Research Article

Nonlinear Stability of the Triangular Libration Points for Radiating and Oblate Primaries in CR3BP in Nonresonance Condition

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This paper investigates the existence of resonance and nonlinear stability of the triangular equilibrium points when both oblate primaries are luminous. The study is carried out near the resonance frequency, satisfying the conditions $\omega_1 = \omega_2$, $\omega_1 = 2\omega_2$, and $\omega_1 = 3\omega_2$ in circular cases by the application of Kolmogorov-Arnold-Moser (KAM) theory. The study is carried out for the various values of radiation pressure and oblateness parameters in general. It is noticed that the system experiences resonance at $\omega_1 = 2\omega_2$, $\omega_1 = 3\omega_2$ for different values of radiation pressures and oblateness parameter. The case $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system. It is found that, except for some values of the radiation pressure, and oblateness parameters and for $\mu \leq \mu_c = 0.0385209$, the triangular equilibrium points are stable.

1. Introduction

The three-body problem, which describes three masses interacting through Newtonian gravity, has attracted many scientists for more than 300 years. The three-body problem is one of the most challenging problems in the history of science. In celestial mechanics the general three-body problem deals with gravitationally interacting astronomical bodies and intends to predict their motions. The restricted three-body problem (R3BP) is a particular case of the general three-body problem. The problem is restricted in the sense that the infinitesimal mass does not influence the motion of two gravitating primaries but is rather influenced by them. In R3BP, there are two possibilities; namely, the two bodies with dominant masses move around their common centre of mass along either circular or elliptic orbits, which lead to the respective circular or elliptic restricted three-body problems (ER3BP). The ER3BP describes the three-dimensional motion of small particle called the infinitesimal mass under the gravitational force of two finite bodies called primaries around their common centre of mass. The

ER3BP generalizes the original circular restricted three-body problems, while some useful properties of circular model still can be satisfied in the elliptical case. The ER3BP describes the dynamical system more accurately as the primaries move along the elliptical orbit.

Modern applications of the three-body problem and restricted three-body problem have been extended to include the Earth, the Moon, and the artificial satellites as well as recently discovered exoplanets. The study of the stability of an elliptical or circular restricted three-body problem of the Hamiltonian system is generally performed by using Kolmogorov-Arnold-Moser (KAM) theorem. The KAM theory is very useful for studying the global stability in three-body problem. The KAM theorem is suitable for the motion undergoing small perturbations which preserves the features of the unperturbed motion.

The nonlinear stability for resonance as well as for the nonresonance cases of the triangular libration points, taking one of the bodies as radiating, was studied by Manju and Choudhry [1]. Kumar and Choudhry [2] investigated the stability of the triangular libration points for nonresonance

as well as resonance case, taking both the bodies as radiating in circular restricted three-body problem in the presence of the third and fourth order resonance. Bhatnagar et al. [3] discussed the nonlinear stability of the triangular equilibrium points in circular restricted three bodies, considering bigger primary as a source of radiation. The nonlinear stability of the triangular Lagrangian points, considering the bigger primary as oblate spheroid in circular case, was examined by Markellos et al. [4]. Recently Narayan and Singh [5] studied the nonlinear stability of higher order for both radiating primaries and found that binary systems are stable.

The detailed description and the behavior of equilibrium points in ER3BP are touched upon by Danby [6], Bennett [7], Szebevely [8], Markeev [9], Selaru and Cucu-Dumitrescu [10], and Hallan and Rana [11]. The influence of the eccentricity of the orbits of the primaries with or without radiation pressure on the existence and stability of the equilibrium points was studied by Györgyey [12], Grebenikov [13], Kumar and Choudhry [2, 14], Markellos [4, 15], Sahoo and Ishwar [16], Roberts [17], Zimovshchikov and Tkhai [18], Ammar [19], Érdi et al. [20], Kumar and Ishwar [21], Singh and Umar [22, 23], Usha et al. [24], and Narayan and Singh [5, 25, 26].

It may be noted that the case when the frequencies are equal to zero or are equal to each other usually corresponds to the boundary of stability of the linear system, since, in the absence of oblateness parameters and radiation pressures the critical value of mass ratio denoted by μ_c is 0.0385209. Also, if it is considered that $\omega_1 > \omega_2 > 0$, then it is found that the inequality $n_1\omega_1 + n_2\omega_2 \neq 0$, provided $|n_1| + |n_2| \leq 4$ (n_j are integers), is invalid for $\omega_1 = 2\omega_2$, $\omega_1 = 3\omega_2$ which gives rise to the resonance cases.

The present paper investigates the existence of the resonance and the stability of the infinitesimal mass about the triangular equilibrium points by taking both primaries as radiating and oblate in nonresonance condition. The existence of the resonance and the stability of infinitesimal near the resonance frequency satisfying the conditions $\omega_1 = \omega_2$, $\omega_1 = 2\omega_2$, $\omega_1 = 3\omega_2$ are studied in the circular cases. The study is carried out at various values of radiation pressures and oblateness parameters.

This paper has been organized in various sections.

Section 1 gives introduction, Section 2 describes the equations of motion of the problem, and Section 3 deals with the characteristics roots and first order stability of the triangular equilibrium points. The existence of resonance is discussed in Section 4, while Section 5 deals with normalization and higher order stability of the libration points. Finally Section 6 summarizes the discussion and conclusion of the paper.

2. Equations of Motion

The differential equations of the motion of the infinitesimal mass in elliptical restricted three-body problem under radiating primaries in pulsating system as given by Narayan and Shrivastava [27] are

$$\begin{aligned} x'' - 2y' &= \phi\Omega_x; \\ y'' - 2x' &= \phi\Omega_y, \end{aligned} \quad (1)$$

where the force function is defined as

$$\begin{aligned} \Omega &= \frac{x^2 + y^2}{2} + \frac{1}{1 + 3((A_1 + A_2)/2)} \left[\frac{(1 - \mu)Q_1}{r_1} \right. \\ &\quad \left. + \frac{\mu Q_2}{r_2} + \frac{(1 - \mu)Q_1 A_1}{2r_1^3} + \frac{\mu A_2 Q_2}{2r_2^3} \right], \\ r_1^2 &= (x + \mu)^2 + y^2; \\ r_2^2 &= (x - 1 + \mu)^2 + y^2; \\ \phi &= \frac{1}{(1 + e \cos f)}, \end{aligned} \quad (2)$$

where $Q_i = 1 - \beta_i$ ($i = 1, 2$) is the radiation pressure, f is a true anomaly of the primaries, A_1, A_2 are the oblateness parameter, e is the eccentricity of the orbits, and μ is the mass ratio defined as

$$\begin{aligned} \mu &= \frac{m_2}{m_1 + m_2}; \\ 1 - \mu &= \frac{m_1}{m_1 + m_2}; \\ 0 &\leq \mu \leq \frac{1}{2}. \end{aligned} \quad (3)$$

The coordinates of the triangular equilibrium points L_4 and L_5 as given by Narayan and Shrivastava [27] are

$$\begin{aligned} x_0 &= \frac{1}{2} - \mu + \frac{A_1}{2} - \frac{A_2}{2} + \frac{\beta_2}{3} - \frac{\beta_1}{3} - \frac{1}{2}A_1\beta_1 + \frac{1}{2} \\ &\quad \cdot A_2\beta_2; \\ y_0 &= \pm \frac{\sqrt{3}}{2} \left[1 - \frac{A_1}{3} - \frac{A_2}{3} - \frac{2\beta_1}{9} - \frac{2\beta_2}{9} - \frac{1}{3}A_1\beta_1 \right. \\ &\quad \left. - \frac{1}{3}A_2\beta_2 \right]. \end{aligned} \quad (4)$$

