



Impulsive Differential Equations and Inclusions

M. BENCHOHRA, J. HENDERSON, and S. NTOUYAS

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Dedication

We dedicate this book to our family members who complete us. In particular, M. Benchohra's dedication is to his wife, Kheira, and his children, Mohamed, Maroua, and Abdelillah; J. Henderson dedicates to his wife, Darlene, and his descendants, Kathy, Dirk, Katie, David, and Jana Beth; and S. Ntouyas makes his dedication to his wife, Ioanna, and his daughter, Myrto.

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Preface

Since the late 1990s, the authors have produced an extensive portfolio of results on differential equations and differential inclusions undergoing impulse effects. Both initial value problems and boundary value problems have been dealt with in their work. The primary motivation for this book is in gathering under one cover an encyclopedic resource for many of these recent results. Having succinctly stated the motivation of the book, there is certainly an obligation to include mentioning some of the all important roles of modelling natural phenomena with impulse problems.

The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of “impulses.” Impulsive differential equations such as

$$x' = f(t, x), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (1)$$

subject to impulse effects

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), \quad k = 1, \dots, m, \quad (2)$$

with $f : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and I_k an impulse operator, have been developed in modelling impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth; in the case when the right-hand side of (1) has discontinuities, differential inclusions such as

$$x'(t) \in F(t, x(t)), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (3)$$

subject to the impulse conditions (2), where $F : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, have played an important role in modelling phenomena, especially in scenarios involving automatic control systems. In addition, when these processes involve hereditary phenomena such as biological and social macrosystems, some of the modelling is done via impulsive functional differential equations such as

$$x' = f(t, x_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (4)$$

subject to (2), and an initial value

$$x(s) = \phi(s), \quad s \in [-r, 0], \quad t \in [0, b], \quad (5)$$

where $x_t(\theta) = x(t + \theta)$, $t \in [0, b]$, and $-r \leq \theta \leq 0$, and $f : ([0, b] \setminus \{t_1, \dots, t_m\}) \times D \rightarrow \mathbb{R}^n$, and D is a space of functions from $[-r, 0]$ into \mathbb{R}^n which are continuous except for a finite number of points. When the dynamics is multivalued, the hereditary phenomena are modelled via impulsive functional differential inclusions such as

$$x'(t) \in F(t, x_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (6)$$

subject to the impulses (2) and the initial condition (5).

An outline of the book as it is devoted to articles published by the authors evolves in a somewhat natural way around addressing issues relating to initial value problems and boundary value problems for both impulsive differential equations and differential inclusions, as well as for both impulsive functional differential equations and functional differential inclusions. Chapter 1 contains fundamental results from multivalued analysis and differential inclusions. In addition, this chapter contains a number of fixed point theorems on which most of the book's existence results depend. Included among these fixed point theorems are those recognized their names: Avery-Henderson, Bohnenblust-Karlin, Covitz and Nadler, Krasnosel'skii, Leggett-Williams, Leray-Schauder, Martelli, and Schaefer. Chapter 1 also contains background material on semigroups that is necessary for the book's presentation of impulsive semilinear functional differential equations.

Chapter 2 is devoted to impulsive ordinary differential equations and scalar differential inclusions, given, respectively, by

$$y' - Ay = By + f(t, y), \quad y' \in F(t, y), \quad (7)$$

each subject to (2), and each satisfies an initial condition $y(0) = y_0$, where A is an infinitesimal generator of a family of semigroups, B is a bounded linear operator from a Banach space E back to itself, and $F : [0, b] \times E \rightarrow 2^E$. Chapter 3 deals with functional differential equations and functional differential inclusions, with each undergoing impulse effects. Also, neutral functional differential equations and neutral functional differential inclusions are addressed in which the derivative of the state variable undergoes a delay. Chapter 4 is directed toward impulsive semilinear ordinary differential inclusions and functional differential inclusions satisfying nonlocal boundary conditions such as $g(y) = \sum_{k=1}^n c_k y(t_k)$, with each $c_k > 0$ and $0 < t_1 < \dots < t_n < b$. Such problems are used to describe the diffusion phenomena of a small amount of gas in a transport tube.

