FLOW INVARIANCE FOR PERTURBED NONLINEAR EVOLUTION EQUATIONS

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ABSTRACT. Let X be a real Banach space, $J = [0, a] \subset \mathbf{R}$, $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ an m -accretive operator and $f : J \times X \rightarrow X$ continuous. In this paper we obtain necessary and sufficient conditions for weak positive invariance (also called viability) of closed sets $K \subset X$ for the evolution system

$$u' + Au \ni f(t, u) \quad \text{on } J = [0, a].$$

More generally, we provide conditions under which this evolution system has mild solutions satisfying time-dependent constraints $u(t) \in K(t)$ on J . This result is then applied to obtain global solutions of reaction-diffusion systems with nonlinear diffusion, e.g. of type

$$u_t = \Delta \Phi(u) + g(u) \quad \text{in } (0, \infty) \times \Omega, \quad \Phi(u(t, \cdot))|_{\partial\Omega} = 0, \quad u(0, \cdot) = u_0$$

under certain assumptions on the set $\Omega \subset \mathbf{R}^n$ the function $\Phi(u_1, \dots, u_m) = (\varphi_1(u_1), \dots, \varphi_m(u_m))$ and $g : \mathbf{R}_+^m \rightarrow \mathbf{R}^m$.

1. INTRODUCTION

Let X be a real Banach space and $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ m -accretive, where $2^X \setminus \emptyset$ denotes the nonempty subsets of X . Given $K : J = [0, a] \rightarrow 2^X \setminus \emptyset$ with closed values $K(t)$ such that $K_A(t) := K(t) \cap \overline{D(A)} \neq \emptyset$ on J and a continuous $f : \text{gr}(K_A) \rightarrow X$, we consider the initial value problem

$$(1) \quad u' + Au \ni f(t, u) \quad \text{on } J, \quad u(0) = x_0.$$

Given any initial value $x_0 \in K_A(0)$, we look for a mild solution u of (1), by which we mean a continuous $u : J \rightarrow X$ such that u is the mild solution of the quasi-autonomous problem

$$u' + Au \ni w(t) \quad \text{on } J, \quad u(0) = x_0,$$

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with $w(t) = f(t, u(t))$ on J ; notice that u then automatically has to satisfy $u(t) \in K_A(t)$ on J , since f is only defined on $\text{gr}(K_A)$. Suppose that (1) has mild solution u and let v be the mild solution of

$$v' + Av \ni f(0, x_0) \quad \text{on } J, \quad v(0) = x_0.$$

By continuity of f and u it follows that

$$\frac{1}{h}|u(h) - v(h)| \leq \frac{1}{h} \int_0^h |f(t, u(t)) - f(0, x_0)| dt \rightarrow 0 \quad \text{as } h \rightarrow 0+,$$

hence

$$\lim_{h \rightarrow 0+} h^{-1} \rho(S_{f(0, x_0)}(h)x_0, K_A(h)) = 0,$$

where $S_z(\cdot)$ denotes the semigroup generated by $-A_z$ with $A_z x := Ax - z$ on $D(A_z) = D(A)$.

By weak positive invariance of $K(\cdot)$ for $u' + Au \ni f(t, u)$ we mean that (1) has a mild solution on $J_\tau = [\tau, a]$ for every $\tau \in [0, a)$ and every initial value $u(\tau) = x_0 \in K_A(\tau)$. The argument given above shows that

$$(2) \quad f(t, x) \in T_K^A(t, x) \quad \text{for all } (t, x) \in \text{gr}(K_A) \text{ with } t < a$$

is a necessary condition for weak positive invariance of $K(\cdot)$, where T_K^A is defined on $\text{gr}(K_A) \cap ([0, a) \times X)$ by

$$T_K^A(t, x) = \{z \in X : \lim_{h \rightarrow 0+} h^{-1} \rho(S_z(h)x, K_A(t+h)) = 0\}.$$

In the special case $A = 0$ this becomes

$$T_K(t, x) = \{z \in X : \lim_{h \rightarrow 0+} h^{-1} \rho(x + hz, K(t+h)) = 0\},$$

and if, in addition, $K(t) \equiv K$ holds then $T_K(t, x) = T_K(x)$ is the Bouligand contingent cone w.r. to K at the point x ; see e.g. §4.1 in [10].

Since all $K(t)$ are closed by assumption, it is also natural to assume that $\text{gr}(K_A)$ is *closed from the left*, i.e.

$$(t_n) \subset J \text{ with } t_n \nearrow t \text{ and } x_n \in K_A(t_n) \text{ with } x_n \rightarrow x \text{ implies } x \in K_A(t);$$

notice that if there are mild solutions u_n with $u_n(t_n) = x_n$, then $K_A(t) \ni u_n(t) \rightarrow x$.

In this situation we will show that the "subtangential condition" (2) is also sufficient, provided the semigroup generated by $-A$ is compact and f satisfies the growth condition

$$(3) \quad |f(t, x)| \leq c(1 + |x|) \quad \text{on } \text{gr}(K_A) \text{ with some } c > 0.$$

In the final section this result is applied to a class of RD-systems including the model problem mentioned in the abstract above, and sufficient conditions for existence of global solutions are obtained.

2. PRELIMINARIES

Let X be a real Banach space with norm $|\cdot|$. Then $\overline{B}_r(x)$ denotes the closed ball in X with center x and radius r , $B_r(x)$ its interior and $\rho(x, B)$ is the distance from x to the set $B \subset X$, with the usual convention $\rho(x, \emptyset) = \infty$. Given $J = [0, a] \subset \mathbb{R}$, we let $C_X(J)$ be the Banach space of all continuous $u : J \rightarrow X$ and $L_X^1(J)$ the Banach space of all equivalence classes (w.r. to equality a.e.) of strongly measurable, Bochner-integrable $w : J \rightarrow X$, both equipped with the usual norms which we denote by $|\cdot|_0$, respectively $|\cdot|_1$. Given an operator $A : X \rightarrow 2^X$, we let $D(A) = \{x \in X : Ax \neq \emptyset\}$, $R(A) = \bigcup_{x \in D(A)} Ax$ and $\text{gr}(A) = \{(x, y) : x \in D(A), y \in Ax\}$ denote the domain, range and graph of A , respectively.

Recall that $A : X \rightarrow 2^X$ is m -accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$ and A is accretive, which means

$$(y - \bar{y}, x - \bar{x})_+ \geq 0 \quad \text{for all } x, \bar{x} \in D(A), y \in Ax \text{ and } \bar{y} \in A\bar{x}.$$

Here $(\cdot, \cdot)_+$ is given by $(z, x)_+ = \max\{x^*(z) : x^* \in \mathcal{F}(x)\}$ where $\mathcal{F} : X \rightarrow 2^{X^*} \setminus \emptyset$ denotes the duality map, i.e. $\mathcal{F}(x) = \{x^* \in X^* : x^*(x) = |x|^2 = |x^*|^2\}$; see e.g. §12.2 in [9].

