APPROXIMATING THE ZEROS OF ACCRETIVE OPERATORS BY THE ISHIKAWA ITERATION PROCESS

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ABSTRACT. Some strong convergence theorems are established for the Ishikawa iteration processes for accretive operators in uniformly smooth Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real Banach space with a dual X^* and normalized duality mapping $J : X \rightarrow 2^{X^*}$, defined by

$$Jx = \{f \in X^* : \langle f, x \rangle = \|f\|\|x\|, \|f\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

It is well known that if X^* is strictly convex, then J is single-valued and such that $J(tx) = tJx$ for all $t \geq 0$, $x \in X$. If X is uniformly smooth, then J is uniformly continuous on bounded subsets of X .

An operator A with domain $D(A)$ and kernel $N(A)$ is said to be “accretive” if, for every $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$(1.1) \quad \langle Ax - Ay, j(x - y) \rangle \geq 0.$$

It is said to be “strongly accretive” if, in addition, there is a strictly increasing function $\psi : R^+ \rightarrow R^+$ such that $\psi(0) = 0$ and

$$(1.2) \quad \langle Ax - Ay, j(x - y) \rangle \geq \psi(\|x - y\|)\|x - y\|.$$

The operator A is “uniformly accretive” if there is a fixed positive constant $k > 0$ such that

$$(1.3) \quad \langle Ax - Ay, j(x - y) \rangle \geq k\|x - y\|.$$

Furthermore, if $N(A) \neq \emptyset$ and the inequalities (1.1), (1.2) and (1.3) hold for any $x \in D(A)$ but $y \in N(A)$, then the corresponding operator A is said

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to be “quasi-accretive”, “strongly quasi-accretive”, and “uniformly quasi-accretive”, respectively. Such operators have been extensively studied and used by various authors (see, e.g., [1]-[3]).

A quasi-accretive operator A is said to satisfy “Condition (I)” if, for any $x \in D(A)$, $p \in N(A)$, and any $j(x - p) \in J(x - p)$ the equality $\langle Ax, j(x - p) \rangle \geq 0$ holds if and only if $Ax = Ap = 0$.

Recently, Xu and Roach [29] studied the characteristic conditions for the convergence of the steepest descent approximation process

$$(*) \quad \begin{cases} x_0 \in X, \\ x_{n+1} = x_n - t_n Ax_n, \quad n \geq 0, \end{cases}$$

where $t_n \in (0, \infty)$, $\sum_{n=0}^{\infty} t_n = \infty$, and $t_n \rightarrow 0$ ($n \rightarrow \infty$), for all $n \geq 0$. They proved the following two theorems.

Theorem A. ([29]) *Let X be a uniformly smooth Banach space and let $A : D(A) = X \rightarrow X$ be a quasi-accretive, bounded operator which satisfies the condition (I). Then, for any initial value $x_0 \in D(A)$, there are positive real numbers $T(x_0)$ such that the steepest descent approximation method $(*)$, with $t_n \leq T(x_0)$ for any n , converges strongly to a solution x^* of the equation $Ax = 0$ if and only if there is a strictly increasing function $\psi : R^+ \rightarrow R^+$, $\psi(0) = 0$, such that*

$$\langle Ax_n - Ax^*, J(x_n - x^*) \rangle \geq \psi(\|x_n - x^*\|)\|x_n - x^*\|.$$

In what follows, $F(T)$ is the fixed point set of the operator T .

Theorem B. ([29]) *Let X be a uniformly convex Banach space, $D \subset X$ a nonempty closed convex subset of X , and $T : D \rightarrow D$ a quasi-nonexpansive mapping (that is, $F(T) \neq \emptyset$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in D$ and $y \in F(T)$). Then, for any initial value $x_0 \in D$, the Mann type iterative process*

$$\begin{cases} x_0 \in D, \\ x_{n+1} = (1 - t_n)x_n + t_n Tx_n, \quad n \geq 0, \quad 0 < t_n < 1, \quad \sum_{n=0}^{\infty} t_n = +\infty, \end{cases}$$

converges strongly to a fixed point x^* of T if and only if there is a strictly increasing function $f : R^+ \rightarrow R^+$, $f(0) = 0$, such that

$$\|x_n - Tx_n\| \geq f(d(x_n, F(T))), \quad n \geq 0.$$

One question arises naturally: Can the Ishikawa type iterative process be extended to the above theorems A and B?

In this paper we give an answer to this question.

To establish our main results, we need some special geometric properties of Banach spaces. Recall that a Banach space X is said to be “uniformly convex” if $\delta_X(\epsilon)$, the modulus of convexity of X , which is defined by

$$\delta_X(\epsilon) = \inf \left\{ 1 - \frac{1}{2}\|x + y\| : \|x\| = 1, \|y\| = 1, \|x - y\| \geq \epsilon \right\},$$

satisfies $\delta_X(0) = 0$ and $\delta_X(\epsilon) > 0$ for any $0 < \epsilon \leq 2$. A Banach space X is said to be “uniformly smooth” if the modulus of smoothness of X , defined by

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}\|x+y\| + \frac{1}{2}\|x-y\| - 1 : \|x\| = 1, \|y\| \leq \tau\right\},$$

satisfies

$$(1.4) \quad \lim_{\tau \rightarrow 0} \frac{\rho_X(\tau)}{\tau} = 0.$$

It is well known that every Hilbert space H , the Lebesgue spaces L^p ($1 < p < \infty$), and the Sobolev spaces W_m^p ($1 < p < \infty$) are uniformly convex and uniformly smooth.