Chapter 5 is focused on positive solutions and multiple positive solutions for impulsive ordinary differential equations and functional differential equations,

including initial value problems as well as boundary value problems for second-order problems such as

$$y'' = f(t, y_t), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \quad (8)$$

subject to impulses

$$\Delta y(t_k) = I_k(y(t_k)), \quad \Delta y'(t_k) = J_k(y(t_k)), \quad k = 1, \dots, m, \quad (9)$$

and initial conditions

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad y'(0) = \eta. \quad (10)$$

Chapter 6 is primarily concerned with boundary value problems for periodic impulsive differential inclusions. Upper- and lower-solution methods are developed for first-order systems and then for second-order systems of functional differential inclusions, $y''(t) \in F(t, y_t)$. For Chapter 7, impulsive differential inclusions satisfying periodic boundary conditions are studied. The problems of interest are termed as being *nonresonant*, because the linear operators involved are invertible in the absence of impulses. The chapter deals with first-order and higher-order nonresonance impulsive inclusions.

Chapter 8 extends the theory of some of the previous chapters to functional differential equations and functional differential inclusions under impulses for which the impulse effects vary with time; that is, $y(t_k^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$. Chapter 9, as well, extends several results of previous chapters on semilinear problems now to semilinear functional differential equations and functional differential inclusions for operators that are nondensely defined on a Banach space.

Chapter 10 ventures into results for second-order impulsive hyperbolic differential inclusions,

$$\begin{aligned} \frac{\partial^2 u(t, x)}{\partial t \partial x} &\in F(t, x, u(t, x)) \quad \text{a.e. } (t, x) \in ([0, a] \setminus \{t_1, \dots, t_m\}) \times [0, b], \\ \Delta u(t_k, x) &= I_k(u(t_k, x)), \quad k = 1, \dots, m, \\ u(t, 0) &= \psi(t), \quad t \in [0, a] \setminus \{t_1, \dots, t_m\}, \quad u(0, x) = \phi(x), \quad x \in [0, b]. \end{aligned} \quad (11)$$

Such models arise especially for problems in biological or medical domains.

The next to last chapter, Chapter 11, addresses some questions for impulsive dynamic equations on time scales. The methods constitute adjustments from those for impulsive ordinary differential equations to dynamic equations on time scales, but these results are the first such results in the direction of impulsive problems on time scales. The final chapter, Chapter 12, is a brief chapter dealing with periodic

boundary value problems for first-order perturbed impulsive systems,

$$\begin{aligned} x' &\in F(t, x(t)) + G(t, x(t)), \quad t \in [0, b] \setminus \{t_1, \dots, t_m\}, \\ x(t_j^+) &= x(t_j^-) + I_j(x(t_j^-)), \quad j = 1, \dots, m, \quad x(0) = x(b), \end{aligned} \quad (12)$$

where both $F, G : ([0, b] \setminus \{t_1, \dots, t_m\}) \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$.

We express our appreciation and thanks to R. I. Avery, A. Boucherif, B. C. Dhage, E. Gatsori, L. Górniewicz, J. R. Graef, J. J. Nieto, A. Ouahab, and Y. G. Sficas for their collaboration in research and to E. Gatsori and A. Ouahab for their careful typing of some parts of this manuscript. We are especially grateful to the Editors-in-Chief of the *Contemporary Mathematics and Applications* book series, R. P. Agarwal and D. O'Regan, for their encouragement of us during the preparation of this volume for inclusion in the series.

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J. Henderson
S. Ntouyas

1

Preliminaries

1.1. Definitions and results for multivalued analysis

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis, which are used throughout this book.

Let (X, d) be a metric space and let Y be a subset of X . We denote

- (i) $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$;
- (ii) $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y \text{ has the property "p"}\}$, where p could be $\text{cl} = \text{closed}$, $b = \text{bounded}$, $\text{cp} = \text{compact}$, $\text{cv} = \text{convex}$, and so forth.

Thus

- (i) $\mathcal{P}_{\text{cl}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- (ii) $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- (iii) $\mathcal{P}_{\text{cv}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$,
- (iv) $\mathcal{P}_{\text{cp}}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- (v) $\mathcal{P}_{\text{cv, cp}}(X) = \mathcal{P}_{\text{cv}}(X) \cap \mathcal{P}_{\text{cp}}(X)$, and so forth.

