Research Article

Relativistic Milne-Eddington Type Solutions with a Variable Eddington Factor for Relativistic Spherical Winds

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Relativistic radiative transfer in a relativistic spherical flow is examined in the fully special relativistic treatment. Under the assumption of a constant flow speed and using a variable (prescribed) Eddington factor, we analytically solve the relativistic moment equations in the comoving frame for several restricted cases, and obtain relativistic Milne-Eddington type solutions. In contrast to the plane-parallel case where the solutions exhibit the exponential behavior on the optical depth, the solutions have power-law forms. In the case of the radiative equilibrium, for example, the radiative flux has a power-law term multiplied by the exponential term. In the case of the local thermodynamic equilibrium with a uniform source function in the comoving frame, the radiative flux has a power-law form on the optical depth. This is because there is an expansion effect (curvature effect) in the spherical wind and the background density decreases as the radius increases.

1. Introduction

The research field of radiative transfer has been developed in astrophysics and atmospheric science [1–13]. Relativistic radiative transfer and relativistic radiation hydrodynamics have been also developed in astrophysics and applied to various energetic phenomena in the universe: nova outbursts, gamma-ray bursts, astrophysical jets, black-hole accretion disks, and black-hole winds. In the subrelativistic regime, some researchers adopted the diffusion approximation in the comoving frame (e.g., [14–17]), or proposed the variable Eddington factor (e.g., [18–24]), or performed the numerical simulations using, for example, the flux-limited diffusion (FLD) approximation (e.g., [25–35]).

In the highly relativistic regime, however, we cannot treat the relativistic radiative transfer properly. Hence, the research and development of the radiative transfer problem in relativistically moving media is now of great importance in this field.

At the present stage, the numerical approach using the FLD approximation is limited within the subrelativistic regime. In addition, even in the subrelativistic regime, the FLD method cannot reproduce the radiative force precisely in the optically thin region [36]. This is because in the

optically thin region the radiative flux vector is not generally parallel to the gradient of the radiation energy density due to the effect of the distant radiation source. On the other hand, the analytical approach is very restricted in the special cases. Indeed, even for the nonrelativistic case only a few analytical solutions have been found, but for the relativistic case little is known. Recently, the relativistic radiative transfer in relativistically moving atmospheres have been investigated from the analytical view points in the plane-parallel case (e.g., [37–40]) and in the spherical case (e.g., [41]).

In Fukue [40], under the assumption of a constant flow speed, the relativistic moment equations in the comoving frame were analytically solved using a variable Eddington factor for several cases, such as the radiative equilibrium (RE) or the local thermodynamic equilibrium (LTE), and the relativistic Milne-Eddington type solutions for the relativistic *plane-parallel* flows have been newly found. In the present study, we also consider the relativistic radiative transfer from the analytical view point. In contrast to Fukue [40], we examine the relativistic moment equations for the *spherical* case, solve the equations under the assumption of a constant flow speed, and obtain several new analytical solutions for the relativistic spherical flows.

In the next section, we describe the radiative moment equations in the comoving frame for spherical flows. In Sections 3 and 4, we show and discuss analytical solutions in the RE and LTE cases, respectively. The final section is devoted to concluding remarks.

2. Relativistic Radiative Transfer Equation

The radiative transfer equations are given in several standard references [6–10, 12, 13, 42–44]. The basic equations for relativistic radiation hydrodynamics are given in, for example, the Appendix E of Kato et al. [43] in general and vertical forms (see also [39, 40]).

2.1. General Form. In a general form, the radiative transfer equation in the *mixed frame*, where the variables in the inertial and comoving frames are used, is expressed as

$$\frac{1}{c} \frac{\partial I}{\partial t} + (\mathbf{l} \cdot \nabla) I = \left(\frac{\nu}{\nu_0} \right)^3 \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi} \right], \quad (1)$$

where c is the speed of light. In the left-hand side, the frequency-integrated specific intensity I and the direction cosine vector $\mathbf{l}$ are quantities measured in the inertial (fixed) frame. In the right-hand side, on the other hand, the mass density ρ , the frequency-integrated mass emissivity j_0 , the frequency-integrated mass absorption coefficient κ_0 , the frequency-integrated mass scattering coefficient σ_0 , the frequency-integrated specific intensity I_0 , and the frequency-integrated radiation energy density E_0 are quantities measured in the comoving (fluid) frame. In this paper, instead of the weakly anisotropic Thomson scattering, we assume that the scattering is isotropic for simplicity.

The Doppler effect, the aberration, and the transformation of the intensities are expressed as

$$\frac{\nu}{\nu_0} = \gamma(1 + \boldsymbol{\beta} \cdot \mathbf{l}_0), \quad (2)$$

$$\mathbf{l} = \frac{\nu_0}{\nu} \left[\mathbf{l}_0 + \left(\frac{\gamma - 1}{\beta^2} \boldsymbol{\beta} \cdot \mathbf{l}_0 + \gamma \right) \boldsymbol{\beta} \right], \quad (3)$$

$$I = \left(\frac{\nu}{\nu_0} \right)^4 I_0, \quad (4)$$

where ν and ν_0 are the frequencies measured in the inertial and comoving frames, respectively, the direction cosine $\mathbf{l}_0$ measured in the comoving frame, $\boldsymbol{\beta} (= \mathbf{v}/c)$ the normalized velocity, $\mathbf{v}$ being the flow velocity, and $\gamma (= 1/\sqrt{1 - \beta^2})$ the Lorentz factor, β being $\mathbf{v}/c$.