If A is m -accretive, the resolvents $J_\lambda := (I + \lambda A)^{-1} : X \rightarrow D(A)$ are non-expansive mappings, i.e. $|J_\lambda x - J_\lambda y| \leq |x - y|$ on $X \times X$, for all $\lambda > 0$. Given $x \in D(A)$ we have $|J_\lambda x - x| \leq \lambda|y|$ for $\lambda > 0$, where y is any element of Ax , which implies $J_\lambda x \rightarrow x$ as $\lambda \rightarrow 0+$ on $\overline{D(A)}$. The resolvents satisfy the so-called resolvent identity

$$J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} J_\lambda x \right) \quad \text{on } X \text{ for all } \lambda, \mu > 0.$$

If A is m -accretive, it generates a semigroup $\{S(t)\}_{t \geq 0}$ of nonexpansive mappings $S(t) : \overline{D(A)} \rightarrow \overline{D(A)}$, given by the so-called exponential formula, i.e.

$$S(t)x = \lim_{n \rightarrow \infty} J_{t/n}^n x \quad \text{for } t \geq 0 \text{ and } x \in \overline{D(A)}.$$

Then $\{S(t)\}_{t \geq 0}$ is called the semigroup generated by $-A$, and it is said to be compact if $\overline{S(t)B}$ is compact for all $t > 0$ and bounded $B \subset \overline{D(A)}$ (i.e. the $S(t)$ are compact maps for $t > 0$). Let us note in passing that $\{S(t)\}_{t \geq 0}$ is compact iff $\{S(t)\}_{t > 0}$ is equicontinuous and J_λ is a compact map for some (or, equivalently, for all) $\lambda > 0$.

Let us also recall some facts concerning the quasi-autonomous problem

$$(4) \quad u' + Au \ni w(t) \quad \text{on } J_\tau = [\tau, a], \quad u(\tau) = x_0,$$

where $\tau \in [0, a)$. For m -accretive A , given any $w \in L_X^1(J_\tau)$ and $x_0 \in \overline{D(A)}$, the initial value problem (4) has a unique mild solution u . This means that $u : J_\tau \rightarrow \overline{D(A)}$ is continuous with $u(\tau) = x_0$ and u is the uniform limit of ϵ -DS-approximate solutions u^ϵ as $\epsilon \rightarrow 0+$. Here, by an ϵ -DS-approximate solution u^ϵ of (4) one means a function u^ϵ with $u^\epsilon(t_0) = x_0$ and $u^\epsilon(t) = x_k$

on $(t_{k-1}, t_k]$ for $k = 1, \dots, m$, where $\tau = t_0 < t_1 < \dots < t_m < a \leq t_m + \epsilon$ with $t_k - t_{k-1} \leq \epsilon$ and the x_k solve the implicit difference scheme

$$\frac{x_k - x_{k-1}}{t_k - t_{k-1}} + Ax_k \ni z_k \quad \text{for } k = 1, \dots, m$$

with $z_1, \dots, z_m \in X$ such that $\sum_{k=1}^m \int_{t_{k-1}}^{t_k} |z_k - w(t)| dt \leq \epsilon$.

In fact, every sequence of such ϵ_m -DS-approximate solutions u^{ϵ_m} converges to u uniformly on $[\tau, a)$ if $\epsilon_m \rightarrow 0+$.

In the sequel $u(\cdot; \tau, x_0, w)$ denotes the mild solution of (4) and we shall use the following property: If $w, \bar{w} \in L_X^1(J_\tau)$ and $x_0, \bar{x}_0 \in \overline{D(A)}$ then

$$|u(t; \tau, x_0, w) - u(t; \tau, \bar{x}_0, \bar{w})| \leq |x_0 - \bar{x}_0| + \int_\tau^t |w(s) - \bar{w}(s)| ds \quad \text{for } t \in J_\tau.$$

In particular, $w_n \rightarrow w$ in $L_X^1(J_\tau)$ implies $u(\cdot; \tau, x_0, w_n) \rightarrow u(\cdot; \tau, x_0, w)$ in $C_X(J_\tau)$. If $w \in L_X^1(J)$ then $u(\cdot; \tau, x_0, w)$ denotes $u(\cdot; \tau, x_0, w|_{J_\tau})$. With these notations, the semigroup property of solutions reads

$$u(t; \tau, x_0, w) = u(t; \bar{\tau}, u(\bar{\tau}; \tau, x_0, w), w) \quad \text{for all } 0 \leq \tau \leq \bar{\tau} \leq t \leq a.$$

If $\tau = 0$ and x_0 is fixed we simply write $u(\cdot; w)$ instead of $u(\cdot; \tau, x_0, w)$.

Let us also note that the autonomous problem, i.e. (4) with $w = 0$, has a unique mild solution for every $x_0 \in \overline{D(A)}$ if A is accretive and satisfies the weak range condition

$$\lim_{h \rightarrow 0+} h^{-1} \rho(x, R(I + hA)) = 0 \quad \text{for all } x \in \overline{D(A)}.$$

In proofs of this result one point is to show that if $(x_k)_{k \geq 0}$ is a solution of the above implicit difference scheme such that $t_k \nearrow t_\infty < a$, then $x_k \rightarrow x_\infty$ for some $x_\infty \in \overline{D(A)}$; this fact will be used later on.

Proofs of all facts mentioned so far can be found in [2] or [4].

Finally, we shall need the following compactness result for mild solutions of (4), which is Theorem 2 in [1].

Lemma 1. *Let X be a real Banach space, $A : X \rightarrow 2^X$ be m -accretive such that $-A$ generates a compact semigroup and let $J = [0, a] \subset \mathbb{R}$, $W = \{w \in L_X^1(J) : |w(t)| \leq \varphi(t) \text{ a.e. on } J\}$ with $\varphi \in L^1(J)$. Then $\{u(\cdot; w) : w \in W\}$ is relatively compact in $C_X(J)$.*

In fact, the assertion of Lemma 1 remains true if W is replaced by any uniformly integrable subset of $L_X^1(J)$; see Theorem 2.3.2 in [23].

3. EXISTENCE OF MILD SOLUTIONS UNDER TIME-DEPENDENT CONSTRAINTS

Our main result concerning problem (1) is

Theorem 1. *Let X be a real Banach space and $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ be m -accretive such that $-A$ generates a compact semigroup. Let $J = [0, a] \subset \mathbb{R}$*

and $K : J \rightarrow 2^X$ be such that $K_A(0) \neq \emptyset$ and $\text{gr}(K_A)$ is closed from the left. Let $f : \text{gr}(K_A) \rightarrow X$ be continuous, satisfying (2) and (3). Then

$$(1) \quad u' + Au \ni f(t, u) \quad \text{on } J, \quad u(0) = x_0$$

has a mild solution for every $x_0 \in K_A(0)$.

Proof. 1. To simplify subsequent arguments, we first reduce to the case when f is bounded on $\text{gr}(K_A)$. For this purpose, let $r(\cdot)$ be the solution of

$$r'(t) = 1 + c(1 + r(t) + |S(t)x_0|) \quad \text{on } J, \quad r(0) = 0,$$

and define

$$\hat{K}(t) := K(t) \cap \overline{B_{r(t)}(S(t)x_0)} \quad \text{and} \quad \hat{K}_A(t) := \hat{K}(t) \cap \overline{D(A)} \quad \text{for } t \in J.$$

Evidently, $x_0 \in \hat{K}_A(0)$, $\text{gr}(\hat{K}_A)$ is closed from the left and f is bounded on $\text{gr}(\hat{K}_A)$. In order to show that (2) also holds for $\hat{K}$ instead of K , let $t \in [0, a)$, $x \in \hat{K}_A(t)$ and $z := f(t, x)$. Due to (2) there are sequences $h_n \rightarrow 0+$ and $e_n \rightarrow 0$ such that

$$S_z(h_n)x + h_ne_n \in K_A(t + h_n) \quad \text{for all } n \geq 1.$$

By means of the estimate

$$\begin{aligned} |S_z(h_n)x + h_ne_n - S(t + h_n)x_0| &\leq \\ |S_z(h_n)x - S(h_n)x| + |x - S(t)x_0| + h_n|e_n| &\leq \\ h_n|f(t, x)| + r(t) + h_n|e_n| &\leq \\ r(t) + h_nc(1 + r(t) + |S(t)x_0|) + h_n|e_n| &\leq r(t + h_n), \end{aligned}$$

which holds if $n \geq 1$ is sufficiently large, this implies

$$S_z(h_n)x + h_ne_n \in \hat{K}_A(t + h_n) \quad \text{for all large } n \geq 1,$$

hence (2) also holds for $\hat{K}$. Consequently, all assumptions of Theorem 1 are also satisfied if K is replaced by $\hat{K}$, and we may therefore assume that f is bounded on $\text{gr}(K_A)$.

