subset \mathbb{R}^3$ be a closed bounded set with piecewise smooth boundary $\partial\Omega$. Denote by $\mathbf{B}$ the Banach space of Lipschitz functions $z : \Omega \to \mathbb{R}$ with the norm

$$\|z\|_{\mathbf{B}} = \sup_{t\in\Omega} |z(t)| + \sup_{\substack{t,\tau\in\Omega \\ t\neq\tau}} \frac{|z(t) - z(\tau)|}{|t - \tau|}, \tag{3.95}$$

where $|t - \tau|$ means the Euclidean distance of points $t, \tau \in \mathbb{R}^3$. Let $\{t_i\}$, $t_i = (\xi_i, \eta_i, \vartheta_i) \in \Omega$, $i = 1, \dots, n$, be a collection of distinct points. As is shown in [50], we can suppose without loss of generality that

$$\max_{k,j=1,\dots,n} |\eta_k - \eta_j| > 0, \qquad \max_{k,j=1,\dots,n} |\vartheta_k - \vartheta_j| > 0. \tag{3.96}$$

Ibidem, there is shown the existence of an angle φ such that all points $(\xi_i, (\eta_i \cos\varphi - \vartheta_i \sin\varphi)) \in \mathbb{R}^2$, $i = 1, \dots, n$, are pairwise different. Let us fix this φ and define the functions

$$y_j^0(\xi, \tau) = \operatorname{Re} \prod_{k\neq j}^{n} \left[\frac{(\xi - \xi_k) + \mathbf{i}(\tau - \tau_k)}{(\xi_j - \xi_k) + \mathbf{i}(\tau_j - \tau_k)} \right]. \tag{3.97}$$

These functions are harmonic and

$$y_j^0(\xi_i,\tau_i) = \delta_{ji} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases} \tag{3.98}$$

Define the system $y_j : \mathbb{R}^3 \to \mathbb{R}$, $j = 1,\dots,n$, by

$$y_j(t) = y_j(\xi,\eta,\vartheta) = y_j^0(\xi,(\eta\cos\varphi - \vartheta\sin\varphi)), \quad j = 1,\dots,n. \tag{3.99}$$

The functions y_j are harmonic (see, e.g., [50]) and

$$y_j(t_i) = \delta_{ji}. \tag{3.100}$$

Define over the space $\mathbf{B}$ the operator Λ:

$$(\Lambda z)(t) = -\frac{1}{4\pi}\int_\Omega \frac{z(s)}{|t-s|}ds. \tag{3.101}$$

For any $z \in \mathbf{B}$, the function $x(t) = (\Lambda z)(t)$ has continuous second derivatives at interior points of Ω and

$$\Delta x \overset{\text{def}}{=} \frac{\partial^2 x}{\partial\xi^2} + \frac{\partial^2 x}{\partial\eta^2} + \frac{\partial^2 x}{\partial\vartheta^2} = z \tag{3.102}$$

(see, e.g., [153]), that is, $\Delta\Lambda z = z$.

Define the space $\mathbf{D} = \mathbf{D}(t_1,\dots,t_n)$ as the space of functions $x : \Omega \to \mathbb{R}$ of the form

$$x = \Lambda z + Y\beta, \tag{3.103}$$

where $z \in \mathbf{B}$, $Y = (y_1,\dots,y_n)$, $\beta = \operatorname{col}(\beta^1,\dots,\beta^n)$.

It follows from (3.102) that $z = \Delta x$. Furthermore,

$$x(t_i) = -\frac{1}{4\pi}\int_\Omega \frac{z(s)}{|t_i-s|}ds + \beta^i = -\frac{1}{4\pi}\int_\Omega \frac{(\Delta x)(s)}{|t_i-s|}ds + \beta^i, \quad i = 1,\dots,n, \tag{3.104}$$

and hence

$$\beta^i = x(t_i) + \frac{1}{4\pi}\int_\Omega \frac{(\Delta x)(s)}{|t_i-s|}ds. \tag{3.105}$$

Thus the operator $\mathcal{J} \overset{\text{def}}{=} \{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ is an isomorphism, $\mathcal{J}^{-1} = [\delta, r]$, where

$$\begin{gathered} \delta x = \Delta x, \\ rx = \operatorname{col}(r^1x,\dots,r^nx), \qquad r^ix = x(t_i) + \frac{1}{4\pi}\int_\Omega \frac{(\Delta x)(s)}{|t_i-s|}ds. \end{gathered} \tag{3.106}$$

D is Banach under the norm

$$\|x\|_{\mathbf{D}} = \|\Delta x\|_{\mathbf{B}} + \|rx\|_{\mathbb{R}^n}. \tag{3.107}$$

The boundary value problems considered in what follows allow the application of the theorems from Chapter 1.

3.4.3. Multipoint boundary value problem for the Poisson equation and its perturbations

Consider in $\mathbf{D}(t_1,\ldots,t_n)$ the boundary value problem

$$\Delta x = f, \qquad x(t_i) = \gamma^i, \quad i = 1,\ldots,n. \tag{3.108}$$

This problem is uniquely solvable for any $f \in \mathbf{B}$ and $\gamma^i \in \mathbb{R}^1$. Really, the representation (3.103) of elements of **D** implies

$$z = f, \qquad x(t_i) = (\Lambda f)(t_i) + \beta^i = \gamma^i, \tag{3.109}$$

hence $\beta^i = \gamma^i - (\Lambda f)(t_i)$. Thus the unique solution $x \in \mathbf{D}$ of (3.108) has the form

$$\begin{aligned} x(t) &= (\Lambda f)(t) + \sum_{i=1}^{n} y_i(t)\left[\gamma^i - (\Lambda f)(t_i)\right] \\ &= -\frac{1}{4\pi}\int_{\Omega}\left[\frac{1}{|t-s|} - \sum_{i=1}^{n}\frac{y_i(t)}{|t_i - s|}\right] f(s)\,ds + \sum_{i=1}^{n} y_i(t)\gamma^i. \end{aligned} \tag{3.110}$$

The integral operator $G : \mathbf{B} \to \mathbf{D}$ defined by

$$(Gf)(t) = -\frac{1}{4\pi}\int_{\Omega}\left[\frac{1}{|t-s|} - \sum_{i=1}^{n}\frac{y_i(t)}{|t_i - s|}\right] f(s)\,ds \tag{3.111}$$

is the Green operator of multipoint problem (3.108) for the Poisson equation.

Remark 3.8. The principal boundary value problem in **D** for the Poisson equation

$$\Delta x = f; \qquad r^i x \equiv x(t_i) + \frac{1}{4\pi}\int_{\Omega}\frac{(\Delta x)(s)}{|t_i - s|}\,ds = \alpha^i, \quad i = 1,\ldots,n, \tag{3.112}$$

is equivalent to multipoint problem (3.108) with

$$\gamma^i = \alpha^i - \frac{1}{4\pi}\int_{\Omega}\frac{f(s)}{|t_i - s|}\,ds. \tag{3.113}$$

Remark 3.9. In the case that a function $x : \Omega \to \mathbb{R}$ is twice continuously differentiable on Ω (Ω is closed) and all t_i are interior, we have, due to the Green formula, (see [153])

$$r^i x = x(t_i) + \frac{1}{4\pi}\int_{\Omega}\frac{(\Delta x)(s)}{|t_i - s|}ds = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{1}{|t_i - s|}\frac{\partial x}{\partial \nu} - x\frac{\partial}{\partial \nu}\frac{1}{|t_i - s|}\right)d_s\sigma, \tag{3.114}$$

where $\partial/\partial\nu$ means differentiation in direction of outward (with respect to Ω) normal, and $d_s\sigma$ is the element of surface $\partial\Omega$.

Remark 3.10. Boundary value problem for the Poisson equation with pointwise inequalities

$$\Delta x = f; \qquad \sum_{i=1}^{n} a_{ji}x(t_i) \leq \alpha^j, \quad j = 1,\ldots,N, \tag{3.115}$$

in the space $\mathbf{D}(t_1,\ldots,t_n)$ is equivalent to the system of linear inequalities

$$\sum_{i=1}^{n} a_{ji}\gamma^i \leq \alpha^j, \quad j = 1,\ldots,N, \tag{3.116}$$

with respect to $\gamma^1,\ldots,\gamma^n$. Every solution $(\overline{\gamma^1},\ldots,\overline{\gamma^n})$ of (3.78) generates a corresponding solution $\overline{x} \in \mathbf{D}$ of (3.115) defined by (3.110).

Now consider a perturbed Poisson equation

$$\mathcal{L}x \stackrel{\text{def}}{=} \Delta x - Tx = f \tag{3.117}$$

with a linear bounded operator $T : \mathbf{D} \to \mathbf{B}$. We assume that the operator $K : \mathbf{B} \to \mathbf{B}$ defined by $K = T\Lambda$ is compact. In such a case, the principal part of $\mathcal{L}$, $Q = \mathcal{L}\Lambda = I - K$, is a canonical Fredholm operator and, by Theorem 1.10, the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is Noether with $\operatorname{ind}\mathcal{L} = n$.

Let $l \stackrel{\text{def}}{=} [l^1,\ldots,l^n] : \mathbf{D} \to \mathbb{R}^n$ be a linear bounded vector functional. The problem

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{3.118}$$

is Fredholm (see Theorem 1.13).

Let the operator T and the vector functional l map the space $\mathbf{C}(\Omega)$ of continuous functions $z : \Omega \to \mathbb{R}$ with the norm $\|z\|_{\mathbf{C}(\Omega)} = \max_{t\in\Omega}|z(t)|$ into spaces $\mathbf{B}$ and $\mathbb{R}^n$, respectively. Note that equality (3.101) defines compact operator $\Lambda : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ (see [153]) and hence, by the continuity of embedding $\mathbf{B}$ into $\mathbf{C}(\Omega)$, the operator $K = T\Lambda : \mathbf{B} \to \mathbf{B}$ is compact. Under these assumptions, multipoint problem

$$\mathcal{L}x = f, \qquad x(t_i) = \gamma^i + l^i x, \quad i = 1,\ldots,n, \tag{3.119}$$

is reducible to an equation of the second kind with a compact operator in $\mathbf{C}(\Omega)$. Namely, representation (3.110) implies that (3.119) is equivalent to

$$x = Vx + g, \tag{3.120}$$

where

$$Vx = GTx + \sum_{i=1}^{n} y_i l^i x, \qquad g = Gf + \sum_{i=1}^{n} y_i \gamma^i. \tag{3.121}$$

Compactness of $V : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ follows from compactness of the operator $G : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$, being the sum of compact $\Lambda : \mathbf{C}(\Omega) \to \mathbf{C}(\Omega)$ and a finite-dimensional operator, and from the continuity of embedding $\mathbf{B}$ into $\mathbf{C}(\Omega)$.

The unique solvability of multipoint problem (3.108) survives under small perturbations T, l, in particular, when $\|V\|_{\mathbf{C}(\Omega)\to\mathbf{C}(\Omega)} < 1$. In this case the Banach principle is applicable to (3.120). In case $l \equiv 0$, the estimate $\|V\|_{\mathbf{C}}(\Omega) \to \mathbf{C}(\Omega) < 1$ holds under the condition

$$\|T\|_{\mathbf{C}(\Omega)\to\mathbf{C}(\Omega)} < \frac{1}{\|d\|_{\mathbf{C}(\Omega)}\left(1 + \sum_{i=1}^{n} \|y_i\|_{\mathbf{C}}\right)}, \tag{3.122}$$

where $d(t) = (1/4\pi)\int_{\Omega} ds/|t-s|$. This follows from the inequality

$$\|Gf\|_{\mathbf{C}(\Omega)} \le \|d\|_{\mathbf{C}(\Omega)}\left(1 + \sum_{i=1}^{n} \|y_i\|_{\mathbf{C}(\Omega)}\right) \cdot \|f\|_{\mathbf{C}(\Omega)}. \tag{3.123}$$

In the general case, effective testing of (3.120) for the unique solvability may be done with the use of the reliable computer experiment presented in Chapter 6.

Conclusively, notice that another space $\mathbf{D}$, being well suitable for consideration of boundary value problems with the Poisson equation and its perturbation, can be constructed with the space $\mathbf{C}^{\alpha}(\Omega)$ of Hölder functions of index α, $0 < \alpha \le 1$, as the space $\mathbf{B}$. The collection $y_1, \dots, y_n$ can be constructed with trigonometric polynomials (see [51]). The case $\Omega \subset \mathbb{R}^m$, $m = 2$ and $m > 3$, meets no difficulties. In the thesis of Bondareva [52] and in [51] as well there are considered in detail the Poisson equation with multipoint inequalities approximating the conditions $\partial x/\partial \nu + cx \ge \alpha$ (or $\alpha \le \partial x/\partial \nu + cx \le \beta$) on $\partial\Omega$, $\Omega \subset \mathbb{R}^2$, and $\Delta x(t) - \int_{\Omega} K(t,s)x(s)ds = f(t)$, $t \in \Omega$, with the conditions

$$\sum_{i=1}^{n} a_{ji} x(t_i) \ge \alpha^i, \quad i = 1, \dots, N. \tag{3.124}$$

For these problems, some criteria and sufficient conditions of the solvability are obtained.

4 Singular equations

4.1. Introduction

The set of functions, in which the solutions of an equation under consideration are to be looked for, sometimes is chosen without a proper reason. An unsuccessful choice of such a set may cause much trouble. We discuss below some reasons and examples related to the question of choosing the proper Banach space, in which it would be suitable to define the notion of the solution of the given equation.

Let $\mathcal{L}x = f$ be an equation with a linear operator $\mathcal{L} : \mathbf{D}_0 \to \mathbf{B}_0$, let $\mathbf{D}_0$ be isomorphic to $\mathbf{B}_0 \times \mathbb{R}^n$, and let $\mathcal{J}_0 = \{\Lambda_0, Y_0\} : \mathbf{B}_0 \times \mathbb{R}^n \to \mathbf{D}_0$ be the isomorphism. If the principal part $\mathcal{L}\Lambda_0 : \mathbf{B}_0 \to \mathbf{B}_0$ of $\mathcal{L}$ is not a Fredholm one, we do not have available standard schemes for investigation of the equation. In this case it is reasonable to call the equation "singular." Nevertheless one may try to construct another space $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^N$ with the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^N \to \mathbf{D}$, so that the principal part $\mathcal{L}\Lambda$ of the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ will be a Fredholm or even invertible operator. Then the equation ceases to be singular (with respect to the chosen space) and one may apply to this equation the theorems of Chapter 1.

Let us note that the property of the principal part of being Fredholm characterizes many intrinsic specifics of the equation. For instance, this property is necessary for unique solvability of any boundary value problem

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{4.1}$$

for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$.

Considering the given equation in various spaces, we change correspondingly the notion of the solution of this equation. The classical theory of differential equations does not use the notions of spaces and operators in these spaces and, in this theory, the investigation of singular equations begins with the definition of the notion of solution as a function that satisfies the equation in one or another sense and possesses certain properties. Thus, the set is chosen, to which the solutions belong. In our reasoning we do, in the same way, choose a Banach space being the domain of the operator $\mathcal{L}$. In addition we offer some recommendation about constructing the space $\mathbf{D}$ such that the operator $\mathcal{L}$ possesses necessary properties.

4.2. The equation $(t-a)(b-t)\ddot{x}(t) - (Tx)(t) = f(t)$

Consider

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \pi(t)\ddot{x}(t) - (Tx)(t) = f(t), \quad t \in [a,b], \tag{4.2}$$

where $\pi(t) = t-a$, $\pi(t) = b-t$ or $\pi(t) = (t-a)(b-t)$, T is a linear bounded operator acting from the space $\mathbf{C}$ of continuous functions $x : [a,b] \to \mathbb{R}^1$ into the space $\mathbf{L}$ of summable functions. Besides we assume that T is compact as an operator acting from the space $\mathbf{W}^1$ of absolutely continuous functions into $\mathbf{L}$. The operator $R : \mathbf{C} \to \mathbf{L}$ of the form

$$(Rx)(t) = \int_a^b x(s)d_s r(t,s) \tag{4.3}$$

studied in Chapter 2 is an example of such T.

The space $\mathbf{W}^2$, which is traditional for equations of the second order, is unacceptable in this case since even the equation $\pi(t)\ddot{x}(t) = 1$ has no solution $x \in \mathbf{W}^2$. We will construct a space $\mathbf{D}_\pi \simeq \mathbf{L}\times\mathbb{R}^2$ in such a way that the operator $\mathcal{L} : \mathbf{D}_\pi \to \mathbf{L}$ is Noether of the index 2 (the principal part $\mathcal{L}\Lambda$ of $\mathcal{L}$ is Fredholm).

4.2.1. The space D_π

Let us show that as $\mathbf{D}_\pi$ we may take the space of functions $x : [a,b] \to \mathbb{R}^1$ that satisfy the following conditions.

(1) x is absolutely continuous on $[a,b]$.

(2) The derivative $\dot{x}$ is absolutely continuous on each $[c,d] \subset (a,b)$.

(3) The product $\pi\ddot{x}$ is summable on $[a,b]$.

It should be noticed that the authors of [112, 113, 130] defined the notion of the solution of the singular ordinary differential equation ($(Tx)(t) = p(t)x(t)$) as a function satisfying (1)–(3).

Denote by $\mathcal{I}_\pi$ the interval (a,b) if $\pi(t) = (t-a)(b-t)$, the interval $[a,b)$ if $\pi(t) = b-t$, and the interval $(a,b]$ if $\pi(t) = t-a$.

Let $\tau \in \mathcal{I}_\pi$ be fixed. Define in the square $[a,b]\times[a,b]$ the function

$$\Lambda_\tau(t,s) = \begin{cases} \dfrac{t-s}{\pi(s)} & \text{if } \tau \le s < t \le b, \\ \dfrac{s-t}{\pi(s)} & \text{if } a \le t < s \le \tau, \\ 0 & \text{in other points of the square } [a,b]\times[a,b]. \end{cases} \tag{4.4}$$

Remark that $\pi(s)\Lambda_\tau(t,s)$ is the Green function of the boundary value problem $\ddot{x}(t) = f(t)$, $x(\tau) = \dot{x}(\tau) = 0$ in the space $\mathbf{W}^2$.

After immediate estimation we obtain

$$0 \le \Lambda_\tau(t,s) \le M, \quad (t,s) \in [a,b]\times[a,b], \tag{4.5}$$

where

$$M=\begin{cases}\max\left\{\dfrac{1}{\tau-a},\dfrac{1}{b-\tau}\right\} & \text{if } \pi(t)=(t-a)(b-t),\\ \max\left\{1,\dfrac{\tau-a}{b-\tau}\right\} & \text{if } \pi(t)=b-t,\\ \max\left\{1,\dfrac{b-\tau}{\tau-a}\right\} & \text{if } \pi(t)=t-a.\end{cases} \tag{4.6}$$

Thus at each $t\in[a,b]$ the product $\Lambda_\tau(t,s)z(s)$ is summable for any summable z.

Next we will show that, for $z\in\mathbf{L}$, the function

$$u(t)\overset{\text{def}}{=}\int_a^b\Lambda_\tau(t,s)z(s)ds\equiv\int_\tau^t\frac{t-s}{\pi(s)}z(s)ds \tag{4.7}$$

belongs to $\mathbf{D}_\pi$.

Since, for $t\in\mathcal{I}_\pi$,

$$\dot{u}(t)=\int_\tau^t\frac{z(s)}{\pi(s)}ds, \tag{4.8}$$

the derivative $\dot{u}$ is absolutely continuous on any $[c,d]\subset\mathcal{I}_\pi$. Further we have, for $t\in[a,b]$,

$$u(t)=\int_\tau^t\frac{t-s}{\pi(s)}z(s)ds=\int_\tau^t\left\{\int_s^t d\xi\right\}\frac{z(s)}{\pi(s)}ds=\int_\tau^t d\xi\int_\tau^\xi\frac{z(s)}{\pi(s)}ds=\int_\tau^t\dot{u}(\xi)d\xi. \tag{4.9}$$

The change of the integration order in the iterated integrals is possible since there exists the finite integral

$$\int_\tau^t\left|\int_s^t d\xi\right|\frac{|z(s)|}{\pi(s)}ds=\int_\tau^t\frac{|t-s|}{\pi(s)}|z(s)|ds. \tag{4.10}$$

Thus, the function u is absolutely continuous on $[a,b]$.

The product $\pi\ddot{u}$ is summable since by virtue of (4.8),

$$\ddot{u}(t)=\frac{z(t)}{\pi(t)} \tag{4.11}$$

a.e. on $[a,b]$.

So the element $x\in\mathbf{D}_\pi$ is defined by

$$x(t)=(\Lambda z)(t)+(Y\beta)(t)=\int_a^b\Lambda(t,s)z(s)ds+\beta^1+\beta^2(t-\tau), \tag{4.12}$$

the space $\mathbf{D}_\pi$ is isomorphic to the product $\mathbf{L}\times\mathbb{R}^2$, isomorphism $\mathcal{J}:\mathbf{L}\times\mathbb{R}^2\to\mathbf{D}_\pi$ is defined by the operator $\mathcal{J}=\{\Lambda,Y\}$. Besides, $\mathcal{J}^{-1}=[\delta,r]$, where $\delta x=\pi\ddot{x}$, $rx=\{x(\tau),\dot{x}(\tau)\}$. The space $\mathbf{D}_\pi$ is Banach under the norm

$$\|x\|_{\mathbf{D}_\pi}=\|\pi\ddot{x}\|_{\mathbf{L}}+|x(\tau)|+|\dot{x}(\tau)|. \tag{4.13}$$

By Theorem 1.10, the operator $\delta:\mathbf{D}_\pi\to\mathbf{L}$ is Noether, $\operatorname{ind}\delta=2$.

Next we will show that $\mathbf{D}_\pi$ is continuously imbedded into the space $\mathbf{W}^1$ of absolutely continuous functions $x:[a,b]\to\mathbb{R}^1$.

Let

$$\|x\|_{\mathbf{W}^1}=\int_a^b|\dot{x}(s)|\,ds+|x(\tau)|. \tag{4.14}$$

For $x\in\mathbf{D}_\pi$, we have

$$\dot{x}(t)=\int_\tau^t\frac{z(s)}{\pi(s)}ds+\dot{x}(\tau),\qquad t\in\mathcal{I}_\pi,\ z=\pi\ddot{x}. \tag{4.15}$$

Let us estimate $\int_a^b|\dot{x}(t)|dt$,

$$\begin{aligned}
\int_a^b\left|\int_\tau^t\frac{z(s)}{\pi(s)}ds\right|dt&\le\int_a^\tau\int_t^\tau\frac{|z(s)|}{\pi(s)}ds\,dt+\int_\tau^b\int_\tau^t\frac{|z(s)|}{\pi(s)}ds\,dt\\
&=\int_a^\tau\int_a^s\frac{|z(s)|}{\pi(s)}dt\,ds+\int_\tau^b\int_s^b\frac{|z(s)|}{\pi(s)}dt\,ds\\
&=\int_a^\tau\frac{s-a}{\pi(s)}|z(s)|\,ds+\int_\tau^b\frac{b-s}{\pi(s)}|z(s)|\,ds\le M\int_a^b\pi(s)|\ddot{x}(s)|\,ds,
\end{aligned}$$

$$\int_a^b|\dot{x}(t)|\,dt\le M\int_a^b\pi(s)|\ddot{x}(s)|\,ds+|\dot{x}(\tau)|(b-a). \tag{4.16}$$

So, for $x\in\mathbf{D}_\pi$, we have the estimate

$$\|x\|_{\mathbf{W}^1}\le M\|\pi\ddot{x}\|_{\mathbf{L}}+|\dot{x}(\tau)|(b-a)+|x(\tau)|\le M_1\|x\|_{\mathbf{D}_\pi}, \tag{4.17}$$

where $M_1=\max\{1,b-a,M\}$.

Since $T:\mathbf{W}^1\to\mathbf{L}$ is compact, the operator $T:\mathbf{D}_\pi\to\mathbf{L}$ is also compact because of the continuous imbedding $\mathbf{D}_\pi\subset\mathbf{W}^1$. Thus the bounded $\mathcal{L}:\mathbf{D}_\pi\to\mathbf{L}$ is Noether of index 2 (the principal part $\mathcal{L}\Lambda:\mathbf{L}\to\mathbf{L}$ is Fredholm). Therefore the theorems of Chapter 1 are applicable to (4.2).

By Theorem 1.31, any Green operator $G:\mathbf{L}\to\mathbf{D}_\pi$ is an integral one, since the operator $\Lambda:\mathbf{L}\to\mathbf{D}_\pi$ in isomorphism (4.12) is integral.

Remark 4.1. As it was noted above, for some $f \in \mathbf{L}$, the equation $\pi\ddot{x} = f$ has no solution in $\mathbf{W}^2$. In the space $\mathbf{D}_\pi$ this equation is solvable for any $f \in \mathbf{L}$. The general solution of the equation has the representation

$$x(t) = \int_a^b \Lambda_\tau(t,s)f(s)ds + c_1 + c_2(t-\tau). \tag{4.18}$$

Remark 4.2. The isomorphism $\{\Lambda, Y\} : \mathbf{L} \times \mathbb{R}^2 \to \mathbf{D}_\pi$ defined by (4.12) is based on the boundary value problem

$$\pi\ddot{x} = z, \qquad x(\tau) = \beta^1, \qquad \dot{x}(\tau) = \beta^2. \tag{4.19}$$

It is natural that the isomorphism $\mathcal{J} : \mathbf{L}\times\mathbb{R}^2 \to \mathbf{D}_\pi$ may be constructed on the base of any other boundary value problem that is uniquely solvable in $\mathbf{D}_\pi$, for instance,

$$\pi\ddot{x} = z, \qquad x(a) = \beta^1, \qquad x(b) = \beta^2. \tag{4.20}$$

This problem is uniquely solvable and the Green function of the problem has the form

$$G_0(t,s) = \begin{cases} -\dfrac{(s-a)(b-t)}{\pi(s)(b-a)} & \text{if } a \le s \le t \le b, \\[2ex] -\dfrac{(t-a)(b-s)}{\pi(s)(b-a)} & \text{if } a \le t < s \le b. \end{cases} \tag{4.21}$$

Thus, the isomorphism $\mathcal{J} : \mathbf{L} \times \mathbb{R}^2 \to \mathbf{D}_\pi$ may be defined by

$$x(t) = \int_a^b G_0(t,s)z(s)ds + \beta^1\frac{b-t}{b-a} + \beta^2\frac{t-a}{b-a}. \tag{4.22}$$

4.2.2. The equation with isotonic T

Everywhere in this point the operator $T : \mathbf{C} \to \mathbf{L}$ is supposed to be isotonic.

The problem

$$\mathcal{L}x = f, \qquad x(\tau) = \alpha^1, \quad \dot{x}(\tau) = \alpha^2, \quad \tau \in \mathcal{I}_\pi \tag{4.23}$$

is equivalent to the equation

$$x = A_\tau x + g \tag{4.24}$$

with $A_\tau = \Lambda T$, $g(t) = (\Lambda f)(t) + \alpha^1 + \alpha^2(t-\tau)$. Any continuous solution of the equation belongs to $\mathbf{D}_\pi$. Therefore we may consider this equation in the space $\mathbf{C}$ of continuous functions $x : [a,b] \to \mathbb{R}^1$. We will denote by $\rho(A_\tau)$ the spectral radius of $A_\tau : \mathbf{C} \to \mathbf{C}$.

Theorem 4.3. *Let $\tau \in \mathcal{I}_\pi$. The following assertions are equivalent.*

(a) *There exists $v \in \mathbf{D}_\pi$ such that*

$$v(t) \geq 0, \quad \varphi(t) \stackrel{\text{def}}{=} (\mathcal{L}v)(t) \geq 0, \quad t \in [a,b], \tag{4.25}$$

and besides

$$\dot{v}(\tau) = 0, \qquad v(\tau) + \int_a^b \Lambda_\tau(t,s)\varphi(s)ds > 0. \tag{4.26}$$

(b) $\rho(A_\tau) < 1$.

(c) *The problem* (4.23) *is uniquely solvable and the Green operator of the problem is antitonic.*

(d) *There exists the solution u of the homogeneous equation $\mathcal{L}x = 0$ such that $\dot{u}(\tau) = 0,\ u(t) > 0,\ t \in [a,b]$.*

Theorem 4.3 is a special case of Theorem A.5. Indeed, all the conditions of the general theorem are fulfilled if

$$\mathcal{L}_0 x = \pi\ddot{x}, \qquad W(t,s) = \Lambda_\tau(t,s), \qquad u_0(t) \equiv \alpha = \text{const} > 0. \tag{4.27}$$

Definition 4.4. Say that (4.2) possesses the property A if the problem (4.23) is uniquely solvable and the Green operator G_τ of the problem is isotonic for each $\tau \in \mathcal{I}_\pi$.

Remark 4.5. The property A is, in a sense, similar to the property P for the regular equation. The property P is guaranteed by Theorem 2.7 based on the uniform boundedness with respect to τ of the Green function of the problems

$$\ddot{x} = f, \qquad x(\tau) = \dot{x}(\tau) = 0, \quad \tau \in [a,b]. \tag{4.28}$$

But for singular problems

$$\pi\ddot{x} = f, \qquad x(\tau) = \dot{x}(\tau) = 0, \quad \tau \in \mathcal{I}_\pi, \tag{4.29}$$

such a uniformity ceases to be true. Nevertheless the uniform boundedness in τ of the operators

$$(A_\tau x)(t) = \int_a^b \Lambda_\tau(t,s)(Tx)(s)ds \tag{4.30}$$

may take place under some conditions, for instance, if the function $(T\mathbf{1})(s)/\pi(s)$ is summable. Here and in what follows the symbol "$\mathbf{1}$" stands for the function that equals 1 identically. In this event the inequality

$$||A_\tau||_{\mathbf{C}\to\mathbf{C}} = ||A_\tau\mathbf{1}||_{\mathbf{C}} < (b-a)\int_a^b \frac{(T\mathbf{1})(s)}{\pi(s)}ds \tag{4.31}$$

holds for all $\tau \in \mathcal{I}_\pi$ if T differs from the null operator. Hence, the inequality

$$\int_a^b \frac{(T\mathbf{1})(s)}{\pi(s)}\,ds \leq \frac{1}{b-a} \tag{4.32}$$

guarantees the property A due to Theorem 4.3 (the implication (b)⇒(a)).

More subtle tests of the property A will be considered below.

Remark 4.6. The equation

$$\pi(t)\ddot{x}(t)-p(t)x(t)=f(t) \tag{4.33}$$

with summable p possesses property A if $p(t)\geq 0$.

The following assertions assume that (4.2) with isotonic T possesses the property A.

Lemma 4.7. *Suppose* $u \in \mathbf{D}_\pi$, $(\mathcal{L}u)(t) \overset{\text{def}}{=} \varphi(t) \geq 0$ $(\varphi(t)\leq 0)$, $t\in[a,b]$, $u(\tau) \overset{\text{def}}{=} c>0$ $(c<0)$, $\dot{u}(\tau)=0$, $\tau\in\mathcal{I}_\pi$. *Then* $u(t)\geq c$, $(u(t)\leq c)$, $t\in[a,b]$.

Proof. Let $\varphi(t)\geq 0$, $c>0$. The function $y=u-c$ satisfies the problem

$$\mathcal{L}y=\psi, \qquad y(\tau)=\dot{y}(\tau)=0 \tag{4.34}$$

with $\psi(t)=\varphi(t)+c(T\mathbf{1})(t)\geq\varphi(t)$. Therefore, $u(t)-c=(G_\tau\psi)(t)\geq 0$. □

Theorem 4.8. *The two-point problem*

$$\mathcal{L}x=f, \qquad x(a)=x(b)=0 \tag{4.35}$$

is uniquely solvable and the Green operator of the problem is antitonic.

Proof. If the problem is not uniquely solvable, the homogeneous problem has a nontrivial solution x. Let $m=\max_{t\in[a,b]}x(t)>0$. Then $x(t)\leq m$, which gives a contradiction to Lemma 4.7.

If the Green operator G is not antitonic, the solution $x=Gf$, for a nonnegative f, has a positive maximum. The contradiction to Lemma 4.7 completes the proof. □

The following two theorems may be proved similarly.

Theorem 4.9. *The boundary value problems*

$$\begin{aligned}
&\mathcal{L}x=f, \quad x(a)=\dot{x}(b)=0, \quad \textit{where } \pi(t)=t-a,\\
&\mathcal{L}x=f, \quad \dot{x}(a)=x(b)=0, \quad \textit{where } \pi(t)=b-t,
\end{aligned} \tag{4.36}$$

are uniquely solvable and their Green operators are antitonic.

Theorem 4.10. *The boundary value problem*

$$\mathcal{L}x = f, \qquad x(a_1) = x(b_1) = 0, \quad a \le a_1 < b_1 \le b, \tag{4.37}$$

is uniquely solvable. Besides, if $f(t) \ge 0$ ($f(t) \not\equiv 0$), the solution x satisfies inequalities $x(t) < 0$ for $t \in (a_1, b_1)$ and $x(t) > 0$ for $t \in [a,b] \setminus [a_1, b_1]$.

Let us formulate the corollary that follows from Theorem 4.8 and Lemma 4.7.

Corollary 4.11. *The solution u_0 of the semihomogeneous problem*

$$\mathcal{L}x = 0, \qquad x(a) = \alpha^1 \ge 0, \qquad x(b) = \alpha^2 \ge 0, \qquad \alpha^1 + \alpha^2 > 0 \tag{4.38}$$

is positive on (a,b).

Proof. The solution u_0 exists by virtue of Theorem 4.8. If u_0 alters its sign, there exists the negative minimum at a point $\tau \in (a,b)$. But this contradicts to Lemma 4.7. $\square$

Definition 4.12. The system $u_1, u_2 \in \mathbf{D}_\pi$ is called nonoscillatory on $[a,b]$ if any nontrivial combination $u = c_1u_1 + c_2u_2$ has on $[a,b]$ not more than one zero, counting the multiple zeros on $\mathcal{I}_\pi$ twice.

Theorem 4.13. *The following assertions are equivalent.*

(a) *The equation $\mathcal{L}x = f$ possesses the property A.*
(b) *Any nontrivial solution of $\mathcal{L}x = 0$ that has a zero on $[a,b]$ does not have zeros of the derivative on $\mathcal{I}_\pi$.*
(c) *The fundamental system of solutions of $\mathcal{L}x = 0$ is nonoscillatory on $[a,b]$.*
(d) *There exists a pair $v_1, v_2 \in \mathbf{D}_\pi$ such that*

$$\begin{gathered} v_1(a) = 0, \qquad v_1(b) > 0, \qquad v_2(a) > 0, \qquad v_2(b) = 0, \\ v_i(t) > 0, \quad \varphi_i(t) \overset{\text{def}}{=} (\mathcal{L}v_i)(t) \ge 0, \quad i = 1,2,\ t \in (a,b), \end{gathered} \tag{4.39}$$

and, besides, $\varphi_1(t) > 0$ ($\varphi_2(t) > 0$) a.e. (a,b), if $\pi(t) = b - t$ ($\pi(t) = (t-a)$).

Proof. The implication (a)⇒(b) follows from Lemma 4.7. Indeed, let the solution u of the equation $\mathcal{L}x = 0$ be such that $\dot{u}(\tau) = 0$, $u(\tau) > 0$, $\tau \in \mathcal{I}_\pi$. By Lemma 4.7, the solution u has no zero on $[a,b]$.

Let (b) be fulfilled. Then any nontrivial solution has no multiple zero on $\mathcal{I}_\pi$. Between two different zeros of u there must exist a zero of the derivative, which is impossible. Thus (b)⇒(c).

To prove (c)⇒(d), consider the problems

$$\begin{aligned} \mathcal{L}x &= 0, \qquad x(a) = 0, \qquad x(b) = 1, \\ \mathcal{L}x &= 0, \qquad x(a) = 1, \qquad x(b) = 0. \end{aligned} \tag{4.40}$$

The problems are uniquely solvable since by virtue of nonoscillation of the fundamental system, the homogeneous problem

$$\mathcal{L}x=0, \qquad x(a)=x(b)=0 \tag{4.41}$$

has only the trivial solution. The solutions u_1 and u_2 of the first and the second problems are positive on (a,b) since they have already a zero apiece on $[a,b]$. Thus $v_1=u_1$ and $v_2=u_2$ satisfy (d) in the case $\pi(t)=(t-a)(b-t)$.

Let $\pi(t)=b-t$. The problem

$$\mathcal{L}x=0, \qquad x(a)=0, \qquad \dot{x}(a)=k \tag{4.42}$$

has a unique solution z_k. It follows from the fact that the homogeneous problem $(k=0)$ has only the trivial solution by virtue of nonoscillation of the fundamental system (the nontrivial solution has no multiple zeros at the point $t=a$). The solution z_k is positive on $(a,b]$ if $k>0$ since it has already a zero at the point $t=a$.

Let $\varphi\in\mathbf{L}$ be fixed, $\varphi(t)>0$, $t\in[a,b]$. Denote by z^{φ} the solution of the problem

$$\mathcal{L}x=\varphi, \qquad x(a)=x(b)=0. \tag{4.43}$$

The sum $z\overset{\text{def}}{=}z^{\varphi}+z_k$ is positive on $(a,b]$ for k large enough since the value $\dot{z}^{\varphi}(a)$ is finite. Thus the functions $v_1=z$ and $v_2=u_2$ satisfy (d).

The case $\pi(t)=t-a$ may be considered similarly.

In order to prove (d)⇒(a), let us show that, for each $\tau\in\mathcal{I}_{\pi}$, there exists a function $v\in\mathbf{D}_{\pi}$ that satisfies condition (a) of Theorem 4.3.

By Lemma 4.7, $\dot{v}_1(t)>0$, $\dot{v}_2(t)<0$, $t\in(a,b)$. So, for each $\tau\in(a,b)$ there exists a positive constant c such that the sum $v=cv_1+v_2$ possesses the property

$$\dot{v}(\tau)=0, \qquad v(\tau)>0, \qquad (\mathcal{L}v)(t)=c\varphi_1(t)+\varphi_2(t)\geq 0, \quad t\in[a,b]. \tag{4.44}$$

Thus the implication (d)⇒(a) is proved for the case $\pi(t)=(t-a)(b-t)$.

If $\pi(t)=b-t$, we must construct in addition the function v that satisfies condition (a) of Theorem 4.3 for $\tau=a$. If $\dot{v}_1(a)=0$, we may put $v=v_1$. If $\dot{v}_1(a)\neq 0$, let $v=v_1-y$, where $y(t)=\dot{v}_1(a)(t-a)$. Then

$$(\mathcal{L}v)(t)=(\mathcal{L}v_1)(t)+(Ty)(t)\geq\varphi_1(t). \tag{4.45}$$

Since $\pi\ddot{v}=\pi\ddot{v}_1=Tv_1+\varphi_1$, we have

$$v(t)=\int_a^t\frac{t-s}{b-s}\left[(Tv_1)(s)+\varphi_1(s)\right]ds>0, \quad t\in(a,b]. \tag{4.46}$$

In the case $\pi(t) = t - a$, $\tau = b$, it might be taken that

$$v(t) = v_2(t) + \dot{v}_2(b)(b - t). \tag{4.47}$$

□

To illustrate Theorem 4.13, let us consider the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} t(1-t)\ddot{x}(t) - p(t)(S_h x)(t) = f(t), \quad p(t) \geq 0, \; t \in [0,1]. \tag{4.48}$$

Letting

$$v_1(t) = (1-t)\ln(1-t) + t, \qquad v_2(t) = t\ln t - t + 1 \tag{4.49}$$

and using the estimates

$$\frac{v_1(t)}{t^2} \leq \frac{(1+t)}{2}, \quad \frac{v_2(t)}{(1-t)^2} \leq 1 - \frac{t}{2}, \quad t \in [0,1], \tag{4.50}$$

we are in a position to formulate by virtue of Theorem 4.13 the following.

Corollary 4.14. *Equation* (4.48) *possesses the property A if the inequalities*

$$\begin{gathered} [1 + h(t)]h^2(t)\sigma_h(t)p(t) \leq 2t, \\ [2 - h(t)][1 - h(t)]^2\sigma_h(t)p(t) \leq 2(1-t) \end{gathered} \tag{4.51}$$

hold for a.e. $t \in [0,1]$.

Here as usual,

$$\sigma_h(t) = \begin{cases} 1 & \text{if } h(t) \in [0,1], \\ 0 & \text{if } h(t) \notin [0,1]. \end{cases} \tag{4.52}$$

The conditions (4.51) hold if

$$\begin{gathered} h^2(t)\sigma_h(t)p(t) \leq t, \\ [1 - h(t)]^2\sigma_h(t)p(t) \leq 1 - t. \end{gathered} \tag{4.53}$$

If besides $\operatorname{ess\,sup}_{t\in[0,1]} p(t)\sigma_h(t) = M < \infty$, the inequalities

$$1 - \sqrt{\frac{1-t}{M}} \leq h(t) \leq \sqrt{\frac{t}{M}} \tag{4.54}$$

yield (4.53).

It should be noticed that the latter inequalities cannot be fulfilled if

$$\varepsilon \leq h(t) \leq 1 - \varepsilon, \quad \varepsilon > 0, \; t \in [0,1]. \tag{4.55}$$

Theorem 4.15. *Equation* (4.48) *does not possess the property A if* $p(t) \geq p_0 = \text{const} > 0$, $h(t) \in [\varepsilon, 1]$, *or* $h(t) \in [0, 1-\varepsilon]$, $\varepsilon > 0$, $t \in [0,1]$.

Proof. Suppose that $h(t) \in [\varepsilon, 1]$, but (4.48) possesses the property A. Then by virtue of Theorem 4.13 there exists $v_1 \in \mathcal{D}_\pi$ such that $v_1(0) = 0$, $v_1(t) > 0$ on $(0,1]$, and

$$t(1-t)\ddot{v}_1(t) = p(t)v_1[h(t)] + \varphi_1(t), \tag{4.56}$$

$\varphi_1(t) \geq 0$, $t \in [0,1]$. Let further $u \in \mathcal{D}_\pi$ be the solution of the problem

$$t(1-t)\ddot{x}(t) = p_0 m, \qquad x(0) = 0, \qquad x(1) = v_1(1), \tag{4.57}$$

where $m = \min_{t\in[\varepsilon,1]} v_1(t)$. Thus,

$$u(t) = p_0 m[(1-t)\ln(1-t) + t\ln t] + v_1(1)t. \tag{4.58}$$

The difference $z = v_1 - u$ does not take positive values since it is the solution of the problem

$$t(1-t)\ddot{x}(t) = \varphi(t) \geq 0, \qquad x(0) = x(1) = 0, \tag{4.59}$$

where $\varphi(t) = p(t)v_1[h(t)] + \varphi_1(t) - p_0 m \geq 0$. Consequently,

$$0 \leq v_1(t) = z(t) + u(t) \leq u(t). \tag{4.60}$$

But $u(t)$ is negative in a neighborhood of zero. The contradiction to the inequalities

$$0 \leq v_1(t) \leq u(t), \quad t \in [0,1], \tag{4.61}$$

completes the proof.

In the case $h(t) \in [0, 1-\varepsilon]$ the proof is similar with replacement of v_1 by v_2 and using as u the solution of the problem

$$t(1-t)\ddot{x}(t) = p_0 m, \qquad x(0) = v_2(0), \qquad x(1) = 0, \tag{4.62}$$

where $m = \min_{t\in[0,1-\varepsilon]} v_2(t)$. □

4.2.3. The general case

Rewrite (4.2) in the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \pi(t)\ddot{x}(t) - (T^+x)(t) + (T^-x)(t) = f(t), \tag{4.63}$$

where $T^+ - T^- = T$, $T^+ : \mathbf{C} \to \mathbf{L}$, and $T^- : \mathbf{C} \to \mathbf{L}$ are isotonic. Denote $\mathcal{L}^+x = \pi\ddot{x} - T^+x$. The equation $\mathcal{L}^+x = f$ was studied in the previous subsection. Let the equation possess the property A. Then the problem

$$\mathcal{L}^+x = f, \qquad x(a) = x(b) = 0 \tag{4.64}$$

is uniquely solvable and the Green operator G^+ of the problem is antitonic (Theorem 4.8).

The general Theorem D.2 allows us to formulate the next Valee-Poussin-like theorem.

Theorem 4.16. *Let the equation $\mathcal{L}^+x = f$ possess the property A. Then the following assertions are equivalent.*

(a) *The problem*

$$\mathcal{L}x = f, \qquad x(a) = x(b) = 0 \tag{4.65}$$

is uniquely solvable and its Green operator is antitonic.

(b) *There exists $v \in \mathbf{D}_\pi$ such that*

$$v(t) > 0, \quad (\mathcal{L}v)(t) \leq 0, \quad t \in (a, b), \tag{4.66}$$

and, besides,

$$v(a) + v(b) - \int_a^b (\mathcal{L}v)(s)ds > 0. \tag{4.67}$$

(c) *The spectral radius of $A \overset{\text{def}}{=} -G^+T^- : \mathbf{C} \to \mathbf{C}$ is less than* 1.

(d) *There exists a positive solution x($x(t) > 0$, $t \in [a, b]$) of the homogeneous equation $\mathcal{L}x = 0$.*

Denote by $G_1^+(G_2^+)$ the Green operator of the problem

$$\begin{gathered} \mathcal{L}^+x = f, \qquad x(a) = \dot{x}(b) = 0 \quad \text{if } \pi(t) = t - a \\ (\mathcal{L}^+x = f,\ \dot{x}(a) = x(b) = 0 \text{ if } \pi(t) = b - t), \end{gathered} \tag{4.68}$$

and let $A_i = -G_i^+T^-$, $i = 1, 2$. A similar assertion may be obtained on the base of Theorem D.2.

Theorem 4.17. *Let the equation $\mathcal{L}^+x = f$ possess the property A. Then the following assertions are equivalent.*

(a) *The problem*

$$\begin{gathered} \mathcal{L}x = f, \qquad x(a) = \dot{x}(b) = 0 \quad \textit{if } \pi(t) = t - a \\ (\mathcal{L}x = f,\ \dot{x}(a) = x(b) = 0 \textit{ if } \pi(t) = b - t) \end{gathered} \tag{4.69}$$

is uniquely solvable and its Green operator is antitonic.

(b) *There exists* $v \in \mathbf{D}_\pi$ *such that*

$$v(t)>0, \quad (\mathcal{L}v)(t)\le 0, \quad t\in(a,b]\ \big(t\in[a,b)\big), \tag{4.70}$$

and, besides,

$$v(a)+\dot{v}(b)-\int_a^b(\mathcal{L}v)(s)ds>0 \qquad \left(\dot{v}(a)+v(b)-\int_a^b(\mathcal{L}v)(s)ds>0\right). \tag{4.71}$$

(c) *The spectral radius of* $A_1:\mathbf{C}\to\mathbf{C}$ $(A_2:\mathbf{C}\to\mathbf{C})$ *is less than* 1.
(d) *There exists a positive solution on* $(a,b]$ *(on* $[a,b)$*) of the problem*

$$\begin{gathered}\mathcal{L}x=0, \qquad x(a)=0, \qquad \dot{x}(b)=1\\ \big(\mathcal{L}x=0,\ \dot{x}(a)=1,\ x(b)=0\big).\end{gathered} \tag{4.72}$$

For the purpose of illustration, let us show that the problem

$$(t-a)(b-t)\ddot{x}(t)-p(t)(S_hx)(t)=f(t), \qquad x(a)=x(b)=0 \tag{4.73}$$

is uniquely solvable and its Green operator is antitonic if

$$\int_a^b\frac{p^+(s)\sigma_h(s)}{\pi(s)}ds\le\frac{1}{b-a}, \tag{4.74}$$

$$\int_a^b p^-(s)\sigma(s)ds\le b-a, \tag{4.75}$$

where $p^+-p^-=p,\ p^+(t),p^-(t)\ge 0$.

Indeed, the inequality (4.74) guarantees, by Remark 4.5, the property A for the equation

$$(\mathcal{L}^+x)(t)\stackrel{\text{def}}{=}(t-a)(b-t)\ddot{x}(t)-p^+(t)(S_hx)(t)=f(t) \tag{4.76}$$

and, consequently (Theorem 4.8), the unique solvability and antitonicity of the Green operator G^+ of the problem

$$\mathcal{L}^+x=f, \qquad x(a)=x(b)=0. \tag{4.77}$$

The inequality (4.75) guarantees the estimate $\rho(A)<1$ of the spectral radius of the operator $A:\mathbf{C}\to\mathbf{C}$ defined by

$$(Ax)(t)=-\int_a^b G^+(t,s)p^-(s)(S_hx)(s)ds. \tag{4.78}$$

Indeed, the function $u(t)\stackrel{\text{def}}{=}(A\mathbf{1})(t)$ is the solution of the problem

$$(t-a)(b-t)\ddot{x}(t)-p^+(t)(S_hx)(t)=-p^-(t)\sigma_h(t), \qquad x(a)=x(b)=0. \tag{4.79}$$

Since u is positive, we obtain the inequality

$$(t-a)(b-t)\ddot{u}(t) \geq p^{-}(t)\sigma_h(t). \tag{4.80}$$

Hence

$$u(t) \leq -\int_a^b G_0(t,s)p^{-}(s)\sigma_h(s)ds \tag{4.81}$$

with $G_0(t,s)$ defined by (4.21).

Using the estimate $|G_0(t,s)| \leq 1/(b-a)$, we conclude that if $p^{-}(t)\sigma_h(t) \not\equiv 0$,

$$\|A\| = \max_{t\in[a,b]} u(t) \leq \max_{t\in[a,b]} \int_a^b |G_0(t,s)|\, p^{-}(s)\sigma_h(s)ds < \frac{1}{b-a}\int_a^b p^{-}(s)\sigma_h(s)ds. \tag{4.82}$$

Hence it follows that $\rho(A) < 1$.

So, by virtue of Theorem 4.16 (the implication (c)⇒(a)) the inequalities (4.74) and (4.75) guarantee the unique solvability and antitonicy of the Green operator of the problem (4.2).

The assumptions of Theorem 4.16 related to equation $\mathcal{L}^{+}x = f$ are too severe if we are interested only in the unique solvability, and the question about the sign of the Green function may be omitted. Using an idea of Lomtatidze (see [139]), E. I. Bravyi obtained the following test of solvability.

Consider the equation

$$(\mathcal{L}x)(t) \overset{\text{def}}{=} (t-a)(b-t)\ddot{x}(t) - (Tx)(t) = f(t), \tag{4.83}$$

and deduce a test of the unique solvability of the problem

$$\mathcal{L}x = f, \qquad x(a) = x(b) = 0. \tag{4.84}$$

Let, as above, $T = T^{+} - T^{-}$, $T^{+} : \mathbf{C} \to \mathbf{L}$ and let $T^{-} : \mathbf{C} \to \mathbf{L}$ be isotonic.

Theorem 4.18. *Problem* (4.84) *has a unique solution* $x \in \mathbf{D}_\pi$ *if*

$$\int_a^b (T^{-}\mathbf{1})(s)ds \leq b-a, \tag{4.85}$$

$$\int_a^b (T^{+}\mathbf{1})(s)ds \leq 2(b-a)\sqrt{1 - \frac{1}{b-a}\int_a^b (T^{-}\mathbf{1})(s)ds}. \tag{4.86}$$

In order to prove the theorem, we will use the following assertion.

Lemma 4.19. *The problem*

$$(t-a)(b-t)\ddot{x}(t)=-(T^{-}x)(t)+f(t), \qquad x(a)=x(b)=0 \tag{4.87}$$

is uniquely solvable and its Green operator G^{-} *is antitonic if the inequality* (4.85) *holds.*

Proof. Denote

$$(Ax)(t)=-\int_a^b G_0(t,s)(T^{-}x)(s)ds, \tag{4.88}$$

with $G_0(t,s)$ defined by (4.21). The operator $A:\mathbf{C}\to\mathbf{C}$ is isotonic. From (4.85) and the estimate

$$\left|G_0(t,s)\right| \le \frac{1}{b-a}, \tag{4.89}$$

we obtain

$$\rho(A)\le \|A\| = \|A\mathbf{1}\|_{\mathbf{C}} < \frac{1}{b-a}\int_a^b (T^{-}\mathbf{1})(s)ds \le 1. \tag{4.90}$$

This and Theorem 4.16 (the implication (c)⇒(a)) imply the assertion of Lemma 4.19. □

Proof of Theorem 4.18. The homogeneous problem (4.84) is equivalent to the equation

$$x=G^{-}T^{+}x \tag{4.91}$$

with antitonic $G^{-}T^{+}:\mathbf{C}\to\mathbf{C}$. Assume that the homogeneous problem have a nontrivial solution u. Then the values of the solution u have varied signs on $[a,b]$. Let u take its maximum and minimum values at the points t^* and t_*. Let further $t^*<t_*$ and let $c\in(t^*,t_*)$ be a zero of u. Denote $M=u(t^*)$, $-m=u(t_*)$. We have the evident inequalities

$$\begin{gathered} -m\le u(t)\le M,\\ -m(T^{+}\mathbf{1})(t)\le (T^{+}u)(t)\le M(T^{+}\mathbf{1})(t),\\ -m(T^{-}\mathbf{1})(t)\le (T^{-}u)(t)\le M(T^{-}\mathbf{1})(t). \end{gathered} \tag{4.92}$$

The function u satisfies the equality

$$(t-a)(b-t)\ddot{u}(t)=(Tu)(t), \quad t\in[a,c], \tag{4.93}$$

and boundary conditions $u(a) = u(c) = 0$. Hence

$$u(t) = \int_a^c G_a(t,s)(Tu)(s)ds, \quad t \in [a,c], \tag{4.94}$$

where (by virtue of (4.21))

$$G_a(t,s) = \begin{cases} -\dfrac{c-t}{(b-s)(c-a)} & \text{if } a \le s \le t \le c, \\ -\dfrac{(t-a)(c-s)}{(s-a)(b-s)(c-a)} & \text{if } a \le t < s \le c. \end{cases} \tag{4.95}$$

Using the estimate

$$0 \ge G_a(t,s) \ge -\frac{c-s}{(c-a)(b-s)} \ge -\frac{1}{b-a}, \quad (t,s) \in [a,c] \times [a,c], \tag{4.96}$$

we obtain

$$\begin{aligned} M &= \int_a^c G_a(t^*,s)\big[(T^+u)(s) - (T^-u)(s)\big]ds \\ &\le M\int_a^c \big|G_a(t^*,s)\big|(T^-\mathbf{1})(s)ds + m\int_a^c \big|G_a(t^*,s)\big|(T^+\mathbf{1})(s)ds \\ &< \frac{M}{b-a}\int_a^c (T^-\mathbf{1})(s)ds + \frac{m}{b-a}\int_a^c (T^+\mathbf{1})(s)ds. \end{aligned} \tag{4.97}$$

Hence

$$M < \frac{(m/(b-a))\int_a^c (T^+\mathbf{1})(s)ds}{1 - (1/(b-a))\int_a^c (T^-\mathbf{1})(s)ds}. \tag{4.98}$$

Similarly, using the equality

$$u(t) = \int_c^b G_b(t,s)(Tu)(s)ds, \quad t \in [c,b], \tag{4.99}$$

where

$$G_b(t,s) = \begin{cases} -\dfrac{(s-c)(b-t)}{(b-s)(s-a)(b-c)} & \text{if } c \le s \le t \le b, \\ -\dfrac{t-c}{(s-a)(b-c)} & \text{if } c \le t < s \le b, \end{cases} \tag{4.100}$$

we conclude that

$$m < \frac{(M/(b-a))\int_c^b (T^+\mathbf{1})(s)ds}{1 - (1/(b-a))\int_c^b (T^-\mathbf{1})(s)ds}. \tag{4.101}$$

This and (4.98) imply the inequality

$$M < M\frac{\int_a^c (T^+\mathbf{1})(s)ds\int_c^b (T^+\mathbf{1})(s)ds}{(b-a)^2\big(1-(1/(b-a))\int_a^c (T^-\mathbf{1})(s)ds\big)\big(1-(1/(b-a))\int_c^b (T^-\mathbf{1})(s)ds\big)}. \tag{4.102}$$

Thus

$$\begin{aligned}
1 &+ \frac{1}{(b-a)^2}\int_a^c (T^-\mathbf{1})(s)ds\int_c^b (T^-\mathbf{1})(s)ds \\
&< \frac{1}{(b-a)^2}\int_a^c (T^+\mathbf{1})(s)ds\int_c^b (T^+\mathbf{1})(s)ds + \frac{1}{b-a}\int_a^c (T^-\mathbf{1})(s)ds \\
&\quad + \frac{1}{b-a}\int_c^b (T^-\mathbf{1})(s)ds \\
&\le \frac{1}{(b-a)^2}\left(\frac{\int_a^c (T^+\mathbf{1})(s)ds + \int_c^b (T^+\mathbf{1})(s)ds}{2}\right)^2 + \frac{1}{b-a}\int_a^b (T^-\mathbf{1})(s)ds \\
&= \frac{1}{(b-a)^2}\left(\frac{\int_a^b (T^+\mathbf{1})(s)ds}{2}\right)^2 + \frac{1}{b-a}\int_a^b (T^-\mathbf{1})(s)ds.
\end{aligned} \tag{4.103}$$

Hence

$$\frac{1}{4(b-a)^2}\left(\int_a^b (T^+\mathbf{1})(s)ds\right)^2 + \frac{1}{b-a}\int_a^b (T^-\mathbf{1})(s)ds > 1. \tag{4.104}$$

Thus, if

$$\frac{1}{4(b-a)^2}\left(\int_a^b (T^+\mathbf{1})(s)ds\right)^2 + \frac{1}{b-a}\int_a^b (T^-\mathbf{1})(s)ds \ge 1, \tag{4.105}$$

which is equivalent to the inequality (4.86), the existence of the nontrivial solution of the homogeneous problem is impossible. Therefore, the problem (4.84) is uniquely solvable. □

Pioneering investigations of singular equations using the idea of choosing a special space $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ for each kind of singularity were published in [136, 207].

The results of Section 4.2 were published in [22].

In the case of ordinary singular differential equation some refined studies were performed in [112, 113].

4.3. Inner singularities

The equations with the coefficient at the leading derivative, which has zeros inside $[a, b]$ were studied in [54, 55]. We illustrate the idea of these works by the example of the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} t\ddot{x}(t) + p(t)(S_h x)(t) = f(t), \quad t \in [a,b], \tag{4.106}$$

where $a < 0 < b$; $p, f \in \mathbf{L}$; and h is a measurable function.

Just as in the previous example, the principal part of $\mathcal{L} : \mathbf{W}^2 \to \mathbf{L}$ is not a Fredholm operator. As the space $\mathbf{D}$, on which it is reasonable to consider the operator $\mathcal{L}$, we take the space of solutions of the three-point impulse model boundary value problem

$$t\ddot{x}(t) = z(t), \qquad x(a) = \beta^1, \qquad x(b) = \beta^2, \qquad x(0) = \beta^3. \tag{4.107}$$

We will suppose that the solution of this problem is a function $x : [a,b] \to \mathbb{R}^1$ whose derivative $\dot{x}$ is absolutely continuous on $[a,0)$ and $[0,b]$ and the product $t\ddot{x}(t)$ is summable on $[a,b]$. Thus, the homogeneous equation $t\ddot{x}(t) = 0$ has three linearly independent solutions

$$\begin{gathered} u_1(t) = \frac{t}{a}\chi_{[a,0)}(t), \qquad u_2(t) = \frac{a-t}{a}\chi_{[a,0)}(t) + \frac{b-t}{b}\chi_{[0,b]}(t), \\ u_3(t) = \frac{t}{b}\chi_{[0,b]}(t), \end{gathered} \tag{4.108}$$

and the nonhomogeneous equation $t\ddot{x}(t) = z(t)$ has a solution for every $z \in \mathbf{L}$. For instance, such a solution is

$$x(t) = (\Lambda z)(t) = \int_a^b \Lambda(t,s)z(s)\,ds, \tag{4.109}$$

where

$$\Lambda(t,s) = \begin{cases} -\dfrac{t(s-a)}{as} & \text{if } a \le s \le t < 0, \\[2mm] -\dfrac{t-a}{a} & \text{if } a \le t < s \le 0, \\[2mm] \dfrac{t-b}{b} & \text{if } 0 \le s \le t \le b, \\[2mm] \dfrac{t(s-b)}{bs} & \text{if } 0 \le t < s \le b, \\[2mm] 0 & \text{at all other points.} \end{cases} \tag{4.110}$$

Since the determinant of the model problem is not equal zero:

$$\begin{vmatrix} u_1(a) & u_2(a) & u_3(a) \\ u_1(b) & u_2(b) & u_3(b) \\ u_1(0) & u_2(0) & u_3(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \tag{4.111}$$

this problem has, for any $\{z,\beta\} \in \mathbf{L}\times\mathbb{R}^3$, the unique solution $x = \Lambda z + Y\beta$, where $\beta = \operatorname{col}\{\beta^1,\beta^2,\beta^3\}$,

$$(Y\beta)(t) = \beta^1 u_1(t) + \beta^2 u_2(t) + \beta^3 u_3(t). \tag{4.112}$$

Let us take $\mathbf{D} = \Lambda\mathbf{L} \oplus Y\mathbb{R}^3$, where $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}\times\mathbb{R}^3 \to \mathbf{D}$ is the isomorphism, the inverse $\mathcal{J}^{-1} = [\delta, r]$ is defined by

$$(\delta x)(t) = t\ddot{x}(t), \qquad rx = \{x(a), x(b), x(0)\}. \tag{4.113}$$

The principal part of $\mathcal{L} : \mathbf{D} \to \mathbf{L}$ has the form $Q = I + K$, where

$$(Kz)(t) = \int_a^b p(t)\Lambda[h(t),s]z(s)ds. \tag{4.114}$$

If the operator $Q : \mathbf{L} \to \mathbf{L}$ has the bounded inverse, the principal boundary value problem

$$\mathcal{L}x = f, \qquad x(a) = \alpha^1, \qquad x(b) = \alpha^2, \qquad x(0) = \alpha^3 \tag{4.115}$$

is uniquely solvable (Theorem 1.5) and the general solution of the equation $\mathcal{L}x = f$ has the representation

$$x(t) = \int_a^b G(t,s)f(s)ds + c_1x_1(t) + c_2x_2(t) + c_3x_3(t), \tag{4.116}$$

where $G(t,s)$ is the Green function of this problem, x_1, x_2, x_3 constitutes a fundamental system of solutions of $\mathcal{L}x = 0$, and c_i are constants.

Consider an example of singularity of another kind. Define the operation θ by

$$(\theta x)(t) = \begin{cases} \ddot{x}(t) & \text{if } t \in [1,2], \\ 0 & \text{if } t \in [0,1), \end{cases} \tag{4.117}$$

and consider the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} (\theta x)(t) + \dot{x}(t) + (Tx)(t) = f(t), \quad t \in [0,2], \tag{4.118}$$

with a linear operator $T : \mathbf{W}^2 \to \mathbf{L}$.

The principal part of the operator $\mathcal{L} : \mathbf{W}^2 \to \mathbf{L}$ is not Fredholm even under the assumption that $T : \mathbf{W}^2 \to \mathbf{L}$ is a compact operator. We will define the operator $\mathcal{L}$ on a wider space $\mathbf{D}$, assuming that T allows an extension onto this space. We will construct the space $\mathbf{D}$ as follows.