The zeroth and first moment equations are, respectively,

$$\begin{aligned} \frac{\partial E}{\partial t} + \frac{\partial F^k}{\partial x^k} &= \rho \gamma (j_0 - \kappa_0 c E_0) - \rho \gamma (\kappa_0 + \sigma_0) \boldsymbol{\beta} \cdot \mathbf{F}_0, \\ \frac{1}{c^2} \frac{\partial F^i}{\partial t} + \frac{\partial P^{ik}}{\partial x^k} &= \rho \gamma \frac{\beta^i}{c} (j_0 - \kappa_0 c E_0) - \rho (\kappa_0 + \sigma_0) \frac{\gamma - 1}{\beta^2} \frac{\beta^i}{c} \\ &\quad \times (\boldsymbol{\beta} \cdot \mathbf{F}_0) - \frac{1}{c} \rho (\kappa_0 + \sigma_0) F_{00}^i, \end{aligned} \quad (5)$$

where the frequency-integrated radiation energy density E , the frequency-integrated radiative flux $\mathbf{F}$, and the frequency-integrated radiation stress P^{ik} are measured in the inertial frame, while those with the subscript 0 are measured in the comoving frame.

As a closure relation, we adopt the Eddington approximation *in the comoving frame*,

$$P_0^{ik} = f^{ik} E_0, \quad (6)$$

where f^{ik} is the Eddington tensor, which is generally a function of the optical depth and flow speed in the relativistic radiative flow.

2.2. Spherical Expression in the Comoving Frame. Let us suppose a relativistic spherical flow, for example, a luminous black-hole wind. In the spherical geometry with the radius r , the transfer equation (1) is expressed as

$$\begin{aligned} \frac{1}{c} \frac{\partial I}{\partial t} + \mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} \\ = \left(\frac{\nu}{\nu_0} \right)^3 \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi} \right], \end{aligned} \quad (7)$$

where μ is the direction cosine in the inertial frame. Inserting the transformation (4) in the left-hand side, this equation (7) becomes

$$\begin{aligned} \frac{\nu}{\nu_0} \left(\frac{1}{c} \frac{\partial I_0}{\partial t} + \mu \frac{\partial I_0}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I_0}{\partial \mu} \right) \\ - 4 \frac{\nu}{\nu_0^2} I_0 \left(\frac{1}{c} \frac{\partial \nu_0}{\partial t} + \mu \frac{\partial \nu_0}{\partial z} + \frac{1 - \mu^2}{r} \frac{\partial \nu_0}{\partial \mu} \right) \\ = \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi} \right]. \end{aligned} \quad (8)$$

To calculate the derivatives of I_0 [9], we apply the chain rules, and, after some manipulations, we have

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{r\mu\nu} &= \frac{\partial}{\partial t} \Big|_{r\mu_0\nu_0} + \frac{\partial \mu_0}{\partial t} \Big|_{r\mu_0\nu_0} \frac{\partial}{\partial \mu_0} + \frac{\partial \nu_0}{\partial t} \Big|_{r\mu_0\nu_0} \frac{\partial}{\partial \nu_0} \\ &= \frac{\partial}{\partial t} \Big|_{r\mu_0\nu_0} - \gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial t} \frac{\partial}{\partial \mu_0} - \gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial t} \frac{\partial}{\partial \nu_0}, \\ \frac{\partial}{\partial r} \Big|_{t\mu\nu} &= \frac{\partial}{\partial r} \Big|_{t\mu_0\nu_0} + \frac{\partial \mu_0}{\partial r} \Big|_{t\mu_0\nu_0} \frac{\partial}{\partial \mu_0} + \frac{\partial \nu_0}{\partial r} \Big|_{t\mu_0\nu_0} \frac{\partial}{\partial \nu_0} \\ &= \frac{\partial}{\partial r} \Big|_{t\mu_0\nu_0} - \gamma^2 (1 - \mu_0^2) \frac{\partial \beta}{\partial r} \frac{\partial}{\partial \mu_0} - \gamma^2 \mu_0 \nu_0 \frac{\partial \beta}{\partial r} \frac{\partial}{\partial \nu_0}, \\ \frac{\partial}{\partial \mu} \Big|_{rt\nu} &= \frac{\partial \mu_0}{\partial \mu} \Big|_{rt\nu_0} \frac{\partial}{\partial \mu_0} + \frac{\partial \nu_0}{\partial \mu} \Big|_{rt\nu_0} \frac{\partial}{\partial \nu_0} \\ &= \gamma^2 (1 + \beta \mu_0)^2 \frac{\partial}{\partial \mu_0} - \gamma^2 \beta (1 + \beta \mu_0) \nu_0 \frac{\partial}{\partial \nu_0}, \end{aligned} \quad (9)$$

where μ is the direction cosine in the comoving frame. In addition, the Doppler shift (2) and the aberration (3) are, respectively, expressed as

$$\begin{aligned}\frac{\nu}{\nu_0} &= \gamma(1 + \beta\mu_0), \\ \mu &= \frac{\mu_0 + \beta}{1 + \beta\mu_0}.\end{aligned}\quad (10)$$