In what follows, by E we will denote a Banach space over the field of real numbers \mathbb{R} and by J a closed interval in \mathbb{R} . We let

$$C(J, E) = \{y : J \rightarrow E \mid y \text{ is continuous}\}. \quad (1.1)$$

We consider the Tchebyshev norm

$$\|\cdot\|_\infty : C(J, E) \rightarrow [0, \infty), \quad (1.2)$$

defined by

$$\|y\|_\infty = \max \{|y(t)|, t \in J\}, \quad (1.3)$$

where $|\cdot|$ stands for the norm in E . Then $(C(J, E), \|\cdot\|_\infty)$ is a Banach space.

Let $N : E \rightarrow E$ be a linear map. N is called *bounded* provided there exists $r > 0$ such that

$$|N(x)| \leq r|x|, \quad \text{for every } x \in E. \quad (1.4)$$

The following result is classical.

2 Impulsive ordinary differential equations & inclusions

2.1. Introduction

For well over a century, differential equations have been used in modeling the dynamics of changing processes. A great deal of the modeling development has been accompanied by a rich theory for differential equations.

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of “impulses.” As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmacokinetics, optimal control, and so forth. Again, associated with this development, a theory of impulsive differential equations has been given extensive attention. Works recognized as landmark contributions include [29, 30, 180, 217], with [30] devoted especially to impulsive periodic systems of differential equations.

Some processes, especially in areas of population dynamics, ecology, and pharmacokinetics, involve hereditary issues. The theory and applications addressing such problems have heavily involved functional differential equations as well as impulsive functional differential equations. The literature devoted to this study is also extensive, with [6, 12–14, 25, 27, 28, 38, 42, 46, 49, 52, 53, 55, 57, 70, 71, 75, 85, 89–91, 94, 95, 117, 130–132, 134, 136, 147, 152, 159, 167, 176, 181, 183, 189, 191, 194, 195, 212, 214, 216, 228] providing a good view of the panorama of work that has been done.

Much attention has also been devoted to modeling natural phenomena with differential equations, both ordinary and functional, for which the part governing the derivative(s) is not known as a single-valued function; for example, a dynamic process governing the derivative $x'(t)$ of a state $x(t)$ may be known only within a set $S(t, x(t)) \subset \mathbb{R}$, and given by $x'(t) \in S(t, x(t))$. A common example of this is observed in a so-called differential inequality such as $x'(t) \leq f(t, x(t))$,

3 Impulsive functional differential equations & inclusions

3.1. Introduction

While the previous chapter was devoted to ordinary differential equations and inclusions involving impulses, our attention in this chapter is turned to functional differential equations and inclusions each undergoing impulse effects. These equations and inclusions have played an important role in areas involving hereditary phenomena for which a delay argument arises in the modelling equation or inclusion. There are also a number of applications in which the delayed argument occurs in the derivative of the state variable, which are sometimes modelled by neutral differential equations or neutral differential inclusions.

This chapter presents a theory for the existence of solutions of impulsive functional differential equations and inclusions, including scenarios of neutral equations, as well as semilinear models. The methods used throughout the chapter range over applications of the Leray-Schauder nonlinear alternative, Schaefer's fixed point theorem, a Martelli fixed point theorem for multivalued condensing maps, and a Covitz-Nadler fixed point theorem for multivalued maps.

3.2. Impulsive functional differential equations

In this section, we will establish existence theory for first- and second-order impulsive functional differential equations. The section will be divided into parts. In the first part, by a nonlinear alternative of Leray-Schauder type, we will present an existence result for the first-order initial value problem

$$y'(t) = f(t, y_t), \quad \text{a.e. } t \in J := [0, T], \quad t \neq t_k, \quad k = 1, \dots, m, \quad (3.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (3.2)$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad (3.3)$$

where $f : J \times \mathcal{D} \rightarrow E$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow E \mid \psi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \psi(s) \text{ and the right limit } \psi(s^+) \text{ exist and } \psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, $I_k \in C(E, E)$ ($k = 1, 2, \dots, m$), and E a real separable Banach space with norm $|\cdot|$. Also, throughout, $J' = J \setminus \{t_1, \dots, t_m\}$.