Lemma 1.1. (Xu and Roach [28]) *Let X be a real uniformly smooth Banach space. Then*

$$(1.5) \quad \|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + K \max\{\|x\| + \|y\|, \frac{c}{2}\} \rho_X(\|y\|),$$

for all $x, y \in X$, where K and c are positive constants.

Remark 1.1. In [19, p. 89] Reich established an inequality analogous to (1.5). Reich’s inequality reads as follows.

Let X be uniformly smooth. Then there is a continuous nondecreasing function $\beta : [0, \infty) \rightarrow [0, \infty)$ such that $\beta(0) = 0, \beta(ct) \leq c\beta(t)$ for $c \geq 1$, and

$$(RI) \quad \|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|)$$

for all x and y in X . ■

We point out that, in some sense, Reich’s inequality (RI) is a special case of inequality (1.5). To see this, take

$$\beta(t) = \begin{cases} K \max\left\{1+t, \frac{c}{2}\right\} \frac{\rho_X(t)}{t}, & \text{if } t > 0, \\ 0, & \text{if } t = 0. \end{cases}$$

It is easy to verify that $\beta : [0, \infty) \rightarrow [0, \infty)$ is a continuous nondecreasing function satisfying $\beta(at) \leq a^2 c_0 \beta(t)$ for $a \geq 1$, where c_0 is a fixed positive constant. For such a function β , inequality (1.5) implies inequality (RI) . Since $\rho_X(t)$ possesses many more nice properties than $\beta(t)$, inequality (1.5) reveals much more information.

Lemma 1.2. (Xu and Roach [28]) *Let X be a real uniformly convex Banach space. Then*

$$(1.6) \quad \|x+y\|^2 \geq \|x\|^2 + 2\langle y, j(x) \rangle + \sigma(x, y),$$

where $j(x) \in Jx$, and

$$\sigma(x, y) = c \int_0^1 \frac{(\|x+ty\| \vee \|x\|)^2}{t} \delta_X\left(\frac{t\|y\|}{2(\|x+ty\| \vee \|x\|)}\right) dt$$

with c a positive constant.

Lemma 1.3. *Let X be a real uniformly convex Banach space. Then*

$$(1.7) \quad \|x - y\|^2 \leq \|x\|^2 - 2 \langle y, j(x - y) \rangle,$$

for all $x, y \in X$.

Proof. Inequality (1.7) follows from the fact that the normalized duality mapping is the subdifferential of $\|x\|^2/2$. ■

Lemma 1.4. *Let $\{\rho_n\}_{n=0}^\infty$ be a nonnegative real sequence satisfying*

$$\rho_{n+1} \leq (1 - \lambda_n)\rho_n + \sigma_n$$

with $\lambda_n \in [0, 1]$, $\sum_{n=0}^\infty \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. Then $\rho_n \rightarrow 0$, as $n \rightarrow \infty$.

Proof. See [20, Theorem, p. 336]. ■

We denote by $B(0, r)$ the open ball with center at zero and radius $r > 0$.

2. MAIN RESULTS

Theorem 2.1. *Let X be a real uniformly smooth Banach space, and let $A : X \rightarrow X$ be a bounded quasi-accretive operator. Assume that there exists a strictly increasing and surjective function $\psi : [0, \infty) = \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\psi(0) = 0$, such that*

$$(2.1) \quad \begin{cases} \langle Ax_0 - Ax^*, J(x_0 - x^*) \rangle \geq \psi(\|x_0 - x^*\|)\|x_0 - x^*\|, \\ \langle Ay_n - Ax^*, J(y_n - x^*) \rangle \geq \psi(\|y_n - x^*\|)\|y_n - x^*\|, \end{cases}$$

for any $x^* \in N(A)$, where $\{x_n\}_{n=0}^\infty$ is defined by

$$(IS) \quad \begin{cases} x_0 \in X, \\ x_{n+1} = x_n - \alpha_n Ay_n - \alpha_n \beta_n Ax_n, \\ y_n = x_n - \beta_n Ax_n, n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences satisfying the following conditions.

- (i) $0 < \alpha_n \leq T(x_0) = \min\left\{\beta, \frac{\psi^{-1}(\|Ax_0\|)}{4M(x_0)}\right\}$ and $0 \leq \beta_n \leq T_1(x_0) = \min\left\{\beta, \frac{\delta}{M(x_0)}, T(x_0)\right\}$ for all $n \geq 0$;
- (ii) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (iii) $\alpha_n \rightarrow 0, \beta_n \rightarrow 0$ as $n \rightarrow \infty$. where

$$M(x_0) = \sup\left\{\|Ay\| : \|y - x_0\| \leq 6\psi^{-1}(\|Ax_0\|)\right\},$$

$$\beta = \max\left\{\beta > 0 : \beta^{-1}\rho_X(2\beta M(x_0)) \leq \frac{\|Ax_0\|\psi^{-1}(\|Ax_0\|)}{K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\}}\right\},$$

and δ is some fixed positive constant such that

$$\|Jx - Jy\| < \frac{\|Ax_0\|\psi^{-1}(\|Ax_0\|)}{2M(x_0)}$$

whenever $\|x - y\| < \delta$ for all $x, y \in B(0, \psi^{-1}(\|Ax_0\|))$.