Using these expressions, after some manipulations, we have the radiative transfer equation in the comoving frame for the spherical flow:

$$\begin{aligned}& \gamma(1 + \beta\mu_0) \frac{1}{c} \frac{\partial I_0}{\partial t} + \gamma(\mu_0 + \beta) \frac{\partial I_0}{\partial r} + \gamma(1 + \beta\mu_0) \frac{1 - \mu_0}{r} \frac{\partial I_0}{\partial \mu_0} \\& + 4\gamma\beta \frac{1 - \mu_0^2}{r} I_0 - \gamma^3(1 + \beta\mu_0) \\& \times \left[(1 - \mu_0^2) \frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0 \right] \frac{1}{c} \frac{\partial \beta}{\partial t} \\& - \gamma^3(\mu_0 + \beta) \left[(1 - \mu_0^2) \frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0 \right] \frac{\partial \beta}{\partial r} \\& = \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi} \right].\end{aligned}\quad (11)$$

Integrating the transfer equation (11) over a solid angle, we have the zeroth and first moment equations in the comoving frame for the spherical flow:

$$\begin{aligned}& \gamma \frac{\partial cE_0}{\partial t} + \gamma \frac{\partial F_0}{\partial r} + \gamma\beta \frac{\partial F_0}{\partial t} + \gamma\beta \frac{\partial cE_0}{\partial r} + \frac{\gamma}{r} [2F_0 + \beta(3cE_0 - cP_0)] \\& + \gamma^3 [2F_0 + \beta(cE_0 + cP_0)] \frac{\partial \beta}{\partial t} \\& + \gamma^3 [2\beta F_0 + (cE_0 + cP_0)] \frac{\partial \beta}{\partial r} \\& = \rho(j_0 - \kappa_0 cE_0), \\& \gamma \frac{\partial F_0}{\partial t} + \gamma \frac{\partial cP_0}{\partial r} + \gamma\beta \frac{\partial cP_0}{\partial t} + \gamma\beta \frac{\partial F_0}{\partial r} + \frac{\gamma}{r} [2\beta F_0 - cE_0 + 3cP_0] \\& + \gamma^3 [2\beta F_0 + (cE_0 + cP_0)] \frac{\partial \beta}{\partial t} \\& + \gamma^3 [2F_0 + \beta(cE_0 + cP_0)] \frac{\partial \beta}{\partial r} \\& = -\rho(\kappa_0 + \sigma_0) F_0,\end{aligned}\quad (12)$$

where E_0 , F_0 , and P_0 are the radiation energy density, the radiative flux, and the radiation pressure in the comoving frame, respectively.

In the present spherical one-dimensional flow, if we assume the pressure isotropy in the comoving frame, the closure relation (6) becomes

$$P_0 = f(\tau, \beta) E_0, \quad (13)$$

where $f(\tau, \beta)$ is the variable (prescribed) Eddington factor, and generally a function of the optical depth, the flow speed, and the velocity gradient [45, 46]. In the plane-parallel flow [40], the following form was adopted:

$$f(\beta) = \frac{1 + 3\beta^2}{3 + \beta^2}, \quad (14)$$

which is 1/3 for $\beta = 0$ and approaches unity as $\beta \rightarrow 1$ [47]. In the present spherical case, we adopt alternative appropriate forms, which are shown later.

It should be noted that historically Auer and Mihalas [48] first used the term a variable Eddington factor (VEF) to express an iterative solver to obtain the Eddington factor (cf. [49]). The Eddington factors which depend on the optical depth should be called *prescribed* or *approximated* Eddington factors, although they are often called a variable Eddington factor [50, 51]. In the previous papers, in order to express the not-constant Eddington factor, we also used the variable Eddington factor, which depends on the optical depth and flow velocity. In this paper we use both terms, variable and prescribed, but both usages express the same meanings; that is, the present Eddington factor is not constant but varies as a function of the optical depth and flow velocity.

2.3. Steady Spherical Flow. Let us further suppose a time-independent steady flow in the radial direction. In this case the transfer equation and moment equations in the comoving frame become

$$\begin{aligned}& \gamma(\mu_0 + \beta) \frac{dI_0}{dr} - \gamma^3(\mu_0 + \beta) \left[(1 - \mu_0^2) \frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0 \right] \frac{d\beta}{dr} \\& + \gamma(1 + \beta\mu_0) \frac{1 - \mu_0^2}{r} \frac{\partial I_0}{\partial \mu_0} + 4\gamma\beta \frac{1 - \mu_0^2}{r} I_0 \\& = \rho \left[\frac{j_0}{4\pi} - (\kappa_0 + \sigma_0) I_0 + \sigma_0 \frac{cE_0}{4\pi} \right],\end{aligned}\quad (15)$$

$$\begin{aligned}& \gamma \frac{dF_0}{dr} + \gamma\beta \frac{dcE_0}{dr} + \gamma^3 [2\beta F_0 + (cE_0 + cP_0)] \frac{d\beta}{dr} \\& + \frac{\gamma}{r} [2F_0 + \beta(3cE_0 - cP_0)] = \rho(j_0 - \kappa_0 cE_0),\end{aligned}\quad (16)$$

$$\begin{aligned}& \gamma \frac{dcP_0}{dr} + \gamma\beta \frac{dF_0}{dr} + \gamma^3 [2F_0 + \beta(cE_0 + cP_0)] \frac{d\beta}{dr} \\& + \frac{\gamma}{r} [2\beta F_0 - cE_0 + 3cP_0] = -\rho(\kappa_0 + \sigma_0) F_0.\end{aligned}\quad (17)$$