4

Impulsive differential inclusions with nonlocal conditions

4.1. Introduction

In this chapter, we will prove existence results for impulsive semilinear ordinary and functional differential inclusions, with nonlocal conditions. Often, nonlocal conditions are motivated by physical problems. For the importance of nonlocal conditions in different fields we refer to [112]. As indicated in [112, 113, 126] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be more descriptive in physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [126], the author used

$$g(y) = \sum_{k=1}^p c_k y(t_k), \quad (4.1)$$

where $c_i, i = 1, \dots, p$ are given constants and $0 < t_1 < t_2 < \dots < t_p \leq b$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (4.1) allows the additional measurements at $t_i, i = 1, \dots, p$.

Nonlocal Cauchy problems for ordinary differential equations have been investigated by several authors, (see, e.g., [103, 113, 114, 202–204, 206, 207]). Nonlocal Cauchy problems, in the case where F is a multivalued map, were studied by Benchohra and Ntouyas [77–79], and Boucherif [103]. Akça et al. [14] initiated the study of a class of first-order semilinear functional differential equations for which the nonlocal conditions and the impulse effects are combined. Again, in this chapter, we will invoke some of our fixed point theorems in establishing solutions for these nonlocal impulsive differential inclusions.

4.2. Nonlocal impulsive semilinear differential inclusions

In this section, we begin the study of nonlocal impulsive initial value problems by proving existence results for the problem

$$y'(t) \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b], \quad t \neq t_k, \quad k = 1, 2, \dots, m, \quad (4.2)$$

5 Positive solutions for impulsive differential equations

5.1. Introduction

Positive solutions and multiple positive solutions of differential equations have received a tremendous amount of attention. Studies have involved initial value problems, as well as boundary value problems, for both ordinary and functional differential equations. In some cases, impulse effects have also been present. The methods that have been used include multiple applications of the Guo-Krasnosel'skii fixed point theorem [158], the Leggett-Williams multiple fixed point theorem [187], and extensions such as the Avery-Henderson double fixed point theorem [26]. Many such multiple-solution works can be found in the papers [6, 8–10, 19, 52, 94, 95, 137, 159, 194].

This chapter is devoted to positive solutions and multiple positive solutions of impulsive differential equations.

5.2. Positive solutions for impulsive functional differential equations

Throughout this section, let $J = [0, b]$, and the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$ are fixed. This section is concerned with the existence of three non-negative solutions for initial value problems for first- and second-order functional differential equations with impulsive effects. In Section 5.2.1, we consider the first-order IVP

$$\begin{aligned} y'(t) &= f(t, y_t), \quad t \in J = [0, b], \quad t \neq t_k, \quad k = 1, \dots, m, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)), \quad k = 1, \dots, m, \\ y(t) &= \phi(t), \quad t \in [-r, 0], \end{aligned} \tag{5.1}$$

where $f : J \times \mathcal{D} \rightarrow \mathbb{R}$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow \mathbb{R}_+ \mid \psi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \psi(s) \text{ and the right limit } \psi(s^+) \text{ exist and } \psi(s^-) = \psi(s)\}$, $\phi \in \mathcal{D}$, $0 < r < \infty$, $I_k : \mathbb{R} \rightarrow \mathbb{R}_+$ ($k = 1, 2, \dots, m$), $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $J' = J \setminus \{t_1, \dots, t_m\}$.

6

Boundary value problems for impulsive differential inclusions

6.1. Introduction

The method of upper and lower solutions has been successfully applied to study the existence of solutions for first-order impulsive initial value problems and boundary value problems. This method generates solutions of such problems, with the solutions located in an order interval with the upper and lower solutions serving as bounds. Moreover, this method, coupled with some monotonicity-type hypotheses, leads to monotone iterative techniques which generate in a constructive way (amenable to numerical treatment) the extremal solutions within the order interval determined by the upper and lower solutions.