Then the Ishikawa iteration process $\{x_n\}_{n=0}^\infty$ defined by (IS) converges strongly to a solution x^* of the equation $Ax = 0$.

Proof. Since $\psi : R^+ \rightarrow R^+$ is strictly increasing and surjective, ψ is certainly bijective. Hence $\psi^{-1}(\|Ax_0\|)$ is well-defined. Let

$$M(x_0) = \sup\{\|Ay\| : \|y - x_0\| \leq 6\psi^{-1}(\|Ax_0\|)\}.$$

Clearly, $\|Ax_0\| \leq M(x_0)$. If $M(x_0) = 0$, then $Ax_0 = 0$, and, by (IS), we know that $y_n = x_n = x_0$. From (2.1) we have

$$0 = \langle Ax_0 - Ax^*, J(x_0 - x^*) \rangle \geq \psi(\|x_0 - x^*\|)\|x_0 - x^*\|,$$

so that $x^* = x_0$ and hence $x_n \rightarrow x^*$ as $n \rightarrow \infty$.

Suppose that $\|Ax_0\| > 0$. Then $M(x_0) > 0$. Since $\frac{\rho_X(\tau)}{\tau}$ is continuous and nondecreasing, and $\frac{\rho_X(\tau)}{\tau} \rightarrow 0$ as $\tau \rightarrow 0$, we can choose the largest β such that

$$\beta^{-1}\rho_X(2\beta M(x_0)) \leq \frac{\psi^{-1}(\|Ax_0\|)\|Ax_0\|}{K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\}}.$$

Let

$$T(x_0) = \min\left\{\beta, \frac{\psi^{-1}(\|Ax_0\|)}{4M(x_0)}\right\}.$$

Since X is uniformly smooth, J is uniformly continuous on the open ball $B(0, \psi^{-1}(\|Ax_0\|))$. Hence, for

$$\epsilon_0 = \frac{\|Ax_0\|\psi^{-1}(\|Ax_0\|)}{2M(x_0)} > 0,$$

there is some fixed $\delta > 0$ such that for all $x, y \in B(0, \psi^{-1}(\|Ax_0\|))$, and $\|x - y\| < \delta$, we have

$$\|Jx - Jy\| < \frac{\|Ax_0\|\psi^{-1}(\|Ax_0\|)}{2M(x_0)}.$$

Let $T_1(x_0) = \min\{\beta, \frac{\delta}{M(x_0)}, T(x_0)\}$. We now consider two possible cases.

Case 1. There exists positive integer n_0 such that

$$(C_1) \quad \|x_n - x^*\| \geq 2\psi^{-1}(\|Ax_0\|)$$

for all $n \geq n_0$.

It follows from (2.1) that $\|x_0 - x^*\| \leq \psi^{-1}(\|Ax_0\|)$. Without loss of generality we may assume that $\|x_{n_0-1} - x^*\| < 2\psi^{-1}(\|Ax_0\|)$. Thus, we have

$$\|x_{n_0-1} - x_0\| \leq \|x_{n_0-1} - x^*\| + \|x^* - x_0\| < 3\psi^{-1}(\|Ax_0\|).$$

So, $\|Ax_{n_0-1}\| \leq M(x_0)$. By (IS) we obtain

$$\begin{aligned} \|y_{n_0-1} - x^*\| &\leq \|x_{n_0-1} - x^*\| + \beta_{n_0-1}\|Ax_{n_0-1}\| \\ &\leq 2\psi^{-1}(\|Ax_0\|) + \psi^{-1}(\|Ax_0\|) \\ &= 3\psi^{-1}(\|Ax_0\|). \end{aligned}$$

Therefore, $\|y_{n_0-1} - x_0\| \leq 4\psi^{-1}(\|Ax_0\|)$ and hence $\|Ay_{n_0-1}\| \leq M(x_0)$. Consequently,

$$\begin{aligned}\|x_{n_0} - x^*\| &\leq \|x_{n_0-1} - x^*\| + \alpha_{n_0-1}\|Ay_{n_0-1}\| + \alpha_{n_0-1}\beta_{n_0-1}\|Ax_{n_0-1}\| \\ &\leq 4\psi^{-1}(\|Ax_0\|),\end{aligned}$$

and $\|x_{n_0} - x_0\| \leq 5\psi^{-1}(\|Ax_0\|)$. So, $\|Ax_{n_0}\| \leq M(x_0)$. Thus, we get that

$$\begin{aligned}\|y_{n_0} - x^*\| &\leq \|x_{n_0} - x^*\| + \beta_{n_0}\|Ax_{n_0}\| \\ &\leq 5\psi^{-1}(\|Ax_0\|).\end{aligned}$$

and $\|y_{n_0} - x_0\| \leq 6\psi^{-1}(\|Ax_0\|)$. Hence $\|Ay_{n_0}\| \leq M(x_0)$. On the other hand,

$$\begin{aligned}\|y_{n_0} - x^*\| &\geq \|x_{n_0} - x^*\| - \beta_{n_0}\|Ax_{n_0}\| \\ &\geq 2\psi^{-1}(\|Ax_0\|) - \psi^{-1}(\|Ax_0\|) \\ &= \psi^{-1}(\|Ax_0\|).\end{aligned}$$