Introducing the optical depth defined by

$$d\tau \equiv -(\kappa_0 + \sigma_0) \rho dr \quad (18)$$

and the scattering albedo,

$$a \equiv \frac{\sigma_0}{\kappa_0 + \sigma_0}, \quad (19)$$

the transfer equation (15) and the moment equations (16) and (17) are finally expressed as

$$\begin{aligned} & \gamma(\mu_0 + \beta) \frac{dI_0}{d\tau} - \gamma(1 + \beta\mu_0) \frac{1 - \mu_0^2}{\rho(\kappa_0 + \sigma_0)r} \frac{\partial I_0}{\partial \mu_0} \\ & - 4\gamma\beta \frac{1 - \mu_0^2}{\rho(\kappa_0 + \sigma_0)r} I_0 \\ & - \gamma^3(\mu_0 + \beta) \left[(1 - \mu_0^2) \frac{\partial I_0}{\partial \mu_0} - 4\mu_0 I_0 \right] \frac{d\beta}{d\tau} \\ & = I_0 - \frac{1}{4\pi} \frac{j_0}{\kappa_0 + \sigma_0} - a \frac{cE_0}{4\pi}, \end{aligned} \quad (20)$$

$$\begin{aligned} & \gamma \frac{dF_0}{d\tau} + \gamma\beta \frac{dcE_0}{d\tau} - \frac{\gamma}{\rho(\kappa_0 + \sigma_0)r} [2F_0 + \beta(3cE_0 - cP_0)] \\ & + \gamma^3 [2\beta F_0 + (cE_0 + cP_0)] \frac{d\beta}{d\tau} \\ & = - \frac{j_0}{\kappa_0 + \sigma_0} + (1 - a)cE_0, \end{aligned} \quad (21)$$

$$\begin{aligned} & \gamma \frac{dP_0}{d\tau} + \gamma\beta \frac{dF_0}{d\tau} - \frac{\gamma}{\rho(\kappa_0 + \sigma_0)r} [2\beta F_0 - cE_0 + 3cP_0] \\ & + \gamma^3 [2F_0 + \beta(cE_0 + cP_0)] \frac{d\beta}{d\tau} \\ & = F_0. \end{aligned} \quad (22)$$

Here, we further introduce the spherical variables by

$$\begin{aligned} L_0 & \equiv 4\pi r^2 F_0, \\ D_0 & \equiv 4\pi r^2 cE_0, \\ Q_0 & \equiv 4\pi r^2 cP_0, \end{aligned} \quad (23)$$

and the moment equations (21) and (22) become

$$\begin{aligned} & \gamma \frac{dL_0}{d\tau} + \gamma\beta \frac{dD_0}{d\tau} - \gamma\beta \frac{D_0 - Q_0}{\rho(\kappa_0 + \sigma_0)r} + \gamma^3 (2\beta L_0 + D_0 + Q_0) \frac{d\beta}{d\tau} \\ & = - \frac{4\pi r^2}{\kappa_0 + \sigma_0} (j_0 - \kappa_0 cE_0), \\ & \gamma \frac{dQ_0}{d\tau} + \gamma\beta \frac{dL_0}{d\tau} - \gamma \frac{Q_0 - D_0}{\rho(\kappa_0 + \sigma_0)r} + \gamma^3 [2L_0 + \beta(D_0 + Q_0)] \frac{d\beta}{d\tau} \\ & = L_0, \end{aligned} \quad (24)$$

and the closure relation (13) is written as

$$Q_0 = f(\tau, \beta) D_0. \quad (25)$$

If we assume the streaming limit of $D_0 = Q_0$ ($f = 1$) with a constant speed in (24), we have the exponential type solutions. In this paper we consider more general cases of $Q_0 = fD_0$.

Using this closure relation (25), the Eddington factor being not yet determined, and the definition of the optical depth (18), the relativistic moment equation (24) is expressed as

$$\begin{aligned} & \gamma \frac{dL_0}{d\tau} + \gamma\beta \frac{dD_0}{d\tau} + \gamma\beta \frac{1-f}{r} D_0 \frac{dr}{d\tau} + \gamma^3 [2\beta L_0 + (1+f)D_0] \frac{d\beta}{d\tau} \\ & = - \frac{4\pi r^2}{\kappa_0 + \sigma_0} (j_0 - \kappa_0 cE_0), \\ & \gamma \frac{d(fD_0)}{d\tau} + \gamma\beta \frac{dL_0}{d\tau} - \gamma \frac{1-f}{r} D_0 \frac{dr}{d\tau} \\ & + \gamma^3 [2L_0 + \beta(1+f)D_0] \frac{d\beta}{d\tau} \\ & = L_0. \end{aligned} \quad (26)$$

