A POINTWISE APPROXIMATION THEOREM FOR LINEAR COMBINATIONS OF BERNSTEIN POLYNOMIALS

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ABSTRACT. Recently, Z. Ditzian gave an interesting direct estimate for Bernstein polynomials. In this paper we give direct and inverse results of this type for linear combinations of Bernstein polynomials.

1. INTRODUCTION

For the Bernstein polynomial

$$(1.1) \quad B_n(f, x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{nk}(x), \quad p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

Berens and Lorentz showed in [1] that

$$(1.2) \quad B_n(f, x) - f(x) = O\left(\left(\frac{1}{\sqrt{n}} \delta_n(x)\right)^\alpha\right) \iff \omega^1(f, t) = O(t^\alpha),$$

where $0 < \alpha < 1$, $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, $\varphi(x) = \sqrt{x(1-x)}$.

Recently, Ditzian [3] gave the following interesting result.

$$|B_n(f, x) - f(x)| \leq C \omega_{\varphi^\lambda}^2(f, n^{-\frac{1}{2}} \varphi(x)^{1-\lambda}), \quad 0 \leq \lambda \leq 1.$$

However, Ditzian did not consider the inverse result in [3]. We did give such an inverse result in [6], where we obtained the following equivalence.

$$B_n(f, x) - f(x) = O((n^{-\frac{1}{2}} \varphi(x)^{1-\lambda})^\alpha) \iff \omega_{\varphi^\lambda}^2(f, t) = O(t^\alpha), \quad 0 < \alpha < 2.$$

In this paper we consider linear combinations of Bernstein polynomials, that is

$$(1.3) \quad B_{n,r}(f, x) = \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f, x),$$

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where n_i and C_i satisfy [5]

$$(1.4) \quad \begin{aligned} (a) \quad & n = n_0 < n_1 < \cdots < n_{r-1} \leq Kn; \quad (b) \quad \sum_{i=0}^{r-1} |C_i(n)| < C \\ (c) \quad & \sum_{i=0}^{r-1} C_i(n) = 1; \quad (d) \quad \sum_{i=0}^{r-1} C_i(n) n_i^{-\rho} = 0, \quad \rho = 1, 2, \dots, r-1. \end{aligned}$$

We recall that

$$(1.5) \quad \omega_{\varphi^\lambda}^r(f, t) = \sup_{0 < h \leq t} \sup_{x \pm \frac{r}{2} h \varphi^\lambda(x) \in [0, 1]} \left| \Delta_{h \varphi^\lambda(x)}^r f(x) \right|$$

is equivalent to the K -functional

$$(1.6) \quad K_{\varphi^\lambda}(f, t^r) = \inf_{g^{(r-1)} \in A.C_{loc}} (\|f - g\|_{C[0, 1]} + t^r \|\varphi^{r\lambda} g^{(r)}\|_{C[0, 1]}).$$

We write $\omega_{\varphi^\lambda}^r(f, t) \sim K_{\varphi^\lambda}(f, t^r)$, i.e., there exists a constant C such that

$$(1.7) \quad C^{-1} K_{\varphi^\lambda}(f, t^r) \leq \omega_{\varphi^\lambda}^r(f, t) \leq C K_{\varphi^\lambda}(f, t^r).$$

Now we state our main result.

Theorem. For $f \in C[0, 1]$, $0 < \alpha < r$, $0 \leq \lambda \leq 1$, we have

$$(1.8) \quad B_{n,r}(f, x) - f(x) = O((n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x))^\alpha) \iff \omega_{\varphi^\lambda}^r(f, t) = O(t^\alpha).$$

Remark. In the case $r = 1$ and $\lambda = 0$ our result is (1.2) of Berens and Lorentz [1].

2. DIRECT THEOREM

In this section we prove the direct part of (1.8). We need the K -functional (see [5], p. 24):

$$(2.1) \quad \overline{K}_{\varphi^\lambda}(f, t^r) = \inf_{g^{(r-1)} \in A.C_{loc}} \{ \|f - g\| + t^r \|\varphi^{r\lambda} g^{(r)}\| + t^{r/(1-\frac{\lambda}{2})} \|g^{(r)}\| \}.$$

which is also equivalent to $\omega_{\varphi^\lambda}^r(f, t)$.

