ITERATIVE SOLUTION OF UNSTABLE VARIATIONAL INEQUALITIES ON APPROXIMATELY GIVEN SETS

Y. I. ALBER, A. G. KARTSATOS, AND E. LITSYN

Abstract. The onvergence and the stability of the iterative regularization method for solving variational inequalities with bounded nonsmooth properly monotone (i.e., degenerate) operators in Banach spaces are studied. All the items of the inequality (i.e., the operator A, the "right hand side" f and the set of constraints Ω) are to be perturbed. The connection between the parameters of regularization and perturbations which guarantee strong convergence of approximate solutions is established. In contrast to previous publications by Bruck, Reich and the first author, we do not suppose here that the approximating sequence is a priori bounded. Therefore the present results are new even for operator equations in Hilbert and Banach spaces. Apparently, the iterative processes for problems with perturbed sets of constraints are being considered for the first time.

1. Introduction and previous results

We consider iterative methods for solving variational inequalities with monotone operators in Banach spaces. A typical example of such problems is the problem of optimization of convex functionals on convex closed sets. Recall that the gradient of a convex functional defined in a Banach space B is a monotone, in general nonlinear and nonsmooth, operator acting from B to the dual space B^*. Particular cases of variational inequalities are problems of solving operator equations in Banach spaces, problems of

1991 *Mathematics Subject Classification.* 49J40, 65K10, 47A55, 47H05, 65L20.

Key words and phrases. Banach spaces, variational inequalities, convex sets, methods of iterative regularization, Lyapunov functionals, metric projection operators, monotone operators, perturbations, Hausdorff distance, convergence, stability.

The research of the first author was supported in part by the Ministry of Science, Grant #3481-1-91, and the Ministry of Absorption Center for Absorption in Science.

Received: October 22, 1995.

finding saddle points of convex-concave functionals, equations with general elliptic operators, etc. [25, 28, 17].

Assume that B is a uniformly convex Banach space. Denote by $\langle \varphi, x\rangle$ the dual product in B, i.e., the pairing between $\varphi \in B^*$ and $x \in B$. Let $\|\cdot\|$ be a norm in B, and $\|\cdot\|_*$ be the norm in B^*. Let Ω be a convex closed set in B, A be a bounded monotone operator from B to B^* with domain $D(A) \subseteq B$ and range $R(A) \subseteq B^*$, $\Omega \subseteq \mathrm{int}D(A)$ ($\mathrm{int}D(A)$ stands for the interior of $D(A)$). Recall that an operator A is called "bounded" if it maps bounded sets of B onto bounded sets of B^*; an operator A is called "monotone" if for every $x_1, x_2 \in B$, $y_1 \in Ax_1$ and $y_2 \in Ax_2$,

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq 0, \tag{1}$$

"strongly monotone" if

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq c\|x_1 - x_2\|^2, \quad c = \mathrm{const} > 0, \tag{2}$$

and "uniformly monotone" if

$$\langle y_1 - y_2, x_1 - x_2\rangle \geq \psi(\|x_1 - x_2\|), \tag{3}$$

where $\psi(t)$ is a continuous nondecreasing positive function for $t > 0$, $\psi(0) = 0$.

If A satisfies some continuity condition (for example, demi-, hemi- continuity) then in the inequalities (1)-(3) we set $y_1 = Ax_1$ and $y_2 = Ax_2$. An operator A is called maximal monotone (in the sense of inclusion) if its graph is a maximal monotone set [19, 25, 28].

We define $A : B \to B^*$ to be a "proper monotone" operator if it is monotone and there is no other strengthening of the condition (1) (for example, to the levels of (2) or (3)).

It is well known [1, 2] that the problem of finding a solution $x^* \in \Omega$ for the variational inequality

$$\langle Ax^* - f, x - x^*\rangle \geq 0, \quad \forall x \in \Omega, \quad f \in B^* \tag{4}$$

with a proper monotone operator A belongs to the class of ill-posed problems. It is unstable with respect to perturbations of all of its terms: the operator A, "the right-hand side" f and the set of constraints Ω. That is why the conventional approximate methods of relaxation type (gradient methods, Newton methods, etc.) do not, in general, converge to the solution x^*, which, moreover, does not necessary exist. In compliance with the concept of ill-posed problems here we assume that there exists a nonempty set N of solutions to the variational inequality (4).