This method has been used only in the context of single-valued impulsive differential equations. We refer to the monographs of Lakshmikantham et al. [180], Samoilenko and Perestyuk [217], the papers of Cabada and Liz [117], Frigon and O'Regan [151], Heikkilä and Lakshmikantham [163], Liu [188], Liz [192, 193], Liz and Nieto [194], and Pierson-Gorez [212]. However, this method has been used recently by Benchohra and Boucherif [35] for the study of first-order initial value problems for impulsive differential inclusions.

Let us mention that other methods like the nonlinear alternative, such as in the papers of Benchohra and Boucherif [34, 35], Frigon and O'Regan [150], and the topological transversality theorem Erbe and Krawcewicz [140], have been used to analyze first- and second-order impulsive differential inclusions. The first part of this chapter presents existence results using upper- and lower-solutions methods to obtain solutions of first-order impulsive differential inclusions with periodic boundary conditions and nonlinear boundary conditions. The last section of the chapter deals with boundary value problems for second-order impulsive differential inclusions.

6.2. First-order impulsive differential inclusions with periodic boundary conditions

This section is devoted to the existence of solutions for the impulsive periodic multivalued problem

7 Nonresonance impulsive differential inclusions

7.1. Introduction

This chapter is devoted to impulsive differential inclusions satisfying periodic boundary conditions. These problems are termed as being *nonresonant*, because the linear operator involved will be invertible in the absence of impulses. The first problem addressed concerns first-order problems. A result from [51] that generalizes a paper by Nieto [199] is presented. The methods used involve the Martelli fixed point theorem (Theorem 1.7) and the Covitz-Nadler fixed point theorem (Theorem 1.11).

The second part of the chapter is focused on a second-order problem, and a result of [55] is obtained which is an extension of the first-order result. Again the method used involves an application of Theorem 1.7. Then, the final section of the chapter is a successful extension of these results to n th order nonresonance problems, which were first established in [63]. Also, an initial value function is introduced for the higher-order consideration.

7.2. Nonresonance first-order impulsive functional differential inclusions with periodic boundary conditions

This section is concerned with the existence of solutions for the nonresonance problem for functional differential inclusions with impulsive effects as

$$y'(t) - \lambda y(t) \in F(t, y_t), \quad t \in J = [0, T], \quad t \neq t_k, \quad k = 1, \dots, m, \quad (7.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (7.2)$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \quad (7.3)$$

$$\phi(0) = y(0) = y(T), \quad (7.4)$$

where $\lambda \neq 0$ and λ is not an eigenvalue of y' , $F : J \times \mathcal{D} \rightarrow \mathcal{P}(E)$ is a compact convex-valued multivalued map, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow E \mid \psi \text{ is continuous everywhere except for a finite number of points } s \text{ at which } \psi(s) \text{ and the right limit } \psi(s^+) \}$

8

Impulsive differential equations & inclusions with variable times

8.1. Introduction

The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [31], Frigon and O'Regan [150, 151], Kaul [173], Kaul et al. [174], and Benchohra et al. [43, 45, 70, 71, 91, 92].

8.2. First-order impulsive differential equations with variable times

This section is concerned with the existence of solutions, for initial value problems (IVP for short), for first-order functional differential equations with impulsive effects

$$\begin{aligned}y'(t) &= f(t, y_t), \quad \text{a.e. } t \in J = [0, T], \quad t \neq \tau_k(y(t)), \quad k = 1, \dots, m, \\y(t^+) &= I_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \dots, m, \\y(t) &= \phi(t), \quad t \in [-r, 0],\end{aligned}\tag{8.1}$$

where $f : J \times \mathcal{D} \rightarrow \mathbb{R}^n$ is a given function, $\mathcal{D} = \{\psi : [-r, 0] \rightarrow \mathbb{R}^n : \psi \text{ is continuous everywhere except for a finite number of points } \bar{t} \text{ at which } \psi(\bar{t}) \text{ and } \psi(\bar{t}^+) \text{ exist, and } \psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in D$, $0 < r < \infty$, $\tau_k : \mathbb{R}^n \rightarrow \mathbb{R}$, $I_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $k = 1, 2, \dots, m$, are given functions satisfying some assumptions that will be specified later.

The main theorem of this section extends the problem (8.1) considered by Benchohra et al. [46] when the impulse times are constant. Our approach is based on Schaefer's fixed point theorem.