Theorem 1. For $f \in C[0, 1]$, $0 \leq \lambda \leq 1$,

$$(2.2) \quad |B_{n,r}(f, x) - f(x)| \leq A \omega_{\varphi^\lambda}^r(f, n^{-\frac{1}{2}} \delta_n(x)^{1-\lambda}).$$

Proof. By (2.1), we may choose $g_n \equiv g_{n,x,\lambda}$ for a fixed x and λ such that

$$(2.3) \quad \|f - g_n\| \leq A_1 \omega_{\varphi^\lambda}^r(f, n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x)),$$

$$(2.4) \quad (n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x))^r \|\varphi^{r\lambda} g_n^{(r)}\| \leq A_2 \omega_{\varphi^\lambda}^r(f, n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x)),$$

$$(2.5) \quad (n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x))^{2r/(2-\lambda)} \|g_n^{(r)}\| \leq A_3 \omega_{\varphi^\lambda}^r(f, n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x)).$$

We recall that [5, p. 134]

$$(2.6) \quad B_{n,r}((\cdot - x)^k, x) = 0, \quad k = 1, 2, \dots, r-1$$

and that [5, p. 138]

$$B_n((\cdot - x)^{2j}, x) = \sum_{m=0}^{j-1} \frac{\varphi(x)^{2(j-m)}}{n^{j-m}} n^{-2m} q_m(x),$$

where $q_m(x)$ are fixed bounded polynomials. Therefore

$$(2.7) \quad B_n((\cdot - x)^{2j}, x) \leq M n^{-j} \delta_n^{2j}(x).$$

From the definition of the $B_{n,r}$ we have

$$|B_{n,r}(f, x) - f(x)| \leq |B_{n,r}(f - g_n, x)| + |f(x) - g_n(x)| + |B_{n,r}(g_n, x) - g_n(x)|$$

$$(2.8) \quad \leq (C + 1) \|f - g_n\| + |B_{n,r}(g_n, x) - g_n(x)|.$$

As in [4, Lemma 5.3], we obtain

$$(2.9) \quad \left| \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du \right| \leq \left| \frac{(t-x)^{r-1}}{\delta_n^{r\lambda}(x)} \int_x^t \delta_n^{r\lambda}(u) g_n^{(r)}(u) du \right|,$$

$$(2.10) \quad \left| \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du \right| \leq \left| \frac{(t-x)^{r-1}}{\varphi^{r\lambda}(x)} \int_x^t \varphi^{r\lambda}(u) g_n^{(r)}(u) du \right|.$$

Using (2.6), (2.7) and (2.9), we get

$$\begin{aligned} & |B_{n,r}(g_n, x) - g_n(x)| \\ &= \left| B_{n,r} \left(\frac{1}{(r-1)!} \int_x^t (t-u)^{r-1} g_n^{(r)}(u) du, x \right) \right| \\ (2.11) \quad & \leq \sum_{i=0}^{r-1} |C_i(n)| \|\delta_n^{r\lambda} g_n^{(r)}\|_{\infty} \frac{1}{(r-1)!} B_{n_i} \left(\frac{|t-x|^r}{\delta_n^{r\lambda}(x)}, x \right) \\ & \leq \sum_{i=0}^{r-1} |C_i(n)| \|\delta_n^{r\lambda} g_n^{(r)}\|_{\infty} \delta_n^{-r\lambda}(x) M_1 n^{-\frac{r}{2}} \delta_n^r(x) \\ & \leq C M_1 n^{-\frac{r}{2}} \delta_n^{r(1-\lambda)}(x) \|\delta_n^{r\lambda} g_n^{(r)}\|_{\infty}. \end{aligned}$$

Similarly, by (2.10) we have

$$(2.12) \quad |B_{n,r}(g_n, x) - g_n(x)| \leq C M_1 n^{-\frac{r}{2}} \delta_n^r(x) \varphi^{-r\lambda}(x) \|\varphi^{r\lambda} g_n^{(r)}\|.$$

If $x \in E_n = [\frac{1}{n}, 1 - \frac{1}{n}]$, then $\delta_n(x) \sim \varphi(x)$, and, by (2.4) and (2.12),

$$\begin{aligned} (2.13) \quad |B_{n,r}(g_n, x) - g_n(x)| & \leq M_2 (n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x))^r \|\varphi^{r\lambda} g_n^{(r)}\| \\ & \leq M_2 A_2 \omega_{\varphi^{\lambda}}^r(f, n^{-\frac{1}{2}} \delta_n^{1-\lambda}(x)). \end{aligned}$$