By a "solution" of (4) we understand an element $x^* \in \Omega$ for which there exists $y \in \bar{A}x^*$ such that the inequality

$$\langle y - f, x - x^*\rangle \geq 0, \quad \forall x \in \Omega,$$

is satisfied. Here $\bar{A}$ is the maximal monotone extension of A on Ω (which always exists by the Zorn's lemma). If A is continuous, then x^* is a classical solution. If A is maximal monotone and $\Omega = B$, then x^* is a solution in the sense of inclusion, i.e., $f \in Ax^*$.

It is well known that N is a convex closed set (see, for instance, [15]). Due to the reflexivity of B, there exists a unique element $\tilde{x}^*$ of minimal norm

$$\|\tilde{x}^*\| = \min_{x \in \mathbf{N}} \quad \|x\|. \tag{5}$$

Moreover, the concept of an ill-posed problem assumes the application of a regularization procedure to the variational inequality with perturbed initial data, to ensure that the convergence of the approximate solutions is stable.

As regularizing (smoothing) operators we consider the one-parameter family

$$S(\alpha, x) = \alpha J x,$$

where α is a positive parameter of regularization and J is a normalized duality mapping. Recall that $J : B \to B^*$ is called the "normalized duality mapping" if

$$\|Jx\|_* = \|x\|, \quad \langle Jx, x\rangle = \|x\|^2. \tag{6}$$

The duality mapping exists in every Banach space. The (generally nonlinear) operator J is always monotone, coercive and bounded. It is also hemicontinuous in smooth reflexive Banach space, uniformly continuous on every bounded set in uniformly smooth Banach space, and uniformly monotone on every bounded set in uniformly convex Banach space [9, 10]. If B is a Hilbert space, then J is the identity operator I. If B is a strictly convex and reflexive along with its dual, and $J^* : B^* \to B$, is the normalized duality mapping in the space B^*, then $J^* = J^{-1}$, where J^{-1} is the inverse of the mapping J. If B is smooth, i.e., the norm of B is differentiable in the sense of Gateaux, then $Jx = \mathrm{grad}(\|x\|^2/2)$. Let us notice that these same properties are shared by the duality mapping J^ν with the gauge function $\nu(t)$:

$$\|J^\nu\|_* = \nu(\|x\|), \qquad \langle Jx, x\rangle = \nu(\|x\|)\|x\|, \tag{7}$$

which also plays an important role in the general theory of monotone and accretive operators.

Assume that instead of A, f, and Ω perturbed values A^h, f^ω and Ω_σ, depending on positive parameters h, ω, and σ respectively are given. Furthermore, let

$$\mathcal{H}_{B^*}(M^h(x), M(x)) \le h\zeta(\|x\|), \tag{8}$$

$$\|f^\omega - f\|_* \le \omega, \tag{9}$$

$$\mathcal{H}_B(\Omega_\sigma, \Omega) \le \sigma. \tag{10}$$

Here $\mathcal{H}_X(Q_1, Q_2)$ is the Hausdorff distance between the sets Q_1 and Q_2 in the space X (see, for example, [2]), $M(x)$ and $M^h(x)$ are the ranges of the operators $\bar{A}$ and $\bar{A}^h$ in x, where $\bar{A}$ and $\bar{A}^h$ stands for the maximal extensions of the corresponding monotone operators, $\zeta(t)$ is continuous non-decreasing

function for $t \geq 0$, bounded on bounded sets. If A and A^h are continuous then (8) is replaced by the inequality

$$\|A^h x - Ax\|_* \leq h\zeta(\|x\|).$$

In what follows we assume that A^h are monotone operators, $D(A^h) = D(A)$ for all $h \geq 0\,(A^0 = A)$, and Ω_σ are convex closed sets for all $\sigma \geq 0$, ($\Omega_0 = \Omega$), $\Omega_\sigma \subset \mathrm{int}D(A)$. Using $S(\alpha, x) = \alpha Jx$ as the smoothing operator we have the following regularized variational inequality:

$$\langle A^h x_\alpha^\Delta + \alpha J x_\alpha^\Delta - f^\omega, x - x_\alpha^\Delta\rangle \geq 0,\ \forall x \in \Omega_\sigma, \Delta = (h, \omega, \sigma), \tag{11}$$

recently studied by Alber and Notik [2, 11].