3. Characteristics Roots and First Order Stability of the Triangular Equilibrium Points

The stability of the elliptical restricted three-body problem is restricted to planar case only. Since the nature of stability about L_4 and L_5 is similar it is sufficient to study the stability only about L_4 . The Hamiltonian as described by Narayan and Shrivastava [27] is given by

$$\begin{aligned} H &= \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \frac{(q_1^2 + q_2^2)e \cos f}{2(1 + e \cos f)} \\ &\quad - \frac{1}{(1 + e \cos f)(1 + 3(A_1 + A_2)/2)} \left\{ \left[\frac{(1 - \mu)Q_1}{r_1} \right. \right. \\ &\quad \left. \left. + \frac{\mu Q_2}{r_2} + \frac{(1 - \mu)Q_1 A_1}{2r_1^3} + \frac{\mu A_2 Q_2}{2r_2^3} \right] \right\}, \end{aligned} \quad (5)$$

where p_1, p_2, q_1, q_2 are the variations in the coordinates L_4 .

By substituting $Q_i = 1 - \beta_i$ ($i = 1, 2$) we have

$$H = \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \frac{(q_1^2 + q_2^2) e \cos f}{2(1 + e \cos f)} - \frac{1}{(1 + e \cos f)(1 + 3(A_1 + A_2)/2)} \left\{ \frac{(1 - \mu)(1 - \beta_1)}{r_1} + \frac{\mu(1 - \beta_2)}{r_2} + \frac{(1 - \mu)(1 - \beta_1)A_1}{2r_1^3} + \frac{\mu A_2(1 - \beta_2)}{2r_2^3} \right\}. \quad (6)$$

Now expand the Hamiltonian function as

$$H = \sum_{K=0}^{\infty} H_K = H_0 + H_1 + H_2 + H_3 + \dots + H_4 + \dots, \quad (7)$$

where H_K is the sum of the terms of the K th degree which is homogeneous in the variables p_1, p_2, q_1, q_2 . H_0 is constant and $H_1 = 0$. Consider

$$H_2 = \frac{p_1^2 + p_2^2}{2} + (p_1 q_2 - p_2 q_1) + \frac{e \cos f}{2(1 + e \cos f) \{1 + 3((A_1 + A_2)/2)\}} (q_1^2 + q_2^2) - \frac{1}{2(1 + e \cos f)} (H_{20} q_1^2 - H_{11} q_1 q_2 - H_{02} q_2^2), \quad (8)$$

where

$$H_{20} = -\left(\frac{-1}{8} + \frac{9}{8} A_1 - \frac{9}{16} A_2 - \frac{73}{24} A_1 \beta_1 + \frac{11}{48} A_2 \beta_2 - \frac{27}{16} \mu A_1 + \frac{27}{16} \mu A_2 + \frac{17}{8} \mu \beta_1 - \frac{7}{9} \mu \beta_2 + \frac{\beta_2}{2} - \frac{13\beta_1}{8} \right), \quad (9)$$

$$H_{11} = \sqrt{3} \left(\frac{3}{4} + \frac{17}{4} A_1 + \frac{7}{8} A_2 + \frac{139}{24} A_1 \beta_1 + \frac{15}{8} A_2 \beta_2 - \frac{41}{8} \mu A_1 - \frac{41}{8} \mu A_2 - \frac{1}{6} \mu \beta_1 + \mu \beta_2 - \frac{3\mu}{2} - \frac{\beta_2}{3} - \frac{7\beta_1}{12} \right), \quad (10)$$

$$H_{02} = \left(\frac{5}{8} + \frac{27}{8} A_1 + \frac{21}{16} A_2 + \frac{107}{16} A_1 \beta_1 + \frac{11}{16} A_2 \beta_2 - \frac{33}{8} \mu A_1 + \frac{33}{16} \mu A_2 - \frac{7}{8} \mu \beta_1 + \frac{11}{8} \mu \beta_2 - \frac{\beta_2}{2} + \frac{7\beta_1}{8} \right), \quad (11)$$

$$H_3 = \frac{-1}{6(1 + e \cos f) [1 + 3(A_1 + A_2)/2]} \left[q_1^3 \left(\frac{21}{8} + \frac{15A_1}{4} - \frac{295}{16} A_2 - \frac{93\beta_1}{4} - \frac{65\beta_2}{6} - \frac{365\beta_1 A_1}{24} - \frac{155\beta_1 A_2}{16} + \mu \left(-\frac{57}{8} - \frac{15A_1}{4} - \frac{305A_2}{16} - \frac{29\beta_1}{4} + \frac{51\beta_2}{8} - \frac{365\beta_1 A_1}{24} + \frac{85\beta_1 A_2}{16} - \frac{17\beta_2 A_1}{8} - \frac{5\beta_2 A_2}{3} \right) \right) + 3q_1^2 q_2 \left(\frac{-3}{8} + \frac{21A_1}{8} - \frac{61}{8} A_2 - \frac{113\beta_1}{24} + \frac{7\beta_2}{4} - \frac{53\beta_1 A_1}{24} + \frac{61\beta_1 A_2}{8} + \mu \sqrt{3} \left(\frac{17A_1}{16} + \frac{9A_2}{16} + \frac{47\beta_1}{6} - \frac{13\beta_2}{4} - \frac{53\beta_1 A_1}{24} - \frac{113\beta_1 A_2}{24} - \frac{91\beta_2 A_1}{16} - \frac{\beta_2 A_2}{48} \right) \right) + 3q_1 q_2^2 \left(\frac{-33}{8} - \frac{315A_1}{8} + \frac{5}{12} A_2 + \frac{149\beta_1}{24} + \frac{5\beta_2}{8} - \frac{265\beta_1 A_1}{8} - \frac{5\beta_1 A_2}{12} + \mu \left(\frac{33}{4} - \frac{245A_1}{8} + \frac{263A_2}{8} + \frac{77\beta_1}{48} - \frac{311\beta_2}{16} - \frac{55\beta_1 A_1}{4} + \frac{3805\beta_1 A_2}{96} + \frac{55\beta_2 A_1}{8} - \frac{1441\beta_2 A_2}{16} \right) \right) + q_2^3 \left(\frac{-9}{8} - \frac{51A_1}{4} - \frac{69}{16} A_2 - \frac{13\beta_1}{8} - \frac{11\beta_2}{4} + \frac{151\beta_1 A_1}{16} + \frac{65\beta_1 A_2}{16} + \mu \left(\frac{195A_1}{16} - \frac{33A_2}{2} + \frac{81\beta_1}{16} + \frac{31\beta_2}{8} - \frac{151\beta_1 A_1}{16} + \frac{117\beta_1 A_2}{32} - \frac{317\beta_2 A_1}{32} + \frac{5\beta_2 A_2}{4} \right) \right) \right], \quad (12)$$