2. We now show which type of ϵ -approximate solutions can be expected for (1), where we start with the usual exploitation of the subtangential condition. Fix $\epsilon \in (0, 1]$. Since $z_0 := f(0, x_0) \in T_K^A(0, x_0)$, there is $h \in (0, \epsilon]$ such that $y_1 := S_{z_0}(h)x_0$ satisfies $\rho(y_1, K_A(h)) \leq \frac{1}{2}\epsilon h$, hence there is $x_1 \in K_A(h)$ such that $|e_0| \leq \epsilon$ for $e_0 := \frac{x_1 - y_1}{h}$. Then, letting $t_0 = 0$ and $t_1 = t_0 + h$,

$$v(t) := S_{z_0}(t - t_0)x_0 + (t - t_0)e_0 \quad \text{on } [t_0, t_1]$$

is a natural candidate as an approximate solution on $[t_0, t_1]$, and we may assume $|v(t) - x_0| \leq \epsilon$ on $[t_0, t_1]$ if $h > 0$ is chosen small enough. Consequently, we get sequences (t_k) , (x_k) , (z_k) and (e_k) by induction such that

$$(5) \quad \begin{aligned} t_k \nearrow t_\infty \leq a, \quad x_k \in K_A(t_k), \quad z_k = f(t_k, x_k), \\ e_k = (x_{k+1} - S_{z_k}(t_{k+1} - t_k)x_k)/(t_{k+1} - t_k), \quad |e_k| \leq \epsilon. \end{aligned}$$

For $k \geq 0$ we then let

$$(6) \quad v(t) = S_{z_k}(t - t_k)x_k + (t - t_k)e_k \quad \text{on } [t_k, t_{k+1}],$$

and may assume $t_{k+1} - t_k \leq \epsilon$ as well as $|v(t) - x_k| \leq \epsilon$ on $[t_k, t_{k+1}]$ by appropriate choice of the t_k . Of course $t_\infty < a$ is possible, and to be able to extend this approximate solution beyond t_∞ we then need (x_k) to be relatively compact.

To see that this is in fact true, let us first show

$$(7) \quad |v(t) - u(t; t_k, x_k, w)| \leq \epsilon(t - t_k) \quad \text{on } [t_k, t_\infty) \quad \text{for all } k \geq 0,$$

where $w \in L_X^1([0, t_\infty])$ is given by $w(t) := z_k$ on $[t_k, t_{k+1}]$; notice that (7) in particular yields $|v(t) - u(t; w)| \leq \epsilon t$ on $[0, t_\infty)$, hence

$$(8) \quad w(t) \in f([J_{t,\epsilon} \times \overline{B}_{\gamma\epsilon}(u(t; w))] \cap \text{gr}(K_A)) \quad \text{a.e. on } [0, t_\infty]$$

with $J_{t,\epsilon} = [t - \epsilon, t] \cap J$ and $\gamma = 1 + a$. Evidently, (7) holds if

$$(9) \quad |v(t) - u(t; t_k, x_k, w)| \leq \epsilon(t - t_k) \quad \text{on } [t_j, t_{j+1}]$$

for all $j \geq k \geq 0$ and (9) is valid for $j = k$, by construction of v . Suppose that (9) holds for fixed $k \geq 0$ and $j = m - 1 \geq k$. Exploitation of

$$u(t; t_k, x_k, w) = u(t; t_m, u(t_m; t_k, x_k, w), z_m) \quad \text{on } [t_m, t_{m+1}]$$

and

$$v(t) = u(t; t_m, x_m, z_m) + (t - t_m)e_m \quad \text{on } [t_m, t_{m+1}]$$

yields

$$\begin{aligned} |v(t) - u(t; t_k, x_k, w)| &\leq |x_m - u(t_m; t_k, x_k, w)| + (t - t_m)|e_m| \\ &\leq (t_m - t_k)\epsilon + (t - t_m)\epsilon \end{aligned}$$

for all $t \in [t_m, t_{m+1}]$, hence (9) holds for $j = m$. By induction (9) is therefore valid for all $j \geq k \geq 0$.

Now, relative compactness of $(x_k) = (v(t_k))$ follows easily, since (7) implies

$$v([0, t_\infty)) \subset C_k + (t_\infty - t_k)\overline{B}_\epsilon(0) \quad \text{for all } k \geq 0,$$

where $C_k := v([0, t_k]) \cup u([t_k, t_\infty]; t_k, x_k, w)$ is relatively compact. Evidently, this also yields relative compactness of $v([0, t_\infty))$.

Therefore, we may define $v(t_\infty) := \lim_{j \rightarrow \infty} x_{k_j}$, where (x_{k_j}) is a convergent subsequence of (x_k) . Then it is easy to check that (7) is still valid on $[t_k, t_\infty]$. Consequently, we are led to consider the set of approximate solutions defined by

$$\begin{aligned} M^\epsilon &= \{(v, w, P, b) : b \in (0, a], \\ &v : [0, b] \rightarrow X \text{ with } v(b) \in K_A(b), v([0, b]) \text{ relatively compact,} \\ &w : [0, b] \rightarrow X \text{ strongly measurable such that (8) holds a.e. on } [0, b], \\ &P \subset [0, b) \text{ with } 0 \in P, b \in \overline{P} \text{ such that } \tau \in P \text{ implies } v(\tau) \in K_A(\tau) \\ &\text{and } |v(t) - u(t; \tau, v(\tau), w)| \leq \epsilon(t - \tau) \text{ on } [\tau, b]\}. \end{aligned}$$

3. By the arguments of step 2 we already know $M^\epsilon \neq \emptyset$, and we want to use Zorn's Lemma to obtain an element of M^ϵ with $b = a$. For this purpose we define a partial ordering on M^ϵ by $(v, w, P, b) \leq (\overline{v}, \overline{w}, \overline{P}, \overline{b})$ if

$$b \leq \overline{b}, \quad v = \overline{v} \text{ on } [0, b], \quad w = \overline{w} \text{ a.e. on } [0, b], \quad P \subset \overline{P}.$$

To be able to apply Zorn's Lemma we have to show that every ordered subset $M \subset M^\epsilon$ has an upper bound in M^ϵ . Let

$$b^* = \sup\{b \in (0, a] : (v, w, P, b) \in M \text{ for some } v, w, P\}.$$

In case the "sup" is actually a "max", i.e. if there is $(v, w, P, b^*) \in M$, we let

$$P^* = \{\tau \in [0, b^*) : \text{there is } (v, w, P, b^*) \in M \text{ with } \tau \in P\}.$$

Evidently, (v, w, P^*, b^*) is an upper bound and $(v, w, P^*, b^*) \in M^\epsilon$ is easy to check.