The following two main problems are naturally posed for the variational inequality (11):

1. Find the connection between parameters h, ω, and σ of the problem and the regularization parameter α, securing convergence of the approximate solutions x_α^Δ to the solution x^* of the initial inequality (4), and their stability.

   The simplest of the known relations is the following: $\Delta/\alpha \to 0$ as $\alpha \to 0$. Namely, under this very condition convergence $x_\alpha^\Delta \to \tilde{x}^*$ is proved in [2, 11]. However, this choice of α is not single-valued and does not guarantee the regularization properties of the approximate solutions to the variational inequality (11). Therefore, the following problem is stated:
2. Justify a single-valued choice of the regularization parameter α providing regularizing properties of the approximate solutions. In other words, the given perturbing parameters $\Delta = (h, \omega, \sigma)$ have to provide a single-valued choice of α, continuously tending to zero along with $\Delta \to 0$. Then the limit relation $x_\alpha^\Delta \to \tilde{x}^*$ is guaranteed.

In this case the method of finding solution x_α^Δ from (11) along with the way to choose α is called the operator regularization, in contrast to the iterative regularization to be discussed in what follows.

The operator method of regularization has got a great deal of interest. For linear equations with exact data in Hilbert spaces it was studied by Lavrentjev in [27] and Lattes and Lions in [26]. Here the regularizing family is especially simple, namely, $S(\alpha, x) = \alpha I$.

Browder [20] and Bakushinskii [17] used a strongly and uniformly monotone operators $M(x)$ in Hilbert spaces:

$$\langle Mx - My, x - y\rangle \geq \|x - y\|\psi(\|x - y\|), \tag{12}$$

$$\psi(t) > 0,\ \ 0 < t < \infty,\ \ \psi(0) = 0,\ \ \psi(0) \to \infty\ \ as\ \ t \to \infty,$$

as the regularizing operators for variational inequalities. Unfortunately, in [20, 17] there are no examples of such operators for Banach spaces. In the work [1] Alber has first applied the duality mappings as regularizing operators for solving equations with exact and approximate data in Banach spaces. The duality mappings do not satisfy the condition (12) even in smooth Banach spaces: generally speaking, they are uniformly monotone

on bounded sets. This makes apparent difficulties in investigation of the methods mentioned above. Nevertheless, duality mappings turn out to be an exceptionally efficient tool for regularization of unstable problems in Banach spaces (operator equations, variational inequalities, problems for calculating values of unbounded operators, convex optimization problems, etc.). This allowed to get the most of the results about convergence and stability [2, 4, 11, 15, 16, 24, 29, 30, 35, 36] which have been known previously only for Hilbert spaces, mainly, in linear cases (cf. [32]). In addition, let us point out that duality mappings have an analytical descriptions for the most important Banach spaces, namely, l^p, L^p, H^p_m, $1 < p < \infty$ [3] and Orlicz spaces with Luxemburg norm [12].

It is a special question how to choose the parameter of regularization. The most popular approach is based on the so called the residual principle. For linear equation and regularization of the variational type in Hilbert spaces, it was first proposed and studied by Ivanov and Morozov (see [23, 32]). For nonlinear problems and regularization of the operator type in Banach spaces, it was first investigated by Alber in [4] (see also [15]). Let us illustrate the idea of this approach by finding the solution $\tilde{x}^*$ to the operator equation $Ax = f$ with properly monotone hemi-continuous operator A acting from a uniformly convex Banach space B to the dual strictly convex space B^*. Let, for simplicity, the equation is defined only with a perturbed RHS f^ω, $\omega > 0$. According to (11) the regularized equation is

$$Ax^\omega_\alpha + \alpha J x^\omega_\alpha = f^\omega, \quad \alpha > 0, \quad \|f^\omega - f\|_* \le \omega. \tag{13}$$