$$H_4 = \frac{-1}{24(1 + e \cos f) \left[1 + \frac{3(A_1 + A_2)}{2} \right]} \left[q_1^4 \left(\frac{-111}{16} - \frac{1155A_1}{16} + \frac{9815}{16} A_2 + \frac{397\beta_1}{2} - \frac{25\beta_2}{2} - \frac{1235\beta_1 A_1}{16} - \frac{1185\beta_1 A_2}{2} + \mu \left(-\frac{1575}{16} - \frac{395A_1}{48} - \frac{29545A_2}{48} + \frac{9269\beta_1}{16} + \frac{439\beta_2}{144} - \frac{1234\beta_1 A_1}{16} - \frac{6325\beta_1 A_2}{12} + \frac{6755\beta_2 A_1}{48} + \frac{605\beta_2 A_2}{48} \right) \right) + 4\sqrt{3} q_1^3 q_2 \left(\frac{-75}{16} - \frac{281A_1}{16} - \frac{21}{2} A_2 + \frac{5\beta_1}{16} + \frac{21\beta_2}{6} + \frac{351\beta_1 A_1}{16} + \frac{21\beta_1 A_2}{2} + \mu \left(\frac{75}{8} \right. \right. \right.$$

$$\begin{aligned}
& + \frac{5669A_1}{208} - \frac{117A_2}{8} - \frac{235\beta_1}{48} - \frac{6121\beta_2}{560} - \frac{7\beta_1A_1}{2} \\
& + \frac{319\beta_1A_2}{32} + \frac{2357\beta_2A_1}{156} - \frac{83\beta_2A_2}{4} \Big) \Big) \\
& + 6q_1^2q_2^2 \left(\frac{123}{16} + \frac{4865A_1}{48} - \frac{105}{8}A_2 + \frac{235\beta_1}{12} \right. \\
& + \frac{75\beta_2}{12} - \frac{4295\beta_1A_1}{48} + \frac{105\beta_1A_2}{8} + \mu \left(-\frac{123}{8} \right. \\
& - \frac{5735A_1}{48} + \frac{1601A_2}{16} - \frac{421\beta_1}{48} - \frac{31\beta_2}{8} \\
& + \frac{4865\beta_1A_1}{48} + \frac{245\beta_1A_2}{32} + \frac{45\beta_2A_1}{8} \\
& \left. \left. - \frac{539\beta_2A_2}{4} \right) \right) + 4\sqrt{3}q_1q_2^3 \left(\frac{135}{16} + \frac{1531A_1}{16} + \frac{15}{4} \right. \\
& \cdot A_2 - \frac{69\beta_1}{16} - \frac{21\beta_2}{5} + \frac{23\beta_1A_1}{2} - \frac{15\beta_1A_2}{4} \\
& + \mu \left(-\frac{135}{8} - \frac{337A_1}{2} - \frac{8997A_2}{112} + \frac{259\beta_1}{16} \right. \\
& + \frac{293\beta_2}{40} + \frac{1391\beta_1A_1}{16} - \frac{3005\beta_1A_2}{32} + \frac{1315\beta_2A_1}{116} \\
& \left. \left. - \frac{175\beta_2A_2}{48} \right) \right) + q_2^4 \left(\frac{9}{16} + \frac{1125A_1}{16} - \frac{1575}{32}A_2 \right. \\
& + \frac{1781\beta_1}{16} - \frac{45\beta_2}{2} - \frac{1045\beta_1A_1}{8} + \frac{1575\beta_1A_2}{32} \\
& + \mu \left(-\frac{495A_1}{8} - \frac{3167A_2}{32} - \frac{1917\beta_1}{16} + 26\beta_2 \right. \\
& + \frac{1045\beta_1A_1}{8} - \frac{14765\beta_1A_2}{96} - \frac{1325\beta_2A_1}{32} \\
& \left. \left. - \frac{6775\beta_2A_2}{48} \right) \right) \Big) \Big].
\end{aligned} \tag{13}$$

Restricting to H_2 alone, the characteristics equation can be given in the form

$$\begin{vmatrix} \lambda^2 - (1 - 2H_{20}) & -2\lambda - H_{11} \\ 2\lambda - H_{11} & \lambda^2 - (1 + 2H_{02}) \end{vmatrix} = 0, \tag{14}$$

FIGURE 1: Correlation between μ and ω_1 and ω_2 with $\beta_1 = 0.01$, $\beta_2 = 0.02$, $A_1 = 0.01$, and $A_2 = 0.02$.

where H_{20} , H_{11} , and H_{02} are given by ((9), (10), and (11)). After further calculations the characteristics equation reduces to the form

$$\begin{aligned}
& \lambda^4 + \lambda^2 + \left\{ \frac{27\mu(1-\mu)}{4} \right\} \\
& \cdot \left\{ 1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right\} = 0.
\end{aligned} \tag{15}$$

If ω_1 and ω_2 are the frequencies then putting $\lambda^2 = -\omega^2$ in (15), roots can be written as

$$\begin{aligned}
\omega_1^2 &= -\lambda_{1,2}^2 = \frac{1}{2} \left[1 + \left\{ 1 - 27\mu(1-\mu) \right. \right. \\
& \cdot \left. \left. \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \right\}^{1/2} \right], \\
\omega_2^2 &= -\lambda_{3,4}^2 = \frac{1}{2} \left[1 - \left\{ 1 - 27\mu(1-\mu) \right. \right. \\
& \cdot \left. \left. \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \right\}^{1/2} \right].
\end{aligned} \tag{16}$$

The correlation between μ and ω_1 and ω_2 is shown in Figures 1–5. It is found that for fixed values of radiation pressure ω_1 increases with an increase in μ whereas ω_2 decreases.

4. Existence of Resonance in Circular Cases

In order to discuss the existence of resonance, we consider the following three cases.

Case 1. The first case is when $\omega_1 = \omega_2$; that is,

$$\lambda_{1,2}^2 = \lambda_{3,4}^2. \quad (17)$$

Solving the above equation we obtain

$$1 - 27\mu(1 - \mu) \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) = 0; \quad (18)$$

that is,

$$27 \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \mu^2 - 27 \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \mu + 1 = 0. \quad (19)$$

Solving for μ we obtain

$$\mu = \frac{9 \pm \sqrt{69} (1 - (125/69)A_1 + (80/69)A_2 + (5/69)\beta_1 - (35/207)\beta_2)}{18}. \quad (20)$$

Since $\mu \leq 1/2$, the positive sign is inadmissible. Hence the region of stability in the first approximation can be written as

$$0 < \mu < \frac{\{9 - \sqrt{69} (1 - (125/69)A_1 + (80/69)A_2 + (5/69)\beta_1 - (35/207)\beta_2)\}}{18}. \quad (21)$$

Thus, the value of μ responsible for stable equilibrium points is given by

$$\mu_c = 0.0385208965 + 0.83601A_1 + 0.535048A_2 - 0.0334405\beta_1 + 0.0780279\beta_2. \quad (22)$$

It is clear from (22) that in the absence of radiation pressures and oblateness parameters the critical value of mass ratio is 0.0385208965 when $\omega_1 = \omega_2$, which usually corresponds to a boundary of the region of stability of the system.