If $x \in E_n^c = [0, \frac{1}{n}] \cup (1 - \frac{1}{n}, 1]$, then $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, and by (2.4), (2.5) and (2.11) we obtain

$$\begin{aligned}
 |B_{n,r}(g_n, x) - g_n(x)| &\leq M_3(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^r (\|\varphi^{r\lambda}g_n^{(r)}\| + \|n^{-\frac{r\lambda}{2}}g_n^{(r)}\|) \\
 &\leq M_4((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^r \|\varphi^{r\lambda}g_n^{(r)}\| \\
 &\quad + (n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^{2r/(2-\lambda)} \|g_n^{(r)}\|) \\
 &\leq M_5\omega_{\varphi^\lambda}^r(f, n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x)).
 \end{aligned}
 \tag{2.14}$$

From (2.3), (2.8), (2.13) and (2.14) we get (2.2). ■

3. INVERSE THEOREM

In this section we prove the inverse part of (1.8).

Theorem 2. For $f \in C[0, 1]$, $0 < \alpha < r$, $0 \leq \lambda \leq 1$, if

$$|B_{n,r}(f, x) - f(x)| \leq B(n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha$$

then

$$\omega_{\varphi^\lambda}^r(f, t) = O(t^\alpha). \tag{3.1}$$

To prove Theorem 2 we need some new notation and some lemmas. We use the following notation.

$$C_0 = \{f \in C[0, 1], f(0) = f(1) = 0\},$$

$$\|f\|_0 = \sup_{x \in (0,1)} |\delta_n^{\alpha(\lambda-1)}(x)f(x)|,$$

$$C_\lambda^0 = \{f \in C_0, \|f\|_0 < \infty\},$$

$$\|f\|_r = \sup_{x \in (0,1)} |\delta_n^{r+\alpha(\lambda-1)}(x)f^{(r)}(x)|,$$

$$C_\lambda^r = \{f \in C_0, \|f\|_r < \infty, f^{(r-1)} \in A.C_{loc}\}.$$

For $f \in C_0$ we define the K -functional as follows:

$$K_\lambda^\alpha(f, t^r) = \inf_{g \in C_\lambda^r} \{\|f - g\|_0 + t^r \|g\|_r\}. \tag{3.2}$$

We also need the following lemmas which will be proved in the next section.

Lemma 3.1. If $n \in \mathbb{N}$, $0 < \alpha < r$, then

$$\|B_n f\|_r \leq B_1 n^{\frac{r}{2}} \|f\|_0 \quad (f \in C_\lambda^0). \tag{3.3}$$

and

$$\|B_n f\|_r \leq B_2 \|f\|_r \quad (f \in C_\lambda^r). \tag{3.4}$$

Lemma 3.2. For $0 < t < \frac{1}{8r}$, $\frac{rt}{2} \leq x \leq 1 - \frac{rt}{2}$ and $0 \leq \beta \leq r$, we have

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \delta_n^{-\beta}(x + \sum_{k=1}^r u_k) du_1 \cdots du_r \leq C(\beta) t^r \delta_n^{-\beta}(x). \tag{3.5}$$

Now we prove (3.1).

Proof of (3.1). Since $B_n(f, x)$ preserves linear functions, we consider only $f \in C_0$ for $r > 1$. If $r = 1$, $f(x) = ax + b$ then $\omega_{\varphi^\lambda}^1(ax + b, t) = a\varphi^\lambda(x)t \leq at^\alpha$ ($0 < \alpha < 1$).

So, we may assume $f \in C_0$. From (3.2) we have

$$(3.6) \quad K_\lambda^\alpha(f, t^r) \leq \|B_{n,r}(f) - f\|_0 + t^r \|B_{n,r}f\|_r$$

and we may choose $g \in C_\lambda^r$ such that

$$(3.7) \quad \|f - g\|_0 \leq 2K_\lambda^\alpha(f, n^{-\frac{1}{2}}) \quad \text{and} \quad n^{-\frac{r}{2}} \|g\|_r \leq 2K_\lambda^\alpha(f, n^{-\frac{1}{2}}).$$