In fact [1], $x^\omega_\alpha \to \tilde{x}^*$ for $\alpha \to 0$ and $\omega/\alpha \to 0$. However, the last relations do not provide the regularization properties of (13) in the sense that they do not guarantee convergence $x^\omega_\alpha \to \tilde{x}^*$ only under the condition $\omega \to 0$. In the principle of the residual we choose $\alpha = \bar{\alpha}$ from the condition

$$r(\bar{\alpha}) = \|Ax^\omega_{\bar{\alpha}} - f^\omega\|_* = \omega^s, \quad 0 < s \le 1. \tag{14}$$

Moreover, $x^\omega_{\bar{\alpha}}$ converge strongly to $\tilde{x}^*$ if $\omega \to 0$ and $0 < s < 1$, and converge weakly if $\omega \to 0$ and $s = 1$. In the both cases as a consequence we get the following limit relations: $\bar{\alpha} \to 0$ for $\omega \to 0$; $\omega/\bar{\alpha} \to 0$ for $0 < s < 1$, and $\omega/\bar{\alpha} <$ const for $s = 1$. This justifies approximating the solution $\tilde{x}^*$ by the solutions $x^\omega_{\bar{\alpha}}$ of (13) for a fixed ω. In this very form the problem of finding $x^* \in N$ is mostly close to the reality, and corresponds to the fundamental principle of selection of the approximate solution in correct (stable) problems: residual in the approximate solution has to be satisfied with the accuracy not exceeding the error of the RHS.

Of special importance in applications are equations and variational inequalities with maximal monotone, possibly multi-valued, operators [7, 19]. Evidently, the residual principle in the form of (14) is not applicable in these cases. The reason is that the residual $r(\bar{\alpha})$ becomes a multi-valued function of $\bar{\alpha}$, but in the proof we require this function to be continuous. In order to overcome this difficulty Alber in [4] proposed significant generalization of the residual principle. The idea of it is as follows: on the solutions $x^{\omega,h}_\alpha$ of

the regularized equation

$$A^h x_\alpha^{\omega,h} + \alpha J x_\alpha^{\omega,h} = f^\omega, \quad \omega > 0, \quad h > 0, \tag{15}$$

with maximal monotone operators A^h, the residual is

$$\|A^h x_\alpha^{\omega,h} - f^\omega\|_* = \alpha \|x_\alpha^{\omega,h}\|. \tag{16}$$

Hence the function $r(\alpha) = \alpha\|x_\alpha^{\omega,h}\|$ is natural to be called the *generalized residual* on approximate solutions $x_\alpha^{\omega,h}$ (in other points it is of no interest). The function $r(\alpha)$ is continuous, single-valued and monotone increasing. Moreover, if A is hemi- (demi-)continuous operator then $r(\alpha)$ coincides with the classical residual $\|Ax_\alpha^{\omega,h} - f^\omega\|_*$. It has been shown in [15] that the generalized residual principle in the form

$$\bar{\alpha}\|x_{\bar{\alpha}}^{\omega,h}\| = (h+\omega)^s, \quad 0 < s \leq 1,$$

where $x_{\bar{\alpha}}^{\omega,h}$ is the solution of (15) for $\alpha = \bar{\alpha}$ is the regularizing algorithm as $\omega, h \to 0$. The statement completely corresponds to the case of continuous operator A. The principle of generalized residual proved to be exceptionally fruitful not only for the problems with maximal but even for discontinuous monotone operators. Moreover, Ryazantseva [35] was first to use this principle for variational inequalities where the classical notion of residual is not defined even in the continuous case (see also [16]). The similar principle of the smoothing functional was studied in detail by Alber and Liskovets [8].

At last, let us mention the papers where under consideration were convergence and stability of the regularization method in the operator form for variational inequalities with perturbed sets Ω. In the most complete form it was studied recently by Alber and Notik [11] and Alber [2]. Different particular cases appeared earlier in Mosco [31], Bakushinskii [18], Liskovets [30], Ryazantseva [36], etc. The principle of the residual for such problems has been obtained by Ryazantseva [36] for Hilbert spaces, for Banach spaces the problem is still open.