After several manipulations and rearrangement, the relativistic moment equation (26) in the comoving frame is finally expressed as

$$\begin{aligned} & \frac{\gamma(f - \beta^2)}{f} \frac{dL_0}{d\tau} + \gamma\beta \left(\frac{1-f^2}{fr} - \frac{1}{f} \frac{df}{dr} \right) D_0 \frac{dr}{d\tau} \\ & + \gamma^3 \left[2\beta \left(1 - \frac{1}{f} \right) L_0 + (1+f) \left(1 - \frac{\beta^2}{f} \right) D_0 \right] \frac{d\beta}{d\tau} \\ & = - \frac{4\pi r^2}{\kappa_0 + \sigma_0} (j_0 - \kappa_0 cE_0) - \frac{\beta}{f} L_0, \\ & \gamma(f - \beta^2) \frac{dD_0}{d\tau} + \gamma \left[\frac{df}{dr} - \frac{(1-f)(1+\beta^2)}{r} \right] D_0 \frac{dr}{d\tau} \\ & + 2\gamma L_0 \frac{d\beta}{d\tau} \\ & = L_0 + \beta \frac{4\pi r^2}{\kappa_0 + \sigma_0} (j_0 - \kappa_0 cE_0). \end{aligned} \quad (27)$$

After we determine the appropriate form of the variable Eddington factor, we can solve the moment equation (27) in some restricted cases.

Before solving the moment equations, we derive a relation between the optical depth τ and radius r . If the flow is steady, as is assumed, the continuity equation for the spherical case is written as

$$4\pi r^2 \rho \gamma \beta c = \dot{M}, \quad (28)$$

where $\dot{M}$ is the constant mass-outflow rate. Using this continuity equation (28), assuming the opacities are constant, and imposing the boundary condition of $\tau = 0$ at $r = \infty$, we can integrate the optical depth (18) to give

$$\tau = \frac{\dot{M}(\kappa_0 + \sigma_0)}{4\pi \gamma \beta c} \frac{1}{r} = \rho(\kappa_0 + \sigma_0)r, \quad (29)$$

which is also written as

$$\frac{\tau}{\tau_c} = \frac{r_c}{r}, \quad (30)$$

where the subscript c denotes some reference position (core radius). It should be noted that the optical depth at the core radius is related to the core radius by

$$\tau_c = \frac{\dot{m} r_g}{2\gamma\beta r_c}, \quad (31)$$

where $\dot{m}$ ($=\dot{M}/\dot{M}_E$) is the mass-outflow rate normalized by the critical rate $\dot{M}_E$ ($= L_E/c^2$), L_E being the Eddington luminosity of the central object, and r_g ($= 2GM/c^2$) is the Schwarzschild radius of the central object. In what follows, we use these relations, if necessary.

3. Radiative Equilibrium

We first consider the case of the radiative equilibrium (RE) without heating and cooling. If the radiative equilibrium holds in the whole flow, and there is no heating or cooling, then $j_0 = \kappa_0 c E_0$, and the relativistic moment equation (27) become

$$\begin{aligned} & \frac{\gamma(f - \beta^2)}{f} \frac{dL_0}{d\tau} + \gamma\beta \left(\frac{1 - f^2}{fr} - \frac{1}{f} \frac{df}{dr} \right) D_0 \frac{dr}{d\tau} \\ & + \gamma^3 \left[2\beta \left(1 - \frac{1}{f} \right) L_0 + (1 + f) \left(1 - \frac{\beta^2}{f} \right) D_0 \right] \frac{d\beta}{d\tau} \\ & = -\frac{\beta}{f} L_0, \end{aligned} \quad (32)$$

$$\begin{aligned} & \gamma(f - \beta^2) \frac{dD_0}{d\tau} + \gamma \left[\frac{df}{dr} - \frac{(1 - f)(1 + \beta^2)}{r} \right] D_0 \frac{dr}{d\tau} \\ & + 2\gamma L_0 \frac{d\beta}{d\tau} \\ & = L_0. \end{aligned} \quad (33)$$

These equations (32) and (33) are rather complicated yet, since they include the velocity gradient term and the derivative of the radius, which are connected with the hydrodynamical equations. Of these, except for the central accelerating region, the wind speed weakly depends on the optical depth and is almost constant in the terminal stage. Hence, as already stated, the flow speed β is assumed to be constant in this paper. On the other hand, the radius-derivative term depends on the optical depth. Indeed, it is expressed as

$$\frac{dr}{d\tau} = -\frac{1}{(\kappa_0 + \sigma_0)\rho} = -\frac{r}{\tau} \propto -\tau^{-2}. \quad (34)$$

Instead, the second term on the left-hand side of (33) can be dropped, if we impose the restricted condition on the Eddington factor as

$$\frac{1 - f^2}{r} - \frac{df}{dr} = 0. \quad (35)$$

This equation (35) is easily integrated to give

$$f = \frac{C(\beta)r^2 - 1}{C(\beta)r^2 + 1}, \quad (36)$$

FIGURE 1: Eddington factor f appropriate for the RE case as a function of the optical depth τ for the various values of the flow speed β . The values of β are 0 to 0.9 in steps of from 0.1 from bottom to top.