In the remaining case there is a sequence $(v_n, w_n, P_n, b_n) \subset M$ with $b_n \nearrow b^*$, hence $P_n \subset P_{n+1}$, $v_{n+1} = v_n$ on $[0, b_n]$ and $w_{n+1} = w_n$ a.e. on $[0, b_n]$ for all $n \geq 1$. We then let

$$P^* = \bigcup_{n \geq 1} P_n, \quad v^*(t) = v_n(t) \text{ on } [0, b_n], \quad w^*(t) = w_n(t) \text{ on } [0, b_n].$$

Suppose for the moment that $v^*([0, b^*))$ is relatively compact. We then let $v^*(b^*) = \lim_{j \rightarrow \infty} v^*(b_{n_j})$ where $(v^*(b_{n_j}))$ is a convergent subsequence of $(v^*(b_n))$, and claim that $(v^*, w^*, P^*, b^*) \in M^\epsilon$ is an upper bound for M . Evidently, (v^*, w^*, P^*, b^*) is an upper bound for M , since $(v, w, P, b) \in M$ implies $b < b_n$, hence $(v, w, P, b) \leq (v_n, w_n, P_n, b_n)$ for some $n \geq 1$. To check that $(v^*, w^*, P^*, b^*) \in M^\epsilon$ is also easy; notice that $\tau \in P^*$ implies $\tau \in P_n$ and $v^*(\tau) = v_n(\tau)$ for all $n \geq n_\tau$. So, it remains to prove relative compactness of $v^*([0, b^*))$. But the latter follows by the corresponding arguments from step 2, where this time we take any sequence $(t_k) \subset P^*$ with $t_k \nearrow b^*$ and $x_k := v^*(t_k)$; notice that (7) then holds with v^* instead of v .

Consequently, there is a maximal element $(v^*, w^*, P^*, b^*) \in M^\epsilon$. Suppose $b^* < a$. We then let $t_0 = b^*$, $x_0 = v^*(b^*)$ and repeat the construction of step 2 to obtain the sequences from (5) and function v from (6). Let

$$\bar{v}(t) = v^*(t) \text{ on } [0, b^*], \quad \bar{v}(t) = v(t) \text{ on } [b^*, t_\infty), \quad \bar{b} = t_\infty,$$

$$\bar{w}(t) = w^*(t) \text{ on } [0, b^*], \quad w^*(t) = z_k \text{ on } [t_k, t_{k+1}], \quad \bar{P} = P^* \cup \{t_k : k \geq 1\}.$$

Then $v([t_0, t_\infty))$ is relatively compact again, and, as before, we let $\bar{v}(t_\infty) := \lim_{j \rightarrow \infty} \bar{v}(t_{k_j})$ for an appropriate subsequence (t_{k_j}) .

To obtain $(\bar{v}, \bar{w}, \bar{P}, \bar{b}) \in M^\epsilon$ we show that $\tau \in P^*$ and $t \in (\tau, t_\infty)$ implies $|\bar{v}(t) - u(t; \tau, \bar{v}(\tau), w)| \leq \epsilon(t - \tau)$; the other cases as well as the remaining properties are rather obvious. Due to (7) and the properties of (v^*, w^*, P^*, b^*) we have

$$\begin{aligned} |\bar{v}(t) - u(t; \tau, \bar{v}(\tau), w)| &\leq \\ |v(t) - u(t; t_0, x_0, w)| + |u(t; t_0, x_0, w) - u(t; t_0, u(t_0; \tau, v^*(\tau), w), w)| &\leq \\ \epsilon(t - t_0) + |v^*(t_0) - u(t_0; \tau, v^*(\tau), w)| &\leq \epsilon(t - \tau), \end{aligned}$$

hence $(\bar{v}, \bar{w}, \bar{P}, \bar{b}) \in M^\epsilon$ with $\bar{b} > b^*$, a contradiction. Consequently, $b^* = a$ for every maximal element of M^ϵ .

4. Given $\epsilon_m \searrow 0$ there are $(v_m, w_m, P_m, a) \in M^{\epsilon_m}$ by steps 2 and 3. Let $u_m = u(\cdot; w_m)$. Since $|w_m(t)| \leq |f|_\infty$ a.e. on J for all $m \geq 1$ and $S(\cdot)$

is compact the sequence (u_m) is relatively compact in $C_X(J)$ by Lemma 1. W.l.o.g. $u_m \rightarrow u_0$ in $C_X(J)$ and $u_0(0) = x_0$. For $t \in (0, a]$ there is $(t_m) \subset [0, t]$ with $t_m \nearrow t$ such that $\rho(u_m(t_m), K_A(t_m)) \leq \epsilon_m$, hence $u_m(t_m) \rightarrow u_0(t)$ implies $u_0(t) \in K_A(t)$ since $\text{gr}(K_A)$ is closed from the left. By (8), for almost all $t \in (0, a]$ we can choose a sequence (t_m) such that $t_m \nearrow t$ and

$$w_m(t) \in f(t_m, \overline{B}_{\gamma\epsilon_m}(u_m(t_m)) \cap K_A(t_m)) \quad \text{for all } m \geq 1,$$

hence for every $\eta > 0$ there is $m_\eta \geq 1$ such that

$$w_m(t) \in f(t_m, \overline{B}_\eta(u_0(t_m)) \cap K_A(t_m)) \quad \text{for all } m \geq m_\eta,$$

and therefore $w_m(t) \rightarrow f(t, u_0(t))$ a.e. on J . Consequently, $w_m \rightarrow f(\cdot, u_0(\cdot))$ in $L_X^1(J)$ which implies $u_0 = \lim_{m \rightarrow \infty} u_m = u(\cdot; f(\cdot, u_0(\cdot)))$, i.e. u_0 is a mild solution of (1). ■

Let us note in passing that the necessary condition $K_A(t) \neq \emptyset$ on J is of course implicitly contained in the assumptions of Theorem 1. Nevertheless, we did not include this condition explicitly, since the reduction to bounded f becomes easier this way.

Notice that compactness of the semigroup generated by $-A$ was only used in the final step to get relative compactness of $(u(\cdot; w_m))$ in $C_X(J)$ via Lemma 1. In the subsequent application to RD-systems the perturbation f has the additional property that, restricted to $\text{gr}(K_A)$,

(10) f maps bounded sets into weakly relatively compact sets.

Since $B := \{u_m(t) : t \in J, m \geq 1\} + \overline{B}_\gamma(0)$ (with u_m as in step 4 of the proof above) is bounded and $w_m(t) \in f([J \times B] \cap \text{gr}(K_A))$ a.e. on J for all $m \geq 1$, it follows from Corollary 2.6 in [13] that (w_m) is weakly relatively compact in $L_X^1(J)$. In this situation the proof of Theorem 1 obviously remains valid if A is such that $w \rightarrow u(\cdot; w)$ maps weakly relatively compact subsets of $L_X^1(J)$ into relatively compact subsets of $C_X(J)$. By the remark following Lemma 1 this property holds if $-A$ generates a compact semigroup. However, the former condition is weaker, in general, and will be useful later on.

Let us record this modification of Theorem 1 as

Theorem 2. *Let X be a real Banach space and $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ be m -accretive such that $\{u(\cdot; w) : w \in W\}$ is relatively compact in $C_X(J)$ for every fixed initial value in $\overline{D(A)}$ whenever $W \subset L_X^1(J)$ is weakly relatively compact. Let $J = [0, a] \subset \mathbb{R}$ and $K : J \rightarrow 2^X$ be such that $K_A(0) \neq \emptyset$ and $\text{gr}(K_A)$ is closed from the left. Let $f : \text{gr}(K_A) \rightarrow X$ be continuous, satisfying (2), (3) and (10). Then*

$$(1) \quad u' + Au \ni f(t, u) \quad \text{on } J, \quad u(0) = x_0$$

has a mild solution for every $x_0 \in K_A(0)$.