Let us take as a model the problem

$$(\mathcal{L}_0x)(t) \stackrel{\text{def}}{=} (\theta x)(t) + \chi_{[0,1)}(t)\dot{x}(t) = z(t), \quad t \in [0,2],$$
$$x(0) = \beta^1, \qquad x(1) = \beta^2, \qquad \dot{x}(1) = \beta^3. \tag{4.119}$$

This problem decays into two problems that are integrable in the explicit form,

$$\dot{x}(t) = z(t), \quad t \in [0,1), \qquad x(0) = \beta^1,$$
$$\ddot{x}(t) = z(t), \quad t \in [1,2], \qquad x(1) = \beta^2, \qquad \dot{x}(1) = \beta^3. \tag{4.120}$$

We may take as the solution of the model problem the function

$$x(t) = \chi_{[0,1)}(t)\left\{\int_0^t z(s)ds + \beta^1\right\}$$
$$+ \chi_{[1,2]}(t)\left\{\int_0^t \chi_{[1,2]}(s)(t-s)z(s)ds + \beta^2 + \beta^3(t-1)\right\}. \tag{4.121}$$

Denote $(\Lambda z)(t) = \int_0^2 \Lambda(t,s)z(s)ds$, where

$$\Lambda(t,s) = \begin{cases} 1 & \text{if } 0 \le s \le t < 1, \\ t-s & \text{if } 1 \le s \le t \le 2, \\ 0 & \text{at all other points.} \end{cases} \tag{4.122}$$

Let, further,

$$(Y\beta)(t) = \beta^1 u_1(t) + \beta^2 u_2(t) + \beta^3 u_3(t), \quad \beta = \operatorname{col}\{\beta^1, \beta^2, \beta^3\},$$
$$u_1(t) = \chi_{[0,1)}(t), \qquad u_2(t) = \chi_{[1,2]}(t), \qquad u_3(t) = \chi_{[1,2]}(t)(t-1). \tag{4.123}$$

The solution of the model problem has the form $x = \Lambda z + Y\beta$.

Next define the space $\mathbf{D}$ by $\mathbf{D} = \Lambda\mathbf{L} \oplus Y\mathbb{R}^3$. This space consists of the functions $x : [0,2] \to \mathbb{R}^1$ with possible discontinuity at the point $t = 1$. These functions are absolutely continuous on $[0,1)$ and have absolutely continuous derivatives on $[1,2]$. $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbb{R}^3 \to \mathbf{D}$ is the isomorphism, $\mathcal{J}^{-1} = [\delta, r]$, where

$$\delta x = \mathcal{L}_0 x, \qquad rx = \{x(0), x(1), \dot{x}(1)\}. \tag{4.124}$$

The norm may be defined by

$$\|x\|_{\mathbf{D}} = \|\mathcal{L}_0 x\|_{\mathbf{L}} + |x(0)| + |x(1)| + |\dot{x}(1)|. \tag{4.125}$$

Since $\mathcal{L}x = \mathcal{L}_0 x + \chi_{[1,2]}\dot{x} + Tx$, we have

$$(Qz)(t) = z(t) + \chi_{[1,2]}(t)\int_0^t \chi_{[1,2]}(s)z(s)ds + (T\Lambda z)(t). \tag{4.126}$$

If the product $T\Lambda : \mathbf{L} \to \mathbf{L}$ is compact, the principal part $Q : \mathbf{L} \to \mathbf{L}$ is canonical Fredholm. If $\|K\|_{\mathbf{L}\to\mathbf{L}} < 1$, where

$$(Kz)(t) = (T\Lambda z)(t) + \chi_{[1,2]}(t)\int_0^t \chi_{[1,2]}(s)z(s)ds, \tag{4.127}$$

the principal boundary value problem

$$\mathcal{L}x = f, \qquad x(0) = \alpha^1, \qquad x(1) = \alpha^2, \qquad \dot{x}(1) = \alpha^3 \tag{4.128}$$

is uniquely solvable and in this case (Theorem 1.6) the homogeneous equation $\mathcal{L}x = 0$ has three-dimensional fundamental system of solutions x_1, x_2, x_3, also the general solution of the equation $\mathcal{L}x = f$ in the space $\mathbf{D}$ has the representation

$$x(t) = \int_0^2 G(t,s)f(s)ds + c_1x_1(t) + c_2x_2(t) + c_3x_3(t), \tag{4.129}$$

where $G(t,s)$ is the Green function of the principal boundary value problem, $c_i =$ const.

Denote $\Delta x(t) = x(t) - x(t-0)$. The constriction $\mathbf{D}_0 = \{x \in \mathbf{D} : \Delta x(1) = 0\}$ of the space $\mathbf{D}$ contains continuous functions only. The homogeneous equation $\mathcal{L}_0 x = 0$ has two linearly independent solutions

$$y_1(t) = 1 - \chi_{[1,2]}(t)(t-1), \qquad y_2(t) = \chi_{[1,2]}(t)(t-1) \tag{4.130}$$

in the space $\mathbf{D}_0$. The equation $\mathcal{L}_0 x = z$ has, for any $z \in \mathbf{L}$, solutions that belong to $\mathbf{D}_0$, for instance,

$$v(t) = \chi_{[0,1)}(t)\int_0^t z(s)ds + \chi_{[1,2]}(t)\left\{\int_0^t \chi_{[1,2]}(s)(t-s)z(s)ds + \int_0^1 z(s)ds\right\}. \tag{4.131}$$

Thus, the general solution of the model equation $\mathcal{L}_0 x = z$ in the space $\mathbf{D}_0$ may be represented in the form

$$x(t) = v(t) + c_1y_1(t) + c_2y_2(t), \tag{4.132}$$

where c_1, c_2 are constants.

Since

$$\begin{vmatrix} y_1(0) & y_2(0) \\ y_1(2) & y_2(2) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0, \tag{4.133}$$

the two-point boundary value problem

$$\mathcal{L}_0 x = z, \qquad x(0) = 0, \qquad x(2) = 0 \tag{4.134}$$

is uniquely solvable in the space $\mathbf{D}_0$. The Green function of this problem, $W(t,s)$, can be constructed by finding the constants $c_1,\ c_2$ in (4.132) so that $x(0) = x(2) = 0$. We have

$$x(t) \stackrel{\text{def}}{=} (Wz)(t) = \int_0^2 W(t,s)z(s)ds, \tag{4.135}$$

where

$$W(t,s) = \begin{cases} 1 & \text{if } 0 \le s \le t < 1, \\ 2-t & \text{if } 1 \le t \le 2,\ 0 \le s < 1, \\ -(2-t)(s-1) & \text{if } 1 \le s \le t \le 2, \\ -(2-s)(t-1) & \text{if } 1 \le t < s \le 2, \\ 0 & \text{at all other points.} \end{cases} \tag{4.136}$$

Notice that it is possible to construct $W(t,s)$ on the base of the representation

$$x(t) = (\Lambda z)(t) + \beta^1 u_1(t) + \beta^2 u_2(t) + \beta^3 u_3(t) \tag{4.137}$$

of the solution (4.119) by demanding the fulfillment of the conditions $x(0) = \Delta x(1) = x(2) = 0$.

Thus, the space $\mathbf{D}_0$ is defined by $\mathbf{D}_0 = W\mathbf{L} \oplus Y_0\mathbb{R}^2$, where

$$(Y_0\beta)(t) = [1 - \chi_{[1,2]}(t)(t-1)]\beta^1 + \chi_{[1,2]}(t)(t-1)\beta^2, \quad \beta = \operatorname{col}\{\beta^1, \beta^2\}, \tag{4.138}$$

$\mathcal{J}_0 = \{W, Y_0\} : \mathbf{L}\times\mathbb{R}^2 \to \mathbf{D}_0$ is the isomorphism, $\mathcal{J}_0^{-1} = [\mathcal{L}_0, r_0],\ r_0 x = \{x(0), x(2)\}$. The two-point boundary value problem

$$\mathcal{L}x = f, \qquad x(0) = \alpha^1, \qquad x(2) = \alpha^2 \tag{4.139}$$

is the principal boundary value problem for the equation $\mathcal{L}x = f$ in the space $\mathbf{D}_0$. This problem is uniquely solvable if and only if the operator $Q = \mathcal{L}W : \mathbf{L} \to \mathbf{L}$ has the bounded inverse.

4.4. The chemical reactor's equation

The mathematical description of some processes in chemical reactors gives rise to the singular boundary value problem

$$\ddot{x}(t) + \frac{k}{t}\dot{x}(t) = f(t,x), \quad t \in [0,1], \qquad \dot{x}(0) = 0, \qquad x(1) = \alpha. \tag{4.140}$$

There is an extensive literature on the subject (see, e.g., [169, 224]). Application of the method of "$\mathcal{L}_1, \mathcal{L}_2$—*quasilinearization*" (see [32]) to this problem demands the tests of solvability of the linear problem

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + \frac{k}{t}\dot{x}(t) + p(t)x(t) = f(t), \quad t \in [0,1], \qquad \dot{x}(0) = 0, \qquad x(1) = \alpha \tag{4.141}$$

and of antitonicy of the Green operator of the problem. We will follow below the scheme suggested by Alves [5].

The value of the operator $\mathcal{L}$ on a function whose derivative differs from zero at the point $t = 0$ (e.g., $x(t) = t$) does not belong to the space of summable functions. Thus it is natural to consider the equation $\mathcal{L}x = f$ in a space where there are no such functions. Let us consider the equation $\mathcal{L}x = f$ in the space $\mathbf{D}$ of functions $x : [0,1] \to \mathbb{R}^1$ defined by

$$x(t) = \int_0^t (t-s)z(s)ds + \beta, \quad \{z,\beta\} \in \mathbf{L}_p \times \mathbb{R}^1, \tag{4.142}$$

where $\mathbf{L}_p$ is the Banach space of functions $z : [0,1] \to \mathbb{R}^1$ under the norm

$$\|z\|_{\mathbf{L}_p} = \left\{ \int_0^1 |z(s)|^p ds \right\}^{1/p}. \tag{4.143}$$

We will demand below the inequalities $p > 1$ and $k > -(p-1)/p$.

The space $\mathbf{D}$ is a finite-dimensional restriction of the space $\mathbf{W}_p^2$ of the functions with absolutely continuous derivative and the second derivative from $\mathbf{L}_p$. Namely

$$\mathbf{D} = \{x \in \mathbf{W}_p^2 : \dot{x}(0) = 0\}. \tag{4.144}$$

Such a space is isomorphic to the product $\mathbf{L}_p \times \mathbb{R}^1$, the isomorphism may be defined, for instance, by

$$\mathcal{J} = \{\Lambda, Y\}, \qquad (\Lambda z)(t) = \int_0^t (t-s)z(s)ds, \qquad Y\beta = \beta, \qquad \mathcal{J}^{-1} = [\delta, r], \qquad \delta x = \ddot{x}, \qquad rx = x(0). \tag{4.145}$$

The equation

$$(Mx)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + \frac{k}{t}\dot{x}(t) = f(t), \quad t \in [0,1], \tag{4.146}$$

is integrable in the explicit form and may be taken as a model one. The principal part $Q_M = M\Lambda$ of the operator $M : \mathbf{D} \to \mathbf{L}_p$ has the form

$$Q_M z = z + Kz, \tag{4.147}$$

where

$$(Kz)(t) = \frac{k}{t}\int_0^t z(s)ds. \tag{4.148}$$

This is the so-called Cezaro operator. As one can make sure by immediate integration of the integral equation $z + Kz = f$, the inverse $Q_M^{-1} = (I+K)^{-1} : \mathbf{L}_p \to \mathbf{L}_p$ has the representation

$$(Q_M^{-1} f)(t) = f(t) - kt^{-(1+k)}\int_0^t s^k f(s)ds \tag{4.149}$$

and is bounded if $k > -(p-1)/p$.

It should be noticed that $K : \mathbf{L}_p \to \mathbf{L}_p$ is not compact and the successive approximations in the case $k \geq 1$ do not converge. One can see the fact by beginning the successive approximations from the element $z_0 = 1$. This phenomenon unexpected from the view-point of the accustomed properties of Volterra integral equations is connected with characteristics of the Cezaro operator $K : \mathbf{L}_p \to \mathbf{L}_p$ whose spectrum was studied in [162].

So, the principal part $Q_M : \mathbf{L}_p \to \mathbf{L}_p$ of $M : \mathbf{D} \to \mathbf{L}_p$ as well the inverse Q_M^{-1} are bounded Volterra. By Theorem 1.16, the Cauchy problem $Mx = f$, $x(0) = 0$ (the principal boundary value problem), is uniquely solvable . By virtue of Theorem 1.17 the fundamental system of $Mx = 0$ is one-dimensional and consists of $x(t) \equiv 1$. Thus the general solution of the equation $Mx = f$ has the form

$$x(t) = (\Lambda Q_M^{-1} f)(t) + \alpha = (C_M f)(t) + \alpha, \tag{4.150}$$

where

$$(C_M f)(t) = \int_0^t C_M(t,s) f(s)ds \tag{4.151}$$

and the Cauchy function $C_M(t,s)$ is defined by

$$C_M(t,s) = t - s - ks^k \int_0^t (t-\tau)\tau^{-(1+k)} d\tau, \quad 0 \leq s \leq t \leq 1. \tag{4.152}$$

Thus,

$$C_M(t,s) = s\ln\frac{t}{s}, \qquad C_M(0,0) = 0 \tag{4.153}$$

in the case $k = 1$ and

$$C_M(t,s) = \frac{s}{k-1}\left[1 - \left(\frac{s}{t}\right)^{k-1}\right], \qquad C_M(0,0) = 0 \tag{4.154}$$

in the case $k \neq 1$.

It should be noticed that the operator $C_M : \mathbf{L}_p \to \mathbf{C}$ defined by (4.151) is compact being the product of the compact $\Lambda : \mathbf{L}_p \to \mathbf{C}$ and the bounded $Q_M^{-1} : \mathbf{L}_p \to \mathbf{L}_p$.

The boundary value problem $Mx = f$, $lx = \alpha$ for any functional l on the space $\mathbf{D}$, such that $l(1) \neq 0$, is uniquely solvable and its Green function $G_M(t,s)$ may be constructed in the explicit form on the base of the equality that follows from Theorem 1.20:

$$G_M(t,s) = C_M(t,s) - \frac{1}{l(\mathbf{1})} l[C_M(\cdot,s)], \tag{4.155}$$

where $C_M(t,s) = 0$ if $a \le t < s \le b$. Thus the Green function of the problem

$$\ddot{x} + \frac{k}{t}\dot{x} = f, \qquad x(1) = 0 \tag{4.156}$$

is defined by

$$G_M(t,s) = \begin{cases} s\ln t & \text{if } 0 \le s \le t \le 1, \\ s\ln s & \text{if } 0 \le t < s \le 1, \end{cases} \tag{4.157}$$

in the case $k = 1$, and by

$$G_M(t,s) = \begin{cases} \dfrac{s^k(t^{1-k} - 1)}{1-k} & \text{if } 0 \le s \le t \le 1, \\ \dfrac{s^k(s^{1-k} - 1)}{1-k} & \text{if } 0 \le t < s \le 1, \end{cases} \tag{4.158}$$

in the case $k \neq 1$.

Next consider the problem

$$(\mathcal{L}x)(t) \overset{\text{def}}{=} \ddot{x}(t) + \frac{k}{t}\dot{x}(t) + p(t)x(t) = f(t), \qquad x(1) = \alpha \tag{4.159}$$

with $p \in \mathbf{L}_p$.

Theorem 4.20. *The following assertions are equivalent.*

(a) *The problem* (4.159) *is uniquely solvable in the space* $\mathbf{D}$ *and its Green operator is antitonic.*

(b) *There exists* $v \in \mathbf{D}$ *such that*

$$v(t) \geq 0, \quad (\mathcal{L}v)(t) \leq 0, \quad t \in [0,1], \tag{4.160}$$

and besides

$$v(1) - \int_0^1 (\mathcal{L}v)(s)ds > 0. \tag{4.161}$$

(c) *There exists the positive solution of the homogeneous equation* $\mathcal{L}x = 0$.

Beforehand, let us prove two auxiliary lemmas using the following designations: $p = p^+ - p^-$, $p^+(t)$, $p^-(t) \geq 0$, $\mathcal{L}_0 x = \ddot{x} + (k/t)\dot{x} - p^- x$.

Lemma 4.21. *The Cauchy problem*

$$\mathcal{L}_0 x = f, \qquad x(0) = \alpha \tag{4.162}$$

is uniquely solvable in the space $\mathbf{D}$, *the Cauchy operator* C_0 *of the equation* $\mathcal{L}_0 x = f$ *is isotonic, and there exists the positive solution* u_0 *of the homogeneous equation* $\mathcal{L}_0 x = 0$.

Proof. The problem (4.162) is equivalent to the equation

$$x = Hx + g, \tag{4.163}$$

where

$$(Hx)(t) = \int_0^t C_M(t,s)p^-(s)x(s)ds, \qquad g(t) = \int_0^t C_M(t,s)f(s)ds + \alpha. \tag{4.164}$$

The operator $H : \mathbf{C} \to \mathbf{C}$ is Volterra, compact, and isotonic. Therefore the problem (4.162) is uniquely solvable and the Cauchy operator of the problem

$$C_0 = (I + H + H^2 + \cdots)C_M \tag{4.165}$$

is also isotonic. The function u_0 as the solution of the equation $x = Hx + \mathbf{1}$ has the representation

$$u_0 = \mathbf{1} + H(\mathbf{1}) + H^2(\mathbf{1}) + \cdots \tag{4.166}$$

and, consequently, $u_0(t) \geq 1,\ t \in [0,1]$. □

Lemma 4.22. *The boundary value problem*

$$\mathcal{L}_0 x = f, \qquad x(1) = 0 \tag{4.167}$$

is uniquely solvable in the space $\mathbf{D}$ *and its Green operator* W *is antitonic.*

Proof. The homogeneous problem $\mathcal{L}_0 x = 0$, $x(1) = 0$ is equivalent to the equation $x = Bx$ with antitonic

$$(Bx)(t) = \int_0^1 G_M(t,s)p^-(s)x(s)ds. \tag{4.168}$$

If we assume that the problem is not uniquely solvable, then it follows that there exists the nontrivial solution y of the problem. In virtue of antitonicity of B, the nontrivial solution varies the sign on $[0,1]$. Thus y has at least two zeros $t = 1$ and $t = \tau$. The case that $\tau = 0$ is impossible since in this case, we have $y(t) \equiv 0$ by Lemma 4.21.

Consider now the regular equation

$$\ddot{x}(t) + \frac{k}{t}\dot{x}(t) - p^-(t)x(t) = 0, \quad t \in [\tau, 1]. \tag{4.169}$$

The fundamental system of this equation is nonoscillatory on $[\tau, 1]$. It follows from Theorem D.1 if we put $v(t) \equiv 1$. Thus, $y(\tau) \neq 0$. The contradiction proves the unique solvability of the problem (4.167).

Assume that the Green operator W of the problem (4.167) is not antitonic and $f(t) \geq 0$ is a function such that the solution $x = Wf$ takes positive values on a set of points from $[0,1]$. If $x(0) = \gamma \geq 0$, the function x, as the solution of the Cauchy problem

$$\mathcal{L}_0 x = f, \qquad x(0) = \gamma, \tag{4.170}$$

does not satisfy the condition $x(1) = 0$ of the boundary value problem. Really, if $f(t) \geq 0$,

$$x(1) = \int_0^1 C_0(1,s)f(s)ds + \gamma > 0 \tag{4.171}$$

since $C_0(1,s) \geq C_M(1,s) > 0$, $s \in (0,1)$, by virtue of (4.165). If $x(0) < 0$, the solution $x = Wf$ has a pair of zeros $\tau_1, \tau_2 \in (0,1]$ such that $x(t) > 0$, $t \in (\tau_1, \tau_2)$. On the segment $[\tau_1, \tau_2]$ the function $x = Wf$ satisfies the two-point boundary value problem

$$\begin{gathered}(Vx)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + \frac{k}{t}\dot{x}(t) - p^-(t)x(t) = f(t), \quad t \in [\tau_1, \tau_2], \\ x(\tau_1) = x(\tau_2) = 0.\end{gathered} \tag{4.172}$$

Since $(V[\mathbf{1}])(t) = -p^-(t) \leq 0$, the solution x of this problem does not take positive values by virtue of Theorem D.1. The contradiction completes the proof. □

Now the proof of Theorem 4.20 follows at once from Theorem D.2. Indeed, Lemmas 4.21 and 4.22 guarantee the fulfillment of the assumptions of this theorem as applied to the problem

$$(\mathcal{L}x)(t) \equiv (\mathcal{L}_0 x)(t) + p^+(t)x(t) = f(t), \qquad x(1) = 0. \tag{4.173}$$

5 Minimization of square functionals

5.1. Introduction

The problem of minimization of functionals is unsolvable in the frame of the classical calculus of variations if the given functional has no minimum on the traditional sets of functions. The question about the suitable choice of the set on which the functional must be defined was posed by Hilbert and, as it was emphasized by the authors of the book of Alekseev et al. [3], therewith each class of functionals must be studied in its own proper space.

The classical calculus of variations usually deals with the functionals of the form

$$\int_a^b (\Phi x)(s)ds \tag{5.1}$$

with a local operator $\Phi : \mathbf{W}^n \to \mathbf{L}$. The results of Chapter 1 enables us to study the functional with more general operator $\Phi : \mathbf{W}^n \to \mathbf{L}$ and replace the space $\mathbf{W}^n$ by a more suitable "own" space $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$.

The scheme proposed below permits approaching a new fashion to the problem of minimization, it extends the capabilities of the calculus of variations and leads to sufficient tests of the existence of the minimum for some classes of functionals in the terms of the problem.

The scheme has been developed on the base of the theory of abstract functional differential equations in the works of the Perm Seminar.

5.2. The criterion for the existence of the minimum of the square functional

Let $\mathbf{D}$ be a Banach space of functions $x : [a,b] \to \mathbb{R}^1$, which is isomorphic to the direct product $\mathbf{L}_2 \times \mathbb{R}^n$, let $\mathbf{L}_2$ be the Banach space of square summable functions $z : [a,b] \to \mathbb{R}^1$, $\|z\|_{\mathbf{L}_2} = \{\int_a^b z^2(s)ds\}^{1/2}$.

Consider the problem on the existence of an element $x \in \mathbf{D}$ at which the square functional

$$\mathcal{I}(x) = \frac{1}{2}\int_a^b \left\{ \sum_{i=1}^{m} (T_{1i}x)(s)(T_{2i}x)(s) + (T_0 x)(s) + \omega(s) \right\} ds \tag{5.2}$$

with additional conditions

$$l^i x = \alpha^i, \quad i = 1,\dots,n, \tag{5.3}$$

reaches the minimum.

Here $T_{ji} : \mathbf{D} \to \mathbf{L}_2$, $j = 1,2$, $i = 1,\dots,m$, and $T_0 : \mathbf{D} \to \mathbf{L}_2$ are linear bounded operators, $l^1,\dots,l^n$ is a system of linear bounded, linearly independent functionals on $\mathbf{D}$, and ω is a summable function.

We will rewrite such a problem in the form

$$\begin{gathered} \mathcal{I}(x) \longrightarrow \min, \\ lx = \alpha, \end{gathered} \tag{5.4}$$

where $l = [l^1,\dots,l^n]$, $\alpha = \{\alpha^1,\dots,\alpha^n\}$.

5.2.1. The reduction of the problem in the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ to a problem in the space $\mathbf{L}_2$

Suppose that the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbb{R}^n \to \mathbf{D}$ is constructed according to a given system of functionals $l^1,\dots,l^n$ on the base of a uniquely solvable boundary value problem

$$\delta x = z, \qquad l^i x = \beta^i, \quad i = 1,\dots,n, \tag{5.5}$$

with linear $\delta : \mathbf{D} \to \mathbf{L}_2$. Such an operator exists by virtue of Theorem 1.22. Thus, Λ is the Green operator of the problem (5.5), $Y = \{y_1,\dots,y_n\}$, $y_1,\dots,y_n$ constitute the fundamental system of the homogeneous equation $\delta x = 0$.

The problem under consideration may be reduced, by means of the substitution

$$x = \Lambda z + Y\alpha, \tag{5.6}$$

to the problem of the minimization of the functional

$$\mathcal{I}_1(z) \stackrel{\text{def}}{=} \mathcal{I}(\Lambda z + Y\alpha) \tag{5.7}$$

over the Hilbert space $\mathbf{L}_2$ without additional conditions. Denoting $Y\alpha = u$, $Q_{ji} = T_{ji}\Lambda$, $Q_0 = T_0\Lambda$, we have

$$
\begin{aligned}
\mathcal{I}(x) &= \mathcal{I}(\Lambda z + u) \overset{\text{def}}{=} \mathcal{I}_1(z) \\
&= \frac{1}{2}\int_a^b \sum_{i=1}^m (Q_{1i}z)(s)(Q_{2i}z)(s)ds \\
&\quad + \frac{1}{2}\int_a^b \sum_{i=1}^m \{(Q_{1i}z)(s)(T_{2i}u)(s) + (Q_{2i}z)(s)(T_{1i}u)(s)\}ds \\
&\quad + \frac{1}{2}\int_a^b \sum_{i=1}^m (T_{1i}u)(s)(T_{2i}u)(s)ds \\
&\quad + \frac{1}{2}\int_a^b \{(Q_0z)(s) + (T_0u)(s)\}ds + \frac{1}{2}\int_a^b \omega(s)ds.
\end{aligned}
\tag{5.8}
$$

Using the equality

$$
\int_a^b (Az)(s)(Bz)(s)ds = \int_a^b (A^*Bz)(s)z(s)ds \tag{5.9}
$$

and denoting

$$
\langle \varphi, \psi \rangle = \int_a^b \varphi(s)\psi(s)ds, \tag{5.10}
$$

we may write

$$
\mathcal{I}_1(z) = \frac{1}{2}\langle Hz, z\rangle - \langle \theta, z\rangle + g, \tag{5.11}
$$

where

$$
\begin{aligned}
H &= \frac{1}{2}\sum_{i=1}^m (Q_{1i}^*Q_{2i} + Q_{2i}^*Q_{1i}), \\
\theta &= -\frac{1}{2}\sum_{i=1}^m (Q_{1i}^*T_{2i} + Q_{2i}^*T_{1i})u - \frac{1}{2}Q_0^*(1), \\
g &= \frac{1}{2}\int_a^b \left\{\sum_{i=1}^m (T_{1i}u)(s)(T_{2i}u)(s) + (T_0u)(s) + \omega(s)\right\}ds.
\end{aligned}
\tag{5.12}
$$

Thus $H : \mathbf{L}_2 \to \mathbf{L}_2$ is a selfadjoint operator, $\theta \in \mathbf{L}_2$, $g = \text{const}$.

Denote

$$
\mathbf{D}_\alpha = \{x \in \mathbf{D} : l^i x = \alpha^i,\ i = 1, \dots, n\}. \tag{5.13}
$$

To state and prove the central assertion on the minimum of the functional (5.2), we will use the following definitions.

A point $x_0 \in \mathbf{D}_\alpha$ ($z_0 \in \mathbf{L}_2$) is called the point of local minimum of functional $\mathcal{I}$ ($\mathcal{I}_1$) if there exists an $\varepsilon > 0$ such that $\mathcal{I}(x) \geq \mathcal{I}(x_0)$ ($\mathcal{I}_1(z) \geq \mathcal{I}_1(z_0)$) for all $x \in \mathbf{D}_\alpha$ ($z \in \mathbf{L}_2$) that satisfy $\|x - x_0\|_{\mathbf{D}} < \varepsilon$ ($\|z - z_0\|_{\mathbf{L}_2} < \varepsilon$). If $\mathcal{I}(x) \geq \mathcal{I}(x_0)$ ($\mathcal{I}_1(z) \geq \mathcal{I}_1(z_0)$) holds for all $x \in \mathbf{D}_\alpha$ ($z \in \mathbf{L}_2$), x_0 (z_0) is called the point of global minimum. The value $\mathcal{I}(x_0)$ ($\mathcal{I}_1(z_0)$) is called local (global) minimum of the functional.

Following the adopted terminology, we will call the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ positive definite one if $\langle Hz, z\rangle \geq 0$ for all $z \in \mathbf{L}_2$. The positive definite operator H is called strictly positive definite if $\langle Hz, z\rangle = 0$ only for $z = 0$.

Denote

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{m}\left(Q_{1i}^{*}T_{2i} + Q_{2i}^{*}T_{1i}\right), \qquad \theta_0 = -\frac{1}{2}Q_0^{*}(\mathbf{1}). \tag{5.14}$$

Theorem 5.1. *Any local minimum of the functional* (5.2) *is the global one.*

A point $x_0 \in \mathbf{D}_\alpha$ is the point of minimum of the functional (5.2) *if and only if*

(a) *x_0 is a solution of the boundary value problem*

$$\mathcal{L}x = \theta_0, \qquad l^i x = \alpha^i, \quad i = 1, \dots, n, \tag{5.15}$$

(b) *the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ defined by* (5.12) *is positive definite.*

Remark 5.2. The equation $\mathcal{L}x = \theta_0$ is naturally called "Euler equation" and the boundary conditions $l^i x = \alpha^i$ correspond to "natural boundary conditions" in the classical calculus of variations.

5.2.2. Proof of Theorem 5.1

As a preliminary we will proof the following auxiliary statements.

Lemma 5.3. *Any local minimum of the functional $\mathcal{I}_1$ on the space $\mathbf{L}_2$ is the global one.*

An element $z_0 \in \mathbf{L}_2$ is the point of minimum of $\mathcal{I}_1$ if and only if the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ defined by (5.12) *is positive definite and z_0 is a solution to the equation $Hz = \theta$.*

Proof. Let z_0 be a point of a local minimum. It means that there exists an $\varepsilon > 0$ such that $\mathcal{I}_1(z) - \mathcal{I}_1(z_0) \geq 0$ for $\|z - z_0\|_{\mathbf{L}_2} < \varepsilon$. Let us fix $\xi \in \mathbf{L}_2$ and let $\gamma_0 > 0$ be a number such that $\|\gamma_0\xi\|_{\mathbf{L}_2} < \varepsilon$. It follows from (5.11) that

$$\mathcal{I}_1(z_0 + \gamma\xi) - \mathcal{I}_1(z_0) = \frac{\gamma^2}{2}\langle H\xi, \xi\rangle + \gamma\langle Hz_0 - \theta, \xi\rangle. \tag{5.16}$$

Due to the condition, the quadratic binomial $(\gamma^2/2)\langle H\xi, \xi\rangle + \gamma\langle Hz_0 - \theta, \xi\rangle$ has no negative value if $\gamma \in (-\gamma_0, \gamma_0)$. It means that this binomial has no negative value for any y. Consequently, z_0 is a point of global minimum. Besides, due to the arbitrary choice of ξ, we obtain from the equality $\langle Hz_0 - \theta, \xi \rangle = 0$ that $Hz_0 - \theta = 0$. Then $\langle H\xi, \xi \rangle \geq 0$ for each $\xi \in \mathbf{L}_2$.

The converse assertion follows from (5.16). □

Lemma 5.4. *If x_0 is the point of a local minimum of the functional* (5.2) *on the set $\mathbf{D}_\alpha$ and $x_0 = \Lambda z_0 + u$, then the point of minimum of the functional $\mathcal{I}_1$ is z_0.*

Proof. Let $\varepsilon > 0$ be such that $\mathcal{I}(x) - \mathcal{I}(x_0) \geq 0$, as soon as $\|x - x_0\|_{\mathbf{D}} < \varepsilon$. Any $x \in \mathbf{D}_\alpha$ has the representation $x = \Lambda z + u$. Since $\|x - x_0\|_{\mathbf{D}} \leq \|\Lambda\| \|z - z_0\|_{\mathbf{L}_2}$, we have

$$\mathcal{I}_1(z) - \mathcal{I}_1(z_0) = \mathcal{I}(x) - \mathcal{I}(x_0) \geq 0 \quad \text{for } \|z - z_0\|_{\mathbf{L}_2} \leq \frac{\varepsilon}{\|\Lambda\|}. \tag{5.17}$$

Hence z_0 is a point of minimum of the functional $\mathcal{I}_1$. □

From the equality $\mathcal{I}(x) - \mathcal{I}(x_0) = \mathcal{I}_1(z) - \mathcal{I}_1(z_0)$ for $x_0 = \Lambda z_0 + u$ and for $x = \Lambda z + u$, it follows at once that x_0 is the point of the global minimum of the functional $\mathcal{I}$ if and only if z_0 is the point of global minimum of the functional $\mathcal{I}_1$. From this and previous lemmas we obtain the following.

Lemma 5.5. *Any local minimum of the functional $\mathcal{I}$ is the global one.*

The functional $\mathcal{I}$ has a point of minimum x_0 on the set $\mathbf{D}_\alpha = \{x \in \mathbf{D} : l^i x = \alpha^i,\ i = 1, \ldots, N\}$ if and only if the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ defined by (5.12) *is positive definite and $Hz = \theta$ has a solution $z_0 \in \mathbf{L}_2$. In this case, $x_0 = \Lambda z_0 + u$.*

Corollary 5.6. *Let the operator H defined by* (5.12) *be of the form $H = I - K$. The functional* (5.2) *with restrictions* (5.3) *has the unique minimum in $\mathbf{D}$ and H is strictly positive definite if $\|K\| < 1$.*

Proof. The inequality $\|K\| < 1$ implies that the equation $Hz = \theta$ has the unique solution z_0. Besides, this inequality guarantees that H is positive definite since

$$\langle Hz, z \rangle = \langle z, z \rangle - \langle Kz, z \rangle \geq \|z\|^2_{\mathbf{L}_2} - \|K\| \|z\|^2_{\mathbf{L}_2} \geq 0. \tag{5.18}$$

By Lemma 5.5, $x_0 = \Lambda z_0 + u$ is the point of the unique minimum of the functional on the set $\mathbf{D}_\alpha$. □

Proof of Theorem 5.1. Let $x_0 \in \mathbf{D}_\alpha$ be the solution of the problem (5.15). There exists $z_0 \in \mathbf{L}_2$ such that $x_0 = \Lambda z_0 + u$. Moreover

$$Hz_0 = \mathcal{L}\Lambda z_0 = \mathcal{L}(x_0 - u) = \theta_0 + \theta - \theta_0 = \theta. \tag{5.19}$$

Consequently, z_0 is the solution to the equation $Hz = \theta$. By virtue of Lemma 5.5, x_0 is the point of minimum.

Conversely, if x_0 is the point of minimum of the functional $\mathcal{I}$ and $x_0 = \Lambda z + u$, z_0 is the point of minimum of the functional $\mathcal{I}_1$. By virtue of Lemma 5.3 the operator H is positive definite and $Hz_0 - \theta = 0$.

Let us show that x_0 is the solution of the problem (5.15). Indeed, $lx_0 = \alpha$,

$$\mathcal{L}x_0 = \mathcal{L}(\Lambda z_0 + u) = Hz_0 - \theta + \theta_0 = \theta_0. \tag{5.20}$$

□

Remark 5.7. In some instances the equation $Hz = \theta$ is more convenient for the study than the boundary value problem (5.15). In such cases there is a good reason to use Lemma 5.5 instead of Theorem 5.1.

5.2.3. A simple example

Consider the functional

$$\mathcal{I}(x) = \frac{1}{2}\int_0^1 \{\dot{x}^2(s) - q(s)\dot{x}(s) - p(s)x(s)\}ds \tag{5.21}$$

with conditions $x(0) = \alpha^1$, $x(1) = \alpha^2$.

If q is absolutely continuous, the classical methods from elementary textbooks is applicable. The classical Euler equation in this case has the form

$$\ddot{x}(t) = \frac{1}{2}[\dot{q}(t) - p(t)] \tag{5.22}$$

and, therefore, the point of minimum is defined by

$$x_0(t) = \frac{1}{2}\int_0^1 W(t,s)[\dot{q}(s) - p(s)]ds + \alpha^1(1-t) + \alpha^2 t, \tag{5.23}$$

where

$$W(t,s) = \begin{cases} -s(1-t) & \text{if } 0 \le s \le t \le 1, \\ -t(1-s) & \text{if } 0 \le t < s \le 1, \end{cases} \tag{5.24}$$

is the Green function of the problem $\ddot{x} = z$, $x(0) = 0$, $x(1) = 0$. Thus

$$\begin{aligned} x_0(t) = {} & \frac{1}{2}\left[\int_0^t q(s)ds - t\int_0^1 q(s)ds - (t-1)\int_0^t sp(s)ds - t\int_t^1 (s-1)p(s)ds\right] \\ & + \alpha^1(1-t) + \alpha^2 t. \end{aligned} \tag{5.25}$$

Note that $\mathcal{I}(x_0) = -p^2/96$ if $\alpha^1 = \alpha^2 = 0$, $p = \text{const}$, $q = \text{const}$.

Next consider the same problem using the scheme above. Let $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^2$ and let $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbb{R}^n \to \mathbf{D}$ be the isomorphism.

Since $T_{11}x = T_{21}x \stackrel{\text{def}}{=} Tx = \dot{x}$, $T_0 x = -q\dot{x} - px$, we have $Q_{11} = Q_{12} = T\Lambda \stackrel{\text{def}}{=} Q$, $H = Q^*Q$. In any case of $\mathbf{D}$ (for every Λ) the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ is positive definite since

$$\langle Hz, z\rangle = \langle Q^*Qz, z\rangle = \langle Qz, Qz\rangle. \tag{5.26}$$

Let $\mathbf{D} = \mathbf{W}_2^2$ be the space of the functions $x : [a,b] \to \mathbb{R}$ with absolutely continuous derivative $\dot{x}$ and $\ddot{x} \in \mathbf{L}_2$. Define the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbb{R}^2 \to \mathbf{W}_2^2$ by

$$(\Lambda z)(t) = \int_0^1 W(t,s)z(s)ds, \qquad (Y\beta)(t) = \beta^1(1-t) + \beta^2 t, \quad \beta = \operatorname{col}\{\beta^1, \beta^2\} \tag{5.27}$$

with $W(t,s)$ defined by (5.24). Thus

$$(\Lambda z)(t) = (t-1)\int_0^t sz(s)ds - t\int_t^1 (1-s)z(s)ds. \tag{5.28}$$

After direct calculations, we have

$$\begin{aligned}
(Qz)(t) &= -\int_t^1 z(s)ds + \int_0^1 sz(s)ds, \qquad (Q^*z)(t) = -\int_0^t z(s)ds + t\int_0^1 z(s)ds,\\
(Q_0z)(t) &= q(t)\int_t^1 z(s)ds - q(t)\int_0^1 sz(s)ds + (1-t)p(t)\int_0^t sz(s)ds\\
&\quad + tp(t)\int_t^1 (1-s)z(s)ds,\\
(Q_0^*z)(t) &= \int_0^t q(s)z(s)ds - t\int_0^1 q(s)z(s)ds + t\int_t^1 (1-s)p(s)z(s)ds\\
&\quad + (1-t)\int_0^t sp(s)z(s)ds,\\
\theta_0(t) &= \frac{1}{2}\left\{-\int_0^t q(s)ds + t\int_0^1 q(s)ds + t\int_t^1 (s-1)p(s)ds + (t-1)\int_0^t sp(s)ds\right\}.
\end{aligned} \tag{5.29}$$

Next $\mathcal{L} = Q^*T$ and the equation $\mathcal{L}x = \theta_0$ takes the form

$$-x(t) + x(0) + tx(1) - tx(0) = \theta_0. \tag{5.30}$$

By Theorem 5.1, the unique point of minimum is again the function (5.25).

Let us notice that after double differentiation, the equation $\mathcal{L}x = \theta_0$ takes the form of the classical Euler equation (5.22).

By immediate differentiation we see that $x_0 \in \mathbf{W}_2^2$ if and only if $p, \dot{q} \in \mathbf{L}_2$. Therefore, the functional (5.21) has no minimum in $\mathbf{W}_2^2$ without this condition.

If we restrict ourselves to the requirement $p, q \in \mathbf{L}_2$, it is natural to look for the minimum in a space being wider than $\mathbf{W}_2^2$.

Consider the problem in the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^2$ of the functions $x : [0,1] \to \mathbb{R}^1$ which are absolutely continuous on $[0,c)$ and $[c,1]$ and such that $\dot{x} \in \mathbf{L}^2$. The isomorphism between $\mathbf{D}$ and $\mathbf{L}_2 \times \mathbb{R}^2$ may be constructed on the base of the impulse boundary value problem

$$\dot{x}(t) = z(t), \qquad x(0) = \beta^1, \qquad x(1) = \beta^2 \tag{5.31}$$

in the space $\mathbf{D}$. The solution of this problem has the form

$$\begin{aligned} x(t) &= (\Lambda z)(t) + (Y\beta)(t) \\ &\overset{\text{def}}{=} \int_0^t z(s)ds - \chi_{[c,1]}(t)\int_0^1 z(s)ds + \beta^1\chi_{[0,c)}(t) + \beta^2\chi_{[c,1]}(t). \end{aligned} \tag{5.32}$$

Then $Qz = Q^*z = z$, $Hz = z$,

$$(Q_0^* z)(t) = \frac{1}{2}\left\{ -q(t)z(t) + \int_c^t p(s)z(s)ds\right\}. \tag{5.33}$$

Let $\alpha^1 = \alpha^2 = 0$. Then

$$\theta(t) = \frac{1}{2}\left\{q(t) - \int_c^t p(s)ds\right\}. \tag{5.34}$$

The solution to $Hz = \theta$ has the form $z_0 = \theta$ and the functional $\mathcal{I}$ has its minimum at the point

$$\begin{aligned} x_0(t) &= (\Lambda\theta)(t) \\ &= \frac{1}{2}\left\{ \int_0^t q(s)ds + \int_0^t sp(s)ds - t\int_c^t p(s)ds \right. \\ &\qquad \left. - \chi_{[c,1]}(t)\left[\int_0^1 q(s)ds + \int_0^1 sp(s)ds - \int_c^1 p(s)ds\right]\right\}. \end{aligned} \tag{5.35}$$

If p and q are constants, then

$$\mathcal{I}(x_0) = -\frac{1}{8}\left[q^2 + pq(2c-1) + p^2\left(c^2 - c + \frac{1}{3}\right)\right]. \tag{5.36}$$

Thus the minimum depends on the position of c, the point of discontinuity. If $q = 0$, $\mathcal{I}(x_0) = -(1/8)p^2(c^2 - c + 1/3)$, $\mathcal{I}(x_0) = -p^2/96$ for $c = 1/2$. If $c \to 0$ or $c \to 1$, then $\mathcal{I}(x_0) \to -p^2/24$.

5.3. The tests of the existence of the minimum of the functional

5.3.1. Some properties of the selfadjoint operators in $\mathbf{L}_2$

It is known that the selfadjoint $H : \mathbf{L}_2 \to \mathbf{L}_2$ is positive definite if and only if the spectrum of H, $\sigma(H)$, does not contain negative numbers: $\sigma(H) \subset [0, +\infty)$. If the operator H is strictly positive definite and moreover is Fredholm, then $\sigma(H) \subset (0, +\infty)$. Indeed, in this event the equation $Hz = 0$ has only the trivial solution and, consequently, there exists the bounded inverse H^{-1}. Thus the number 0 is a regular value to H.

For the selfadjoint H arising in studying the functional (5.2), the situation $H = H_0 - H_1$ is typical where H_0 and H_1 are bounded, selfadjoint, positive definite, there exists the bounded inverse H_0^{-1} and H_1 is compact. It should be noticed that in this case H is Fredholm as the sum of an invertible operator and a compact one. Let us dwell on such a situation.

It is known that there exists the only square root $\sqrt{H_0}$ (such that $(\sqrt{H_0})^2 = H_0$) for the positive definite H_0 and, besides, $\sqrt{H_0}$ permutes with H_0 and with any other operator that is permutable with H_0 (see [108]).

Lemma 5.8. *Let H_0 be positive definite and there exists the bounded inverse H_0^{-1}. Then there exists the bounded inverse $(\sqrt{H_0})^{-1}$ and, besides, $(\sqrt{H_0})^{-1} = \sqrt{H_0^{-1}}$.*

Proof. H_0^{-1} is positive definite. Consequently, there exists $\sqrt{H_0^{-1}}$ and it permutes with H_0^{-1}. Since H_0^{-1} permutes with H_0 and $\sqrt{H_0}$, the operators $\sqrt{H_0^{-1}}$ and $\sqrt{H_0}$ permute with each other. The product of positive definite operators is positive definite. Hence it follows, by the uniqueness of the square root, that

$$\sqrt{H_0}\sqrt{H_0^{-1}} = \sqrt{H_0^{-1}}\sqrt{H_0} = \sqrt{\left(\sqrt{H_0^{-1}}\sqrt{H_0}\right)^2} = \sqrt{H_0 H_0^{-1}} = I. \tag{5.37}$$

□

By Lemma 5.8, for any $z \in \mathbf{L}_2$, there exists a unique $y \in \mathbf{L}_2$ such that $z = \sqrt{H_0^{-1}}y$. Therefore,

$$\begin{aligned} \langle Hz, z\rangle &= \langle H_0 z, z\rangle - \langle H_1 z, z\rangle \\ &= \left\langle H_0\sqrt{H_0^{-1}}y, \sqrt{H_0^{-1}}y\right\rangle - \left\langle H_1\sqrt{H_0^{-1}}y, \sqrt{H_0^{-1}}y\right\rangle \\ &= \left\langle \sqrt{H_0}y, \sqrt{H_0^{-1}}y\right\rangle - \left\langle \sqrt{H_0^{-1}}H_1\sqrt{H_0^{-1}}y, y\right\rangle \\ &= \langle (I-K)y, y\rangle. \end{aligned} \tag{5.38}$$

The operator $K = \sqrt{H_0^{-1}}H_1\sqrt{H_0^{-1}}$ is selfadjoint and compact. Besides this operator is positive definite. Indeed, since H_1 is positive definite,

$$\langle Kz, z\rangle = \left\langle H_1\sqrt{H_0^{-1}}z, \sqrt{H_0^{-1}}z\right\rangle \geq 0. \tag{5.39}$$

Lemma 5.9. *The following assertions are equivalent.*

(a) *The operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ is strictly positive definite ($\langle Hz, z\rangle > 0$ for each $z \in \mathbf{L}_2$, $z \neq 0$).*
(b) $\sigma(H) \subset (0, +\infty)$.
(c) $\rho(K) < 1$.
(d) $\rho(H_1 H_0^{-1}) < 1$.
(e) $\rho(H_0^{-1} H_1) < 1$.

Proof. The implication (a)⇒(b) was proved above since H is Fredholm.

The implication (b)⇒(a) follows from the fact that $m = \inf_{\|z\|_{\mathbf{L}_2}=1} \langle Hz, z\rangle$ is the point of the spectrum of H. Therefore $m > 0$ and

$$\langle Hz, z\rangle = \|z\|^2_{\mathbf{L}_2} \left\langle H\left[\frac{z}{\|z\|_{\mathbf{L}_2}}\right], \frac{z}{\|z\|_{\mathbf{L}_2}}\right\rangle \geq \|z\|^2_{\mathbf{L}_2} m > 0 \tag{5.40}$$

for $z \neq 0$.

The implication (a)⇒(c). By virtue of (a) and the representation

$$I - K = \sqrt{H_0^{-1}} H \sqrt{H_0^{-1}}, \tag{5.41}$$

the operator $I - K$ is strictly positive. Since this operator is Fredholm, $\sigma(I - K) \subset (0, +\infty)$. This with the fact that K is positive definite implies $\sigma(K) \subset [0, 1)$. Therefore $\rho(K) < 1$.

The implication (c)⇒(a). From (c) it follows, like in the proof of (b)⇒(a), that $I - K$ is strictly positive definite. From the equality

$$\langle Hz, z\rangle = \left\langle (I - K)\sqrt{H_0^{-1}} z, \sqrt{H_0^{-1}} z\right\rangle, \tag{5.42}$$

it follows that H is strictly positive definite.

The implication (c)⇒(d)⇒(e)⇒(c). The operators K, $H_1 H_0^{-1}$, and $H_0^{-1} H_1$ are compact. Between the sets of solutions v, y, and z of the equations

$$\lambda v = Kv, \qquad \lambda y = H_1 H_0^{-1} y, \qquad \lambda z = H_0^{-1} H_1 z, \tag{5.43}$$

there exist for each λ the one-to-one mappings defined by

$$\begin{aligned} v &= \sqrt{H_0^{-1}} y, & y &= \sqrt{H_0} v; \\ z &= \sqrt{H_0^{-1}} v, & v &= \sqrt{H_0} z; \\ y &= H_0 z, & z &= H_0^{-1} y. \end{aligned} \tag{5.44}$$

Therefore

$$\rho(K) = \rho(H_1 H_0^{-1}) = \rho(H_0^{-1} H_1). \tag{5.45}$$

□

5.3.2. De la Vallee-Poussin-like theorem

Theorem D.1 which is called as de la Vallee-Poussin like one, was of certain importance in Chapters 2 and 4. This theorem on equivalence of a set of assertions, connected with linear equations, contains an assertion on the existence of the unique minimum of a functional. The generalization of the mentioned results may be formulated on the base of Theorem D.2.

Let $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ be a space of functions $x : [a,b] \to \mathbb{R}^1$, which is continuously embedded into the space $\mathbf{C}$ of continuous functions. The operator $\Lambda : \mathbf{L}_2 \to \mathbf{D}$ that defines the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbb{R}^n \to \mathbf{D}$ is supposed to be isotonic (antitonic). Consider the functional

$$F = \mathcal{I} - F_1 \tag{5.46}$$

in the space $\mathbf{D}$. Here $\mathcal{I}$ is defined by (5.2),

$$F_1(x) = \int_a^b \sum_{i=1}^{\mu} (T_i x)^2(s)\,ds, \tag{5.47}$$

$T_i : \mathbf{C} \to \mathbf{L}_2$, $i = 1,\ldots,\mu$, are linear bounded operators such that the products $T_i\Lambda : \mathbf{L}_2 \to \mathbf{L}_2$ are compact.

Denote

$$\mathcal{L} = \mathcal{L}_0 - T, \tag{5.48}$$

where

$$\mathcal{L}_0 = \frac{1}{2}\sum_{i=1}^{m}\Big\{(T_{1i}\Lambda)^* T_{2i} + (T_{2i}\Lambda)^* T_{1i}\Big\}, \quad T = \sum_{i=1}^{\mu}(T_i\Lambda)^* T_i. \tag{5.49}$$

Thus, the operators $\mathcal{L}_0 : \mathbf{D} \to \mathbf{L}_2$ and $\mathcal{L} : \mathbf{D} \to \mathbf{L}_2$ define the Euler equations for the functionals $\mathcal{I}$ and F. In the case that, among the functionals $l^1,\ldots,l^n$, there are functionals such that $l^i x \overset{\text{def}}{=} x(v_i)$, $v_i \in [a,b]$, we denote by $\{v\}$ the set of the points v_i. Otherwise the symbol $\{v\}$ denotes the empty set.

We will assume that the functional $\mathcal{I}$ with restrictions (5.3) has the minimum, the operator $T : \mathbf{C} \to \mathbf{L}_2$ is isotonic (antitonic), the boundary value problem

$$\mathcal{L}_0 x = f, \qquad l^i x = 0, \quad i = 1,\ldots,n, \tag{5.50}$$

is uniquely solvable, and the Green operator W of the problem is isotonic (antitonic). We will assume also that the homogeneous equation $\mathcal{L}_0 x = 0$ has a positive solution u_0 $(u_0(t) > 0,\ t \in [a,b] \setminus \{v\})$.

Define $A : \mathbf{C} \to \mathbf{C}$ by $A = WT$. The operator A is isotonic and compact by the assumptions above.

Theorem 5.10. *The following assertions are equivalent.*

(a) *The functional F with the conditions* (5.3) *has a unique minimum in* $\mathbf{D}$.

(b) *There exists $v \in \mathbf{D}$ such that*

$$v(t) > 0, \quad r(t) \stackrel{\text{def}}{=} (W\varphi)(t) + g(t) > 0, \quad t \in [a,b] \setminus \{v\}, \tag{5.51}$$

where $\varphi = \mathcal{L}v$, g is the solution of the semihomogeneous problem

$$\mathcal{L}_0 x = 0, \qquad l^i x = l^i v, \quad i = 1,\dots,n. \tag{5.52}$$

(c) $\rho(A) < 1$.

(d) *The boundary value problem*

$$\mathcal{L}x = f, \qquad l^i x = 0, \quad i = 1,\dots,n, \tag{5.53}$$

is uniquely solvable and the Green operator of the problem is isotonic (antitonic).

(e) *The homogeneous equation $\mathcal{L}x = 0$ has a positive solution u ($u(t) > 0$, $t \in [a,b] \setminus \{v\}$) such that $l^i u = l^i u_0$, $i = 1,\dots,n$.*

By Theorem D.2, we need only to prove the implications (a)⇒(c) and (c)⇒(a). As shown previously we will notice the following.

There exists the bounded inverse H_0^{-1} to $H_0 \stackrel{\text{def}}{=} \mathcal{L}_0\Lambda$ by virtue of the unique solvability of the problem (5.50). It follows from the fact that there exists the one-to-one mapping $x = \Lambda z$, $z = \delta x$ between the solutions $x \in \mathbf{D}$ of the problem and the solutions $z \in \mathbf{L}_2$ of the equation $H_0 z = f$. Besides the operator H_0 is positive definite by virtue of Theorem 5.1 and the existence of a minimum of $\mathcal{I}$.

The operator $H_1 \stackrel{\text{def}}{=} T\Lambda$ is compact. Consequently, $H = H_0 - H_1$ is Fredholm. The homogeneous Euler problem

$$\mathcal{L}x = 0, \qquad l^i x = 0, \quad i = 1,\dots,n, \tag{5.54}$$

is equivalent to

$$x = Ax \tag{5.55}$$

in the space of continuous functions, because any continuous solution of the latter equation belongs to the space $\mathbf{D}$. Between the set of solutions x of the equation $x = Ax$ (of the homogeneous Euler problem) and the set of the solutions z of the equation

$$\mathcal{L}\Lambda z \equiv (H_0 - H_1)z = 0, \tag{5.56}$$

which can be rewritten in the form

$$z = H_0^{-1} H_1 z, \tag{5.57}$$

there exists the one-to-one mapping $x = \Lambda z$, $z = \delta x$. The same mapping holds for the solutions of the equations

$$\lambda x = Ax, \qquad \lambda z = H_0^{-1} H_1 z \tag{5.58}$$

for each λ. Therefore $\rho(A) = \rho(H_0^{-1}H_1)$ since $A : \mathbf{C} \to \mathbf{C}$ and $H_0^{-1}H_1 : \mathbf{L}_2 \to \mathbf{L}_2$ are both compact.

Let us prove now the implication (a)⇒(c). From (a) and Theorem 5.1 it follows that H is positive definite and the uniqueness of the trivial solution of the equation $Hz = 0$ takes place. Therefore the number 0 is not a point of the spectrum of H. Consequently, $\sigma(H) \subset (0, +\infty)$. From this by virtue of Lemma 5.12 (the implication (b)⇒(e)), we have (c).

Implication (c)⇒(a). Since $\rho(H_0^{-1}H_1) = \rho(A)$, the operator H is strictly positive definite by virtue of Lemma 5.9. Besides H is Fredholm. Therefore there exists the bounded inverse H^{-1}. Thus we have (a).

5.3.3. Examples

Example 5.11. The paper [231] was devoted to the problem

$$\begin{gathered}\frac{1}{2}\int_0^\omega \left\{\frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p(s)x^2(s)\right\} ds \longrightarrow \min, \\ x(0) - x(\omega) = \alpha.\end{gathered} \tag{5.59}$$

The results of the paper were obtained by means of the methods of classical calculus of variations, which met some difficulties due to the term $x^2(\omega)/\omega$. To illustrate the new approach to the minimization of functionals, we will consider the functional of a more general form

$$\begin{gathered}\frac{1}{2}\int_0^\omega \left\{\frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p(s)x[h(s)]x[g(s)] + \mu(s)\dot{x}(s) + \nu(s)x(s)\right\} ds, \\ x(\xi) = \varphi(\xi) \quad \text{if } \xi \notin [0,\omega],\end{gathered} \tag{5.60}$$

with periodic condition $x(0) - x(\omega) = \alpha$. Assume that $p, \mu, \nu \in \mathbf{L}_2$, the functions h and g are measurable, and the initial function $\varphi : (-\infty, +\infty) \setminus [0, \omega] \to \mathbb{R}^1$ is piecewise continuous.

Using the notations S_h and φ^h introduced in Subsection 2.2.1, rewrite the functional (5.60) in the form

$$\begin{aligned}&\frac{1}{2}\int_0^\omega \left\{\frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p(s)(S_h x)(s)(S_g x)(s)\right\} ds \\ &\quad - \frac{1}{2}\int_0^\omega p(s)\{\varphi^g(s)(S_h x)(s)s + \varphi^h(s)(S_g x)(s) + \varphi^h(s)\varphi^g(s) + \mu(s)\dot{x}(s) + \nu(s)x(s)\} ds.\end{aligned} \tag{5.61}$$

It is natural to look for the point of minimum of such a functional in the space $\mathbf{W}_2^1$ of absolutely continuous functions $x : [0,\omega] \to \mathbb{R}^1$ with $\dot{x} \in \mathbf{L}_2$. We will construct the isomorphism $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbb{R}^1 \to \mathbf{W}_2^1$ on the base of the general solution $x = \Lambda z + Y\beta$ of the model boundary value problem

$$(\delta x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + \frac{x(\omega)}{\omega} = z(t), \qquad rx \stackrel{\text{def}}{=} x(0) - x(\omega) = \beta. \tag{5.62}$$

One can see directly

$$(Y\beta)(t) = \left(2 - \frac{t}{\omega}\right)\beta, \qquad (\Lambda z)(t) = \int_0^\omega \Lambda(t,s)z(s)ds, \tag{5.63}$$

where

$$\Lambda(t,s) = \begin{cases} 2 - \dfrac{t}{\omega} & \text{if } 0 \le s \le t \le \omega, \\ 1 - \dfrac{t}{\omega} & \text{if } 0 \le t < s \le \omega. \end{cases} \tag{5.64}$$

Assume that $\Lambda(t,s) = 0$ outside the square $[0,\omega] \times [0,\omega]$.

First let us dwell on the problem about the minimum of the truncated functional

$$\begin{gathered} \frac{1}{2}\int_0^\omega \left\{\dot{x}^2(s) - p(s)(S_h x)(s)(S_g x)(s) + \frac{x^2(\omega)}{\omega}\right\}ds \longrightarrow \min, \\ x(0) - x(\omega) = \alpha. \end{gathered} \tag{5.65}$$

We have

$$\begin{gathered} (T_{11}x)(t) = (T_{21}x)(t) = \dot{x}(t), \qquad (T_{12}x)(t) = -p(t)(S_h x)(t), \\ (T_{22}x)(t) = (S_g x)(t), \qquad T_{13}x = T_{23}x = \frac{1}{\sqrt{\omega}}x(\omega), \\ Q_{11}z = Q_{11}^* z = Q_{21}z = Q_{21}^* z = z - \frac{1}{\omega}\int_0^\omega z(s)ds, \\ (Q_{12}z)(t) = -p(t)(S_h\Lambda z)(t) = -p(t)\int_0^\omega \Lambda[h(t),s]z(s)ds, \\ (Q_{12}^* z)(t) = -\int_0^\omega p(s)\Lambda[h(s),t]z(s)ds, \end{gathered}$$

$$(Q_{22}z)(t) = (S_g\Lambda z)(t) = \int_0^\omega \Lambda[g(t),s]z(s)ds,$$

$$(Q_{22}^*z)(t) = \int_0^\omega \Lambda[g(s),t]z(s)ds,$$

$$Q_{13}z = Q_{23}z = Q_{13}^*z = Q_{23}^*z = \frac{1}{\sqrt{\omega}}\int_0^\omega z(s)ds,$$

$$\theta(t) = \theta_0(t) \equiv 0,$$

$$(\mathcal{L}x)(t) = \dot{x}(t) + x(\omega) - \frac{1}{2}\int_0^\omega p(s)\{\Lambda[g(s),t](S_hx)(s) + \Lambda[h(s),t](S_gx)(s)\}ds. \tag{5.66}$$

Let us represent $\mathcal{L}$ in the form

$$\mathcal{L}x = \delta x - Px, \tag{5.67}$$

where

$$(Px)(t) = \frac{1}{2}\int_0^\omega p(s)\{\Lambda[g(s),t](S_hx)(s) + \Lambda[h(s),t](S_gx)(s)\}ds + x(\omega)\left(\frac{1}{\omega} - 1\right). \tag{5.68}$$

The operator $P : \mathbf{W}_2^1 \to \mathbf{L}_2$ is compact. It follows from the compactness of the integral operator with the kernel $p(s)\Lambda[g(s),t]$ as the operator acting in the space $\mathbf{L}_2$ and the boundedness of S_h as the operator acting from $\mathbf{W}_2^1$ into $\mathbf{L}_2$. Let us represent the operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ in the form

$$Hz = \mathcal{L}\Lambda z = z - Kz, \tag{5.69}$$

where $K = P\Lambda$.

The operator $H : \mathbf{L}_2 \to \mathbf{L}_2$ is a Fredholm one because of the compactness of the operator $K : \mathbf{L}_2 \to \mathbf{L}_2$. Therefore, the existence-uniqueness of the point of minimum of the functional $\mathcal{I}$ does not depend on its linear summands and the number α. The summands and α define the right-hand side of the equation $Hz = \theta$ and does not influence the construction of H.

Thus by studying the problem on existence and uniqueness of the minimum of the functional (5.60) with restrictions $x(0) - x(\omega) = \alpha$ it is sufficient to consider the problem (5.65) for the truncated functional

$$\frac{1}{2}\int_0^\omega \left[\frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p(s)(S_hx)(s)(S_gx)(s)\right]ds. \tag{5.70}$$

By Corollary 5.6, the condition $\|K\| < 1$ is sufficient for the existence and the uniqueness of the minimum of the functional (5.70) with the restrictions $x(0) - x(\omega) = \alpha$.

The boundary value problem (5.15) for the functional (5.70) has the form

$$\mathcal{L}x = 0, \qquad rx = \alpha, \tag{5.71}$$

where $\mathcal{L}$ is defined by (5.67). Denote

$$A = \Lambda P. \tag{5.72}$$

Then the homogeneous problem is equivalent to the equation $x = Ax$ in the space $\mathbf{W}_2^1$. Any continuous solution of the equation $x = Ax$ belongs to $\mathbf{W}_2^1$ by virtue of the property of Λ. At each λ there is the one-to-one mapping $z = \delta x$, $x = \Lambda z$ between the set of solutions $x \in \mathbf{C}$ of the equation $\lambda x = Ax$ and the set of solutions $z \in \mathbf{L}_2$ of the equation $\lambda z = Kz$. Thus the spectra of the compact operators $A : \mathbf{C} \to \mathbf{C}$ and $K : \mathbf{L}_2 \to \mathbf{L}_2$ coincide.

Let, as usual,

$$\sigma_r(t) = \begin{cases} 1 & \text{if } r(t) \in [0, \omega], \\ 0 & \text{if } r(t) \notin [0, \omega]. \end{cases} \tag{5.73}$$

Then

$$\|A\| \leq \int_0^\omega \Lambda(t,s)\left[\frac{1}{2}\int_0^\omega p(\tau)\{\Lambda[g(\tau),s]\sigma_h(\tau)+\Lambda[h(\tau),s]\sigma_g(\tau)\}d\tau + x(\omega)\left(\frac{1}{\omega}-1\right)\right]ds. \tag{5.74}$$

Since $\int_0^\omega \Lambda(t,s)ds = \omega$ and

$$\begin{aligned} &\int_0^\omega |p(\tau)|\{\Lambda[g(\tau),s]\sigma_h(\tau) + \Lambda[h(\tau),s]\sigma_g(\tau)\}d\tau \\ &\qquad \leq \int_0^\omega |p(\tau)|\sigma_h(\tau)\sigma_g(\tau)\left(4 - \frac{g(\tau)+h(\tau)}{\omega}\right)d\tau, \end{aligned} \tag{5.75}$$

the inequality $\|A\| < 1$ (and, consequently, the inequality $\|K\| < 1$) is guaranteed by the estimate

$$\int_0^\omega |p(s)|\sigma_h(s)\sigma_g(s)\left[4 - \frac{g(s)+h(s)}{\omega}\right]ds \leq 2. \tag{5.76}$$

Thus by virtue of Corollary 5.6 the condition (5.76) is sufficient for the existence of the unique minimum in the space $\mathbf{W}_2^1$ of the functional (5.70) with the representation $x(0) - x(\omega) = \alpha$.