where $C(\beta)$ is an integration constant, and generally a function of the constant flow speed β . We impose the boundary condition at the core radius such as

$$f = \frac{1 + 3\beta^2}{3 + \beta^2} \quad \text{at } r = r_c, \quad (37)$$

and the appropriate Eddington factor requested to the present case finally becomes

$$f = \frac{2\gamma^2(1 + \beta^2)\hat{r}^2 - 1}{2\gamma^2(1 + \beta^2)\hat{r}^2 + 1} = \frac{2\gamma^2(1 + \beta^2) - \hat{\tau}^2}{2\gamma^2(1 + \beta^2) + \hat{\tau}^2}, \quad (38)$$

where $\hat{r} = r/r_c$ and $\hat{\tau} = \tau/\tau_c$. This variable Eddington factor (38) satisfies the condition $f \rightarrow 1/3$ when $r \rightarrow r_c$ and $\beta \rightarrow 0$, and $f \rightarrow 1$ when $r \rightarrow \infty$ ($\tau \rightarrow 0$) or $\beta \rightarrow 1$. The behavior of this Eddington factor is shown in Figure 1.

Under these restrictive conditions, after several manipulations, (32) and (33) become

$$\frac{dL_0}{d\tau} = -\Gamma L_0, \quad (39)$$

$$\gamma \frac{1}{g} \frac{d}{d\tau} [g(f - \beta^2) D_0] = L_0. \quad (40)$$

In these equations,

$$\Gamma \equiv \frac{\beta}{\gamma(f - \beta^2)} \quad (41)$$

is a function of the flow speed and the optical depth, and it becomes

$$\Gamma = \frac{\gamma\beta}{1 + \beta^2} \frac{2\hat{\tau}^2}{2 - \hat{\tau}^2} + \gamma\beta \quad (42)$$

FIGURE 2: Comoving luminosity for relativistic spherical flows in the RE case without heating and cooling. The values of β are from 0 to 0.9 in steps of 0.1 from top to bottom. The optical depth τ_c at the core radius is set to be 10.

for the Eddington factor (38), while g is the curvature factor defined by

$$\ln g \equiv - \int_{\tau_c}^{\tau} \frac{(1-f)(1+\beta^2)}{(f-\beta^2)r} \frac{dr}{d\tau} d\tau' \quad (43)$$

and becomes in the present case

$$g = \frac{\hat{\tau}^3}{2 - \hat{\tau}^2}. \quad (44)$$

Since the index Γ is analytically expressed by the optical depth, the differential equation (39) can analytically integrate to give the comoving luminosity L_0 . Imposing the boundary condition of L_s at $\tau = 0$, we finally have the comoving luminosity for the RE case:

$$\frac{L_0}{L_s} = \left(\frac{\sqrt{2} - \hat{\tau}}{\sqrt{2} + \hat{\tau}} \right)^b \exp \left(\frac{1}{\sqrt{2}\gamma^2} b \hat{\tau} \right), \quad (45)$$

where

$$b = \frac{\sqrt{2}\gamma\beta}{1 + \beta^2} \tau_c. \quad (46)$$

The analytical solutions of the comoving luminosity (45) are shown in Figure 2 as a function of the optical depth for several values of the flow speed. The values of β are from 0 to 0.9 in steps of 0.1.

Although the comoving luminosity (45) has an exponential term, the power-law behavior is dominant in this case. In the nonrelativistic limit of $\beta \rightarrow 0$, $b \rightarrow 0$ and the solution reduces to

$$\frac{L_0}{L_s} \sim 1 - \frac{b}{\sqrt{2}} \hat{\tau} \sim 1 - \beta\tau. \quad (47)$$

In the extremely relativistic limit of $\beta \rightarrow 1$, on the other hand, $b \rightarrow \gamma\tau_c/\sqrt{2}$ and the solution reduces to

$$\frac{L_0}{L_s} \sim \left(\frac{\sqrt{2} - \hat{\tau}}{\sqrt{2} + \hat{\tau}} \right)^b. \quad (48)$$

In contrast to this comoving luminosity, it is still difficult to obtain analytical solutions of the spherical radiation energy density D_0 . Even in the extremely relativistic limit, we cannot obtain the analytical solution for D_0 .

4. Local Thermodynamic Equilibrium

Next, we consider the case of the local thermodynamic equilibrium (LTE) with a uniform source function. If the local thermodynamic equilibrium (LTE) holds in the comoving frame,

$$\frac{j_0}{4\pi} = \kappa_0 B_0, \quad (49)$$

where $B_0 (= \sigma T_0^4/\pi)$ is the frequency-integrated blackbody intensity in the comoving frame, T_0 being the blackbody temperature and generally a function of the height r or the optical depth τ , but assumed to be constant in what follows.

In this case the relativistic moment equation (27) become

$$\begin{aligned} & \frac{\gamma(f - \beta^2)}{f} \frac{dL_0}{d\tau} + \gamma\beta \left(\frac{1 - f^2}{fr} - \frac{1}{f} \frac{df}{dr} \right) D_0 \frac{dr}{d\tau} \\ & + \gamma^3 \left[2\beta \left(1 - \frac{1}{f} \right) L_0 + (1 + f) \left(1 - \frac{\beta^2}{f} \right) D_0 \right] \frac{d\beta}{d\tau} \\ & = - \frac{\kappa_0}{\kappa_0 + \sigma_0} (W_0 - D_0) - \frac{\beta}{f} L_0, \\ & \gamma(f - \beta^2) \frac{dD_0}{d\tau} + \gamma \left[\frac{df}{dr} - \frac{(1 - f)(1 + \beta^2)}{r} \right] \\ & \times D_0 \frac{dr}{d\tau} + 2\gamma L_0 \frac{d\beta}{d\tau} \\ & = L_0 + \beta \frac{\kappa_0}{\kappa_0 + \sigma_0} (W_0 - D_0), \end{aligned} \quad (50)$$

where

$$W_0 \equiv 16\pi^2 r^2 B_0 \quad (51)$$

is the spherical source function.