Since ψ is increasing, we have

$$\psi(\|y_{n_0} - x^*\|) \geq \|Ax_0\|.$$

Using the inequality (1.5), we obtain

$$\begin{aligned}(2.2) \quad &\|x_{n_0+1} - x^*\|^2 \\ &= \|x_{n_0} - x^* - \alpha_{n_0}(Ay_{n_0} - Ax^*) - \alpha_{n_0}\beta_{n_0}(Ax_{n_0} - Ax^*)\|^2 \\ &\leq \|x_{n_0} - x^*\|^2 - 2\alpha_{n_0} \langle Ay_{n_0} - Ax^*, J(x_{n_0} - x^*) \rangle \\ &\quad + K \max\{\|x_{n_0} - x^*\| + \alpha_{n_0}\|Ay_{n_0}\| + \alpha_{n_0}\beta_{n_0}\|Ax_{n_0}\|, \frac{c}{2}\} \\ &\quad \rho_X \alpha_{n_0} (\|Ay_{n_0}\| + \beta_{n_0}\|Ax_{n_0}\|) \\ &\quad - 2\alpha_{n_0} \langle Ay_{n_0} - Ax^*, J(x_{n_0} - x^*) - J(y_{n_0} - x^*) \rangle \\ &\quad - 2\alpha_{n_0} \langle Ay_{n_0} - Ax^*, J(y_{n_0} - x^*) \rangle \\ &\quad + K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \rho_X (2\alpha_{n_0}M(x_0)) \\ &\leq \|x_{n_0} - x^*\|^2 + 2\alpha_{n_0}a_{n_0} - 2\alpha_{n_0}\psi(\|y_{n_0} - x^*\|)\|y_{n_0} - x^*\| \\ &\quad + K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \rho_X (2\alpha_{n_0}M(x_0)),\end{aligned}$$

where $a_{n_0} = \langle Ay_{n_0} - Ax^*, J(x_{n_0} - x^*) - J(y_{n_0} - x^*) \rangle$.

Noting that $\|x_{n_0} - y_{n_0}\| \leq \beta_{n_0}\|Ax_{n_0}\| < \delta$, we know that

$$\begin{aligned}(2.3) \quad &|a_{n_0}| \leq \|Ay_{n_0}\| \|J(x_{n_0} - x^*) - J(y_{n_0} - x^*)\| \\ &\leq M(x_0) \frac{\|Ax_0\|\psi^{-1}(\|Ax_0\|)}{2M(x_0)} \\ &= \frac{1}{2}\|Ax_0\|\psi^{-1}(\|Ax_0\|).\end{aligned}$$

Substituting (2.3) in (2.2) yields

$$\begin{aligned}
\|x_{n_0+1} - x^*\|^2 &\leq \|x_{n_0} - x^*\|^2 + \alpha_{n_0} \|Ax_0\| \psi^{-1}(\|Ax_0\|) \\
&\quad - 2\alpha_{n_0} \|Ax_0\| \psi^{-1}(\|Ax_0\|) \\
&\quad + K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \rho_X(2\alpha_{n_0} M(x_0)) \\
&\leq \|x_{n_0} - x^*\|^2 - \alpha_{n_0} (\|Ax_0\| \psi^{-1}(\|Ax_0\|) \\
&\quad - K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \rho_X(2\beta M(x_0)) \beta^{-1}) \\
&\leq \|x_{n_0} - x^*\|^2.
\end{aligned}$$

In the same way, we can prove that

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\| \leq \cdots \leq \|x_{n_0} - x^*\|$$

for all $n \geq n_0$. Hence $\lim_{n \rightarrow \infty} \|x_n - x^*\|$ exists, and let

$$l = \lim_{n \rightarrow \infty} \|x_n - x^*\|.$$

Now we want to show that $l = 0$. If not, assume that $l > 0$, then, by (IS), we have

$$\begin{aligned}
l &= \lim_{n \rightarrow \infty} (\|x_n - x^*\| - \beta_n \|Ax_n\|) \\
&\leq \lim_{n \rightarrow \infty} \inf \|y_n - x^*\| \leq \lim_{n \rightarrow \infty} \sup \|y_n - x^*\| \\
&\leq \lim_{n \rightarrow \infty} (\|x_n - x^*\| + \beta_n \|Ax_n\|) = l.
\end{aligned}$$

So, $\lim_{n \rightarrow \infty} \|y_n - x^*\| = l$.

We can choose positive integer N_1 such that $\|y_n - x^*\| > \frac{l}{2}$ and $\psi(\|y_n - x^*\|) > \psi(\frac{l}{2})$ for all $n > N_1$.

From (IS) we know that $\{Ay_n\}$ is a bounded sequence.