In the case $h(t) = g(t)$, we may obtain a more subtle result.

Let $p = p^+ - p^-$, $p^+(t) \geq 0$, $p^-(t) \geq 0$ and let us rewrite the problem (5.65) in the form

$$\frac{1}{2}\int_0^\omega \left[\frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p^+(s)\left(S_h x\right)^2(s)\right]ds + \frac{1}{2}\int_0^\omega \left(T_2 x\right)^2(s)ds \longrightarrow \min, \quad x(0) - x(\omega) = \alpha, \tag{5.77}$$

where

$$T_2 x = \sqrt{p^-} S_h x. \tag{5.78}$$

The operator H defined for the problem (5.77) by (5.12) is the sum $H = H_1 + H_2$, where $H_2 = (T_2\Lambda)^* T_2\Lambda$ is compact and positive definite. The compactness of H_2 follows from the compactness of Λ, as an operator acting from $\mathbf{L}_2$ into $\mathbf{C}$ (see [229]), and from the boundedness of $T : \mathbf{C} \to \mathbf{L}_2$. The operator $H_1 = I - K_1$ is defined by (5.12) for the problem

$$\frac{1}{2}\int_0^\omega \left\{\frac{x^2(\omega)}{\omega} + \dot{x}^2(s) - p^+(s)\left(S_h x\right)^2(s)\right\}ds \longrightarrow \min, \quad x(0) - x(\omega) = \alpha. \tag{5.79}$$

Thus here $K_1 = P_1\Lambda$, and by virtue of (5.68),

$$(P_1 x)(t) = \int_0^\omega p^+(s)\sigma_h(s)\Lambda[h(s), t](S_h x)(s)ds + x(\omega)\left(\frac{1}{\omega} - 1\right). \tag{5.80}$$

Define $A_1 : \mathbf{C} \to \mathbf{C}$ by $A_1 = \Lambda P_1$. As it was shown above, the spectra of compact operators $K_1 : \mathbf{L}_2 \to \mathbf{L}_2$ and $A_1 : \mathbf{C} \to \mathbf{C}$ coincide. Since $\|K_1\| = \rho(K_1) = \rho(A_1)$, the condition $\|A_1\| < 1$ yields the estimate $\|K_1\| < 1$. Therefore by virtue of Corollary 5.6 the operator H_1 is strictly positive if $\|A_1\| < 1$. At that case the Fredholm $H = H_1 + H_2$ is also strictly positive definite and, consequently, is invertible. The estimate (5.76), as applied to the problem (5.79), guarantees the inequality $\|A_1\| < 1$ and has the form

$$\int_0^\omega p^+(s)\sigma_h(s)\left[2 - \frac{h(s)}{\omega}\right]ds \leq 1. \tag{5.81}$$

Thus, the latter inequality is sufficient for the existence of the minimum of the functional (5.70) with restriction $x(0) - x(\omega) = \alpha$ in the case $h(t) = g(t)$.

Theorem 5.10 is suitable to the problem (5.79). Indeed, rewrite this problem in the form

$$F(x) = \mathcal{I}(x) - F_1(x) \longrightarrow \min, \quad x(0) - x(\omega) = \alpha, \tag{5.82}$$

where

$$\mathcal{I}(x) = \frac{1}{2}\int_0^{\omega}\left[\frac{x^2(\omega)}{\omega} + \dot{x}^2(s)\right]ds,$$
$$F_1(x) = \frac{1}{2}\int_0^{\omega}(T_1x)^2(s)ds, \quad T_1x = \sqrt{p^+}S_hx. \tag{5.83}$$

First we consider the problem

$$\mathcal{I}(x) \longrightarrow \min,$$
$$x(0) - x(\omega) = \alpha. \tag{5.84}$$

By the above scheme (in the case $p(t) \equiv 0$) we obtain

$$\mathcal{L}_0x = \delta x - P_0x = \dot{x} + \frac{x(\omega)}{\omega} - \frac{x(\omega)}{\omega} + x(\omega) = \dot{x} + x(\omega),$$
$$H_0z = \mathcal{L}_0\Lambda z = \delta\Lambda z - P_0\Lambda z = z - \frac{1}{\omega}\int_0^{\omega} z(s)ds \stackrel{\text{def}}{=} z - K_0z, \tag{5.85}$$
$$A_0x = \Lambda P_0x = \int_0^{\omega}\Lambda(t,s)\left[\frac{x(\omega)}{\omega} - x(\omega)\right]ds = x(\omega) - \omega x(\omega).$$

As was shown above, the spectra of $K_0 : \mathbf{L}_2 \to \mathbf{L}_2$ and $A_0 : \mathbf{C} \to \mathbf{C}$ coincide. If $\omega < 1$, $\|K_0\| = \rho(A_0) < 1$ since $\|A_0\| = 1 - \omega < 1$. Therefore the operator H_0 is positive definite and has the inverse H_0^{-1} if $\omega < 1$. Consequently, the boundary value problem

$$\mathcal{L}_0x = f, \qquad x(0) - x(\omega) = 0 \tag{5.86}$$

is uniquely solvable for each $f \in \mathbf{L}_2$. The Green operator $W = \Lambda(I - K_0)^{-1}$ of the problem is isotonic. The equation $\mathcal{L}_0x = 0$ has the positive solution $u_0 = 1 + 1/\omega - t/\omega$. Thus all the conditions of the general Theorem 5.10 are fulfilled and, consequently, we can formulate the following.

Theorem 5.12. *Let $\omega < 1$. Then the following assertions are equivalent.*

(a) *The problem* (5.79) *has the unique solution in the space $\mathbf{W}_2^1$.*

(b) *There exists $v \in \mathbf{W}_2^1$ such that*

$$v(t) > 0, \quad (\mathcal{L}v)(t) \geq 0, \quad t \in [0,\omega], \tag{5.87}$$

and, besides, $v(0) \geq v(\omega)$, $v(0) - v(\omega) + \int_0^{\omega}(\mathcal{L}v)(s)ds > 0$.

(c) *The spectral radius of the operator $A_1 : \mathbf{C} \to \mathbf{C}$ is less than* 1.

(d) *The homogeneous equation $\mathcal{L}x = 0$ has a solution u such that $u(t) > 0$, $t \in [0,\omega]$, $u(0) > u(\omega)$.*

Here $A_1 = \Lambda P_1$,

$$(P_1x)(t) = \int_0^\omega p^+(s)\Lambda[h(s),t](S_hx)(s)ds + x(\omega)\left(\frac{1}{\omega} - 1\right),$$
$$\mathcal{L}x = \delta x - P_1x. \tag{5.88}$$

In the paper [231] it was shown that the problem (5.59) in the case of $p(t) \equiv 1$ has the unique minimum if

$$\omega < \arcsin\frac{4}{5}. \tag{5.89}$$

From the estimate (5.50) for $p(t) \equiv 1$, we obtain only $\omega \le 2/3$. The inequality (5.89) can be established putting

$$v(t) = \cos t + \sin t\frac{1 - \cos\omega}{\sin\omega} \tag{5.90}$$

in the assertion (b) of Theorem 5.12. Then under condition (5.89), we have

$$v(t) > 0, \quad (\mathcal{L}v)(t) = 1 - \sin\omega - \frac{(1 - \cos\omega)^2}{\sin\omega} > 0, \quad t \in [0,\omega]. \tag{5.91}$$

If $\omega = \arcsin(4/5)$, the function v is a solution to the homogeneous problem $\mathcal{L}x = 0$, $x(0) - x(\omega) = 0$. Thus, the estimate (5.89) is the best possible one.

Example 5.13. Kudryavtsev (see [123, 124]) considered the problem

$$\frac{1}{2}\int_0^1 \{[s(1 - s)\ddot{x}(s)]^2 - p(s)x^2(s)\}ds \longrightarrow \min,$$
$$x(0) = \alpha^1, \qquad x(1) = \alpha^2. \tag{5.92}$$

The author saw a difficulty of the problem in the fact that the Euler equation is singular. We will consider a more general problem

$$\frac{1}{2}\int_0^1 \{[s(1 - s)\ddot{x}(s)]^2 - p(s)(S_hx)(s)(S_gx)(s)\}ds \longrightarrow \min,$$
$$x(0) = \alpha^1, \qquad x(1) = \alpha^2 \tag{5.93}$$

with measurable h and g and $p \in \mathbf{L}_2$. For the space $\mathbf{D}$ we will choose the analog of the space $\mathbf{D}_\pi$ constructed above in Section 4.2. Replacing the space $\mathbf{L}$ by $\mathbf{L}_2$ we will denote the space by $\mathbf{D}_\pi^2$. Thus the space $\mathbf{D}_\pi^2$ consists of the functions $x : [0,1] \to \mathbb{R}^1$ with the following properties.

(1) x is absolutely continuous on $[0,1]$.

(2) The derivative $\dot{x}$ is absolutely continuous on each $[c,d] \subset (0,1)$.

(3) The product $t(1 - t)\ddot{x}(t)$ is square integrable.

The space $\mathbf{D}_\pi^2$ is defined by $\mathbf{D}_\pi^2 = \Lambda\mathbf{L}_2 \oplus Y\mathbb{R}^2$, where

$$(\Lambda z)(t) = \int_0^1 \Lambda(t,s)z(s)ds, \qquad (Y\beta)(t) = (1-t)\beta^1 + t\beta^2, \quad \beta = \operatorname{col}\{\beta^1,\beta^2\},$$
$$\Lambda(t,s) = \begin{cases} \dfrac{t-1}{1-s} & \text{if } 0 \le s \le t \le 1, \\ -\dfrac{t}{s} & \text{if } 0 \le t < s \le 1. \end{cases} \tag{5.94}$$

We will suppose that $\Lambda(t,s) = 0$ outside the square $[0,1]\times[0,1]$. It should be noticed that $\Lambda(t,s)$ is the Green function of the boundary value problem

$$t(1-t)\ddot{x}(t) = z(t), \qquad x(0) = \beta^1, \qquad x(1) = \beta^2 \tag{5.95}$$

in the space $\mathbf{D}_\pi^2$ and, besides, $\Lambda(t,s)s(1-s) = G_0(t,s)$, where $G_0(t,s)$ is the Green function of the problem

$$\ddot{x} = z, \qquad x(0) = x(1) = 0 \tag{5.96}$$

in the space $\mathbf{W}_2^2$ of the functions with square integrable the second derivative,

$$G_0(t,s) = \begin{cases} -s(1-t) & \text{if } 0 \le s \le t \le 1, \\ -t(1-s) & \text{if } 0 \le t < s \le 1. \end{cases} \tag{5.97}$$

Thus $\{\Lambda, Y\}^{-1} = [\delta, r]$, where

$$(\delta x)(t) = t(1-t)\ddot{x}(t), \qquad rx = \{x(0),x(1)\}. \tag{5.98}$$

According to the general scheme, we have

$$\begin{gathered} T_{11} = T_{21} = \delta, \qquad Q_{11} = Q_{21} = Q_{11}^* = Q_{21}^* = I, \\ (T_{12}x)(t) = -p(t)(S_h x)(t), \qquad (T_{22}x)(t) = (S_g x)(t), \\ (Q_{12}z)(t) = -p(t)\int_0^1 \Lambda[h(t),s]z(s)ds, \\ (Q_{22}z)(t) = \int_0^1 \Lambda[g(t),s]z(s)ds, \\ (Q_{12}^*z)(t) = -\int_0^1 p(s)\Lambda[h(s),t]z(s)ds, \\ (Q_{22}^*z)(t) = \int_0^1 \Lambda[g(s),t]z(s)ds, \qquad \theta_0(t) \equiv 0, \\ H = I + \frac{1}{2}[Q_{12}^*Q_{22} + Q_{22}^*Q_{12}] = I - K, \end{gathered} \tag{5.99}$$

where

$$(Kz)(t) = \int_0^1 K(t,s)z(s)ds,$$
$$K(t,s) = \frac{1}{2}\int_0^1 p(\tau)\{\Lambda[h(\tau),t]\Lambda[g(\tau),s] + \Lambda[g(\tau),t]\Lambda[h(\tau),s]\}d\tau, \tag{5.100}$$
$$\mathcal{L}x = \frac{1}{2}\sum_{i=1}^{2}(Q_{1i}^{*}T_{2i} + Q_{2i}^{*}T_{1i})x \overset{\text{def}}{=} \delta x - Px,$$

where

$$(Px)(t) = \frac{1}{2}\int_0^1 p(s)\{\Lambda[h(s),t](S_g x)(s) + \Lambda[g(s),t](S_h x)(s)\}ds. \tag{5.101}$$

Λ as the operator acting from $\mathbf{L}_2$ into $\mathbf{C}$ is compact (see [229]). The operators $T_{12} : \mathbf{C} \to \mathbf{L}_2$ and $T_{22} : \mathbf{C} \to \mathbf{L}_2$ are bounded. Therefore the operators $Q_{12} = T_{12}\Lambda : \mathbf{L}_2 \to \mathbf{L}_2$, $Q_{22} = T_{22}\Lambda : \mathbf{L}_2 \to \mathbf{L}_2$, and, consequently, $K : \mathbf{L}_2 \to \mathbf{L}_2$ are compact. The problem (5.15) has the form

$$\mathcal{L}x = 0, \qquad x(0) = \alpha^1, \qquad x(1) = \alpha^2. \tag{5.102}$$

This is equivalent to the equation

$$\Lambda\mathcal{L}x \overset{\text{def}}{=} x - Ax = u \tag{5.103}$$

in the space $\mathbf{C}$. Here $u(t) = (1-t)\alpha^1 + t\alpha^2$, $A = \Lambda P : \mathbf{C} \to \mathbf{C}$ is compact. Thus, the problem (5.102) is uniquely solvable if and only if $I - A$ has the inverse.

The equalities $z = \delta x$, $x = \Lambda z$ establish the one-to-one mapping between the set of solutions $x \in \mathbf{C}$ of the equation $\lambda x = Ax$ and the set of solutions $z \in \mathbf{L}_2$ of the equation $\lambda z = Kz$. Therefore, the spectra of the compact operators $A : \mathbf{C} \to \mathbf{C}$ and $K : \mathbf{L}_2 \to \mathbf{L}_2$ coincide.

The inequality $\|K\| < 1$ guarantees by virtue of Corollary 5.6 the existence of the unique point of minimum. We have: $\|K\| = \rho(K) = \rho(A) \le \|A\|_{\mathbf{C}\to\mathbf{C}}$. Since $|\Lambda(t,s)| \le 1$, $\rho(A) < 1$ if

$$\int_0^1 |p(s)|\{\sigma_h(s) + \sigma_g(s)\}ds \le 2. \tag{5.104}$$

This test of the existence of the unique minimum may be sharpened in the case that $h(t) = g(t)$.

Let $p = p^+ - p^-$, $p^+(t) \ge 0$, $p^-(t) \ge 0$, $h(t) = g(t)$. First we consider the problem

$$F(x) = \frac{1}{2}\int_0^1 \Big\{[s(1-s)\ddot{x}(s)]^2 - p^+(s)(S_h x)^2(s)\Big\}ds \longrightarrow \min, \qquad x(0) = \alpha^1, \qquad x(1) = \alpha^2, \tag{5.105}$$

and apply Theorem 5.10. Denote

$$\mathcal{I}(x) = \frac{1}{2}\int_0^1 \left[s(1-s)\ddot{x}(s)\right]^2 ds, \qquad T_1 x = \sqrt{\frac{p^+}{2}} S_h x. \tag{5.106}$$

Under such a notation

$$F(x) = \mathcal{I}(x) - \int_0^1 (T_1 x)^2(s)\,ds. \tag{5.107}$$

All the conditions of Theorem 5.10 are fulfilled in the event of the problem (5.105). Indeed, T_1 is isotonic. Let $\mathcal{L}_0 = \delta$, $W = \Lambda$. The equation $\mathcal{L}_0 x = 0$ has a positive solution.

In the case of (5.105) the operator $\mathcal{L}$ defined by (5.100) has the form

$$(\mathcal{L}x)(t) = t(1-t)\ddot{x}(t) - (P^+ x)(t), \tag{5.108}$$

where

$$(P^+ x)(t) = \int_0^1 p^+(s)\Lambda[h(s),t](S_h x)(s)\,ds. \tag{5.109}$$

Let us set $v(t) = t(1-t)$ in the assertion (b) of Theorem 5.10. Then

$$(\mathcal{L}v)(t) \le 0 \tag{5.110}$$

if

$$\int_0^1 p^+(s)\sigma_h(s)\,ds \le 2. \tag{5.111}$$

Indeed,

$$\Lambda(s,t)s(1-s) = G_0(s,t), \tag{5.112}$$

where $G_0(t,s)$ is defined by (5.97). The estimate

$$0 > G_0(s,t) > -t(1-t), \quad t,s \in (0,1),\ t \ne s, \tag{5.113}$$

holds. Therefore

$$\Lambda[h(s),t]h(s)[1-h(s)] \ge -t(1-t) \tag{5.114}$$

and, consequently,

$$(\mathcal{L}v)(t) \le -2t(1-t) + t(1-t)\int_0^1 p^+(s)\sigma_h(s)\,ds, \quad t \in [0,1]. \tag{5.115}$$

This inequality is strict on a set of positive measure. By Theorem 5.10 (the implication (b)⇒(a)), the problem (5.105) under the condition (5.111) has a unique solution in $\mathbf{D}_\pi$ and, besides (the implication (b)⇒(c)), $\rho(A^+) < 1$, where $A^+ : \mathbf{C} \to \mathbf{C}$ is defined by $A^+ = \Lambda P^+$.

Let us turn back to the problem (5.93) and assume that $h(t) = g(t)$. Rewrite the problem in the form

$$\begin{gathered} F(x) + \int_0^1 \left(T_2 x\right)^2(s)ds \longrightarrow \min, \\ x(0) = \alpha^1, \qquad x(1) = \alpha^2, \end{gathered} \tag{5.116}$$

where

$$T_2 x = \sqrt{\frac{p^-}{2}} S_h x. \tag{5.117}$$

Let H be the operator defined by (5.12) for the problem (5.116). Then $H = H_0 + H_2$, where

$$H_0 = I - \left(T_1\Lambda\right)^* T_1\Lambda \stackrel{\text{def}}{=} I - K^+ \tag{5.118}$$

is the operator defined by (5.99) for the problem (5.105). The operator $H_2 = (T_2\Lambda)^* T_2\Lambda$ is compact and positive definite. Under the assumption (5.111) we have $\rho(A^+) < 1$ by virtue of Theorem 5.10 (the implication (b)⇒(c) at $\nu(t) = t(1-t)$). Consequently, as above, $\|K^+\| < 1$. From this, by Corollary 5.6, the operator H_0 is strictly positive definite. Therefore the Fredholm operator $H = H_0 + H_2$ is also strictly positive definite and, consequently, invertible. Thus the condition (5.111) guarantees by virtue of Lemma 5.5 the existence of the unique solution of the problem (5.93) in the event $h(t) = g(t)$.

The approach to the problem of minimization of functionals on the base of the theory of abstract functional differential equations was developed by the Perm Seminar in 1987–1993. Pioneering results in such a direction were published in [75] and discussed in the survey [19].

The general assertions about the existence of a minimum of square functionals under linear boundary conditions in $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ were given in the surveys [20, 21] (see also [39]).

The case omitted in this chapter, when the number of linear boundary conditions of the minimization problem differs from n, was thoroughly studied in [89] (see also [39, 94]).

The assertions of Section 5.3.1 were proved in the unpublished lecture by Hargelia at the Perm Seminar in 1999.

6 Constructive study of linear problems (using computer algebra in the study of linear problems)

6.1. Introduction

In the theory of functional differential equations, the equations possessing the property that a solution set of the equation admits a finite-dimensional parameterization are of special interest. Such a parameterization provides a way to reduce many of the problems of functional differential equations to the problems of finite-dimensional analysis. The principal problem with the practical implementation of this idea is the lack of an exact and explicit description of the finite-dimensional object to analyze. The situation is more simple in case we are interested in rough properties of the original problem (say, the unique solvability of a Fredholm boundary value problem), which are preserved under small perturbations. In this case we can use an approximate description of a solution set if the approximation is reasonably accurate. The basis of the constructive study of linear problems we are concerned with in this chapter is the special technique of an approximate description of the solution set to the linear functional differential equation with a guaranteed error bound. This technique is used in parallel with the special theorems, the conditions of which can be verified in the course of the reliable computing experiment due to the modern mathematical packages (Maple, Mathematica, e.g.). Notice that sometimes (e.g., when known sufficient conditions of the solvability of the boundary value problem are inapplicable) the constructive approach can give only a chance to obtain the result.

In Section 6.2, a constructive scheme of testing the abstract linear boundary value problem for the unique solvability is proposed. Next some details of computer aided implementation are described as applied to the boundary value problems in the space of absolutely continuous functions (Section 6.3); in the space of piecewise absolutely continuous functions (the case of impulse boundary value problems)—Section 6.4; and to a class of singular boundary value problems (Section 6.5). Sections 6.6, 6.7 are devoted to some other problems, the efficient study of which uses the modern computer-assisted technique.

6.2. General theorem on the solvability of the boundary value problem

Following the notations and the terms of Chapter 1, consider the linear boundary value problem for the abstract functional differential equation

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{6.1}$$

with linear operators $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ and $l = [l^1, \ldots, l^n] : \mathbf{D} \to \mathbb{R}^n$, assuming as usual that an isomorphism $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ is defined by the operators

$$\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \longrightarrow \mathbf{D}, \qquad \mathcal{J}^{-1} = [\delta, r] : \mathbf{D} \longrightarrow \mathbf{B} \times \mathbb{R}^n. \tag{6.2}$$

In this chapter, we suppose the principal boundary value problem

$$\mathcal{L}x = f, \qquad rx = \alpha \tag{6.3}$$

to be uniquely solvable for any $f \in \mathbf{B}$, $\alpha \in \mathbb{R}^n$. Recall (see Theorems 1.11, 1.16) that in such a case we have $\dim\ker \mathcal{L} = n$, and a necessary and sufficient condition for the unique solvability of problem (6.1) is

$$\det lX \neq 0, \tag{6.4}$$

where $X = (x_1, \ldots, x_n)$ is a fundamental vector of the homogeneous equation $\mathcal{L}x = 0$, $lX \overset{\text{def}}{=} (l^i x_j)$, $i, j = 1, \ldots, n$. Here and in what follows we deal with the fundamental vector such that

$$\mathcal{L}X = 0, \qquad rX = E, \tag{6.5}$$

where E is the identity $n \times n$ matrix (see Theorem 1.16 and (1.53)). Since in actual practice approximate elements of the matrix lX are only available, reliable testing of the criterion (6.4) requires specialized theorems, techniques, and algorithms.

The techniques of the study of problem (6.1) for the unique solvability, which are proposed below, are based on the following simple consideration. If we could find an invertible $n \times n$ matrix Γ such that

$$\|lX - \Gamma\| < \frac{1}{\|\Gamma^{-1}\|}, \tag{6.6}$$

then the matrix lX is invertible too, and hence problem (6.1) is uniquely solvable. We will look for Γ in the form $\Gamma = \bar{l}X^a$, where $\bar{l} : \mathbf{D} \to \mathbb{R}^n$ is a vector functional close to l; a matrix X^a with the columns from $\mathbf{D}$ satisfies the equality $rX^a = E$, gives for an operator $\overline{\mathcal{L}} : \mathbf{D} \to \mathbf{B}$ close to $\mathcal{L}$ a sufficiently small defect $\Delta \overset{\text{def}}{=} \overline{\mathcal{L}}X^a$, and hence is an approximation for the fundamental vector X. The proximity of $\bar{l}$ and l, $\overline{\mathcal{L}}$ and $\mathcal{L}$ as well as the smallness of Δ, which guarantee the unique solvability of (6.1), are defined by Theorem 6.1 given below.

Denote by x_i^a, $i = 1,\dots,n$, the columns of X^a, $\Delta_i \overset{\text{def}}{=} \overline{\mathcal{L}}x_i^a$, $G_0 : \mathbf{B} \to \mathbf{D}$ is the Green operator of the principal boundary value problem (6.3). In this chapter, the norm $|\cdot|$ in $\mathbb{R}^n$ is defined by $|\alpha| = \max_{1\le i\le n}|\alpha^i|$ for $\alpha = \operatorname{col}\{\alpha^1,\dots,\alpha^n\}$; for $n\times n$ matrix $A = \{a_{ij}\}$ we define $\|A\| = \max_{1\le i\le n}\sum_{j=1}^n |a_{ij}|$; for $\alpha,\beta \in \mathbb{R}^n$ the inequality $\alpha \le \beta$ means $\alpha^i \le \beta^i$, $i = 1,\dots,n$; $\left|\alpha\right| \overset{\text{def}}{=} \operatorname{col}\{|\alpha^1|,\dots,|\alpha^n|\}$; $\left|A\right| \overset{\text{def}}{=} \{|a_{ij}|\}$.

Let us define constants λ, g_0, M^a, μ_i^a, ν_i^a, δ_i^a, $i = 1,\dots,n$, by the inequalities

$$\begin{gathered}\lambda \ge \|l\|, \qquad g_0 \ge \|G_0\|, \qquad M^a \ge \|(\bar{l}X^a)^{-1}\|,\\ \mu_i^a \ge |(l-\bar{l})x_i^a|, \qquad \nu_i^a \ge \|(\mathcal{L}-\overline{\mathcal{L}})x_i^a\|_{\mathbf{B}}, \qquad \delta_i^a \ge \|\Delta_i\|_{\mathbf{B}}.\end{gathered} \tag{6.7}$$

Theorem 6.1. *Let operators $\overline{\mathcal{L}}$ and $\bar{l}$ and a vector X^a be such that the matrix $\bar{l}X^a$ is invertible, and*

$$\sum_{i=1}^n \mu_i^a + \lambda g_0 \sum_{i=1}^n (\nu_i^a + \delta_i^a) < \frac{1}{M^a}. \tag{6.8}$$

Then the boundary value problem (6.1) *is uniquely solvable for any $f \in \mathbf{B}$, $\alpha \in \mathbb{R}^n$.*

Proof. Take the estimate

$$\|lX - \bar{l}X^a\| \le \|(l-\bar{l})X^a\| + \|l(X-X^a)\| \le \sum_{i=1}^n \mu_i^a + \lambda\sum_{i=1}^n \|x_i - x_i^a\|_{\mathbf{D}}. \tag{6.9}$$

Next

$$x_i - x_i^a = G_0\{(\mathcal{L}-\overline{\mathcal{L}})x_i^a - \Delta_i\}, \quad i = 1,\dots,n, \tag{6.10}$$

which implies that

$$\|x_i - x_i^a\|_{\mathbf{D}} \le g_0(\nu_i^a + \delta_i^a). \tag{6.11}$$

Thus under the conditions of the theorem,

$$\|lX - \bar{l}X^a\| < \frac{1}{\|(\bar{l}X^a)^{-1}\|} \tag{6.12}$$

and, by the theorem on invertible operator (see, e.g., [100, Theorem 3.6.3]), (6.4) holds. □

Actual constructing of a matrix $\bar{l}X^a$ and reliable testing inequality (6.8) have become possible with the development of modern computer-assisted techniques and appropriate software. These techniques place certain requirements upon operators $\overline{\mathcal{L}}$ and $\bar{l}$. Because of this, we enter below the special classes of the so-called *computable* operators and functions. In the framework of this classes, it has been possible to formulate analogs of Theorem 6.1 such that their conditions can be checked by the computer in the course of the reliable computing experiment that follows the scheme:

(1) constructing operators $\overline{\mathcal{L}}$ and $\bar{l}$ approximating $\mathcal{L}$ and l, respectively;
(2) constructing an approximate fundamental vector X^a of the equation $\overline{\mathcal{L}}x = 0$;
(3) constructing the matrix $\bar{l}X^a$;
(4) checking the invertibility of matrix $\bar{l}X^a$;
(5) constructing the inverse matrix $(\bar{l}X^a)^{-1}$;
(6) finding constants involved in inequality (6.8);
(7) checking inequality (6.8).

In the case that the realization of this scheme does not establish the fulfillment of (6.8) (say, $\bar{l}X^a$ is not invertible or (6.8) does not hold) the sequence of procedures (1)–(7) is executed again with a higher accuracy of the approximation to operators $\mathcal{L}$, l and vector X. The computing experiment as a whole consists in many times repeating the procedure (1)–(7) with successive increase in the accuracy of the mentioned approximation. It is either finished with the result (establishing the fulfillment (6.8)) or terminated with no result. Let us be concerned briefly with the conditions providing that the computing experiment gives the result, theoretically, for any uniquely solvable problem (6.1) (the detailed consideration of such conditions with the proofs and corresponding estimates is given in the monograph Rumyantsev [196]). Denote by $\{\overline{\mathcal{L}}_k\}$, $\{\bar{l}_k\}$, $\{X_k^a\}$, $\{\Delta^k\}$ the sequences of the approximating operators, the approximate fundamental vectors and the defects, respectively. If $\overline{\mathcal{L}}_k \to \mathcal{L}$ and $\bar{l}_k \to l$ uniformly as $k \to \infty$ and $\Delta^k \to 0$ as $k \to \infty$ in components in $\mathbf{B}$, then the existence k_0, such that for $\overline{\mathcal{L}} = \overline{\mathcal{L}}_{k_0}$, $\bar{l} = \bar{l}_{k_0}$, $X^a = X_{k_0}^a$, $\Delta = \Delta^{k_0}$ inequality (6.8) holds, follows immediately from the theorem on invertible operator. Conditions for the uniform convergences $\overline{\mathcal{L}}_k \to \mathcal{L}$ and $\bar{l}_k \to l$ can be too stringent for some concrete spaces $\mathbf{B}$ and classes of operators $\overline{\mathcal{L}}$, $\bar{l}$. From the inequalities (6.7), (6.8) we notice that under the condition of the strong convergence $\overline{\mathcal{L}}_k \to \mathcal{L}$, $\bar{l}_k \to l$ it is sufficient for the existences $\overline{\mathcal{L}}$, $\bar{l}$, and X^a, which satisfy (6.3), that $X_k^a \to X$ in components in $\mathbf{D}$. In view of Lemma 4.1.3 of Azbelev et al. [32], the latter condition is fulfilled under the condition that the principal boundary value problems

$$[\overline{\mathcal{L}}_k, r]x = \{f, 0\}, \quad k = 1, 2, \ldots, \tag{6.13}$$

are uniquely solvable, and for each $f \in \mathbf{B}$ their solutions v_k are uniformly bounded: $\sup_k \|v_k\|_{\mathbf{D}} < \infty$. Some conditions of the uniform boundedness of solutions to the sequence of the Cauchy problems in the case of the space of absolutely continuous functions are formulated in Section 4.3, Azbelev et al. [32]. Conditions for the strong convergence of the composition operators sequence are given by Theorems C.10–C.15.

6.3. BVP in the space of absolutely continuous functions

6.3.1. Notation and definitions

Let $\mathbf{L}^n = \mathbf{L}^n[0,1]$ be the space of summable functions $z : [0,1] \to \mathbb{R}^n$, let $\|z\|_{\mathbf{L}^n} = \int_0^1 |z(s)|ds$, let $\mathbf{D}^n = \mathbf{D}^n[0,1]$ be the space of absolutely continuous functions $x : [0,1] \to \mathbb{R}^n$, and let $\|x\|_{\mathbf{D}^n} = \|\dot{x}\|_{\mathbf{L}^n} + |x(0)|$. For a fixed set of points $0 = t_0 < t_1 < \cdots < t_{m+1} = 1$, we denote by $\mathbf{DS}^n(m)$ the space $\mathbf{DS}[0, t_1, \ldots, t_m, 1]$ (see Section 3.2); next

$$(Vz)(t) = \int_0^t z(s)ds, \quad t \in [0,1]. \tag{6.14}$$

For any linear bounded operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ with the principal part $Q : \mathbf{L}^n \to \mathbf{L}^n$, define the linear bounded operator $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ by the equality

$$(\widetilde{\mathcal{L}}y)(t) = (Q\dot{y})(t) + A(t)y(0), \tag{6.15}$$

where $A(t) = (\mathcal{L}E)(t)$.

For any linear bounded vector functional $l : \mathbf{D}^n \to \mathbb{R}^n$ with the representation

$$lx = \Psi x(0) + \int_0^1 \Phi(s)\dot{x}(s)ds, \tag{6.16}$$

where Ψ is constant $n \times n$ matrix, the elements of $n \times n$ matrix Φ are measurable and essentially bounded on $[0,1]$, the linear bounded vector functional $\widetilde{l} : \mathbf{DS}^n(m) \to \mathbb{R}^n$ we define by

$$\widetilde{l}y = \Psi y(0) + \int_0^1 \Phi(s)\dot{y}(s)ds. \tag{6.17}$$

We suppose in what follows that the space $\mathbf{DS}^n(m) = \mathbf{DS}^n[0, t_1, \ldots, t_m, 1]$ is constructed in relation to the system of rational points t_i, $i = 1, \ldots, m$. Denote

$$\mathcal{E}_i = [t_{i-1}, t_i), \quad i = 1, \ldots, m; \qquad \mathcal{E}_{m+1} = [t_m, 1]; \qquad \mathcal{E}_0 = (-\infty, 0); \tag{6.18}$$

χ_i is the characteristic function of the set $\mathcal{E}_i$.

Definition 6.2. A function $y \in \mathbf{DS}^n(m)$ is said to possess the property $\mathcal{C}$ (is computable) if its components as well as the components of functions $\dot{y}$ and Vy take rational values at any rational value of the argument.

For example, functions of the form

$$y(t) = \sum_{i=1}^{m+1} \chi_i(t)p_i(t), \tag{6.19}$$

where the components of the vector functions $p_i : [0,1] \to \mathbb{R}^n$, $i = 1,\dots,m+1$, are polynomials with rational coefficients, possess the property $\mathcal{C}$.

Denote by $\mathcal{P}_m^n$ the set of all $y \in \mathbf{DS}^n(m)$ having the form (6.19).

Definition 6.3. A function $h \in \mathcal{P}_m^n$, $h = \operatorname{col}\{h^1,\dots,h^n\}$, is said to possess the property Δ_q if, for every $j = 1,\dots,m+1$, there exists a vector

$$q = \operatorname{col}\{q^1,\dots,q^n\}, \quad 0 \le q^i \le j,\ i = 1,\dots,n, \tag{6.20}$$

such that $h^i(t) \in \mathcal{E}_{q^i}$, $i = 1,\dots,n$, as $t \in \mathcal{E}_j$.

The property Δ_q takes place, for example, for functions $h \in \mathcal{P}_m^n$ with the components h^i, $i = 1,\dots,n$, satisfying the inequality $h^i(t) \le t$, $t \in [0,1]$, and being piecewise constant rational-valued functions.

Definition 6.4. A linear bounded operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ is said to possess the property $\mathcal{C}$ (is computable) if the operator $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ constructed by formula (6.15) maps any element of $\mathcal{P}_m^n$ in an element of this set.

In the case $\mathcal{L}x = \dot{x} - Px$, the operator $\mathcal{L}$ is computable under the condition that the columns of P are functions of the form (6.19). The operator $\mathcal{L}$,

$$(\mathcal{L}x)(t) = \dot{x}(t) - P(t)x_h(t), \quad t \in [0,1], \tag{6.21}$$

is computable if, for instance, the columns of P have the form (6.19) and the function $h \in \mathcal{P}_m^1$ possesses the property Δ_q.

Definition 6.5. A linear bounded vector functional $l : \mathbf{D}^n \to \mathbb{R}^n$ is said to have *the property* $\mathcal{C}$ (is *computable*) if the vector functional $\tilde{l} : \mathbf{DS}^n(m) \to \mathbb{R}^n$ defined by (6.19) takes, for every $y \in \mathcal{P}_m^n$, the value $\tilde{l}y$ being the vector with the rational components.

6.3.2. Boundary value problem for the system of ordinary differential equations

Consider the boundary value problem

$$\begin{aligned} (\mathcal{L}x)(t) &\stackrel{\text{def}}{=} \dot{x}(t) - P(t)x(t) = f(t), \quad t \in [0,1], \\ lx &\stackrel{\text{def}}{=} \Psi x(0) + \int_0^1 \Phi(s)\dot{x}(s)\,ds = \alpha. \end{aligned} \tag{6.22}$$

Here $P(t) = \{p_{ij}(t)\}_1^n$, $p_{ij} \in \mathbf{L}^1$; $f \in \mathbf{L}^n$; $\Psi = \{\psi_{ij}\}_1^n$; and $\Phi(t) = \{\varphi_{ij}(t)\}_1^n$, $\varphi_{ij} : [0,1] \to \mathbb{R}^1$, are piecewise continuous functions with possible discontinuities of the first kind at fixed points $\tau_1,\dots,\tau_{\overline{m}}$, $0 < \tau_1 < \cdots < \tau_{\overline{m}} < 1$, and continuous on the right at these points; $\alpha \in \mathbb{R}^n$.

The problem (6.94) is approximated in the following way. Let each of the points τ_j, $j = 1,\dots,\overline{m}$, be in correspondence with a pair of rational points t_{2j-1}, t_{2j} such that $t_{2j-1} < \tau_j < t_{2j}$ and $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$. Denote, as above, $\mathcal{E}_i = [t_{i-1}, t_i)$, $i = 1,\dots,m-1$, $\mathcal{E}_m = [t_m, 1]$; χ_i is the characteristic function of $\mathcal{E}_i$, $i = 1,\dots,m$. Further we define the space $\mathbf{DS}^n[0,t_1,\dots,t_{m-1},1] = \mathbf{DS}^n(m-1)$ over the system of the points $t_1,\dots,t_{m-1}$. Denote $\mathcal{J}_1 = \{1,3,\dots,2\overline{m}+1\}$, $\mathcal{J}_2 = \{2,4,\dots,2\overline{m}\}$.

On the sets $\mathcal{E}_i$, $i \in \mathcal{J}_1$, the functions ϕ_j^k are approximated by the polynomials ${}^i\phi_{jk}^a$ with rational coefficients; on the sets $\mathcal{E}_i$, $i \in \mathcal{J}_2$, functions ${}^i\phi_{jk}^a$ are taken as zero. By ${}^i\phi_{jk}^v$ we denote the rational error bounds of the approximation:

$$\begin{aligned} {}^i\phi_{jk}^v &\geq \left|\phi_j k(t) - {}^i\phi_{jk}^a(t)\right|, \quad t \in \mathcal{E}_i,\ i \in \mathcal{J}_1, \\ {}^i\phi_{jk}^v &\geq \left|\phi_j k(t)\right|, \quad t \in \mathcal{E}_i,\ i \in \mathcal{J}_2. \end{aligned} \tag{6.23}$$

Let us approximate functions p_{jk} over the sets $\mathcal{E}_i$, $i = 1,\dots,m$, by polynomials ${}^ip_{jk}^a$ with rational coefficients and denote by ${}^ip_{jk}^v$ rational error bounds of the approximation:

$${}^ip_{jk}^v \geq \int_{t_{i-1}}^{t_i} \left|p_{jk}(s) - {}^ip_{jk}^a(s)\right| ds, \quad i = 1,\dots,m. \tag{6.24}$$

Denote $P_a^i(t) = \{{}^ip_{jk}^a(t)\}_1^n$, $\Phi_a^i(t) = \{{}^i\phi_{jk}^a(t)\}_1^n$, $P_v^i = \{{}^ip_{jk}^v\}_1^n$, $\Phi_v^i = \{{}^i\phi_{jk}^v\}_1^n$. Define matrices P_a and Φ_a by the equalities

$$P_a(t) = \sum_{i=1}^m P_a^i(t)\chi^i(t), \qquad \Phi_a(t) = \sum_{i=1}^m \Phi_a^i(t)\chi_i(t). \tag{6.25}$$

Next, let us approximate numbers ψ_{jk} by rational numbers ψ_{jk}^a and define rational error bounds ψ_{jk}^v of the approximation: $\psi_{jk}^v \geq |\psi_{jk} - \psi_{jk}^a|$. Denote $\Psi_a = \{\psi_{jk}^a\}_1^n$, $\Psi_v = \{\psi_{jk}^v\}_1^n$. Thus, the boundary value problem

$$\begin{aligned} (\overline{\mathcal{L}}x)(t) &\equiv \dot{x}(t) - P_a(t)x(t) = f(t), \quad t \in [0,1], \\ \bar{l}x &\equiv \Psi_a x(0) + \int_0^1 \Phi_a \dot{x}(s)\,ds = \alpha, \end{aligned} \tag{6.26}$$

approximates problem (6.22).

By the construction, $\overline{\mathcal{L}} : \mathbf{D}^n \to \mathbf{L}^n$ is linear bounded operator with the invertible principal part and the property $\mathcal{C}$; $\bar{l} : \mathbf{D}^n \to \mathbb{R}^n$ is linear bounded vector functional with the property $\mathcal{C}$ as well.

Construct an approximate fundamental matrix X_a of the system

$$\dot{x}(t) - P_a(t)x(t) = 0, \quad t \in [0,1], \tag{6.27}$$

in the following way.

The fundamental matrix $\overline{X}$ of system (6.27) is the solution of the Cauchy problem $\dot{X} = P_a(t)X(t)$, $t \in [0,1]$, $X(0) = E$ being the collection of the following problems:

$$\dot{y}^i(t) - P_a(t)y^i(t) = 0, \quad t \in [0,1],\ y^i(0) = e_i, \tag{6.28}$$

$i = 1,\ldots,n$, where e_i is the ith column of the identity matrix. We define approximate solution y_a^i of problem (6.28) by the equality

$$y_a^i(t) = \sum_{i=1}^{m} {}^j y_a^i(t)\chi_j(t), \tag{6.29}$$

where ${}^j y_a^i(t)$, $t \in \mathcal{E}_j$, is an approximate solution of the Cauchy problem

$$\begin{gathered} {}^j\dot{y}^i(t) - P_a^j(t)\,{}^j y^i(t) = 0, \quad t \in \mathcal{E}_j, \\ {}^j y^i(t_{j-1}) = {}^{j-1}y_a^i(t_{j-1}) + \varepsilon_i^j, \end{gathered} \tag{6.30}$$

${}^0y_a^i(0) = e_i$, $\varepsilon_i^1 = 0$, $i = 1,\ldots,n$, $j = 1,\ldots,m$, ${}^j y_a^i(t) = 0$ when $t \notin \mathcal{E}_j$.

Note that the putting of a deviation vector, ε_i^j, enables us to take as an initial value of the solution at every next interval $\mathcal{E}_j$ a rational number, which has lower number of figures in decimal notation than the number ${}^{j-1}y_a^i(t_{j-1})$ does.

The components of the approximate solution, ${}^j y_a^i(\cdot) = \operatorname{col}\{{}_1^j y_a^i(\cdot),\ldots,{}_n^j y_a^i(\cdot)\}$ are defined as the segments of the Taylor series of the exact solution:

$${}_q^j y_a^i = {}_q^{j-1} y_a^i(t_{j-1}) + \varepsilon_i^j + \sum_{r=1}^{\nu} {}_r^q c_i^j (t - t_{j-1})^r, \quad q = 1,\ldots,n. \tag{6.31}$$

The coefficients ${}_r^q c_i^j$ are found by the indefinite coefficients method. The desired matrix X_a is defined by the equality

$$X_a(t) = \sum_{j=1}^{m} X_a^j(t)\chi_j(t) \tag{6.32}$$

with

$$X_a^j(\cdot) = \{{}^j y_a^1(\cdot),\ldots,{}^j y_a^n(\cdot)\}. \tag{6.33}$$

By the construction, $y_a^i \in \mathcal{P}_{m-1}^n$, $i = 1,\ldots,m$, and

$$y_a^i(t) = \int_0^t \dot{y}_a^i(s)ds + \sum_{j=1}^{m-1} \varepsilon_i^j \chi_{[t_j,1]}(t) + e_i. \tag{6.34}$$

Let X be the fundamental matrix of the equation $\dot{x} - Px = 0$ and let x_i be its ith column. Construct a matrix X_ν such that

$$X_\nu(t) \geq \int_0^t \left\| \dot{X}(s) - \dot{X}_a(s) \right\| ds, \quad t \in [0,1]. \tag{6.35}$$

Denote $\omega_i(t) = x_i(t) - y_a^i(t)$, $t \in [0,1]$, $i = 1,\dots,n$. Then $\omega_i \in \mathbf{DS}^n(m-1)$, $\omega_i(0) = 0$, $i = 1,\dots,n$, $\omega_i(t_j) - \omega_i(t_j - 0) = \varepsilon_j$, $j = 1,\dots,m-1$. The error ω_i, $i = 1,\dots,n$, satisfies the equation

$$\dot{\omega}_i(t) - P(t)\omega_i(t) = \left[P(t) - P_a(t)\right] y_a^i(t) + \mu_i(t), \quad t \in [0,1], \tag{6.36}$$

where $\mu_i(\cdot) = \operatorname{col}\{\mu_i^1(\cdot),\dots,\mu_i^n(\cdot)\}$ is the defect

$$\mu_i(t) = -\dot{y}_a^i(t) + P_a(t) y_a^i(t). \tag{6.37}$$

Let $t \in \mathcal{E}_j$, $j = 1,\dots,m$. The following estimates hold:

$$\begin{aligned} &\int_{t_{j-1}}^t \left\| \mu_i(s) \right\| ds \\ &\quad \leq \operatorname{col}\left\{ t_j^\Delta \left(\int_{t_{j-1}}^{t_j} \left| \mu_i^1(s) \right|^2 ds \right)^{1/2}, \dots, t_j^\Delta \left(\int_{t_{j-1}}^{t_j} \left| \mu_i^n(s) \right|^2 ds \right)^{1/2} \right\} \\ &\quad \stackrel{\text{def}}{=} {}^j\beta_1^i, \quad t_j^\Delta = \sqrt{t_j - t_{j-1}}, \\ &\left\| y_a^i(t) \right\| \leq \left\| {}^j y_a^i(t_{j-1}) \right\| + \int_{t_{j-1}}^t \left\| {}^j \dot{y}_a^i(s) \right\| ds \leq \left\| {}^j y_a^i(t_{j-1}) \right\| + {}^j x_N^i, \end{aligned} \tag{6.38}$$

where

$${}^j x_N^i \geq \operatorname{col}\left\{ t_j^\Delta \left(\int_{t_{j-1}}^{t_j} \left| {}_1^j \dot{y}_a^i(s) \right|^2 ds \right)^{1/2}, \dots, t_j^\Delta \left(\int_{t_{j-1}}^{t_j} \left| {}_n^j \dot{y}_a^i(s) \right|^2 ds \right)^{1/2} \right\}. \tag{6.39}$$

Hence

$$\begin{aligned} &\int_{t_{j-1}}^t \left\| P(s) - P_a^j(s) \right\| \left\| y_a^i(s) \right\| ds \leq P_\nu^j \left\{ \left\| {}^j y_a^i(t_{j-1}) \right\| + {}^j x_N^i \right\} \stackrel{\text{def}}{=} {}^j\beta_2^i, \\ &\qquad \int_{t_{j-1}}^t \left\| P(s) \right\| \left\| \omega_i(s) \right\| ds \leq \int_{t_{j-1}}^t \left\| P(s) \right\| \int_{t_{j-1}}^s \left\| \dot{\omega}_i(\tau) \right\| d\tau\, ds \\ &\qquad\qquad + \{P_N^j + P_\nu^j\} \sum_{q=1}^{j-1} \left({}^q x_\nu^i + \left\| \varepsilon_i^q \right\| \right), \end{aligned} \tag{6.40}$$

where

$$
\begin{aligned}
&{}^{q}x_v^i \geq \int_{t_{q-1}}^{t_q} \left\| \dot{\omega}_i(s) \right\| ds, \quad q = 1,\ldots,j-1, \qquad P_N^j = \{{}^{j}p_{kq}^N\}_1^n,\\
&{}^{j}p_{kq}^N \geq t_j^{\Delta}\left(\int_{t_{j-1}}^{t_j} \left|{}^{j}p_{kq}^a(s)\right|^2 ds\right)^{1/2}, \quad k,q = 1,\ldots,n.
\end{aligned}
\tag{6.41}
$$

Let ${}^{j}\beta_3^i$ and ${}^{j}\beta_4^i$ be vectors with the rational components such that

$$
\begin{aligned}
&{}^{j}\beta_3^i \geq \{P_N^j + P_v^j\} \sum_{q=1}^{j-1} \left({}^{q}x_v^i + \left\|\varepsilon_i^q\right\|\right),\\
&{}^{j}\beta_4^i \geq {}^{j}\beta_1^i + {}^{j}\beta_2^i + {}^{j}\beta_3^i.
\end{aligned}
\tag{6.42}
$$

The obtained estimates and equation (6.36) imply

$$
\int_{t_{j-1}}^{t} \left\|\dot{\omega}_i(s)\right\| ds \leq \int_{t_{j-1}}^{t} \left\|P(s)\right\| \int_{t_{j-1}}^{s} \left\|\dot{\omega}_i(\tau)\right\| d\tau\, ds + {}^{j}\beta_4^i. \tag{6.43}
$$

Thus

$$
\int_{t_{j-1}}^{t} \left|\dot{\omega}_i(s)\right| ds \leq \int_{t_{j-1}}^{t} ||P(s)|| \int_{t_{j-1}}^{s} \left|\dot{\omega}_i(\tau)\right| d\tau\, ds + \left|{}^{j}\beta_4^i\right|. \tag{6.44}
$$

Due to the Gronwall-Bellman lemma we obtain conclusively

$$
\int_{t_{j-1}}^{t} \left|\dot{\omega}_i(s)\right| ds \leq \left|{}^{j}\beta_4^i\right| \exp\left\{\int_{t_{j-1}}^{t_j} ||P(s)|| ds\right\}. \tag{6.45}
$$

Denote by ${}_0^jx_v^i$ a rational number which majorizes the right-hand side of this inequality, define an n-dimensional vector ${}^{j}x_v^i = \operatorname{col}\{{}_0^jx_v^i,\ldots,{}_0^jx_v^i\}$ and the matrix

$$
X_v^j = \{{}^{j}x_v^1,\ldots,{}^{j}x_v^n\}. \tag{6.46}
$$

Then the desired matrix X_v is defined by

$$
X_v(t) = \sum_{j=1}^{m} \chi_j(t) X_v^j. \tag{6.47}
$$

Define the matrix C_a with the rational elements by the equality

$$
C_a = \Psi_a + \sum_{j=1}^{m} \int_{t_{j-1}}^{t_j} \Phi_a^j(s)\dot{X}_a^j(s)ds. \tag{6.48}
$$

To invert the matrix C_a, one can apply the compact Gauss scheme (see, e.g., Bakhvalov [45]) in the frames of the rational arithmetic that allows constructing exactly C_a^{-1} for any invertible C_a.

The estimate $\left| lX - C_a \right| \le C_v$, holds, where

$$C_v = \Psi_v + \sum_{j=1}^{m} \{\Phi_M^j X_v^j + \Phi_v^j X_N^j + \Phi_v^j X_v^j\}, \tag{6.49}$$

$$X_N^j = \{{}^j x_N^1, \dots, {}^j x_N^n\}, \qquad \Phi_M^j = \{{}^j \varphi_{kq}^M\}, \tag{6.50}$$

$${}^j\varphi_{kq}^M \ge \left| {}^j\varphi_{kq}^a(t_{j-1}) \right| + t_j^\Delta \left(\int_{t_{j-1}}^{t_j} \left| {}^j\dot{\varphi}_{kq}^a(s) \right|^2 ds \right)^{1/2} \ge \left| {}^j\varphi_{kq}^a(t) \right|, \quad t \in \mathcal{E}_j. \tag{6.51}$$

Problem (6.22) is uniquely solvable if C_a is invertible and

$$||C_v|| < \frac{1}{||C_a^{-1}||}. \tag{6.52}$$

Therefore the following analog of Theorem 6.1 is proved.

Theorem 6.6. *Let computable operators $\overline{\mathcal{L}}$, $\bar{l}$ in the approximate problem* (6.26) *and matrix X_a with the computable elements defined by* (6.32) *be such that the matrix C_a defined by* (6.48) *is invertible and the inequality* (6.52) *is fulfilled, where C_v is defined by* (6.49). *Then boundary value problem* (6.22) *is uniquely solvable for any $f \in \mathbf{L}^n$ and $\alpha \in \mathbb{R}^n$.*

Now Scheme (1)–(7) takes the following form:

(1) constructing an approximate problem (6.26) with computable operators $\overline{\mathcal{L}}$ and $\bar{l}$;
(2) constructing matrices X_a and X_v defined by equalities (6.32) and (6.47);
(3) constructing the matrix C_a defined by (6.48) and inverting it;
(4) constructing a matrix C_v defined by (6.49) and checking the test (6.52).

Example 6.7. Let us investigate the following boundary value problem for the solvability:

$$\dot{x}(t) - \begin{bmatrix} -2 & 2t \ln\left(1 + \frac{1}{10}t\right) & 0 \\ 0 & \frac{1}{9}t - 2 & \frac{8}{9}t \\ t \cdot \exp\left(-\frac{1}{8}t\right) & 0 & -2 - t \end{bmatrix} x(t) = f(t), \quad t \in [0,1], \tag{6.53}$$

$$\frac{1}{\ln 2}x^1(0) + x^2(0) = \alpha^1, \qquad x^2(1) - \sqrt{10}\,x^3(0) = \alpha^2, \tag{6.54}$$

$$\int_0^1 \cos\left(\frac{1}{5}s\right)x^3(s)ds = \alpha^3. \tag{6.55}$$

Here $x(t) = \operatorname{col}\{x^1(t), x^2(t), x^3(t)\}$.

Let $0 = t_0 < t_1 = 1$ (the points τ_j are absent). Take

$$\begin{aligned}
{}^1p_{12}^a(t) &= 2t\left(\frac{1}{10}t - \frac{1}{200}t^2 + \frac{1}{3\cdot 10^3}t^3 - \frac{1}{4\cdot 10^4}t^4 + \frac{1}{5\cdot 10^5}t^5\right),\\
{}^1p_{12}^v &= \frac{1}{24\cdot 10^6},\\
{}^1p_{31}^a(t) &= t - \frac{1}{8}t^2 + \frac{1}{128}t^3 - \frac{1}{3072}t^4 + \frac{1}{98304}t^5,\\
{}^1p_{31}^v &= \frac{1}{27525120}.
\end{aligned} \tag{6.56}$$

The rest of the elements of the coefficient matrix do not need to be approximated and the corresponding elements of P_v are equal zero.

Boundary conditions (6.54) can be written in the form

$$\Psi x(0) + \int_0^1 \Phi(s)\dot{x}(s)ds = \alpha, \tag{6.57}$$

where

$$\Psi = \begin{bmatrix} \frac{1}{\ln 2} & 1 & 0 \\ 0 & 1 & -\sqrt{10} \\ 0 & 0 & 5\sin\frac{1}{5} \end{bmatrix}, \qquad \Phi(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5\left\{\sin\frac{1}{5} - \sin\left(\frac{1}{5}t\right)\right\} \end{bmatrix}. \tag{6.58}$$

Take

$$\begin{aligned}
\psi_{11}^a &= \frac{1442695}{1000000}, & \psi_{11}^v &= 5\cdot 10^{-8},\\
\psi_{23}^a &= -\frac{3162278}{1000000}, & \psi_{23}^v &= 4\cdot 10^{-7},\\
\psi_{33}^a &= \frac{74501}{75000}, & \psi_{33}^v &= \frac{1}{78750000},\\
{}^1\varphi_{33}^a(t) &= -t + \frac{1}{150}t^3 - \frac{1}{7500}t^5 + \frac{74501}{75000}, & {}^1\varphi_{33}^v &= 10^{-7}.
\end{aligned} \tag{6.59}$$

With a computer program realizing the constructive scheme for the study of boundary value problem (6.22) the unique solvability of (6.53), (6.54) is established. In this example we have

$$5 \cdot 10^{-3} < \frac{1}{||C_a^{-1}||} < 6 \cdot 10^{-3}, \qquad 2 \cdot 10^{-4} < ||C_v|| < 3 \cdot 10^{-4}. \tag{6.60}$$

6.3.3. Boundary value problem for the differential system with concentrated delay

Consider the boundary value problem

$$\begin{aligned} (\mathcal{L}x)^i(t) &\stackrel{\text{def}}{=} \dot{x}^i(t) + \sum_{j=1}^{n} p_{ij}(t)x^j[h_{ij}(t)] = f^i(t), \quad t \in [0,1], \\ x^i(\xi) &= 0 \quad \text{if } \xi \notin [0,1],\ i = 1,\dots,n; \\ lx &\stackrel{\text{def}}{=} \Psi x(0) + \int_0^1 \Phi(s)\dot{x}(s)ds = \alpha. \end{aligned} \tag{6.61}$$

Here

$$\begin{gathered} p_{ij} \in \mathbf{L}^1, \qquad f^i \in \mathbf{L}^1, \qquad \Psi = \{\psi_{ij}\}_1^n; \\ \Phi(t) = \{\varphi_{ij}(t)\}_1^n, \qquad \varphi_{ij} : [0,1] \longrightarrow \mathbb{R}^1, \qquad h_{ij} : [0,1] \longrightarrow \mathbb{R}^1 \end{gathered} \tag{6.62}$$

are piecewise continuous functions with possible breaks of the first kind at fixed points $\tau_1,\dots,\tau_{\overline{m}}$, $0 < \tau_1 < \cdots < \tau_{\overline{m}} < 1$, being continuous on the right at these points; $h_{ij}(t) \le t$; $\alpha \in \mathbb{R}^n$.

The approximating of problem (6.61) is done in the following way. Add to the set of points $\tau_1,\dots,\tau_{\overline{m}}$ the zeros of the functions h_{ij}, $i, j = 1,\dots,n$, and assume in what follows that the set $0 < \tau_1 < \cdots < \tau_{\overline{m}} < 1$ includes the break points of the functions φ_{ij}, h_{ij} and the zeros of h_{ij} as well. Next, as in Section 6.3.2, we construct a collection of rational points t_i, $0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$, taking into our consideration the sets $\mathcal{E}_i$ and their characteristic functions χ_i, $i = 1,\dots,m$. The space $\mathbf{DS}^n(m-1)$ is defined by the system $t_1,\dots,t_{m-1}$. Denote by $\mathcal{Z}^1$ the union of the sets $\mathcal{E}_i$ which do not include points $\tau_1,\dots,\tau_{\overline{m}}$, $\mathcal{Z}^2$ which is the union of the sets $\mathcal{E}_i$ including the break points of functions φ_{jk}, h_{jk}, $j,k = 1,\dots,n$, $\mathcal{Z}^3$ is the union of the sets $\mathcal{E}_i$ including the zeros of functions h_{jk} $j,k = 1,\dots,n$.

On the sets $\mathcal{E}_i \subset \mathcal{Z}^1$, we define ${}^ih^a_{jk}(t) \equiv -1$ if $h_{jk}(t) < 0$, $t \in \mathcal{E}_i$. If $h_{jk}(t) \ge 0$, $t \in \mathcal{E}_i$, then functions h_{jk} are approximated by polynomials ${}^ih^a_{jk}$ with rational coefficients. On the sets $\mathcal{E}_i \subset \mathcal{Z}^2 \cup \mathcal{Z}^3$, we define functions ${}^ih^a_{jk}$ being identically zero. Denote by ${}^ih^v_{jk}$ rational-valued error bounds

$${}^ih^v_{jk} \ge \left| h_{jk}(t) - {}^ih^a_{jk}(t) \right|, \quad t \in \mathcal{E}_i,\ i = 1,\dots,m,\ j,k = 1,\dots,n. \tag{6.63}$$

Define ${}^i h^a_{jk}(t) = 0$ if $t \notin \mathcal{E}_i$. The functions h^a_{jk} are defined by the equalities

$$h^a_{jk}(t) = \sum_{i=1}^{m} {}^i h^a_{jk}(t)\chi_i(t), \quad j,k = 1,\dots,n. \tag{6.64}$$

We require that the functions h^a_{jk} possess the property Δ_q. This requirement does not mean any additional restrictions concerning functions h_{jk}. Actually, the requirement is fulfilled as soon as we approximate functions h_{jk} on $\mathcal{E}_i \subset \mathcal{Z}^1$ by piecewise rational-valued functions ${}^i h^a_{jk}$.

On the sets $\mathcal{E}_i \subset \mathcal{Z}^1 \cup \mathcal{Z}^3$, we approximate functions φ_{qr}, $q,r = 1,\dots,n$, by polynomials ${}^i\varphi^a_{qr}$ with rational coefficients. On the sets $\mathcal{E}_i \subset \mathcal{Z}^2$, we suppose functions ${}^i\varphi^a_{qr}$ to be identically zero. Next define rational-valued error estimates ${}^i\varphi^v_{qr}$ by the inequality

$${}^i\varphi^v_{qr} \geq \left|\varphi_{qr}(t) - {}^i\varphi^a_{qr}(t)\right|, \quad t \in \mathcal{E}_i,\ i = 1,\dots,m. \tag{6.65}$$

In the same way as in Section 6.3.2, define matrices Φ^i_a, Φ^i_v, Φ_a, Φ_v, Ψ_a, and Ψ_v; functions ${}^i p^a_{qr}$, p^a_{qr}; constants ${}^i p^v_{qr}$, $q,r = 1,\dots,n$, $i = 1,\dots,m$. The boundary value problem

$$\begin{aligned}(\overline{\mathcal{L}}x)^i(t) &\overset{\text{def}}{=} \dot{x}^i(t) + \sum_{j=1}^{n} p^a_{ij}(t)x^j\left[h^a_{ij}(t)\right] = f^i(t), \quad t \in [0,1],\\ x^i(\xi) &= 0 \quad \text{if } \xi \notin [0,1],\ i = 1,\dots,n;\end{aligned} \tag{6.66}$$

$$\bar{l}x \overset{\text{def}}{=} \Psi_a x(0) + \int_0^1 \Phi_a(s)\dot{x}(s)\,ds = \alpha \tag{6.67}$$

approximates problem (6.61). By the construction, $\overline{\mathcal{L}} : \mathbf{D}^n \to \mathbf{L}^n$ is a linear bounded operator with invertible principal part and it possesses the property $\mathcal{C}$; $\bar{l} : \mathbf{D}^n \to \mathbb{R}^n$ is a linear bounded vector functional with the property $\mathcal{C}$.

An approximate fundamental matrix X_a of the homogeneous equation $\overline{\mathcal{L}}x = 0$ will be constructed in the following way.