Thus, the operator method of regularizing variational inequalities in Banach spaces has been developed in satisfactory completeness. However, this method by itself does not give a final solution to the problem, - for every fixed α it requires solving a nonlinear problem of type (15). In contrast to this the iterative regularization method gives successive approximations to the solution $\tilde{x}^*$ on every step of the iterative process. Here the parameters of the regularization are preassigned. Let us emphasize that the residual principle in the methods of iterative type has been studied in Hilbert spaces for linear equations and for iterative sequences of the form $x_{n+1} = x_n - \alpha(Ax_n - f)$ (see Vainikko [37]).

1. Main result

The process of proving convergence and stability of iterative regularization includes the operator regularization of the form

$$\langle Az_n + \alpha J z_n - f, x - z_n\rangle \geq 0, \quad \forall x \in \Omega_{\sigma_n}. \tag{17}$$

The method of iterative regularization which we consider can be described by the following iterative procedure:

$$Jx_{n+1} = Jx_n - \varepsilon_n(y^{h_m}\bar{x}_n + \alpha_m J\bar{x}_n - f^{\omega_m} + q_m),\ y^{h_m} \in A^{h_m}, \tag{18}$$

$$q_m = (\alpha_m + \alpha_m\|\bar{x}_n\| + \|y^{h_m}\bar{x}_n - f^{\omega_m}\|_*)\frac{J(x_n - \bar{x}_n)}{\|x_n - \bar{x}_n\|}, \tag{19}$$

$$m = n + n_0, \quad n_0 = \text{const} \geq 0, \quad x_n = J^*Jx_n, \quad x_1 \in B. \tag{20}$$

Here $\bar{x}_n = P_\Omega x_n$ is the metric projection of x_n on Ω. Since Ω is a convex closed set in a reflexive space then for arbitrary $x_n \in B$ there exists the unique $\bar{x}_n \in \Omega$ [3].

Analogously to (8) and (9) assume that $\Omega_{\sigma_n} \subset \text{int}D(A)$ are convex sets, $D(A^{h_n}) = D(A)$ and for $x \in D(A)$ the following inequality holds:

$$\mathcal{H}_{B^*}(\bar{A}x, \bar{A}^{h_n}x) \leq \zeta(\|x\|)h_n, \quad 0 \leq h_n \leq \bar{h}, \tag{21}$$

$$\|f^{\omega_n} - f\|_* \leq \omega_n, \quad 0 \leq \omega_n \leq \bar{\omega}, \tag{22}$$

$$\mathcal{H}_B(\Omega_{\sigma_n}, \Omega) \leq \sigma_n, \quad 0 \leq \sigma_n \leq \bar{\sigma}, \quad \sigma_{n+1} \leq \sigma_n. \tag{23}$$

The method of iterative regularization was first studied by Bruck and Bakushinskii in Hilbert spaces (see [21, 17]). They used the following scheme of the proof: according to the inequality

$$\|x_{n+1} - \tilde{x}^*\| \leq \|z_{n+1} - \tilde{x}^*\| + \|x_{n+1} - z_{n+1}\|, \tag{24}$$

on the fist stage convergence to $\tilde{x}^*$ of the solutions z_n to the variational inequality

$$\langle Az_n + \alpha_n z_n - f, x - z_n\rangle \geq 0, \quad \forall x \in \Omega$$

was established. On the second stage the limit relation

$$\lim_{n\to\infty} \|x_n - z_n\| = 0, \tag{25}$$

was proved. Here

$$x_{n+1} = P_\Omega(x_n - \varepsilon_n(A^{h_n}x_n + \alpha_n x_n - f^{\omega_n})), \quad n = 1, 2, \ldots \tag{26}$$

Such a scheme stems from Alber [5] where stabilization of the method of continuous regularization

$$\frac{d\,x(t)}{d\,t} + A(t)x(t) + \alpha(t)x(t) = f(t), \quad x(t_0) = x_0, \quad t_0 \leq t < \infty, \tag{27}$$

in Hilbert space H for solution of the operator equation $Ax = f$ with conditions

$$\|f(t) - f\|_H \leq \omega(t),$$
$$\|A(t) - A\|_H \leq h(t),$$

was studied.