Case 2. It is when $\omega_1 = 2\omega_2$; that is,

$$\lambda_{1,2}^2 = 4\lambda_{3,4}^2; \quad (23)$$

that is,

$$\left[1 + \left\{ 1 - 27\mu(1 - \mu) \cdot \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \right\}^{1/2} \right]$$

Case 3. It is when $\omega_1 = 3\omega_2$; that is,

$$\lambda_{1,2}^2 = 9\lambda_{3,4}^2; \quad (26)$$

that is,

$$\left[1 + \left\{ 1 - 27\mu(1 - \mu) \cdot \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \right\}^{1/2} \right] = 9 \left[1 - \left\{ 1 - 27\mu(1 - \mu) \cdot \left(1 - \frac{125}{6}A_1 + \frac{40}{3}A_2 + \frac{5}{6}\beta_1 - \frac{35}{18}\beta_2 \right) \right\}^{1/2} \right]. \quad (27)$$

FIGURE 2: Correlation between μ and ω_1 and ω_2 with $\beta_1 = 0.01$, $\beta_2 = 0.02$, $A_1 = 0.03$, and $A_2 = 0.04$.

FIGURE 5: Correlation between μ and ω_1 and ω_2 with $\beta_1 = 0.01$, $\beta_2 = 0.02$, $A_1 = 0.001$, and $A_2 = 0.002$.

FIGURE 3: Correlation between μ and ω_1 and ω_2 with $\beta_1 = 0.01$, $\beta_2 = 0.02$, $A_1 = 0.04$, and $A_2 = 0.05$.

FIGURE 4: Correlation between μ and ω_1 and ω_2 with $\beta_1 = 0.01$, $\beta_2 = 0.02$, $A_1 = 0.05$, and $A_2 = 0.06$.

FIGURE 6: Correlation between μ and A_1 and $\beta_1 = 0.01$, $\beta_2 = 0.02$, and $A_2 = 0.01$.

Solving the above equation for μ the resonance value is obtained as

$$\mu_{03} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{9}{675(1 - 125A_1/6 + 40A_2/3 + 5\beta_1/6 - 35\beta_2/18)}} \quad (28)$$

Table 1 shows the values of μ corresponding to $\omega_1 = \omega_2$, $\omega_1 = 2\omega_2$, and $\omega_1 = 3\omega_2$. Figures 6–10 show correlation between μ and A_1 by varying β_1 , β_2 , and A_2 . Correlation between μ and β_1 is depicted in Figure 11 by varying β_2 and taking $A_1 = 0.01$ and $A_2 = 0.02$. Figure 12 shows combined figures between μ and β_1 by taking $\beta_2 = 0.01$, $A_1 = 0.01$, and $A_2 = 0.01, 0.02, 0.03$.

TABLE 1: Values of μ_{01} , μ_{02} , and μ_{03} for different values of radiation pressure and oblateness parameter.

β_1	β_2	A_1	A_2	μ_{01}	μ_{02}	μ_{03}
0.01	0.01	0.01	0.02	0.0367	0.032	0.0129
0.01	0.01	0.02	0.03	0.0397	0.0250	0.0139
0.01	0.01	0.04	0.05	0.0473	0.0297	0.0165
0.01	0.02	0.01	0.02	0.0374	0.0236	0.0131
0.01	0.02	0.02	0.03	0.0405	0.0255	0.0142
0.01	0.02	0.04	0.05	0.0485	0.0305	0.0169
0.01	0.03	0.01	0.02	0.0367	0.0232	0.0129
0.01	0.03	0.02	0.03	0.0441	0.0278	0.0154
0.01	0.03	0.04	0.05	0.0498	0.0312	0.173
0.01	0.04	0.01	0.02	0.0390	0.0246	0.0137
0.01	0.04	0.02	0.03	0.0423	0.0266	0.0148
0.01	0.04	0.04	0.05	0.0511	0.0321	0.0178
0.01	0.05	0.01	0.02	0.0398	0.0251	0.0139
0.01	0.05	0.02	0.03	0.0433	0.0272	0.0151
0.01	0.05	0.04	0.05	0.0525	0.0329	0.0182
0.01	0.06	0.01	0.02	0.0406	0.0256	0.0142
0.01	0.06	0.02	0.03	0.0443	0.0279	0.0155
0.01	0.06	0.04	0.05	0.0540	0.0338	0.0187
0.01	0.07	0.01	0.02	0.0415	0.0262	0.0145
0.01	0.07	0.02	0.03	0.0453	0.0285	0.0158
0.01	0.07	0.04	0.05	0.0556	0.0348	0.0193
0.01	0.08	0.01	0.02	0.0425	0.0267	0.0149
0.01	0.08	0.02	0.03	0.0465	0.0292	0.0162
0.01	0.08	0.04	0.05	0.0573	0.0358	0.0198
0.01	0.09	0.01	0.02	0.0434	0.0273	0.0152
0.01	0.09	0.02	0.03	0.0476	0.0299	0.0166
0.01	0.09	0.04	0.05	0.0590	0.0369	0.0204
0.87	0.79	0.01	0.02	0.1835	0.1074	0.0572
0.87	0.79	0.02	0.03	0.3131	0.1648	0.0846
0.87	0.79	0.04	0.05	0.5-1.1902i	0.5-0.9037i	0.5-0.5916i
0.72	0.29	0.01	0.02	0.0351	0.0221	0.0123
0.72	0.29	0.02	0.03	0.0378	0.0238	0.0133
0.72	0.29	0.04	0.05	0.0409	0.0258	0.0143
0.62	0.39	0.01	0.02	0.0476	0.0299	0.0166
0.62	0.39	0.02	0.03	0.0527	0.0331	0.0183
0.62	0.39	0.04	0.05	0.0671	0.0418	0.231
0.52	0.49	0.01	0.02	0.0742	0.0461	0.0254
0.52	0.49	0.02	0.03	0.0875	0.0540	0.0296
0.52	0.49	0.04	0.05	0.1367	0.0823	0.0445
0.82	0.29	0.01	0.02	0.0325	0.0205	0.0115
0.82	0.29	0.02	0.03	0.0348	0.0220	0.0122
0.82	0.29	0.04	0.05	0.0405	0.0255	0.0142
0.92	0.29	0.01	0.02	0.0303	0.0192	0.0107
0.92	0.29	0.02	0.03	0.0323	0.0204	0.0114
0.92	0.29	0.04	0.05	0.0371	0.0234	0.0130

FIGURE 7: Correlation between μ and A_1 and $\beta_1 = 0.01$, $\beta_2 = 0.02$, and $A_2 = 0.02$. FIGURE 8: Correlation between μ and A_1 and $\beta_1 = 0.01$, $\beta_2 = 0.02$, and $A_2 = 0.03$.

5. Normalization and Higher Order Stability of the Libration Points

In order to investigate the stability the Hamiltonian H is normalized by Birkhoff's method to the following form:

$$H = \omega_1 t_1 - \omega_2 t_2 + c_{20} t_1^2 + c_{11} t_1 t_2 + c_{02} t_2^2 + o(t_1 + t_2)^{5/2}, \quad (29)$$

FIGURE 9: Correlation between μ and A_1 and $\beta_1 = 0.01$, $\beta_2 = 0.02$, and $A_2 = 0.04$.