In several applications it happens that for an appropriate choice of the $K(t)$ these sets are positively invariant for the resolvents of A . Then it is

helpful to know that the subtangential condition can be separated, by which we mean that

$$(11) \quad \begin{aligned} J_\lambda K(t) &\subset K(t) \text{ for } \lambda > 0, t \in [0, a) \text{ and} \\ f(t, x) &\in T_K(t, x) \text{ for } t \in [0, a), x \in K_A(t) \end{aligned}$$

implies (2) if $gr(K_A)$ is closed from the left. We do not have a simple direct proof of this fact, but it is not difficult to show that (11) implies the "weak range condition"

$$(12) \quad \lim_{h \rightarrow 0+} h^{-1} \rho(x + hf(t, x), (I + hA)(K(t + h) \cap D(A))) = 0$$

for $t \in [0, a), x \in K_A(t)$,

and the latter in turn implies (2). This is the content of

Lemma 2. *Let X be a real Banach space and $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ be m -accretive. Let $J = [0, a] \subset \mathbb{R}$, $K : J \rightarrow 2^X$ with $gr(K_A)$ closed from the left and $f : gr(K_A) \rightarrow X$ be continuous.*

(a) *Then (12) implies (2).*

(b) *Then (11) implies (2).*

Proof. 1. To obtain (a), let $t_0 \in [0, a)$ and $x_0 \in K_A(t_0)$. Evidently, (2) holds if for every $\eta > 0$ there is $\delta = \delta_\eta \in (0, \eta]$ such that

$$\rho(S_{f(t_0, x_0)}(\delta)x_0, K_A(t_0 + \delta)) \leq 3\eta\delta.$$

The idea is to construct local ϵ -DS-approximate solutions for

$$(13) \quad u' + Au \ni f(t, u) \quad \text{on } [t_0, t_0 + d], \quad u(t_0) = x_0,$$

and to compare them to corresponding ϵ -DS-approximate solutions for

$$(14) \quad v' + Av \ni f(t_0, x_0) \quad \text{on } [t_0, t_0 + d], \quad v(t_0) = x_0.$$

Given $\eta \in (0, 1]$, fix $r \in (0, a - t_0)$ such that $|f(t, x) - f(t_0, x_0)| \leq \eta$ for all $t \in [t_0, t_0 + r]$, $x \in \overline{B}_r(x_0) \cap K_A(t)$ and let $\epsilon \in (0, r)$ with $\epsilon \leq 1$. Exploitation of (12) yields $h_k \in (0, \epsilon]$ and $e_k \in X$ with $|e_k| \leq \epsilon$ such that

$$(15) \quad x_{k+1} := J_{h_k}(x_k + h_k(f(t_k, x_k) + e_k)) \in K_A(t_{k+1}) \quad \text{for } k \geq 0$$

where $t_{k+1} := t_k + h_k$. Given these h_k we also let

$$(16) \quad \bar{x}_{k+1} := J_{h_k}(\bar{x}_k + h_k f(t_0, x_0)) \quad \text{for } k \geq 0, \quad \bar{x}_0 := x_0.$$

Since all J_{h_k} are nonexpansive it follows by induction that

$$(17) \quad \begin{aligned} |x_k - \bar{x}_k| &\leq (t_k - t_0)(\epsilon + \max_{j=1, \dots, k-1} |f(t_j, x_j) - f(t_0, x_0)|), \\ |\bar{x}_k - x_0| &\leq (t_k - t_0)|f(t_0, x_0)| + |J_{h_{k-1}} \cdots J_{h_0} x_0 - x_0|, \end{aligned}$$

hence

$$(18) \quad |x_k - x_0| \leq (t_k - t_0)(2 + |f(t_0, x_0)| + |y|) + 2|x_0 - x|$$

for all $(x, y) \in A$ as long as $t_k - t_0 \leq r$ and $|x_k - x_0| \leq r$. Let $x \in D(A)$ with $|x_0 - x| \leq r/4$, $y \in Ax$ and $d = \frac{1}{2}r(2 + |f(t_0, x_0)| + |y|)^{-1}$, where we may assume $d \leq \eta$. Then (18) yields $|x_k - x_0| \leq r$ for all $k \geq 1$ such that $t_k \leq t_0 + d$.

To obtain an ϵ -DS-approximate solution for (13) from (15), we have to show

that the h_k can be chosen such that $t_m \geq t_0 + d$ for some $m \geq 1$. This can be achieved by the usual trick: For $t \in [0, a)$ and $x \in K_A(t)$ let

$$\varphi_\epsilon(t, x) = \sup\{h \in (0, \epsilon] : \rho(x + hf(t, x), (I + hA)(K(t + h) \cap D(A))) \leq \epsilon h\}$$

and choose $h_k \geq \frac{1}{2}\varphi_\epsilon(t_k, x_k)$, say, in each step. Suppose $t_k \nearrow t_\infty \leq t_0 + d$. Given $j \geq 0$ we then let $\bar{x}_k$ be given by (16), but starting at $k = j$ instead of $k = 0$ (i.e., $\bar{x}_j = x_j$). Since (16) means $\bar{x}_{k+1} = J_{h_k}^z \bar{x}_k$ where J_λ^z is the resolvent of A_z with $z := f(t_0, x_0)$, we know that $(\bar{x}_k)$ is a Cauchy sequence. Hence

$$|x_{k+l} - x_k| \leq (t_{k+l} - t_j)(\epsilon + 1) + (t_k - t_j)(\epsilon + 1) + |\bar{x}_{k+l} - \bar{x}_k|$$

for all $l \geq 1$, $k > j \geq 0$ shows that (x_k) is a Cauchy sequence too. Consequently, $x_k \rightarrow x_\infty \in K_A(t_\infty)$ as $k \rightarrow \infty$ and therefore

$$\lim_{(t,x) \rightarrow (t_\infty, x_\infty)} \varphi_\epsilon(t, x) \leq \lim_{k \rightarrow \infty} \varphi_\epsilon(t_k, x_k) \leq 2 \lim_{k \rightarrow \infty} h_k = 0.$$

This is a contradiction, since we will show

$$(19) \quad \lim_{(s,y) \rightarrow (t-, x)} \varphi_\epsilon(s, y) > 0 \quad \text{for all } t \in [0, a), x \in K_A(t).$$

For this purpose, choose $h \geq \frac{1}{2}\varphi_{\epsilon/3}(t, x) > 0$ and $e \in B_{\epsilon/2}(0)$ such that

$$x + h(f(t, x) + e) \in (I + hA)(K(t + h) \cap D(A)).$$

Given $t_n \nearrow t$ and $x_n \in K_A(t_n)$ with $x_n \rightarrow x$, let $h_n = h + t - t_n \geq h$. Then

$$J_{h_n}(x + h(f(t, x) + e)) \in K(t + h) \cap D(A) = K(t_n + h_n) \cap D(A).$$

Using the resolvent identity and letting $z := x + h(f(t, x) + e)$, we get

$$J_{h_n} z = J_{h_n} \left(z + \frac{t - t_n}{h} (z - J_{h_n} z) \right),$$

hence

$$z + \frac{t - t_n}{h} (z - J_{h_n} z) \in (I + h_n A)(K(t_n + h_n) \cap D(A)) =: R_n$$

and therefore

$$\begin{aligned} \rho(x_n + h_n f(t_n, x_n), R_n) &\leq |x - x_n| + h |f(t, x) - f(t_n, x_n)| + \\ &\quad (t - t_n)(|f(t_n, x_n)| + |z - J_{h_n} z|/h) + \epsilon \frac{h}{2} \leq \epsilon h \leq \epsilon h_n \end{aligned}$$

for all large $n \geq 1$, i.e. $\lim_{n \rightarrow \infty} \varphi_\epsilon(t_n, x_n) \geq h > 0$ and consequently (19) holds.