Let $\overline{X}$ be the fundamental matrix of $\overline{\mathcal{L}}x = 0$ and let $y_k = \operatorname{col}\{y^1_k,\dots,y^n_k\}$ be its the kth column, $k = 1,\dots,n$. The vector function y_k is the solution of the Cauchy problem

$$\begin{gathered}\dot{y}^i_k(t) + \sum_{j=1}^{m} p^a_{ij}(t)y^j_k\left[h^a_{ij}(t)\right] = 0, \quad t \in [0,1],\\ y^i_k(\xi) = 0 \quad \text{if } \xi \notin [0,1],\ i = 1,\dots,n,\\ y^i_k(0) = \delta^i_k, \quad i,k = 1,\dots,n,\end{gathered} \tag{6.68}$$

where δ_k^i is the Kronecker symbol. Define an approximate solution $y_a^k = \operatorname{col}\{{}_1y_a^k, \dots, {}_ny_a^k\}$ of problem (6.68) by the equality

$$y_a^k(t) = \sum_{q=1}^{m} {}^q y_a^k(t)\chi_q(t), \qquad {}^q y_a^k(\cdot) = \operatorname{col}\{{}_1^q y_a^k(\cdot), \dots, {}_n^q y_a^k(\cdot)\}. \tag{6.69}$$

Here ${}^q y_a^k(t) = 0$ as $t \notin \mathcal{E}_q$; if $t \in \mathcal{E}_q$, then ${}_i^q y_a^k$ is the ith component of approximate solution to the Cauchy problem

$$\dot{y}^i(t) + \sum_{j=1}^{n} {}^q p_{ij}^a(t) y^j\left[{}^q h_{ij}^a(t)\right] = 0, \quad t \in \mathcal{E}_q, \tag{6.70}$$

$$y^j\left[{}^q h_{ij}^a(t)\right] = \begin{cases} {}_j^s y_a^k\left[{}^q h_{ij}^a(t)\right] & \text{if } {}^q h_{ij}^a(t) \in \mathcal{E}_s,\ s < q, \\ 0 & \text{if } {}^q h_{ij}^a(t) < 0, \end{cases} \tag{6.71}$$

$$y^i(t_{q-1}) = {}_i^{q-1} y_a^k(t_{q-1}) + {}^i\varepsilon_k^q, \qquad {}_i^0 y_a^k(0) = \delta_k^i. \tag{6.72}$$

Let ${}^q h_{ij}^a(t) \in \mathcal{E}_r$, $r < q$, as $t \in \mathcal{E}_q$. Then the superpositions $y^j[{}^q h_{ij}^a(t)]$ are known functions and the components of ${}^q y_a^k$ can be found by immediate integrating:

$${}_i^q y_a^k(t) = {}_i^{q-1} y_a^k(t_{q-1}) + {}^i\varepsilon_k^q - \int_{t_{q-1}}^{t} \sum_{j=1}^{n} {}^q p_{ij}^a(s) y^j\left[{}^q h_{ij}^a(s)\right] ds, \quad i = 1, \dots, n. \tag{6.73}$$

Let ${}^q h_{ij}^a(t) \in \mathcal{E}_q$ as $t \in \mathcal{E}_q$, then we take for ${}_i^q y_a^k(\cdot)$, $i = 1, \dots, n$, a segment of the power series

$${}_i^q y_a^k(t) = \sum_{j=1}^{\nu} {}_j^i c_k^q (t - t_{q-1})^j + {}_i^{q-1} y_a^k(t_{q-1}) + {}^i\varepsilon_k^q, \tag{6.74}$$

whose coefficients ${}_j^i c_k^q$ are to be found by the indefinite coefficients method.

Thus the matrix X_a is defined by

$$X_a(t) = \sum_{q=1}^{m} \chi_q(t) X_a^q(t), \quad \text{where } X_a^q(t) = \{{}_i^q y_a^j(t)\}_1^n. \tag{6.75}$$

By the constructing, ${}_i y_a^j \in \mathbf{DS}^1(m-1)$ and

$${}_i y_a^j(t) = \int_0^t {}_i\dot{y}_a^j(s)\,ds + \sum_{q=1}^{m-1} {}^i\varepsilon^q \chi_{[t_q,1]}(t) + \delta_j^i. \tag{6.76}$$

Denote by X the fundamental matrix of the equation $\mathcal{L}x = 0$ and by x_k its kth column. Now construct a matrix X_ν such that

$$X_\nu(t) \geq \int_0^t \left\| \dot{X}(s) - \dot{X}_a(s) \right\| ds, \quad t \in [0,1]. \tag{6.77}$$

For this purpose, denote $\omega_k(t) = x_k(t) - y_a^k(t)$, $t \in [0,1]$, $\omega_k = \operatorname{col}\{\omega_k^1, \dots, \omega_k^n\}$. By the definition, $\omega_k^i \in \mathbf{DS}^1(m-1)$, $k, i = 1, \dots, n$, and

$$\omega_k^i(t) = \int_0^t \dot{\omega}_k^i(s)ds + \sum_{q=1}^{m-1} {}^i\varepsilon_k^q \chi_{[t_q,1]}(t). \tag{6.78}$$

Let us define functions μ_k^i, $i, k = 1, \dots, n$, by

$$\mu_k^i(t) = \sum_{q=1}^{m} {}^q\mu_k^i(t)\chi_q(t), \tag{6.79}$$

where

$$\begin{gathered}
{}^q\mu_k^i(t) = \begin{cases} 0 & \text{if } t \notin \mathcal{E}_q, \\ -{}_i^q\dot{y}_a^k(t) - \displaystyle\sum_{j=1}^{n} {}^q p_{ij}^a(t) {}_j^q z_k^i(t) & \text{if } t \in \mathcal{E}_q, \end{cases} \\
{}_j^q z_k^i(t) = \begin{cases} {}_j^r y_a^k\left[{}^q h_{ij}^a(t)\right] & \text{if } {}^q h_{ij}^a(t) \in \mathcal{E}_r,\ 0 < r \leq q, \\ 0 & \text{if } {}^q h_{ij}^a(t) < 0 \text{ or } t \notin \mathcal{E}_q. \end{cases}
\end{gathered} \tag{6.80}$$

The error ω_k satisfies the system

$$\begin{gathered}
\dot{\omega}_k^i(t) + \sum_{j=1}^{n} p_{ij}(t)\omega_k^j\left[h_{ij}(t)\right] + \sum_{j=1}^{n} \left[p_{ij}(t) - p_{ij}^a(t)\right]{}_i y_a^k\left[h_{ij}(t)\right] \\
+ \sum_{j=1}^{n} p_{ij}^a(t)\left\{{}_j y_a^k\left[h_{ij}(t)\right] - {}_j z_k^i(t)\right\} = \mu_k^i(t), \quad t \in [0,1]; \\
\omega_k^j(\xi) = 0 \quad \text{if } \xi \notin [0,1]; \qquad {}_j y_a^k(\xi) = 0 \quad \text{if } \xi \notin [0,1]; \\
{}_j z_k^i(t) = \sum_{q=1}^{m} {}_j^q z_k^i(t)\chi_q(t), \quad k, i, j = 1, \dots, n.
\end{gathered} \tag{6.81}$$

As it is shown in [198], the solution of this system, being of the form (6.78), possesses the property that the function w_k defined by $w_k(t) = \int_{t_{q-1}}^t |\dot{\omega}_k(s)| ds$ satisfies on $\mathcal{E}_q$ the integral inequality

$$w_k(t) \leq \int_{t_{q-1}}^t \vartheta_q(s)\omega_k(s)ds + \gamma_k, \tag{6.82}$$

where function ϑ_q and constant γ_k are defined efficiently according to the parameters of approximate problem (6.66), (6.67). Hence, by the Gronwall-Bellman lemma, we have

$$\int_{t_{q-1}}^{t} |\dot{\omega}_k(s)|\, ds \le \gamma_k \exp\left\{\int_{t_{q-1}}^{t_q} \vartheta_q(s)ds\right\}, \quad t \in \mathcal{E}_q. \tag{6.83}$$

Denote by ${}^{q}x_{\nu}^{k}$ a rational-valued majorant to the right-hand side of inequality (6.83), by X_{ν}^{q} the $n \times n$ matrix with the kth column

$$\operatorname{col}\{{}^{q}x_{\nu}^{k}, \dots, {}^{q}x_{\nu}^{k}\}. \tag{6.84}$$

Then the desired matrix X_ν is defined by the equality

$$X_\nu(t) = \sum_{q=1}^{m} \chi_q(t) X_\nu^q, \quad t \in [0,1]. \tag{6.85}$$

Let $X_N^q = \{{}_i^q x_N^j\}_1^n$, where constant ${}_i^q x_N^j$ is a rational-valued majorant of

$$\left\{(t_q - t_{q-1}) \int_{t_{q-1}}^{t_q} |{}_i^q \dot{y}_a^j(s)|^2 ds\right\}^{1/2}. \tag{6.86}$$

By matrices X_ν^q and X_N^q we define $n \times n$ matrix C_ν as follows: each its element is a rational majorant of the corresponding element of the matrix

$$\Psi_\nu + \sum_{q=1}^{m} \{\Phi_M^q X_\nu^q + \Phi_\nu^q [X_N^q + X_\nu^q]\}. \tag{6.87}$$

By the constructing, $C_\nu \ge \left| lX - C_a \right|$, where C_a is defined by the equality

$$C_a = \Psi_a + \sum_{q=1}^{m} \int_{t_{q-1}}^{t_q} \Phi_a^q(s) \dot{X}_a^q(s) ds. \tag{6.88}$$

Problem (6.61) is uniquely solvable when the matrix C_a is invertible and the condition

$$||C_\nu|| < \frac{1}{||C_a^{-1}||} \tag{6.89}$$

holds.

Thus the following analog of Theorem 6.1 is obtained.

Theorem 6.8. *Let computable operators $\overline{\mathcal{L}}$ and $\bar{l}$ in approximate problem* (6.66), (6.67) *and the matrix X_a with computable elements defined by* (6.75) *be such that* *the matrix C_a defined by* (6.88) *is invertible and the condition* (6.89) *is fulfilled with the matrix C_v defined by* (6.87). *Then boundary value problem* (6.61) *is uniquely solvable for any $f \in \mathbf{L}^n$ and $\alpha \in \mathbb{R}^n$.*

Example 6.9. Let us study the problem

$$\begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \end{bmatrix} + \begin{bmatrix} 2t-3 & 4t-3+2t^3 \\ 2t\chi_2(t) & \dfrac{2}{t+2}\sin\dfrac{t+2}{6} \end{bmatrix} \begin{bmatrix} x^1\left[\dfrac{2}{3}t\chi_1(t)\right] \\ x^2\left[\dfrac{1}{2}\left(t^4-\dfrac{1}{2}\right)\right] \end{bmatrix} = \begin{bmatrix} f^1(t) \\ f^2(t) \end{bmatrix}, \quad t \in [0,1], \tag{6.90}$$

$$\begin{gathered} x^i(\xi) = 0, \quad \xi \notin [0,1],\ i = 1,2, \\ x^1(1) - \frac{1}{10}x^2(0) = \alpha^1, \\ \frac{1}{5}x^1(0) + \int_0^1 (2s-1)x^2(s)ds = \alpha^2 \end{gathered} \tag{6.91}$$

for the unique solvability. Here χ_1 and χ_2 are the characteristic functions of the segments $[0, \sqrt{2}/2)$ and $[\sqrt{2}/2, 1]$, respectively ($\tau_1 = \sqrt{2}/2$).

Let $0 = t_0 < t_1 < t_2 < t_3 = 1$ with $t_1 = 0.7071067811$, $t_2 = t_1 + 10^{-10}$. Notice that $t_1 < \sqrt{2}/2 < t_2$. Define the parameters of the approximate problem as follows:

$$\begin{gathered} p_{11}^a(t) = 2t-3, \qquad p_{12}^a(t) = 4t-3+2t^3, \\ {}^1p_{22}^a(t) = {}^2p_{12}^a(t) = 0, \qquad {}^3p_{21}^a(t) = 2t, \\ p_{22}^a = \frac{1}{6} - \frac{(t+2)^2}{1296} + \frac{(t+2)^4}{6^5\cdot 5!} - \frac{(t+2)^6}{6^7\cdot 7!}, \\ p_{11}^v = p_{12}^v = {}^1p_{21}^v = {}^3p_{21}^v = 0, \qquad {}^2p_{21}^v = t_2^2 - t_1^2, \\ {}^1p_{22}^v = 443361\cdot 10^{-15}, \qquad {}^2p_{22}^v = 10^{-15}, \qquad {}^3p_{22}^v = \frac{90199}{125}\cdot 10^{-12}, \\ {}^1h_{11}^a(t) = {}^1h_{21}^a(t) = \frac{2}{3}t, \qquad {}^2h_{11}^a(t) = {}^3h_{11}^a(t) = {}^2h_{21}^a(t) = {}^3h_{21}^a(t) = 0, \\ {}^1h_{11}^v = {}^3h_{11}^v = {}^1h_{21}^v = {}^3h_{21}^v = 0, \qquad {}^2h_{11}^v = {}^2h_{21}^v = \frac{2}{3}t_2, \\ h_{12}^a(t) = h_{22}^a(t) = \frac{1}{2}\left(t^2+\frac{1}{2}\right)(t^2-t_1^2), \end{gathered}$$

$$
\begin{gathered}
{}^2h_{12}^{\nu} = {}^2h_{22}^{\nu} = \frac{1}{2}\left(t_2^2 + \frac{1}{2}\right)\left(\frac{1}{2} - t_1^2\right), \\
{}^3h_{12}^{\nu} = {}^3h_{22}^{\nu} = \frac{3}{4}\left(t_2^2 - \frac{1}{2}\right), \\
\Psi_a = \Psi = \begin{bmatrix} 1 & -\dfrac{1}{10} \\ \dfrac{1}{5} & 0 \end{bmatrix}, \qquad \Phi_a(t) = \Phi(t) = \begin{bmatrix} 1 & 0 \\ 0 & t - t^2 \end{bmatrix}.
\end{gathered}
\tag{6.92}
$$

The unique solvability of problem (6.90), (6.91) is proved in the computer-assisted way by the constructive scheme for the study of problem (6.61).

In this example

$$
0.53 < \frac{1}{||C_a^{-1}||} < 0.54, \qquad 10^{-5} < ||C_\nu|| < 2 \cdot 10^{-5}. \tag{6.93}
$$

6.4. BVP in the space of piecewise absolutely continuous functions

Consider the general linear boundary value problem (see (3.12))

$$
\widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}y = \alpha \tag{6.94}
$$

with linear bounded operators $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ and $\widetilde{l} = [\widetilde{l}^1, \ldots, \widetilde{l}^{n+mn}] : \mathbf{DS}^n(m) \to \mathbb{R}^{n+mn}$, following the notation of Section 6.3. Recall (see Section 3.2) that $\mathbf{DS}^n(m) \simeq \mathbf{L}^n \times \mathbb{R}^{n+mn}$ if

$$
\mathcal{J} = \{\Lambda, Y\}, \qquad (\Lambda z)(t) = \int_0^t z(s)ds, \qquad (Y\beta)(t) = Y(t)\beta, \tag{6.95}
$$

where

$$
Y(t) = \left(E, \chi_{[t_1,1]}(t)E, \ldots, \chi_{[t_m,1]}(t)E\right); \tag{6.96}
$$

$$
\begin{gathered}
\mathcal{J}^{-1} = [\delta, r], \qquad \delta y = \dot{y}, \qquad ry = \operatorname{col}\left(y(0), \Delta y(t_1), \ldots, \Delta y(t_m)\right), \\
\Delta y(t_i) = y(t_i) - y(t_i - 0).
\end{gathered}
\tag{6.97}
$$

In what follows in this section, we assume that the operator $Q = \widetilde{\mathcal{L}}\Lambda : \mathbf{L}^n \to \mathbf{L}^n$ has the bounded inverse operator Q^{-1}. In this case the principal boundary value problem (6.3)

$$
\widetilde{\mathcal{L}}y = f, \qquad ry = \alpha \tag{6.98}
$$

is uniquely solvable for any $f \in \mathbf{L}^n$ and $\alpha \in \mathbb{R}^{n+mn}$, and the fundamental vector $\mathcal{Y} = (y_1, \ldots, y_{n+mn})$ ($r\mathcal{Y} = E$) of the homogeneous equation $\widetilde{\mathcal{L}}y = 0$ is the solution of the problem (6.5):

$$\widetilde{\mathcal{L}}\mathcal{Y} = 0, \qquad r\mathcal{Y} = E. \tag{6.99}$$

Let us demonstrate that finding the elements y_i of the fundamental vector is reduced to solving $n + mn$ the Cauchy problems for an equation in the space $\mathbf{D}^n$. Denote

$$X(t) = (x_1, \dots, x_{n+mn}) = \mathcal{Y}(t) - Y(t). \tag{6.100}$$

Clearly $\widetilde{\mathcal{L}}X = -\widetilde{\mathcal{L}}Y$ and $rX = 0$. This and the representation (3.9),

$$\widetilde{\mathcal{L}}y = Q\dot{y} + A_0 y(0) + \sum_{i=1}^{m} A_i \Delta y(t_i), \tag{6.101}$$

imply that each element x_i, $i = 1, \dots, n + mn$, of X is the solution of the problem

$$(\widetilde{\mathcal{L}}x_i)(t) = -a_i(t), \quad t \in [0,1],\ x_i(0) = 0, \qquad \Delta x_i(t_k) = 0, \quad k = 1, \dots, m, \tag{6.102}$$

where $a_i(t)$ is the ith column of the matrix $A(t) = (A_0(t), A_1(t), \dots, A_m(t))$,

$$A_0 = \widetilde{\mathcal{L}}E, \qquad A_i = \widetilde{\mathcal{L}}(\chi_{[t_i,1]}E). \tag{6.103}$$

Denoting by $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ the restriction of the operator $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ on the space $\mathbf{D}^n$,

$$\mathcal{L}x = Q\dot{x} + A_0 x(0), \tag{6.104}$$

we can define the element x_i as the solution of the Cauchy problem

$$(\mathcal{L}x_i)(t) = -a_i(t), \quad t \in [0,1],\ x_i(0) = 0. \tag{6.105}$$

After solving $n + mn$ such problems we obtain the fundamental vector $\mathcal{Y}$:

$$\mathcal{Y}(t) = Y(t) + X(t). \tag{6.106}$$

In view of the above consideration, the constructive study of boundary value problems for the unique solvability in the space $\mathbf{DS}^n(m)$ requires only some minimum and evident modification to the corresponding procedures used in the case of the space $\mathbf{D}^n$ (see Section 6.4). To illustrate the said, consider in the space $\mathbf{DS}^n(m)$ the boundary value problem (compare with (6.61))

$$
\begin{aligned}
(\widetilde{\mathcal{L}}y)^i(t) &\stackrel{\text{def}}{=} \dot{y}^i(t) + \sum_{j=1}^{n} p_{ij}(t)y_j[h_{ij}(t)] = f^i(t), \quad t \in [0,1],\\
y^i(\xi) &= 0 \quad \text{if } \xi \notin [0,1],\ i = 1,\dots,n,
\end{aligned}
\tag{6.107}
$$

$$
\widetilde{l}y \stackrel{\text{def}}{=} \int_0^1 \Phi(s)\dot{y}(s)ds + \Psi_0 y(0) + \sum_{k=1}^{m} \Psi_k \Delta y(t_k) = \alpha. \tag{6.108}
$$

The parameters of this problem are to be approximated within the class of computable function in the same way as it was described in Subsection 6.3.3. It holds true for the constant matrices $\Psi_1,\dots,\Psi_m$ too, which are not included in the description of problem (6.61). Here criterion (6.4) has the form $\det \widetilde{l}\mathcal{Y} \neq 0$. Notice that

$$
\widetilde{l}\mathcal{Y} = \widetilde{l}Y + \widetilde{l}X = \sum_{k=0}^{m} \Psi_k + \int_0^1 \Phi(s)\dot{X}(s)ds, \tag{6.109}
$$

thus, for the constructive checking of the criterion, we can use the considerations of Subsection 6.3.3 taking into account the case when the restriction of $\widetilde{\mathcal{L}}$ onto $\mathbf{D}^n$ has the same form as in (6.61).

6.5. Boundary value problem for a singular equation

The key condition for the applicability of constructive Theorem 6.1 is the unique solvability of the principal boundary value problem (6.3) for any $f \in \mathbf{B}$ and $\alpha \in \mathbb{R}^n$. In case this condition is fulfilled, the main problem of the constructive study of the general boundary value problem is the construction of an approximate fundamental vector with sufficiently high guaranteed accuracy (step 2 of the scheme (1)–(7)). In all above-considered cases of applying the general scheme, the principal boundary value problem was taken as the Cauchy problem. In this section, we consider a possibility of constructive studying of the principal boundary value problem different from the Cauchy problem as well as constructing an approximate fundamental vector as applied to the equation

$$
(\mathcal{L}x)(t) \stackrel{\text{def}}{=} t(1-t)\ddot{x}(t) + p(t)(S_h x)(t) = f(t), \quad t \in [0,1], \tag{6.110}
$$

with a given measurable function h and summable p, f. We consider equation (6.110) in the space $\mathbf{D}_\pi \simeq \mathbf{L} \times \mathbb{R}^2$ entered in Section 4.2. $\mathbf{D}_\pi$ is the space of all functions $x : [0,1] \to \mathbb{R}^1$ possessing the properties as follows:

(1) function x is absolutely continuous on $[0,1]$,
(2) the derivative $\dot{x}$ is absolutely continuous on every $[c,d] \subset (0,1)$,
(3) the product $t(1-t)\ddot{x}(t)$ is summable on $[0,1]$.

We use the isomorphism (4.22) $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L} \times \mathbb{R}^2 \to \mathbf{D}_\pi$,

$$(\Lambda z)(t) = \int_0^1 \Lambda(t,s)z(s)ds, \qquad \Lambda(t,s) = \begin{cases} \dfrac{(t-1)}{(1-s)}, & 0 \le s \le t \le 1, \\ -\dfrac{t}{s}, & 0 \le t < s \le 1, \end{cases} \tag{6.111}$$
$$(Y\beta)(t) = (1-t)\beta^1 + t\beta^2, \quad \beta = \operatorname{col}\{\beta^1, \beta^2\}.$$

In this case $\mathcal{J}^{-1} = [\delta, r]$,

$$(\delta x)(t) = t(1-t)\ddot{x}(t), \qquad rx = \operatorname{col}\{x(0), x(1)\}. \tag{6.112}$$

The norm in the space $\mathbf{D}_\pi$ is defined by

$$\|x\|_{\mathbf{D}_\pi} = \|\delta x\|_{\mathbf{L}} + |x(0)| + |x(1)|. \tag{6.113}$$

Under such isomorphism, the principal boundary value problem is the problem

$$\mathcal{L}x = f, \qquad x(0) = \beta^1, \qquad x(1) = \beta^2. \tag{6.114}$$

As is demonstrated in Section 4.2.1, the operator $\mathcal{L} : \mathbf{D}_\pi \to \mathbf{L}$ is Noether, and

$$(Qz)(t) \stackrel{\text{def}}{=} (\mathcal{L}\Lambda z)(t) = z(t) - (Kz)(t), \tag{6.115}$$

where $K : \mathbf{L} \to \mathbf{L}$ is defined by

$$(Kz)(t) = \int_0^1 K(t,s)z(s)ds \tag{6.116}$$

with the kernel

$$K(t,s) = -p(t)\Lambda[h(t),s]. \tag{6.117}$$

Equation (6.110) can be written in the form

$$\begin{aligned} t(1-t)\ddot{x}(t) - \int_0^1 K(t,s)[s(1-s)\ddot{x}(s)]ds + p(t)[1-h(t)]\sigma_h(t)x(0) \\ + p(t)h(t)\sigma_h(t)x(1) = f(t), \quad t \in [0,1], \end{aligned} \tag{6.118}$$

where

$$\sigma_h(t) = \begin{cases} 1 & \text{if } h(t) \in [0,1], \\ 0 & \text{if } h(t) \notin [0,1]. \end{cases} \tag{6.119}$$

Here and in what follows the function $\Lambda(t,s)$ is equal to zero outside the square $[0,1] \times [0,1]$. The operator $K : \mathbf{L} \to \mathbf{L}$ is compact (Theorem B.1) and, hence, $Q : \mathbf{L} \to \mathbf{L}$ is a canonical Fredholm operator. The invertibility of this operator is a criterion of the unique solvability of principal boundary value problem (6.114) for any $f \in \mathbf{L}^1$, $\beta^1, \beta^2 \in \mathbb{R}$ (Theorem 1.16). The standard conditions for the invertibility of Q of the form $\|K\|_{\mathbf{L}\to\mathbf{L}} < 1$ or $\rho(K) < 1$ can be too rough to be useful for the study of concrete boundary value problems. Our constructive approach enables us to extend essentially the possibilities for establishing the invertibility of $I - K$.

Fix $\varepsilon \in (0,1)$. Let

$$\widetilde{K}(t,s) = \sum_{i=1}^{N} u_i(t)v_i(s) \tag{6.120}$$

be a degenerate kernel with measurable essentially bounded functions v_i and summable u_i such that

$$\operatorname*{ess\,sup}_{s\in[0,1]} \int_0^1 \left| K(t,s) - \widetilde{K}(t,s) \right| dt \le \varepsilon. \tag{6.121}$$

Next, let the $N \times N$ matrix

$$E - A, \qquad A - \{a_{ij}\}, \quad a_{ij} = \int_0^1 v_i(t)u_j(t)dt, \ i,j = 1,\dots,N, \tag{6.122}$$

be invertible.

Denote by b_{ij} the elements of the matrix $B = (E - A)^{-1}$ and

$$\widetilde{H}(t,s) = \sum_{j=1}^{N}\sum_{i=1}^{N} u_i(t)b_{ij}v_j(s). \tag{6.123}$$

The function $\widetilde{H}(t,s)$ is the resolvent kernel of $\widetilde{K}(t,s)$: for each $f \in \mathbf{L}$, the unique solution of the equation

$$z(t) - \int_0^1 \widetilde{K}(t,s)z(s)ds = f(t), \quad t \in [0,1], \tag{6.124}$$

is the function

$$\widetilde{z}(t) = f(t) + \int_0^1 \widetilde{H}(t,s)f(s)ds. \tag{6.125}$$

Let d be such that

$$\operatorname*{ess\,sup}_{s} \int_0^1 \left| \widetilde{H}(t,s) \right| dt \le d. \tag{6.126}$$

If the inequality

$$\varepsilon < \frac{1}{1+d} \tag{6.127}$$

holds, then due to the theorem on invertible operator (see, e.g., [100, Theorem 3.6.3]) the operator $I - K$ has the bounded inverse and the principal boundary value problem (6.117) is uniquely solvable for any $f \in \mathbf{L}$, $\beta^1, \beta^2 \in \mathbb{R}$. Notice that inequality (6.127) can be checked with the computing experiment if the functions u_i, v_i, $i = 1, \ldots, N$, are computable.

Assuming condition (6.127) is fulfilled, consider the question on constructing an approximation to fundamental vector $X = (x_1, x_2)$ of the equation $\mathcal{L}x = 0$ ($\mathcal{L}X = 0$, $rX = E$) with a guaranteed error bound. The element x_1 is the solution of the problem

$$\mathcal{L}x = 0, \qquad x(0) = 1, \qquad x(1) = 0 \tag{6.128}$$

and has the representation

$$x_1(t) = (\Lambda z_1)(t) + (1 - t), \tag{6.129}$$

where $z_1(t)$ is the solution of the equation

$$z(t) - \int_0^1 K(t,s)z(s)ds = -p(t)[1 - h(t)]\sigma_h(t), \quad t \in [0,1], \tag{6.130}$$

$$\sigma_h(t) = \begin{cases} 1 & \text{if } h(t) \in [0,1], \\ 0 & \text{if } h(t) \notin [0,1]. \end{cases} \tag{6.131}$$

For x_2 we have the problem

$$\mathcal{L}x = 0, \qquad x(0) = 0, \qquad x(1) = 1 \tag{6.132}$$

and, therefore,

$$x_2(t) = (\Lambda z_2)(t) + t, \tag{6.133}$$

where $z_2(t)$ is the solution of the equation

$$z(t) - \int_0^1 K(t,s)z(s)ds = -p(t)h(t)\sigma_h(t), \quad t \in [0,1]. \tag{6.134}$$

Denoting by $\widetilde{z}_1$ ($\widetilde{z}_2$) the solution of (6.130) (of (6.134)), where the kernel $K(t,s)$ is replaced by $\widetilde{K}(t,s)$, we obtain due to the known estimate (see, e.g., [100, Theorem 3.6.3]), taking place under the conditions of the invertible operator theorem, the following inequalities:

$$\begin{aligned}\int_0^1 |z_1(t)-\widetilde{z}_1(t)|\,dt &\le \varepsilon\frac{(1+d)^2}{1-\varepsilon(1+d)}\int_0^1 |p(t)|(1-h(t))\sigma_h(t)dt \stackrel{\text{def}}{=} \varepsilon_1,\\ \int_0^1 |z_2(t)-\widetilde{z}_2(t)|\,dt &\le \varepsilon\frac{(1+d)^2}{1-\varepsilon(1+d)}\int_0^1 |p(t)|h(t)\sigma_h(t)dt \stackrel{\text{def}}{=} \varepsilon_2.\end{aligned} \tag{6.135}$$

These estimates allow us to obtain a guaranteed error bound for the approximate fundamental vector $(\widetilde{x}_1,\widetilde{x}_2)$.

Now consider the boundary value problem for equation (6.110) with the general linear boundary conditions

$$l^i x = \alpha^i, \quad i=1,2. \tag{6.136}$$

The linear bounded functional $l^i : \mathbf{D}_\pi \to \mathbb{R}$ has the representation

$$l^i x = \int_0^1 \varphi_i(t)t(1-t)\ddot{x}(t)dt + \psi_{i1}x(0) + \psi_{i2}x(1), \tag{6.137}$$

where function φ_i is measurable and essentially bounded on $[0,1]$, $\psi_{i,1},\psi_{i,2}$ = const.

A criterion of the unique solvability of (6.110), (6.136) is the invertibility of the matrix $\{l^i x_j\}$, $i,j = 1,2$. Estimates (6.135) together with the possibility of constructing functions $\widetilde{z}_1$ and $\widetilde{z}_2$ allow us to check efficiently this criterion. Indeed, the presentation (6.137) implies

$$l^i x_j = \int_0^1 \varphi_i(t)z_j(t)dt + \psi_{ij}. \tag{6.138}$$

Denote

$$\theta = \{\vartheta_{ij}\}, \quad \vartheta_{ij} = \int_0^1 \varphi_i(t)\widetilde{z}_j(t)\,dt + \psi_{ij},\ i,j=1,2. \tag{6.139}$$

The estimates (6.135) imply

$$|\vartheta_{ij} - l^i x_j| \le \varepsilon_j \cdot q_i, \tag{6.140}$$

where $q_i \ge \operatorname{ess\,sup}_{t\in[0,1]} |\varphi_i(t)|$.

If the matrix θ is invertible and

$$\max\{q_1,q_2\}(\varepsilon_1+\varepsilon_2) < \frac{1}{||\theta^{-1}||}, \tag{6.141}$$

then the matrix $\{l^i x_j\}$ of problem (6.110), (6.136) is invertible too, that is, this problem is uniquely solvable.

Notice that the computer experiment realizing the above scheme allows us to recognize the unique solvability for any uniquely solvable problem such that its kernel $K(t,s)$ can be approximated with such an accuracy as we wish by the kernels $\widetilde{K}(t,s)$ with computable functions u_i, v_i and the functions φ_1, φ_2 can be approximated by computable functions with any required accuracy in the uniform metric.

Example 6.10. Consider problem (6.110), (6.136) with the functional l^i defined by (6.137) and $p(t) \equiv 10$,

$$h(t) = \begin{cases} t - \dfrac{1}{4}, & t \in [0, 0.25),\ \varphi_1(t) = \chi_{[0.26,0.49]\cup[0.51,0.74]}(t) \\ \dfrac{100}{8}t - 3, & t \in [0.25, 0.26),\ \varphi_1(t) = \chi_{[0.26,0.49]\cup[0.51,0.74]}(t), \\ \dfrac{1}{4}, & t \in [0.26, 0.49),\ \varphi_2(t) = \chi_{[0.51,0.74]}(t) \\ 25t - 12, & t \in [0.49, 0.51),\ \varphi_2(t) = \chi_{[0.51,0.74]}(t), \\ \dfrac{3}{4}, & t \in [0.51, 0.74),\ \psi_{11} = 6,\ \psi_{12} = -0.6, \\ \dfrac{100}{8}t - \dfrac{17}{2}, & t \in [0.74, 0.75),\ \psi_{11} = 6,\ \psi_{12} = -0.6, \\ t + \dfrac{1}{4}, & t \in [0.75, 1],\ \psi_{21} = -4.5,\ \psi_{22} = 13.6. \end{cases} \tag{6.142}$$

We will follow the above scheme of the study. Define the kernel $\widetilde{K}(t,s)$ by (6.120), where

$$\begin{aligned} u_1(t) &= -10\chi_{[1/4,1/2]}(t), \qquad u_2(t) = -10\chi_{[1/2,3/4]}(t), \\ v_1(s) &= -\frac{3}{4(1-s)}\chi_{[0,1/4]}(s) - \frac{1}{4s}\chi_{[1/4,1]}(s), \\ v_2(s) &= -\frac{1}{4(1-s)}\chi_{[0,3/4]}(s) - \frac{3}{4s}\chi_{[3/4,1]}(s). \end{aligned} \tag{6.143}$$

Here ε, the error bound of the approximation to $K(t,s)$ (6.114) defined by (6.121), is no greater than 0.2. The resolvent kernel $\widetilde{H}(t,s)$ is defined by (6.123), where $b_{11} = b_{22} = 1.123$, $b_{12} = b_{21} = -1.554$, and $d < 1.1$. Next

$$\begin{aligned} \widetilde{z}_1(t) &= -10[1 - h(t)]\chi_{[1/4,3/4]}(t) + 1.7\chi_{[1/4,1/2]}(t) + 26.7\chi_{[1/2,3/4]}(t), \\ \widetilde{z}_2(t) &= -10h(t)\chi_{[1/4,3/4]}(t) + 16.6\chi_{[1/4,1/2]}(t) - 8.4\chi_{[1/2,3/4]}(t), \end{aligned} \tag{6.144}$$

and $\varepsilon_1 + \varepsilon_2 \le 7.604$ (see (6.135)). The matrix θ (6.139) is defined by

$$\theta = \begin{pmatrix} 10.232 & -1.014 \\ 1.066 & 9.943 \end{pmatrix}. \tag{6.145}$$

Thus

$$\theta^{-1} = \frac{1}{10.232 \cdot 9.943 + 1.066 \cdot 1.014} \begin{pmatrix} 9.943 & 1.014 \\ -1.066 & 10.232 \end{pmatrix}, \tag{6.146}$$

and $\|\theta^{-1}\| < 0.113$, $1/\|\theta^{-1}\| > 8.5$.

Since in this example $q_1 = q_2 = 1$, inequality (6.141)

$$\varepsilon_1 + \varepsilon_2 \le 7.604 < 8.5 < \frac{1}{\|\theta^{-1}\|} \tag{6.147}$$

holds and, consequently, the boundary value problem under consideration is uniquely solvable for any $f \in \mathbf{L}$ and $\alpha^1, \alpha^2 \in \mathbb{R}$.

In conclusion it may be said that in this example, $\|K\|_{\mathbf{L}\to\mathbf{L}} > 3$.

6.6. The Cauchy matrix and a posteriori error bounds

Efficiency in realizing the scheme (1)–(7) depends essentially on fineness of guaranteed error bounds for the approximate fundamental vector. For a posteriori error bounds obtained by computing (or estimating) the defect, this fineness is defined either by the accuracy in solving the corresponding operator (most often, integral) inequality for the error or by the exactness of the estimate for the norm of the Green operator G_0 to the principal boundary value problem (6.3) (see Theorem 6.1).

Consider the possibility of constructing the mentioned a posteriori error bounds as applied to the equation

$$\mathcal{L}x = f, \tag{6.148}$$

with linear bounded operator $\mathcal{L} : \mathbf{D}^n \to \mathbf{L}^n$ (in this section $\mathbf{L}^n = \mathbf{L}^n[0,T]$, $\mathbf{D}^n = \mathbf{D}^n[0,T]$, $(\Lambda z)(t) = \int_0^t z(s)ds$, $(Y\alpha)(t) = E\alpha$, $\delta x = \dot{x}$, $rx = x(0)$) having the principal part $Q : \mathbf{L}^n \to \mathbf{L}^n$ of the form

$$(Qz)(t) = z(t) - \int_0^t K(t,s)z(s)ds, \tag{6.149}$$

where the elements $k^{ij}(t,s)$ of the kernel $K(t,s)$ are measurable in the triangle $0 \le s \le t \le T$ and satisfy the inequalities

$$|k^{ij}(t,s)| \le \mu(t), \quad \mu \in \mathbf{L}^1,\ i,j = 1,\dots,n. \tag{6.150}$$

Recall that in this event equation (6.148) covers as the special cases the equations with concentrated or distributed delay (see Section 2.2). The operator Q has the bounded inverse Q^{-1}:

$$(Q^{-1}z)(t) = z(t) + \int_0^t H(t,s)z(s)ds, \tag{6.151}$$

where $H(t,s)$ is the resolvent kernel for $K(t,s)$. The principal boundary value problem (here the Cauchy problem)

$$\mathcal{L}x = f, \quad x(0) = \alpha \tag{6.152}$$

is uniquely solvable for any $f \in \mathbf{L}^n$ and $\alpha \in \mathbb{R}^n$. Problem (6.5) for the fundamental vector $X = (x_1, \dots, x_n)$ has the form

$$\mathcal{L}X = 0, \quad X(0) = E. \tag{6.153}$$

Thus the column x_i is the solution of the problem

$$\mathcal{L}x = 0, \quad x(0) = e_i, \tag{6.154}$$

where e_i is the ith column of the identity $n \times n$ matrix. The solution x of the problem (6.152) has the form

$$x(t) = X(t)\alpha + \int_0^t C(t,s)f(s)ds, \tag{6.155}$$

where $C(t,s)$ is the Cauchy matrix possessing the following properties (see Subsection 2.2.3):

$$C_t'(t,s) = H(t,s), \quad 0 \le s \le t \le T, \tag{6.156}$$

$$C_t'(t,s) = \int_s^t C_t'(t,\tau)K(\tau,s)d\tau + K(t,s), \quad 0 \le s \le t \le T, \tag{6.157}$$

$$C(t,s) = E + \int_s^t C_\tau'(\tau,s)d\tau, \quad 0 \le s \le t \le T. \tag{6.158}$$

Let $X^a = (x_1^a, \dots, x_n^a)$ be an approximation of the fundamental vector:

$$\mathcal{L}x_i^a = \Delta_i, \quad x_i^a(0) = e_i. \tag{6.159}$$

A simple way of constructing the estimate of the error $y_i = (x_i - x_i^a)$ is as follows. For $z_i = \dot{y}_i$ we have

$$z_i(t) = \int_0^t K(t,s)z_i(s)ds + \Delta_i(t), \tag{6.160}$$

that implies

$$|z_i(t)| \le \int_0^t \|K(t,s)\| \cdot |z_i(s)|\,ds + |\Delta_i(t)|, \tag{6.161}$$

and, by the theorem on integral inequality, we obtain

$$|z_i(t)| \le m(t)\int_0^t \exp\left(\int_s^t m(\tau)d\tau\right)|\Delta_i(s)|\,ds + |\Delta_i(t)|, \tag{6.162}$$

where $m(t) = \|\{\mu^{ij}(t)\}\|$, $\mu^{ij}(t) = \mu(t)$, $i, j = 1\dots, n$. Hence

$$|y_i(t)| \le \int_0^t m(\tau)\exp\left(\int_0^\tau m(\xi)d\xi\right)\int_0^\tau |\Delta_i(s)|\,ds\,d\tau + \int_0^t |\Delta_i(s)|\,ds, \quad t \in [0,T]. \tag{6.163}$$

The presence of exponential factor in the right-hand side of (6.163) indicates that even for modest values n and T (say, $n = 10$, $T = 5$) estimate (6.163) can be highly overstated with respect to actual values of error.

Consider now an alternate way of constructing an error bound. This way is based on constructing an approximation $\widetilde{C}(t,s)$ of the Cauchy matrix $C(t,s)$ such that the norm $\|C - \widetilde{C}\|_{\mathbf{L}^n \to \mathbf{L}^n_\infty}$ is no greater than a given ε_C; here $C, \widetilde{C} : \mathbf{L}^n \to \mathbf{L}^n_\infty$ are linear integral Volterra operators with kernels $C(t,s)$ and $\widetilde{C}(t,s)$, respectively. In this event, for $|y_i(t)|$ we have the estimate

$$|y_i(t)| \le \int_0^t \|\widetilde{C}(t,s)\| \cdot |\Delta_i(s)|\,ds + \varepsilon_C \cdot \int_0^T |\Delta_i(s)|\,ds, \quad t \in [0,T]. \tag{6.164}$$

This estimate is essentially more accurate than (6.163) if it is possible to construct a sufficiently good approximation $\widetilde{C}(t,s)$. We will describe an efficiently realizable way of constructing such an approximation under the condition that the kernel $K(t,s)$ admits a piecewise constant approximation being as accurate as we wish. This way can be extended to more wide classes of kernels.

Split the segment $[0,T]$ on $N+1$ equal parts by the points $0 = t_0 < t_1 < \cdots < t_{N+1} = T$ and denote $t_{i+1} - t_i = \hbar$.

Next, on every square

$$\Box_{ij} \overset{\text{def}}{=} (t_i, t_{i+1}) \times (t_{j-1}, t_j), \quad i = 1,\dots,N,\ j = 1,\dots,i, \tag{6.165}$$

we replace the matrix $K(t,s)$ by the constant matrix K_{ij} and assume constant $n \times n$ matrices ΔK_{ij} to be known such that

$$|K(t,s) - K_{ij}| \le \Delta K_{ij}, \quad (t,s) \in \Box_{ij},\ i = 1,\dots,N,\ j = 1,\dots,i. \tag{6.166}$$

Here the symbol $|A|$ for a matrix $A = \{a^{ij}\}$ means the matrix $\{|a^{ij}|\}$. Denote

$$\eta_i(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}], \\ 0, & t \notin [t_i, t_{i+1}], \end{cases} \qquad i = 0, 1, \ldots, N.$$

$$\Gamma = \begin{pmatrix} E & 0 & 0 & \cdots & 0 \\ -\hbar K_{22} & E & 0 & \cdots & 0 \\ -\hbar K_{32} & -\hbar K_{33} & E & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\hbar K_{N2} & -\hbar K_{N3} & -\hbar K_{N4} & \ldots & E \end{pmatrix}, \quad \Gamma^{-1} = \{B_{ij}\},\ i, j = 1, \ldots, N,$$

$$\widetilde{K}(t,s) = K_{ij}, \quad (t,s) \in \square_{ij},\ i = 1, \ldots, N;\ j = 1, \ldots, i. \tag{6.167}$$

The resolvent kernel $\widetilde{H}(t,s)$ for $\widetilde{K}(t,s)$ can be found in the explicit form (see, e.g., Maksimov et al. [151]):

$$\widetilde{H}(t,s) = \sum_{i=1}^{N} \eta_i(t) \sum_{k=1}^{i} \Xi_{ik} \eta_{k-1}(s), \tag{6.168}$$

where

$$\Xi_{ik} = \sum_{j=k}^{i} B_{ij} K_{jk}. \tag{6.169}$$

Define the matrices $\widetilde{C}'_t(t,s)$ and $\widetilde{C}(t,s)$ by the equalities

$$\widetilde{C}'_t(t,s) \stackrel{\text{def}}{=} \widetilde{H}(t,s), \qquad \widetilde{C}(t,s) = E + \int_s^t \widetilde{H}(\tau,s)\, d\tau. \tag{6.170}$$

Also define the linear operators $K, \widetilde{K}, \widetilde{H}, \Delta K : \mathbf{L}^n \to \mathbf{L}^n$ as integral operators with the kernels $K(t,s)$ $\widetilde{K}(t,s)$, $\widetilde{H}(t,s)$ and $[\widetilde{K}(t,s) - K(t,s)]$, respectively.

The inequality

$$q \stackrel{\text{def}}{=} ||\Delta K(I + \widetilde{H})||_{\mathbf{L}^n \to \mathbf{L}^n} < 1 \tag{6.171}$$

allows us, applying the theorem on invertible operator, to obtain the estimate

$$\|C - \widetilde{C}\|_{\mathbf{L}^n \to \mathbf{L}^n_\infty} \le \|H - \widetilde{H}\|_{\mathbf{L}^n \to \mathbf{L}^n} \le \frac{q}{1-q} \|I + \widetilde{H}\|_{\mathbf{L}^n \to \mathbf{L}^n}. \tag{6.172}$$

Thus under (6.171) we can replace the constant ε_C in (6.164) by the right-hand side of (6.172). Notice that constants q and $\|I + \widetilde{H}\|_{\mathbf{L}^n \to \mathbf{L}^n}$ can be calculated efficiently with computer.

Example 6.11. Consider the Cauchy problem

$$\begin{aligned} \dot{x}(t) - p(t)x_h(t) = 1, \quad t \in [0,5], \\ x(0) = 0, \end{aligned} \tag{6.173}$$

where $p(t) = \eta_1(t) - 2\eta_2(t) - 2\eta_3(t) + 3\eta_4(t) - \eta_6(t) - \eta_7(t) + 4\eta_8(t) + 4\eta_9(t)$,

$$\begin{aligned} h(t) &= 0.4\eta_1(t) + 0.9\eta_2(t) + 0.1\eta_3(t) + 0.7\eta_4(t) \\ &\quad - \eta_5(t) + 0.2\eta_6(t) + \eta_7(t) + 2\eta_8(t) + 3\eta_9(t), \\ \eta_i(t) &= \chi_{[0.5i;0.5(i+1)]}(t), \quad i = 1,\dots,9. \end{aligned} \tag{6.174}$$

Let $x^a(t)$ be an approximate solution of (6.173) giving the defect $\Delta(t)$ with the estimate

$$|\Delta(t)| \le \varepsilon, \quad t \in [0,5]. \tag{6.175}$$

Here a posteriori estimate (6.163) has the form

$$|x^a(t) - x(t)| \le \frac{5}{9}(e^9 - 1)\varepsilon, \quad t \in [0,5]. \tag{6.176}$$

The estimate obtained with (6.172) is as follows:

$$|x^a(t) - x(t)| \le 165\varepsilon, \quad t \in [0,5] \tag{6.177}$$

(in this case the estimate $\|\widetilde{H}\|_{\mathbf{L}^1\to\mathbf{L}^1} \le 10$ is used, it is obtained in the way as it was described above).

6.7. Other applications of the constructive approach

Considering the problems in this section, we restrict ourselves to brief description of the scheme for reducing an original problem to a finite-dimensional one and to discussing some details of the realization of the constructive approach, and present some illustrative examples.

6.7.1. Boundary value problems with boundary inequalities

Consider the problem

$$\widetilde{\mathcal{L}}y = f, \qquad \widetilde{l}y \le \beta, \quad \beta \in \mathbb{R}^N, \tag{6.178}$$

following the notations and the assumptions of Section 6.4 with respect the operator $\widetilde{\mathcal{L}} : \mathbf{DS}^n(m) \to \mathbf{L}^n$ and the components $\widetilde{l}^i : \mathbf{DS}^n(m) \to \mathbb{R}^1$ of the vector functional $\widetilde{l} : \mathbf{DS}^n(m) \to \mathbb{R}^N$. Point out that here the number of inequalities, N, in the boundary conditions is fixed and does not connect with the dimension n and the number m of the possible break-points.

The general solution of equation $\widetilde{\mathcal{L}}y = f$ has the form

$$y(t) = \mathcal{Y}(t)\alpha + g(t), \tag{6.179}$$

where $\alpha \in \mathbb{R}^{n+mn}$, $\mathcal{Y} = (y_1, \ldots, y_{n+mn})$ $(r\mathcal{Y} = E)$ is the fundamental vector of the homogeneous equation $\widetilde{\mathcal{L}}y = 0$, and g is the solution of the principal boundary value problem

$$\widetilde{\mathcal{L}}y = f, \qquad ry = 0. \tag{6.180}$$

In the view of considerations of Sections 6.3, 6.4, and 6.6 we suppose $N \times (n+mn)$ matrices $\underline{M}$, $\overline{M}$ and vectors $\underline{d}, \overline{d} \in \mathbb{R}^N$ to be known such that

$$\underline{M} \leq \widetilde{l}\mathcal{Y} \leq \overline{M}, \tag{6.181}$$

$$\underline{d} \leq \widetilde{l}g \leq \overline{d}. \tag{6.182}$$

Theorem 6.12. *Let there exist a vector $c \in \mathbb{R}^{n+mn}$ with nonnegative components such that the system of linear inequalities*

$$\overline{M}\gamma \leq \beta + \underline{M}c - \overline{d} \tag{6.183}$$

has a nonnegative solution $\overline{\gamma} \in \mathbb{R}^{n+mn}$. Then problem (6.178) *has a solution $\overline{y} \in$ $\mathbf{DS}^n(m)$ being the solution of the principal boundary value problem*

$$\widetilde{\mathcal{L}}y = f, \quad ry = \overline{\gamma} - c. \tag{6.184}$$

Proof. Let $\overline{\gamma} \in \mathbb{R}^{n+mn}$, $\overline{\gamma} \geq 0$ be such that

$$\overline{M}\overline{\gamma} \leq \beta + \underline{M}c - \overline{d}. \tag{6.185}$$

This implies due to (6.181), (6.182) that

$$\begin{aligned} \widetilde{l}\mathcal{Y} \cdot \overline{\gamma} &\leq \beta + \widetilde{l}\mathcal{Y} \cdot c - \widetilde{l}g, \\ \widetilde{l}\mathcal{Y} \cdot (\overline{\gamma} - c) &\leq \beta - \widetilde{l}g. \end{aligned} \tag{6.186}$$

Substituting $\overline{\alpha} = \overline{\gamma} - c$ in (6.179), we conclude that

$$\overline{y}(t) = \mathcal{Y}(t) \cdot \overline{\alpha} + g(t) \tag{6.187}$$

satisfies boundary conditions (6.1). Since $rg = 0$, we have

$$r\overline{y} = r\overline{\alpha} = r(\overline{\gamma} - c). \tag{6.188}$$

□

To investigate system (6.183), one can use the standard possibilities of Maple.

Example 6.13. Consider on the segment $[0,5]$ the system

$$\begin{aligned}
\dot y_1(t) &= -\lambda(t)\times y_1(t)+\mu(t)\times K(t)\times v_0,\\
\dot y_2(t) &= 0.041y_1(t)-0.231y_2(t)+\big(2.7L(t)-0.73KV(t)-273.3t+3664.3\big)\cdot v_0,\\
\dot y_3(t) &= 0.0443y_1(t)-0.1041y_3(t)\\
&\quad+\big(0.5727L(t)-0.45888KA(t)+0.9853t-220\big)\cdot v_0,\\
\dot y_4(t) &= 0.02957y_1(t)+0.1823y_2(t)+0.346y_3(t)\\
&\quad-0.643y_4(t)-(329.17+25.68t)\cdot v_0,\\
\dot y_5(t) &= 0.0834y_1(t)+0.0938y_2(t)+0.01304y_3(t)\\
&\quad-0.04845y_5(t)-(5.636t+88.643)\cdot v_0,
\end{aligned} \tag{6.189}$$

where $v_0 = 0.001$, $\lambda(t) = 0.03+0.2t$, $\mu(t) = 0.98+0.3t$,

$$\begin{gathered}
K(t) = 356.36+32.997t-0.223t^2,\qquad L(t) = 561.86+8.13t-0.497t^2,\\
KV(t) = 19.45+4.81t,\qquad KA(t) = 0.58+6.13t,
\end{gathered} \tag{6.190}$$

with 23 boundary conditions

$$\begin{gathered}
0\le y_4(5)\le 1,\qquad 1\le y_5(5)\le 2,\qquad y_2(5)\le 1,\qquad y_3(5)\le 1,\\
\int_0^5 2.635y_1(s)ds\le 50,\qquad 0.5\le y_1(0)\le 5,\qquad 0.5\le y_2(0)\le 5,\\
0.5\le y_3(0)\le 0.8,\qquad 0.5\le y_4(0)\le 5,\qquad 0.5\le y_5(0)\le 5,\\
\Delta y_i(2)\le 0,\quad \Delta y_i(3)\le 0,\quad \Delta y_i(4.8)\le 0,\quad i=1,2.
\end{gathered} \tag{6.191}$$

For this problem in the space $\mathbf{DS}^5[0,2,3,4.8,5]$ the unique solvability is established and there are found the initial values of all components of a solution (admissible trajectory):

$$y_1(0)=0.5,\qquad y_2(0)=0.5,\qquad y_3(0)=0.5,\qquad y_4(0)=0.5,\qquad y_5(0)=0.5, \tag{6.192}$$

as well as the values of all jumps:

$$\begin{aligned}
&\Delta y_1(2)=-1.21,\qquad \Delta y_1(3)=-0.4,\qquad \Delta y_1(4.8)=-0.8,\\
&\Delta y_2(2)=-8.2,\qquad \Delta y_2(3)=-4.4,\qquad \Delta y_2(4.8)=-6.1.
\end{aligned} \tag{6.193}$$

Notice that the problem (6.189), (6.191) is a model of the so-called problem of impulsive control for an ecological situation (see [148]), where y_1 is the volume of equipment funds of the region's industry, y_2 is the substance dispersion level of the water resources, y_3 is the substance dispersion level of atmosphere, y_4 is the sick rate of respiration organs (of population), and y_5 is the sick rate of digestion organs. All values are presented in some conventional units of measurement.

6.7.2. The control problem

Turn back to the control problem considered in Subsection 2.2.4:

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) + \int_0^t d_sR(t,s)x(s) = v(t) + (Bu)(t), \quad t \in [0,T], \tag{6.194}$$

$$x(0) = \alpha, \qquad lx \stackrel{\text{def}}{=} \Psi x(0) + \int_0^T \Phi(s)\dot{x}(s)ds = \beta, \tag{6.195}$$

assuming that the parameters of the problem admit a sufficiently accurate approximation within the class of computable operators and functions. It is required to find a control $u : [0,T] \to \mathbb{R}^r$, $u \in \mathbf{L}_2^r$ ($\mathbf{L}_2^r$ is the space of r-vector functions square summable on $[0,T]$) such that the boundary value problem (6.194), (6.195) with such u has the solution $x \in \mathbf{D}^n$. As is shown in Subsection 2.2.4, a criterion of the solvability of the control problem is the invertibility of the $n \times n$ matrix

$$M = \int_0^T [B^*\theta](\tau)[B^*\theta]^\top(\tau)d\tau. \tag{6.196}$$

Here $B^* : (\mathbf{L}^n)^* \to (\mathbf{L}_2^r)^*$ is the adjoint operator to $B : \mathbf{L}_2^r \to \mathbf{L}^n$,

$$\theta(s) = \Phi(s) + \int_s^T \Phi(\tau)C'_\tau(\tau,s)d\tau, \tag{6.197}$$

$C(t,s)$ is the Cauchy matrix of the operator $\mathcal{L}$, and $\cdot^\top$ is the symbol of transposition.

A matrix M_a approximating M with the accuracy M_v,

$$\left\| M - M_a \right\| \leq M_v, \tag{6.198}$$

can be constructed on the base of approximations of operator B^*, matrix $\Phi(s)$ and matrix $C(t,s)$ by a computable operator B_a^*, matrix $\Phi_a(s)$ with computable elements, and matrix $\widetilde{C}(t,s)$, respectively. Therewith we can construct $\widetilde{C}(t,s)$ approximating $C(t,s)$ with a guaranteed error bound in the way described in Section 6.5.

The invertibility of M under the estimate (6.198) with invertible M_a is provided by the inequality

$$||M_v|| < \frac{1}{||M_a^{-1}||}. \tag{6.199}$$

For the case $lx \stackrel{\text{def}}{=} x(T)$, some details of the constructive study of (6.194), (6.195) as well as illustrative examples can be found in [118].

The constructive study of the control problem for the solvability can be done easier in the case $(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) - P(t)x(t)$, $(Bu)(t) \stackrel{\text{def}}{=} B(t)u(t)$, and $lx \stackrel{\text{def}}{=} x(T)$, with respect to the classical setting of the control problem.

Consider some details of the constructive study of such a problem as applied to the system

$$\dot{x}(t) - P(t)x(t) = v(t) + B(t)u(t), \quad t \in [0,1], \tag{6.200}$$

where the elements of $n \times n$ matrix P and function v are summable on $[0,1]$, the elements $b_{ij} : [0,1] \to \mathbb{R}^1$ of $n \times n$ matrix B are piecewise continuous functions with possible breaks of the first kind at the fixed points $\tau_1, \ldots, \tau_{\overline{m}}$ and being continuous from the right at these points. In such a case the matrix M has the form

$$M = \int_0^1 C(1,s)B(s)B^\top(s)C^\top(1,s)ds. \tag{6.201}$$

The Cauchy matrix $C(t,s)$ of the system

$$\dot{x}(t) - P(t)x(t) = 0, \quad t \in [0,1], \tag{6.202}$$

has the form $C(t,s) = X(t)X^{-1}(s)$, where X is the fundamental matrix of (6.202). Thus the invertibility of M (6.201) is equivalent to the invertibility of the matrix

$$W = \int_0^1 \Gamma(s)\Gamma^\top(s)ds, \quad \Gamma(s) = X^{-1}(s)B(s). \tag{6.203}$$

Denote $Y(s) = X^{-1}(s)$. In the way described in Subsection 6.3.3 we define the system of rational points $0 = t_0 < t_1 < \cdots < t_m = 1$; the matrices P_a^i, P_v^i; the matrices Y_a^i, Y_v^i, Y_N^i (similarly to the matrices X_a^i, X_v^i, X_N^i); and the matrices B_a^i, B_v^i, B_M^i (similarly to the matrices Φ_a^i, Φ_v^i, Φ_M^i); $i = 1, \ldots, m$.

Next, define matrices

$$\begin{aligned} \Gamma_a^i(t) &= Y_a^i(t)B_a^i(t), \\ \Gamma_v^i(t) &= \left\{\left\| Y_a^i(t_{i-1}) \right\| + Y_N^i\right\}B_v^i + Y_v^i\{B_M^i + B_v^i\}, \end{aligned} \tag{6.204}$$

such that $\Gamma_v^i \geq \left\| \Gamma(t) - \Gamma_a^i(t) \right\|$, $t \in [t_{i-1}, t_i)$, $i = 1, \ldots, m$.

Finally define matrices

$$W_a = \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \Gamma_a^i(s)[\Gamma_a^i(s)]^\top ds,$$
$$W_v = \sum_{i=1}^{m} \Gamma_N^i[\Gamma_v^i]^\top + \Gamma_v^i[\Gamma_N^i]^\top + (t_i - t_{i-1})\Gamma_v^i[\Gamma_v^i]^\top, \tag{6.205}$$

such that

$$W_v \geq \left| W - W_a \right|. \tag{6.206}$$

The control problem for the system (6.200) is solvable if matrix W_a is invertible and

$$||W_v|| < \frac{1}{||W_a^{-1}||}. \tag{6.207}$$

Example 6.14. With the computer program realizing the proposed scheme, the solvability of the control problem is established for system (6.200), where

$$n = 3, \qquad r = 2, \qquad \mathcal{E}_1 = [0; 0,5), \qquad \mathcal{E}_2 = [0,5; 0,75),$$
$$\mathcal{E}_3 = [0,75; 1], \qquad \mathcal{E}_4 = [0; 0,75), \qquad \mathcal{E}_5 = [0,5; 1],$$
$$p_{11}(t) = p_{22}(t) = p_{33}(t) = t, \quad t \in [0,1],$$
$$p_{12}(t) = \begin{cases} \frac{1}{5}t^2, & t \in \mathcal{E}_1, \\ \frac{1}{12}t, & t \in \mathcal{E}_2, \\ 0, & t \in \mathcal{E}_3, \end{cases} \qquad p_{23}(t) = \begin{cases} \frac{1}{8}t, & t \in \mathcal{E}_1, \\ -\frac{1}{8}t, & t \in \mathcal{E}_2, \\ 0, & t \in \mathcal{E}_3, \end{cases}$$
$$p_{13}(t) = \begin{cases} 0, & t \in \mathcal{E}_4, \\ \frac{1}{10}t, & t \in \mathcal{E}_3, \end{cases} \qquad p_{21}(t) = \begin{cases} 0, & t \in \mathcal{E}_1, \\ -\frac{1}{7}t, & t \in \mathcal{E}_5, \end{cases}$$
$$p_{31}(t) = \begin{cases} -\frac{1}{5}t, & t \in \mathcal{E}_1, \\ 0, & t \in \mathcal{E}_5, \end{cases} \qquad p_{32}(t) = \begin{cases} 0, & t \in \mathcal{E}_1, \\ -\frac{1}{6}t, & t \in \mathcal{E}_5, \end{cases}$$
$$g_{12}(t) = g_{21}(t) = g_{32}(t) = 0, \quad g_{31}(t) = t^3, \quad t \in [0,1],$$

$$g_{11}(t) = \begin{cases} t + \dfrac{1}{2}, & t \in \mathcal{E}_1, \\ -t^2, & t \in \mathcal{E}_5, \end{cases} \qquad g_{22}(t) = \begin{cases} \dfrac{1}{2}t, & t \in \mathcal{E}_4, \\ t^5, & t \in \mathcal{E}_3. \end{cases} \tag{6.208}$$

In this example,

$$5 \cdot 10^{-2} < \frac{1}{||W_a^{-1}||} < 6 \cdot 10^{-2}, \qquad 7 \cdot 10^{-5} < ||W_v|| < 8 \cdot 10^{-5}. \tag{6.209}$$

6.7.3. The study of asymptotic properties of the solutions to the delay systems

Following the paper [160, 161], consider the Cauchy problem for the system of differential equations with concentrated delay

$$\begin{gathered} \dot{x}(t) - P(t)x[h(t)] = f(t), \quad t \in [0, \infty), \\ x(\xi) = \varphi(\xi) \quad \text{if } \xi < 0, \end{gathered} \tag{6.210}$$

$$x(0) = \alpha. \tag{6.211}$$

We assume that the elements of $n \times n$ matrix P and n-vector function $f : [0, \infty) \to \mathbb{R}^n$ are T-periodic ($T > 0$) and summable on the period, function $h : [0, \infty) \to \mathbb{R}^1$ has the form $h(t) = t - \Delta(t)$, $0 \le \Delta(t) \le T$, where $\Delta : [0, \infty) \to [0, T]$ is T-periodic, piecewise continuous with possible breaks of the first kind at fixed points $\tau_1, \dots, \tau_m$ and continuous on the right at these points; initial function $\varphi : [-T, 0) \to \mathbb{R}^n$ is such that the function

$$\varphi^h(t) = \begin{cases} 0 & \text{if } h(t) \ge 0, \\ \varphi[h(t)] & \text{if } h(t) < 0, \end{cases} \tag{6.212}$$

is measurable and essentially bounded on $[0, \infty)$.

Definition 6.15. The solution $x(t, \alpha)$ of problem (6.210), (6.211) is said to be stabilizable to a T-periodic function $y : [0, \infty) \to \mathbb{R}^n$ if

$$\lim_{N \to \infty} \max_{t \in [NT, (N+1)T]} \left| x(t, \alpha) - y(t) \right| = 0. \tag{6.213}$$

Below, a scheme of the study of system (6.210) for the stabilizability of its solution to a T-periodic function is described.

Denote by $C(t, s)$ the Cauchy matrix of the system

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \dot{x}(t) - P(t)x_h(t) = f(t), \quad t \in [0, T], \tag{6.214}$$

where

$$x_h(t) = \begin{cases} x[h(t)] & \text{if } h(t) \in [0, T], \\ 0 & \text{if } h(t) < 0. \end{cases} \tag{6.215}$$

Define the operators A and B acting in the space $\mathbf{C}[0, T]$ of continuous functions $z : [0, T] \to \mathbb{R}^n$ by the equality

$$(Az)(t) = C(t, 0)z(T), \tag{6.216}$$

$$(Bz)(t) = \int_0^t C(t, s)P(s)z_\eta(s)ds, \tag{6.217}$$

where $\eta(t) = h(t) + T$.