FIGURE 3: Eddington factor f appropriate for the LTE case as a function of the optical depth τ for the various values of the flow speed β . The values of β are from 0 to 0.9 in steps of 0.1 from bottom to top.

These equations (50) and (51) can be rearranged as

$$\begin{aligned}
 & \frac{\gamma(f - \beta^2)}{f} \frac{dL_0}{d\tau} + \left[\frac{\gamma\beta}{f} \left(\frac{1 - f^2}{r} - \frac{df}{dr} \right) \frac{dr}{d\tau} - \frac{\kappa_0}{\kappa_0 + \sigma_0} \right] D_0 \\
 & + \gamma^3 \left[2\beta \left(1 - \frac{1}{f} \right) L_0 + (1 + f) \left(1 - \frac{\beta^2}{f} \right) D_0 \right] \frac{d\beta}{d\tau} \\
 & = - \frac{\kappa_0}{\kappa_0 + \sigma_0} W_0 - \frac{\beta}{f} L_0, \\
 & \gamma(f - \beta^2) \frac{dD_0}{d\tau} \\
 & + \left\{ \gamma \left[\frac{df}{dr} - \frac{(1 - f)(1 + \beta^2)}{r} \right] \frac{dr}{d\tau} + \beta \frac{\kappa_0}{\kappa_0 + \sigma_0} \right\} \\
 & \times D_0 + 2\gamma L_0 \frac{d\beta}{d\tau} \\
 & = L_0 + \beta \frac{\kappa_0}{\kappa_0 + \sigma_0} W_0.
 \end{aligned} \tag{52}$$

Equation (52) is yet too complicated to solve analytically.

Hence, in order to simplify these equations by dropping the second terms on the left-hand sides of equation (52), we impose the following two conditions:

$$\begin{aligned}
 & \frac{\gamma\beta}{f} \left(\frac{1 - f^2}{r} - \frac{df}{dr} \right) \frac{dr}{d\tau} - \frac{\kappa_0}{\kappa_0 + \sigma_0} = 0, \\
 & \gamma \left[\frac{df}{dr} - \frac{(1 - f)(1 + \beta^2)}{r} \right] \frac{dr}{d\tau} + \beta \frac{\kappa_0}{\kappa_0 + \sigma_0} = 0.
 \end{aligned} \tag{53}$$

Eliminating $\kappa_0/(\kappa_0 + \sigma_0)$ from (53), we obtain the differential equation for the variable Eddington factor f ,

$$\frac{df}{dr} + \frac{f - 1}{r} = 0, \tag{54}$$

as long as $dr/d\tau \neq 0$.

This equation (54) is easily integrated to give

$$f = 1 - \frac{C(\beta)}{\hat{r}}, \tag{55}$$

where $C(\beta)$ is an integration constant and generally a function of the constant flow speed β . We impose the boundary condition at the core radius such as

$$f = \frac{1 + 3\beta^2}{3 + \beta^2} \quad \text{at } r = r_c, \tag{56}$$

and the appropriate Eddington factor requested to the present case finally becomes

$$f = 1 - \frac{2(1 - \beta^2)}{3 + \beta^2} \frac{1}{\hat{r}} = 1 - \frac{2(1 - \beta^2)}{3 + \beta^2} \hat{\tau}, \tag{57}$$

where $\hat{r} = r/r_c$ and $\hat{\tau} = \tau/\tau_c$. This variable Eddington factor (57) satisfies the condition: $f \rightarrow 1/3$ when $r \rightarrow r_c$ and $\beta \rightarrow 0$, and $f \rightarrow 1$ when $r \rightarrow \infty$ ($\tau \rightarrow 0$) or $\beta \rightarrow 1$. The behavior of this Eddington factor is shown in Figure 3.

Under these restrictive conditions, after several manipulations, equation (52) becomes

$$\frac{dL_0}{d\tau} = -\Gamma L_0 - \Delta \frac{\kappa_0}{\kappa_0 + \sigma_0} W_0, \tag{58}$$

$$\frac{dD_0}{d\tau} = \frac{\Gamma}{\beta} L_0 + \Gamma \frac{\kappa_0}{\kappa_0 + \sigma_0} W_0, \tag{59}$$

where

$$\Gamma \equiv \frac{\beta}{\gamma(f - \beta^2)} = \frac{\gamma\beta(3 + \beta^2)}{3 + \beta^2 - 2\hat{\tau}}, \tag{60}$$

$$\Delta \equiv \frac{f}{\gamma(f - \beta^2)} = \gamma \frac{3 + \beta^2 - 2(1 - \beta^2)\hat{\tau}}{3 + \beta^2 - 2\hat{\tau}},$$

respectively, in the present case.