By the assumption of Theorem 2, one has

$$(3.8) \quad \|B_{n,r}(f, x) - f(x)\|_0 \leq Bn^{-\frac{\alpha}{2}}.$$

Using Lemma 3.1 and (3.7) we have

$$(3.9) \quad \begin{aligned} \|B_{n,r}(f)\|_r &\leq \|B_{n,r}(f - g)\|_r + \|B_{n,r}g\|_r \\ &\leq M(n^{\frac{r}{2}} \|f - g\|_0 + \|g\|_r) \\ &\leq 2Mn^{\frac{r}{2}} K_\lambda^\alpha(f, n^{-\frac{r}{2}}). \end{aligned}$$

From (3.6), (3.8) and (3.9) we obtain

$$K_\lambda^\alpha(f, t^r) \leq M_1(n^{-\frac{\alpha}{2}} + t^r n^{\frac{r}{2}} K_\lambda^\alpha(f, n^{-\frac{r}{2}}))$$

and this implies, via the Berens-Lorentz lemma [1], that if $\alpha < r$ then

$$(3.10) \quad K_\lambda^\alpha(f, t^r) \leq M_2 t^\alpha.$$

On the other hand, notice that $\delta_n^{\alpha(1-\lambda)}(x)$ is concave. So, we have, for $g \in C_\lambda^r$,

$$(3.11) \quad \begin{aligned} \left| \Delta_{t\varphi^\lambda(x)}^r g(x) \right| &\leq \|g\|_0 \sum_{j=0}^r \binom{r}{j} \delta_n^{\alpha(1-\lambda)} \left(x + (j - \frac{r}{2}) t\varphi^\lambda(x) \right) \\ &\leq \|g\|_0 2^r \delta_n^{\alpha(1-\lambda)}(x). \end{aligned}$$

Using Lemma 3.2 for $g \in C_\lambda^r$, $0 < t\varphi^\lambda(x) < \frac{1}{8r}$ and $\frac{rt\varphi^\lambda(x)}{2} \leq x \leq 1 - \frac{rt\varphi^\lambda(x)}{2}$, we have

$$(3.12) \quad \begin{aligned} &\left| \Delta_{t\varphi^\lambda(x)}^r g(x) \right| \\ &\leq \left| \int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} \cdots \int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} g^{(r)}(x + u_1 + \cdots + u_r) du_1 \cdots du_r \right| \\ &\leq \|g\|_r \int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} \cdots \int_{-\frac{t}{2}\varphi^\lambda(x)}^{\frac{t}{2}\varphi^\lambda(x)} \delta_n^{-r+\alpha(1-\lambda)}(x + u_1 + \cdots + u_r) du_1 \cdots du_r \\ &\leq M_3 t^r \varphi^{r\lambda}(x) \delta_n^{-r+\alpha(1-\lambda)}(x) \|g\|_r \leq M_3 t^r \delta_n^{(\alpha-r)(1-\lambda)}(x) \|g\|_r. \end{aligned}$$

From (3.9)-(3.12), $0 < t\varphi^\lambda(x) < \frac{1}{8r}$, $\frac{rt\varphi^\lambda(x)}{2} \leq x \leq 1 - \frac{rt\varphi^\lambda(x)}{2}$ and by choosing an appropriate g , we have

$$\begin{aligned} \left| \Delta_{t\varphi^\lambda(x)}^r f(x) \right| &\leq \left| \Delta_{t\varphi^\lambda(x)}^r (f - g)(x) \right| + \left| \Delta_{t\varphi^\lambda(x)}^r g(x) \right| \\ &\leq M_4 \delta_n^{\alpha(1-\lambda)}(x) \left\{ \|f - g\|_0 + t^r \delta_n^{r(\lambda-1)}(x) \|g\|_r \right\} \\ &\leq 2M_4 \delta_n^{\alpha(1-\lambda)} K_\lambda^\alpha \left(f, \frac{t^r}{\delta_n^{r(1-\lambda)}(x)} \right) \\ &\leq 2M_4 \delta_n^{\alpha(1-\lambda)}(x) \frac{t^\alpha}{\delta_n^{\alpha(1-\lambda)}(x)} \\ &= M_5 t^\alpha. \end{aligned}$$

The proof of (3.1) is complete.