As is shown in [160], in case the spectral radius $\rho(A + B)$ of $A + B$ is less than one, every solution $x(t, \alpha)$ of (6.210) is stabilizable to the T-periodic function y being the T-periodic extension on $[0, \infty)$ of the solution $z(t)$ of the equation

$$z = (A + B)z + g, \tag{6.218}$$

with $g(t) = \int_0^t C(t, s) f(s)ds$.

At first consider the case $h([0, T]) \subset [0, T]$. As it takes place, $B = 0$, and the condition $\rho(A) < 1$ is equivalent to the following condition. The spectral radius of the monodromy matrix $X(T) \equiv C(T, 0) : \mathbb{R}^n \to \mathbb{R}^n$ is less than one. The way proposed in Section 6.3.3 allows us to construct an approximate monodromy matrix $X_a(T) = \{x_{ij}^a\}_1^n$ and a matrix $X_\nu(T) = \{x_{ij}^\nu\}_1^n$ such that $X_\nu(T) \geq \left|X(T) - X_a(T)\right|$. Write the characteristic equation for $X(T)$:

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0, \tag{6.219}$$

where the constant $(-1)^i c_i$, $i = 1, \ldots, n$, is equal to the sum of all the ith order principal minors of $X(T)$. As is known, the condition $\rho(X(T)) < 1$ is equivalent to the condition that $|\lambda_i| < 1$ for all roots λ_i, $i = 1, \ldots, n$, of equation (6.219). In [217] one can find a way of constructing efficient criteria of the fulfillment of the inequalities $|\lambda_i| < 1$, $i = 1, \ldots, n$. For example as $n = 3$, such criterion has the form

$$\begin{aligned} &c_1 + c_2 + c_3 + 1 > 0, && 1 - c_1 + c_2 - c_3 > 0, \\ &3 + 3c_3 - c_2 - c_1 > 0, && 1 - c_3^2 + c_3c_1 - c_2 > 0. \end{aligned} \tag{6.220}$$

With the inequalities $x_{ij}^a - x_{ij}^\nu \leq x_{ij} \leq x_{ij}^a + x_{ij}^\nu$ and due to interval arithmetic (see, e.g., Alefeld and Herzberger [2]) we can find numbers c_i^a, c_i^ν, such that $c_i^a - c_i^\nu \leq c_i \leq c_i^a + c_i^\nu$, and hence check condition (6.220).

Example 6.16. With a special program implementing the scheme proposed above, there is established the stabilizability of all solutions of the system

$$\begin{bmatrix} \dot{x}^1(t) \\ \dot{x}^2(t) \\ \dot{x}^3(t) \end{bmatrix} + \begin{bmatrix} 0.5+0.505[-(t-k)] & 0 & -0.5(t-k) \\ -0.25(t-k) & 0.25(t-k) & 0 \\ 0 & -\frac{1}{3}(t-k) & 1-\frac{1}{3}(t-k) \end{bmatrix} \times \begin{bmatrix} x^1[(t-k)^2+k] \\ x^2(t) \\ x^3[(t-k)^2+k] \end{bmatrix} = 0, \quad t \in [k,k+1),\ k = 0,1,\dots. \tag{6.221}$$

Remark 6.17. In [48] it is shown that the conditions $h([0,T]) \subset [0,T]$ and $\rho(X(T)) < 1$ imply the existence of positive numbers $\mathcal{N}$ and α such that

$$||C(t,s)|| \leq \mathcal{N} \exp\{-\alpha(t-s)\}, \quad 0 \leq s \leq t < \infty. \tag{6.222}$$

Consider one further case when the inequality $\rho(A+B) < 1$ can be efficiently checked. Let $h(t) \equiv c \in (-T,0)$ on the set $\mathcal{H} = \{t \in [0,T] : h(t) \notin [0,T]\}$. As it takes place, we have for the operator B defined by (6.217) the representation

$$(Bz)(t) = \mathcal{D}(t)z(d), \tag{6.223}$$

where $\mathcal{D}(t) = \int_0^t C(t,s)P(s)\chi_{\mathcal{H}}(s)ds$, $\chi_{\mathcal{H}}$ is the characteristic function of $\mathcal{H}$, $d = c+T$; and the equation (6.218) takes the form

$$z(t) = C(t,0)z(T) + \mathcal{D}(t)z(d) + g(t), \quad t \in [0,T]. \tag{6.224}$$

In this event the condition $\rho(A+B) < 1$ is fulfilled, if the spectral radius of the $2n \times 2n$ matrix

$$\mathcal{F} = \begin{pmatrix} C(T,0) & \mathcal{D}(T) \\ C(d,0) & \mathcal{D}(d) \end{pmatrix} \tag{6.225}$$

is less than one. The elements of the matrices $C(T,0)$ and $C(d,0)$ can be approximated in the way described above. To approximate the elements of $\mathcal{D}(T)$ and $\mathcal{D}(d)$, we can use the approximation of the Cauchy matrix (see Section 6.6). Thus one can consider that, for each element of $\mathcal{F}$, there is found a sufficiently small interval with the rational end-points that include this element. Further it makes it possible to check the condition $\rho(\mathcal{F}) < 1$ in the way used above in the case $B = 0$.

The proposed scheme of the study can be naturally extended to the case of system with a finite number of delays and a prehistory concentrated at a finite number of points:

$$(Bz)(t) = \mathcal{D}_1(t)z(d_1) + \cdots + \mathcal{D}_\nu(t)z(d_\nu), \tag{6.226}$$

$d_i \in [0,T]$.

Notice conclusively that the effectively computable estimates of the rate of stabilization,

$$\theta(N, \alpha) = \max_{t \in [NT, (N+1)T]} |x(t, \alpha) - y(t)|, \tag{6.227}$$

(see Definition 6.15) are given in [161].

The questions of theoretical validating the computer-assisted study of various classes of equations (ordinary differential, partial differential, integral, operator equations) occupy an important place in the current literature. See, for instance, the book by Kaucher and Miranker [110], first published 1984, and [68, 85, 158, 159, 173, 174]. The presentation in this chapter is based on the works [147–151, 192, 195–197].

7 Nonlinear equations

7.1. Introduction

Let, as previously, $\mathbf{D}$ be a Banach space that is isomorphic to the direct product $\mathbf{B} \times \mathbb{R}^n$, and let isomorphism $\mathcal{J}^{-1} : \mathbf{D} \to \mathbf{B} \times \mathbb{R}^n$ be defined by $\mathcal{J}^{-1}x = [\delta, r]x$.

The study of the equation $\delta x = Fx$ with nonlinear operator F defined on the space $\mathbf{D}$ or on a certain set of this space meets a lot of difficulties and any rich in content theory is possible to develop only for special narrow classes of such equations. The second part of the book [32] in Russian is devoted to the boundary value problems for nonlinear equations. We will restrict ourselves below to a survey of the results of the mentioned book and by the proofs of some assertions which are most actual from our point of view.

The first section of the chapter is devoted to equations with monotone operators. The theorems of solvability of quasilinear problems in the section are based on the reduction of the boundary value problem to the equation $x = Hx$ with monotone (isotonic or antitonic) operator H on the appropriate semiordered space. Some suitable choice of such a space permits investigating certain singular boundary value problems. The schemes and constructions of this section are based on the results of Chapters 1, 2, and 4.

In case $F : \mathbf{D} \to \mathbf{B}$ is continuous compact, the equation $\delta x = Fx$ allows us to apply some theorems of functional analysis. This is why such equations may be studied by certain standard methods. It should be noted that some equations quite different at first sight may have one and the same set of solutions. By various transformations, keeping the set of solutions, one can come from a given equation to another which is equivalent to the initial one. The reduction of the given equation to the equivalent one, but more convenient for investigation, is an ordinary mode for studying new equations. Such a mode however demands solving some complicated auxiliary equations. Nevertheless the aim of investigation may be attained by only establishing the fact of the solvability. In other words, it suffices sometimes to establish only the fact of the reducibility of the equation to the desired form. Due to such circumstance, the class of "reducible" equations as well as the problem of "reducibility" of equations have a special place in the theory of functional differential equations.

The equation $\delta x = Fx$ is called reducible on a set M of the space $\mathbf{D}$ if there exists a continuous compact operator $F_0 : M \to \mathbf{B}$ such that the equations $\delta x = Fx$ and $\delta x = F_0 x$ are equivalent (the sets of solutions of $\delta x = Fx$ and $\delta x = F_0 x$, which belong to M, coincide). The problem of reducibility is discussed in Section 7.3.

An approach to the problem of a priori estimates of solutions is described in Section 7.4. This approach is based on the notion of a priori inequality. Some theorems on the solvability of nonlinear boundary value problems for reducible functional differential equations are obtained in Section 7.5 making use of the mentioned a priori estimates.

Section 7.6 is devoted to the problem of minimization of nonlinear functionals that generalize the square functionals considered in Chapter 5.

The ideas on the reducibility of equations find applications in the theory of stochastic functional differential equations. Some results on such a question are presented in Section 7.7 written by A. V. Ponosov.

7.2. Equations with monotone operators

7.2.1. Theorems on "forks"

On a semiordered set $\mathbf{X}$ of a linear space, consider the equation

$$x = Hx. \tag{7.1}$$

Assume that the operator $H : \mathbf{X} \to \mathbf{X}$ permits the representation $Hx = P(x,x)$, where $P(x,x) : \mathbf{X}\times\mathbf{X} \to \mathbf{X}$ does not decrease with respect to the first argument and does not increase with respect to the second one ($P(\alpha,\beta)$ is isotonic with respect to α and antitonic with respect to β). Let there exist a pair $u, v \in \mathbf{X}$ that composes a "fork:"

$$u < z, \quad u \leq P(u,z), \quad z \geq P(z,u). \tag{7.2}$$

Denote

$$[u,z] = \{x \in X : u \leq x \leq z\}. \tag{7.3}$$

The order interval $[u,z]$ is a convex set. From (7.2) it follows that the operator H maps the interval $[u,z]$ into itself:

$$u \leq P(u,z) \leq P(x,x) = Hx \leq P(z,u) \leq z. \tag{7.4}$$

If $\mathbf{X}$ is a Banach space under a norm such that $[u,z]$ is bounded and closed, the complete continuity (i.e., both continuity and compactness) of $H : [u,z] \to \mathbf{X}$ guarantees, by Schauder theorem, that there exists a solution $x \in [u,z]$ of $x = Hx$.

It is useful for applications of the given "fork scheme" that a wide class of operators H permits a decomposition $H = H_1 + H_2$ with isotonic H_1 and antitonic

H_2 (see, e.g., [36, 44, 132]). For such an operator H, inequalities (7.2) take the form

$$u < z, \quad u \leq H_1 u + H_2 z, \quad z \geq H_1 z + H_2 u. \tag{7.5}$$

In particular, if H is antitonic, (7.2) takes the form

$$u < z, \quad u \leq Hz, \quad z \geq Hu. \tag{7.6}$$

Consider for illustration the problem on positive solution of the equation

$$x = Kx + f \tag{7.7}$$

with antitonic K.

Theorem 7.1. *Let $f(t) \geq 0$, $Kf \geq 0$, $K(0) = 0$, $f + Kf \geq 0$, and $K : [f + Kf, f] \to$ **X** be continuous compact and antitonic. Then the equation $x = Kx + f$ has a positive solution $x \in [f + Kf, f]$.*

Proof. Due to the scheme given above, it suffices to put $z = f$, $u = f + Kf$, $Hx = Kx + f$. □

Let us estimate by Theorem 7.1 the length of the interval where there is defined a positive solution of the Cauchy problem

$$\begin{gathered} \ddot{x}(t) + p(t)x^\gamma(kt) = 0, \qquad x(0) = 0, \qquad \dot{x}(0) = \alpha > 0, \\ \gamma > 0, \qquad 0 < k \leq 1, \end{gathered} \tag{7.8}$$

with summable $p(t) \geq 0$. The Cauchy problem in $\mathbf{W}^2$ is equivalent to the equation $x = Kx + f$ in the space $\mathbf{C}$ of continuous functions, where

$$(Kx)(t) = \int_0^t (s - t)p(s)x^\gamma(ks)ds, \qquad f(t) = \alpha t. \tag{7.9}$$

By Theorem 7.1, the latter equation has a positive solution on $(0, b)$ if

$$\alpha^{\gamma-1}k^\gamma \int_0^t (t - s)s^\gamma p(s)ds \leq t, \quad t \in (0, b). \tag{7.10}$$

For $\gamma = 2$ this inequality holds on $(0, b)$ if

$$\int_0^b p(s)ds \leq \frac{27}{4b^2\alpha k^2}. \tag{7.11}$$

In case $\gamma = 1$ there exists a positive solution on $(0, b)$, if

$$\int_0^b p(s)ds \leq \frac{4}{bk}. \tag{7.12}$$

The latter inequality in case $k = 1$ is a well known test by Lyapunov-Zhukovskii for nonoscillation of the equation $\ddot{x}(t) + p(t)x(t) = 0$.

Below we give a simple variant of the theorem by Tarskii-Birkhof-Kantorovich (see [109]) in the form that is convenient for our purposes.

Theorem 7.2. *Let there exist a pair $u, z \in \mathbf{X}$ such that*

$$u < z, \quad u \leq Hu, \quad z \geq Hz. \tag{7.13}$$

Let, further, the operator $H : [u, z] \to \mathbf{X}$ be continuous compact and isotonic. Then the successive approximations $\{x^i\}$, $x^{i+1} = Hx^i$, $x^0 = z$ ($x^0 = u$) converge to the solution $\overline{x}$ ($\underline{x}$) of the equation $x = Hx$; $\bar{x}$, $\underline{x}$ belong to $[u, z]$, and for each solution $x \in [u, z]$, the inequality $\underline{x} \leq x \leq \bar{x}$ holds.

Proof. The operator H maps the interval $[u, z]$ into itself. Therefore there exists at least one solution $x \in [u, z]$. Let x be such a solution. The sequence $\{x^i\}$, $x^{i+1} = Hx^i$, $x^0 = z$ is decreasing and bounded below by x since H maps $[x, z]$ into itself. The sequence $\{x^i\}$ is compact and monotone. Therefore there exists $\bar{x} = \lim_{i \to \infty} x^i$. Since $\bar{x}$ is a solution, the inequality $\bar{x} \geq x$ for any solution $x \in [u, z]$ is proved.

The proof for $\underline{x}$ is analogous. □

7.2.2. Reduction of the boundary value problem to an equation with isotonic (antitonic) operator

Let $\mathbf{X}$ and $\mathbf{B}$ be semiordered Banach spaces, and $u, z \in \mathbf{X}$, $u < z$. We say that an operator $F : [u, z] \to \mathbf{B}$ satisfies the condition $\mathcal{L}^1_{[u,z]}$ ($\mathcal{L}^2_{[u,z]}$) if the representation

$$Fx = T^1 x + M^1 x \qquad \left(Fx = T^2 x + M^2 x\right) \tag{7.14}$$

is possible, where $M^1 : [u, z] \to \mathbf{B}$ ($M^2 : [u, z] \to \mathbf{B}$) is isotonic (antitonic) and $T^1 : \mathbf{X} \to \mathbf{B}$ ($T^2 : \mathbf{X} \to \mathbf{B}$) is linear.

Consider the boundary value problem

$$\mathcal{L}x = Fx, \qquad lx = \alpha \tag{7.15}$$

with linear operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$, linear bounded vector functional $l : \mathbf{D} \to \mathbb{R}^n$, and nonlinear operator $F : \mathbf{D} \to \mathbf{B}$. Let $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ and let $\mathbf{D}$, $\mathbf{B}$ be semiordered Banach spaces.

If F satisfies condition $\mathcal{L}^i_{[u,z]}$, problem (7.15) might be rewritten in the form

$$\mathcal{L}^i x = M^i x, \qquad lx = \alpha, \tag{7.16}$$

where $\mathcal{L}^i = \mathcal{L} - T^i$. Let the linear problem

$$\mathcal{L}^i x = \varphi, \qquad lx = \alpha \tag{7.17}$$

be uniquely solvable for any $\varphi \in \mathbf{B}$ and α, and let the Green operator G^i of the problem be isotonic (antitonic). Problem (7.16) is equivalent to the equation

$$x = G^i M^i x + g \tag{7.18}$$

in the space $\mathbf{D}$. Here g is a solution to the semihomogeneous problem

$$\mathcal{L}^i x = 0, \qquad lx = \alpha. \tag{7.19}$$

Under proper assumptions the operator

$$H \overset{\text{def}}{=} G^i M^i + g \tag{7.20}$$

will be isotonic (antitonic).

Let $[u,z] = \{x \in \mathbf{C} : u(t) \leq x(t) \leq z(t),\ t \in [a,b]\}$ and let $N[u,z] \to \mathbf{L}$ be a Nemytskii operator defined by

$$(Nx)(t) = f(t, x(t)). \tag{7.21}$$

In many cases the Nemytskii operator satisfies the condition $\mathcal{L}^1_{[u,z]}$ ($\mathcal{L}^2_{[u,z]}$) with an operator $T^1 : [u,z] \to \mathbf{L}$ ($T^2 : [u,z] \to \mathbf{L}$) of the form

$$(T^1 x)(t) = p^1(t)x(t) \qquad ((T^2 x)(t) = p^2(t)x(t)), \tag{7.22}$$

where $p^1 \in \mathbf{L}$ ($p^2 \in \mathbf{L}$). In such a case we will say that $N : [u,z] \to \mathbf{L}$ satisfies the condition $\mathcal{L}^1_{[u,z]}$ ($\mathcal{L}^2_{[u,z]}$) with the coefficient p^1 (p^2). It will be so, for instance, if

$$p^1(t) \leq \frac{\partial}{\partial y} f(t,y) \quad \left(\frac{\partial}{\partial y} f(t,y) \leq p^2(t)\right) \tag{7.23}$$

for $y \in [\min_{t\in[a,b]} u(t),\ \max_{t\in[a,b]} z(t)]$.

Indeed, let

$$\frac{\partial f(t,y)}{\partial y} \geq p^1(t), \quad t \in [a,b],\ y \in [m,M]. \tag{7.24}$$

The function $M^1(t,y) \overset{\text{def}}{=} f(t,y) - p^1(t)y$ does not decrease in y:

$$\frac{\partial M(t,y)}{\partial y} = \frac{\partial f(t,y)}{\partial y} - p^1(t) \geq 0. \tag{7.25}$$

Therefore $N : [u,z] \to \mathbf{L}$ satisfies the condition $\mathcal{L}^1_{[u,z]}$ with the coefficient p^1.

If a Nemytskii operator is Lipschitz with the coefficient $p \in \mathbf{L}$, then it satisfies both of the conditions $\mathcal{L}^1_{[u,z]}$ and $\mathcal{L}^2_{[u,z]}$ with $p^1 = -p$ and $p^2 = p$, respectively. Remark that a Nemytskii operator that satisfies simultaneously the conditions $\mathcal{L}^1_{[u,v]}$ and $\mathcal{L}^2_{[u,v]}$ with p^1 and p^2 is Lipschitz (see [125]).

The Nemytskii operator is a factor in many constructions of the operator F. For instance, if $[u,z] \subset \mathbf{C}$ and $F : [u,z] \to \mathbf{L}$ has the form

$$(Fx)(t) = f(t, x_h(t)), \tag{7.26}$$

then $F = NS_h$. If the function

$$M^1(t,y) \stackrel{\text{def}}{=} f(t,y) - p^1(t)y \qquad (M^2(t,y) \stackrel{\text{def}}{=} f(t,y) - p^2(t)y) \tag{7.27}$$

does not decrease (increase) in the second argument for $y \in [\min\{u(t),\ t \in [a,b]\}, \max\{z(t),\ t \in [a,b]\}]$, then $F : [u,z] \to \mathbf{L}$, defined by (7.26), satisfies the condition $\mathcal{L}^1_{[u,z]}$ ($\mathcal{L}^2_{[u,z]}$). Namely,

$$\begin{gathered}(Fx)(t) = p^1(t)x_h(t) + M^1(t, x_h(t)),\\ ((Fx)(t) = p^2(t)x_h(t) + M^2(t, x_h(t))).\end{gathered} \tag{7.28}$$

7.2.3. Nagumo-like theorems

Let $\mathbf{B}$ be a Banach space of measurable functions $z : [a,b] \to \mathbb{R}^1$, let the space $\mathbf{D}$ of $x : [a,b] \to \mathbb{R}^1$ be isomorphic to $\mathbf{B} \times \mathbb{R}^n$, and also $\mathbf{D} \subset \mathbf{C}$.

Suppose $\mathbf{D}$, such that any Green operator $G : \mathbf{B} \to \mathbf{D}$, as an operator acting into the space $\mathbf{C}$ of continuous functions ($G : \mathbf{B} \to \mathbf{C}$), is compact. By Remark 1.24, this assumption is fulfilled if the compactness of the Green operator $G : \mathbf{B} \to \mathbf{C}$ for a certain problem is established. For instance, if $\mathbf{D}$ is the space $\mathbf{W}^n$ of functions $x : [a,b] \to \mathbb{R}^1$ with absolutely continuous derivatives of the order up to $(n-1)$, then in case $n \geq 2$, the compactness property is fulfilled for the Cauchy operator $C : \mathbf{L}_p \to \mathbf{C}$, $1 \leq p \leq \infty$,

$$(Cz)(t) = \int_a^t \frac{(t-s)^{n-1}}{(n-1)!} z(s)\,ds \tag{7.29}$$

(i.e., for the Green operator of the Cauchy problem $x^{(n)} = z$, $x^{(k)}(a) = 0$, $k = 0, \ldots, n-1$). In case $n = 1$, the operator $C : \mathbf{L} \to \mathbf{C}$ is not compact (see [229]). Therefore none of the Green operators $G : \mathbf{L} \to \mathbf{C}$ is compact if $n = 1$.

Let $u, z \in \mathbf{D}$, $u(t) < z(t)$, $t \in [a,b]$, $[u,z] = \{x \in \mathbf{C} : u(t) \leq x(t) \leq z(t),\ t \in [a,b]\}$, and let $F : [u,z] \to \mathbf{B}$ be continuous and bounded. Consider the boundary value problem

$$(\mathcal{L}x)(t) = (Fx)(t), \qquad lx = \alpha, \tag{7.30}$$

where $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is a linear bounded operator, $l : \mathbf{D} \to \mathbb{R}^n$ is a linear bounded vector functional. Denote

$$\begin{gathered}\omega^u(t) = (\mathcal{L}u)(t) - (Fu)(t),\\ \omega^z(t) = (\mathcal{L}z)(t) - (Fz)(t).\end{gathered} \tag{7.31}$$

Theorem 7.3. *Let the following conditions be fulfilled.*

(1) $\omega^u(t) \le 0$, $\omega^z(t) \ge 0$ $(\omega^u(t) \ge 0,\ \omega^z(t) \le 0)$, $t \in [a,b]$.

(2) *The operator* $F : [u,z] \to \mathbf{B}$ *satisfies the condition* $\mathcal{L}^1_{[u,z]}$ $(\mathcal{L}^2_{[u,z]})$ *with the continuous and bounded* $M^1 : [u,z] \to \mathbf{B}$ $(M^2 : [u,z] \to \mathbf{B})$.

(3) *The boundary value problem*

$$\mathcal{L}^1 x \stackrel{\text{def}}{=} \mathcal{L}x - T^1 x = f, \qquad lx = 0$$
$$\left(\mathcal{L}^2 x \stackrel{\text{def}}{=} \mathcal{L}x - T^2 x = f,\ lx = 0\right) \tag{7.32}$$

is uniquely solvable and its Green operator G^1 (G^2) *is isotonic (antitonic).*

(4) *For the solutions* g_u, g, g_z *of the homogeneous equation* $\mathcal{L}^1 x = 0$ $(\mathcal{L}^2 x = 0)$, *satisfying the boundary conditions* $lx = lu$, $lx = \alpha$, *and* $lx = lz$, *respectively, the inequalities*

$$g_u(t) \le g(t) \le g_z(t), \quad t \in [a,b], \tag{7.33}$$

hold.

Then the problem (7.30) *has a solution* $x \in [u,z]$. *If the solution is not unique, then there exists a pair of solutions* $\underline{x}, \overline{x} \in [u,z]$ *such that any solution* $x \in [u,z]$ *of* (7.30) *satisfies the inequalities*

$$\underline{x}(t) \le x(t) \le \overline{x}(t), \quad t \in [a,b]. \tag{7.34}$$

Proof. It suffices to consider the case of the condition $\mathcal{L}^2_{[u,z]}$, another case is similar.

Let the condition $\mathcal{L}^2_{[u,z]}$ be fulfilled. By the scheme proposed in the foregoing subsection, problem (7.30) is equivalent to the equation $x = Hx$ in the space $\mathbf{D}$. Here $Hx = G^2M^2x + g$. We will consider this equation on $[u,z] \subset \mathbf{C}$ (it is possible as any one of continuous solutions of this equation belongs to $\mathbf{D}$). The operator $H : [u,z] \to \mathbf{B}$ is isotonic and continuous compact. Since $z = G^2M^2z + G^2\omega^z + g_z$ and $u = G^2M^2u + G^2\omega^u + g_u$, we have $z \ge Hz$, $u \le Hu$. Addressing Theorem 7.2 completes the proof. □

Remark 7.4. Proof of Theorem 7.3 makes use of the compactness and continuity of $G^2M^2 : \mathbf{C} \to \mathbf{C}$. This property of G^2M^2 ensures by the above assumption that the Green operator of every boundary value problem in the space $\mathbf{D}$ is compact as the operator acting into $\mathbf{C}$. Such assumption is needless when the operator $M^2 : \mathbf{C} \to \mathbf{B}$ is continuous and compact.

Remark 7.5. If in addition to conditions (1)–(4) of Theorem 7.3 the condition $\mathcal{L}^1_{[u,z]}$ $(\mathcal{L}^2_{[u,z]})$ is fulfilled and the Green operator G^1 (G^2) of the problem

$$\mathcal{L}^2 x = f, \qquad lx = 0 \qquad (\mathcal{L}^1 x = f,\ lx = 0) \tag{7.35}$$

is also antitonic (isotonic), then the solution $x \in [u,z]$ is unique.

Indeed, from the assumption of the existence of the ordered pair $\underline{x}, \overline{x}$ of solutions we obtain that the right-hand side and the left-hand side of the equality

$$\overline{x} - \underline{x} = G^2(M^2\overline{x} - M^2\underline{x}) \qquad (\overline{x} - \underline{x} = G^1(M^1\overline{x} - M^1\underline{x})) \tag{7.36}$$

have different signs.

Example 7.6. Consider the problem

$$\dddot{x}(t) = -2x_h^2(t) + 60, \quad x(0) = x(1) = \dot{x}(1) = 0, \quad t \in [0,1], \tag{7.37}$$

in the space W_p^3, $1 \le p \le \infty$. The function $h : [0,1] \to \mathbb{R}^1$ is assumed to be measurable.

Assertion 7.7. Problem (7.37) has a unique solution $x \in \mathbf{W}_p^3$ such that $0 \le x(t) \le 10t(1-t)^2$.

Proof. Let us put $u(t) = 0$, $z(t) = 10t(1-t)^2$ and examine the fulfillment of the conditions of Theorem 7.3. We have

$$\omega^u(t) = -60 < 0, \qquad \omega^z(t) = 2z^2(t) \ge 0. \tag{7.38}$$

Denote $f(y) = -2y^2 + 60$, $Fx = f(x_h)$. Since $df/dy \ge -4M$ for $y \in [0,M]$, where $M = \max_{t\in[0,1]} z(t) = 40/27$, the operator $F : [u,z] \to \mathbf{L}_\infty$ satisfies the condition $\mathcal{L}^1_{[u,z]}$:

$$(Fx)(t) = p^1 x_h(t) + M^1(t, x_h(t)), \tag{7.39}$$

where $p^1 = -4M$, the function $M^1(t,y)$ does not decrease in y.

For the problem

$$(\mathcal{L}^1 x)(t) \stackrel{\text{def}}{=} \dddot{x}(t) - p^1 x_h(t) = \varphi(t), \qquad x(0) = x(1) = \dot{x}(1) = 0, \tag{7.40}$$

condition (2.214) is fulfilled. Hence this problem is uniquely solvable and its Green operator is isotonic. Next, $g_u = g = g_z = 0$. Thus, by Theorem 7.3, problem (7.37) has a solution $x \in [u,z]$. By Remark 7.5, this solution is unique. Indeed, F is antitonic. Hence $\mathcal{L}^2 x \equiv \dddot{x}$. The problem

$$(\mathcal{L}^2 x)(t) \stackrel{\text{def}}{=} \dddot{x}(t) = \varphi(t), \qquad x(0) = x(1) = \dot{x}(1) = 0 \tag{7.41}$$

is uniquely solvable and its Green function

$$G^2(t,s) = \begin{cases} \dfrac{1}{2}s(1-t)^2 & \text{if } 0 \le s \le t \le 1, \\ \dfrac{1}{2}t(1-s)(2s - t - ts) & \text{if } 0 \le t < s \le 1, \end{cases} \tag{7.42}$$

is nonnegative. □

Example 7.8. In the elasticity theory, the problem

$$\ddot{x}(t) + \frac{3}{t}\dot{x}(t) = -\frac{2}{x^2(t)}, \quad \dot{x}(0) = 0, \quad x(1) = \alpha > 0, \quad t \in [0,1]. \tag{7.43}$$

arises. Conditions of the solvability of this problem are established by Stuart (see, e.g., [100]). Making use of Theorem 7.3, we present here a refinement of Stuart's result given by Alves [5].

On the base of the results of Section 4.4, the problem should be considered in the space $\mathbf{D} \simeq \mathbf{L}_p \times \mathbb{R}^1$, $1 < p < \infty$, the elements of which are defined by

$$x(t) = \int_0^1 (t-s)z(s)ds + \beta, \quad \{z,\beta\} \in \mathbf{L}_p \times \mathbb{R}^1. \tag{7.44}$$

Thus $\mathbf{D} = \{x \in \mathbf{W}_p^2 : \dot{x}(0) = 0\}$.

Assertion 7.9. Problem (7.43) has a unique solution $x \in \mathbf{D}$ satisfying the inequalities

$$\alpha \le x(t) \le \frac{1}{4\alpha^2}(1-t^2) + \alpha, \quad t \in [0,1]. \tag{7.45}$$

Proof. Denoting

$$(\mathcal{L}x)(t) = \ddot{x}(t) + \frac{3}{t}\dot{x}(t), \qquad (Fx)(t) = -\frac{2}{x^2(t)}, \tag{7.46}$$

write the problem (7.43) in the form

$$\mathcal{L}x = Fx, \qquad x(1) = \alpha. \tag{7.47}$$

Make use of Theorem 7.3. Putting $u(t) = \alpha$, $z(t) = (1/4\alpha^2)(1-t^2) + \alpha$, we have

$$u(t) \le z(t), \qquad \omega^u(t) = \frac{2}{\alpha^2} > 0, \qquad \omega^z(t) = -\frac{2}{\alpha^2} + \frac{2}{z^2(t)} < 0. \tag{7.48}$$

The Nemytskii operator $F : [u,z] \to \mathbf{L}_p$ satisfies the condition $\mathcal{L}^2_{[u,z]}$ with $p^2 = 4/\alpha^3$ as $dF(y)/dy \le 4/\alpha^3$ for $y \ge \alpha$. Consider the auxiliary linear problem

$$(\mathcal{L}^2 x)(t) \overset{\text{def}}{=} \ddot{x}(t) + \frac{3}{t}\dot{x}(t) - \frac{4}{\alpha^3}x(t) = \varphi(t), \qquad x(1) = 0. \tag{7.49}$$

For $v(t) = (1/4\alpha^2)(1-t)$, we have statement (c) of Theorem 4.20. Namely,

$$(\mathcal{L}^2 v)(t) = -\frac{1}{\alpha^2} - \frac{1}{\alpha^5}(1-t^2) < 0, \qquad v(1) - \int_0^1 (\mathcal{L}^2 v)(t)dt > 0. \tag{7.50}$$

Hence, problem (7.49) is uniquely solvable and its Green operator G^2 is antitonic. Since, besides, $g_u = g = g_z$, all the conditions of Theorem 7.3 are fulfilled. So problem (7.43) has a solution $x \in \mathbf{D}$ such that $u(t) \leq x(t) \leq z(t)$.

Operator $F : [u,z] \to \mathbf{L}_p$ is isotonic and, hence, it satisfies the condition $\mathcal{L}^1_{[u,z]}$ with $p^1 = 0$. The Green function $G^1(t,s)$ of problem $\mathcal{L}^1 x = \varphi$, $x(1) = 0$ was constructed in Section 4.4:

$$G^1(t,s) = \begin{cases} -\dfrac{s^3}{2t^2}\left(1-t^2\right) & \text{if } 0 \leq s \leq t \leq 1, \\ -\dfrac{s}{2}\left(1-s^2\right) & \text{if } 0 \leq t < s \leq 1. \end{cases} \tag{7.51}$$

This function takes no positive value. Thus the solution is unique by Remark 7.5. □

An analogous proof of the solvability of the problem

$$\ddot{x}(t) + \frac{1}{t}\dot{x}(t) = \beta \exp\left(-\frac{1}{x(t)}\right), \qquad \dot{x}(0) = 0, \qquad x(1) = 0 \tag{7.52}$$

in the space $\mathbf{D} = \{x \in \mathbf{W}^2_p : \dot{x}(0) = 0\}$ is presented in [5]. Such a problem arises in the study of processes in chemical reactor.

Consider the problem

$$\pi(t)\ddot{x}(t) = f\left[t, x(t)\right], \quad x(a) = \alpha^1, \quad x(b) = \alpha^2, \quad t \in [a,b], \tag{7.53}$$

where $\pi(t)$ is one of the following functions: $\pi(t) = t-a$, $\pi(t) = b-t$, $\pi(t) = (t-a)(b-t)$. We take as the space $\mathbf{D}$ the space similar to $\mathbf{D}_\pi$ (see Section 4.2.1) replacing in its definition the space $\mathbf{L}$ by the space $\mathbf{L}_p$, $1 < p < \infty$. Denote this space by $\mathbf{D}^p_\pi$. By Remark 1.24, the Green operator $G_p : \mathbf{L}_p \to \mathbf{D}^p_\pi$ of any boundary value problem in the space $\mathbf{D}^p_\pi$, as an operator mapping into the space $\mathbf{C}$, is compact. This follows from the fact that, for instance, the Green operator $G_0 : \mathbf{L}_p \to \mathbf{C}$ of problem $\pi\ddot{x} = z$, $x(a) = x(b) = 0$ is compact [229]. Recall that the Green function of this problem is defined by (4.21). Note that the operator G_0 is antitonic.

Assume that there exists a pair $u, z \in \mathbf{D}^p_\pi$ such that

$$\begin{gathered} u(t) < z(t), \quad t \in (a,b), \\ u(a) \leq \alpha^1 \leq z(a), \qquad u(b) \leq \alpha^2 \leq z(b), \\ \pi(t)\ddot{u}(t) - f\left[t, u(t)\right] \stackrel{\text{def}}{=} \omega^u(t) \geq 0, \\ \pi(t)\ddot{z}(t) - f\left[t, z(t)\right] \stackrel{\text{def}}{=} \omega^z(t) \leq 0, \quad t \in [a,b]. \end{gathered} \tag{7.54}$$

Recall that $[u,z] = \{x \in \mathbf{C} : u(t) \leq x(t) \leq z(t),\ t \in [a,b]\}$.

Theorem 7.10. *Let the Nemytskii operator* $F : [u,z] \to \mathbf{L}_p$, $1 < p < \infty$, *defined by* $(Fx)(t) = f[t,x(t)]$ *be continuous, bounded, and satisfies the condition* $\mathcal{L}^2_{[u,z]}$ *with* p^2. *Then there exists a solution* $x \in [u,z]$ *of* (7.53).

Proof. By the condition $\mathcal{L}^2_{[u,z]}$, we can write the equation $\pi\ddot{x} = f(t,x)$ in the form

$$(\mathcal{L}^2 x)(t) \overset{\text{def}}{=} \pi(t)\ddot{x}(t) - p^2(t)x(t) = M^2[t,x(t)], \tag{7.55}$$

where $M^2(t,y)$ does not increase in y. Therefore

$$\mathcal{L}^2[z-u] = M^2(t,z) - M^2(t,u) + \omega^z - \omega^u \overset{\text{def}}{=} \varphi, \tag{7.56}$$

where $\varphi(t) \le 0$. Thus $v = z - u$ satisfies the equality

$$(\mathcal{L}^2 v)(t) = \varphi(t) \le 0. \tag{7.57}$$

Note that without loss of generality we can put $\varphi(t) \not\equiv 0$. Indeed, otherwise, v satisfies the homogeneous equation $\pi\ddot{x} - p^2 x = 0$. In this case we take another coefficient p_0^2 $(p_0^2(t) \ge p^2(t))$ of the condition $\mathcal{L}^2_{[u,z]}$ such that v ceases to be a solution of the homogeneous equation.

For the problem

$$\mathcal{L}^2 x = f, \qquad x(a) = x(b) = 0, \tag{7.58}$$

the statement (a) of Theorem D.2 with $W = G_0$ holds. Therefore this problem is uniquely solvable and its Green operator G^2 is antitonic, this with Theorem 7.3 completes the proof. □

Example 7.11. Consider the problem

$$\ddot{x}(t) = q(t)\ln x(t), \quad x(0) = \alpha^1, \quad x(1) = \alpha^2, \quad t \in [0,1], \tag{7.59}$$

$0 \le \alpha^i \le 1$, $i = 1,2$, under the assumption that coefficient q is summable on $[0,1]$ and $\operatorname{ess\,inf}_{t\in[0,1]} q(t) = \alpha > 0$.

We consider problem (7.59) in the space $\mathbf{D}^p_\pi$, $1 < p < \infty$, $\pi(t) = t(1-t)$.

Assertion 7.12. For every α^1, α^2 problem (7.59) has a unique solution $x \in \mathbf{D}^p_\pi$ such that

$$\beta t(1-t) \le x(t) \le 1, \quad t \in [0,1], \tag{7.60}$$

where β satisfies the inequality $2\beta + \ln(\beta/4) \le 0$.

Proof. Let us write (7.59) in the form

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} t(1-t)\ddot{x}(t) = (Nx)(t), \qquad x(0) = \alpha^1, \qquad x(1) = \alpha^2, \tag{7.61}$$

where

$$(Nx)(t) = t(1-t)q(t)\ln x(t), \tag{7.62}$$

and apply Theorem 7.10. Putting $u(t) = \beta t(1-t)$ and $z(t) \equiv 1$, we have

$$\omega^u(t) = -2\beta - q(t)\ln\left[\beta t(1-t)\right] \geq -\left(2\beta + \ln\frac{\beta}{4}\right) \geq 0, \qquad \omega^z(t) \equiv 0. \tag{7.63}$$

The Nemytskii operator $N : [u,z] \to \mathbf{L}_p$ satisfies the condition $\mathcal{L}^2_{[u,v]}$ with $p^2(t) = (1/\beta)q(t)$. Indeed, the operator $M^2 : [u,z] \to \mathbf{L}_p$ defined by

$$(M^2x)(t) = q(t)t(1-t)\ln x(t) - \frac{1}{\beta}q(t)x(t), \tag{7.64}$$

is antitonic, as for $x_1, x_2 \in [u,z]$, $x_1(t) \leq x_2(t)$, $t \in [0,1]$, we have

$$\begin{aligned}(M^2x_2)(t) - (M^2x_1)(t) &= q(t)\left\{t(1-t)\ln\frac{x_2(t)}{x_1(t)} - \frac{1}{\beta}x_1(t)\left[\frac{x_2(t)}{x_1(t)} - 1\right]\right\} \\ &\leq \frac{1}{\beta}q(t)\left[u(t) - x_1(t)\right]\left[\frac{x_1(t)}{x_2(t)} - 1\right] \leq 0, \quad t \in (0,1).\end{aligned} \tag{7.65}$$

By Theorem 7.10, problem (7.59) has a solution $x \in [0,z]$. This solution is unique due to Remark 7.5, as N is isotonic and the Green operator G^1 of the problem

$$t(1-t)\ddot{x}(t) = \varphi(t), \qquad x(0) = x(1) = 0 \tag{7.66}$$

is antitonic. □

7.3. Reducibility of equations

Mathematical description of many problems is often realized in the form of functional equation, for instance, differential one. Equations arising in applications are, as a matter of fact, a kind of a picture composed by means of mathematical symbols. Any investigation of functional equation demands a definition of the notion of solution. In other words, we are forced to define the functional space in which the equation must be considered. As such a space we offer the Banach one $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$. This space might be considered as a generalization of the space of absolutely continuous functions. The above theory of linear equations $\mathcal{L}x = f$ in the space $\mathbf{D} \simeq \mathbf{B} \times \mathbb{R}^n$ assumes the operator $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ to be Noether of the index n. This assumption guarantees the existence of a finite-dimensional parameterization of the set of all solutions and provides reducibility of the equation to the form $\delta x = Px + f$ with a compact operator $P : \mathbf{D} \to \mathbf{B}$. In the case of nonlinear equation $\delta x = Fx$ we also assume that the equation may be reduced to an equivalent $\delta x = F_0 x$ with continuous compact $F_0 : \mathbf{D} \to \mathbf{B}$. As we will demonstrate below, the reducibility of the nonlinear equation may be guaranteed, as in the linear case, by the property of the set of all solutions to admit a finite-dimensional parameterization. In the general case the necessary and sufficient condition for reducibility is local compactness of the set of all solutions.

7.3.1. Reducibility in the space D of absolutely continuous functions

The next example may explain the essence of reducibility.

Consider the linear equation

$$\dot{x} = Fx \stackrel{\text{def}}{=} (S_g + K)\dot{x} + Ax(a) + f \tag{7.67}$$

with compact operator $K : \mathbf{L} \to \mathbf{L}$ and a composition operator $S_g : \mathbf{L} \to \mathbf{L}$ (see Appendix C). Since nonzero operator $S_g : \mathbf{L} \to \mathbf{L}$ is never compact (Theorem C.9), the operator $F : \mathbf{D} \to \mathbf{L}$ cannot be compact. Suppose there exists the bounded inverse $(I - S_g)^{-1}$. Applying this operator to both sides of (7.67) rewritten in the form

$$(I - S_g)\dot{x} = K\dot{x} + Ax(a) + f, \tag{7.68}$$

we obtain the equivalent equation

$$\dot{x} = K_0\dot{x} + A_0 x(a) + f_0 \tag{7.69}$$

with compact operator $K_0 = (I - S_g)^{-1}K$. Thus (7.67) is reducible on the space $\mathbf{D}$. If in addition the operator $I - K_0$ is also invertible, we may apply $(I - K_0)^{-1}$ to both sides of the equation

$$(I - K_0)\dot{x} = A_0 x(a) + f_0, \tag{7.70}$$

and obtain a very simple integrable equation

$$\dot{x} = A_1 x(a) + f_1 \tag{7.71}$$

with a finite-dimensional operator.

A similar hierarchy of equivalent equations (the given equation $\dot{x} = Fx$, the equation $\dot{x} = F_0 x$ with continuous compact F_0, the equation $\dot{x} = F_1 x$ with finite-dimensional F_1) might be established in some nonlinear cases. As an example, we construct the first step of such a hierarchy for the equation

$$\begin{aligned} \dot{x}(t) &= f\big(t, x[h(t)], \dot{x}[g(t)]\big), \quad t \in [a,b], \\ x(\xi) &= \varphi(\xi), \quad \dot{x}(\xi) = \psi(\xi) \quad \text{if } \xi \notin [a,b]. \end{aligned} \tag{7.72}$$

With the notation of Section 2.2 (see (2.23)) this equation may be rewritten in the form

$$\dot{x}(t) = (Fx)(t) \stackrel{\text{def}}{=} f_1\big(t, (S_h x)(t), (S_g \dot{x})(t)\big) \tag{7.73}$$

under natural assumptions on operators $S_h : \mathbf{D} \to \mathbf{L}$ and $S_g : \mathbf{L} \to \mathbf{L}$ (see Appendix C). Suppose that the auxiliary functional equation

$$y(t) = f_1\big(t, u(t), (S_g y)(t)\big) \tag{7.74}$$

is uniquely solvable in $\mathbf{L}$ for every $u \in \mathbf{L}$. Then there exists the operator $H : \mathbf{L} \to \mathbf{L}$ such that the solution of the auxiliary equation has the representation $y = Hu$ and, hence, (7.72) is equivalent to the equation

$$\dot{x} = F_0 x \stackrel{\text{def}}{=} H S_h x. \tag{7.75}$$

The operator $S_h : \mathbf{D} \to \mathbf{L}$ is compact (see Appendix C), therefore $F_0 : \mathbf{D} \to \mathbf{L}$ is continuous compact if, for instance, H is continuous.

Let the boundary value problem

$$\dot{x} = Fx, \qquad lx = \alpha \tag{7.76}$$

be correctly solvable (uniquely solvable with the solution depending continuously on α). Then the equation $\dot{x} = Fx$ is reducible. Indeed, denote by $\varphi(t, \alpha)$ the derivative of the solution of the problem (7.76). Then we obtain the equivalent equation

$$\dot{x}(t) = \big(F_0 x\big)(t) \stackrel{\text{def}}{=} \varphi(t, lx) \tag{7.77}$$

with continuous compact $F_0 : \mathbf{D} \to \mathbf{L}$. Let us demonstrate that in this case the general solution of $\dot{x} = Fx$ depends on arbitrary constant vector $\alpha \in \mathbb{R}^n$.

As is shown in Section 2.2 (equality (2.32)), for any linear bounded vector functional $l : \mathbf{D} \to \mathbb{R}^n$ with linearly independent components it is possible to construct the linear bounded operator $W_l : \mathbf{L} \to \{x \in \mathbf{D} : lx = 0\}$ such that it has the bounded inverse and is represented by

$$\big(W_l z\big)(t) = \int_a^b W(t,s) z(s)\, ds. \tag{7.78}$$

Let

$$lx \equiv \Psi x(a) + \int_a^b \Phi(s)\dot{x}(s)ds, \tag{7.79}$$

then

$$W(t,s) = \begin{cases} E - U(t)\Phi(s) & \text{for } a \le s \le t \le b, \\ -U(t)\Phi(s) & \text{for } a \le t < s \le b. \end{cases} \tag{7.80}$$

Here the $n \times n$ matrix U with the columns from $\mathbf{D}$ is such that $\det U(a) \neq 0$, $lU = E$. The substitution

$$x(t) = \big(W_l z\big)(t) + U(t)\alpha \tag{7.81}$$

establishes the one-to-one mapping between the set of solutions $x \in \mathbf{D}$ of problem (7.76) and the set of solutions $z \in \mathbf{L}$ of the equation

$$z = \Omega_\alpha z, \tag{7.82}$$

where $\Omega_\alpha : \mathbf{L} \to \mathbf{L}$ is defined by

$$\big(\Omega_\alpha z\big)(t) = \int_a^b \dot{U}(t)\Phi(s)z(s)ds - \dot{U}(t)\alpha - F\{W_l z + U\alpha\}(t). \tag{7.83}$$

Let us write the solution of (7.82) in the form $z(t) = \theta(t,\alpha)$. Then the general solution of the equation $\dot{x} = Fx$ has the form

$$x(t) = \int_a^b W(t,s)\theta(s,\alpha)ds + U(t)\alpha. \tag{7.84}$$

Thus the correct solvability of the problem (7.76) guarantees the property that the set of all the solutions of the equation $\dot{x} = Fx$ admits an n-dimensional parameterization. If $lx \stackrel{\text{def}}{=} x(a)$, then

$$W_l z = \int_a^t z(s)ds \tag{7.85}$$

and the general solution takes the form

$$x(t) = \int_a^t \theta(s,\alpha)ds + \alpha. \tag{7.86}$$

Let us dwell on two examples.

The Cauchy problem $\dot{x} = Fx$, $x(a) = \alpha$ for the equation

$$\dot{x}(t) = (Fx)(t) \overset{\text{def}}{=} f[t, x_h(t), \dot{x}_g(t)], \quad t \in [a,b], \tag{7.87}$$

is correctly solvable if $h(t) = t - \tau_1$, $g(t) = t - \tau_2$, where τ_1 and τ_2 are positive constants and the superposition $f[t, u(t), v(t)]$ is summable for any measurable and essentially bounded $u : [a,b] \to \mathbb{R}^n$ and summable $v : [a,b] \to \mathbb{R}^n$. It follows from the fact that in this event the "step-by-step method" of construction of the solution to the Cauchy problem is applicable.

Let us return to problem (7.76) under the assumption that the linear problem

$$\dot{x} = z, \qquad lx = 0 \tag{7.88}$$

is uniquely solvable. Let, further, G be the Green operator of this problem, and $g = \|G\|_{\mathbf{L}\to\mathbf{D}}$. Problem (7.76) is equivalent to the equation

$$x = GFx + r \tag{7.89}$$

in the space $\mathbf{D}$. If there exists a constant k such that

$$||Fx_1 - Fx_2||_{\mathbf{L}} \le k||x_1 - x_2||_{\mathbf{D}} \tag{7.90}$$

for any pair $x_1, x_2 \in \mathbf{D}$, then the inequality $gk < 1$ permits applying the Banach principle. In this case the equation $\dot{x} = Fx$ is reducible and the set of the solutions permits the n-dimensional parameterization.

In [32], one can find some tests of reducibility of the equation $\dot{x} = Fx$ in the space $\mathbf{D}$ of absolutely continuous functions.

A specific place in applications is occupied by the equations with retarded argument

$$\dot{x}(t) = f(t, x_h(t)), \quad h(t) \le t, \tag{7.91}$$

and their generalization in the form

$$\dot{x}(t) = (Fx)(t) \tag{7.92}$$

with Volterra operator F (Volterra operator is understood in sense of the definition by Tikhonov [215]). Sometimes such equations are called equations with aftereffect.

Let $\mathbf{X}$ and $\mathbf{Y}$ be sets of measurable functions $x : [a,b] \to \mathbb{R}^n$ and $y : [a,b] \to \mathbb{R}^n$. The operator $\Phi : \mathbf{X} \to \mathbf{Y}$ is called Volterra, if for each $c \in (a,b]$ and any $x_1, x_2 \in \mathbf{X}$ such that $x_1(t) = x_2(t)$ a.e. on $[a,c]$ the equality $(\Phi x_1)(t) = (\Phi x_2)(t)$ holds a.e. on $[a,c]$.

To avoid difficulties connected with generalizations we will restrict ourselves to (7.92) in the space $\mathbf{D} \simeq \mathbf{L} \times \mathbb{R}^n$ of absolutely continuous $x : [a,b] \to \mathbb{R}^n$.

As in the linear case (see Section 2.2.3), the Volterra property of F permits considering (7.92) and its solutions on any $[a,c] \subset [a,b]$ and ignoring the values $x(t)$ and $(Fx)(t)$ for $t > c$. In other words, an absolutely continuous function, $x : [a,c] \to \mathbb{R}^n$, is called the solution to (7.92) if $\dot{x}(t) = (Fx)(t)$ a.e. on $[a,c]$.

The foundations of the theory of ordinary differential equation are theorems on local solvability of the Cauchy problem, on extendability of the solution of this problem and about connectedness and compactness of the set of solutions. All these theorems keep in the general case of "*Volterra-reducible*" equations.

Definition 7.13. Equation (7.92) is called Volterra-reducible if there exists a continuous compact Volterra operator $F_0 : \mathbf{D} \to \mathbf{L}$ such that for each $c \in (a,b]$, the set of solutions on $[a,c]$ of (7.92) and the set of solutions on $[a,c]$ of the equation

$$\dot{x} = F_0 x \tag{7.93}$$

coincide.

If the Cauchy problem

$$\dot{x} = Fx, \qquad x(a) = \alpha \tag{7.94}$$

for (7.92) is uniquely solvable and the solution is continuous in α (in this case we say that the problem is correctly solvable), the equation $\dot{x} = Fx$ is Volterra-reducible. Equation (7.72) under natural assumptions is Volterra-reducible if $h(t) \leq t$ and there exists a constant $\tau > 0$ such that $t - g(t) \geq \tau$. To prove it, it suffices to repeat the transformations of (7.72) used in Section 7.3.1 and to note that the existence of the continuous Volterra operator $H : \mathbf{L} \to \mathbf{L}$ is ensured by the nilpotency of $S_g : \mathbf{L} \to \mathbf{L}$.

In [32] some conditions are given which guarantee Volterra reducibility of (7.92).

Let us give the basic theorems on properties of the Volterra-reducible equations.

Theorem 7.14. *Let* (7.92) *be Volterra-reducible on* $[a,b]$. *Then for each* $\alpha \in \mathbb{R}^n$, *there exists a* $c \in (a,b]$ *such that the set of solutions to the Cauchy problem*

$$\dot{x} = Fx, \qquad x(a) = \alpha \tag{7.95}$$

defined on $[a,c]$ *is nonempty.*

Proof. The substitution $x(t) = \alpha + \int_a^t z(s)ds$ reduces the problem $\dot{x} = F_0 x$, $x(a) = \alpha$ for the reduced equation to the equation $z = \Omega z$ with a continuous compact Volterra operator $\Omega : \mathbf{L} \to \mathbf{L}$ defined by

$$(\Omega z)(t) = \left(F_0\left(\alpha + \int_a^{(\cdot)} z(s)ds\right)\right)(t). \tag{7.96}$$

Denote $\mathcal{B}_b^\gamma = \{z \in \mathbf{L}[a,b] : \|z\|_{\mathbf{L}[a,b]} \le \gamma\}$. All the elements of precompact set $\Omega\mathcal{B}_b^\gamma$ have the equipotentially absolutely continuous norms. Therefore there exists a $c \in (a,b]$ such that

$$\int_a^c |y(s)|\,ds \le \gamma, \quad y \in \Omega\mathcal{B}_b^\gamma. \tag{7.97}$$

The operator $\Omega : \mathbf{L}[a,c] \to \mathbf{L}[a,c]$ maps the set $\mathcal{B}_c^\gamma$ into itself. Reference to the Schauder principle completes the proof. □

Theorem 7.15. *Let* (7.92) *be Volterra-reducible on* $[a,b]$ *and let* x *be a solution of* (7.95) *defined on* $[a,c] \subset [a,b)$. *Then there exists a* $c_1 \in (c,b]$ *such that problem* (7.95) *has on* $[a,c_1]$ *at least one solution* x_1 *such that* $x_1(t) = x(t)$ *on* $[a,c]$.

Proof. As in the proof of Theorem 7.10, reduce (7.95) to equivalent equation $z = \Omega z$. Define the operator $\Omega_c : \mathbf{L}[a,b] \to \mathbf{L}[a,b]$ by

$$\Omega_c z = \Omega z_c, \tag{7.98}$$

where

$$z_c(t) = \begin{cases} \dot{x}(t) & \text{if } t \in [a,c], \\ z(t) & \text{if } t \in (c,b]. \end{cases} \tag{7.99}$$

The operator Ω_c is completely continuous. Fix $\gamma > 0$ and denote

$$\mathcal{B}_b^\gamma = \Big\{z \in \mathbf{L}[a,b] : \|z\|_{\mathbf{L}[a,b]} \le \gamma + \|\dot{x}\|_{\mathbf{L}[a,c]}\Big\}. \tag{7.100}$$

All the elements of precompact set $\Omega_c\mathcal{B}_b^\gamma$ have the equipotentially absolutely continuous norms. Therefore there exists a $c_1 > c$ such that

$$\int_c^{c_1} |y(s)|\,ds \le \gamma, \quad y \in \Omega_c\mathcal{B}_b^\gamma. \tag{7.101}$$

The operator Ω_c maps $\mathcal{B}_{c_1}^\gamma$ into itself. By the Schauder principle, there exists a solution to $z = \Omega_c z$ defined on $[a,c_1]$. Obviously, $z(t) = \dot{x}(t)$ on $[a,c]$. The function $x_1(t)$ defined on $[a,c_1]$ by $x_1(t) = \alpha + \int_a^t z(s)ds$ is a solution (on this segment) of problem (7.95) and $x_1(t) = x(t)$ on $[a,c]$. □

The significance of the condition of Theorem 7.15 can be demonstrated by the following example by S. A. Gusarenko. For the equation

$$\begin{gathered} \dot{x}(t) = (Fx)(t) \overset{\text{def}}{=} 3\sqrt{x\Big[t - \sqrt{x(t)}\Big]}, \quad t \ge 0, \\ x(\xi) = 0 \quad \text{if } \xi < 0, \end{gathered} \tag{7.102}$$

the solution x of the problem $\dot{x} = Fx$, $x(0) = \alpha > 0$ is defined only on $[0, \sqrt{\alpha}]$: the graph of $x(t)$ stops at the point $\{\sqrt{\alpha}, \alpha\}$. In this example the operator $F : \mathbf{D} \to \mathbf{L}$ is not continuous. More details on such equations are presented in [95].

Let us give a theorem of compactness and connectedness of the set of solutions to the Cauchy problem for the Volterra-reducible equation.

Theorem 7.16 (see [58]). *Let $\alpha \in \mathbb{R}^n$ be fixed and let there exist a constant m such that for any $c \in [a, b]$, the uniform a priori estimate*

$$\|x\|_{\mathbf{D}^n[a,c]} \leq m, \quad c \in (a, b], \tag{7.103}$$

holds for all solutions to (7.95) *defined on* $[a, c]$. *Let, further,* (7.92) *be Volterra-reducible. Then the set of all defined solutions on* $[a, b]$ *to* (7.95) *is nonempty, compact (in itself), and connected in* $\mathbf{D}$.

7.3.2. Reducibility of the abstract equation

The papers [91, 92] are devoted to general assertions about reducibility of the abstract equation $\delta x = Fx$. We produce below some of them.

Consider the equation

$$\Phi x = g \tag{7.104}$$

with the operator Φ acting from a Banach space $\mathbf{X}$ into a Banach space $\mathbf{X}_1$; $g \in \mathbf{X}_1$. Let $\mathcal{Y}$ let be a subset of a Banach space $\mathbf{Y}$, let the intersection $\mathbf{Y} \cap \mathbf{X}$ be nonempty.

Equation (7.104) is called $(\mathcal{Y}, \mathbf{Y})$-reducible if there exists a continuous compact operator $\Pi : \mathbf{Y} \to \mathbf{Y}$ such that the set of solutions of (7.104), which belongs to $\mathcal{Y}$, coincides with the set of solutions of the equation $x = \Pi x$.

The $(\mathbf{Y}, \mathbf{Y})$-reducible equation is called $\mathbf{Y}$-reducible as well as $\mathbf{X}$-reducible which is called reducible.

Example 7.17. The equation

$$\dot{x}(t) - (\mathbf{F}x)(t) = 0, \quad t \in [a, b], \tag{7.105}$$

is $\mathbf{C}$-reducible to the equation

$$x(t) = x(a) + \int_a^t (\mathbf{F}_0 x)(s)\,ds \tag{7.106}$$

($\mathbf{C}$ is the space of continuous functions $y : [a, b] \to \mathbb{R}^n$) if the operator $\mathcal{F} : \mathbf{D} \to \mathbf{L}$ may be extended to a continuous operator $F : \mathbf{C} \to \mathbf{L}$ whose values on elements of any ball $\{y \in \mathbf{C} : \|y\|_{\mathbf{C}} \leq \rho\}$ are bounded by a summable u_ρ : $\|(Fy)(t)\|_{\mathbb{R}^n} \leq u_\rho(t)$. $\mathbf{C}$-reducible equations were studied in [145].

Let, as above, the space $\mathbf{D}$ be isomorphic to the direct product $\mathbf{B} \times \mathbb{R}^n$, let $\mathcal{J} = \{\Lambda, Y\} : \mathbf{B} \times \mathbb{R}^n \to \mathbf{D}$ be an isomorphism, $\mathcal{J}^{-1} = [\delta, r]$. Rewrite the abstract functional differential equation in the form

$$\Phi x \equiv \delta x - Fx = 0. \tag{7.107}$$

Theorem 7.18. *Equation* (7.107) *is* $\mathbf{D}$*-reducible if and only if there exists a continuous compact operator* $F_0 : \mathbf{D} \to \mathbf{B}$ *such that the set of all solutions of the equation*

$$\delta x = F_0 x \tag{7.108}$$

coincides with the set of all solutions of (7.107).

Proof. Let (7.107) be $\mathbf{D}$-reducible to the equation $x = \Pi x$ with continuous compact $\Pi : \mathbf{D} \to \mathbf{D}$. Define the continuous compact $\theta : \mathbf{D} \to \mathbf{B}$ by

$$\theta x = \frac{\left(\|rx\|_{\mathbb{R}^n} + 2\|\delta x\|_{\mathbf{B}}\right) z}{\|z\|_{\mathbf{B}}} \tag{7.109}$$

with fixed $z \in \mathbf{B}$, $z \neq 0$. Define the continuous compact $F_0 : \mathbf{D} \to \mathbf{B}$ by

$$F_0 x = \delta \Pi x + \theta(x - \Pi x). \tag{7.110}$$

The equations $\delta x = F_0 x$ and $x = \Pi x$ are equivalent. Indeed, any solution x_0 of the equation $x = \Pi x$ satisfies $\delta x = F_0 x$. Conversely, let x_0 be a solution to $\delta x = \delta \Pi x + \theta(x - \Pi x)$. Then

$$\left\|\delta\left(x_0 - \Pi x_0\right)\right\|_{\mathbf{B}} = \left\|r\left(x_0 - \Pi x_0\right)\right\|_{\mathbb{R}^n} + 2\left\|\delta\left(x_0 - \Pi x_0\right)\right\|_{\mathbf{B}}. \tag{7.111}$$

Consequently $\|x_0 - \Pi x_0\|_{\mathbf{D}} = 0$. Thus x_0 is a solution to $x = \Pi x$.

Let (7.107) be equivalent to (7.108) with continuous compact $F_0 : \mathbf{D} \to \mathbf{B}$. Equation (7.95) is equivalent to the equation $x = \Pi x$ with continuous compact $\Pi : \mathbf{D} \to \mathbf{D}$ defined by

$$\Pi x = \Lambda F_0 x + Y r x. \tag{7.112}$$

□

Lemma 7.19. *The set* $\mathcal{R}$ *of all solutions of a reducible equation is closed.*

Proof. The set $\mathcal{R}$ of all solutions of (7.104) is also the set of all solutions of the equation $x = \Pi x$ with continuous compact $\Pi : \mathbf{X} \to \mathbf{X}$. Let $x_0 = \lim_{k \to \infty} x_k$, $x_k \in \mathcal{R}$, $k = 1, 2, \ldots$. Then

$$x_0 = \lim_{k \to \infty} \Pi x_k = \Pi x_0. \tag{7.113}$$

Therefore $x_0 \in \mathcal{R}$. □

Theorem 7.20. *Let the space* $\mathbf{X}$ *be finite-dimensional. Then the property of being closed of the set of all solutions to* (7.104) *is sufficient and necessary for reducibility of this equation.*

Proof. Let the set $\mathcal{R}$ be closed. Then (7.104) is reducible to the equation $x = x + \rho(x,\mathcal{R})z$, where $\rho(x,\mathcal{R})$ is the distance between the point x and the set $\mathcal{R}$, $z \neq 0$ is a fixed element of $\mathbf{X}$. □

The equation $x\varphi(x) = 0$ with $\varphi(x) = \max\{0, \|x\|_{\mathbf{X}} - 1\}$ gives an example of nonreducible equation with a closed set of solutions. Indeed, the ball $\overline{\mathcal{B}}_1 = \{x \in \mathbf{X} : \|x\|_{\mathbf{X}} \le 1\}$ is the set of solutions. The assumption of the compactness of $\overline{\mathcal{B}}_1$ contradicts the equality $\overline{\mathcal{B}}_1 = \Pi\overline{\mathcal{B}}_1$ for continuous compact $\Pi : \mathbf{X} \to \mathbf{X}$.