FIGURE 10: Correlation between μ and A_1 and $\beta_1 = 0.01$, $\beta_2 = 0.02$, and $A_2 = 0.05$.

where

$$2t_i = p_i^2 + q_i^2, \quad i = 1, 2. \quad (30)$$

If H_2 is Hamiltonian of 2nd order, defined by (8), (9), (10), and (11) and is of a positive definite form, then the equilibrium position is stable by virtue of Lyapunov theorem [28]. Otherwise the problem of stability can be solved by KAM theorem as given by Arnold [29, 30]. To apply KAM

FIGURE 11: Correlation between β_1 and μ for $A_1 = 0.01$, $A_2 = 0.02$, and $\beta_2 = 0.02, 0.04, 0.06, 0.08$.

FIGURE 12: Combined figure between μ and β_1 ($\beta_2 = 0.01$, $A_1 = 0.01$, and $A_2 = 0.01, 0.02, 0.03$).

theorem, linear canonical transformation of variations as given by Manju and Choudhry [1] is used which is given as

$$(q_1, q_2, p_1, p_2) = (q'_1, q'_2, p'_1, p'_2)N, \quad (31)$$

where

$$N = \begin{bmatrix} a_1 & a_1 c_1 & -a_1 c_1 & a_1 (1 - \omega_1^2 b_1) \\ a_2 & a_2 c_2 & -a_2 c_2 & a_2 (1 - \omega_2^2 b_2) \\ 0 & a_1 b_1 & a_1 (1 - b_1) & a_1 c_1 \\ 0 & -a_2 b_2 & -a_2 (1 - b_2) & -a_2 c_2 \end{bmatrix}, \quad (32)$$

where

$$\begin{aligned}
 a_1 &= \frac{1}{2} \left(\frac{2l_1}{\omega_1^2 - 1/2} \right)^{1/2}; \\
 a_2 &= \frac{1}{2} \left(\frac{2l_2}{\omega_2^2 - 1/2} \right)^{1/2}, \\
 l_1 &= 1 + \omega_1^2 + 2H_{02}; \\
 l_2 &= 1 + \omega_2^2 + 2H_{02} \\
 b_1 &= \frac{2}{l_1}; \\
 b_2 &= \frac{2}{l_2}; \\
 c_1 &= \frac{-H_{11}}{l_1}; \\
 c_2 &= \frac{-H_{11}}{l_2}.
 \end{aligned} \tag{33}$$

The transformation (31) reduces the Hamiltonian to the following form:

$$\begin{aligned}
 H &= \frac{1}{2} (p_1'^2 + \omega_1^2 q_1'^2) - \frac{1}{2} (p_2'^2 + \omega_2^2 q_2'^2) \\
 &+ \sum_{\alpha+\gamma=3}^{\infty} h_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} q_1'^{\alpha_1} q_2'^{\alpha_2} p_1'^{\gamma_1} p_2'^{\gamma_2},
 \end{aligned} \tag{34}$$

where $\alpha = \alpha_1 + \alpha_2$; $\gamma = \gamma_1 + \gamma_2$.

The Hamiltonian H_3 can be expanded in the following form:

$$\begin{aligned}
 H_3 &= H_{0003} p_2^3 + H_{0030} p_1^3 + H_{0300} q_2^3 + H_{3000} q_1^3 \\
 &+ H_{2100} q_1^2 q_2 + H_{2010} q_1^2 p_1 + \dots + H_{1011} q_1 p_1 p_2.
 \end{aligned} \tag{35}$$

Similarly H_4 can be expanded as

$$\begin{aligned}
 H_4 &= H_{0004} p_2^4 + H_{0040} p_1^4 + H_{0400} q_2^4 + H_{4000} q_1^4 + \dots \\
 &+ H_{1111} q_1 q_2 p_1 p_2.
 \end{aligned} \tag{36}$$

The coefficient of third and fourth order terms of $h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}$ and $h_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}$ given by Kumar and Choudhry [2] can be listed as

$$\begin{aligned}
 h_{0030} &= a_1^3 b_1^3 H_{0300}; \\
 h_{3000} &= a_1^3 (H_{3000} + c_1 H_{2100} + c_1^2 H_{2100} + c_1^3 H_{0300});
 \end{aligned}$$

$$\begin{aligned}
 h_{1020} &= a_1^3 b_1^2 (H_{1200} + 3c_1 H_{0300}); \\
 h_{2010} &= a_1^3 b_1 (H_{2100} + 2c_1 H_{1200} + 3c_1^2 H_{0300}); \\
 h_{0120} &= a_1^2 a_2 b_1^2 (H_{1200} + 3c_2 H_{0300}); \\
 h_{2001} &= -a_1^2 a_2 b_2 (H_{2100} + 2c_1 H_{1200} + 3c_1^2 H_{0300}); \\
 h_{1011} &= -2a_1^2 a_2 b_1 b_2 (H_{1200} + 3c_1 H_{0300}); \\
 h_{1110} &= 2a_1^2 a_2 b_1 [H_{2100} + (c_1 + c_2) H_{1200} + 3c_1 c_2 H_{0300}]; \\
 h_{0021} &= -3a_1^2 b_1^2 b_2 H_{0300}; \\
 h_{2100} &= a_1^2 a_2 [3H_{3000} + (2c_1 + c_2) H_{2100} \\
 &\quad + c_1 (c_1 + 2c_2) H_{1200} + 3c_1^2 c_2 H_{0300}]; \\
 h_{1002} &= a_1 b_1^2 a_2^2 (H_{1200} + 3c_1 H_{0300}); \\
 h_{0210} &= a_1 a_2^2 b_1 (H_{2100} + 2c_2 H_{1200} + 3c_2^2 H_{0300}); \\
 h_{0012} &= 3a_1 a_2^2 b_2^2 b_1 H_{0300}; \\
 h_{1200} &= a_2^2 a_1 [3H_{3000} + (c_1 + 2c_2) H_{2100} \\
 &\quad + c_2 (2c_1 + c_2) H_{1200} + 3c_1 c_2^2 H_{0300}]; \\
 h_{0111} &= -2a_1^2 a_2^2 b_1 b_2 (H_{1200} + 3c_2 H_{0300}); \\
 h_{1101} &= -2a_2^2 a_1 b_2 [H_{2100} + (c_1 + c_2) H_{1200} \\
 &\quad + 3c_1 c_2 H_{0300}]; \\
 h_{0102} &= a_2^3 b_2^2 (H_{1200} + 3c_2 H_{0300}); \\
 h_{0201} &= -a_2^3 b_2 (H_{2100} + 2c_2 H_{1200} + 3c_2^2 H_{0300}); \\
 h_{0102} &= a_2^3 b_2^2 (H_{1200} + 3c_2 H_{0300}); \\
 h_{0003} &= -a_2^3 b_2^3 H_{0300}; \\
 h_{0300} &= a_2^3 (H_{3000} + c_2 H_{2100} + c_2^2 H_{1200} + c_2^3 H_{0300}); \\
 h_{0040} &= a_1^4 b_1^4 H_{0400}; \\
 h_{4000} &= a_1^4 (H_{4000} + c_1 H_{3100} + c_1^2 H_{2200} + c_1^3 H_{1300} \\
 &\quad + c_1^4 H_{0400}); \\
 h_{2020} &= a_1^4 b_1^2 (H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400}); \\
 h_{0022} &= 6a_1^2 a_2^2 b_1^2 b_2^2 H_{0400}; \\
 h_{2200} &= a_1^2 a_2^2 [6H_{4000} + 3(c_1 + c_2) H_{3100} \\
 &\quad + (c_1^2 + 4c_1 c_2 + c_2^2) H_{2200} + 3c_1 c_2 (c_1 + c_2) H_{1300} \\
 &\quad + 6c_1^2 c_2^2 H_{0400}]; \\
 h_{2002} &= a_1^2 a_2^2 b_2^2 (H_{2200} + 3c_1 H_{1300} + 6c_1^2 H_{0400});
 \end{aligned}$$