Let $M_1 = \sup_{n > 0} \{\|Ay_n\|\}$, $M_2 = \sup_{n > 0} \{\|x_n - x^*\|\}$, $M_3 = \sup_{n > 0} \{\|Ay_n\|\}$, $M_4 = M_1 + M_3$, and $M_5 = K \max\{M_2 + M_4, \frac{c}{2}\}$. Again, using the inequality (1.5), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\alpha_n \langle Ay_n - Ax^*, J(x_n - x^*) \rangle \\
&\quad - 2\alpha_n \beta_n \langle Ax_n - Ax^*, J(x_n - x^*) \rangle \\
&\quad + K \max\{\|x_n - x^*\| + \alpha_n \|Ay_n\| \\
&\quad + \alpha_n \beta_n \|Ax_n\|, \frac{c}{2}\} \rho_X \alpha_n (\|Ay_n\| + \beta_n \|Ax_n\|) \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n \|Ay_n\| \|J(x_n - x^*) - J(y_n - x^*)\| \\
&\quad - 2\alpha_n \psi(\|y_n - x^*\|) \|y_n - x^*\| \\
&\quad + K \max\{M_4 + M_2, \frac{c}{2}\} \rho_X (2\alpha_n M_4) \\
&\leq \|x_n - x^*\|^2 + 2\alpha_n b_n - \alpha_n l \psi(\frac{l}{2}) + M_5 \rho_X (2\alpha_n M_4) \\
&= \|x_n - x^*\|^2 + 2\alpha_n b_n - \alpha_n l \psi(\frac{l}{2}) + M_5 \rho_X (\alpha_n M_4),
\end{aligned}$$

where $b_n = M_1 \|J(x_n - x^*) - J(y_n - x^*)\|$. Observing that $\|y_n - x_n\| = \beta_n \|Ax_n\| \rightarrow 0$ as $n \rightarrow \infty$, we see that $b_n \rightarrow 0$ as $n \rightarrow \infty$, since J is uniformly continuous on bounded sets of X .

At this point we choose positive integer N_2 such that

$$2b_n + \frac{M_5 \rho_X(\alpha_n M_4)}{\alpha_n} < \frac{l}{2} \psi\left(\frac{l}{2}\right),$$

for all $n \geq N_2$. Then (2.4) yields

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|^2 - \alpha_n \frac{l}{2} \psi\left(\frac{l}{2}\right),$$

thus

$$\frac{l}{2} \psi\left(\frac{l}{2}\right) \alpha_n \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

for all $n \geq N_2$, and hence $\frac{l}{2} \psi\left(\frac{l}{2}\right) \sum_{n=N_2}^{\infty} \alpha_n \leq \|x_{N_2} - x^*\|^2$, which contradicts

$\sum_{n=0}^{\infty} \alpha_n = \infty$. So, $l = 0$. From (C_1) we see that $0 \geq 2\psi^{-1}(\|Ax_0\|)$. Hence $\psi^{-1}(\|Ax_0\|) = 0$ and $\|Ax_0\| = 0$, which contradicts $\|Ax_0\| > 0$. This contradiction shows that Case 1 is impossible.

Case 2. There exists an infinite subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$(C_2) \quad \|x_{n_k} - x^*\| < 2\psi^{-1}(\|Ax_0\|).$$

We are going to show

$$\|x_{n_k+m} - x^*\| \leq 2\psi^{-1}(\|Ax_0\|)$$

for all positive integers $m \geq 1$.

First of all, we prove that

$$\|x_{n_k+1} - x^*\| \leq 2\psi^{-1}(\|Ax_0\|)$$

If not, $\|x_{n_k+1} - x^*\| > 2\psi^{-1}(\|Ax_0\|)$. Observing

$$\|x_{n_k} - x_0\| \leq \|x_{n_k} - x^*\| + \|x^* - x_0\| \leq 3\psi^{-1}(\|Ax_0\|),$$

we have $\|Ax_{n_k}\| \leq M(x_0)$. Hence

$$\|y_{n_k} - x^*\| \leq \|x_{n_k} - x^*\| + \beta_{n_k} \|Ax_{n_k}\| \leq 3\psi^{-1}(\|Ax_0\|)$$

and

$$\|y_{n_k} - x_0\| \leq \|y_{n_k} - x^*\| + \|x^* - x_0\| \leq 4\psi^{-1}(\|Ax_0\|),$$

so that $\|Ay_{n_k}\| \leq M(x_0)$.

On the other hand, by (IS) we have

$$\begin{aligned}
\|y_{n_k} - x^*\| &\geq \|x_{n_k} - x^*\| - \beta_{n_k} \|Ax_{n_k}\| \\
&\geq \|x_{n_k+1} - x^*\| - \alpha_{n_k} \|Ay_{n_k}\| \\
&\quad - \beta_{n_k} \|Ax_{n_k}\| - \alpha_{n_k} \beta_{n_k} \|Ax_{n_k}\| \\
&\geq 2\psi^{-1}(\|Ax_0\|) - \frac{1}{4}\psi^{-1}(\|Ax_0\|) \\
&\quad - \frac{1}{4}\psi^{-1}(\|Ax_0\|) - \frac{1}{2}\psi^{-1}(\|Ax_0\|) \\
&= \psi^{-1}(\|Ax_0\|).
\end{aligned}$$

Hence $\psi(\|y_{n_k} - x^*\|) \geq \|Ax_0\|$.