Theorem 7.21. *Let* $\mathcal{Y}$ *be a bounded closed set in the space* $\mathbf{Y}$*. Then* (7.104) *is* $(\mathcal{Y},\mathbf{Y})$*-reducible if and only if the set of solutions of* (7.104) *belonging to* $\mathcal{Y}$ *is compact in* $\mathbf{Y}$.

Proof. Let (7.104) be $(\mathcal{Y},\mathbf{Y})$-reducible to the equation $x = \Pi x$. Then the compactness of the set $\mathcal{R}$ of solutions belonging to $\mathcal{Y}$ follows from the equality $\mathcal{R} = \Pi\mathcal{R}$.

Let the set $\mathcal{R}$ be compact, $x_0 \in \mathcal{R}$. Since the closed convex hull $\overline{\mathrm{co}}\mathcal{R}$ of the set $\mathcal{R}$ is compact in $\mathbf{Y}$ [119], there exists a continuous compact projector $\mathcal{P}$ from $\mathbf{Y}$ into $\overline{\mathrm{co}}\mathcal{R}$ [119]. Thus, we may define the operator $\Pi : \mathbf{Y} \to \overline{\mathrm{co}}\mathcal{R}$ by

$$\Pi x = \frac{x_0\rho(x,\mathcal{R}) + \mathcal{P}x}{1 + \rho(x,\mathcal{R})}, \tag{7.114}$$

where $\rho(x,\mathcal{R})$ is the distance between the point x and the set $\mathcal{R}$. This operator is continuous compact and the equality

$$\|\Pi x - x\|_{\mathbf{Y}} = \frac{\rho(x,\mathcal{R})}{1+\rho(x,\mathcal{R})}\|x - x_0\|_{\mathbf{Y}} \tag{7.115}$$

holds. Consequently, the set of solutions of the equation $x = \Pi x$ coincides with $\mathcal{R}$ and (7.104) is $(\mathcal{Y},\mathbf{Y})$-reducible to the equation $x = \Pi x$. □

Let $\mathcal{B}_1 = \{x \in \mathbf{X} : \|x\|_{\mathbf{X}} < 1\}$. Define continuous operators $\Gamma : \mathbf{X} \to \mathcal{B}_1$ and $\Gamma^{-1} : \mathcal{B}_1 \to \mathbf{X}$ by

$$\Gamma x = \frac{x}{1 + \|x\|_{\mathbf{X}}}, \qquad \Gamma^{-1}x = \frac{x}{1 - \|x\|_{\mathbf{X}}}. \tag{7.116}$$

Corollary 7.22. *Equation* (7.104) *is reducible if and only if the set* $\mathcal{R}$ *of all solutions of the equation is closed and the set* $\Gamma\mathcal{R} \subset \mathbf{X}$ *is precompact.*

Proof. If (7.104) is reducible, the fact that $\mathcal{R}$ is closed follows from Lemma 7.19 and the compactness of $\Gamma\mathcal{R}$ follows from the continuous compactness of the operator $\Gamma\Pi\Gamma^{-1} : \mathcal{B}_1 \to \mathcal{B}_1$ and the equality $\Gamma\mathcal{R} = (\Gamma\Pi\Gamma^{-1})\Gamma\mathcal{R}$. Let $\mathcal{R}$ be closed, and let $\Gamma\mathcal{R}$ be compact. By Theorem 7.21, there exists a continuous compact operator

$$\Pi_0 : \overline{\mathcal{B}}_1 \longrightarrow \overline{\mathrm{co}}\Gamma\mathcal{R} \tag{7.117}$$

such that the set of all solutions of the equation $y = \Pi_0 y$ coincides with $\overline{\Gamma\mathcal{R}}$. Then (7.104) is reducible to the equation $x = \Gamma^{-1}\Pi_0\Gamma x$. □

Theorem 7.21 implies some sufficient tests of reducibility of (7.104).

Test 1. Let the set $\mathcal{Y}$ be compact (in itself), and let the set of all solutions of (7.104) belonging to $\mathcal{Y}$ be closed. Then (7.104) is $(\mathcal{Y}, \mathbf{Y})$-reducible.

Test 2. Equation (7.104) is reducible if it has a finite set of solutions.

Test 3. Let the operator $\Omega : \mathbf{X} \to \mathbf{X}$ be such that its kth iteration Ω^k is continuous compact. Let, further, the set of all solutions of the equation

$$x = \Omega x \tag{7.118}$$

be closed. Then this equation is reducible.

Following [1], we will say that, on a bounded set of the space $\mathbf{X}$, a measure ψ of noncompactness is defined and that the operator $\Omega : \mathbf{X} \to \mathbf{X}$ is ψ-condensing if $\psi(\Omega\mathcal{X}) < \psi(\mathcal{X})$ for any bounded noncompact set $\mathcal{X} \subset \mathbf{X}$.

Test 4. Let the operator $\Omega : \mathbf{X} \to \mathbf{X}$ be ψ-condensing. Then (7.118) is $(X, \mathbf{X})$-reducible for any bounded closed X.

We say that the set of solutions of (7.104) admits a finite-dimensional parameterization if the set is homeomorphic to a closed subset of a finite-dimensional space (two sets are called homeomorphic if there is a continuous one-to-one mapping between them).

Theorem 7.23. *Let the set of all solutions of* (7.104) *admit a finite-dimensional parameterization. Then the equation is reducible.*

Proof. Denote by $\Theta : \mathbb{R}^m \to \mathbf{X}$ a homeomorphism between a closed subset $\mathcal{U} \subset \mathbb{R}^m$ and the set $\mathcal{R} \subset \mathbf{X}$ of all solutions of (7.104). Since the set $\mathcal{U}$ is closed, so the set $\mathcal{R}$ is denoted by

$$\Gamma x = \frac{x}{1 + \|x\|_{\mathbf{X}}}, \qquad \Gamma_0\alpha = \frac{\alpha}{1 + \|\alpha\|_{\mathbb{R}^n}}, \tag{7.119}$$

the operators $\Gamma : \mathbf{X} \to \mathbf{X}$ and $\Gamma_0 : \mathbb{R}^m \to \mathbb{R}^m$. Consider a sequence $\{x_k\} \subset \Gamma\mathcal{R}$. Since $x_k = \Gamma\theta\Gamma_0^{-1}\alpha_k$, where $\{\alpha_k\} \subset \Gamma\mathcal{U}$ and $\Gamma\theta\Gamma_0^{-1}$ is a homeomorphism between $\Gamma_0\mathcal{U}$ and $\Gamma\mathcal{R}$, it is possible to extract from bounded $\{\alpha_k\}$ and $\{x_k\}$ convergent sequences. Therefore the set $\Gamma\mathcal{R}$ is precompact. Hence, by Theorem 7.21, (7.104) is reducible. □

In contrast to the linear case the nonlinear reducible equation does not always permit a finite-dimensional parameterization, see [32].

Theorem 7.24. *Let $\mathcal{M}$ be a closed subset of $\mathbb{R}^m$ and let $\mathcal{R}$ be the set of all the solutions of* (7.104). *The following assertions are equivalent.*

(a) *There exists a homeomorphism between $\mathcal{R}$ and $\mathcal{M}$.*

(b) *There exists a continuous vector functional $\varphi : \mathbf{X} \to \mathbb{R}^m$ such that $\varphi\mathcal{R} \subset \mathcal{M}$ and the system of equation*

$$\Phi x = g, \qquad \varphi x = \alpha \tag{7.120}$$

is correctly solvable for any $\alpha \in \mathcal{M}$.

(c) *Equation* (7.104) *is reducible to the equation $x = \theta\varphi x$ with continuous $\varphi : \mathbf{X} \to \mathbb{R}^m$, $\theta : \mathbb{R}^m \to \mathbf{X}$, and, besides, $\alpha = \varphi\theta\alpha$ for $\alpha \in \mathcal{M}$.*

Proof. (a)⇒(b). Denote by θ a homeomorphism between $\mathcal{R}$ and $\mathcal{M}$. Let the vector functional $\varphi : \mathbf{X} \to \mathbb{R}^m$ be a continuous extension of the vector functional $\theta^{-1} : \mathcal{R} \to \mathcal{M}$. Since the solution of system (7.120) belongs to $\mathcal{R}$, the system is equivalent to

$$\Phi x = g, \qquad \theta^{-1} x = \alpha. \tag{7.121}$$

Hence system (7.120) has a unique solution $x = \theta\alpha$ for any $\alpha \in \mathcal{R}$ and the solution continuously depends on α.

(b)⇒(c). If $x \in \mathcal{R}$, then $\varphi x \in \mathcal{M}$. Consequently, $x = \theta\varphi x$. If $x = \theta\varphi x$, then $\varphi x = \varphi\theta\varphi x$, $\varphi x \in \mathcal{M}$, $x \in \mathcal{R}$.

(c)⇒(a). Let φ_0 be a restriction of the vector functional φ to the set $\mathcal{R}$. Then $\theta = \varphi_0^{-1} : \mathcal{M} \to \mathcal{R}$ is a homomorphism between $\mathcal{R}$ and $\mathcal{M}$. □

Section 2.5 was devoted to linear abstract equations with generalized Volterra operators. Let us consider briefly the nonlinear case of equation with abstract Volterra operators.

Define in the Banach space $\mathbf{X}$ a family of linear and bounded in common operators $P^v : \mathbf{X} \to \mathbf{X}$, $v \in [0,1]$, such that

$$\begin{gathered} P^v P^u = P^{\min(v,u)} \quad \text{for } v, u \in [0,1], \\ \lim_{u \to v} P^u x = P^v x \quad \text{for } x \in \mathbf{X},\ v \in [0,1], \\ P^0 x = 0, \quad P^1 x = x \quad \text{for } x \in \mathbf{X}. \end{gathered} \tag{7.122}$$

The operator $F : \mathbf{X} \to \mathbf{X}$ is said to be $\mathcal{B}$-Volterra if $P^v F P^v = P^v F$ for any $v \in [0,1]$.

Consider the nonlinear equation

$$\Phi x = g \tag{7.123}$$

with $\Phi : \mathbf{X} \to \mathbf{X}$, $g \in \mathbf{X}$.

The element $x^\nu \in \mathbf{X}$, $\nu \in (0,1)$, is called a local solution to (7.123) if $P^\nu x^\nu = x^\nu$, $P^\nu \Phi x^\nu = P^\nu g$. Equation (7.123) is called $\mathcal{B}$-Volterra-reducible if it is reducible to the equation $x = \Pi x$ with $\mathcal{B}$-Volterra operator $\Pi : \mathbf{X} \to \mathbf{X}$ and the sets of local solutions of the equations $x = \Pi x$ and (7.123) coincide. Local solution x^ν of (7.123) is said to be continuable (extendable) if there exists a local solution x^u of (7.123) such that $u \in (\nu, 1)$, $P^\nu x^u = x^\nu$.

We say that the property A is fulfilled if there exists a sequence of linear bounded in common operators $\theta_k : \mathbf{X} \to \mathbf{X}$, $k = 1, 2, \ldots$, such that

$$\begin{gathered} \lim_{k\to\infty} \theta_k x = x \quad \text{for } x \in \mathbf{X}, \\ P^\nu \theta_k = P^\nu \theta_k P^{\nu - 1/k} \quad \text{for } \nu > \frac{1}{k}, \\ P^\nu \theta_k = 0 \quad \text{for } \nu \le \frac{1}{k}. \end{gathered} \tag{7.124}$$

Theorem 7.25 (see [91, 92]). *Let* (7.123) *be $\mathcal{B}$-Volterra-reducible. Then*

(1) *equation* (7.123) *has at least one local solution;*
(2) *any local solution of* (7.123) *is extendable;*
(3) *if the set of all local solutions of* (7.123) *is bounded, the set of solutions of* (7.123) *is nonempty and compact;*
(4) *if the set of all local solutions of* (7.123) *is bounded and property A is fulfilled, the set of solutions of* (7.123) *is connected.*

7.4. A priori inequalities

Any existence theorem based on fixed point principles assumes the presence of an a priori estimate of possible solution. It is well known that it is difficult to establish a priori estimates even in the case of differential equations. As for the case of functional differential equations, the difficulties increase (see, e.g., [56]) and the literature thereof). This is why we do not attempt to discuss the problem for abstract functional differential equation. We will restrict ourselves to the space of absolutely continuous functions and offer an approach to the problem on the base of the concept of "a priori inequality."

7.4.1. The concept of a priori inequality

The next argument may illustrate the idea of a priori inequality.

Let the equation $\dot{x} = Fx$ be reducible to the form

$$\dot{x}(t) = \varphi(t, x(a)). \tag{7.125}$$

As it was shown in Section 7.3.1, such a reducibility is possible if the Cauchy problem $\dot{x} = Fx$, $x(a) = \alpha$ is correctly solvable.

Consider the boundary value problem

$$\dot{x} = Fx, \qquad lx = \beta, \tag{7.126}$$

where

$$lx \overset{\text{def}}{=} \Psi x(a) + \int_a^b \Phi(s)\dot{x}(s)ds, \qquad \det \Psi \neq 0. \tag{7.127}$$

The general solution of the equation $\dot{x} = Fx$ has the form

$$x(t) = \alpha + \int_a^t \varphi(s,\alpha)ds. \tag{7.128}$$

Applying the functional l to both sides of the latter equality we get

$$\gamma \overset{\text{def}}{=} \Psi^{-1}\beta = \alpha + \Psi^{-1}\int_a^b \Phi(s)\varphi(s,\alpha)ds \overset{\text{def}}{=} \alpha + \theta\alpha. \tag{7.129}$$

In such a way we have reduced problem (7.126) to the algebraic equation

$$\alpha + \theta\alpha = \gamma \tag{7.130}$$

with respect to α. From this equality we have

$$|Q\alpha| \leq ||\Psi^{-1}|| \int_a^b ||\Phi(s)|| \cdot |\varphi(s,\alpha)| ds. \tag{7.131}$$

Here and below in this section $|\cdot|$ denotes the norm in $\mathbb{R}^n$ with the property of monotonicity. Namely, for $\alpha = \operatorname{col}(\alpha_1,\dots,\alpha_n)$, $\beta = \operatorname{col}(\beta_1,\dots,\beta_n)$ the inequalities $|\alpha_i| \leq |\beta_i|$, $i = 1,\dots,n$, imply $|\alpha| \leq |\beta|$. The symbol $\|\cdot\|$ denotes the norm of $n \times n$ matrix concordant with $|\cdot|$.

Thus, by the presence of the estimate of the form

$$|\varphi(s,\alpha)| \leq m(s,\alpha), \tag{7.132}$$

we can establish solvability of (7.130). For instance, let the function $m(s,\alpha)$ be bounded or be such that

$$\lim_{\alpha\to\infty} \frac{m(s,\alpha)}{\alpha} = 0 \tag{7.133}$$

holds. Then (7.130) and, consequently, problem (7.126) are solvable.

As an example consider the boundary value problem

$$\dot{x}(t) = (Fx)(t) \overset{\text{def}}{=} f(t,(S_h x)(t),(S_g \dot{x})(t)), \qquad lx = \beta. \tag{7.134}$$

Assume that $t - h(t) \geq \text{const} > 0$, $t - g(t) \geq \text{const} > 0$ (or, what is more general, assume that the operators $S_h : \mathbf{D} \to \mathbf{L}$, $S_g : \mathbf{L} \to \mathbf{L}$ are Volterra and nilpotent). Under such an assumption the equation $\dot{x} = Fx$ is reducible to the form $\dot{x}(t) = \varphi(t, x(a))$, since the solution of the Cauchy problem $\dot{x} = Fx$, $x(a) = \alpha$ may be constructed by the "step-by-step method." Let, further,

$$lx = \Psi x(a) + \int_a^b \Phi(s)\dot{x}(s)ds, \qquad \det \Psi \neq 0,$$
$$\lim_{\|x\|_{\mathbf{D}} \to \infty} \frac{\|Fx\|_{\mathbf{L}}}{\|x\|_{\mathbf{D}}} = 0. \tag{7.135}$$

The latter assumptions guarantee the reducibility of problem (7.134) to (7.130) and the solvability of this equation.

Definition 7.26. For the equation

$$\dot{x} = Fx, \tag{7.136}$$

a *canonical a priori inequality* is said to be fulfilled in the ball with radius r if there exists a function $m : [a, b] \times [0, r] \to [0, \infty)$ such that $m(\cdot, s)$ is summable at each $s \in [0, r]$ and the inequality

$$|\dot{x}(t)| \leq m(t, |x(a)|) \tag{7.137}$$

holds for any solution x of (7.136) with $|x(a)| \leq r$.

If there exists a function $m : [a, b] \times [0, \infty) \to [0, \infty)$ such that $m(\cdot, s)$ is summable at each $s \in [0, \infty)$ and (7.137) holds for any solution of (7.136), we say that for (7.136), a canonical a priori inequality is fulfilled.

It should be remarked that the assumption about an a priori inequality does not assume any existence of solutions at all and means only the fact that (7.136) has no solutions that violate (7.137).

In case (7.136) is reducible to the canonical form

$$\dot{x}(t) = \varphi(t, x(a)), \tag{7.138}$$

the function $m(t, s)$ is a majorant for the right-hand side of the equation

$$|\varphi(t, x(a))| \leq m(t, |x(a)|). \tag{7.139}$$

Let us consider a scheme of using a priori inequalities to illustrate the connection between the presence of such an inequality, reducibility of the equation, and solvability of the Cauchy problem.

Let (7.136) be equivalent to the equation

$$\dot{x} = F_0 x \tag{7.140}$$

with continuous compact $F_0 : \mathbf{D} \to \mathbf{L}$.

Definition 7.27. A priori inequality (7.137) fulfilled on the ball with radius r is said to possess the property Λ if it holds for all solutions x_λ, $|x_\lambda(a)| \leq r$, of the family of the equations

$$\dot{x} = \lambda F_0 x, \quad \lambda \in [0,1]. \tag{7.141}$$

Let a canonical a priori inequality on the ball with radius r be fulfilled for solutions of (7.136) and possess the property Λ. Then, by Leray-Schauder theorem, the Cauchy problem

$$\dot{x} = Fx, \qquad x(a) = \alpha, \quad |\alpha| \leq r, \tag{7.142}$$

has at least one solution $x \in \mathbf{D}$.

Indeed, the substitution

$$x(t) = \alpha + \int_a^t z(s)ds \tag{7.143}$$

reduces (7.141) to the form

$$z = \lambda F_0\left(\alpha + \int_a^{(\cdot)} z(s)ds\right) \tag{7.144}$$

with continuous compact operator $\Omega : \mathbf{L} \to \mathbf{L}$,

$$\Omega z = \lambda F_0\left(\alpha + \int_a^{(\cdot)} z(s)ds\right). \tag{7.145}$$

By virtue of the a priori inequality, the a priori estimate

$$||z_\lambda||_{\mathbf{L}} \leq \int_a^b m(s,|\alpha|)ds \tag{7.146}$$

holds for any $\lambda \in [0,1]$.

Hence, by Leray-Schauder theorem, the equation $z = \Omega z$ has a solution z_1 and consequently

$$x(t) = \alpha + \int_a^t z_1(s)ds \tag{7.147}$$

is a solution of the Cauchy problem.

If inequality (7.137) is fulfilled on the ball of any radius and possesses property Λ, then the Cauchy problem $\dot{x} = Fx$, $x(a) = \alpha$ is solvable for any $\alpha \in \mathbb{R}^n$.

It is difficult to check the presence of property Λ if an explicit form of the operator F_0 is unknown. We offer below an effective method to overcome this difficulty.

7.4.2. A scheme of construction of a priori inequality and its realization with the majorant Cauchy problem

A series of researches of the Perm Seminar about construction of a priori inequalities on the base of one-sided as well as two-sided estimates of the operator F_0 is systematized in [32]. We will restrict ourselves below to the following scheme.

Let the estimate

$$|(Fx)(t)| \leq (\mathcal{M}(|x(a)|, |\dot{x}(\cdot)|))(t) \tag{7.148}$$

hold, where the operator $\mathcal{M}$ acts from the space $\mathbb{R}^1 \times \mathbf{L}^1$ into the space $\mathbf{L}^1$ of summable scalar functions and, besides, is isotonic in each argument. Then for any solution x to (7.136) we have

$$|\dot{x}(t)| \leq (\mathcal{M}(|x(a)|, |\dot{x}(\cdot)|))(t). \tag{7.149}$$

Let further the Chaplygin-like theorem be valid for the *majorant equation*

$$z = \mathcal{M}(v, z) \tag{7.150}$$

in the space $\mathbf{L}^1$: if the inequality

$$\xi \leq \mathcal{M}(v, \xi) \tag{7.151}$$

holds for $\xi \in \mathbf{L}^1$, then the estimate $\xi(t) \leq z(t, v)$ for the solution $z(t, v)$ of (7.150) is valid. Putting $\xi(t) = |\dot{x}(t)|$, we get (7.137) from (7.149). Namely

$$|\dot{x}(t)| \leq m(t, |x(a)|) \stackrel{\text{def}}{=} (\mathcal{M}(|x(a)|, z(\cdot, |x(a)|)))(t). \tag{7.152}$$

In case

$$\mathcal{M}(v, z)(t) = \omega\left(t, v + \int_a^t z(s)ds\right), \tag{7.153}$$

the substitution $\zeta(t) = v + \int_a^t z(s)ds$ reduces (7.150) to the Cauchy problem

$$\dot{\zeta}(t) = \omega(t, \zeta), \qquad \zeta(a) = v. \tag{7.154}$$

For such a problem, the Chaplygin theorem on a differential inequality [44] holds. If the equation $\dot{\zeta} = \omega(t,\zeta)$ is solvable in an explicit form, the problem of construction of the a priori inequality has its solution. However the function ω is convenient as a majorant to the Nemytskii operator only. The more complicated operators are expected to have more complicated majorants. The difficulties arising on the way of construction of such majorants are connected with the fact that we are forced to deal with integro-functional inequalities instead of the well known integral inequalities. Below we construct the estimate of all the solutions of inequality (7.149) on the base of a special majorant Cauchy problem. Here we will be in need of the following auxiliary assertion. Below we denote by $\mathbf{L}^1_\infty$ the space of measurable and essentially bounded functions $z : [a,b] \to \mathbb{R}^1$.

Lemma 7.28. *Let $B : \mathbf{L}^1 \to \mathbf{L}^1$ be a linear isotonic Volterra operator and let the function $v \in \mathbf{L}^1_\infty$ be nonnegative. Then for any nonnegative $y \in \mathbf{L}^1$, the inequality*

$$\int_a^t v(s)(By)(s)ds \le \int_a^t \kappa(s)y(s)ds, \quad t \in [a,b], \tag{7.155}$$

holds with

$$\kappa(t) = \frac{d}{dt}\int_a^b v(s)\Big(B\chi_{[a,t]}\Big)(s)ds, \tag{7.156}$$

$\chi_{[a,t]}$ is the characteristic function of the segment $[a,t]$.

Proof. The integral $\int_a^t v(s)(By)(s)ds$ represents by each fixed $t \in [a,b]$ a linear functional on the space of functions summable on $[a,t]$. From this and the Volterra-property of B we obtain the representation

$$\int_a^t v(s)(By)(s)ds = \int_a^t K(t,s)y(s)ds, \tag{7.157}$$

where the kernel $K(t,s)$ is essentially bounded for each fixed t.

The inequality

$$K(t,s) \le K(b,s) \tag{7.158}$$

holds for each $t \in [a,b]$ a.e. on $[a,t]$. Indeed, assume the converse: there exist t_0 and a set $\Delta \subset [a,t_0]$ of positive measure such that

$$K(t_0,s) > K(b,s), \quad s \in \Delta. \tag{7.159}$$

Denote by χ_Δ the characteristic function of the set Δ. It is obvious that

$$\mathcal{I} = \int_\Delta \big(K(b,s) - K(t_0,s)\big)ds < 0. \tag{7.160}$$

On the other hand,

$$\begin{aligned}\mathcal{I} &= \int_a^b K(b,s)\chi_\Delta(s)ds - \int_a^{t_0} K(t_0,s)\chi_\Delta(s)ds \\ &= \int_a^b v(s)\Big(B\chi_\Delta\Big)(s)ds - \int_a^{t_0} v(s)\Big(B\chi_\Delta\Big)(s)ds \geq 0.\end{aligned} \tag{7.161}$$

The contradiction proves inequality (7.158). It remains to observe that

$$\int_a^t K(b,s)ds = \int_a^b K(b,s)\chi_{[a,t]}(s)ds = \int_a^b v(s)\Big(B\chi_{[a,t]}\Big)(s)ds. \tag{7.162}$$

□

Let us note that

$$\kappa(t) = \big(B^* v\big)(t) \tag{7.163}$$

since

$$\int_a^b v(s)(By)(s)ds = \int_a^b (B^* v)(s)y(s)ds. \tag{7.164}$$

Consider the inequality

$$z \leq \mathcal{M}(v,z) \stackrel{\text{def}}{=} BN\mathcal{M}_1(v,z). \tag{7.165}$$

Here $B : \mathbf{L}^1 \to \mathbf{L}^1$ is linear Volterra isotonic, the operator $\mathcal{M}_1$ acts from $\mathbb{R}^1 \times \mathbf{L}^1$ into a linear space Ξ of measurable functions $\xi : [a,b] \to \mathbb{R}^1$ and is defined by

$$\mathcal{M}_1(v,z)(t) = q(t)v + u(t)\int_a^t v(s)z(s)ds \tag{7.166}$$

with nonnegative $v \in \mathbf{L}^1_\infty$, $q, u \in \Xi$, $q(t) \leq u(t)$ a.e. on $[a,b]$, $N : \Xi \to \mathbf{L}^1$ is the operator of Nemytskii, $(N\xi)(t) = \omega(t,\xi(t))$, $\omega(t,\cdot)$ is continuous and nondecreasing. The notion of the solution of inequality (7.165) on $[a,c] \subset [a,b]$ for fixed $v \geq 0$ is defined as follows. The solution is nonnegative summable on $[a,c]$ function z such that

$$z(t) \leq \mathcal{M}\big(v,z^c\big)(t) \tag{7.167}$$

a.e. on $[a,c]$. Here z^c is a summable on $[a,b]$ function such that $z^c(t) = z(t)$ a.e. on $[a,c]$.

To construct the majorant Cauchy problem to inequality (7.165), we define the function $\Omega : [a,b] \times [0,\infty) \to [0,\infty)$ by

$$\Omega(t,y) = \omega\big(t,u(t)y\big)\big(B^* v\big)(t). \tag{7.168}$$

Definition 7.29. The Cauchy problem

$$\dot{y} = \Omega(t, y), \qquad y(a) = \beta \tag{7.169}$$

is said to have the upper solution $y(t, \beta)$ on $[a, b]$ if y is a solution such that for each $c \in (a, b]$, any solution y_c of the equation $\dot{y} = \Omega(t, y)$, defined on $[a, c)$ and satisfying the initial condition $y_c(a) = \beta$, satisfies the inequality $y_c(t) \leq y(t)$, $t \in [a, c)$.

Lemma 7.30. *Let problem* (7.169) *have an upper solution* $y(t, \beta)$ *on* $[a, b]$. *Let, furthermore,* $v \leq \beta$, *and let* z *be a solution on* $[a, c] \subset [a, b]$ *to inequality* (7.165). *Then the inequality*

$$z(t) \leq (BN\xi_v)(t) \tag{7.170}$$

with $\xi_v(t) = u(t)y(t, v)$ *holds a.e. on* $[a, c]$.

Proof. Let z be a solution to inequality (7.165) defined on $[a, c]$,

$$z^c(t) = \begin{cases} z(t) & \text{if } t \in [a, c], \\ 0 & \text{if } t \notin [a, c]. \end{cases} \tag{7.171}$$

It is clear that z^c is the solution to inequality (7.165) defined on $[a, b]$. The inequality

$$\eta(t) \leq u(t) \int_a^t v(s)(BN\eta)(s)ds + q(t)v \tag{7.172}$$

for $\eta(t) = \mathcal{M}_1(v, z^c)(t)$ holds a.e. on $[a, b]$.

From this, we obtain for $\zeta(t) = \eta(t)/u(t)$ the inequality

$$\zeta(t) \leq \int_a^t v(s)[BN(\zeta \cdot u)](s)ds + v. \tag{7.173}$$

By Lemma 7.28,

$$\zeta(t) \leq \int_a^t (B^* v)(s)N(\zeta \cdot u)(s)ds + v. \tag{7.174}$$

Denote the right-hand side of the latter inequality by w. It is clear that $w \in \mathbf{D}^1$, $\dot{w} = \Omega(t, \zeta) \leq \Omega(t, w)$, $w(a) = v$. By virtue of the theorem of Chaplygin on differential inequality we get the estimate $w(t) \leq y(t, v)$. Hence

$$\zeta(t) = \frac{\eta(t)}{u(t)} \leq y(t, v), \qquad \eta(t) \leq u(t)y(t, v). \tag{7.175}$$

From this, by virtue of isotonicy of the right-hand side of (7.165), we obtain

$$z^c(t) \le \big[BN\big(u(\cdot)y(\cdot,v)\big)\big](t) \tag{7.176}$$

a.e. on $[a,b]$. □

Lemma 7.30 permits realizing the construction of a canonical a priori inequality for (7.136) under the assumption that $F : \mathbf{D} \to \mathbf{L}$ satisfies the condition

$$\big|(Fx)(t)\big| \le (B_1NMx)(t) + (B_2|\dot{x}|)(t), \quad t \in [a,b],\ x \in \mathbf{D}. \tag{7.177}$$

Here $B_1, B_2 : \mathbf{L}^1 \to \mathbf{L}^1$ are linear isotonic Volterra operators, the spectral radius of B_2 is less than one; the operator M acts from $\mathbf{D}$ into a linear space Ξ of measurable functions $\xi : [a,b] \to \mathbb{R}^1$ and is defined by

$$(Mx)(t) = q(t)\big|x(a)\big| + u(t)\int_a^t v(s)\big|\dot{x}(s)\big|\,ds, \tag{7.178}$$

where $v \in \mathbf{L}^1_\infty$ is nonnegative, $q, u \in \Xi$, $q(t) \le u(t)$ a.e. on $[a,b]$, $N : \Xi \to \mathbf{L}^1$ is the operator of Nemytskii, and $(N\xi)(t) = \omega(t,\xi(t))$, $\omega(t,\cdot)$ is continuous and does not decrease.

It should be observed that in case $q(t) = u(t) = v(t) = 1$ the operator M majorizes the operator M_1 of the form $(M_1x)(t) = \max_{s\in[a,t]} |x(s)|$:

$$\max_{s\in[a,t]}\big|x(s)\big| \le \big|x(a)\big| + \int_a^t \big|\dot{x}(s)\big|\,ds. \tag{7.179}$$

Any solution of (7.136) satisfies the inequality

$$|\dot{x}| \le B_1NMx + B_2|\dot{x}|. \tag{7.180}$$

Definition 7.31. The equation

$$\dot{y} = \Omega(t,y) \tag{7.181}$$

is said to be a *majorant equation* that corresponds to inequality (7.177) if the function $\Omega : [a,b] \times [0,\infty) \to [0,\infty)$ is defined by

$$\Omega(t,y) = \omega\big(t,u(t)y\big)\Big[B_1^*\big(I - B_2^*\big)^{-1}v\Big](t). \tag{7.182}$$

The Cauchy problem

$$\dot{y} = \Omega(t,y), \qquad y(a) = \beta \tag{7.183}$$

is said to be the *majorant Cauchy problem*.

Lemma 7.30 permits solving the problem about estimation of all the solutions of inequality (7.177) with respect to their initial values and, in such a way, to get the a priori inequality. Namely the following assertion holds.

Lemma 7.32. *Let* $\beta \geq 0$ *and let the majorant Cauchy problem have the upper solution* $y(t,\beta)$ *defined on* $[a,b]$*. Let, furthermore,* x *be defined on* $[a,c] \subset [a,b]$ *solution of inequality* (7.180) *such that* $|x(a)| \leq \beta$*. Then, a.e. on* $[a,c]$,

$$|\dot{x}(t)| \leq m(t, |x(a)|), \tag{7.184}$$

with

$$m(t,v) = \left\{(I - B_2)^{-1} B_1 N z_v\right\}(t), \quad z_v(t) = u(t)y(t,v). \tag{7.185}$$

Proof. Denote $|\dot{x}(t)| = z(t)$, $|x(a)| = v$. Then $Mx = \mathcal{M}_1(v,z)$ and we may use Lemma 7.30. □

From Lemma 7.32 we obtain the following assertion on a priori inequality for solutions of (7.136).

Theorem 7.33. *Let the operator* F *satisfy the condition* (7.177)*. Let, furthermore,* $\beta \geq 0$*, and let majorant Cauchy problem* (7.183) *have the upper solution* $y(t,\beta)$ *defined on* $[a,b]$*. Then for any solution of* (7.136) *canonical, a priori inequality* (7.137) *is fulfilled on the ball with radius* β*. Here the function* m *is defined by equality* (7.185).

It should be remarked that the function $m(t,v)$ in a priori inequality (7.137) does not decrease in v if the inequality is constructed on the base of the majorant Cauchy problem.

Next consider the conditions under which a priory inequalities have the property Λ.

Denote by $\mathbf{Z}$, $\mathbf{Z}_1$, $\mathbf{Z}_2$ the linear spaces of measurable functions defined on $[a,b]$.

Definition 7.34. The operator $F : \mathbf{D} \to \mathbf{L}$ is said to satisfy *condition* H if there exist the operators $\theta : \mathbf{D} \to \mathbf{Z}_1$, $\Sigma : \mathbf{L} \to \mathbf{Z}_2$, $\mathcal{H} : \theta\mathbf{D} \times \Sigma\mathbf{L} \to \mathbf{L}$, $H : \theta\mathbf{D} \to \mathbf{L}$ such that the operator F may be represented in the form

$$Fx = \mathcal{H}(\theta x, \Sigma\dot{x}), \tag{7.186}$$

the product $H\theta : \mathbf{D} \to \mathbf{L}$ is continuous compact, and the function $y = Hz$ is the unique solution to the equation

$$y = \mathcal{H}(z, \Sigma y) \tag{7.187}$$

for each $z \in \theta\mathbf{D}$.

It should be noticed that (7.136) is reducible to the form (7.140) with $F_0 = H\theta$ if $F : \mathbf{D} \to \mathbf{L}$ satisfies condition H. We obtain the case of the reducibility to the canonical form if $\theta x \equiv x(a)$.

Let $F : \mathbf{D} \to \mathbf{L}$ satisfy condition H and, besides, the inequality

$$\left| \mathcal{H}(\theta x, \Sigma y) \right| \leq B_1 N M x + B_2 |y| \tag{7.188}$$

with the operators B_1, B_2, N, and M defined as in (7.177) holds for each $x \in \mathbf{D}$ and $y \in \mathbf{L}$.

Definition 7.35. The majorant Cauchy problem (7.183) constructed according to the operators B_1, B_2, N, and M is called the *majorant Cauchy problem* relevant to inequality (7.188).

Theorem 7.36. *Let $F : \mathbf{D} \to \mathbf{L}$ satisfy the conditions H and* (7.188). *Let, furthermore, the relevant majorant Cauchy problem* (7.183) *have the upper solution $y(t,\beta)$ defined on $[a,b]$ for $\beta \geq 0$. Then the canonical a priori inequality* (7.137) *for the solutions to* (7.136) *holds, where the function m is defined by* (7.185). *This a priori inequality has the property Λ.*

Proof. Since

$$|Fx| = \left| \mathcal{H}(\theta x, \Sigma \dot{x}) \right| \leq B_1 N M x + B_2 |\dot{x}|, \tag{7.189}$$

inequality (7.137) holds, by Theorem 7.33, for all the solutions of the equation

$$\dot{x} = \lambda H \theta x \tag{7.190}$$

such that $|x(a)| \leq \beta$ and $\lambda = 1$. Inequality (7.137) is obvious for $\lambda = 0$. Next let $\lambda \in (0,1)$. Any solution of the equation $\dot{x} = \lambda H \theta x$ is a solution to the equation

$$\dot{x} = \lambda \mathcal{H}\left(\theta x, \Sigma \frac{1}{\lambda} \dot{x}\right) \tag{7.191}$$

by the definition of the operator H. On the other hand, any solution of the latter equation satisfies the inequality

$$|\dot{x}| \leq \lambda \left| \mathcal{H}\left(\theta x, \Sigma \frac{1}{\lambda} \dot{x}\right) \right|. \tag{7.192}$$

Therefore,

$$|\dot{x}| \leq \lambda B_1 N M x + \lambda B_2 \left| \frac{1}{\lambda} \dot{x} \right| \leq B_1 N M x + B_2 |\dot{x}|. \tag{7.193}$$

This gives the required result by virtue of Lemma 7.32. □

7.5. Nonlinear boundary value problems

A priori inequalities together with certain tests of reducibility allow us to formulate some theorems on solvability of the boundary value problem

$$\dot{x}(t) = (Fx)(t), \quad t \in [a,b]; \qquad \eta x = 0 \tag{7.194}$$

with continuous vector functional $\eta : \mathbf{D} \to \mathbb{R}^n$.

The proofs of such theorems follow two schemes suggested below on the base of a priori inequalities. The first one deals with the equation $\dot{x} = Fx$ being reducible to the canonical form

$$\dot{x}(t) = \varphi\big(t, x(a)\big). \tag{7.195}$$

The second scheme uses the condition H.

If the equation $\dot{x} = Fx$ is reducible to the form (7.195), solvability of problem (7.194) is equivalent to solvability of the equation

$$\eta\left[\alpha + \int_a^{(\cdot)} \varphi(s,\alpha)ds\right] = 0 \tag{7.196}$$

with respect to α. Rewrite the latter equation in the form

$$\alpha = B\alpha \tag{7.197}$$

with continuous $B : \mathbb{R}^n \to \mathbb{R}^n$,

$$B\alpha = \alpha - \eta\left[\alpha + \int_a^{(\cdot)} \varphi(s,\alpha)ds\right]. \tag{7.198}$$

Any solution α_0 of (7.197) corresponds to the solution x of problem (7.194), which coincides with the solution of the Cauchy problem

$$\dot{x} = Fx, \qquad x(a) = \alpha_0. \tag{7.199}$$

The effectiveness of such a reduction of the infinite-dimensional problem (7.194) to the finite-dimensional one (7.197) depends on the information given about the function $\varphi(t,\alpha)$. An important information of the form

$$\big|\varphi(t,\alpha)\big| \leq m(t,|\alpha|) \tag{7.200}$$

gives a priori inequality (7.137).

Let the functional $\mu : \mathbf{L}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ do not decrease in the first argument, and

$$\big|x(a) - \eta x\big| \leq \mu\big(\big|\dot{x}(\cdot)\big|, \big|x(a)\big|\big) \quad \forall x \in \mathbf{D}. \tag{7.201}$$

Then the operator B has a fixed point and consequently problem (7.194) is solvable under any of the following conditions:

(i) the set of positive solutions to the inequality

$$\delta \le \mu[m(\cdot,\delta),\delta] \tag{7.202}$$

is bounded;

(ii) the functions $m(t,\cdot)$ and $\mu(z,\cdot)$ do not decrease and there exists $\delta > 0$ such that

$$\delta \ge \mu[m(\cdot,\delta),\delta]. \tag{7.203}$$

Condition (7.202) is fulfilled if

$$\overline{\lim_{\delta\to\infty}}\frac{1}{\delta}\mu[m(\cdot,\delta),\delta] < 1. \tag{7.204}$$

It should be noted that condition (7.203) guarantees the solvability of problem (7.194) as well as in the case when the Cauchy problem $\dot{x} = Fx$, $x(a) = \alpha$ is correctly solvable for all α such that $|\alpha| \le \delta$ and equality (7.137) holds on the ball with radius δ.

Consider the second scheme of using a priori inequalities. Let $F : \mathbf{D} \to \mathbf{L}$ satisfy condition H (see Definition 7.34). Then the equation $\dot{x} = Fx$ is equivalent to the equation

$$x(t) = x(a) + \int_a^t (H\theta x)(s)ds \tag{7.205}$$

with continuous compact $H\theta : \mathbf{D} \to \mathbf{L}$ and problem (7.194) is equivalent to the equation

$$x = \Pi x \stackrel{\text{def}}{=} x(a) - \eta x + \int_a^{(\cdot)} (H\theta x)(s)ds. \tag{7.206}$$

The operator $\Pi : \mathbf{D} \to \mathbf{D}$ is continuous compact if the continuous vector functional $\eta : \mathbf{D} \to \mathbb{R}^n$ is bounded on every ball. In this case the Leray-Schauder theorem may be used. By this theorem, (7.206) has a solution if there exists a general a priori estimate of all the solutions x_λ of the family of the equations

$$x = \lambda\Pi x, \tag{7.206$_\lambda$}$$

that is, if there exists $d > 0$ such that

$$||x_\lambda||_{\mathbf{D}} \le d, \quad \lambda \in [0,1]. \tag{7.207}$$

The main difficulty in getting such an estimate arises in the case when the explicit form of the operator $F : \mathbf{D} \to \mathbf{L}$ is unknown.

The a priori estimate of the solution of (7.206) might be obtained as follows. Any solution x of (7.206) is a solution to the equation $\dot{x} = Fx$. Therefore

$$|\dot{x}(t)| \leq m(t, |x(a)|). \tag{7.208}$$

On the other hand, $\eta x = 0$. Therefore we have in addition to (7.208) the inequality

$$|x(a)| \leq \mu(|\dot{x}(\cdot)|, |x(a)|) \tag{7.209}$$

if we have the majorant μ (7.201). Thus, if x is a solution to problem (7.194), then the norm $|x(a)|$ satisfies the inequality

$$\delta \leq \mu[m(\cdot, \delta), \delta]. \tag{7.210}$$

Condition (7.202) guarantees the existence of δ_0 such that $\delta_0 \geq \delta$ for any $\delta > 0$ that satisfies inequality (7.210). In this case we have $|x(a)| \leq \delta_0$ and

$$\|x\|_{\mathbf{D}} \leq \delta_0 + \sup_{\delta \in [0, \delta_0]} \|m(\cdot, \delta)\|_{\mathbf{L}^1}. \tag{7.211}$$

Thus the required estimate (7.207) is obtained for $\lambda = 1$. Without additional assumptions with respect to inequality (7.137), it is impossible to get the estimate (7.207) for $\lambda \in (0, 1)$ in the general case. But such an estimate may be obtained if inequality (7.137) possesses the property Λ. Indeed, in this case the a priori inequality (7.208) holds for the solutions of the family $\dot{x} = \lambda H\theta x$, $\lambda \in (0, 1)$. Since the inequality $\lambda|x(a) - \eta x| \leq \mu(|\dot{x}(\cdot)|, |x(a)|)$ follows from (7.201) for any λ, the initial value $x(a)$ of the solution x of (7.206) satisfies inequality (7.210) for any λ. Thus we obtain the a priori estimate (7.211) under condition (7.202).

In [32] there are presented some theorems on the solvability of boundary value problems on the base of a priori inequalities. We restrict ourselves to the following assertion.

Theorem 7.37. *Let the operator F satisfy the conditions H and* (7.188). *Let, furthermore, the corresponding majorant Cauchy problem* (7.183) *have, for a $\beta \geq 0$, the upper solution $y(t, \beta)$ defined on $[a, b]$. If in addition*

$$|x(a) - \eta x| \leq \beta \quad \forall x \in \mathbf{D}, \tag{7.212}$$

then problem (7.194) *has at least one solution $x \in \mathbf{D}$.*

Proof. Problem (7.194) is equivalent to (7.206) with continuous compact $\Pi : \mathbf{D} \to \mathbf{D}$. The common a priori estimate $\|x\|_{\mathbf{D}} \leq \beta + \|m(\cdot, \beta)\|_{\mathbf{L}^1}$ holds for any solution of the equation $x = \lambda \Pi x$ for any $\lambda \in [0, 1]$. □

As evidenced by the foregoing in Section 7.4, the a priori inequality of the canonical form is especially adopted for the Cauchy problem or the boundary value problem whose boundary conditions are some perturbations of the initial condition. In the general case every nonlinear boundary value problem demands to find a form of a priori inequality such that it allows us to obtain the required a priori estimate from the boundary condition.

As an example consider the boundary value problem

$$\sqrt{t}\ddot{x} = g(x) \overset{\text{def}}{=} \begin{cases} x^{3/2} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad t \in [0, \tau], \tag{7.213}$$

$$x(0) = \alpha, \quad \dot{x}(\tau) = \frac{x(\tau) + d}{\tau}, \quad \alpha,\ d \geq 0. \tag{7.214}$$

Such a problem for the Thomas-Fermi equation arises in the statistical theory of the atom (see [84]).

Rewrite the problem in the form

$$\dot{x} = y, \qquad \dot{y} = \frac{1}{\sqrt{t}} g(x), \quad t \in [0, \tau], \tag{7.215}$$

$$x(0) = \alpha, \qquad y(\tau) = \frac{x(\tau) + d}{\tau}. \tag{7.216}$$

From the second equation of (7.215), we get

$$\int_0^\tau \sqrt{s}\dot{y}(s) y(s) ds = \int_0^\tau g(x(s)) y(s) ds. \tag{7.217}$$

The left-hand side of the latter equality takes the form

$$\int_0^\tau \sqrt{s}\dot{y}(s) y(s) ds = \frac{1}{2}\sqrt{\tau} y^2(\tau) - \frac{1}{4}\int_0^\tau y^2(s) \frac{ds}{\sqrt{s}} \leq \sqrt{\tau} y^2(\tau) \tag{7.218}$$

after integration by parts. On the other hand,

$$\int_0^\tau g(x(s)) y(s) ds = \int_\alpha^{x(\tau)} g(\xi) d\xi \leq \frac{2}{5} x^{5/2}(\tau) - \frac{2}{5}\alpha^{5/2}. \tag{7.219}$$

Thus

$$|x(\tau)| \leq \left\{\alpha^{5/2} + \frac{5}{4}\sqrt{\tau} y^2(\tau)\right\}^{2/5}. \tag{7.220}$$

The latter a priori inequality with the boundary condition $y(\tau) = (x(\tau) + d)/\tau$ leads to the a priori estimate for $|y(\tau)|$. Indeed,

$$|y(\tau)| \leq \frac{\{\alpha^{5/2} + (5/4)\sqrt{\tau} y^2(\tau)\}^{2/5} + d}{\tau}. \tag{7.221}$$

Since the right-hand side of the latter inequality has the sublinear growth, there exists $m > 0$ such that $|y(\tau)| \leq m$. This estimate allows us to obtain an a priori estimate for $\|y\|_{\mathbf{C}[0,\tau]}$ and $\|x\|_{\mathbf{C}[0,\tau]}$ and to establish the solvability of problem (7.213), (7.214). Following this scheme of obtaining a priori estimate, we can also obtain conditions of the solvability of the generalized Thomas-Fermi problem

$$\begin{gathered} q(t)\ddot{x} = f(x), \quad t \in [0,\tau], \\ x(0) = \alpha, \qquad \dot{x}(\tau) = \varphi[x(\tau)] + \psi(x) \end{gathered} \tag{7.222}$$

under the following assumptions. The function $q : [0,\tau] \to \mathbb{R}^1$ is nonnegative-valued absolutely continuous, does not decrease, and $\int_0^\tau (dt/q(t)) < \infty$; and the continuous function $f : \mathbb{R}^1 \to \mathbb{R}^1$ does not take negative values for $x \geq 0$ and $f(x) \equiv 0$ for $x < 0$; $\alpha > 0$; the function $\varphi : \mathbb{R}^1 \to \mathbb{R}^1$ is continuous; and the functional $\psi : \mathbf{W}^2 \to \mathbb{R}^1$ is continuous and bounded: $|\psi(x)| \leq \gamma$ for all $x \in \mathbf{W}^2$.

Theorem 7.38. *Suppose that the equation*

$$\xi = \lambda(\tau - \nu)\big[\varphi(\xi) + \psi(x)\big] \tag{7.223}$$

has no negative solution ξ for any $x \in \mathbf{W}^2$, $\lambda \in (0,1]$, $\nu \in (0,\tau)$, and the inequality

$$\int_\alpha^T f(s)ds \geq \eta_1(T) - \eta_2(\alpha) \tag{7.224}$$

holds for any $T \geq 0$. Here continuous $\eta_i : [0,\infty) \to [0,\infty)$, $i = 1,2$, are such that all the positive solutions ξ of the scalar inequality

$$\eta_1(\xi) \leq \frac{1}{2}q(\tau)\big\{\,|\varphi(\xi)\,| + \gamma\big\}^2 + \eta_2(\alpha) \tag{7.225}$$

are bounded by one and the same positive m.

Then problem (7.222) *has at least one solution $x \in \mathbf{W}^2$.*

7.6. Sufficient conditions for minimum of functionals

The Euler boundary value problems for square functional on the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ with linear restrictions was considered in Chapter 5, where the problem was reduced by immediate W-substitution to a problem without restriction on the space $\mathbf{L}_2$. Below we consider perturbations of the square functional on $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$

and again reduce the problem by means of W-substitution to the problem without restrictions on the space $\mathbf{L}_2$.

7.6.1. The main assertion

Suppose $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ is the space of functions $x : [a,b] \to \mathbb{R}^1$, $\mathcal{J} = \{\Lambda, Y\} : \mathbf{L}_2 \times \mathbb{R}^n \to \mathbf{D}$ is an isomorphism, $\mathcal{J}^{-1} = [\delta, r]$, $\|x\|_{\mathbf{D}} = \|\delta x\|_{\mathbf{L}_2} + \|rx\|_{\mathbb{R}^n}$, $\mathbf{D}_\alpha = \{x \in \mathbf{D} : rx = \alpha\}$. Let the functional

$$\mathcal{I}(x) = \int_a^b \Big\{ \big[(\delta x)(s)\big]^2 - f\big(s, (T_1 x)(s), \dots, (T_m x)(s)\big) \Big\} ds, \tag{7.226}$$

with linear bounded $T_i : \mathbf{D} \to \mathbf{L}_2$ be defined on an open set $\mathcal{D} \subset \mathbf{D}$. We will consider the problem on existence of a minimum of the functional on the set $\Omega = \mathcal{D} \cap \mathbf{D}_\alpha$, (thus we take into account the restriction $rx = \alpha$). We denote this problem by

$$\mathcal{I}(x) \longrightarrow \min, \quad x \in \Omega. \tag{7.227}$$

We say that a point $x_0 \in \Omega$ is a point of local minimum in problem (7.227) if there exists $\varepsilon > 0$ such that $\mathcal{I}(x) \geq \mathcal{I}(x_0)$ for all $x \in \Omega$, $\|x - x_0\|_{\mathbf{D}} < \varepsilon$. Let us construct by W-substitution

$$x = \Lambda z + Y\alpha \tag{7.228}$$

an auxiliary functional

$$\mathcal{I}_1(z) = \mathcal{I}(\Lambda z + Y\alpha) \tag{7.229}$$

on the space $\mathbf{L}_2$. This functional possesses the following property: if $\mathcal{I}(x_1) \geq \mathcal{I}(x_2)$ for the pair $x_1, x_2 \in \Omega$, then $\mathcal{I}_1(z_1) \geq \mathcal{I}_1(z_2)$, where $z_1 = \delta x_1$, $z_2 = \delta x_2$ (and vice versa), because of

$$\mathcal{I}_1(z_1) - \mathcal{I}_1(z_2) = \mathcal{I}(\Lambda z_1 + Y\alpha) - \mathcal{I}(\Lambda z_2 + Y\alpha) = \mathcal{I}(x_1) - \mathcal{I}(x_2). \tag{7.230}$$

This implies that if x_0 is a point of local minimum in problem (7.227), then $z_0 = \delta x_0$ is a point of local minimum of the functional $\mathcal{I}_1$. And vice versa, if z_0 is a point of local minimum of the functional $\mathcal{I}_1$, then $x_0 = \Lambda z_0 + Y\alpha$ is a point of local minimum in problem (7.227).

Further, we denote $A_i = T_i Y$, $Q_i = T_i \Lambda$; $Q_i^* : \mathbf{L}_2 \to \mathbf{L}_2$ is the adjoint operator to $Q_i : \mathbf{L}_2 \to \mathbf{L}_2$,

$$
\begin{gathered}
f_i(t, y_1, \dots, y_m) = \frac{\partial}{\partial y_i} f(t, y_1, \dots, y_m), \\
f_{ij}(t, y_1, \dots, y_m) = \frac{\partial}{\partial y_j} f_i(t, y_1, \dots, y_m), \\
(F_i x)(t) = f_i(t, (T_1 x)(t), \dots, (T_m x)(t)), \\
g_x^{ij}(t) = f_{ij}(t, (T_1 x)(t), \dots, (T_m x)(t)), \\
R_x z = \frac{1}{2} \sum_{i,j=1}^{m} Q_i^* \left[g_x^{ij} \cdot Q_j z \right], \\
H_x z = z - R_x z, \\
\Phi x = \frac{1}{2} \sum_{i=1}^{m} Q_i^* F_i x.
\end{gathered}
\tag{7.231}
$$

Suppose that $f_{ij}(t, \cdot)$, $i, j = 1, \dots, m$, $t \in [a, b]$, are continuous in the domain of definition; the operator Φ acting in the space $\mathbf{L}_2$ is defined on $\mathcal{D}$, is continuous, and is bounded; the linear operator $R_x : \mathbf{L}_2 \to \mathbf{L}_2$ is defined and bounded at each $x \in \mathcal{D}$, besides, it is continuous with respect to x as a mapping from $\mathcal{D}$ into the Banach space of linear bounded operators acting in the space $\mathbf{L}_2$.

The boundary value problem

$$
\delta x = \Phi x, \qquad r x = \alpha \tag{7.232}
$$

is called the Euler problem.

Theorem 7.39. *The point $x_0 \in \Omega$ is a point of local minimum in problem* (7.227) *if*

(a) *x_0 is a solution of the Euler problem;*

(b) *the operator $H_{x_0} : \mathbf{L}_2 \to \mathbf{L}_2$ defined by* (7.231) *is strongly positive definite: there exists $\gamma > 0$ such that*

$$
\int_a^b (H_{x_0} \xi)(s) \xi(s) ds \geq \gamma \|\xi\|_{\mathbf{L}_2}^2 \tag{7.233}
$$

for all $\xi \in \mathbf{L}_2$.

Consider as a preliminary the following well-known assertion (see, e.g., [4, 83]).

Lemma 7.40. *Let the functional $\mathcal{I}_1$ be defined in a neighborhood of the point z_0 and let it have the second derivative by Frechet at this point. The point z_0 is a point of local minimum of the functional $\mathcal{I}_1$ if $\mathcal{I}_1'(z_0) = 0$ and the second differential possesses the property of the strict positivity: there exists a positive μ such that $\mathcal{I}_1''(z_0)(\xi,\xi) \geq \mu\|\xi\|^2_{\mathbf{L}_2}$ for all $\xi \in \mathbf{L}_2$.*

Proof. Let $\mathcal{I}_1'(z_0) = 0$ and $\mathcal{I}_1''(z_0)(\xi,\xi) \geq \mu\|\xi\|^2_{\mathbf{L}_2}$.

We have

$$\mathcal{I}_1(z_0+\xi) - \mathcal{I}_1(z_0) = \mathcal{I}_1'(z_0)\xi + \frac{1}{2}\mathcal{I}_1''(z_0)(\xi,\xi) + \omega(z_0,\xi), \tag{7.234}$$

where $\lim_{\xi\to 0}(\omega(z_0,\xi)/\|\xi\|^2_{\mathbf{L}_2}) = 0$. Let $\varepsilon > 0$ be chosen such that, for $\|\xi\|_{\mathbf{L}_2} < \varepsilon$,

$$|\omega(z_0,\xi)| \leq \frac{1}{4}\mu\|\xi\|^2_{\mathbf{L}_2}. \tag{7.235}$$

Then

$$\begin{aligned} \mathcal{I}_1(z_0+\xi) - \mathcal{I}_1(z_0) &= \frac{1}{2}\mathcal{I}_1''(z_0)(\xi,\xi) + \omega(z_0,\xi) \\ &\geq \left(\frac{1}{2}\mu - \frac{1}{4}\mu\right)\|\xi\|^2_{\mathbf{L}_2} = \frac{1}{4}\mu\|\xi\|^2_{\mathbf{L}_2} \geq 0. \end{aligned} \tag{7.236}$$

Consequently, z_0 is a point of local minimum. □

Proof of Theorem 7.39. Rewrite the functional $\mathcal{I}_1$ in the form

$$\mathcal{I}_1(z) = \int_a^b \{z^2(s) - f(s,(Q_1z)(s) + A_1(s)\alpha,\dots,(Q_mz)(s) + A_m(s)\alpha)\}ds. \tag{7.237}$$

The Frechet differential at the point z_0 has the form

$$\begin{aligned} &\mathcal{I}_1'(z_0)\xi \\ &= \int_a^b \left\{2z_0(s)\xi(s) - \sum_{i=1}^m f_i(s,(Q_1z_0)(s)+A_1(s)\alpha,\dots,(Q_mz_0)(s)+A_m(s)\alpha)(Q_i\xi)(s)\right\}ds. \end{aligned} \tag{7.238}$$

Taking into consideration the equality

$$\int_a^b \xi_1(s)(Q_i\xi_2)(s)ds = \int_a^b (Q_i^*\xi_1)(s)\xi_2(s)ds \tag{7.239}$$

at all $\xi_1,\xi_2 \in \mathbf{L}_2$, we obtain

$$\mathcal{I}_1'(z_0)\xi = \int_a^b \left\{2z_0(s) - \sum_{i=1}^m [Q_i^*F_i(\Lambda z_0 + Y\alpha)](s)\xi(s)\right\}ds. \tag{7.240}$$

This implies that $\mathcal{I}_1'(z_0) = 0$ if z_0 is a solution to the equation

$$z = \frac{1}{2}\sum_{i=1}^{m} Q_i^* F_i(\Lambda z + Y\alpha), \tag{7.241}$$

that is, $x_0 = \Lambda z_0 + Y\alpha$ is a solution to boundary value problem (7.232).

Next

$$\begin{aligned}
&\mathcal{I}_1''(z_0)(\xi,\xi)\\
&\quad= \int_a^b \Big\{2\xi(s) - \sum_{i,j=1}^{m} \big[Q_i^*\big(f_{ij}(\cdot, Q_1 z_0 + A_1\alpha, \dots, Q_m z_0 + A_m\alpha)\cdot Q_j\xi\big)\big](s)\Big\}\xi(s)\,ds\\
&\quad= \int_a^b \Big\{2\xi(s) - \sum_{i,j=1}^{m} \big[Q_i^*\big(f_{ij}(\cdot, T_1(\Lambda z_0 + Y\alpha), \dots, T_m(\Lambda z_0 + Y\alpha))\cdot Q_j\xi\big)\big](s)\Big\}\xi(s)\,ds\\
&\quad= \int_a^b \Big\{2\xi(s) - \sum_{i,j=1}^{m} \big[Q_i^*\big(f_{ij}(\cdot, T_1 x_0, \dots, T_m x_0)\cdot Q_j\xi\big)\big](s)\Big\}\xi(s)\,ds\\
&\quad= \int_a^b \Big\{2\xi(s) - \sum_{i,j=1}^{m} \Big[Q_i^*\Big(g_{x_0}^{ij}\cdot Q_j\xi\Big)\Big](s)\Big\}\xi(s)\,ds\\
&\quad= 2\int_a^b \{\xi(s) - (R_{x_0}\xi)(s)\}\xi(s)\,ds = 2\int_a^b (H_{x_0}\xi)(s)\xi(s)\,ds \ge 2\gamma\|\xi\|^2_{\mathbf{L}_2}.
\end{aligned} \tag{7.242}$$

The reference to Lemma 7.40 completes the proof. □

The estimate

$$\|R_{x_0}\|_{\mathbf{L}_2\to\mathbf{L}_2} < 1 \tag{7.243}$$

guarantees the fulfillment of condition (b) of Theorem 7.39. Indeed,

$$\begin{aligned}
\int_a^b (H_{x_0}\xi)(s)\xi(s)\,ds &= \int_a^b \xi^2(s)\,ds - \int_a^b (R_{x_0}\xi)(s)\xi(s)\,ds \ge \|\xi\|^2_{\mathbf{L}_2} - \|R_{x_0}\|_{\mathbf{L}_2\to\mathbf{L}_2}\|\xi\|^2_{\mathbf{L}_2}\\
&= \Big(1 - \|R_{x_0}\|_{\mathbf{L}_2\to\mathbf{L}_2}\Big)\|\xi\|^2_{\mathbf{L}_2}.
\end{aligned} \tag{7.244}$$

Now suppose that R_{x_0} may be decomposed as $R_{x_0}^+ - R_{x_0}^-$ with positive definite $R_{x_0}^+$ and $R_{x_0}^-$. Then

$$\begin{aligned}
\int_a^b (H_{x_0}\xi)(s)\xi(s)\,ds &= \int_a^b \xi^2(s)\,ds - \int_a^b (R_{x_0}^+\xi)(s)\xi(s)\,ds + \int_a^b (R_{x_0}^-\xi)(s)\xi(s)\,ds\\
&\ge \|\xi\|^2_{\mathbf{L}_2} - \|R_{x_0}^+\|_{\mathbf{L}_2\to\mathbf{L}_2}\|\xi\|^2_{\mathbf{L}_2} = \Big(1 - \|R_{x_0}^+\|_{\mathbf{L}_2\to\mathbf{L}_2}\Big)\|\xi\|^2_{\mathbf{L}_2}.
\end{aligned} \tag{7.245}$$

Consequently, the estimate

$$||R^+_{x_0}||_{\mathbf{L}_2\to\mathbf{L}_2} < 1 \tag{7.246}$$

as well guarantees the strong positivity of the operator $H_{x_0} : \mathbf{L}_2 \to \mathbf{L}_2$. In the case $m = 1$ we may decompose the functional $g^{11}_{x_0}(t) = f_{11}(t, (T_1x_0)(t))$ as $g^+_{x_0} - g^-_{x_0}$ with $g^+_{x_0}(t) \geq 0$, $g^-_{x_0}(t) \geq 0$. Then

$$R^+_{x_0}z = \frac{1}{2}Q^*_1\left(g^+_{x_0} \cdot Q_1z\right), \qquad R^-_{x_0}z = \frac{1}{2}Q^*_1\left(g^-_{x_0} \cdot Q_1z\right). \tag{7.247}$$

If the explicit form of the solution x_0 of the Euler problem is known, we may guarantee by estimate (7.243) or (7.254) the existence of a local minimum (and even calculate its value). If we know that the solution x_0 of the Euler problem exists and there is available a proper estimate of x_0 ($x_0 \in \omega \subset \mathcal{D}$), then the existence of a local minimum will be guaranteed by at least one of the estimates $\sup_{x\in\omega} \|R_x\|_{\mathbf{L}_2\to\mathbf{L}_2} < 1$ or $\sup_{x\in\omega} \|R^+_x\|_{\mathbf{L}_2\to\mathbf{L}_2} < 1$.

Thus we are in a position to formulate the following corollary from Theorem 7.39.

Corollary 7.41. *Let x_0 be a solution of the Euler problem, let the set $M \subset \mathcal{D}$ be such that $x_0 \in M$, and let at least one of the estimates* $\sup_{x\in M} \|R_x\|_{\mathbf{L}_2\to\mathbf{L}_2} < 1$ *and* $\sup_{x\in M} \|R^+_x\|_{\mathbf{L}_2\to\mathbf{L}_2} < 1$ *be fulfilled. Then x_0 is a point of local minimum in problem* (7.227).