$$\begin{aligned}
h_{0220} &= a_1^2 a_2^2 b_1^2 (H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}); \\
h_{0004} &= a_2^4 b_2^4 H_{0400}; \\
h_{0400} &= a_2^4 (H_{4000} + c_2 H_{3100} + c_2^2 H_{2200} + c_2^3 H_{1300} \\
&\quad + c_2^4 H_{0400}); \\
h_{0202} &= a_2^4 b_2^2 (H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}); \\
h_{0013} &= -4a_1 a_2^3 b_1 b_2^3 H_{0400}; \\
h_{1300} &= a_1 a_2^3 [4H_{4000} + (c_1 + 3c_2) H_{3100} + 4c_1 c_2^3 H_{0400} \\
&\quad + 2c_2 (c_1 + c_2) H_{2200} + c_2^2 (3c_1 + c_2) H_{1300}]; \\
h_{1102} &= a_1 a_2^3 b_2^2 [2H_{2200} + 3(c_1 + c_2) H_{1300} \\
&\quad + 12c_1 c_2 H_{0400}]; \\
h_{0211} &= -2a_2^3 a_1 b_1 b_2 [H_{2200} + 3c_2 H_{1300} + 6c_2^2 H_{0400}]; \\
h_{0112} &= 3a_1 a_2^3 b_1 b_2^2 (H_{1300} + 4c_1 H_{0400}); \\
h_{1003} &= -a_1 a_2^3 b_2^3 (H_{1300} + 4c_1 H_{0400}); \\
h_{1201} &= -a_1 a_2^3 b_2 [3H_{3100} + 2(c_1 + 2c_2) H_{2200} \\
&\quad + 3c_2 (2c_1 + c_2) H_{1300} + 12c_1 c_2^2 H_{0400}]; \\
h_{0310} &= a_1 a_2^3 b_1 (H_{3100} + 2c_2 H_{2200} + 3c_2^2 H_{1300} \\
&\quad + 4c_2^2 H_{0400}).
\end{aligned} \tag{37}$$

Again the Hamiltonian is reduced to a more convenient form which is suitable for further investigation by using the following canonical transformation:

$$\begin{aligned}
q_1' &= \frac{1}{2} q_1'' + \frac{i}{\omega_1} p_1''; \\
p_1' &= \frac{1}{2} i \omega_1 q_1'' + p_1''; \\
q_2' &= -\frac{1}{2} q_2'' + \frac{1}{\omega_2} p_2''; \\
p_2' &= -\frac{1}{2} i \omega_2 q_2'' + i p_2''.
\end{aligned} \tag{38}$$

Thus, the Hamiltonian (34) may be written as

$$\begin{aligned}
H &= i \omega_1 q_1'' p_1'' + i \omega_2 q_2'' p_2'' \\
&\quad + \sum_{\alpha+\gamma=3}^{\infty} h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} q_1''^{\alpha_1} q_2''^{\alpha_2} p_1''^{\gamma_1} p_2''^{\gamma_2}.
\end{aligned} \tag{39}$$

As explained by Kumar and Choudhry [2], consider the following.

If $h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} = x_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} + i y_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}$ then

$$\begin{aligned}
x_{0030} &= h_{0030} - \frac{1}{\omega_1^2} h_{2010}; \\
y_{0030} &= \frac{1}{\omega_1} h_{1020} - \frac{1}{\omega_1^3} h_{3000}; \\
x_{1020} &= -\frac{1}{2} h_{1020} - \frac{3}{2\omega_1^2} h_{3000}; \\
y_{1020} &= \frac{3\omega_1}{2} h_{0030} + \frac{1}{2\omega_1} h_{2010}; \\
x_{0120} &= -\frac{\omega_2}{2} h_{0021} + \frac{1}{2\omega_1} h_{1110} + \frac{\omega_2}{2\omega_1} h_{2001}; \\
y_{0120} &= -\frac{1}{2} h_{0120} - \frac{\omega_2}{2\omega_1} h_{1011} + \frac{1}{2\omega_1^2} h_{2100}; \\
x_{1011} &= -\omega_1 h_{0021} - \frac{1}{\omega_1} h_{2001}; \\
y_{0021} &= h_{0021} + \frac{1}{\omega_1 \omega_2} h_{1110} - \frac{1}{\omega_1^2} h_{2001}; \\
x_{1002} &= -\frac{\omega_1}{2\omega_2} h_{0111} - \frac{1}{2} h_{1002} + \frac{1}{2\omega_2^2} h_{1200}; \\
y_{1002} &= -\frac{\omega_1}{2} h_{0012} + \frac{\omega_1}{2\omega_2^2} h_{0210} + \frac{1}{2\omega_2^2} h_{1101}; \\
x_{0012} &= -h_{0012} + \frac{1}{\omega_2^2} h_{0210} - \frac{1}{\omega_1 \omega_2} h_{1101}; \\
y_{0012} &= \frac{1}{\omega_2} h_{0111} - \frac{1}{\omega_1} h_{1002} + \frac{1}{\omega_1 \omega_2^2} h_{1200}; \\
x_{0111} &= \frac{\omega_2}{\omega_1} h_{1002} + \frac{1}{\omega_1 \omega_2} h_{1200}; \\
y_{0111} &= -\omega_2 h_{0012} - \frac{1}{\omega_2} h_{0210}; \\
x_{0201} &= \frac{-\omega_2}{4} h_{0102} - \frac{3}{4\omega_2} h_{0300}; \\
y_{0201} &= \frac{-3\omega_2^2}{4} h_{0003} + \frac{1}{4} h_{0201}; \\
x_{0003} &= \frac{-1}{\omega_2} h_{0102} + \frac{1}{\omega_2^3} h_{0300}; \\
y_{0003} &= -h_{0003} + \frac{1}{\omega_2^2} h_{0201}.
\end{aligned} \tag{40}$$

The other ten coefficients $h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}$ of third order terms of (39) are obtained by the formula

$$\begin{aligned}
&h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} \\
&= (y_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} + i x_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}) \left(\frac{-\omega_1}{2} \right)^{\gamma_1 - \alpha_1} \left(\frac{-\omega_2}{2} \right)^{\gamma_2 - \alpha_2}.
\end{aligned} \tag{41}$$

If we denote $(\alpha_1 - \gamma_1) + (\alpha_2 - \gamma_2) = a_{\alpha\gamma}$ and consider $|\alpha_1 - \gamma_1| + |\alpha_2 - \gamma_2| = 3$ and if it is found that $a_{\alpha\gamma} \neq$ an integer, then it implies that resonance of the third order is absent (Kumar and Choudhry [14]).