Using the inequality (1.5), we get

$$\begin{aligned}
\|x_{n_k+1} - x^*\|^2 &\leq \|x_{n_k} - x^*\|^2 - 2\alpha_{n_k} \langle Ay_{n_k} - Ax^*, J(x_{n_k} - x^*) \rangle \\
&\quad - 2\alpha_{n_k} \beta_{n_k} \langle Ax_{n_k} - Ax^*, J(x_{n_k} - x^*) \rangle \\
&\quad + K \max\{\|x_{n_k} - x^*\| \\
&\quad + \alpha_{n_k} \|Ay_{n_k}\| + \alpha_{n_k} \beta_{n_k} \|Ax_{n_k}\|, \frac{c}{2}\} \rho_X(2\alpha_{n_k} M(x_0)) \\
&\leq \|x_{n_k} - x^*\|^2 + \alpha_{n_k} \|Ay_{n_k}\| \|J(x_{n_k} - x^*) - J(y_{n_k} - x^*)\| \\
&\quad - 2\alpha_{n_k} \psi(\|y_{n_k} - x^*\|) \|y_{n_k} - x^*\| \\
&\quad + K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \rho_X(2\alpha_{n_k} M(x_0)) \\
&\leq \|x_{n_k} - x^*\|^2 + 2\alpha_{n_k} M(x_0) \frac{\|Ax_0\| \psi^{-1}(\|Ax_0\|)}{2M(x_0)} \\
&\quad - 2\alpha_{n_k} \psi^{-1}(\|Ax_0\|) \|Ax_0\| \\
&\quad + K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \rho_X(2\alpha_{n_k} M(x_0)) \\
&\leq \|x_{n_k} - x^*\|^2 - \alpha_{n_k} (\|Ax_0\| \psi^{-1}(\|Ax_0\|) \\
&\quad - K \max\{6\psi^{-1}(\|Ax_0\|), \frac{c}{2}\} \frac{\rho_X(2\beta M(x_0))}{\beta}) \\
&\leq \|x_{n_k} - x^*\|^2.
\end{aligned}$$

Hence, $\|x_{n_k+1} - x^*\| \leq \|x_{n_k} - x^*\| < 2\psi^{-1}(\|Ax_0\|)$, which is a contradiction.

By induction, we can prove that

$$\|x_{n_k+m} - x^*\| \leq 2\psi^{-1}(\|Ax_0\|)$$

for all $m \geq 1$. Hence $\{x_n\}$ is a bounded sequence, so are $\{Ax_n\}$, $\{y_n\}$ and $\{Ay_n\}$.

Let $\alpha = \inf_{n \geq 0} \|y_n - x^*\|$. Then $\alpha = 0$. If not, assume that $\alpha > 0$. Then $\psi(\|y_n - x^*\|) \geq \psi(\alpha) > 0$.

Again, using (1.5), we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
& \leq \|x_n - x^*\|^2 - 2\alpha_n \langle Ay_n - Ax^*, J(x_n - x^*) \rangle \\
& \quad - 2\alpha_n \beta_n \langle Ax_n - Ax^*, J(x_n - x^*) \rangle \\
& \quad + K \max\{\|x_n - x^*\| + \alpha_n \|Ay_n\| \\
& \quad + \alpha_n \beta_n \|Ax_n\|, \frac{c}{2}\} \rho_X(\alpha_n (\|Ay_n\| + \beta_n \|Ax_n\|)) \\
& \leq \|x_n - x^*\|^2 + 2\alpha_n \|Ay_n\| \|J(x_n - x^*) - J(y_n - x^*)\| \\
(2.4) \quad & - 2\alpha_n \psi(\|y_n - x^*\|) \|y_n - x^*\| \\
& \quad + K \max\{M_4 + M_2, \frac{c}{2}\} \rho_X(2\alpha_n M_4) \\
& \leq \|x_n - x^*\|^2 + 2\alpha_n M_1 \|J(x_n - x^*) - J(y_n - x^*)\| \\
& \quad - 2\alpha_n \alpha \psi(\alpha) + c_1 \rho_X(\alpha_n) \\
& \leq \|x_n - x^*\|^2 - \alpha_n (2\alpha \psi(\alpha) \\
& \quad - 2M_1 \|J(x_n - x^*) - J(y_n - x^*)\| - c_1 \frac{\rho_X(\alpha_n)}{\alpha_n}),
\end{aligned}$$

where c_1 is some positive constant.

Since $\|J(x_n - x^*) - J(y_n - x^*)\| \rightarrow 0$ and $\frac{\rho_X(\alpha_n)}{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, we can choose a positive integer N_3 such that

$$2M_1 \|J(x_n - x^*) - J(y_n - x^*)\| + c_1 \frac{\rho_X(\alpha_n)}{\alpha_n} < \alpha \psi(\alpha)$$

for all $n \geq N_3$. Thus (2.5) yields

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - \alpha \psi(\alpha) \alpha_n, \quad n \geq N_3.$$

Hence $\alpha \psi(\alpha) \sum_{n=N_3}^{\infty} \alpha_n \leq \|x_{N_3} - x^*\|^2$, which contradicts with $\sum_{n=0}^{\infty} \alpha_n = \infty$. This contradiction shows that $\alpha = 0$. Consequently, there exists an infinite subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \rightarrow x^*$ as $j \rightarrow \infty$, and hence $x_{n_j} \rightarrow x^*$ as $j \rightarrow \infty$. As in the proof of the boundedness for $\{x_n\}$ we can prove that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. ■

Remark 2.1. In the same way, as [29], we can prove that if $x_n \rightarrow x^* \in N(A)$ as $n \rightarrow \infty$ and $A : X \rightarrow X$ is quasi-accretive and satisfies condition (I), then there exists a strictly increasing function $\psi : [0, \infty) \rightarrow [0, \infty)$, $\psi(0) = 0$, such that

$$\langle Ay_n - Ax^*, J(y_n - x^*) \rangle \geq \psi(\|y_n - x^*\|) \|y_n - x^*\|.$$

But, such a function ψ is not surjective. From our Theorem 1, we can deduce the sufficiency of the Theorem 1 of [29]. In fact, the proof of the sufficiency of Theorem 1 of [29] has some mistakes. The authors of [29] did not require that the function ψ be surjective. Since x_0 is arbitrarily chosen, it is possible

that $\|Ax_0\| \notin R(\psi)$ (the range of ψ). In this case, $\psi^{-1}(\|Ax_0\|)$ is not well defined.