7.6.2. Effective tests

Problem (7.232) is equivalent to the equation $x = \Psi x$, where $\Psi x = \Lambda\Phi x + Y\alpha$.

Theorem 7.42. *Let a set $M \subset \mathcal{D}$ be nonempty, closed, and convex, and let*

$$\sup_{x\in M} ||R_x||_{\mathbf{L}_2\to\mathbf{L}_2} < 1. \tag{7.248}$$

Then the Euler problem has a unique solution $x_0 \in M$, this solution is a point of local minimum in problem (7.227) *and the unique point of minimum of the functional on $M : \mathcal{I}(x) > \mathcal{I}(x_0)$ for all $x \in M$, $x \neq x_0$.*

Proof. Under the conditions of the theorem the operator $\Psi : M \to M$ is contractive. Indeed, if $x_1, x_2 \in M$, then

$$\Psi x_2 - \Psi x_1 = \Lambda(\Phi x_2 - \Phi x_1), \qquad ||\Psi x_2 - \Psi x_1||_{\mathbf{D}} = ||\Phi x_2 - \Phi x_1||_{\mathbf{L}_2}. \tag{7.249}$$

With the Taylor formula (see [60, Chapter 1, Theorem 5.6.1]), we obtain

$$\begin{aligned}\Phi x_2 - \Phi x_1 &= \int_0^1 \Phi'\left(x_1 + \tau\left(x_2 - x_1\right)\right)\left(x_2 - x_1\right) d\tau \\ &= \int_0^1 \Phi'\left(x_1 + \tau\left(x_2 - x_1\right)\right)\Lambda\delta\left(x_2 - x_1\right) d\tau \\ &= \int_0^1 R_{x_1+\tau(x_2-x_1)}\delta\left(x_2 - x_1\right) d\tau.\end{aligned} \tag{7.250}$$

Therefore

$$\begin{aligned}\left|\left|\Psi x_2 - \Psi x_1\right|\right|_{\mathbf{D}} &= \left|\left| \int_0^1 R_{x_1+\tau(x_2-x_1)}\delta\left(x_2 - x_1\right) d\tau\right|\right|_{\mathbf{L}_2} \\ &\leq \sup_{x\in M}\left|\left|R_x\right|\right|_{\mathbf{L}_2\to\mathbf{L}_2}\cdot\left|\left|x_2 - x_1\right|\right|_{\mathbf{D}}.\end{aligned} \tag{7.251}$$

Thus, by virtue of the Banach principle, there exists a unique solution $x_0 \in M$ to the equation $x = \Psi x$ being equivalent to the Euler problem. By virtue of Corollary 7.41, x_0 is a point of local minimum in problem (7.227).

Next we prove the uniqueness of the minimum of the functional.

Let $x \in M$, $x \neq x_0$. Then $\mathcal{I}(x) - \mathcal{I}(x_0) = \mathcal{I}_1(\delta x) - \mathcal{I}_1(\delta x_0)$, $\xi = \delta x - \delta x_0 \neq 0$. Again, using the Taylor formula, we get

$$\begin{aligned}\mathcal{I}(x) - \mathcal{I}\left(x_0\right) &= \mathcal{I}_1'\left(\delta x_0\right)\xi + \frac{1}{2}\int_0^1 (1-\tau)\mathcal{I}_1''\left(\delta x_0 + \tau\xi\right)(\xi,\xi) d\tau \\ &= \int_0^1 (1-\tau)\int_a^b \left\{\xi(s) - \left(R_{x_0+\tau(x-x_0)}\xi\right)(s)\right\}\xi(s) ds\, d\tau \\ &\geq \int_0^1 (1-\tau) d\tau\left(1 - \sup_{x\in M}\left|\left|R_x\right|\right|_{\mathbf{L}_2\to\mathbf{L}_2}\right)\|\xi\|_{\mathbf{L}_2}^2 > 0.\end{aligned} \tag{7.252}$$

□

In the case when the operator Ψ is defined on the space $\mathbf{C}$ of continuous functions, the equation $x = \Psi x$ may be considered, as it was often practiced above, in the space $\mathbf{C}$. It follows from the fact that any continuous solution to this equation belongs to $\mathbf{D}$ and thus it is a solution of the Euler problem.

Theorem 7.43 (Theorem 7.42 bis). *Let a nonempty set $M \subset \mathbf{C}$ be closed and convex, and let the operator $\Psi : M \to \mathbf{C}$ be completely continuous. If Ψ maps the set M into itself and*

$$\sup_{x\in M\cap\Omega}\left|\left|R_x\right|\right|_{\mathbf{L}_2\to\mathbf{L}_2} < 1, \tag{7.253}$$

then there exists a unique solution $x_0 \in M$ of the Euler problem and this solution is the unique point of minimum of the functional $\mathcal{I}$ on the set $M \cap \Omega$: $\mathcal{I}(x) > \mathcal{I}(x_0)$ for all $x \in M \cap \Omega$, $x \neq x_0$.

Proof. The equation $x = \Psi x$, which is equivalent to the Euler problem, has at least one solution $x_0 \in M$ by the Schauder principle. This solution is a point of local minimum in problem (7.227) by virtue of Corollary 7.41. Similarly to the proof of Theorem 7.42 we get the inequality $\mathcal{I}(x) > \mathcal{I}(x_0)$ for $x \in M \cap \Omega$, $x \neq x_0$. From this inequality we get as well the uniqueness of the solution of the Euler problem.

The following lemma is sometimes useful to get the estimate $\|R_x\|_{\mathbf{L}_2 \to \mathbf{L}_2} < 1$ because the lemma reduces the problem to well known estimates of the spectral radius of operators in the space **C**. We assume below that the space **D** is continuously embedded into the space **C** and that $T_i(\mathbf{C}) \subset \mathbf{L}_2$, $i = 1, \dots, m$.

Denote by A_y the derivative by Frechet of the operator Ψ in the point y. Thus

$$A_y x = \frac{1}{2}\Lambda \sum_{i,j=1}^{m} Q_i^* \left(g_y^{ij} \cdot T_i x\right). \tag{7.254}$$

Assume that the linear operator A_y, for a fixed $y \in \mathcal{D}$, is acting in the space **C** and is bounded. Denote by $\rho(A_y)$ its spectral radius. □

Lemma 7.44. *Let the operators $A_y : \mathbf{C} \to \mathbf{C}$ and $R_y : \mathbf{L}_2 \to \mathbf{L}_2$ be completely continuous. Then*

$$||R_y||_{\mathbf{L}_2 \to \mathbf{L}_2} = \rho(A_y). \tag{7.255}$$

Proof. The spectra of completely continuous operators A_y and R_y coincide since the equations $x = \Lambda z$ and $z = \delta x$ establish the one-to-one mapping between the set of solutions x of the equation $\lambda x = A_y x$ and the set of solutions z of the equation $\lambda z = R_y z$. Consequently,

$$\rho(A_y) = \rho(R_y) = ||R_y||_{\mathbf{L}_2 \to \mathbf{L}_2}. \tag{7.256}$$

The latter equality holds due to the fact that the operator R_y is selfadjoint. □

Let $M = [u, v] = \{x \in \mathbf{C} : u(t) \leq x(t) \leq v(t),\ t \in [a, b]\}$, $u, v \in \mathcal{D}$. Assume the following: for any $y \in M \cap \mathcal{D}$, (7.254) defines a linear bounded operator A_y in the space **C**; the operator $A_y : \mathbf{C} \to \mathbf{C}$ is isotonic and besides, for any $x \in M$, the operator $y \to A_y x$ is isotonic: $(A_{y_1}x)(t) \leq (A_{y_2}x)(t)$, if $y_1(t) \leq y_2(t)$, $t \in [a, b]$. Under such assumptions there holds the following.

Theorem 7.45. *Assume that the operator $\Psi : M \to \mathbf{C}$ is completely continuous, isotonic (antitonic), and the inequalities*

$$\begin{aligned} &\eta_u(t) \overset{\text{def}}{=} (\Psi u)(t) - u(t) \geq 0, \qquad \eta_v(t) \overset{\text{def}}{=} v(t) - (\Psi v)(t) \geq 0 \\ &\big(\eta_u(t) \overset{\text{def}}{=} (\Psi v)(t) - u(t) \geq 0, \qquad \eta_v(t) \overset{\text{def}}{=} v(t) - (\Psi u)(t) \geq 0\big) \end{aligned} \tag{7.257}$$

are fulfilled at $t \in [a, b]$, and besides at least one of the functions η_u or η_v has zeros on $[a, b]$. Then the set M contains a point of a local minimum in problem (7.227).

Proof. The completely continuous operator Ψ maps the set M into itself. Therefore a solution $x_0 \in M$ of the Euler problem does exist. Suppose for definiteness that the operator Ψ is isotonic and $\eta_v(t) > 0$ on $[a, b]$. Then

$$\begin{aligned} v(t) - x_0(t) > (\Psi v)(t) - (\Psi x_0)(t) &= \int_0^1 \left[\Psi'\left(x_0 + \tau(v - x_0)\right)(v - x_0)\right](t)d\tau \\ &= \int_0^1 \left(A_{x_0+\tau(v-x_0)}(v - x_0)\right)(t)d\tau \geq \left(A_{x_0}(v - x_0)\right)(t) \geq 0. \end{aligned} \tag{7.258}$$

Thus

$$v(t) - x_0(t) > 0, \quad v(t) - x_0(t) > \left(A_{x_0}(v - x_0)\right)(t), \quad t \in [a, b]. \tag{7.259}$$

From this it follows by Lemma A.1 that $\rho(A_{x_0}) < 1$.

To complete the proof, we refer to Lemma 7.44 and Theorem 7.39.

The rest of the cases are proved similarly. □

Remark 7.46. The condition of the strict inequalities $\eta_v(t) > 0$ or $\eta_u(t) > 0$ may be weakened by using Theorem A.3 and some specific characters of the vector functional r.

7.6.3. Examples

Consider the functional

$$\mathcal{I}(x) = \int_0^1 \left\{\left[(\delta x)(s)\right]^2 - p(s)\ln\left[(Tx)(s) + 1\right] - q(s)(Tx)(s)\right\}ds \tag{7.260}$$

with linear homogeneous restrictions $rx = 0$.

Assume that the space $\mathbf{D} \simeq \mathbf{L}_2 \times \mathbb{R}^n$ as well as the isomorphism $\mathcal{J} = \{\Lambda, Y\}$ are fixed, the space $\mathbf{D}$ is continuously embedded into $\mathbf{C}$, the operator Λ in the representation of the isomorphism is isotonic or antitonic, the linear operator $T : \mathbf{C} \to \mathbf{L}_\infty$ is bounded and isotonic, $p, q \in \mathbf{L}_2$.

The functional is defined on the set

$$\mathcal{D} = \left\{x \in \mathbf{D} : x(t) > \frac{-1}{\|T\|_{\mathbf{C}\to\mathbf{L}_\infty}},\ t \in [0, 1]\right\}, \quad \Omega = \mathcal{D} \cap \mathbf{D}_0. \tag{7.261}$$

The Euler problem takes the form

$$\delta x = \frac{1}{2}Q^*\left(\frac{p}{Tx + 1} + q\right), \quad rx = 0. \tag{7.262}$$

This problem is equivalent to the equation

$$x = \Psi x \stackrel{\text{def}}{=} \frac{1}{2}\Lambda Q^*\left(\frac{p}{Tx + 1} + q\right). \tag{7.263}$$

The operator R_x is defined by

$$R_x z = \frac{1}{2} Q^* (g_x \cdot Qz), \tag{7.264}$$

where

$$g_x(t) = -\frac{p(t)}{[(Tx)(t)+1]^2}. \tag{7.265}$$

Consider two cases: the case (A): $p(t) \geq 0$, $q(t) \geq 0$; the case (B): $p(t) \leq 0$, $p(t)+q(t) \geq 0$.

In the case (A), we have $g_x(t) \leq 0$ for $x(t) \geq 0$. Therefore $R_x^+ = 0$ for such x and, if there exists a nonnegative solution of the Euler problem, then this solution is a point of local minimum in problem (7.227). Let $u(t) \equiv 0$, $v(t) = (1/2)[\Lambda Q^*(p+q)](t)$, $M = [u,v] = \{x \in \mathbf{C} : u(t) \leq x(t) \leq v(t),\ t \in [0,1]\}$. The antitonic $\Psi : \mathbf{C}^+ \to \mathbf{C}^+$, where $\mathbf{C}^+ = \{x \in \mathbf{C} : x(t) \geq 0,\ t \in [0,1]\}$, maps M into itself since $u(t) \leq v(t)$, $u(t) \leq (\Psi v)(t)$, $v(t) \geq (\Psi u)(t)$ (see Section 7.2.1). If the operator $\Psi : M \to \mathbf{C}$ is completely continuous, then, by virtue of the Schauder principle, there exists a solution $x_0 \in M$ of (7.263).

In the case (B), let $u(t) \equiv 0$, $v(t) = (1/2)[\Lambda Q^* q](t)$, $M = [u,c] \subset \mathbf{C}$. The isotonic operator Ψ maps M into itself since $u(t) \leq v(t)$, $u(t) \leq (\Psi u)(t)$, $v(t) \geq (\Psi v)(t)$. From the inequality

$$g_x(t) = -\frac{p(t)}{[(Tx)(t)+1]^2} \leq -p(t) \quad \text{for } x(t) \geq 0, \tag{7.266}$$

it follows that

$$||R_x||_{\mathbf{L}_2 \to \mathbf{L}_2} \leq \frac{1}{2} ||Q^*||_{\mathbf{L}_2 \to \mathbf{L}_2} \cdot ||g_x||_{\mathbf{L}_2} \cdot \|Q\|_{\mathbf{L}_2 \to \mathbf{L}_2} \leq \frac{1}{2} \|Q\|^2_{\mathbf{L}_2 \to \mathbf{L}_2} \cdot \|p\|_{\mathbf{L}_2} \tag{7.267}$$

for any $x \in M \cap \Omega$. Thus, the inequality

$$\frac{1}{2} \|Q\|^2_{\mathbf{L}_2 \to \mathbf{L}_2} \cdot \|p\|_{\to \mathbf{L}_2} < 1 \tag{7.268}$$

guarantees by virtue of Theorem 7.43 the existence of a point $x_0 \in M \cap \Omega$ of local minimum in problem (7.227) and besides $\mathcal{I}(x) > \mathcal{I}(x_0)$ for all $x \in M \cap \Omega$, $x \neq x_0$.

Let us dwell on concrete realizations of the space $\mathbf{D}$ and the operators δ and T.

(1) Let $\mathbf{D}$ be the space of absolutely continuous $x : [0,1] \to \mathbb{R}^1$ with the derivative belonging to $\mathbf{L}_2$; $\delta x = \dot{x}$, $rx = x(0)$, $\|x\|_{\mathbf{D}} = \|\dot{x}\|_{\mathbf{L}_2} + |x(0)|$. The space $\mathbf{D}$ is continuously embedded into $\mathbf{C}$. Let further $(Tx)(t) = x(\lambda t)$, $\lambda \in (0,1]$. Thus

$$\mathcal{I}(x) = \int_0^1 \left\{ [\dot{x}(s)]^2 - p(s) \ln [x(\lambda s)+1] - q(s) x(\lambda s) \right\} ds. \tag{7.269}$$

The functional $\mathcal{I}$ is defined on the set $\mathcal{D} = \{x \in \mathbf{D} : x(t) > -1,\ t \in [0,1]\}$ since $\|T\|_{\mathbf{C}\to\mathbf{L}_\infty} = 1$.

We have

$$(\Lambda z)(t) = \int_0^1 z(s)ds,$$
$$(Qz)(t) = \int_0^{\lambda t} z(s)ds, \tag{7.270}$$
$$(Q^*z)(t) = \int_0^1 \chi(t,s)z(s)ds,$$

where $\chi(t,s)$ is the characteristic function of the set

$$\{(t,s) \in [0,1] \times [0,1] : t \le \lambda s\};$$

$$\|Q\|_{\mathbf{L}_2\to\mathbf{L}_2} \le \sqrt{\frac{\lambda}{2}}, \qquad (\Lambda Q^*z)(t) = \int_0^1 K(t,s)z(s)ds,$$
$$K(t,s) = \begin{cases} \lambda s & \text{if } t \in [0,\lambda],\ s \in \left[0, \dfrac{t}{\lambda}\right],\ \text{if } t \in (\lambda,1],\ s \in [0,1], \\ t & \text{if } t \in [0,\lambda],\ s \in \left(\dfrac{t}{\lambda}, 1\right]. \end{cases} \tag{7.271}$$

The operator $\Psi : \mathbf{C}^+ \to \mathbf{C}$ is completely continuous since the operator $Q : \mathbf{L}_2 \to \mathbf{L}_2$ is completely continuous. In the case (A) $(p(t) \ge 0,\ q(t) \ge 0)$ a solution x_0 of the Euler problem exists, $x_0 \in [0,v]$, where

$$v(t) = \frac{1}{2}\int_0^1 K(t,s)[p(s) + q(s)]ds. \tag{7.272}$$

This solution is a point of local minimum in problem (7.227).

In the case (B) $(p(t) \le 0,\ p(t) + q(t) \ge 0)$, under the condition

$$\int_0^1 p^2(s)ds < \frac{16}{\lambda^2}, \tag{7.273}$$

a solution x_0 of the Euler problem exists, it belongs to the segment $[0,v]$, where

$$v(t) = \frac{1}{2}\int_0^1 K(t,s)q(s)ds. \tag{7.274}$$

This solution x_0 is a point of local minimum in problem (7.227) and besides $\mathcal{I}(x) > \mathcal{I}(x_0)$ for all $x \in [0,v] \cap \Omega,\ x \ne x_0$.

(2) Let $\mathbf{D} = \mathbf{D}^2_\pi$. The space $\mathbf{D}^2_\pi$ is defined in Subsection 5.3.3 (see Example 5.13).

Assume

$$(\delta x)(t) = t(1-t)\ddot{x}(t), \qquad rx = \{x(0), x(1)\}. \tag{7.275}$$

Then (see Subsection 4.2.1, Remark 4.2)

$$(\Lambda z)(t) = \int_0^1 \Lambda(t,s)z(s)ds, \tag{7.276}$$

where

$$\Lambda(t,s) = \begin{cases} -\dfrac{1-t}{1-s} & \text{if } 0 \le s \le t \le 1, \\ -\dfrac{t}{s} & \text{if } 0 \le t < s \le 1, \end{cases} \tag{7.277}$$

$$\Lambda(t,s) \le 0, \qquad |\Lambda(t,s)| \le 1.$$

The element $x \in \mathbf{D}_\pi^2$ has the representation

$$x(t) = \int_0^1 \Lambda(t,s)\pi(s)\ddot{x}(s)ds + (1-t)x(0) + tx(1), \tag{7.278}$$

where $\pi(s) = s(1-s)$. Define the norm in $\mathbf{D}_\pi^2$ by

$$\|x\|_{\mathbf{D}_\pi^2} = \|\pi\ddot{x}\|_{\mathbf{L}_2} + |x(0)| + |x(1)|. \tag{7.279}$$

The space $\mathbf{D}_\pi^2$ is continuously imbedded into the space $\mathbf{C}$. It follows from the inequality

$$\begin{aligned} |x(t)| &\le \int_0^1 |\Lambda(t,s)||\pi(s)\ddot{x}(s)|ds + (1-t)|x(0)| + t|x(1)| \\ &\le \|\pi\ddot{x}\|_{\mathbf{L}_2} + |x(0)| + |x(1)| = \|x\|_{\mathbf{D}_\pi^2}. \end{aligned} \tag{7.280}$$

Define the operator T by

$$(Tx)(t) = (S_h x)(t) \stackrel{\text{def}}{=} \begin{cases} x[h(t)] & \text{if } h(t) \in [0,1], \\ 0 & \text{if } h(t) \notin [0,1], \end{cases} \tag{7.281}$$

where $h : [0,1] \to \mathbb{R}^1$ is measurable and such that $T \neq 0$; $\|T\|_{\mathbf{C}\to\mathbf{L}_\infty} = 1$. Thus

$$\mathcal{I}(x) = \int_0^1 \Big\{[s(1-s)\ddot{x}(s)]^2 - p(s)\ln[(S_h x)(s) + 1] - q(s)(S_h x)(s)\Big\}ds. \tag{7.282}$$

The functional $\mathcal{I}$ is defined on the set $\mathcal{D} = \{x \in \mathbf{D}_\pi^2 : x(t) > -1,\ t \in [0,1]\}$. In the case

$$(Qz)(t) = \int_0^1 \Lambda[h(t), s]z(s)ds, \qquad (Q^*z)(t) = \int_0^1 \Lambda[h(s), t]z(s)ds \tag{7.283}$$

(we suppose $\Lambda(t,s) = 0$ outside $[0,1] \times [0,1]$). The operator $Q : \mathbf{L}_2 \to \mathbf{L}_2$ is completely continuous,

$$(\Lambda Q^*z)(t) = \int_0^1 K(t,s)z(s)ds, \tag{7.284}$$

with

$$K(t,s) = \int_0^1 \Lambda(t,\tau)\Lambda[h(s),\tau]d\tau. \tag{7.285}$$

In the case (A) a solution x_0 of the Euler problem exists, $x_0 \in [0,v]$, where

$$v(t) = \frac{1}{2}\int_0^1 K(t,s)[p(s) + q(s)]ds. \tag{7.286}$$

This solution is a point of local minimum in problem (7.227).

In the case (B) under the condition

$$\int_0^1 p^2(s)\sigma(s)ds < 4, \tag{7.287}$$

where

$$\sigma(s) = \begin{cases} 1 & \text{if } h(s) \in [0,1], \\ 0 & \text{if } h(s) \notin [0,1], \end{cases} \tag{7.288}$$

a solution x_0 of the Euler problem exists, $x_0 \in [0,v]$,

$$v(t) = \int_0^1 K(t,s)q(s)ds; \tag{7.289}$$

this solution is a point of local minimum in problem (7.227) and besides $\mathcal{I}(x) > \mathcal{I}(x_0)$ for all $x \in [0,v] \cap \Omega$, $x \neq x_0$.

The schemes of forks and $\mathcal{L}^1$, $\mathcal{L}^2$-quasilinearization were used seemingly for the first time by researchers at the Izhevsk seminar in the middle of the 1950s (see, e.g., [43, 81, 111]).

The ideas of the reducibility as well as a priori inequalities were put forth in [31] and were developed in [145].

The a priori inequalities were used in some special cases in [128, 207, 208, 232, 235].

Rumyantsev [194, 196] extended the method of a priori inequalities onto the problems with pulse perturbations (in the space $\mathbf{DS}(m)$, see Section 3.2).

The detailed proof of Theorem 7.38 can be found in [143].

The presentation of Section 7.6 follows basically [27, 28].

7.7. Reducible stochastic functional differential equations

7.7.1. Notation and preliminary results

A[1] stochastic functional differential equation studied in this section is as follows:

$$dx_t = Fx_t dz_t, \quad t \in [0, T], \tag{7.290}$$

where z_t $(0 \le t \le T)$ is an arbitrary, not necessarily continuous, semimartingale (for all relevant definitions see, e.g., [79, 105, 107]). We will also assume that F is a nonlinear operator depending on the trajectories $x(s),\ 0 \le s \le t$. The nature of this dependence will be specified in examples below.

If one wants to extend the deterministic theory of reducible functional differential equation to the stochastic case, one will face the following problem: bounded sets of solutions to stochastic differential equations are normally noncompact. That is why one cannot expect compactness of solution sets in any equivalent equation, and hence one will not be able to apply the deterministic technique based on compact operators, the Schauder fixed point theorem, and so forth.

At the same time, a more detailed analysis shows (see, e.g., [177–180, 182]) that one can go over to an equivalent operator equation where the involved operators have another convenient properties. These are proved to be *locality* and *tightness*.

We will mostly use notation and terminology from [105].

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P),\ 0 \le t \le T$, be a stochastic basis with the "usual" conditions: $(\mathcal{F}_t)$ is a right-continuous filtration; both $\mathcal{F}_t$ and $\mathcal{F}$ contain P-null sets, an m-dimensional semimartingale $z_t = (z_t^j)_{j=1,\dots,m}$ is defined on $[0, T]$; its predictable characteristics form the triplet (B, C, ν). This implies, in particular, that $B_t = (B_t^j)_{j=1,\dots,m}$ is a $(\mathcal{F}_t)$-predictable m-dimensional stochastic process with nondecreasing components and $C = (C_t^{jk})_{j,k=1,\dots,m}$ is a predictable nonnegative matrix, ν is a predictable random measure on $[0, T] \times \mathbb{R}^m$ [106].

Semimartingales constitute a class of most general (in a sense) stochastic processes that can serve as "integrating functions" for stochastic integrals, like functions of bounded variation for Stieltjes integrals. In order to be able to describe the corresponding classes of stochastic integrands, we have to introduce a nondecreasing predictable process $\lambda_t,\ 0 \le t \le T$, defined by

$$\lambda_t = \sum_{j \le m} \left(C_t^{jj} + \operatorname*{var}_{s \in [0,t]} B_s^j \right) + \int_{\mathbb{R}^m} \nu\big([0, t] \times dx\big) \{1 \wedge |x|^2\} \tag{7.291}$$

[1] This section is written by A. V. Ponosov.

($x \wedge y$ stands for $\min\{x, y\}$). Denote by α_t and β_t the Radon derivatives of the λ_t-absolutely continuous functions B and $C + E$ with respect to λ, respectively. Here $E = (E_t^{jk})_{j,k=1,m}$, $E_t^{jk} = \int_{\mathbb{R}^m} \nu([0,t]\times dx)x^i x^j I_{\{|x|\le 1\}}$, and x^i are the coordinate functions in $\mathbb{R}^m$. That is,

$$B_t^j = \int_0^t \alpha_s^j d\lambda_s; \qquad C_t^{jk} + E_t^{jk} = \int_0^t \beta_s^{jk} d\lambda_s. \tag{7.292}$$

It is also convenient to introduce the following function:

$$\Gamma_p(t,u) = |u\alpha_t|^p + |u\beta_t u^T|^p, \tag{7.293}$$

which will be used in the sequel.

Let us describe the functional spaces we are going to deal with. The first space $\mathbf{k}$ ("constants") contains all $\mathcal{F}_0$-measurable random variables. After identifying P-equivalent functions and endowing $\mathbf{k}$ with the topology of convergence in probability we get a linear metric space. The second space Λ_p ("integrands"), defined for $1 \le p < \infty$, consists of row vectors $H = (H^1, \dots, H^m)$ with predictable components, for which

$$\|H\|_{\Lambda_p}^p \stackrel{\text{def}}{=} \int_0^T |H_s\alpha_s|^p d\lambda_s + \left(\int_0^T |H_s\beta_s H_s^T| d\lambda_s\right)^{1/2} < \infty \quad \text{a.s.} \tag{7.294}$$

Identifying H_1 and H_2 if $\|H_1 - H_2\|_{\Lambda_p} = 0$ a.s. yields a linear space with the metric $E(\|H_1 - H_2\|_{\Lambda_p} \wedge 1)$.

Using Jacod's description of z_t-integrable stochastic processes (see, e.g., [106]) one can easily see that for each $H \in \Lambda_p$ the stochastic integral $\int_0^t H_s dz_s$ does exist and for each t determines a continuous operator from the space Λ_p to the space $\mathbf{k}$.

We define now the third space ("solutions") by

$$\mathbf{S}_p = \left\{x : x_t - x_0 = \int_0^t H_s dz_s,\ x_0 \in \mathbf{k},\ H \in \Lambda_p\right\}. \tag{7.295}$$

If we identify *indistinguishable* stochastic processes (see, e.g., [105]), we get the following.

Proposition 7.47. *Under the above identifications, the isomorphism*

$$\mathbf{S}_p \simeq \Lambda_p \times \mathbf{k} \tag{7.296}$$

given by $x_t = \int_0^t H_s dz_s + x_0$ *holds.*

Remark 7.48. Using (7.296) we can equip the space $\mathbf{S}_p$ with the direct product topology. This topology is slightly stronger than the Emery topology of the semimartingale space $\mathbf{S}$ studied in [80]: being a linear subspace of $\mathbf{S}$, the space $\mathbf{S}_p$ is closed with respect to its own topology.

Remark 7.49. Using the deterministic terminology we can call $\mathbf{k}$ a "space of initial data," $\mathbf{S}_p$ a "space of solutions," and Λ_p a "space of (abstract) derivatives." It will be shown later that these spaces play the same role in the stochastic theory as the spaces $\mathbb{R}$, $\mathbf{L}_p$, and $\mathbf{D}_p$ (the latter space consists of absolutely continuous functions with p-summable derivatives) do in the deterministic theory.

Remark 7.50. Being mostly dealing with vector processes we will in the sequel use the notion $\mathbf{X}^n$ for the space of n-columns with components belonging to a given space $\mathbf{X}$. For example, $\mathbf{S}_p^n$ will denote the space of n-dimensional semimartingales with components from $\mathbf{S}_p$, and so forth.

An important property of the spaces $\mathbf{D}_p$ is that they admit a compact imbedding in $\mathbf{L}_q$ (if p is arbitrary, and $q < \infty$) and in $\mathbf{C}$ (if $p > 1$). It is this property, which, based on the theory of compact operators, provides some basic features of deterministic functional differential equations, like solvability, continuous dependence on initial data, and so forth. Unfortunately, the imbedding of the spaces $\mathbf{S}_p$ in the spaces Λ_q is never compact. Wishing, however, to understand, as in the deterministic theory, which properties of solutions are crucial for *stochastic functional differential equations*, we should find out what kind of imbedding we have in the stochastic case.

Analysis shows that compactness should be replaced by *tightness*. Recall that a set Q of random points in a metric space $\mathbf{M}$ (i.e., mappings from Ω to $\mathbf{M}$) is called *tight* if for any $\varepsilon > 0$ there exists a compact set $K \subset \mathbf{M}$, such that $P\{\omega \mid x(\omega) \notin K\} < \varepsilon$ as soon as $x \in Q$.

We find it also convenient to use the following notation. For a metric space $\mathbf{M}$ the space $\widetilde{\mathbf{M}}$ consists of all $(\mathcal{F}_t)$— adapted random points in $\mathbf{M}$. The space $\widetilde{\mathbf{M}}$ will be endowed with the metric $E(\|x-y\|_X \wedge 1)$.

Before formulating an exact result on embedding, let us notice that instead of the space $\mathbf{C}$ of continuous functions we have to consider more general functional spaces, as solutions of (7.290) can be discontinuous. That is why we introduce the space $\widetilde{\mathbf{D}}$ consisting of $(\mathcal{F}_t)$— adapted stochastic processes with trajectories belonging to the space $\mathbf{D}$ of right-continuous functions having left-hand limits at any point. The space $\mathbf{D}$ is equipped with the sup-norm, while $\widetilde{\mathbf{D}}$, as above, will be endowed with the metric $E(\|x-y\|_X \wedge 1)$.

The following theorem was proved in [177–180] for the case of Ito integrals. Modifying slightly the proof and using standard estimates for stochastic integrals with respect to semimartingales (see [105] or [106]), we obtain the following.

Theorem 7.51. (**A**) *For $p > 1$ each set, which is bounded in $\mathbf{S}_p$, is at the same time tight in $\widetilde{\mathbf{D}}$.*

(**B**) *For $p \geq 1$, $q < \infty$ each set, which is bounded in $\mathbf{S}_p$, is at the same time tight in $\widetilde{\mathbf{L}_q}$.*

Theorem 7.51 shows that operators that map bounded sets into tight ones can be of interest in stochastic analysis. This justifies the following.

Definition 7.52. An operator h, defined in a linear space of random points, is said to be *tight* if (1) it transforms bounded sets into tight ones and (2) it is uniformly continuous on each tight subset of its domain.

One can easily observe that if Ω shrinks into a single point (= no randomness), then this definition describes nothing, but usual compact operators, because in this case tight sets equal compact sets.

Yet, the tightness property is too general for our purposes. Indeed, every nonlinear, bounded, and uniformly continuous operator defined in **k** is tight, while **k** is an infinite-dimensional Frechét space. That is why more assumptions on operators are needed.

As it was explained in the works [177, 178, 182], it is quite natural to take into account a "trajectorial" nature of stochastic differential equations. This intuitive concept can be formalized in the following manner (see, e.g., [176, 177, 209]).

Definition 7.53. An operator h is said to be *local* if for any x, y from its domain and for any $A \in \mathcal{F}$, the equality $x(\omega) = y(\omega)$ ($\omega \in A$ a.s.) implies the equality $(hx)(\omega) = (hy)(\omega)$ ($\omega \in A$ a.s).

Examples of local operators:

(1) the superposition operator generated by a random operator $A(\omega)$, that is,

$$(hx)(\omega) \equiv A(\omega, x(\omega)); \tag{7.297}$$

(2) stochastic integrals with respect to arbitrary semimartingales;

(3) combinations of (1) and (2), like sums, products, compositions, pointwise limits, and so forth.

Below we suggest a theory of reducible stochastic functional differential equations based on operators, which are both local and tight.

7.7.2. Properties of reducible stochastic functional differential equations

We define Volterra reducibility in a way that is similar to one used in the deterministic theory. We assume that (7.290) includes a nonlinear operator F acting from "the space of solutions" $\mathbf{S}_p^n$ to "the space of abstract derivatives" Λ_p^n for some p. The choice of the spaces has been explained before. The operator F is assumed to be Volterra: for any $t \in (0, T]$, $x_s = y_s$ a.s. for all $s \in [0, t]$, implies $Fx_s = Fy_s$ a.s. for all $s \in [0, t]$.

Definition 7.54. Equation (7.290) with a nonlinear operation F is *Volterra-reducible* (in $\mathbf{S}_p^n$) to a stochastic functional differential equation

$$dx_t = F_0 x_t dz_t, \quad t \in [0, T], \tag{7.298}$$

with a local, tight, and Volterra operator $F : \mathbf{S}_p^n \to \Lambda_p^n$, if for each stopping time $\tau : \Omega \to [0, T]$, the sets of $\mathbf{S}_p$-solutions to (7.290) and (7.298) are identical within each random interval $[0, \tau)$.

Recall that a stopping time τ with respect to a filtration $\mathcal{F}_t$ is a random variable satisfying $\tau^{-1}(A \cap [0, t]) \in \mathcal{F}_t$ for any Borel subset A of real numbers and any $t \geq 0$.

Remark 7.55. We are studying here *Volterra-reducible stochastic equations*, only. Therefore we will usually omit the word "Volterra" in our considerations.

In this section we are going to look at some basic properties of general reducible equations (7.290).

But first of all we will give a more accurate definition of a solution to (7.298). The challenge here, compared to deterministic equations, is to cover *weak solutions*, that is, solutions defined on extended probability spaces, as it is well known that the original probability space may be unsuitable.

Definition 7.56. A stochastic basis $(\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)$ is called a (regular) splitting of the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ if there exists a $(\mathcal{F}^*, \mathcal{F})$-measurable surjective mapping $c : \Omega^* \to \Omega$ such that

(1) $P^* c^{-1} = P$;

(2) $c^{-1}(\mathcal{F}_t) \subset \mathcal{F}_t^*$ (for all t);

(3) $z_t c$ is again a semimartingale on $(\Omega^*, \mathcal{F}^*, \mathcal{F}_t^*, P^*)$ with the same local characteristics as z_t.

The third property implies, in particular, that the Wiener process remains Wiener for any regular splitting. Also other integrators preserve their probabilistic properties (see [106, 107] for details).

One can easily construct the new "solution space" $\mathbf{S}_p^{n*}$ and the new "space of derivatives" Λ_p^{n*} being nothing, but the spaces $\mathbf{S}_p^n$ and Λ_p^n, respectively, being related to the new stochastic basis. Clearly, $\mathbf{S}_p^{n*} \supset \mathbf{S}_p^n$ and $\Lambda_p^{n*} \supset \Lambda_p^n$. It can also be proved that there exists the only continuous and local (in the sense of Definition 7.53) extension $F_0^* : \mathbf{S}_p^{n*} \to \Lambda_p^{n*}$ of the operator $F_0 : \mathbf{S}_p^n \to \Lambda_p^n$. This extension will necessarily be tight (see [175]).

Definition 7.57. A weak solution $x(t)$ of (7.298) is a stochastic process, which (A) is defined on a regular splitting of the original stochastic basis, (B) belongs to the space $\mathbf{S}_p^{n*} \supset \mathbf{S}_p^n$, and (C) satisfies

$$dx_t = F^* x_t d(z_t c) \tag{7.299}$$

on a random interval $[0, \tau]$.

Remark 7.58. The notation $z_t c$ shows that we integrate with respect to "the same" semimartingale z_t, just redefined for the new probability space (like a function $g(u)$ of one variable u, which can be regarded as a v-independent function of two variables u, v).

Now we are able to formulate basic properties of reducible stochastic functional differential equations.

Theorem 7.59. *Assume that* (7.298) *is reducible in the sense of Definition 7.54.*

(1) *For any initial condition $x_0 = \kappa \in \mathbf{k}^n$, there exists at least one weak solution to* (7.298) *defined on a random interval $[0, \tau]$, where $\tau > 0$ a.s. is a stopping time ("property of local solvability").*

(2) *Any weak solution of* (7.298)*, satisfying $x_0 = \kappa$, can be extended up to either the terminal point T or an explosion time ("property of extension of solutions").*

In the next two properties, it is assumed that all solutions of (7.298) *reach the terminal point T.*

(3) *Any set of solutions of* (7.298)*, which is bounded in $\mathbf{S}_p^n$, is also tight in $\mathbf{S}_p^n$, and hence tight in $\widetilde{\mathbf{D}}^n$ ("tightness property of solutions").*

(4) *If the solutions of* (7.298) *satisfy the property of pathwise uniqueness for any initial value $\kappa \in \mathbf{k}^n$, then all the solutions will be strong (i.e., they will be defined on the original stochastic basis); moreover, the solutions will continuously (in $\mathbf{S}_p^n$-topology) depend on $\kappa \in \mathbf{k}^n$ ("pathwise uniqueness property").*

Proof. Consider the following operator equation in the space Λ_p^n:

$$H = \Phi H, \quad \text{where } \Phi H \stackrel{\text{def}}{=} F\left(\kappa + \int_0^{(\cdot)} H_s dz_s\right). \tag{7.300}$$

By assumption, Φ is local, tight, and Volterra. We can therefore apply the fixed point theorem for local tight operators [177, 178], which states that if an operator h with these two properties has an invariant ball in a space of random points that satisfies the so-called "Π-property" (see below), then h has at least one weak fixed point.

The "Π-property" in a functional space $\mathbf{Y}$ says that there exists a sequence of random finite-dimensional Volterra projections $P^m : \mathbf{Y} \to \mathbf{Y}$, which strongly converges to the identity operator in $\mathbf{Y}$.

In [179] it was proved that the space Λ_p^n satisfies the "Π-property." Thus, the only thing that should be verified is the existence of an invariant ball. This can be done by making use of the technique suggested in [181] where properties (1)–(4) where proved for stochastic functional differential equations driven by the Wiener process. We first find a random Volterra retraction $\pi : \Lambda_p^n \to \mathcal{B}^n$, where $\mathcal{B}$ is the "unit ball" in Λ_p given by

$$\mathcal{B} = \left\{ x \in \Lambda_p : \int_0^T \Gamma_p(t, x_t)\, d\lambda_t \leq 1 \right\}, \tag{7.301}$$

and Γ_p is defined by (7.293). Without loss of generality we may assume that $n = 1$.

Put

$$\pi x_t = x_t \left(I_{\{(t,\omega):\Gamma_p(t,\omega,x_t(\omega))<1\}} + \gamma_x I_{\{(t,\omega):\tau(\omega)=t\}} \right), \tag{7.302}$$

where $\tau \stackrel{\text{def}}{=} \inf\{t : \int_0^t \Gamma_p(s, x_s)\, d\lambda_s \geq 1\}$, and $\gamma_x \geq 0$ is chosen in such a way that $\gamma_x = 0$ if $\tau = +\infty$ and

$$\int_0^\tau \Gamma_p(s, \pi x_s)\, d\lambda_s \leq 1. \tag{7.303}$$

Let us show that such a γ_x does exist.

Consider the equation

$$au^2 + bu + c = 1, \tag{7.304}$$

where $a = |x\beta_t x|^p \Delta\lambda_\tau$, $b = |x\alpha_t|^p \Delta\lambda_\tau$, $c = \int_0^\tau \Gamma_p(s, x_s)\, d\lambda_s$.

Since $a, b, c \geq 0$ and $c \leq 1$, this equation has only one positive solution, say, u_x. Putting $\gamma_x = {u_x}^{1/p}$ we have

$$\begin{aligned} \int_0^\tau \Gamma_p(s, \pi x_s)\, d\lambda_s &= \int_0^\tau \Gamma_p(s, x_s)\, d\lambda_s + \Gamma_p(\tau, \gamma_x x_\tau)\Delta\lambda_\tau \\ &= \gamma_x^p b + \gamma_x^{2p} a + c = 1, \end{aligned} \tag{7.305}$$

and therefore

$$\int_0^\tau \Gamma_p(s, \pi x_s)\, d\lambda_s = 1 \quad \text{if } \tau < \infty, \tag{7.306}$$

or

$$\int_0^\tau \Gamma_p(s, \pi x_s)\, d\lambda_s < 1 \quad \text{if } \tau = \infty \tag{7.307}$$

(of course, all the entries here may depend on the random variable ω).

Clearly, γ_x is a continuous function of the variable $x \in \Lambda_p$. We claim that (i) $\pi(\Lambda_p) \subset \mathcal{B}$, and (ii) $\pi x = x$ if $x \in \mathcal{B}$, (iii) $x^{(k)} \to x$ implies $\pi x^{(k)} \to \pi x$. To see this, we notice that (i) follows directly from (7.303), (ii) can be deduced from the following implications: $\{\int_0^T \Gamma_p(s, x_s)\, d\lambda_s < 1\} \Rightarrow \{\tau = +\infty\} \Rightarrow \{\gamma_x = 0 \text{ and } [0, \tau) \cap [0, T] = [0, T]\} \Rightarrow \{\pi x_t = x_t\}$, while (iii) follows from continuity of the integration and continuity of γ_x with respect to x.

Let us continue to study (7.298). Consider the operator equation

$$H = \pi\Phi H \tag{7.308}$$

in the space Λ_p^n. By the above-mentioned fixed point theorem for local tight operators from [177–180], there exists at least one weak solution of the latter equation, belonging, in general, to the extended space Λ_p^{n*}.

Putting $\tau = \inf\{t : \|(\pi\Phi)^* H^* I_{[0,t]}\|_{\Lambda_p^{n*}} \geq 1\}$ gives a predictable stopping time with the property $P^*\{\tau > 0\} = 1$, because

$$\lim_{t\to+0} \left|\left|(\pi\Phi)^* H^* I_{[0,t]}\right|\right|_{\Lambda_p^{n*}} = 0. \tag{7.309}$$

As τ is predictable, there exists another stopping time η, for which $0 < \eta < \tau$. Then we have $\pi(\Phi^* H^*)I_{[0,\eta]} = F^* H^* I_{[0,\eta]}$. Now taking into account that Φ is local (in the sense of Definition 7.53) and Volterra, we obtain

$$\Phi^*\left(H^* I_{[0,\eta]}\right) = \left(\Phi^* H^*\right) I_{[0,\eta]} = \pi\left(\Phi^* H^*\right) I_{[0,\eta]} = H^* I_{[0,\eta]}, \tag{7.310}$$

so that $x_t^* = \kappa + \int_0^t H_s^* d(z_s c)$ becomes a weak solution of (7.298), defined on the random interval $[0, \eta]$. This proves property (1).

To prove property (2) we, by induction, construct a sequence of weak solutions $x_t^{(k)} \equiv x_0 + \int_0^t H_s^{(k)} d(z_s c^{(k)})$ defined on some random intervals $[0, \eta^{(k)})$, respectively. Here $c^{(k)} : \Omega^{(k)} \to \Omega$ stands for the kth splitting mapping corresponding to a weak solution $H_t^{(k)}$ of the equation

$$H = \Phi^{(k)} H \stackrel{\text{def}}{=} \pi^{(k)} \Phi\left(H I_{[\eta^{(k-1)}, T]} + H^{(k-1)} I_{[0,\eta^{(k-1)})}\right), \tag{7.311}$$

$\pi^{(k)}$ being the Volterra retraction onto the set $\{\Gamma_p(t, H_t) \leq k\}$. We also put $\Omega^{(0)} = \Omega$, $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}_t^{(0)} = \mathcal{F}_t$, $P^{(0)} = P$ for the sake of convenience.

At least one weak solution $H^{(k)}$ of (7.311) does exist due to the fixed point theorem for local tight operators, and we, as before, can define a stopping time $\eta^{(k)}$ if we put

$$\eta^{(k)} \stackrel{\text{def}}{=} \inf\left\{t : \left|\left|\Phi^{(k)} H^* I_{[0,t]}\right|\right|_{\Lambda_p^{n*}} \geq k\right\}. \tag{7.312}$$

Using the definition of $\Phi^{(k)}$ we have

$$H^{(k)}\big|_{[0,\eta^{(k-1)})} = H^{(k-1)} c^{(k,k-1)}, \tag{7.313}$$

where $c^{(k,k-1)} : \Omega^{(k)} \to \Omega^{(k-1)}$ relates two splittings to each other, and

$$\left(\Omega^{(k)}, \mathcal{F}^{(k)}, \mathcal{F}_t^{(k)}, P^{(k)}\right) \tag{7.314}$$

is the splitting, where the process $H^{(k)}$ is defined.

The sequence of the constructed splitting forms therefore a projective family of probability spaces. Its projective limit, called in the sequel $\{\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, \bar{P}\}$, inherits the property of regularity. Putting $\eta = \sup_k\{\eta^{(k)}\}$, $\bar{H}|_{[0,\eta^{(k)})} \overset{\text{def}}{=} H^{(k)}\bar{c}^{(k)}$, and $\bar{x}_t \overset{\text{def}}{=} \kappa + \int_0^t \bar{H}_s d(z_s \bar{c}^{(0)})$, where the projections $\bar{c}^{(k)} : \bar{\Omega} \to \Omega^{(k)}$ $(k \geq 0)$ are projective limits of the sequence $c^{(r,k)} : \Omega^{(r)} \to \Omega^{(k)}$ as $r \to \infty$, we get a predictable stopping time and solutions of (7.300) and (7.298), respectively. Moreover, it immediately follows from the definition of $\Phi^{(k)}$ that

$$\bar{P}\{||\bar{H}I_{[0,\eta)}||_{\Lambda_p^n} = +\infty\} + \bar{P}\{\eta = T\} = 1, \tag{7.315}$$

and therefore

$$\bar{P}\{||\bar{x}I_{[0,\eta)}||_{\tilde{D}_p^n} = +\infty\} + \bar{P}\{\eta = T\} = 1. \tag{7.316}$$

This completes the proof of property (2).

The proofs of the third and the fourth properties are similar to those given in [181] and are omitted here. □

Corollary 7.60. *Stochastic functional differential equation* (7.290) *is reducible to the so-called "canonical form"*

$$dx_t = f(t, x_0)dz_t \tag{7.317}$$

if (A) equation (7.290) *is reducible in the sense of Definition 7.54, (B) its (weak) solutions are pathwise unique, and (C) for any bounded set B of initial data X_0 and any positive ε, there is a constant $c = c(B, \varepsilon)$ such that each local solution of* (7.290) *satisfies the following a priori estimate:*

$$P^*\{\|x\|_{S_p^{n*}} > c\} < \varepsilon. \tag{7.318}$$

7.7.3. Specific classes of reducible stochastic functional differential equations driven by semimartingales

7.7.3.1. Ordinary stochastic differential equations driven by semimartingales

Consider the equation

$$dx_t = f(t, x_{t-})dz_t, \tag{7.319}$$

where $x_{t-} \overset{\text{def}}{=} \lim_{s\to t-0} x(s)$ (the limit always exists for all $x \in \mathbf{S}_p^n$, see, e.g., [79]). Assume that

(A1) $f : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is continuous in $x \in \mathbb{R}^n$, predictable in $(t, \omega) \in [0, T] \times \Omega$, and takes values in the space of matrices with m columns and n rows (further denoted by $\mathbb{R}^{m\times n}$);

(A2) for any $R > 0$ there is a stochastic process φ_t^R, which is integrable with respect to λ_t and satisfies

$$\Gamma_p(t, f(t,x)) \leq \varphi_t^R \quad \text{a.s. } (\forall t \in [0,T],\ |x| \leq R), \tag{7.320}$$

where $\Gamma_p(t,u)$ is given by (7.293).

Remark 7.61. As before, we omit the random variable ω, when describing stochastic processes. Strictly speaking, however, the function f in (7.319) as well as similar functions below are assumed to depend on ω, that is, $f = f(t,\omega,x)$, and so forth.

Remark 7.62. By definition, $|\alpha_s| \leq 1$, $|\beta_s| \leq 1$, so that (7.320) is implied by the following more simple inequality:

$$|f(t,x)|^{2p} \leq \varphi_t^R \quad \text{a.s. } (\forall t \in [0,T],\ |x| \leq R). \tag{7.321}$$

However, the latter estimate may be too restrictive in applications. For example, Ito equations with a p-integrable drift coefficient and a $2p$-integrable diffusion coefficient do not, in general, satisfy (7.321), but it can be shown that they fit in (7.320).

Theorem 7.63. *Under assumptions (A1)-(A2),* (7.319) *is reducible in the sense of Definition 7.54.*

Remark 7.64. Clearly, the case $p = 1$ gives the least restrictive estimate, and at the same time the biggest space of solutions, namely $\mathbf{S}_1^n$. The smaller solution space is needed, the stronger estimates are required.

Proof of Theorem 7.63. The operator F, $(Fx_t)(\omega) = f(t,\omega,x_{t-}(\omega))$, is random, and the generating function $f(t,\omega,x)$ is continuous in x. Therefore, F is local, Volterra, and uniformly continuous on tight subsets as the operator from $\widetilde{\mathbf{D}}^n$ to Λ_p^n (the latter is ensured by (A2)). Clearly, the same is true for F considered as an operator from $\mathbf{S}_p^n$ to Λ_p^n, because $\mathbf{S}_p^n$ has a stronger topology. Due to Theorem 7.39 bounded subsets of the space $\mathbf{S}_p^n$ $(p > 1)$ are tight in $\widetilde{\mathbf{D}}^n$. It is also straightforward that F maps tight subsets of $\widetilde{\mathbf{D}}^n$ into tight subsets of Λ_p^n. Hence, F is tight as an operator from $\mathbf{S}_p^n$ to Λ_p^n for $p > 1$.

For $p = 1$ we observe that for each $R > 0$ the operator $F_R x_t = f(t, \pi_R(x_{t-}))$ can be extended to the space $\widetilde{\mathbf{L}}_2^n$ as a continuous random operator taking values in Λ_p^n. Here π_R is the retraction of $\mathbb{R}^n$ onto the ball $\{x \in \mathbb{R}^n : |x| \leq R\}$. Repeating the above argument for the case $p > 1$ and applying part **B** of Theorem 7.51, we get the tightness of F_R as the operator from $\mathbf{S}_1^n$ to Λ_1^n. It remains now to notice that any bounded subset of $\mathbf{S}_1^n$ satisfies the condition

$$\forall \epsilon > 0 \quad \exists R_\epsilon \quad \text{such that} \quad P\{\|x\|_{D^n} > R_\epsilon\} < \epsilon, \tag{7.322}$$

and the result follows. □

Remark 7.65. A slight modification of the above proof shows that Theorem 7.59 remains valid for the case of the so-called "Dolean-Protter equation"

$$dx_t = g(t, x_{t-})dz_t \tag{7.323}$$

with $g : [0,T] \times \Omega \times \mathbf{D}^n \to \mathbb{R}^n$ satisfies conditions similar to (A1)-(A2), where one has to replace $\mathbb{R}^n$ by $\mathbf{D}^n$ and f by g assuming g to be of Volterra type.

7.7.3.2. Stochastic delay equations

Consider the equation

$$dx_t = f(t, x_{t-}, Tx_t)dz_t, \tag{7.324}$$

$$x_s = \varphi_s, \quad s < 0. \tag{7.325}$$

Here $Tx_t = \int_{-\infty}^{t} d_s\mathcal{R}(t,s)x_s$.

Assume that

(B1) $f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m\times n}$ is continuous in $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ and predictable in $(\omega,t) \in [0,T] \times \Omega$;

(B2) for any $R > 0$ the following estimate holds:

$$\Gamma_p(t, f(t,x,y)) \le \varphi_t^R + c|y|^q \quad (c \ge 0,\ p,q \ge 1), \tag{7.326}$$

where φ_t^R is the same as in (A2), $y \in R$, $t \in [0,T]$, $|x| \le R$, and Γ_p is given by (7.293);

(B3) the kernel $\mathcal{R} : [0,T] \times [-\infty,T] \times \Omega \to \mathbb{R}^{n\times n}$ is $\mathcal{B} \otimes \mathcal{P}$-measurable (where $\mathcal{B}$ and $\mathcal{P}$ stand for the σ-algebra of Borel and predictable sets, respectively; we put $\mathcal{F}_s = \mathcal{F}_0$ for $s < 0$ by definition) and satisfies the following condition:

$$\int_0^T \left[\operatorname*{var}_{s\in[0,T]} \mathcal{R}^{ij}(t,s,\omega)\right]^{2q} d\lambda_t(\omega) < \infty \quad a.s.; \tag{7.327}$$

(B4) φ is $\mathcal{B} \otimes \mathcal{F}_0$-measurable and locally bounded on $(-\infty,0) \times \Omega$ stochastic process.

Theorem 7.66. *Under assumptions (B1)–(B4),* (7.324)*, supplied with the "prehistory condition"* (7.325)*, is reducible in the sense of Definition 7.54.*

Proof. Notice that due to (B1)–(B4) the operator g defined by

$$g(t, x_{t-}) = f\left(t, x_{t-}, \int_0^t d_s\mathcal{R}(t,s)x_s + \psi_t\right) \tag{7.328}$$

satisfies (A1)-(A2), where $\mathbb{R}^n$ is replaced by $\mathbf{D}^n$. Here $\psi_t = \int_{-\infty}^{0} d_s\mathcal{R}(t,s)\varphi_s$. According to Remark 7.49 we may apply Theorem 7.59. $\square$

7.7.3.3. Integrodifferential equations

Consider the equation

$$dx_t = f(t, x_t, Ux_t)dz_t. \tag{7.329}$$

Here U is a nonlinear stochastic integral operator of the form

$$Ux_t = \psi + \int_0^t H(t, s, x_s)dz_t. \tag{7.330}$$

Such equations driven by the Wiener process were studied in [155] under Lipschitz conditions.

We assume that

(C1) f satisfies (B1);

(C2) $\Gamma_p(t, f(t, x, y)) \leq \varphi_t^R$ for any $R > 0$, where ($p \geq 1$, φ^R is the same as in (A1), $t \in [0, T]$, $x, y \in \mathbb{R}^n$, $|x|, |y| \leq R$, and Γ_p is given by (7.293);

(C3) the functions $H^{ij} : [0, T] \times [0, T] \times \mathbb{R}^n \to \mathbb{R}$ ($i = 1, \dots, n$, $j = 1, \dots, m$) are absolutely continuous in t, while their first derivatives with respect to t are $\mathcal{B} \otimes \mathcal{P}$-measurable in (t, s, ω) and continuous in x;

(C4) the following estimates hold for any $R > 0$:

$$\begin{gathered} |H^{ij}(s, s, \omega, x)| \leq Q_R(t, s, \omega); \qquad \left\| \frac{\partial H^{ij}}{\partial t}(t, s, \omega, x) \right\| \leq K_R(t, s, \omega), \\ \int_0^T d\lambda_s \left(Q_R^2(s, \omega) + \int_s^T K_R^2(\tau, s, \omega)d\tau \right) < \infty \quad \text{a.s.}; \end{gathered} \tag{7.331}$$

(C5) $\psi : [0, T] \times \Omega \to \mathbb{R}^n$ is predictable and locally bounded.

Conditions (C3)–(C5) can be weakened if z has independent increments (see, e.g., [106] for the definition):

(C3a) H^{ij} are $\mathcal{B} \otimes \mathcal{P}$-measurable in (t, s, ω) and continuous in x;

(C4a) $|H^{ij}(t, s, \omega, x)| \leq Q_R(t, s, \omega)$ ($|x| \leq R$, $R > 0$) and

$$\int_0^T \int_0^T |Q_R(t, s)|^q d\lambda_t\, d\lambda_s < \infty \quad \text{a.s.} \tag{7.332}$$

for some $q > 2$;

(C5a) ψ is predictable and q-summable in t.

But then we have to replace (C2) with

(C2a) $\Gamma_p(t, f(t, x, y)) \leq \varphi_t^R + c|x|^q$ ($c \geq 0$ a.s., $|x| \leq R$, $R > 0$).

Theorem 7.67. *Under assumptions (C1)–(C5) or, if z_t has independent increments, under assumptions (C1), (C2a)–(C5a),* (7.329) *is reducible in the sense of Definition 7.53.*

Proof. Making use of the proofs of Theorems 7.66 and 7.67, we notice that it is sufficient to show that the integral operator

$$Ax_t = \int_0^t H(t,s,x_s)\,dz_s \tag{7.333}$$

is uniformly continuous on tight subsets of the space $\widetilde{\mathbf{D}}^n$ and takes values in $\widetilde{\mathbf{L}}^n_\infty(\lambda)$ (for conditions (C3)-(C4), or in $\widetilde{\mathbf{L}}^n_q(\lambda)$ (for conditions (C3a)-(C4a). Both cases can be treated in a similar way, so we prove the first statement.

The semimartingale z_t can be represented as a sum:

$$z_t = B_t + \beta_t + z_t I_{\{|\Delta z|>1\}}(t), \tag{7.334}$$

where B_t is the first local characteristic of z_t, while the second term is a local martingale (see [106], or [79]).

The integral operator

$$A_1 x_t = \int_0^t H(t,s,x_s)\,d\big(B_T + z_t I_{\{|\Delta z|>1\}}(t)\big) \tag{7.335}$$

will then be a random Stieltjes integral operator driven by a process which is absolutely continuous with respect to λ_t. The integrand satisfies the following estimate:

$$\begin{aligned} \big|H^{ij}(t,s,x)\big| &\le \big|H^{ij}(s,s,x)\big| + \left|\int_s^t \frac{\partial H^{ij}}{\partial \tau}(\tau,s,x)\,d\tau\right| \\ &\le Q_R(s) + \int_s^t K_R(\tau,s)\,d\tau \in \mathbf{L}_1(\lambda) \quad \text{a.s.} \end{aligned} \tag{7.336}$$

Hence, the operator A_1 maps $\widetilde{\mathbf{D}}^n$ to $\widetilde{\mathbf{L}}^n_\infty(\lambda)$, being uniformly continuous on tight subsets of its domain.

The integral operator

$$A_2 x_t = \int_0^t H(t,s,x_s)\,d\beta_s \tag{7.337}$$

can be represented as follows (see, e.g., [184]):

$$A_2 x_t = \int_0^t H(s,s,x_s)\,d\beta_s + \int_0^t d\tau \int_0^\tau H'_\tau(\tau,s,x_s)\,d\beta_s. \tag{7.338}$$

For any predictable stopping time $T_n \le T$ we have

$$E\sup_{t\le T_n} \big|A_2 x_t\big|^2 \le 2E\sup_{t\le T_n}\left(\int_0^t H(s,s,x_s)\,d\beta_s\right)^2 + 2E\sup_{t\le T_n}\left(\int_0^t d\tau \int_0^\tau H'_\tau(\tau,s,x_s)\,d\beta_s\right)^2. \tag{7.339}$$

Using standard technique of estimating stochastic integrals with respect to semi-martingales, we obtain

$$E\sup_{t\le T_n}\left|A_2x_t\right|^2 \le K_1\left(E\int_0^{T_n} Q_R^2(s)d\lambda_s+\left(\int_0^{T_n} Q_R(s)d\lambda_s\right)^2\right)\\ +K_2E\int_0^{T_n} d\lambda_s\int_s^T K_R^2(\tau,s)d\tau, \tag{7.340}$$

so that $P\{\sup_{t\le T}|A_2x_t|^2 \ge K_\epsilon\}<\epsilon$ for arbitrary $\epsilon>0$ and sufficiently large K_ϵ. On the other hand, the process $\xi_t=\int_0^t H(t,s,x_s)d\beta_s$ admits a $\mathcal{F}_{t-}\otimes\mathcal{B}$-measurable version, hence it is equivalent to a predictable process. We have just proven that $A_2(\widetilde{\mathcal{L}}^n_\infty(\lambda))\subset\widetilde{\mathbf{L}}^n_\infty(\lambda)$.

A similar reasoning implies the estimate

$$E\sup_{t\le T_n}\left|A_2x_t-A_2y_t\right|^2 \le C\left(E\int_0^{T_n}\left[H(s,s,x_s)-H(s,s,y_s)\right]^2d\lambda_s\right.\\ \left.+E\int_0^{T_n} d\lambda_s\int_s^T\left[H'_\tau(\tau,s,x_s)-H'_\tau(\tau,s,y_s)\right]^2d\tau\right). \tag{7.341}$$

Consider now the random integral operator defined by

$$\mathcal{I}(x_t,y_t)(\omega)=\int_0^t G(s,\omega,x_s,y_s)d\lambda_s, \tag{7.342}$$

where the kernel G is equal to either $\int_s^T(H'_\tau(\tau,s,x)-H'_\tau(\tau,s,y))^2d\tau$, or $[H(s,s,x)-H(s,s,y)]^2$.

By our assumptions, $\mathcal{I}(\omega)$ is continuous a.a. all ω as an operator from $\mathbf{L}^{2n}_\infty(\lambda)$ to $\mathbf{L}^n_\infty(\lambda)$. Hence the operator $\mathcal{I}$, as a superposition operator from $\widetilde{\mathbf{L}}^{2n}_\infty(\lambda)$ to $\widetilde{\mathbf{L}}^n_\infty(\lambda)$, will be uniformly continuous on tight sets. In particular, given a tight set $Q\subset\widetilde{\mathbf{L}}^n_\infty(\lambda)$, we have

$$P-\lim_{\delta\to+0}\sup\left|\int_0^t G(s,x_s,y_s)d\lambda_s\right|=0, \tag{7.343}$$

where sup is taken over $t\in[0,T]$, $x,y\in Q$, $E(\|x-y\|_{\mathbf{D}^n}\wedge 1)\le\delta$. In other words, for some exhaustive (i.e., going a.s. to infinity) sequence of predictable stopping times,

$$\lim_{\delta\to+0}\sup\left|\int_0^{t\wedge T_n} G(s,x_s,y_s)d\lambda_s\right|=0. \tag{7.344}$$

Making use of the estimate (7.341) we obtain that

$$\lim_{\delta\to+0}\sup\left|A_2x_{t\wedge T_n}-A_2y_{t\wedge T_n}\right|^2=0. \tag{7.345}$$

It exactly means that A_2 is continuous on the set Q.

Let us complete the proof of Theorem 7.67. If $p > 1$, then the result immediately follows from Theorem 7.39. If $p = 1$, we proceed in a similar way replacing first the operator A by a "truncated" operator, as it was done in the course of the proof of Theorem 7.63. $\square$

7.7.3.4. Neutral stochastic functional differential equations

We will only consider one particular kind of neutral functional differential equations, which is rather illustrative and comparatively simple from the technical point:

$$dx_t = f(t, Tx_t, Sx_t)dz_t, \tag{7.346}$$

where

$$Tx_t = \int_{-\infty}^{t} d_s\mathcal{R}(t,s)x_s, \qquad Sx_t = \int_{-\infty}^{t} Q(t,s)dx_s, \tag{7.347}$$

supplied with "a prehistory:"

$$x_s = \xi_s, \quad s < 0. \tag{7.348}$$

Some other kinds of stochastic neutral equations can be found in [115, 181, 183].