Thus we get ϵ -DS-approximate solutions u^ϵ, v^ϵ for (13), (14) having the values $x_k, \bar{x}_k$ on $(t_{k-1}, t_k]$ for $k = 1, \dots, m$, respectively, and $t_m < t_0 + d \leq t_m + \epsilon$. Moreover, by (15) and (17),

$$\rho(\bar{x}_k, K_A(t_k)) \leq$$

$$(t_k - t_0) \left(\epsilon + \sup\{|f(t_0, x_0) - f(t, x)| : t \in [t_0, t_0 + r], x \in K_A(t) \cap \bar{B}_r(x_0)\} \right)$$

for $k = 1, \dots, m$, hence $\rho(\bar{x}_k, K_A(t_k)) \leq (t_k - t_0)(\epsilon + \eta)$. Given $\epsilon \rightarrow 0+$ we have $v^\epsilon(t) \rightarrow S_{f(t_0, x_0)}(t - t_0)x_0$ uniformly on $[t_0, t_0 + d)$. Now notice that

the choice of $d > 0$ above was in fact independent of $\epsilon \in (0, 1]$. Therefore, we find $\epsilon \in (0, \eta]$ such that $t_m - t_0 \geq d - \epsilon \geq d/2$ and

$$|v^\epsilon(t) - S_{f(t_0, x_0)}(t - t_0)x_0| \leq \frac{1}{2}\eta d \quad \text{on } [t_0, t_m].$$

Let $\delta = t_m - t_0$. Then

$$\rho(S_{f(t_0, x_0)}(\delta)x_0, K_A(t_0 + \delta)) \leq \eta\delta + \rho(\bar{x}_m, K_A(t_m)) \leq 3\eta\delta,$$

hence (2) holds.

2. In the situation of (b) let $t \in [0, a)$ and $x \in K_A(t)$. Then, given $\epsilon > 0$, there is $h \in (0, \epsilon]$ and $e \in X$ with $|e| \leq \epsilon$ such that $x + h(f(t, x) + e) \in K(t + h)$, hence

$$J_h(x + h(f(t, x) + e)) \in K(t + h) \cap D(A).$$

Consequently,

$$\rho(x + hf(t, x), (I + hA)(K(t + h) \cap D(A))) \leq h\epsilon$$

and therefore (12) holds. By step 1 of this proof the latter implies (2). ■

Theorem 1 and Theorem 2 together with Lemma 2 obviously imply

Corollary 1. *Let X be a real Banach space, $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ be m -accretive, $J = [0, a] \subset \mathbb{R}$, $K : J \rightarrow 2^X$ with $K_A(0) \neq \emptyset$ and $\text{gr}(K_A)$ closed from the left. Let $f : \text{gr}(K_A) \rightarrow X$ be continuous, satisfying (3). In addition, assume that $-A$ generates a compact semigroup, or f satisfies (10) and $\{u(\cdot; w) : w \in W\}$ is relatively compact in $C_X(J)$ for every fixed initial value in $\overline{D(A)}$ whenever $W \subset L_X^1(J)$ is weakly relatively compact. Then*

$$(1) \quad u' + Au \ni f(t, u) \quad \text{on } J, \quad u(0) = x_0$$

has a mild solution for every $x_0 \in K_A(0)$ if also (11) or (12) holds.

Additional information is contained in the following

Remarks. 1. In the situation of Theorem 1 but without the growth condition on f we still get existence of a local solution of (1). This follows by application of Theorem 1 with J and K replaced by $\hat{J} = [0, b]$ and $\hat{K} : \hat{J} \rightarrow 2^X$ with $\hat{K}(t) = K(t) \cap \overline{B}_{tM}(S(t)x_0)$, respectively, where $b \in (0, a]$ and $M > 1$ are chosen such that $|f(t, x)| \leq M - 1$ on $\hat{J} \times \overline{B}_r(x_0)$ for $r := bM + \max_{[0, b]} |S(t)x_0 - x_0|$.

If f is locally Lipschitz on $\text{gr}(K_A)$, we may choose $\hat{J}$ and $\hat{K}$ above such that, in addition, f is Lipschitz on $\text{gr}(\hat{K}_A)$. Exploitation of the latter yields convergence of the ϵ_m -approximate solutions u_m from step 4 of the proof of Theorem 1, without using any compactness property of A . Evidently, this yields a local solution which can be extended up to a noncontinuable solution of (1). Moreover, this solution is unique. To summarize, problem (1) with $x_0 \in K_A(0)$ has a unique noncontinuable solution if A is m -accretive, $K : J = [0, a] \rightarrow 2^X$ is such that $\text{gr}(K_A)$ is closed from the left and $f : \text{gr}(K_A) \rightarrow X$ is locally Lipschitz, satisfying (2).

2. Problem (1) has been considered in [22] in case $K(t) \equiv K$ is "semi locally closed". The "subtangential condition" used there is much stronger than (2): for closed K it essentially becomes

$$\lim_{h \rightarrow 0+} \sup \{ h^{-1} \rho(S_{f(t,x)}(h)x, K) : (t, x) \in J \times K \} = 0.$$

Semilinear cases have been studied e.g. in [21], [19] and [6]. In the first paper the linear part A is allowed to depend on time with varying domains $D(A(t))$ and existence of mild solutions is obtained under a necessary subtangential condition and a compactness assumption, either on the evolution system generated by the linear part or on the perturbation. In [19] multivalued perturbations are considered and existence of mild solutions is proven for compact semigroups under a strong subtangential condition. In [6], §7 it is shown that the latter result remains true under the necessary subtangential condition, and that the additional assumption on the semigroup can be replaced by a compactness assumption on the perturbation.

3. Let us note that for dissipative, not necessarily continuous $f : D(f) \subset X \rightarrow X$ and $K(t) \equiv K$ the invariance results of [20] can be applied to $A - f$. In particular, in this situation Theorem 2 of [20] implies that for accretive A problem (1) has a mild solution if for every $x \in K_A := K \cap \overline{D(A)}$ and $\epsilon > 0$ there is $h \in (0, \epsilon]$, $x_h \in D(A) \cap D(f)$ and $y_h \in Ax_h$ such that

$$|x - x_h + h(f(x_h) - y_h)| \leq h\epsilon \quad \text{and} \quad \rho(x_h, K_A) \leq h\epsilon.$$

In case $D(f) = K$ this is just the weak range condition for $A - f$, and it becomes (12) if, in addition, f is continuous bounded and $\overline{K \cap D(A)} = K_A$.

4. APPLICATION TO REACTION-DIFFUSIONS-SYSTEMS: GLOBAL EXISTENCE OF SOLUTIONS

Let us start with the model problem

$$(20) \quad u_t = \Delta \Phi(u) + g(u) \quad \text{in } (0, \infty) \times \Omega, \quad \Phi(u(t, \cdot))|_{\partial\Omega} = 0, \quad u(0, \cdot) = u_0,$$

where $\Omega \subset \mathbb{R}^n$ is open bounded with smooth boundary, $\Phi(u_1, \dots, u_m) = (\varphi_1(u_1), \dots, \varphi_m(u_m))$ with $\varphi_k : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R}_+^m \rightarrow \mathbb{R}^m$.