Let us start by defining what we mean by a solution of problem (8.1).

Definition 8.1. A function $y \in \Omega \cap AC((t_k, t_{k+1}), \mathbb{R})$, $k = 0, \dots, m$, is said to be a solution of (8.1) if y satisfies the equation $y'(t) = f(t, y_t)$ a.e. on J , $t \neq \tau_k(y(t))$, $k = 1, \dots, m$, and the conditions $y(t^+) = I_k(y(t))$, $t = \tau_k(y(t))$, $k = 1, \dots, m$, and $y(t) = \phi(t)$ on $[-r, 0]$.

9

Nondensely defined impulsive differential equations & inclusions

9.1. Introduction

This chapter deals with semilinear functional differential equations and functional differential inclusions involving linear operators that are nondensely defined on a Banach space. This chapter extends several previous results of this book that were devoted to semilinear problems with densely defined operators. Some of the results of this chapter were first presented in the work by Benchohra et al. [76].

9.2. Nondensely defined impulsive semilinear differential equations with nonlocal conditions

In this section, we will prove existence results for an evolution equation with nonlocal conditions of the form

$$y'(t) = Ay(t) + F(t, y(t)), \quad t \in J := [0, T], \quad t \neq t_k, \quad k = 1, \dots, m, \quad (9.1)$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \quad (9.2)$$

$$y(0) + g(y) = y_0, \quad (9.3)$$

where $A : D(A) \subset E \rightarrow E$ is a nondensely defined closed linear operator, $F : J \times E \rightarrow E$ is continuous, $g : C(J', E) \rightarrow E$, $(J' = J \setminus \{t_1, \dots, t_m\})$, $I_k : E \rightarrow \overline{D(A)}$, $k = 1, \dots, m$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h \rightarrow 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \rightarrow 0^+} y(t_k - h)$, and E is a separable Banach space with norm $|\cdot|$.

As indicated in [112, 115, 126] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be applied to physics with better effect than the classical initial condition $y(0) = y_0$. For example, in [126], the author used

$$g(y) = \sum_{k=1}^p c_i y(t_i), \quad (9.4)$$

where c_i , $i = 1, \dots, p$, are given constants and $0 < t_1 < t_2 < \dots < t_p \leq T$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, (9.4) allows the additional measurements at t_i , $i = 1, \dots, p$.

10 Hyperbolic impulsive differential inclusions

10.1. Introduction

In this chapter, we will be concerned with the existence of solutions for second-order impulsive hyperbolic differential inclusions in a separable Banach space. More precisely, we will consider impulsive hyperbolic differential inclusions of the form

$$\begin{aligned} \frac{\partial^2 u(t, x)}{\partial t \partial x} &\in F(t, x, u(t, x)), \quad \text{a.e. } (t, x) \in J_a \times J_b, \quad t \neq t_k, \quad k = 1, \dots, m, \\ \Delta u(t_k, x) &= I_k(u(t_k, x)), \quad k = 1, \dots, m, \\ u(t, 0) &= \psi(t), \quad t \in J_a, \quad u(0, x) = \phi(x), \quad x \in J_b, \end{aligned} \quad (10.1)$$

where $J_a = [0, a]$, $J_b = [0, b]$, $F : J_a \times J_b \times E \rightarrow \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $\phi \in C(J_a, E)$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = a$, $I_k \in C(E, E)$ ($k = 1, \dots, m$), $\Delta u|_{t=t_k} = u(t_k^+, y) - u(t_k^-, y)$, $u(t_k^+, y) = \lim_{(h,x) \rightarrow (0^+, y)} u(t_k + h, x)$ is the right limit and $u(t_k^-, y) = \lim_{(h,x) \rightarrow (0^+, y)} u(t_k - h, x)$ is left limit of $u(t, x)$ at (t_k, x) , and E is a real separable Banach space with norm $|\cdot|$.

In the last few years impulsive differential and partial differential equations have become the object of increasing investigation in some mathematical models of real world phenomena, especially in biological or medical domain; see the monographs by Bařnov and Simeonov [29], Lakshmikantham et al. [180], Samoilenko and Perestyuk [217].