The method of iterative regularization of operator equation $Ax = f$

$$x_{n+1} = x_n - \varepsilon_n(A^{h_n}x_n + \alpha_n x_n - f^{\omega_n}), \quad n = 1, 2, \dots$$

is the difference approximation of the differential equation (27) according to the simplest Euler scheme.

In application to variational inequalities the following fundamental statement strikes: a point $x^* \in \Omega \subset H$ is a solution of the variational inequality

$$\langle Ax - f, \xi - x\rangle \geq 0, \quad \forall \xi \in \Omega,$$

if and only if x^* is a solution of the operator equation

$$x = P_\Omega(x - \varepsilon(Ax - f)) \tag{28}$$

with an arbitrary $\varepsilon > 0$. Successive approximations for the regularized equation (28) with perturbed data give the iterative sequence (26).

In Banach spaces iterative regularization was considered in [6] only for nonlinear equations $Ax = f$ with monotone operators $A : B \to B^*$. Transition from Hilbert to Banach spaces is not trivial due to the following reasons. First of all, using conventional Lyapunov functionals $V_1(x) = \|x - x^*\|^2$ or $V_2(x) = \|Jx - Jx^*\|_*^2$ leads to unnatural a priori assumptions being outside the framework of the theory of monotone operators. To avoid this it was proposed in [6] (and earlier in [9]) to use in Banach spaces a new Lyapunov functional

$$V(Jx, x^*) = (\|Jx\|_*^2 - 2\langle Jx, x^*\rangle) + \|x^*\|^2)/2, \tag{29}$$

that does not require any a priori structure assumptions but monotonicity of the operator A. However, using such functional essentially complicates the proof, and the question of relations between $V(Jx, x^*)$ and the conventional functional $V_1(x)$ becomes of major importance since the final results should be stated in terms of $V_1(x)$.

Second, it turned out that in Banach spaces it is necessary to use the geometry of these spaces in order to investigate convergence and stability of iterative processes. In Hilbert spaces there is no such problem, more precisely, the geometry of Hilbert spaces is trivial enough, and it is automatically employed in squaring the norm $\|x - x^*\|$ in the Lyapunov functional $V_1(x)$, which is equivalent using the parallelogram equality [22]

$$\|x + y\|_H^2 + \|x - y\|_H = 2\|x\|_H^2 + 2\|y\|_H^2.$$

Third, in Hilbert spaces the operator $(A+\alpha I)$ of (24) is strongly monotone all over the space, in contrast to Banach spaces the operator $(A + \alpha J)$ is uniformly monotone only on the bounded sets. This is essentially used in the proof of uniform boundedness of approximate iterations and turns out to be the most difficult point in the theory of iterative methods.

Now let us summarize the main characteristics which distinct this paper from previous works:

1. In [6] iterative regularization of operator equations in all the space B was studied in the form

$$Jx_{n+1} = Jx_n - \varepsilon_n(y^{h_n} + \alpha_n Jx_n - f^{\omega_n}), \quad y^{h_n} \in A^{h_n}x_n, \tag{30}$$

$$x_n = J^*Jx_n, \quad n = 0, 1, \dots, \quad x_0 \in B.$$

In the present paper we study the iterative regularization method for variational inequalities which uses the operation of metric projection on Ω under the sign of the operator A (cf. [13]) and the additional "penalty" term (19).
2. In [6] it was a priori assumed that the sequence (30) is bounded (the same in [21, 34]). Here we omit this requirement. Therefore, the results obtained in this paper turn out to be new even for operator equations in Banach spaces.
3. In this paper we are first to consider stability of iterative processes for variational inequalities in relation to perturbation of the set Ω.