Since the index Γ is analytically expressed by the optical depth, the solution of the homogeneous part of (58), where W_0 is set to be 0, is analytically obtained as

$$\frac{L_0}{L_s} = (1 - p\hat{\tau})^q, \tag{61}$$

where

$$p \equiv \frac{2}{3 + \beta^2}, \tag{62}$$

$$q \equiv \frac{\gamma\beta(3 + \beta^2)}{2} \tau_c.$$

FIGURE 4: Comoving luminosity for relativistic spherical flows in the LTE case with a uniform source function. The values of β are from 0 to 0.9 in steps of 0.1 from top to bottom. The optical depth τ_c at the core radius is set to be 10, and the value of the source function $[\kappa_0/(\kappa_0 + \sigma_0)]W_0/L_s$ is set to be unity. The solid curves represent the present analytical solutions, while the dashed ones mean the solutions of the homogeneous part.

When the spherical source function W_0 is uniform and $\kappa_0/(\kappa_0 + \sigma_0)$ is also constant, the analytical solution of (58) can be obtained after some manipulations as

$$\frac{L_0}{L_s} = (1 - p\hat{\tau})^q + \frac{\gamma\tau_c}{1-q} \left[\left(\frac{3+\beta^2}{2} - \frac{\beta}{\gamma\tau_c} \right) - \frac{\hat{\tau}}{\gamma^2} \right] \frac{\kappa_0}{\kappa_0 + \sigma_0} \frac{W_0}{L_s}. \quad (63)$$

The analytical solutions of the comoving luminosity (63) are shown in Figure 4 as a function of the optical depth for several values of the flow speed. The values of β are from 0 to 0.9 in steps of 0.1.

In the LTE case the comoving luminosity (63) has the power-law form. In the nonrelativistic limit of $\beta \rightarrow 0$, $p \rightarrow 2/3$, and $q \rightarrow 3\beta\tau_c/2$, and the solution becomes a linear function of τ . In the extremely relativistic limit of $\beta \rightarrow 1$, on the other hand, $p \rightarrow 1/2$ and $q \rightarrow 2\gamma\tau_c$, the solution reduces to

$$\frac{L_0}{L_s} \sim (1 - p\hat{\tau})^q - \frac{\kappa_0}{\kappa_0 + \sigma_0} \frac{W_0}{L_s}. \quad (64)$$

In contrast to the RE case, we can obtain the analytical solutions of the spherical radiation energy density D_0 . However, it is rather complicated, and we omit the expression for D_0 .

5. Concluding Remarks

In this paper, we have examined the relativistic radiative transfer in the relativistic spherical flows in the fully special

relativistic treatment. Under the assumption of a constant flow speed and using a variable Eddington factor $f(\tau, \beta)$, we have analytically solved the relativistic moment equations written in the comoving frame for RE and LTE cases and found new analytical solutions for several restricted situations. In both RE and LTE cases, the radiative flux decreases with the optical depth in the power-law manner, while the radiative flux has the exponential behavior in the plane-parallel case [40].

We here clarify the *essential difference* for the relativistic radiative transfer between the plane-parallel and spherical cases; the former is the exponential type, and the latter is the power-law manner. Since the original transfer equation is the linear differential equation, the exponential behavior is natural, but there arised two different types. This essential difference is roughly understood as follows.

In the relativistic plane-parallel flow [40], where we have assumed the constant flow speed, the density is also constant; there is no expansion effect. The index Γ is also constant. In this case, the natural exponential behavior emerges and the analytical solutions exhibit the exponential behavior on the optical depth. In the relativistic spherical flow in the present case, where we have also assumed the constant flow speed, the density decreases as the radius increases due to the geometrical effect; there is an expansion effect. The index Γ is no longer constant but varies as a function of r (or τ). As a result, the natural exponential behavior is lost, and the analytical solutions exhibit the power-law behavior.

From the view point of the background density variation, this difference is somewhat similar to the growth of the density fluctuation of the gravitational instability in the static interstellar space and the expanding universe. Namely, in the static background, where the background density is constant, the density fluctuation increases exponentially [52], whereas it increases in a power-law manner in the expanding universe, where the background density decreases with time [53]. Hence, we can guess that, even in the plane-parallel case, there may be power-law type solutions if the flow is accelerated and the density decreases as the optical depth decreases.

In order to research the physical problem, the analytical approach has several advantages. First, the analytical solutions can often reveal the essential properties of the radiative transfer problem. In the present case, we can clarify the exponential versus power-law type behavior and its causes. Secondly, they can clarify the restrictions of the assumptions and/or crucial problems inherent in the formalism. In the present case, in order to avoid the critical point in the basic equations with a traditional constant Eddington factor, we use a variable Eddington factor, which approaches unity in the limit of $\tau \rightarrow 0$ or $\beta \rightarrow 1$ (cf. [24, 54]). Finally, they can help us to check the precision and the validity of the numerical code for the radiative transfer problem. Particularly, in the recent research of the radiation transfer problem on the black-hole accretion using the ART code [36], the FLD approximation often adopted in the radiation hydrodynamical simulations cannot reproduce the radiative force in the optically thin region. Hence, the new numerical method should be developed for the multidimensional

radiation hydrodynamical simulations, and the analytical solutions like the present case would be useful for such codes in the future.

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