Introduce the following hypotheses:

(D1) f satisfies (B1);

(D2) for all $R > 0$ $\Gamma_p(t, f(t,x,y)) \le \varphi_t^R + |x|^q$ ($p > 1$, $q \ge 1$, $t \in [0,T]$, $x, y \in \mathbb{R}^n$, $|y| \le R$, Γ_p is given by (7.293));

(D3) $\mathcal{R}$ satisfies (B3);

(D4) Q is absolutely continuous in t, while its derivative Q_t' satisfies the measurability conditions (B3) and, in addition, the following estimate:

$$\int_0^T \left(\int_s^T |Q_\tau'(\tau,s)|^2 d\tau\right)^r d\lambda_s + \int_0^T |Q(s,s)|^{2r} d\lambda_s < \infty \quad \text{a.s.}; \tag{7.349}$$

(D5) $\int_{-\infty}^0 Q(t,s)d\varphi_s$ exists and locally bounded on time intervals.

Theorem 7.68. *Under assumptions (D1)–(D5)* (7.346) *with the "prehistory" condition,* (7.347) *is reducible in the sense of Definition 7.54.*

Proof. The crucial point is the tightness of the operator S_0 defined by $S_0x_t = \int_0^t Q(t,s)dx_s$ as a mapping from $\mathbf{S}_p^n$ to $\widetilde{\mathbf{D}}^n$. The following estimate can easily be derived from the proof of the preceding theorem:

$$E\sup_{0\le t\le T_n} |S_0x_t|^2 \le E\left(\int_0^{T_n}\left(\int_s^T |Q_\tau'(\tau,s)|^2 d\tau\right)^r d\lambda_s\right)^{1/r} + E\left(\int_0^{T_n} |Q(s,s)|^{2r} d\lambda_s\right)^{1/r} \tag{7.350}$$

for some exhaustive sequence $\{T_n\}$ of predictable stopping times. This means that for any $\varepsilon > 0$ there exists a number N for which $P\{T_n < T\} < \varepsilon$ for all $n \geq N$. Thus, S_0 becomes a bounded linear operator from $\mathbf{S}_p^n$ to $\widetilde{\mathbf{D}}^n$. Approximation of the kernel Q by kernels Q^n, which are finite sums of the form $\sum_j a_j(t)b_j(s)$, gives, after using estimates similar to (7.350), a uniform approximation of the operator S_0 by finite-dimensional random (and hence tight) operators acting from S_p^n to $\widetilde{D}^n$. This implies tightness of the operator S_0 as well.

Now combining the proof of Theorem 7.59 with the fact just established we get the required result. $\square$

8 Appendices

A. On the spectral radius estimate of a linear operator

Consider the problem on the estimate of the spectral radius $\rho(A)$ of a linear operator A in the space $\mathbf{C}$ of continuous functions $x:[a,b]\to\mathbb{R}^1$, $\|x\|_{\mathbf{C}}=\max_{t\in[a,b]}|x(t)|$.

Since $\rho(A)<1$ if $\|A\|_{\mathbf{C}\to\mathbf{C}}<1$, for the isotonic A the estimate $\rho(A)<1$ follows from

$$\|A\|_{\mathbf{C}\to\mathbf{C}} = \max_{t\in[a,b]} (A[1])(t) < 1. \tag{A.1}$$

A sharper estimate may be obtained due to the following well known assertion.

Lemma A.1. *Let a linear bounded $A:\mathbf{C}\to\mathbf{C}$ be isotonic. The estimate $\rho(A)<1$ is valid if and only if there exists a $v\in\mathbf{C}$ such that*

$$v(t)>0,\quad r(t)\stackrel{\text{def}}{=} v(t)-(Av)(t)>0,\quad t\in[a,b]. \tag{A.2}$$

Proof. Necessity is rather obvious if we take as v the solution of the equation $x-Ax=1$.

Sufficiency. Define in the space $\mathbf{C}$ the norm $\|\cdot\|_{\mathbf{C}}^{v}$ equivalent to the norm $\|\cdot\|_{\mathbf{C}}$ by

$$\|x\|_{\mathbf{C}}^{v} = \max_{t\in[a,b]} \frac{|x(t)|}{v(t)}. \tag{A.3}$$

Since

$$\|A\|_{\mathbf{C}\to\mathbf{C}}^{v} = \sup_{\|x\|_{\mathbf{C}}^{v}\le 1} \|Ax\|_{\mathbf{C}}^{v} \le \max_{t\in[a,b]} \frac{(Av)(t)}{v(t)} < 1, \tag{A.4}$$

$\rho(A)\le\|A\|_{\mathbf{C}\to\mathbf{C}}^{v}<1$. □

The requirement of the strict inequalities $v(t)>0$, $r(t)>0$ on the whole of $[a,b]$ involves certain difficulties in some applications of Lemma A.1, for instance, as applied to multipoint boundary value problems. Thus it is natural that some works (see, e.g., [29, 101, 117]) have been devoted to weakening the conditions of Lemma A.1 at the expense of some additional requirements to the operator A. The Islamov theorem (see [104]) covered the previous results. But studying the mentioned work demands some sophisticated knowledge in the theory of functions, and occasionally it is difficult to check the conditions of the Islamov theorem. Besides, the theorem assumed the weak compactness of A which prevents the application to some singular problems. Below we offer simple assertions as an addition to Islamov's theorem.

We say that the linear operator $A : \mathbf{C} \to \mathbf{C}$ possesses *the property M* if $(Ax)(t) > 0,\ t \in [a,b]$, for each $x \in \mathbf{C}$ such that $x(t) \geq 0,\ x(t) \not\equiv 0$.

If A possess the property M, some conditions with respect to the defect r may be weakened as the following theorem shows. Notice that the inequalities $v(t) \geq 0$, $r(t) \geq 0$ in this case imply that $v(t) > 0$ on $[a,b]$. Therefore, the M-property does not permit weakening the condition of Lemma A.1 with respect to the inequality $v(t) > 0$.

Theorem A.2. *Let a linear bounded $A : \mathbf{C} \to \mathbf{C}$ possess the property M. Then $\rho(A) < 1$ if and only if there exists $v \in \mathbf{C}$ such that*

$$v(t) > 0, \quad r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad r(t) \not\equiv 0, \quad t \in [a,b]. \tag{A.5}$$

Proof

Sufficiency. If $r(t) > 0$ on $[a,b]$, $\rho(A) < 1$ by virtue of Lemma A.1. If $r(t) \geq 0$, we apply the operator A to both sides of $v - Av = r$:

$$(Av)(t) - (A^2 v)(t) = (Ar)(t) > 0, \quad t \in [a,b]. \tag{A.6}$$

Therefore, $v(t) - (A^2 v)(t) > 0$. Consequently, applying Lemma A.1 to A^2, we get $\rho(A^2) < 1$. Since $\rho(A) = [\rho(A^2)]^{1/2}$, $\rho(A) < 1$.

Necessity follows from Lemma A.1. □

We say that the linear operator $A : \mathbf{C} \to \mathbf{C}$ possesses *the property N* if there exists a finite number of the points $v_1, \ldots, v_m \in [a,b]$ such that $(Ax)(v_i) = 0,\ i = 1, \ldots, m$, for each $x \in \mathbf{C}$.

Theorem A.3. *Let a linear bounded isotonic $A : \mathbf{C} \to \mathbf{C}$ possesses the property N. Then $\rho(A) < 1$ if and only if there exists $v \in \mathbf{C}$ such that*

$$v(t) > 0, \quad r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) > 0, \quad t \in [a,b] \setminus \{v_1, \ldots, v_m\}. \tag{A.7}$$

Proof

Sufficiency. Denote $v_\varepsilon = v + \varepsilon$, $\varepsilon > 0$, $r_\varepsilon \stackrel{\text{def}}{=} v_\varepsilon - Av_\varepsilon = r + \varepsilon\psi$, where $\psi = \mathbf{1} - A[\mathbf{1}]$. Thus $v_\varepsilon(t) > 0,\ t \in [a,b]$, $r_\varepsilon(t) > 0$ on $[a,b]$ since $\psi(v_i) = 1,\ i = 1, \ldots, m$. Consequently, $\rho(A) < 1$ by virtue of Lemma A.1.

Necessity is obvious. □

We say that the linear operator $A : \mathbf{C} \to \mathbf{C}$ possesses *the property MN*, if it possesses the property N and $(Ax)(t) > 0$, $t \in [a,b] \setminus \{v_1,\dots,v_m\}$ for each $x \in \mathbf{C}$ such that $x(t) \geq 0$, $x(t) \not\equiv 0$.

Theorem A.4. *Let a linear bounded $A : \mathbf{C} \to \mathbf{C}$ possesses the property MN. Then $\rho(A) < 1$ if and only if there exists $v \in \mathbf{C}$ such that*

$$v(t) > 0, \quad r(t) \stackrel{\text{def}}{=} v(t) - (Av)(t) \geq 0, \quad r(t) \not\equiv 0, \quad t \in [a,b] \setminus \{v_1,\dots,v_m\}. \tag{A.8}$$

Proof

Sufficiency. If $r(t) > 0$, $t \in [a,b] \setminus \{v_1,\dots,v_m\}$, we have $\rho(A) < 1$ by virtue of Theorem A.3. In the other case we get $v(t) - (A^2v)(t) > 0$ by applying the operator A to both sides of $v - Av = r$. Therefore,

$$v(t) - (A^2v)(t) > 0, \quad t \in [a,b] \setminus \{v_1,\dots,v_m\}. \tag{A.9}$$

Hence $\rho(A^2) < 1$ by virtue of Theorem A.3 and, consequently, $\rho(A) < 1$.

Necessity is obvious. □

Consider some estimates of the solution of the equation

$$x + Ax = f. \tag{A.10}$$

Theorem A.5. *Let $A : \mathbf{C} \to \mathbf{C}$ be a linear bounded isotonic operator, $f \in \mathbf{C}$, $\theta = f - Af$. Let, furthermore, at least one of the following conditions hold.*

(a) $f(t) > 0$, $\theta(t) > 0$, $t \in [a,b]$.

(b) *The operator A possesses the property M and $f(t) > 0$, $\theta(t) \geq 0$, $\theta(t) \not\equiv 0$, $t \in [a,b]$.*

(c) *The operator A possesses the property N and $f(t) > 0$, $\theta(t) > 0$, $t \in [a,b] \setminus \{v_1,\dots,v_n\}$.*

(d) *The operator A possesses the property MN and $f(t) > 0$, $\theta(t) \geq 0$, $\theta(t) \not\equiv 0$, $t \in [a,b] \setminus \{v_1,\dots,v_n\}$.*

Then (A.10) *has a unique solution $x \in \mathbf{C}$ and, for the solution, the estimate*

$$f(t) - (Af)(t) \leq x(t) \leq f(t), \quad t \in [a,b], \tag{A.11}$$

holds.

Proof. By the previous assertions, we have $\rho(A) < 1$ from each of the conditions (a), (b), (c), and (d). Applying the operator $I - A$ to both sides of (A.10), we obtain the equation $x - A^2x = \theta$ with isotonic A^2, $\rho(A^2) < 1$, which is equivalent to (A.10). Therefore,

$$x(t) = \theta(t) + (A^2\theta)(t) + (A^4\theta)(t) + \cdots \geq \theta(t) = f(t) - (Af)(t) \geq 0. \tag{A.12}$$

Besides $x(t) = f(t) - (Ax)(t) \leq f(t)$. □

B. A compactness condition for a linear integral operator in the space of summable functions

Consider the integral operator

$$(Kz)(t) = \int_a^b K(t,s)z(s)ds \tag{B.1}$$

on the space $\mathbf{L}$ of summable functions $z : [a,b] \to \mathbb{R}^n$ with the norm $\|z\|_{\mathbf{L}} = \int_a^b |z(s)|ds$.

Theorem B.1 (see [32, 33, 141, 144]). *Let the elements $k_{ij}(t,s)$ of the $n \times n$ matrix $K(t,s)$ be measurable on the square $[a,b]\times[a,b]$, let the functions $k_{ij}(t,s)$ for almost every $t \in [a,b]$ have at each point $s \in [a,b]$ finite one-sided limits, and let there exist a summable v such that $|k_{ij}(t,s)| \le v(t)$ for each $s \in [a,b]$. Then the operator K defined by* (B.1) *is acting in* $\mathbf{L}$ *and is compact.*

Proof. It is sufficient to consider the case of the scalar operator

$$(Kz)(t) = \int_a^b k(t,s)z(s)ds. \tag{B.2}$$

Let us show that

$$\lim_{\tau\to 0}\int_a^b |k(t,s) - k(t+\tau,s)|\,dt = 0 \tag{B.3}$$

uniformly with respect to $s \in [a,b]$ (we assume that $k(t,s)$ is equal to zero outside of the square $[a,b]\times[a,b]$). Assume the contrary. Then for some $\varepsilon > 0$ and for each sequence $\{\delta_\nu\}$, $\delta_\nu > 0$, $\delta_\nu \to 0$, there exist sequences $\{s_\nu\}$ and $\{\tau_\nu\}$, $|\tau_\nu| < \delta_\nu$, such that

$$I_\nu = \int_a^b |k(t,s_\nu) - k(t+\tau_\nu,s_\nu)|\,dt \ge \varepsilon. \tag{B.4}$$

Denote a monotone subsequence of $\{s_\nu\}$ again by $\{s_\nu\}$. Let $\lim_{\nu\to\infty} s_\nu = s_0$, $\rho(t) = \lim_{s_\nu\to s_0} k(t,s_\nu)$. Then ρ is summable under the conditions of the theorem. On the other hand,

$$\begin{aligned} I_\nu &\le \int_a^b |k(t,s_\nu) - \rho(t)|\,dt \\ &\quad + \int_a^b |\rho(t) - \rho(t+\tau_\nu)|\,dt + \int_a^b |\rho(t+\tau_\nu) - k(t+\tau_\nu,s_\nu)|\,dt \\ &\le 2\int_a^b |k(t,s_\nu) - \rho(t)|\,dt + \int_a^b |\rho(t) - \rho(t+\tau_\nu)|\,dt. \end{aligned} \tag{B.5}$$

The first term of the estimate above may be made as small as is wished with the increase of v by virtue of the Lebesgue theorem, as well the second term may be made small since ρ is integrable.

To prove the compactness of the operator K we will check the M. Riesz compactness criterion for the set $K\mathcal{B} \subset \mathbf{L}$, where $\mathcal{B} \subset \mathbf{L}$ is a bounded set. For $y = Kz$, $z \in \mathcal{B}$, we have

$$|y(t)| \leq v(t) \int_a^b |z(s)| ds. \tag{B.6}$$

Next, the inequality

$$\int_a^b |y(t) - y(t+\tau)| dt \leq \sup_{s \in [a,b]} \int_a^b |k(t,s) - k(t+\tau,s)| dt \cdot \int_a^b |z(s)| ds \tag{B.7}$$

and (B.3) imply that

$$\lim_{\tau \to 0} \int_a^b |y(t) - y(t+\tau)| dt = 0 \tag{B.8}$$

uniformly in $y \in K\mathcal{B}$. Therefore, the elements of the set $K\mathcal{B}$ are uniformly bounded and mean equicontinuous. Thus the set $K\mathcal{B}$ is compact in $\mathbf{L}$. □

C. The composition operator

C.1. The conditions of the continuity of the composition operator

The composition operator S_g on the set of the functions $z : [a,b] \to \mathbb{R}^1$ is defined by

$$(S_g z)(t) = \begin{cases} z[g(t)] & \text{if } g(t) \in [a,b], \\ 0 & \text{if } g(t) \notin [a,b]. \end{cases} \tag{C.1}$$

In case the mapping $g : [a,b] \to \mathbb{R}^1$ is measurable, the operator S_g is acting from the space $\mathbf{D}$ of absolutely continuous functions $x : [a,b] \to \mathbb{R}^n$ in the space $\mathbf{L}$ of summable functions $z : [a,b] \to \mathbb{R}^n$ ($\|x\|_{\mathbf{D}} = |x(a)| + \|\dot{x}\|_{\mathbf{L}}$, $\|z\|_{\mathbf{L}} = \int_a^b |z(s)| ds$) and is compact. It follows from the representation that

$$(S_g x)(t) = \int_a^b \chi_g(t,s)\dot{x}(s) ds + \chi_g(t,a)x(a), \tag{C.2}$$

where $\chi_g(t,s)$ is the characteristic function of the set $\{(t,s) \in [a,b] \times [a,b] : a \leq s \leq h(t) \leq b\}$, and Theorem B.1.

In order to state the conditions of the continuous acting of S_g in the space $\mathbf{L}$, we will turn our attention to the following. The function g on the set $\{t \in [a,b] : g(t) \notin [a,b]\}$ may be defined arbitrarily because the values of g such that $g(t) \notin [a,b]$ do not influence the values of S_g since they do not take part in the construction of S_g. For $e \subset [a,b]$, denote $g^{-1}(e) \stackrel{\text{def}}{=} \{t \in [a,b] : g(t) \in e\}$, mes denotes the Lebesgue measure.

Theorem C.1 (see [78]). *The operator S_g is continuously acting in the space $\mathbf{L}$ if and only if*

$$\sup_{\substack{e \subset [a,b] \\ \operatorname{mes} e > 0}} \frac{\operatorname{mes} g^{-1}(e)}{\operatorname{mes} e} = M < \infty. \tag{C.3}$$

As this takes place, $\|S_g\|_{\mathbf{L} \to \mathbf{L}} = M$.

It should be remarked that it is necessary for satisfying (C.3) that

$$\operatorname{mes}(e) = 0 \Longrightarrow \operatorname{mes} g^{-1}(e) = 0 \tag{C.4}$$

for every set $e \subset [a,b]$.

The condition (C.4) is fulfilled if, for example, the function g is piecewise strictly monotone and has absolutely continuous inverse on each interval of monotonicity (see, e.g., [166]).

The role of the condition (C.4) (the so called nonhovering condition of the graph of the function g) is the following. Elements of the space $\mathbf{L}$ are classes of equivalent functions. Thus it is necessary for action of the operator S_g in the space $\mathbf{L}$ that the operator maps equivalent functions into equivalent ones. The condition (C.4) is necessary for such an action.

Indeed, let $e \subset [a,b]$, $\operatorname{mes} e = 0$, but $\operatorname{mes} g^{-1}(e) > 0$, then for a pair of equivalent $y_1, y_2 : [a,b] \to \mathbb{R}^n$ such that $y_1(t) \neq y_2(t)$ at $t \in e$, we obtain $y_1[g(t)] \neq y_2[g(t)]$ for each $t \in g^{-1}(e)$, that is, on the set of positive measure. Next assume that there exist a couple of $y_1, y_2 : [a,b] \to \mathbb{R}^n$ and a set $e \subset [a,b]$ of positive measure such that $g(e) \subset [a,b]$ but $y_1[g(t)] \neq y_2[g(t)]$ for $t \in e$. Denote $e_1 = g(e)$ and let $\operatorname{mes} e_1 > 0$. We obtain the contradiction $y_1(t) \neq y_2(t)$ on the set of positive measure. If $\operatorname{mes} e_1 = 0$, the condition (C.4) implies that the set $g^{-1}(e_1) = e$ has also zero measure.

It should be remarked that if the condition of nonhovering is not satisfied, the composition $y[g(t)]$ may turn out to be nonmeasurable for measurable g and y. One can find the corresponding examples in [99, 114]. The condition (C.4) guarantees measurability of any function $y[g(t)]$ for measurable g and y. Indeed, any equivalence class, containing a measurable y, contains a Borel measurable function z (see, e.g., [206]). The function $S_g z$ is measurable [116]. The condition (C.3) guarantees the equivalence of $S_g z$ and $S_g y$. Therefore $S_g y$ is also measurable.

M. E. Drakhlin and T. K. Plyshevskaya replaced (C.3) by the equivalent condition that is sometimes easier for verifying. This condition may be formulated as follows.

The set $g^{-1}(e)$ is measurable under the condition (C.3) for every measurable $e \subset [a,b]$. Denote

$$\operatorname{mes} g^{-1}(e) = \mu_g(e). \tag{C.5}$$

By the Radon-Nicodym theorem [78], there exists a summable function $v : [a,b] \to \mathbb{R}^1$ such that

$$\mu_g(e) = \int_e v(s)ds \tag{C.6}$$

for each measurable $e \subset [a,b]$. This v is called the Radon-Nicodym derivative of the set function $\mu_g(e)$ with respect to the Lebesgue measure. This derivative is denoted by $d\mu_g/dm$. It is relevant to note that

$$v(s) = \frac{d\mu_g}{dm}(s) = \lim_{\operatorname{mes} e \to 0} \frac{\mu_g(e)}{\operatorname{mes} e}, \tag{C.7}$$

where e is a segment from $[a,b]$ containing the point s. The condition (C.4) is necessary for the existence of summable v.

Theorem C.2 (Theorem C.1 bis, see [76]). *The operator S_g is continuously acting in the space* $\mathbf{L}$ *if and only if*

$$\operatorname*{ess\,sup}_{s \in [a,b]} v(s) = M < \infty. \tag{C.8}$$

As this takes place, $\|S_g\|_{\mathbf{L} \to \mathbf{L}} = M$.

Proof. To complete the proof of the Theorems C.1 and C.2, we will state the equality

$$\sup_{\substack{e \subset [a,b] \\ \operatorname{mes} e > 0}} \frac{\mu_g(e)}{\operatorname{mes} e} = \operatorname*{ess\,sup}_{s \in [a,b]} v(s). \tag{C.9}$$

Let (C.3) be fulfilled and assume that $\operatorname{ess\,sup}_{s \in [a,b]} v(s) > M$. Then there exists a set e of positive measure such that $v(s) > M$ at $s \in e$. For such a set

$$\frac{\mu_g(e)}{\operatorname{mes} e} = \frac{1}{\operatorname{mes} e} \int_e v(s)ds > M, \tag{C.10}$$

which yields a contradiction with (C.3). Thus

$$\operatorname*{ess\,sup}_{s \in [a,b]} v(s) \le \sup_e \frac{\mu_g(e)}{\operatorname{mes} e}. \tag{C.11}$$

Conversely, if (C.3) holds,

$$\frac{\mu_g(e)}{\operatorname{mes} e} = \frac{1}{\operatorname{mes} e}\int_e v(s)ds \le M \tag{C.12}$$

for each set $e \subset [a,b]$ of positive measure. Hence

$$\sup_e \frac{\mu_g(e)}{\operatorname{mes} e} \le \operatorname*{ess\,sup}_{s\in[a,b]} v(s). \tag{C.13}$$

Suppose that the operator S_g acts continuously in the space $\mathbf{L}$ and $\|S_g\|_{\mathbf{L}\to\mathbf{L}} = N$. Define a function $y : [a,b] \to \mathbb{R}^n$ by

$$y(s) = \begin{cases} c & \text{if } s \in e, \\ 0 & \text{if } s \in [a,b] \setminus e, \end{cases} \tag{C.14}$$

where $e \subset [a,b]$ is a set of positive measure, $c \in \mathbb{R}^n$ is a fixed nonzero vector. For such a function y we obtain from the inequality

$$\|S_g y\|_{\mathbf{L}} \le N\|y\|_{\mathbf{L}} \tag{C.15}$$

that $|c|\mu_g(e) \le N|c| \operatorname{mes} e$. Hence $M = \sup_e(\mu_g(e)/\operatorname{mes} e) \le N$.

A positive operator acting in the space $\mathbf{L}$ is continuous [120]. Therefore, it remains to show that under (C.8) the value of the operator S_g on each summable function is summable.

We have

$$\|S_g y\|_{\mathbf{L}} = \int_{g^{-1}([a,b])} |y[g(s)]|\, ds = \int_a^b |y(s)| \frac{d\mu_g}{dm}(s)ds \tag{C.16}$$

(for obtaining the latter equality, the formula of change of variables from [78] has been used). Thus

$$\|S_g y\|_{\mathbf{L}} \le \operatorname*{ess\,sup}_{s\in[a,b]} \frac{d\mu_g}{dm}(s)\|y\|_{\mathbf{L}}. \tag{C.17}$$

Hence $N = \|S_g\|_{\mathbf{L}\to\mathbf{L}} \le M$. □

The derivative $d\mu_g/dm$ can efficiently be calculated for a wide class of g. For this calculation, it is suitable to use the equality $(d\mu_g/dm)(s) = (d/ds)\mu_g([a,s])$, that holds a.e. on $[a,b]$ if (C.4) is fulfilled [206]. It should be remarked that for a strictly monotone $g : [a,b] \to \mathbb{R}^1$ we have

$$\frac{d}{ds}\mu_g([a,s]) = \begin{cases} \left|\dfrac{dg^{-1}}{ds}(s)\right| & \text{a.e. over } [a,b] \cap g([a,b]), \\ 0 & \text{at the other points of } [a,b]. \end{cases} \tag{C.18}$$

Example C.3. Let $g(t) = (1/2)t^2,\ t \in [0,1]$, then

$$\frac{d\mu_g}{dm}(s) = \frac{d}{ds}\mu_g([a,s]) = \begin{cases} \dfrac{1}{\sqrt{2s}} & \text{if } s \in \left(0, \dfrac{1}{2}\right], \\ 0 & \text{if } s \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{C.19}$$

In this case, (C.8) is not fulfilled and, consequently, S_g is not continuous in $\mathbf{L}$.

Example C.4. Let

$$g(t) = \begin{cases} \dfrac{1}{2}\sqrt{t} & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ t & \text{if } t \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{C.20}$$

Then

$$\frac{d\mu_g}{dm}(s) = \begin{cases} 8s & \text{if } s \in \left[0, \dfrac{1}{2\sqrt{2}}\right], \\ 0 & \text{if } s \in \left(\dfrac{1}{2\sqrt{2}}, \dfrac{1}{2}\right], \\ 1 & \text{if } s \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{C.21}$$

In this case, the operator S_g acts continuously in the space $\mathbf{L}$ and $\|S_g\|_{\mathbf{L}\to\mathbf{L}} = 2\sqrt{2}$.

Let, further, $g : [a,b] \to \mathbb{R}^1$ be piecewise strictly monotone and I_i, $i = 1,2,\dots,k$, be intervals of its monotonicity ($\cup_i I_i = [a,b]$, $I_i \cap I_j = \varnothing$ at $i \neq j$). Denote by g_i the constriction of g to I_i. In this case

$$\frac{d}{ds}\mu_g([a,s]) = \sum_{i=1}^{k} \frac{d}{ds}\mu_{g_i}([a,s]), \tag{C.22}$$

and, if the functions g_i^{-1} are absolutely continuous, we have

$$\frac{d\mu_g}{dm}(s) = \sum_{i=1}^{k} \frac{d}{ds}\mu_{g_i}([a,s]), \tag{C.23}$$

where

$$\frac{d}{ds}\mu_{g_i}([a,s]) = \begin{cases} \left|\dfrac{dg_i^{-1}}{ds}(s)\right| & \text{a.e. over } [a,b] \cap g_i(I_i), \\ 0 & \text{at the other points of } [a,b]. \end{cases} \tag{C.24}$$

Example C.5. Let

$$g(t) = \begin{cases} t^2 & \text{if } t \in \left[0, \dfrac{1}{2}\right], \\ 1 - t & \text{if } t \in \left(\dfrac{1}{2}, 1\right]. \end{cases} \tag{C.25}$$

We have

$$\frac{d}{ds}\mu_{g_1}([a, s]) = \begin{cases} \dfrac{1}{2\sqrt{s}} & \text{if } s \in \left(0, \dfrac{1}{4}\right], \\ 0 & \text{if } s \in \left(\dfrac{1}{4}, 1\right], \end{cases} \tag{C.26}$$

$$\frac{d}{ds}\mu_{g_2}([a, s]) = \begin{cases} 1 & \text{if } s \in \left[0, \dfrac{1}{2}\right], \\ 0 & \text{if } s \in \left(\dfrac{1}{2}, 1\right], \end{cases} \tag{C.27}$$

$$\frac{d\mu_g}{dm}(s) = \begin{cases} 1 + \dfrac{1}{2\sqrt{s}} & \text{if } s \in \left(0, \dfrac{1}{4}\right], \\ 1 & \text{if } s \in \left(\dfrac{1}{4}, \dfrac{1}{2}\right), \\ 0 & \text{if } s \in \left[\dfrac{1}{2}, 1\right]. \end{cases} \tag{C.28}$$

The principal part Q of the operator $\mathbf{L}$ in the case of the so called "neutral" equation with $\mathcal{L} : \mathbf{D} \to \mathbf{L}$, where $\mathbf{D}$ is the space of absolutely continuous functions, has the form $Q = I - S - K$, where I is the identity operator, $K : \mathbf{L} \to \mathbf{L}$ is compact, and $S : \mathbf{L} \to \mathbf{L}$ is defined by

$$(Sz)(t) = B(t)(S_g z)(t). \tag{C.29}$$

If the elements of $n \times n$ matrix B are measurable and essentially bounded on $[a, b]$ and $S_g : \mathbf{L} \to \mathbf{L}$ is continuous, the product $S = BS_g$ is also continuous. In some cases it is possible to establish the continuous action of S under some weaker assumptions.

Suppose $\int_{g^{-1}([a,b])} \|B(\tau)\| d\tau < \infty$, $e \subset [a, b]$ is a measurable set and

$$\mu_g^B(e) = \int_{g^{-1}(e)} ||B(\tau)|| d\tau. \tag{C.30}$$

If $g : [a, b] \to \mathbb{R}^1$ satisfies (C.4), $\operatorname{mes} e = 0 \Rightarrow \mu_g^b(e) = 0$ and, by virtue of the Radon-Nicodym theorem (see [78, page 194], there exists a summable function

$v^B : [a,b] \to \mathbb{R}^1$ such that

$$\mu_g^B(e) = \int_e v^B(s)ds. \tag{C.31}$$

The function $\mu_g^B(e)$ is called the Radon-Nicodym derivative of the set function μ_g^B with respect to the Lebesgue measure and is denoted by $d\mu_g^B/dm$. It should be noticed that (see [206])

$$\frac{d\mu_g^B}{dm}(s) = \frac{d}{dt}\mu_g^B([a,s]). \tag{C.32}$$

The relations

$$\int_a^b |(Sy)(s)|ds \leq \int_{g^{-1}([a,b])} ||B(s)|| \, |(S_g z)(s)|ds = \int_a^b |z(s)| \frac{d\mu_g^B}{dm}(s)ds \tag{C.33}$$

imply the following assertion.

Theorem C.6 (see [59, 72]). *Let*

$$\operatorname*{ess\,sup}_{s\in[a,b]} \frac{d\mu_g^B}{dm}(s) = M < \infty. \tag{C.34}$$

Then the operator S acts continuously in the space $\mathbf{L}$, *and besides* $\|S\|_{\mathbf{L}\to\mathbf{L}} = M$.

Example C.7. Let $(Sy)(t) = (1/\sqrt{t})y(\sqrt{t}),\ t \in [0,1]$. In this case,

$$\frac{d\mu_g^B}{dm}(s) = \frac{d}{ds}\int_{g^{-1}([0,s])} \frac{1}{\sqrt{\tau}}d\tau = \frac{d}{ds}\int_0^{s^2} \frac{1}{\sqrt{\tau}}d\tau = 2 \tag{C.35}$$

and, consequently, the operator S acts continuously in the space $\mathbf{L}$ and besides $\|S\|_{\mathbf{L}\to\mathbf{L}} = 2$.

Example C.8 (see [59]). Let $(Sy)(t) = ty(t^2),\ t \in [0,1]$. In this case,

$$\frac{d\mu_g^B}{dm}(s) = \frac{d}{ds}\int_{g^{-1}([0,s])} \tau\, d\tau = \frac{d}{ds}\int_0^{\sqrt{s}} \tau\, d\tau = \frac{1}{2}. \tag{C.36}$$

Thus $\|S\|_{\mathbf{L}\to\mathbf{L}} = 1/2$.

Any continuous linear operator has a general differential integral representation (see [109]) according to which we have

$$(Sy)(t) = \frac{d}{dt}\int_a^b Q(t,s)y(s)ds. \tag{C.37}$$

Bykadorov [59] has shown that

$$\begin{aligned} Q(t,s) &= \frac{d}{ds}\int_a^t B(\tau)\Theta_g(s,\tau)d\tau, \\ \frac{d\mu_g^B}{dm}(s) &= \frac{d}{ds}\int_a^b ||B(\tau)||\Theta_g(s,\tau)d\tau, \end{aligned} \tag{C.38}$$

where $\Theta_g(s,\tau)$ is the characteristic function of the set

$$\{(s,\tau)\in[a,b]\times[a,b] : a \le g(\tau) \le s \le b\}. \tag{C.39}$$

The following assertion explains some difficulties arising in studying the neutral equations.

Theorem C.9 (see [76]). *The operator $S_g : \mathbf{L}\to\mathbf{L}$ is not compact if it differs from the zero operator.*

Proof. Let $e = g([a,b])\cap[a,b]$ and $M = \operatorname{mes} e$. $M \neq 0$ if S_g is not the zero operator. Divide the set e into subsets e_1 and e_2 in such a way that $e_1\cup e_2 = e$, $e_1\cap e_2 = \varnothing$, and $\mu_{g_1}(e_1) = \mu_{g_2}(e_2) = (1/2)M$. It is possible since $\mu(t) = \mu_g([a,t]) = \int_a^t \nu(s)ds$ is continuous, $\mu(a) = 0$, $\mu(b) = M$, and we take, for instance, $e_1 = [a,t_1]\cap g([a,b])$ if $\mu(t_1) = (1/2)M$, $e_2 = e\setminus e_1$. The sets e_i, $i = 1,2$, will be divided into disjoint e_{i1} and e_{i2} such that $e_{i1}\cup e_{i2} = e_i$ and $\mu_g(e_{i1}) = \mu_g(e_{i2}) = (1/2^2)M$. Continuing the subdivision process, we construct the sets $e_{\alpha_1,\ldots,\alpha_k}$, $k = 1,2,\ldots$, where the index α_k has values 1 or 2, so that

$$\begin{gathered} e_{\alpha_1,\ldots,\alpha_{k-1},1}\cup e_{\alpha_1,\ldots,\alpha_{k-1},2} = e_{\alpha_1,\ldots,\alpha_{k-1}}, \\ e_{\alpha_1,\ldots,\alpha_{k-1},1}\cap e_{\alpha_1,\ldots,\alpha_{k-1},2} = \varnothing, \\ \mu_g\left(e_{\alpha_1,\ldots,\alpha_k}\right) = \frac{1}{2^k}M. \end{gathered} \tag{C.40}$$

Now the sequence of the sets Φ_k will be constructed as follows: $\Phi_1 = e_1$, $\Phi_2 = e_{11}\cup e_{21}$, $\Phi_3 = e_{111}\cup e_{121}\cup e_{211}\cup e_{221}$, and so on. Thus $\mu_g(\Phi_k) = (1/2)M$ and $\operatorname{mes}\{g^{-1}(\Phi_k)\cap g^{-1}(\Phi_l)\} = (1/4)M$ for each k and l, $k\neq l$. Let $Y = \{y\in\mathbf{L} : \|y\|_{\mathbf{L}} \le r\}$ be an arbitrary ball and let $\{y_k\}\subset Y$ be the sequence where $y_k(t) = c\chi_k(t)$, $\chi_k(t)$ is the characteristic function of the set Φ_k and $c\in\mathbb{R}^n$ $(c\neq 0)$ is chosen so that $y_k\in Y$. Let us demonstrate that the sequence $\{S_g y_k\}$ does not contain any subsequence converging by measure (and all the more by the norm of $\mathbf{L}$). Indeed, for

$$t\in\{g^{-1}(\Phi_k)\setminus g^{-1}(\Phi_l)\}\cup\{g^{-1}(\Phi_l)\setminus g^{-1}(\Phi_k)\} \tag{C.41}$$

we have $|(S_g y_k)(t) - (S_g y_l)(t)| = |c|$ for any k and l, $k \neq l$. Thus

$$\operatorname{mes}\{t \in [a,b] : |(S_g y_k)(t) - (S_g y_l)(t)| = |c|\} = \frac{1}{2}M. \tag{C.42}$$

□

C.2. Conditions for the strong convergence of a sequence of composition operators

In this subsection, we consider the question on the strong convergence of a sequence of composition operators $S_{h^k} : \mathbf{D} \to \mathbf{L}$ ($S_{g^k} : \mathbf{L} \to \mathbf{L}$) to an operator $S_{h^0} : \mathbf{D} \to \mathbf{L}$ ($S_{g^0} : \mathbf{L} \to \mathbf{L}$). In what follows, we assume these operators to be continuous.

Denote

$$\begin{aligned} y_k(t) &= (S_{h^k}x)(t), \\ e_h &= \{t \in [a,b] : h^0(t) \in [a,b]\}, \\ e_k &= \{t \in [a,b] : \chi_k(t) \neq \chi_0(t)\}, \end{aligned} \tag{C.43}$$

where χ_k is the characteristic function of the set

$$\{t \in [a,b] : h^k(t) \in [a,b]\}, \quad k = 0,1,\ldots. \tag{C.44}$$

Thus for $t \in e_k$ we have either $y_k(t) = x[h^k(t)]$, $y_0(t) = 0$, or $y_k(t) = 0$, $y_0(t) = x[h^0(t)]$.

Theorem C.10. *The sequence of operators $S_{h^k} : \mathbf{D} \to \mathbf{L}$ converges strongly (i.e., at every point $x \in \mathbf{D}$) to the operator S_{h^0} if and only if the following conditions are fulfilled:*

(a) *the sequence $\{h^k(t)\}$ converges to $h^0(t)$ by measure on the set e_h;*

(b) $\lim_{k\to\infty} \operatorname{mes} e_k = 0$.

Proof

Sufficiency. Show that $y_k(t) \xrightarrow{\text{mes}} y_o(t)$, $t \in e_h$. Let $\varepsilon > 0$ be fixed, and let $\delta > 0$ be chosen so that $|x(\xi_1) - x(\xi_2)| < \varepsilon$ as $|\xi_1 - \xi_2| < \delta$. Next let $e'_k = \{t \in e_h \setminus e_k : |h^k(t) - h^0(t)| \geq \delta\}$. Since $|h^k(t) - h^0(t)| < \delta$ as $t \in e_h \setminus (e_h \cup e'_k)$, we have $|y_k(t) - y_0(t)| = |x[h^k(t)] - x[h^0(t)]| < \varepsilon$ for the same t. Now $\{t \in [a,b] : |y_k(t) - y_0(t)| \geq \varepsilon\} \subset e_k \cup e'_k$ and $\operatorname{mes}(e_k \cup e'_k) \to 0$ as $k \to \infty$ imply the convergence $y_k(t) \to y_0(t)$ by measure.

Let

$$\begin{aligned} e_k^1 &= \{t \in e_k : h^k(t) \in [a,b]\}, \\ e_k^2 &= \{t \in e_k : h^0(t) \in [a,b]\}, \quad k = 1,2,\ldots. \end{aligned} \tag{C.45}$$

Then,

$$\|y_k - y_0\|_{\mathbf{L}} = \int_{e_h \setminus e_k} |y_k(s) - y_0(s)|\,ds + \int_{e_k^1} |y_k(s)|\,ds + \int_{e_k^2} |y_0(s)|\,ds. \tag{C.46}$$

The first of the integrals tends to zero as $k \to \infty$ by virtue of the Lebesgue theorem, the second as well as the third tends to zero, being estimated by the value $\max_{t\in[a,b]} |x(t)| \operatorname{mes} e_k$.

Necessity. By setting $x(t) = \operatorname{col}\{t, 0, \dots, 0\}$, we obtain

$$|y_k^1(t) - y_0^1(t)| = \begin{cases} |h^k(t) - h^0(t)| & \text{if } t \in e_h \setminus e_k, \\ |h^k(t)| & \text{if } t \in e_k^1, \\ |h^0(t)| & \text{if } t \in e_k^2. \end{cases} \tag{C.47}$$

Therefore, $h^k(t) \xrightarrow{\text{mes}} h^0(t)$ on e_h and $\operatorname{mes} e_k \to 0$. □

Consider now the sequence of operators $T_k : \mathbf{D} \to \mathbf{L}$ defined by

$$(T_k x)(t) = P^k(t)(S_{h^k} x)(t) \tag{C.48}$$

measurable on $[a, b]$ elements of $n \times n$ matrices P^k.

Theorem C.11. *Let the following conditions be fulfilled:*

(a) $\|P^k(t) - P^0(t)\| \to 0$ *by measure on* e_h *as* $k \to \infty$;

(b) *there exists a summable function* $\rho : [a, b] \to \mathbb{R}^1$ *such that*

$$\|P^k(t)\| \le \rho(t), \quad t \in [a, b],\ k = 0, 1, \dots; \tag{C.49}$$

(c) $h^k(t) \to h^0(t)$ *by measure on* e_h *as* $k \to \infty$;

(d) $\lim_{k\to\infty} \operatorname{mes} e_k = 0$.

Then $\lim_{k\to\infty} \|T_k x - T_0 x\|_{\mathbf{L}} = 0$ *for any* $x \in \mathbf{D}$.

Proof. The following inequality is valid:

$$\begin{aligned} \|T_k x - T_0 x\|_{\mathbf{L}} \le \int_a^b &\|P^k(s) - P^0(s)\|\,|y_k(s)|\,ds \\ &+ \int_a^b \|P^0(s)\|\,|y_k(s) - y_0(s)|\,ds = \mathcal{I}_1^k + \mathcal{I}_2^k. \end{aligned} \tag{C.50}$$

Consider each of the integrals individually:

$$\mathcal{I}_1^k = \int_{e_h \setminus e_k} ||P^k(s) - P^0(s)|| \cdot |y_k(s)| ds + \int_{e_k^1} ||P^k(s) - P^0(s)|| |y_k(s)| ds \\ + \int_{e_2^k} ||P^k(s) - P^0(s)|| |y_0(s)| ds. \tag{C.51}$$

The first term in the right-hand side of the latter equality tends to zero as $k \to \infty$ by virtue of the Lebesgue theorem. The second and the third are estimated by

$$2 \max_{t \in [a,b]} |x(t)| \int_{e_k} \rho(s) ds. \tag{C.52}$$

In the same way with the equality

$$\mathcal{I}_2^k = \int_{e_h \setminus e_k} ||P^0(s)|| |y_k(s) - y_0(s)| ds + \int_{e_k^1} ||P^0(s)|| |y_k(s)| ds + \int_{e_k^2} ||P^0(s)|| |y_0(s)| ds, \tag{C.53}$$

one can establish that $\lim_{k\to\infty} \mathcal{I}_2^k = 0$. □

Remark C.12. The conditions (c) and (d) of Theorem C.11 can be replaced by the weaker ones:

(c′) $h^k(t) \to h^0(t)$ by measure on $\{t \in e_h : \|P^0(t)\| \neq 0\}$;

(d′) $\lim_{k\to\infty} \operatorname{mes}\{t \in e_h : \|P^0(t)\| \neq 0\} = 0$.

Conditions for the strong convergence of a sequence $\{S_{g^k}\}$, $S_{g^k} : \mathbf{L} \to \mathbf{L}$, have been established by M. E. Drakhlin and T. K. Plyshevskaya. Here we formulate the corresponding results without proof. Denote

$$\begin{aligned} \mathcal{E}_g &= \{t \in [a,b] : g^0(t) \in [a,b]\}, \\ \mathcal{E}_k &= \{t \in [a,b] : \chi_k(t) \neq \chi_0(t)\}, \end{aligned} \tag{C.54}$$

where χ_k is the characteristic function of the set

$$\{t \in [a,b] : g^k(t) \in [a,b]\}, \quad k = 0, 1, \ldots. \tag{C.55}$$

Theorem C.13 (see [76]). *The sequence of operators $S_{g^k} : \mathbf{L} \to \mathbf{L}$ converges to S_{g^0} at each point of the space* $\mathbf{L}$ *if and only if the following conditions are fulfilled:*

(a) *$g^k(t) \to g^0(t)$ by measure on $\mathbf{E}_g$;*

(b) $\lim_{k\to\infty} \operatorname{mes} \mathcal{E}_k = 0$;

(c) *the norms of the operators S_{g^k}, $k = 1, 2, \ldots$, are uniformly bounded.*

Remark C.14. In [77], it is shown that the strong convergence of the operators $S_{g^k} : \mathbf{L} \to \mathbf{L}$ to $S_{g^0} : \mathbf{L} \to \mathbf{L}$ is equivalent to the weak convergence of these operators to $S_{g^0} : \mathbf{L} \to \mathbf{L}$.

Consider now the sequence of operators $S_k : \mathbf{L} \to \mathbf{L}$ defined by

$$(S_k y)(t) = B^k(t)(S_{g^k} y)(t), \tag{C.56}$$

where the elements of $n \times n$ matrices B^k are measurable and essentially bounded on $[a,b]$. It is not difficult to show that the sequence $\{S_k\}$ converges to S_0 at every point $y \in \mathbf{L}$ under the following conditions: $\lim_{k\to\infty} \|S_{g^k} y - S_{g^0} y\|_{\mathbf{L}} = 0$ for every $y \in \mathbf{L}$; $\|B^k(t) - B^0(t)\| \to 0$ by measure on $\mathcal{E}_g$ as $k \to \infty$ with the addition that there exists an essentially bounded function $\rho : [a,b] \to \mathbb{R}^1$ such that $\|B^k(t)\| \leq \rho(t),\ t \in [a,b]$, for all k.

The following theorem gives a more subtle test for the strong convergence of $\{S_k\}$.

Denote

$$\mu_{g^k}^{B^k}(e) = \int_{(g^k)^{-1}(e)} ||B^k(s)||ds, \tag{C.57}$$

where $\nu_k(s) = (d\mu_{g^k}^{B^k}/dm)(s)$ is the Radon-Nicodym derivative of function $\mu_{g^k}^{B^k}$ with respect to the Lebesgue measure (see [78]).

Theorem C.15 (see [73]). *Let the following conditions be fulfilled:*

(a) $\|B^k(t) - B^0(t)\| \to 0$ *by measure on* $\mathcal{E}_g$ *as* $k \to \infty$*;*

(b) $\sup_k \operatorname{ess\,sup}_{s\in[a,b]} \nu_k(s) < \infty$*;*

(c) $g^k(t) \to g^0(t)$ *by measure on* $\{t \in \mathcal{E}_g : \|B^0(t)\| \neq 0\}$*;*

(d) $\lim_{k\to\infty} \operatorname{mes}\{t \in \mathcal{E}_k : \|B^0(t)\| \neq 0\} = 0$.

Then $\lim_{k\to\infty} \|S_k y - S_0 y\|_{\mathbf{L}} = 0$ *for any* $y \in \mathbf{L}$.

D. Vallee-Poussin-like theorem

An exclusive place in the theory of differential equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + q(t)\dot{x}(t) + p(t)x(t) = 0 \tag{D.1}$$

is occupied by the interval $[a,b]$ on which any nontrivial solution has at most one zero. Such an interval is called *the nonoscillation interval* (of the solutions of $\mathcal{L}x = 0$).

From the Sturm theorem on separation of zeros of solutions to $\mathcal{L}x = 0$, it follows that $[a,b]$ is the nonoscillation interval if and only if there exists a positive solution of $\mathcal{L}x = 0$ on $[a,b]$.

Vallee-Poussin criterion [69]. An interval $[a,b]$ *is the nonoscillation interval for* $\mathcal{L}x = 0$ *if and only if there exists a function* $v : [a,b] \to \mathbb{R}^1$ *with absolutely continuous derivative* $\dot{v}$ *such that*

$$v(t) \geq 0, \quad (\mathcal{L}v)(t) \leq 0 \quad \text{for } t \in [a,b], \qquad v(a) + v(b) - \int_a^b (\mathcal{L}v)(s)ds > 0. \tag{D.2}$$

A proper choice of the comparison function v permits getting on the base of the Vallee-Poussin criterion the estimates of the length of the nonoscillation interval. It should be noted that the Vallee-Poussin criterion was known before the mentioned estimates.

The interest to the nonoscillation interval may be explained by a link between the existence of a positive solution and many actual problems. Such a link is established by the next theorem.

Theorem D.1. *The following assertions are equivalent.*

(a) $[a,b]$ *is the nonoscillation interval.*

(b) *There exists a* $v:[a,b]\to\mathbb{R}^1$ *with absolutely continuous derivative* $\dot v$ *such that*

$$v(t)\ge 0,\quad (\mathcal{L}v)(t)\le 0,\quad t\in[a,b],\qquad v(a)+v(b)-\int_a^b(\mathcal{L}v)(s)ds>0. \tag{D.3}$$

(c) *The Cauchy function* $C(t,s)$ *of the equation* $\mathcal{L}x=f$ *is strictly positive in the triangle* $a\le s<t\le b$.

(d) *The two-point boundary value problem*

$$\mathcal{L}x=f,\qquad x(a)=0,\qquad x(b)=0 \tag{D.4}$$

is uniquely solvable for each summable f, *and the Green function of this problem is strictly negative in the square* $(a,b)\times(a,b)$.

It should be noticed that due to well-known results of Jacobi, the list of the equivalent assertions of Theorem D.1 may be added by the following assertion.

(e) *There exists a unique function* $v:[a,b]\to\mathbb{R}^1$ *with absolutely continuous derivative* $\dot v$ *on which the functional*

$$\int_a^b\exp\int_a^s q(\tau)d\tau\left[\dot x^2(s)-p(s)x^2(s)\right]ds \tag{D.5}$$

with the conditions $x(a)=0$, $x(b)=0$ *reaches its minimum.*

The equivalence of (a) and (c) had been established by Wilkins [223].

The equivalence of (a) and (d) had been established by Pack [168].

It is natural to name Theorem D.1 after Vallee-Poussin. A wide series of investigations by various authors has been devoted to the extension of this theorem upon equations of higher order and delay differential equations. Below we offer a general assertion on the base of which it is possible to prove some variants of Vallee-Poussin-like theorems for a wide class of functional differential equations.

Let **B** be a Banach space of measurable $z:[a,b]\to\mathbb{R}^n$, let **C** be a Banach space of continuous $x:[a,b]\to\mathbb{R}^n$, let **D** be a Banach space of $x:[a,b]\to\mathbb{R}^n$

isomorphic to the direct product $\mathbf{B} \times \mathbb{R}^n$, and, besides, let the elements of $\mathbf{D}$ be continuous on $[a, b]$.

Consider the boundary value problem

$$\mathcal{L}x = f, \qquad lx = \alpha \tag{D.6}$$

under the following assumptions.

The linear $\mathcal{L} : \mathbf{D} \to \mathbf{B}$ is bounded and Noether with $\operatorname{ind} \mathbf{L} = n$. The components $l^1, \dots, l^n$ of the vector functional $l = [l^1, \dots, l^n]$ are linear bounded functionals on the space $\mathbf{D}$. If there are functionals among l^i such that $l^i x = x(v_i)$, $v_i \in [a, b]$, the set of all such points v_i is denoted by $\{v\}$; otherwise the symbol $\{v\}$ indicates the empty set.

Further we assume that the decomposition $\mathcal{L} = \mathcal{L}_0 - T$ holds, where $T : \mathbf{C} \to \mathbf{B}$ is bounded and isotonic (antitonic) and the linear $\mathcal{L}_0 : \mathbf{D} \to \mathbf{B}$ possesses the following properties.

(1) The boundary value problem

$$\mathcal{L}_0 x = f, \qquad lx = \alpha \tag{D.7}$$

has a unique solution $x \in \mathbf{D}$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$ and besides the Green operator W of this problem is isotonic (antitonic).

(2) There exists $\psi \in \mathbf{B}$ such that $\psi(t) \geq 0$ ($\psi(t) \leq 0$) a.e. on $[a, b]$ and $(W\psi)(t) > 0$ for $t \in [a, b] \setminus \{v\}$.

(3) There exists a positive solution u_0 ($u_0(t) > 0$ for $t \in [a, b] \setminus \{v\}$) of the homogeneous equation $\mathcal{L}_0 x = 0$.

As for these properties, it should be noticed that in some cases the properties (1) and (3) are equivalent. For instance, in the case of the problem

$$\ddot{x}(t) + p(t)x(t) = f(t), \qquad x(a) = \alpha^1, \qquad x(b) = \alpha^2, \tag{D.8}$$

such equivalence follows from Theorem D.1. In the general case, such a connection may be absent as it is demonstrated by the example of the problem

$$\ddot{x}(t) + x(t) = f(t), \qquad x\left(\frac{b-a}{2}\right) = \alpha^1, \qquad \dot{x}\left(\frac{b-a}{2}\right) = \alpha^2 \tag{D.9}$$

if $\pi < b - a < 2\pi$. Indeed, the solution $u(t) = c_1 \cos t + c_2 \sin t$ is not positive on $[a, b]$ if $b - a > \pi$. But the Green operator is isotonic because the Green function $W(t, s)$ is defined by

$$W(t,s) = \begin{cases} \sin(t-s) & \text{if } \dfrac{b-a}{2} \leq s \leq t \leq b, \\ -\sin(t-s) & \text{if } a \leq t < s \leq \dfrac{b-a}{2}, \\ 0 & \text{at the other points of the square } [a,b] \times [a,b]. \end{cases} \tag{D.10}$$

The condition (2) is fulfilled, for instance, if the Green operator W is acting from the space $\mathbf{W}^n$ of $(n-1)$ times differentiable functions with absolutely continuous derivative of the order $n-1$ into the space $\mathbf{L}$ of summable functions. The Green operator $W : \mathbf{L} \to \mathbf{W}^n$ is integral (Section 2.3):

$$(Wf)(t) = \int_a^b W(t,s)f(s)ds. \tag{D.11}$$

As ψ we may take any $\psi \in \mathbf{L}$ such that $\psi(t) > 0$ ($\psi(t) < 0$) a.e. on $[a,b]$. Indeed, for such ψ the equality $(W\psi)(\tau) = 0$ is possible only for $\tau \in \{v\}$.

It is relevant to note that condition (2) is fulfilled for not all the integral Green operators: if $W : \mathbf{L} \to \mathbf{D}$, $\mathbf{D} = \{x \in \mathbf{W}^n : x(\xi) = 0,\ \xi \in [a,b]\}$, the equality $(Wf)(\xi) = 0$ holds for any $f \in \mathbf{L}$.

Denote $A = WT$. The problem (D.6) is equivalent to the equation

$$x = Ax + y, \tag{D.12}$$

where y is the solution of the model problem $\mathcal{L}_0 x = f$, $lx = \alpha$. The equation may be considered in the space $\mathbf{C}$ since the values of A on continuous functions belong to $\mathbf{D}$. Denote by $\rho(A)$ the spectral radius of $A : \mathbf{C} \to \mathbf{C}$.

Theorem D.2. *The following assertions are equivalent.*

(a) *There exists $v \in \mathbf{D}$ such that*

$$v(t) > 0, \quad r(t) \stackrel{\text{def}}{=} (W\varphi)(t) + g(t) > 0, \quad t \in [a,b] \setminus \{v\}, \tag{D.13}$$

where $\varphi = \mathcal{L}v$, g is the solution of the semihomogeneous problem

$$\mathcal{L}_0 x = 0, \qquad lx = lv. \tag{D.14}$$

(b) $\rho(A) < 1$.

(c) *The problem* (D.6) *has a unique solution $x \in \mathbf{D}$ for each $\{f, \alpha\} \in \mathbf{B} \times \mathbb{R}^n$ and, besides, the Green operator G of this problem is isotonic (antitonic).*

(d) *The homogeneous equation $\mathcal{L}x = 0$ has a positive solution u ($u(t) > 0$ for $t \in [a,b] \setminus \{v\}$) such that $lu = lu_0$.*

Proof. Let (a) hold. The function v as a solution of $\mathcal{L}x = \varphi$ satisfies (D.12) for $y = W\varphi + g = r$. Therefore, $v - Av = r$. If $\{v\}$ is empty, the implication (a)⇒(b) follows from Lemma A.1, if $\{v\}$ is not empty, the implication follows from Theorem A.3.

Since $\rho(A) < 1$, the implication (b)⇒(c) follows from the fact that the solution x of (D.12) exists and may be presented in the form

$$x = y + Ay + A^2 y + \cdots. \tag{D.15}$$

Hence

$$G = (I + A + A^2 + \cdots)W. \tag{D.16}$$

To prove the implication (c)⇒(a) let us use $\psi \in \mathbf{B}$ such that $\psi(t) \geq 0$ ($\psi(t) \leq 0$), $(W\psi)(t) > 0$ for $t \in [a,b] \setminus \{v\}$. Since G is isotonic (antitonic), the function $v = G\psi$ does not take negative values. Therefore, $v(t) = (Av)(t) + (W\psi)(t) > 0$ and $r(t) = (W\psi)(t) > 0,\ t \in [a,b] \setminus \{v\}$.

The implication (b)⇒(d) follows from the fact that the estimate $\rho(A) < 1$ guarantees the unique solvability of the problem (D.6) and the representation in the form

$$u = u_0 + Au_0 + A^2u_0 + \cdots \tag{D.17}$$

of the solution u to the semihomogeneous problem $\mathcal{L}x = 0$, $lx = lu_0$. Therefore, $u(t) \geq u_0(t),\ t \in [a,b]$.

Taking $v = u$, we get the implication (d)⇒(a). □

Remark D.3. If W and G are integral operators, the list of the equivalent assertions of Theorem C.10 may be added by the one on the fact that $G(t,s) \geq W(t,s)$ ($G(t,s) \leq W(t,s)$), where $W(t,s)$ and $G(t,s)$ are the kernels of the integral operators.

Remark D.4. In the case of the second-order differential equation the assertion (d) on the existence of a positive solution is equivalent to the assertion of nonoscillation.

Remark D.5. The proof of the equivalence of the assertions (a), (b), and (c) does not use the property (3).

Turning back to the equation

$$(\mathcal{L}x)(t) \stackrel{\text{def}}{=} \ddot{x}(t) + q(t)\dot{x}(t) + p(t)x(t) = f(t), \tag{D.18}$$

we will prove Theorem D.1 assuming the coefficients q and p to be summable.

Proof of Theorem D.1. First consider an auxiliary lemma using the specific character of ordinary differential equations.

Lemma D.6. *The following assertions are equivalent.*

(a) *$[a,b]$ is the nonoscillation interval for $\mathcal{L}x = 0$.*

(b) *The Cauchy function $C(t,s)$ of the equation $\mathcal{L}x = f$ is strictly positive in the triangle $a \leq s < t \leq b$.*

(c) *The two-point boundary value problem*

$$\mathcal{L}x = f, \qquad x(a) = x(b) = 0 \tag{D.19}$$

is uniquely solvable and the Green function $G(t,s)$ of the problem is strictly negative in the square $(a,b) \times (a,b)$.

Proof.

$$C(t,s) = \frac{1}{W(s)} \begin{vmatrix} u_1(s) & u_2(s) \\ u_1(t) & u_2(t) \end{vmatrix}, \quad a \le s \le t \le b, \tag{D.20}$$

where u_1, u_2 constitute the fundamental system for $\mathcal{L}x = 0$, W is the Wronskian of the system. From this representation of $C(t,s)$, it is clear that the section $C_s(t) = C(t,s)$ for each fixed $s \in [a,b)$ is a solution of $\mathcal{L}x = 0$ on $[s,b]$. Now (a)⇒(b) follows from the conditions $C_s(s) = 0$, $(d/ds)C_s(s) = 1$ which guarantee $C_s(t) > 0$ on $(s,b]$. The implication (b)⇒(a) follows from Sturm's theorem on the separation of zeros and the inequality $C_a(t) > 0$, $t \in (a,b]$.

Theorem 2.4 yields the following properties of Green function $G(t,s)$. For each fixed $s \in (a,b)$ the section $g_s(t) = G(t,s)$ of the Green function is continuous on $[a,b]$, on each interval $[a,s)$ or $(s,b]$ the section g_s satisfies the homogeneous equation $\mathcal{L}x = 0$ and the boundary conditions $g_s(a) = g_s(b) = 0$. Besides,

$$\dot{g}_s(s+0) - \dot{g}_s(s-0) = 1. \tag{D.21}$$

Let (a) be fulfilled, let $s \in (a,b)$ be fixed, let v_s be the solution of the problem

$$\mathcal{L}x = 0, \qquad x(s) = 0, \qquad \dot{x}(s) = 1, \tag{D.22}$$

and let u_s be the solution of the two-point problem

$$\mathcal{L}x = 0, \qquad x(a) = 0, \qquad x(b) = v_s(b). \tag{D.23}$$

The latter problem is uniquely solvable since $v_s(t) > 0$. The section g_s may be represented as

$$g_s(t) = \chi_{[s,b]}(t)v_s(t) - u_s(t), \tag{D.24}$$

where $\chi_{[s,b]}$ is the characteristic function of the interval $[s,b]$. The section g_s has zeros only at the points a and b by virtue of the nonoscillation. Thus $g_s(t) < 0$, $t \in (a,b)$, whence (c) follows.

Let (c) be fulfilled. The solution y of the problem

$$\mathcal{L}x = 0, \qquad x(a) = 0, \qquad \dot{x}(a) = 1 \tag{D.25}$$

is proportional to the function g_s on the interval $[a,s]$. The assumption that $y(b) = 0$ leads to contradiction: under such an assumption y is a nontrivial solution of the homogeneous problem

$$\mathcal{L}x = 0, \qquad x(a) = x(b) = 0. \tag{D.26}$$

Thus $y(t) < 0$ on $(a, b]$. Hence (a) follows by Sturm's theorem. Thus Lemma D.6 is proved. □

Next let us show that the conditions of Theorem D.2 are fulfilled for the equation $\mathcal{L}x = f$.

Let $p = p^+ - p^-$, $p^+(t), p^-(t) \geq 0$, $\mathcal{L}_0 x = \ddot{x} + q\dot{x} - p^- x$. Then

$$\mathcal{L}x = \mathcal{L}_0 x + p^+ x. \tag{D.27}$$

The equation $Mx \overset{\text{def}}{=} \ddot{x} + q\dot{x} = 0$ has the solution $x(t) \equiv 1$. Therefore, by the Sturm theorem, $[a, b]$ is an nonoscillation interval for the equation. By Lemma D.6, the Cauchy function $C_M(t, s)$ for the equation $Mx = f$ is positive. The Cauchy problem

$$\mathcal{L}_0 x \overset{\text{def}}{=} \ddot{x} + q\dot{x} - p^- x = 0, \qquad x(a) = 1, \qquad \dot{x}(a) = 0 \tag{D.28}$$

is equivalent to the equation $x = Kx + \mathbf{1}$ with isotonic Volterra operator

$$(Kx)(t) = \int_a^t C_M(t, s) p^-(s) x(s) ds. \tag{D.29}$$

Thus the solution u_0 of the latter equation is positive:

$$u_0(t) = \mathbf{1} + \big(K[\mathbf{1}]\big)(t) + \big(K^2[\mathbf{1}]\big)(t) + \cdots \geq 1, \tag{D.30}$$

and so $[a, b]$ is the nonoscillation interval for the model equation $\mathcal{L}_0 x = 0$. It follows, by Lemma D.6, that the model problem

$$\mathcal{L}_0 x = f, \qquad x(a) = x(b) = 0 \tag{D.31}$$

is uniquely solvable and the Green function $W(t, s)$ of the problem is strictly negative in the square $(a, b) \times (a, b)$. Besides there exists a solution of the equation $\mathcal{L}_0 x = 0$ such that $u_0(t) > 0$ for $t \in (a, b)$, $u_0(a) + u_0(b) > 0$. □

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