Denote by $\delta_B(\varepsilon)$, $\varepsilon \in [0, 2]$ the modulus of convexity, by $\rho_B(\tau)$, $\tau \geq 0$, the modulus of smoothness of B [22], $g_B(\varepsilon) = \delta_B(\varepsilon)/\varepsilon$. Assume that the inverse function $g_B^{-1}(\cdot)$ exists. It is known that $g_B(\varepsilon)$ and $g_B^{-1}(\cdot)$ are nondecreasing, and $\delta_B(\varepsilon)$ and $\rho_B(\tau)$ are increasing functions, $\delta_B(0) = \rho_B(0) = 0$. We also consider the function $\mu(t) = d_1t^2 + d_2\rho_B(t)$, which is increasing along with its inverse $\mu^{-1}(\cdot)$ (d_1 and d_2 are positive constants). It is possible to assume that $\delta_B(\varepsilon) \geq c\varepsilon^\gamma$, where $\gamma \geq 2$ and c is a positive constant. This assumption is not too restrictive since, on one hand, a direct calculation of $\delta_B(\varepsilon)$ in the spaces l^p, L^p, W_m^p, $1 < p < \infty$, and in some Orlicz spaces, shows that it allows the corresponding lower estimate

$$\delta_B(\varepsilon) \geq (p-1)\varepsilon^2/16, \quad 1 < p \leq 2,$$

$$\delta_B(\varepsilon) \geq p^{-1}(\varepsilon/2)^p, \quad p \geq 2.$$

On the other hand, the Pisier theorem [33] states that the same result up to isomorphism is valid for arbitrary uniformly convex Banach space. Thus, without loss of generality, we assume the following: there exists continuous on $[0, 2]$ convex increasing function $\tilde{\delta}_B(\varepsilon)$, $\tilde{\delta}_B(0) = 0$, such that

$$\delta_B(\varepsilon) \geq \tilde{\delta}_B(\varepsilon). \tag{31}$$

Our aim is to prove the following result.

Theorem 1. *Let B be an uniformly convex Banach space, B^* be its dual. Moreover, assume that*

1. *$\tilde{x}^*$ is the solution of the variational inequality* (4) *of minimal norm;*
2. *The operator $A : B \to 2^{B^*}$ is monotone bounded with φ-arbitrary growth order, i.e.*

$$\|y\|_* \leq \varphi(\|x\|), \quad \forall y \in Ax, \tag{32}$$

   where $\varphi(t)$ is a continuous nondecreasing function for $t \geq 0$.
3. *The perturbations of the operator A, the element f and the convex set Ω are given with errors and satisfy* (21)-(23).

4. *In the method* (18)-(19) *the initial approximation* x_0 *satisfies the inequality* $V(Jx_0, z_{n_0}) \le R_0$, *where* R_0 *is an arbitrary constant,* z_{n_0} *is the solution of the variational inequality*

$$\langle Az + \alpha_{n_0} Jz - f, x - z\rangle \ge 0, \ \forall x \in \Omega_{\sigma_{n_0}}, \tag{33}$$

and n_0 *satisfies* (62).
5. $\sum_{n=1}^{\infty} \alpha_n \varepsilon_n = \infty$.
6. *There exist constants* $a > 0$, *and* $b > 0$, *such that*

$$\frac{\sigma_n}{\alpha_n} \le a, \quad \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n} \le b,$$

and for $n \to \infty$

$$\begin{gathered} \alpha_n \to 0, \quad \frac{\rho_{B^*}(\varepsilon_n)}{\alpha_n \varepsilon_n} \to 0, \quad \frac{\omega_n + h_n + \varepsilon_n + \sigma_n}{\alpha_n} \to 0, \\ g_B^{-1}\left(\sqrt{\frac{\sigma_n}{\alpha_n}} + \frac{|\alpha_n - \alpha_{n+1}|}{\alpha_n}\right) / \alpha_n \varepsilon_n \to 0. \end{gathered} \tag{34}$$

Then the sequence $\{x_n\}$ *generated by the iterative method* (18)-(19) *converges strongly to* $\tilde{x}^*$.

2. Auxiliary lemma

As a preliminary, let us show validity of the following "proximity lemma".