To be able to apply the results from section 3 we need some information concerning the abstract formulation of the scalar nonlinear diffusion equation

$$(21) \quad v_t = \Delta \varphi(v) \quad \text{in } (0, T) \times \Omega, \quad \varphi(v(t, \cdot))|_{\partial\Omega} = 0, \quad v(0, \cdot) = v_0,$$

where Ω is as above and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous increasing with $\varphi(0) = 0$. Define $A : D(A) \subset L^1(\Omega) \rightarrow L^1(\Omega)$ by

$$(22) \quad \begin{aligned} Au &= -\Delta \varphi(u), \\ D(A) &= \{u \in L^1(\Omega) : \varphi(u) \in W_0^{1,1}(\Omega), \Delta \varphi(u) \in L^1(\Omega)\}. \end{aligned}$$

Then (21) corresponds to the autonomous problem $u' + Au \ni 0$. Let us collect some basic facts concerning A . Recall that $Q : L^1(\Omega) \rightarrow L^1(\Omega)$ is called order-preserving if $u \leq \bar{u}$ a.e. on Ω implies $Qu \leq Q\bar{u}$ a.e. on Ω .

Lemma 3. *Let $\Omega \subset \mathbb{R}^n$ be open bounded with smooth boundary, $X = L^1(\Omega)$, $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be continuous increasing with $\varphi(0) = 0$ and A be given by (22). Then the following holds.*

- (a) *A is m -accretive with $\overline{D(A)} = X$.*
- (b) *$J_\lambda : X \rightarrow X$ is order-preserving for all $\lambda > 0$, and $J_\lambda u \leq |u|_\infty$ if $u \geq 0$.*
- (c) *In addition, let φ be strictly increasing. Then $\{u(\cdot; w) : w \in W\}$ is relatively compact in $C_X(J)$ for every fixed initial value whenever $W \subset L^1_X(J)$ is weakly relatively compact.*

Assertion (a) and the first part of (b) are contained in Théorème 2.1 in [3], while the second part of (b) is a consequence of the same theorem combined with Corollaire 2.2. in [3]. Assertion (c) is Theorem 1 in [11].

To reformulate (20) as an abstract evolution system we let

$$\begin{aligned} X &= L^1(\Omega)^m \quad \text{with } |u| = |u_1|_1 + \dots + |u_m|_1, \\ Au &= -\Delta\Phi(u) = (-\Delta\varphi_1(u_1), \dots, -\Delta\varphi_m(u_m)), \\ D(A) &= \{u \in X : \varphi_k(u_k) \in W_0^{1,1}(\Omega), \Delta\varphi_k(u_k) \in L^1(\Omega) \text{ for } k = 1, \dots, m\}, \\ f : D(f) \subset X^+ &\rightarrow X \quad \text{defined by } f(u)(x) = g(u(x)) \text{ on } \Omega, \end{aligned}$$

where $X^+ = \{u \in X : u_k \geq 0 \text{ a.e. on } \Omega \text{ for } k = 1, \dots, m\}$ is the positive cone in X and $D(f) = \{u \in X^+ : f(u) \in X\}$; notice that $L^\infty(\Omega)_+^m \subset D(f)$. Suppose that the φ_k are continuous and strictly increasing with $\varphi_k(0) = 0$. Then A is m -accretive with $A(0) = 0$, all J_λ are order-preserving w.r. to the partial ordering induced by X^+ on X (i.e. $u \leq v$ if $v - u \in X^+$) and $J_\lambda u \leq u$ if $u(x) = \alpha \in \mathbb{R}_+^m$ a.e. on Ω . This implies

$$J_\lambda K \subset K \quad \text{for all } \lambda > 0 \text{ and } K = \{u \in X : 0 \leq u \leq \bar{u}\} \text{ with } \bar{u} \equiv \alpha \in \mathbb{R}_+^m,$$

i.e. such "rectangles" are positively invariant under J_λ . Moreover, due to Lemma 3(c) the operator A satisfies the compactness assumption imposed in Corollary 1. Therefore, it is natural to look for "tubes" of type $C(t) = [0, c(t)]$ with $c : \mathbb{R}_+ \rightarrow \mathbb{R}_+^m$ such that $\text{gr}(C)$ is weakly positively invariant for $y' = g(y)$. By Corollary 1 in [5] this holds if $g(y) \in T_C(t, y)$ for all $t \geq 0$, $y \in C(t)$. For this special $C(\cdot)$ the latter condition means

$$(23) \quad \begin{aligned} t \geq 0, y \in C(t) \text{ with } y_k &= 0 && \text{implies} && g_k(y) \geq 0 \\ t \geq 0, y \in C(t) \text{ with } y_k &= c_k(t) && \text{implies} && g_k(y) \leq D^+ c_k(t), \end{aligned}$$

where D^+ denotes the upper right Dini derivative; see Chapter 9.1 in [6]. The first part of (23) is a natural assumption if g models a chemical reaction, and to find an admissible upper bound $c(\cdot)$ we consider the initial value problem

$$(24) \quad y' = \hat{g}(y) \quad \text{on } \mathbb{R}_+, \quad y(0) = y_0 \in \mathbb{R}_+^m,$$

with

$$(25) \quad \hat{g}_k(y) := \max\{g_k(z) : 0 \leq z \leq y, z_k = y_k\};$$

notice that $g = \hat{g}$ on $\mathbb{R}_+^m$ iff g is quasimonotone w.r. to $\mathbb{R}_+^m$. Let $\hat{y}(\cdot; y_0)$ be any solution of (24) with $[0, T)$ being its maximal interval of existence. Then $C(\cdot) = [0, \hat{y}(\cdot)]$ is weakly positively invariant for $y' = g(y)$ on $[0, T)$.

By application of Corollary 1 we therefore get

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be open bounded, $X = L^p(\Omega)^m$ with $m \geq 1$, $p \in [1, \infty)$ and $A : D(A) \subset X \rightarrow 2^X \setminus \emptyset$ be m -accretive with $0 \in A(0)$ such that all J_λ are order-preserving with $J_\lambda u \leq u$ if $u \equiv \alpha \in \mathbb{R}_+^m$. Suppose also that $\{u(\cdot; w) : w \in W\}$ is relatively compact in $C_X(J)$ for every fixed initial value in $\overline{D(A)}$ whenever $W \subset L_X^1(J)$ is weakly relatively compact. Let $g : \mathbb{R}_+^m \rightarrow \mathbb{R}^m$ be continuous with $g_k(y) \geq 0$ if $y_k = 0$ and $f : D(f) \subset X^+ \rightarrow X$ be defined by $f(u)(x) = g(u(x))$ on Ω . Then the abstract RD-system*

$$u' + Au \ni f(u) \quad \text{on } \mathbb{R}_+, \quad u(0) = u_0$$

has a global mild solution in X for $0 \leq u_0 \in L^\infty(\Omega)^m \cap \overline{D(A)}$, if (24) with $\hat{g}$ from (25) has a global solution for $y_0 = (|u_{0,1}|_\infty, \dots, |u_{0,m}|_\infty)$.

Proof. Let $c(\cdot)$ denote the global solution of (24), $C(t) := [0, c(t)]$ and

$$K(t) := \{u \in X : u(x) \in C(t) \text{ a.e. on } \Omega\}.$$

Notice first that $C(\cdot)$ is continuous w.r. to d_H and bounded on $[0, a]$ for every $a > 0$. Let $1 \leq p < \infty$. Then

$$\rho(u, K(t))^p = \int_\Omega \rho(u(x), C(t))^p dx \quad \text{for every } u \in X,$$

which follows from the fact that $P_{C(t)}(u(\cdot))$ has a measurable selection for every $u \in X$ by Proposition 3.2 in [10], where $P_{C(t)}$ denotes the metric projection onto $C(t)$.