Remark 2.2. From our Theorem 1, we can deduce the relevant results of Tan and Xu [23] and Chidume [6-11].

Remark 2.3. We would like to point out that Theorem 2.1 is closely related to the well known strong convergence theorems in [4], [18]. Although they can't be deduced directly from Theorem 2.1, we can yield those results with our new approach. The detailed discussion of the relationship between Theorem 2.1 and the corresponding strong convergence theorems will be presented in a subsequent paper.

In the sequel, we prove the convergence theorems of the Ishikawa iteration processes for quasi-nonexpansive operators.

Let C be a nonempty bounded closed convex subset of Banach space X . An operator $T : C \rightarrow C$ is said to be quasi-nonexpansive, if the fixed point set $F(T)$ of T is nonempty, and

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x \in C$ but $y \in F(T)$.

The operator T is said to satisfy "Condition (A)" if there is a nondecreasing function $f : R^+ \rightarrow R^+$, $f(0) = 0$, $f(r) > 0$ for all $r > 0$, such that

$$\|x - Tx\| \geq f(d(x, F(T))), x \in C,$$

where $d(x, F(T)) = \inf\{\|x - z\| : z \in F(T)\}$ (see, e.g., [16]).

We study the following Ishikawa iteration process:

$$(I') \begin{cases} x_0 \in C \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n, n > 0 \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences satisfying:

- (i) $0 < \alpha_n, \beta_n < 1$;
- (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (iii) $\alpha_n \rightarrow 0, \beta_n \rightarrow 0$ as $n \rightarrow \infty$.

Let $A = I - T$. Then (I') yields

$$(I'') \begin{cases} x_0 \in C, \\ x_{n+1} = x_n - \alpha_n Ay_n - \alpha_n \beta_n Ax_n, \\ y_n = x_n - \beta_n Ax_n, n > 0. \end{cases}$$

We also need the following lemmas.

Lemma 2.1. For any $y \in F(T)$, $\lim_{n \rightarrow \infty} \|x_n - y\|$ exists.

Proof. From (I') , we have

$$\begin{aligned}
 \|x_{n+1} - y\| &\leq (1 - \alpha_n)\|x_n - y\| + \alpha_n\|y_n - y\| \\
 &\leq (1 - \alpha_n)\|x_n - y\| + \alpha_n((1 - \beta_n)\|x_n - y\| + \beta_n\|Tx_n - Ty\|) \\
 &\leq (1 - \alpha_n)\|x_n - y\| + \alpha_n\|x_n - y\| \\
 &= \|x_n - y\|. \blacksquare
 \end{aligned}$$

Lemma 2.2. (Xu and Roach [29]) *Let X be a real uniformly convex Banach space and let C be a nonempty closed convex subset of X . Assume $T : C \rightarrow C$ is quasi-nonexpansive. Let $A = I - T$. Then*

$$\langle Ax - Ay, j(x - y) \rangle \geq \frac{c}{2}\|x - y\|^2 \int_0^{\frac{\|Ax - Ay\|}{2\|x - y\|}} \frac{\delta_X(\epsilon)}{\epsilon} d\epsilon,$$

where $j(x - y) \in J(x - y)$, $\delta_X(\epsilon)$ is the modulus of convexity of X , and c is a fixed positive constant.

Now we prove:

Theorem 2.2. *Let X be a real uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of X . Assume that $T : C \rightarrow C$ is quasi-nonexpansive. If T satisfies the condition (A), then the Ishikawa type iteration sequence $\{x_n\}_{n=0}^\infty$ defined by (I'') converges strongly to some fixed point of T .*

Proof. We consider the following two possible cases.

Case a. $\inf_{n>0} \|x_n - y\| = 0$, $y \in F(T)$.

In this case, there exists subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow y$ as $j \rightarrow \infty$. By Lemma 2.1, we know that $x_n \rightarrow y$ as $n \rightarrow \infty$.

Case b. $\inf_{n>0} \|x_n - y\| = \alpha > 0$, $y \in F(T)$.

In this case, we again consider two possible cases.

Case b₁. $\inf_{n>0} \|Ax_n\| = 0$.

In this case, there exists subsequence $\{Ax_{n_j}\}$ of $\{Ax_n\}$ such that $Ax_{n_j} \rightarrow 0$ as $j \rightarrow \infty$. Since T satisfies the condition (A) we have

$$\|Ax_{n_j}\| = \|x_{n_j} - Tx_{n_j}\| \geq f(d(x_{n_j}, F(T))).$$

Hence $f(d(x_{n_j}, F(T))) \rightarrow 0$ as $j \rightarrow \infty$.