Using Birkhoff's transformation $(q_j'', p_j'') \rightarrow (q_j''', p_j''')$, all the third order terms from the Hamiltonian (39) are nullified provided third order resonance does not occur. This transformation is introduced by means of the generating function which is given as follows:

$$s = q_1'' p_1''' + q_2'' p_2''' + s_3 + s_4, \quad (42)$$

where

$$\begin{aligned} q_i''' &= q_i'' + \frac{\partial s_3}{\partial p_i'''} + \frac{\partial s_4}{\partial p_i'''}; \\ p_i''' &= p_i'' + \frac{\partial s_3}{\partial q_i''} + \frac{\partial s_4}{\partial q_i''}; \quad (i = 1, 2). \end{aligned} \quad (43)$$

From (39) and (43), expanding and equating the terms of the same degree on the two sides, we obtain

$$\begin{aligned} H_2'(q_1'', q_2'', p_1''', p_2''') &= H_2(q_1'', q_2'', p_1''', p_2'''); \\ H_3'(q_1'', q_2'', p_1''', p_2''') &= \sum_{i=1}^2 \left[-\frac{\partial s_3}{\partial p_i'''} \frac{\partial H_2'}{\partial p_i'''} + \frac{\partial s_3}{\partial q_i''} \frac{\partial H_2}{\partial p_i'''} \right] \\ &\quad + H_3(q_1'', q_2'', p_1''', p_2'''); \\ \sum_{i=1}^2 \left[-\frac{\partial s_4}{\partial p_i'''} \frac{\partial H_2'}{\partial q_i''} + \frac{\partial s_4}{\partial q_i''} \frac{\partial H_2}{\partial p_i'''} \right] &+ K_4 = 0; \\ H_4' &= \sum_{i=1}^2 \left[-\frac{\partial s_3}{\partial p_i'''} \frac{\partial H_3'}{\partial q_i''} + \frac{\partial s_3}{\partial q_i''} \frac{\partial H_3}{\partial p_i'''} \right] + H_4 - K_4, \end{aligned} \quad (44)$$

where K_4 is the term other than the homogeneous ones in $q_1 p_1$ and $q_2 p_2$. In (44) the new variables q_1''' and p_2''' can be replaced by q_1'' and p_2'' on both sides of (44) by implicit function theorem.

Since our system is autonomous $\partial s_3 / \partial t = \partial s_4 / \partial t = 0$.

If we put

$$\begin{aligned} H_3 &= \sum_{\alpha+\gamma=3}^{\infty} h_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\gamma_1} p_2'''^{\gamma_2}, \\ s_3 &= \sum_{\alpha+\gamma=3}^{\infty} g_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} q_1'''^{\alpha_1} q_2'''^{\alpha_2} p_1'''^{\gamma_1} p_2'''^{\gamma_2}, \end{aligned} \quad (45)$$

using (44), we get

$$g_{\alpha_1 \alpha_2 \gamma_1 \gamma_2} = \frac{i h'_{\alpha_1 \alpha_2 \gamma_1 \gamma_2}}{(\alpha_1 - \gamma_1) \omega_1 + (\alpha_2 - \gamma_2) \omega_2}. \quad (46)$$

With the help of third equation of (44), the new Hamiltonian, inclusive of the fourth order terms, is given as

$$\begin{aligned} H' &= i \omega_1 q_1''' p_1''' + i \omega_2 q_2''' p_2''' - c_{20} (q_1''' p_1''')^2 \\ &\quad + c_{11} (q_1''' p_1''') (q_2''' p_2''') - c_{02} (q_2''' p_2''')^2 + \dots, \end{aligned} \quad (47)$$

where

$$\begin{aligned} K_4 &= H_4(q_1''', q_2''', p_1''', p_2''') - h'_{2020} q_1'''^2 p_1'''^2 \\ &\quad - h'_{1111} (q_1''' p_1''') (q_2''' p_2''') \\ &\quad - h'_{0202} q_2'''^2 p_2'''^2; \\ c_{20} &= -h'_{2020} - \frac{3\omega_1^2}{8} (x_{0030}^2 + y_{0030}^2) \\ &\quad - \frac{3}{2} (x_{1020}^2 + y_{1020}^2) + \frac{1}{2} (x_{1011}^2 + y_{1011}^2) \\ &\quad - \frac{\omega_1^2}{2\omega_2(2\omega_1 - \omega_2)} (x_{0120}^2 + y_{0120}^2) \\ &\quad + \frac{\omega_1^2 \omega_2}{8(2\omega_1 + \omega_2)} \frac{3}{2} (x_{0021}^2 + y_{0021}^2), \\ c_{11} &= -h'_{1111} - \frac{2\omega_2^2}{\omega_1(\omega_1 - 2\omega_2)} (x_{1002}^2 + y_{1002}^2) \\ &\quad + \frac{\omega_1 \omega_2^2}{2(\omega_1 + 2\omega_2)} (x_{0012}^2 + y_{0012}^2) \\ &\quad - \frac{\omega_2 \omega_1^2}{2(2\omega_1 + \omega_2)} (x_{0021}^2 + y_{0021}^2) \\ &\quad - \frac{2\omega_1^2}{(2\omega_1 - \omega_2)} (x_{0120}^2 + y_{0120}^2) \\ &\quad + 2(x_{0111} x_{1020} + y_{0111} y_{1020}) \\ &\quad - \frac{4}{\omega_2} (x_{0201} y_{1011} + x_{1011} y_{0201}); \\ c_{02} &= h'_{0202} + \frac{3\omega_2^2}{8} (x_{0003}^2 + y_{0003}^2) \\ &\quad + \frac{6}{\omega_2^2} (x_{0201}^2 + y_{0201}^2) \\ &\quad - \frac{\omega_2^2}{2\omega_1(\omega_1 - 2\omega_2)} (x_{1002}^2 + y_{1002}^2) \\ &\quad - \frac{1}{2} (x_{0111}^2 + y_{0111}^2) \\ &\quad - \frac{\omega_1 \omega_2^2}{8(\omega_1 + 2\omega_2)} (x_{0012}^2 + y_{0012}^2); \end{aligned}$$

$$\begin{aligned}
h'_{2020} &= -\frac{3}{2}\omega_1^2 h_{0040} - \frac{3}{2\omega_1^2} h_{4000} - \frac{1}{2} h_{2020}; \\
h'_{1111} &= \omega_1 \omega_2 h_{0022} + \frac{1}{\omega_1 \omega_2} h_{2200} + \frac{\omega_1}{\omega_2} h_{0220} + \frac{\omega_2}{\omega_1} h_{2002}; \\
h'_{0202} &= -\frac{3}{2}\omega_2^2 h_{0004} - \frac{3}{2\omega_2^2} h_{0400} - \frac{1}{2} h_{0202};
\end{aligned} \tag{48}$$

now, KAM theorem is applied which is best suited for the problem which is being defined and is stated as follows.

If the Hamiltonian of the perturbed motion satisfy the following given conditions mentioned below as:

(1) The characteristics equation of the system with H_2 has pure imaginary roots and the frequencies ω_1, ω_2 satisfy the inequality

$$\begin{aligned}
n_1 \omega_1 + n_2 \omega_2 &\neq 0; \\
\text{when } |n_1| + |n_2| &\leq 4 \text{ } (n_j \text{ are integers}).
\end{aligned} \tag{49}$$

(2) Consider $D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0$, as given by Kumar and Choudhry [2].

If all the above mentioned conditions are satisfied, then the equilibrium points are stable.