To check that $gr(K_A)$ is closed from the left, let $t_k \nearrow t$ and $u_k \in K_A(t_k)$ with $u_k \rightarrow u \in X$. Passing to an appropriate subsequence, we may assume $u_k(x) \rightarrow u(x)$ a.e. on Ω which implies $u(x) \in C(t)$ a.e. on Ω , hence $u \in K_A(t)$. Since g is continuous and bounded on $C([0, a])$ it follows that f is continuous and bounded on $K([0, a])$ with $f(K([0, a])) \subset X$ weakly relatively compact. For $t \geq 0$ and $y \in C(t)$ with $y_k = c_k(t)$ we have $g_k(y) \leq \hat{g}_k(c(t)) = c'_k(t)$ by construction of $\hat{g}$. Since the first part of (23) holds by assumption, this implies $g(y) \in T_C(t, y)$. Due to $c'_k(t) = D_+ c_k(t)$ we even get

$$\lim_{h \rightarrow 0+} h^{-1} \rho(y + hg(y), C(t+h)) = 0,$$

hence

$$h^{-1} \rho(u + hf(u), K(t+h)) = \left(\int_\Omega [h^{-1} \rho(u(x) + hg(u(x)), C(t+h))]^p dx \right)^{1/p}$$

together with the dominated convergence theorem imply $f(u) \in T_K(t, u)$ for $t \geq 0$, $u \in K(t)$.

Evidently, $J_\lambda K(t) \subset K(t)$ for all $t \geq 0$ and $\lambda > 0$. Therefore (1) with $f(t, u) := f(u)$ has a mild solution u_k on $[0, k]$ for every $k \geq 1$, by Corollary 1. Let (u_{k_j}) be a subsequence of (u_k) which converges uniformly on bounded intervals to a mild solution u of (1). Then u is a global mild solution of the abstract RD-system. ■

Due to Lemma 3 this result applies to the model problem (20). This yields

Corollary 2. *Let $\Omega \subset \mathbb{R}^n$ be open bounded with smooth boundary, and $\varphi_k : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and strictly increasing with $\varphi_k(0) = 0$ for $k = 1, \dots, m$. Let $g : \mathbb{R}_+^m \rightarrow \mathbb{R}^m$ be continuous with $g_k(y) \geq 0$ if $y_k = 0$. Then problem (20) has a global mild solution in $L^1(\Omega)^m$ for $0 \leq u_0 \in L^\infty(\Omega)^m$, if (24) with $\hat{g}$ from (25) has a global solution for $y_0 = (|u_{0,1}|_\infty, \dots, |u_{0,m}|_\infty)$.*

Remarks. 4. A similar invariance approach is used in [17] to obtain global existence for RD-systems with linear diffusion and smooth quasimonotone reaction terms.

In [16] the model problem (20) is considered for $m = 2$, but with Dirichlet boundary conditions replaced by the mixed boundary conditions $\frac{\partial \varphi_k}{\partial \nu}(u_k) + \alpha_k \varphi_k(u_k) = 0$ in $(0, \infty) \times \partial\Omega$, where the α_k are sufficiently smooth nonnegative functions. In this paper weak solutions of (20) are obtained in the following situation: Either $\varphi_k \equiv 0$ or $\varphi_k(0) = \varphi'_k(0) = 0$ and $\varphi_k(r), \varphi'_k(r), \varphi''_k(r) > 0$ for $r > 0$. Furthermore g is assumed to be smooth and quasimonotone w.r. to $\mathbb{R}_+^2$ with $g(0) = 0$ such that for every $y \in \mathbb{R}_+^2$ there is $\bar{y} \geq y$ with $g(\bar{y}) \leq 0$. Notice that the latter assumption on g implies positive invariance of $[0, \bar{y}]$ for $y' = g(y) = \hat{g}(y)$, hence every solution of (24) exists globally.

5. Under the assumptions imposed on φ in Lemma 3(c) the semigroup generated by $-A$ (with A from (22)) need not be compact, but compactness of the semigroup is guaranteed if, in addition, φ is continuously differentiable on $\mathbb{R} \setminus \{0\}$ such that $\varphi'(r) \geq c|r|^{\gamma-1}$ on $\mathbb{R} \setminus \{0\}$ with some $c > 0$ and $\gamma > \max\{0, \frac{n-2}{n}\}$; see Lemma 2.6.2 in [23].

In the special case $\varphi(r) = |r|^{\gamma-1}r$, which corresponds to the porous medium equation, the condition $\gamma > \max\{0, \frac{n-2}{n}\}$ is optimal in the sense that the semigroup is not compact for $0 < \gamma < \frac{n-2}{n}$. This is a consequence of Theorem 8 in [8]; see Remark 11 there.

6. In chemical applications $g(\cdot)$ will usually be of special type (see e.g. [14]). In the simplest case $m = 2$ a typical reaction term, corresponding to the chemical reaction $\alpha A + \beta B \rightarrow P$, is given by the so-called Freundlich kinetics

$$g(y) = (-\alpha k y_1^\alpha y_2^\beta, -\beta k y_1^\alpha y_2^\beta) \quad \text{with } \alpha, \beta, k > 0;$$

here α , respectively β is the order of the reaction w.r. to A , respectively B and k is the rate constant. Evidently $\hat{g}(y) = 0$ on $\mathbb{R}_+^2$, hence (24) has global solutions for every $\alpha, \beta > 0$.

In case of a mixed order reversible reaction $\alpha A \rightleftharpoons \beta B$ one has

$$g(y) = (\alpha(k_2 y_2^\beta - k_1 y_1^\alpha), \beta(k_1 y_1^\alpha - k_2 y_2^\beta)) \quad \text{with } \alpha, \beta, k_1, k_2 > 0.$$

Here g is quasimonotone with $g(k_1^{-1/\alpha} r^\beta, k_2^{-1/\beta} r^\alpha) = 0$ for all $r > 0$, hence (24) has global solutions for every $\alpha, \beta > 0$.

Finally, if g is given by

$$g(y) = (-y_1^\alpha y_2^\beta, y_1^\alpha y_2^\beta) \quad \text{with } \alpha, \beta > 0,$$

we get $\hat{g}(y) = (0, y_1^\alpha y_2^\beta)$, hence (24) has global solutions provided $\beta \leq 1$. Here as well as in the first example the use of Theorem 3, say, is to provide existence of solutions in cases when g is not locally Lipschitz, rather than to obtain global existence.

In case of linear diffusion, more precisely if $\varphi_k(r) = d_k r$ with $d_k > 0$ for $k = 1, 2$, existence of global solutions of (20) with g as in the last example above and $u_0 \in L^\infty(\Omega)^2$ with $u_0 \geq 0$ was established in [18] for $\alpha = 1$ and arbitrary $\beta \geq 1$ by different techniques; see also [15].

Additional difficulties occur e.g. in the case of so-called zero-order reactions, for instance g from one of the examples above in the limit case $\alpha = 0$, for the reaction term is then discontinuous. In this situation it is appropriate to replace g by a certain multivalued "regularization" G . Corresponding RD-systems of type (20) have been considered recently in [12], where existence of local mild solutions was proven; see also Chapter 3.4 in [23] and [7].

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