4. PROOFS OF THE LEMMAS

Proof of Lemma 3.1. We first prove (3.3). Suppose that $E_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. For $x \in E_n^c = (0, \frac{1}{n}) \cup (1 - \frac{1}{n}, 1)$, we have [5] by Hölder's inequality

$$\begin{aligned} \left| B_n^{(r)}(f, x) \right| &= \left| \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} \bar{\Delta}_{\frac{1}{n}}^r f\left(\frac{k}{n}\right) p_{n-r,k}(x) \right| \\ &\leq L n^r \|f\|_0 \sum_{k=0}^{n-r} \sum_{j=0}^r \binom{r}{j} \delta_n^{\alpha(1-\lambda)} \left(\frac{k+r-j}{n} \right) p_{n-r,k}(x) \\ &\leq L_1 n^r \|f\|_0 \left(\sum_{k=0}^{n-r} \sum_{j=0}^r \delta_n^{2r} \left(\frac{k+r-j}{n} \right) p_{n-r,k}(x) \right)^{\alpha(1-\lambda)/2r}. \end{aligned}$$

For $n > 4r$ we have

$$\begin{aligned} \sum_{k=0}^{n-r} \varphi^{2r} \left(\frac{k+1}{n} \right) p_{n-r,k}(x) &= \left(\sum_{k=0}^{2r} + \sum_{k=2r+1}^{n-3r} + \sum_{k=n-3r+1}^{n-r} \right) \varphi^{2r} \left(\frac{k+1}{n} \right) p_{n-r,k}(x) \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Obviously $I_1 + I_3 \leq 2((3r+1)!)^r/n^r$ and, by simple computation, we have

$$\begin{aligned} I_2 &= \sum_{k=2r+1}^{n-3r} \left(\frac{k+1}{n} \right)^r \left(\frac{n-k-1}{n} \right)^r \frac{(n-r)!}{(n-r-k)!k!} x^k (1-x)^{n-r-k} \\ &= x^r (1-x)^r \sum_{k=2r+1}^{n-3r} \frac{(n-r) \cdots (n-3r+1)}{n^{2r}} \cdot \frac{(k+1)^r}{k \cdots (k-r+1)} \\ &\quad \cdot \frac{(n-k-1)^r}{(n-r-k) \cdots (n-2r-k+1)} p_{n-3r,k-r}(x) \\ &\leq 2 \times 2^r \times 3^r \varphi^{2r}(x). \end{aligned}$$

By this and $\delta_n^{2r}(\frac{k+r-j}{n}) \sim \varphi^{2r}(\frac{k+1}{n}) + (\frac{1}{n})^r$, one has

$$\left| B_n^{(r)}(f, x) \right| \leq L_2 n^r \|f\|_0 \delta_n^{\alpha(1-\lambda)}(x).$$

Recalling that $x \in E_n^c$ implies $\delta_n(x) \sim \frac{1}{\sqrt{n}}$, we see that

$$(4.1) \quad \delta_n^{r+\alpha(\lambda-1)}(x) \left| B_n^{(r)}(f, x) \right| \leq L_3 n^{\frac{r}{2}} \|f\|_0.$$

For $x \in E_n$ we use the expression (cf. [5])

$$B_n^{(r)}(f, x) = (x(1-x))^{-r} \sum_{i=0}^r Q_i(x, n) n^i \sum_{k=0}^n \left(\frac{k}{n} - x\right)^i f\left(\frac{k}{n}\right) p_{n,k}(x)$$

with $Q_i(n, x)$ a polynomials in $nx(1-x)$ of degree $[(r-i)/2]$ with nonconstant bounded coefficients. Thus,

$$\left| (x(1-x))^{-r} Q_i(x, n) n^i \right| \leq L_4 \left(\frac{n}{x(1-x)} \right)^{(r+i)/2} \quad \text{for } x \in E_n.$$

If $x \in E_n$ then $\delta_n(x) \sim \varphi(x)$ and, recalling that $\delta_n^2(\frac{k}{n}) \sim \varphi^2(\frac{k}{n}) + \frac{1}{n}$, we have by using Hölder inequality twice

$$\begin{aligned} \left| B_n^{(r)}(f, x) \right| &\leq L_5 \sum_{i=0}^r \left(\frac{n}{\varphi^2(x)} \right)^{(r+i)/2} \sum_{k=0}^n \left| \frac{k}{n} - x \right|^i \delta_n^{\alpha(1-\lambda)} \left(\frac{k}{n} \right) p_{n,k}(x) \|f\|_0 \\ &\leq L_5 \|f\|_0 \sum_{i=0}^r \left(\frac{n}{\varphi^2(x)} \right)^{(r+i)/2} \left(\sum_{k=0}^n \left(\frac{k}{n} - x \right)^{2i} p_{n,k}(x) \right)^{\frac{1}{2}} \\ &\quad \left(\sum_{k=0}^n \left(\varphi^2 \left(\frac{k}{n} \right) + \frac{1}{n} \right)^r p_{n,k}(x) \right)^{\frac{\alpha(1-\lambda)}{2r}}. \end{aligned}$$