Since $f : R^+ \rightarrow R^+$ is nondecreasing and $f(r) > 0$ for all $r > 0$, so, $d(x_{n_j}, F(T)) \rightarrow 0$ as $j \rightarrow \infty$. By Lemma 2.1, we see that $\lim_{n \rightarrow \infty} d(x_n, F(T))$ exists. Hence $d(x_n, F(T)) \rightarrow 0$.

At this point we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{p_k\} \subset F(T)$, respectively, such that

$$\|x_{n_k} - p_k\| < 2^{-k} \text{ for all } k \geq 1.$$

By Lemma 2.1, we see that

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < 2^{-k},$$

so that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|x_{n_{k+1}} - p_{k+1}\| + \|x_{n_{k+1}} - p_k\| \\ &\leq 2^{-(k+1)} + 2^{-k} \\ &< 2^{-k+1} \end{aligned}$$

for all $k \geq 1$. Hence $\{p_k\}$ must be a Cauchy sequence. Thus we can assume $p_k \rightarrow p$ as $k \rightarrow \infty$.

Since $F(T)$ is closed, we know that $p \in F(T)$. Therefore, $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$ and hence $x_n \rightarrow p$ as $n \rightarrow \infty$, since $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists.

Case b₂. $\inf_{n>0} \|Ax_n\| = r > 0$.

Let $Q = \sup_{n>0} \|x_n - y\| > 0$, $y \in F(T)$. Then

$$0 < \frac{r}{2Q} \leq \frac{\|Ax_n - Ay\|}{2\|x_n - y\|} \leq 1.$$

Applying Lemma 2.2 we obtain

$$\begin{aligned} (2.5) \quad \langle Ax_n - Ay, j(x_n - y) \rangle &\geq \frac{c}{2} \|x_n - y\|^2 \int_0^{\frac{r}{2Q}} \frac{\delta_X(\epsilon)}{\epsilon} d\epsilon \\ &\geq k \|x_n - y\|^2, \end{aligned}$$

where

$$0 < k < \min\left\{1, \frac{c}{2} \int_0^{\frac{r}{2Q}} \frac{\delta_X(\epsilon)}{\epsilon} d\epsilon\right\}.$$

Using Lemma 1.3, (I'') and (2.6) we have

$$\begin{aligned} (2.6) \quad &\|x_{n+1} - y\|^2 \\ &\leq \|x_n - y\|^2 - 2\alpha_n \langle Ay_n - Ay, j(x_{n+1} - y) \rangle \\ &\quad - 2\alpha_n \beta_n \langle Ax_n, j(x_{n+1} - y) \rangle \\ &\leq \|x_n - y\|^2 - 2\alpha_n \langle Ay_n - Ax_{n+1}, j(x_{n+1} - y) \rangle \\ &\quad - 2\alpha_n \beta_n \langle Ax_n, j(x_{n+1} - y) \rangle \\ &\quad - 2\alpha_n \langle Ax_{n+1} - Ay, j(x_{n+1} - y) \rangle \\ &\quad - 2\alpha_n \beta_n \langle Ax_n, j(x_{n+1} - y) \rangle \\ &\leq \|x_n - y\|^2 - 2\alpha_n c_n - 2\alpha_n k \|x_{n+1} - y\|^2 \\ &\quad - 2\alpha_n \beta_n \langle Ax_n, j(x_{n+1} - y) \rangle, \end{aligned}$$

where $c_n = \langle Ay_n - Ax_{n+1}, j(x_{n+1} - y) \rangle$.

We show $c_n \rightarrow 0$ as $n \rightarrow \infty$. Indeed, since

$$\begin{aligned} (2.7) \quad &\|Ay_n - Ax_{n+1}\| \\ &\leq 2\|y_n - x_{n+1}\| \\ &\leq 2\beta_n \|Ax_n\| + 2\alpha_n \|Ay_n\| + 2\alpha_n \beta_n \|Ax_n\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

and $\|j(x_{n+1} - y)\| = \|x_{n+1} - y\|$ is bounded, $c_n \rightarrow 0$ as $n \rightarrow \infty$. Let $d_n = \langle Ax_n, j(x_{n+1} - y) \rangle$ and

$$\sigma_n = \frac{-2\alpha_n(c_n + \beta_n d_n)}{1 + 2k\alpha_n}.$$

From (2.7) we get

$$\begin{aligned} \|x_{n+1} - y\|^2 &\leq \frac{1}{1 + 2k\alpha_n} \|x_n - y\|^2 + \sigma_n \\ &= (1 - 2k\alpha_n + \frac{4k^2\alpha_n^2}{1 + 2k\alpha_n}) \|x_n - y\|^2 + \sigma_n \\ &\leq (1 - k\alpha_n) \|x_n - y\|^2 + \sigma_n. \end{aligned}$$

Set $\rho_n = \|x_n - y\|^2$, $\lambda_n = k\alpha_n$. Then, $\lambda_n \in [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$. By Lemma 1.4, we see that $\rho_n \rightarrow 0$ as $n \rightarrow \infty$, i.e., $x_n \rightarrow y$ as $n \rightarrow \infty$, which contradicts with $\inf_{n>0} \|x_n - y\| = \alpha > 0$.

From the above discussion, we know that $\{x_n\}$ converges strongly to some fixed point of T . ■

In the same way, we can prove

Theorem 2.3. *Let X and C be as in Theorem 2.2, and let $T : C \rightarrow C$ be a quasi-nonexpansion with $T(C)$ compact. Then the conclusion of Theorem 2.2 is still true.*

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