In the last three decades several papers have been devoted to the study of hyperbolic partial differential equations with local and nonlocal initial conditions; see for instance [113, 115, 182] and the references cited therein. For similar results with set-valued right-hand side, we refer to the papers by Byszewski and Papageorgiou [116], Papageorgiou [208], and Benchohra and Ntouyas [33, 81, 83, 84].

Here we will present three existence results for problem (10.1) in the cases when F has convex and nonconvex values. In the convex case, an existence result will be given by means of the nonlinear alternative of Leray-Schauder type for multivalued maps. In the nonconvex, case two results will be presented. The first

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Impulsive dynamic equations on time scales

11.1. Introduction

In recent years dynamic equations on time scales have received much attention. We refer to the books by Agarwal and O'Regan [7], Bohner and Peterson [101, 102], and Lakshmikantham et al. [184], and the papers by Anderson [15, 18], Agarwal et al. [2, 3, 5], Bohner and Guseinov [100], Bohner and Eloe [99], and Erbe and Peterson [141, 142].

The time scales calculus has a tremendous potential for applications in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics, neural networks, social sciences, as is pointed out in the monographs of Aulbach and Hilger [24], Bohner and Peterson [101, 102], and Lakshmikantham et al. [184].

The existence of solutions of boundary value problem on a time scale was recently studied by Agarwal and O'Regan [7], Anderson [16, 17], Henderson [166], and Sun and Li [223]. In this chapter, dynamic equations on time scales are considered for both impulsive initial value problems and impulsive boundary value problems. The results here are based on work from [72, 165].

11.2. Preliminaries

We will introduce some basic definitions and facts from the time scale calculus that we will use in the sequel.

A time scale \mathbb{T} is a nonempty closed subset of \mathbb{R} . It follows that the jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad (11.1)$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If \mathbb{T} has a right-scattered minimum m , define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If \mathbb{T} has a left-scattered maximum M , define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The notations $[c, d]$, $[c, d)$, and so

12

On periodic boundary value problems of first-order perturbed impulsive differential inclusions

12.1. Introduction

In this chapter, we study the existence of solutions to periodic nonlinear boundary value problems for first-order Carathéodory impulsive ordinary differential inclusions with convex multifunctions. Given a closed and bounded interval $J := [0, T]$ in \mathbb{R} , and given the impulsive moments t_1, t_2, \dots, t_p with $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = T$, $J' = J \setminus \{t_1, t_2, \dots, t_p\}$, $J_j = (t_j, t_{j+1})$, consider the following periodic boundary value problem for impulsive differential inclusions (IDI):

$$x'(t) \in F(t, x(t)) + G(t, x(t)) \quad \text{a.e. } t \in J', \quad (12.1)$$

$$x(t_j^+) = x(t_j^-) + I_j(x(t_j^-)), \quad (12.2)$$

$$x(0) = x(T), \quad (12.3)$$

where $F, G : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ are impulsive multifunctions, $I_j : \mathbb{R} \rightarrow \mathbb{R}$, $j = 1, 2, \dots, p$, are the impulse functions, and $x(t_j^+)$ and $x(t_j^-)$ are, respectively, the right and the left limits of x at $t = t_j$.

Let $C(J, \mathbb{R})$ and $L^1(J, \mathbb{R})$ denote the space of continuous and Lebesgue integrable real-valued functions on J . Consider the Banach space

$$X := \{x : J \rightarrow \mathbb{R} : x \in C(J', \mathbb{R}), x(t_j^+), x(t_j^-) \text{ exist}, x(t_j^-) = x(t_j), j = 1, 2, \dots, p\}, \quad (12.4)$$

equipped with the norm $\|x\| = \max\{|x(t)| : t \in J\}$, and the space

$$Y := \{x \in X : x \text{ is differentiable a.e. on } (0, T), x' \in L^1(J, \mathbb{R})\}. \quad (12.5)$$

By a solution of (12.1)–(12.3), we mean a function x in $Y_T := \{v \in Y : v(0) = v(T)\}$ that satisfies the differential inclusion (12.1) and the impulsive conditions (12.2).

Our aim is to provide sufficient conditions to the multifunctions F , G and the impulsive functions I_j that insure the existence of solutions of problem IDI (12.1)–(12.3).

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