The value of D is calculated with the help of the following formula:

$$D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2, \tag{50}$$

where c_{20}, c_{11} , and c_{02} are defined in (48).

Table 2 shows the values D for different values of radiation pressures and oblateness parameters.

6. Discussion and Conclusion

The stability of the triangular equilibrium points in CR3BP is investigated when both primaries are radiating and oblate, under the nonresonance case using KAM theory. Recently Narayan and Singh [5] studied nonlinear stability considering both primaries as radiating in ER3BP and found that the binary systems are stable. It has been observed that in general the stability character remains the same even if oblateness factor is considered apart from radiation factor in circular cases. The following observation has been made assuming both primaries as radiating and oblateness:

- (i) It is found that $\omega_1 = \omega_2$ corresponds to the boundary region of the stability for the system, whereas the other two cases $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ correspond to the resonant cases. It has been observed that resonance of the third and the fourth order exists for all values of $\mu, \beta_1, \beta_2, A_1$, and A_2 taken.
- (ii) It is noticed that for fixed values of radiation pressures and by varying oblateness parameters ω_1 increases by increasing μ whereas ω_2 decreases and becomes equal to the critical value (i.e., $\mu = \mu_c$).
- (iii) From Table 2 it is clear that for $\mu = 0.038, \beta_1 = 0.72, \beta_2 = 0.49, A_1 = 0.04, A_2 = 0.05$ and $\beta_1 = 0.62,$

TABLE 2: Values of D at different values of eccentricity for different values of radiation pressure and oblateness parameter.

μ	β_1	β_2	A_1	A_2	Values of D
0.001	0.87	0.29	0.01	0.02	-3.1202×10^9
0.001	0.87	0.29	0.02	0.03	-4.8371×10^9
0.001	0.87	0.29	0.04	0.05	-1.1754×10^7
0.005	0.82	0.29	0.01	0.02	-2.7755×10^7
0.005	0.82	0.29	0.02	0.03	-4.2670×10^7
0.005	0.82	0.29	0.04	0.05	-6.5897×10^8
0.010	0.72	0.29	0.01	0.02	-3.8443×10^6
0.010	0.72	0.29	0.02	0.03	-5.8280×10^6
0.010	0.72	0.29	0.04	0.05	-1.3814×10^7
0.015	0.62	0.39	0.01	0.02	-1.5405×10^6
0.015	0.62	0.39	0.02	0.03	-2.4736×10^6
0.015	0.62	0.39	0.04	0.05	-6.9668×10^6
0.025	0.52	0.49	0.01	0.02	-5.7810×10^5
0.025	0.52	0.49	0.02	0.03	-1.0457×10^6
0.025	0.52	0.49	0.04	0.05	-4.5781×10^6
0.035	0.92	0.29	0.01	0.02	-7.8501×10^4
					$+1.1316 \times 10^4 i$
					-1.4121×10^5
0.035	0.92	0.29	0.02	0.03	$+1.4800 \times 10^4 i$
0.035	0.92	0.29	0.04	0.05	-9.9125×10^4
0.038	0.72	0.49	0.01	0.02	6.8892×10^5
0.038	0.72	0.49	0.02	0.03	-4.8483×10^6
0.038	0.72	0.49	0.04	0.05	-2.3721×10^6
0.038	0.62	0.39	0.01	0.02	6.9437×10^4
0.038	0.62	0.39	0.02	0.03	3.2725×10^5
0.038	0.62	0.39	0.04	0.05	-2.5116×10^6
0.001	0.01	0.01	0.01	0.02	-3.2004×10^7
0.001	0.01	0.01	0.02	0.03	-5.4067×10^7
0.001	0.01	0.01	0.04	0.05	-1.5692×10^8
0.005	0.01	0.02	0.01	0.02	-3.1222×10^5
0.005	0.01	0.02	0.02	0.03	-5.2414×10^7
0.005	0.01	0.02	0.04	0.05	-1.5044×10^6
0.010	0.01	0.03	0.01	0.02	-3.0522×10^5
0.010	0.01	0.03	0.02	0.03	-8.5874×10^4
0.010	0.01	0.03	0.04	0.05	-2.3880×10^5
0.015	0.01	0.04	0.01	0.02	-2.2203×10^4
0.015	0.01	0.04	0.02	0.03	-3.5166×10^4
0.015	0.01	0.04	0.04	0.05	-9.2124×10^4
0.025	0.01	0.05	0.01	0.02	-4.5169×10^5
0.025	0.01	0.05	0.02	0.03	-3.3881×10^4
0.025	0.01	0.05	0.04	0.05	-3.2969×10^4
0.030	0.01	0.06	0.01	0.02	3.4841×10^4
0.030	0.01	0.06	0.02	0.03	1.7250×10^4
0.030	0.01	0.06	0.04	0.05	-4.8836×10^4
0.035	0.01	0.07	0.01	0.02	492.1240
0.035	0.01	0.07	0.02	0.03	2.5112×10^4
0.035	0.01	0.07	0.04	0.05	5.7581×10^5
0.038	0.01	0.08	0.01	0.02	38.4651
0.038	0.01	0.08	0.02	0.03	1.0408×10^3
0.038	0.01	0.08	0.04	0.05	3.9251×10^4

$\beta_2 = 0.39, A_1 = 0.01, A_2 = 0.02$, there is change in the sign of value of D .

This change in sign suggests that for fixed $\mu = 0.038$ and by slightly changing radiation pressure and oblateness parameters D vanishes, where KAM theory is not applicable. It is also concluded that except at this point the equilibrium points are stable. Consider the following:

- (i) It is clear from Table 2 that for $\mu = 0.025$, $\beta_1 = 0.01$, $\beta_2 = 0.05$, $A_1 = 0.04$, $A_2 = 0.05$ and $\mu = 0.030$, $\beta_1 = 0.01$, $\beta_2 = 0.06$, $A_1 = 0.01$, $A_2 = 0.02$ again there is change in sign of D . This suggests that for slight variation in radiation pressure and oblateness parameters and for $0.025 \leq \mu \leq 0.030$, the value of D must be zero.
- (ii) It is observed that for $\mu = 0.030$, $\beta_1 = 0.01$, $\beta_2 = 0.06$, $A_1 = 0.02$, $A_2 = 0.03$ and $\mu = 0.030$, $\beta_1 = 0.01$, $\beta_2 = 0.06$, $A_1 = 0.04$, $A_2 = 0.05$ D changes sign. This implies that when $\mu = 0.030$, $\beta_1 = 0.01$, and $\beta_2 = 0.06$ and for slight variation in oblateness parameters, D vanishes.
- (iii) Also, it is found that when $\mu = 0.038$, $\beta_1 = 0.62$, $\beta_2 = 0.39$, $A_1 = 0.02$, $A_2 = 0.03$ and $\mu = 0.038$, $\beta_1 = 0.01$, $\beta_2 = 0.06$, $A_1 = 0.04$, $A_2 = 0.05$, there is change in the sign of D .

From the above discussion the following conclusion can be drawn.

In all the above four cases considered, it is noticed that KAM theory fails. From the above discussion it is also clear that by varying one of the parameters and taking another parameter as fixed, there may be many cases where KAM theory fails and their motion can be found numerically which needs further investigation. It is observed that, except for some values of radiation pressure and oblateness parameters and for $\mu < 0.0385209$ the triangular equilibrium points are stable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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