Proceeding as in (4.1), we obtain

$$\sum_{k=0}^n \left(\varphi^2 \left(\frac{k}{n} \right) + \frac{1}{n} \right)^r p_{n,k}(x) \leq C \delta_n^{2r}(x).$$

It is known that for $m \in N$ [5]

$$\left| \sum_{k=0}^n \left(\frac{k}{n} - x \right)^{2m} p_{n,k}(x) \right| \leq L_6 n^{-m} \varphi^{2m}(x) \quad \text{for } x \in E_n.$$

Consequently, with $x \in E_n$, $\delta_n(x) \sim \varphi(x)$ and

$$\begin{aligned} (4.2) \quad \left| B_n^{(r)}(f, x) \right| &\leq L_6 \|f\|_0 \sum_{i=0}^r \left(\frac{n}{\varphi^2(x)} \right)^{(r+i)/2} \left(\frac{\varphi^{2i}(x)}{n^i} \right)^{\frac{1}{2}} \varphi^{\alpha(1-\lambda)}(x) \\ &\leq L_6 (r+1) \left(\frac{n}{\varphi^2(x)} \right)^{r/2} \varphi^{\alpha(1-\lambda)}(x) \|f\|_0. \end{aligned}$$

Hence, using $\delta_n(x) \sim \varphi(x)$ we obtain (3.1) for $x \in E_n$. We complete our proof by using (4.1) and (4.2).

Proof of (3.4). We recall that [5]

$$(4.3) \quad \left| B_n^{(r)}(f, x) \right| \leq n^r \sum_{k=0}^{n-r} \left| \bar{\Delta}_{\frac{1}{n}}^r f \left(\frac{k}{n} \right) \right| p_{n-r,k}(x).$$

For $0 < k < n - r$, by [5, p. 155],

$$\frac{k}{n} \left(1 - \frac{k}{n} \right) \leq R \left(\frac{k}{n} + y \right) \left(1 - \frac{k}{n} - y \right), \quad 0 < y < \frac{r}{n},$$

therefore $\delta_n \left(\frac{k}{n} \right) \leq R_1 \delta_n \left(\frac{k}{n} + y \right)$. Hence we have, as in [5],

$$(4.4) \quad \begin{aligned} \left| \bar{\Delta}_{\frac{1}{n}}^r f \left(\frac{k}{n} \right) \right| &\leq R_2 n^{-r+1} \int_0^{\frac{r}{n}} \left| f^{(r)} \left(\frac{k}{n} + u \right) \right| du \\ &\leq R_2 n^{-r+1} \|f\|_r \int_0^{\frac{r}{n}} \delta_n^{-r+\alpha(1-\lambda)} \left(\frac{k}{n} + u \right) du \\ &\leq R_2 n^{-r+1} \|f\|_r \frac{r}{n} \delta_n^{-r+\alpha(1-\lambda)} \left(\frac{k}{n} \right). \end{aligned}$$

For $k = 0$, note that $u \in (0, \frac{r}{n})$ implies $\delta_n(u) \sim \frac{1}{\sqrt{n}}$. Thus, we have

$$(4.5) \quad \begin{aligned} \left| \bar{\Delta}_{\frac{1}{n}}^r f(0) \right| &\leq R_3 \int_0^{\frac{r}{n}} u^{r-1} |f^{(r)}(u)| du \\ &\leq R_3 \|f\|_r \int_0^{\frac{r}{n}} u^{r-1} \delta_n^{-r+\alpha(1-\lambda)}(u) du \\ &\leq R_4 n^{-(r+\alpha(1-\lambda))/2} \|f\|_r. \end{aligned}$$

Similarly for $k = n - r$ we have

$$(4.6) \quad \left| \bar{\Delta}_{\frac{1}{n}}^r f \left(\frac{n-r}{n} \right) \right| \leq R_5 n^{-(r+\alpha(1-\lambda))/2} \|f\|_r.$$

From (4.3)-(4.6) we get, with $\delta_n(\frac{k}{n}) \sim \delta_n(\frac{k+1}{n-r+2})$,

$$(4.7) \quad \begin{aligned} &\left| \delta_n^{r+\alpha(\lambda-1)}(x) B_n^{(r)}(f, x) \right| \\ &\leq R_6 n^r \|f\|_r \delta_n^{r+\alpha(\lambda-1)}(x) \left[n^{\frac{-r+\alpha(\lambda-1)}{2}} (p_{n-r,0}(x) + p_{n-r,n-r}(x)) \right. \\ &\quad \left. + \sum_{k=1}^{n-r-1} n^{-r} \delta_n^{-r+\alpha(\lambda-1)} \left(\frac{k}{n} \right) p_{n-r,k}(x) \right]. \end{aligned}$$

By a simple computation, it is easy to get (cf. [2])

$$\sum_{k=1}^{n-r-1} \left(\frac{n}{k} \right)^r p_{n-r,k}(x) \leq C \frac{1}{x^r},$$

and

$$\sum_{k=1}^{n-r-1} \left(\frac{n}{n-k} \right)^r p_{n-r,k}(x) \leq C \frac{1}{(1-x)^r}.$$

Hence

$$\begin{aligned}
 (4.8) \quad & \sum_{k=1}^{n-r-1} \varphi^{-2r} \left(\frac{k}{n} \right) p_{n-r,k}(x) \\
 & \leq 2^r \sum_{k=1}^{n-r-1} \left(\left(\frac{n}{k} \right)^r + \left(\frac{n}{n-k} \right)^r \right) p_{n-r,k}(x) \leq C_1 \varphi^{-2r}(x).
 \end{aligned}$$

Note that $\delta_n(\frac{k}{n}) \sim \max\{\varphi(\frac{k}{n}), \frac{1}{\sqrt{n}}\}$, and that by (4.7) and (4.8) we have

$$\begin{aligned}
 & \left| \delta_n^{r+\alpha(\lambda-1)}(x) B_n^{(r)}(f, x) \right| \\
 & \leq R_6 \|f\|_r \delta_n^{r+\alpha(\lambda-1)}(x) \left[n^r (p_{n-r,0}(x) + p_{n-r,n-r}(x)) \right. \\
 & \quad \left. + \sum_{k=1}^{n-r-1} \delta_n^{-2r} \left(\frac{k}{n} \right) p_{n-r,k}(x) \right]^{\frac{r-\alpha(1-\lambda)}{2r}} \\
 & \leq R_7 \|f\|_r \delta_n^{r+\alpha(\lambda-1)}(x) \left(\min \left\{ n^r, \sum_{k=1}^{n-r-1} \varphi^{-2r} \left(\frac{k}{n} \right) p_{n-r,k}(x) \right\} \right)^{\frac{r-\alpha(1-\lambda)}{2r}} \\
 & \leq R_8 \|f\|_r \delta_n^{r+\alpha(\lambda-1)}(x) \delta_n^{-r+\alpha(1-\lambda)}(x) = R_8 \|f\|_r.
 \end{aligned}$$

Now we have proved the inequality (3.4). This finishes the proof of Lemma 3.1.

Proof of Lemma 3.2. It is known that, for $0 < t < \frac{1}{8r}$, $\frac{rt}{2} \leq x \leq 1 - \frac{rt}{2}$ (cf. [7]),

$$\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \varphi^{-r}(x + u_1 + \cdots + u_r) du_1 \cdots du_r \leq C t^r \varphi^{-r}(x).$$

Using this and Hölder's inequality we obtain

$$\begin{aligned}
 & \int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \delta_n^{-r+\alpha(1-\lambda)}(x + u_1 + \cdots + u_r) du_1 \cdots du_r \\
 & \leq C_1 \left(\int_{-\frac{t}{2}}^{\frac{t}{2}} \cdots \int_{-\frac{t}{2}}^{\frac{t}{2}} \min \left\{ \varphi^{-r}(x + u_1 + \cdots + u_r), n^{\frac{r}{2}} \right\} du_1 \cdots du_r \right)^{\frac{r-\alpha(1-\lambda)}{r}} t^{\alpha(1-\lambda)} \\
 & \leq C_2 \left(t^r \min \left\{ \varphi^{-r}(x), n^{\frac{r}{2}} \right\} \right)^{\frac{r-\alpha(1-\lambda)}{r}} t^{\alpha(1-\lambda)} \\
 & \leq C_3 t^r \delta_n^{-r+\alpha(1-\lambda)}(x),
 \end{aligned}$$

which is the stated result.

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