Lemma 1. *Suppose that*

(i) B *is a uniformly convex Banach space with the modulus of convexity* $\delta_B(\varepsilon)$;
(ii) *the sequences* $\{z_1\}$ *and* $\{z_2\}$ *of the solutions to the variational inequalities*

$$\langle T_1 z_1 + \alpha_1 J z_1, x - z_1\rangle \ge 0, \quad \forall x \in \Omega_1 \subset B, \quad z_1 \in \Omega_1, \quad \alpha_1 > 0,$$

$$\langle T_2 z_2 + \alpha_2 J z_2, x - z_2\rangle \ge 0, \quad \forall x \in \Omega_2 \subset B, \quad z_2 \in \Omega_2, \quad \alpha_2 > 0,$$

are bounded for all α_1 *and* α_2 *respectively, i.e.* $\|z_1\| \le M_1, \|z_2\| \le M_1$;
(iii) *the operator* T_1 *is monotone and bounded on the sets* $\{z_1\}$ *and* $\{z_2\}$, *i.e.* $\|T_1 z_1\|_* \le M_2$ *and* $\|T_1 z_2\|_* \le M_2$, $M_2 = \text{const.}$;
(iv) $\mathcal{H}_{B^*}(T_1(z), T_2(z)) \le \omega\zeta(\|z\|)$, *where* z *is an arbitrary element of* $\{z_2\}$;
(v) $\Omega_1 \subset D$ *and* $\Omega_2 \subset D$ *are convex closed sets,* $\mathcal{H}_B(\Omega_1, \Omega_2) \le \sigma_0$; $D = D(T_1) = D(T_2)$.

Then the following estimate is valid:

$$\|z_1 - z_2\| \le c_1 g_B^{-1}\left(c_3\sqrt{\frac{\sigma_0}{\alpha_1}} + c_4 \frac{|\alpha_1 - \alpha_2|}{\alpha_1} + c_5 \frac{\omega}{\alpha_1}\right), \tag{35}$$

where

$$\begin{gathered} c_1 \le 2\max\{1, M_1\}, \\ c_2 = 2M_2 + \alpha_1 M_1 + \alpha_2 M_1 + \omega\zeta(M_1), \\ c_3 = \max\{1, 2^{-1} L c_2\}, \quad 1 < L < 3.18, \\ c_4 = 2c_1 L M_1, \end{gathered}$$

$$c_5 = 2c_1 L\zeta(M_1),$$

Proof. We consider the expression

$$\begin{aligned} D &= \langle T_1 z_1 + \alpha_1 J z_1 - T_2 z_2 - \alpha_2 J z_2, z_1 - z_2\rangle \\ &= \langle T_1 z_1 - T_1 z_2 + \alpha_1 (J z_1 - J z_2) + T_1 z_2 - T_2 z_2 + (\alpha_1 - \alpha_2) J z_2, z_1 - z_2\rangle. \end{aligned}$$

By virtue of monotonicity of T_1,

$$\langle T_1 z_1 - T_1 z_2, z_1 - z_2\rangle \geq 0$$

In [9, 10] an estimate of the adjoint mapping was given:

$$\langle J z_1 - J z_2, z_1 - z_2\rangle \geq (2L)^{-1}\delta_B(\|z_1 - z_2\|/c_1),$$

Then we have

$$\begin{aligned} D &\geq \alpha_1 (2L)^{-1}\delta_B(\|z_1 - z_2\|/c_1) - \|T_1 z_2 - T_2 z_2\|_* \|z_1 - z_2\| \\ &\quad - |\alpha_1 - \alpha_2| \|z_2\| \|z_1 - z_2\| \\ &\geq -\|z_1 - z_2\|(\omega\zeta(\|M_1\|) + M_1|\alpha_1 - \alpha_2|) \\ &\quad + \alpha_1 (2L)^{-1}\delta_B(\|z_1 - z_2\|/c_1). \end{aligned} \tag{36}$$

On the other hand, since $\mathcal{H}_B(\Omega_1, \Omega_2) \leq \sigma_0$, then for $z_2 \in \Omega_2$ there exists $\tilde{z_1} \in \Omega_1$ such that $\|z_2 - \tilde{z_1}\| \leq \sigma_0$ and

$$\begin{aligned} \langle T_1 z_1 + \alpha_1 J z_1, z_1 - z_2\rangle &= \langle T_1 z_1 + \alpha_1 J z_1, z_1 - \tilde{z}_1 + \tilde{z}_1 - z_2\rangle \\ &= \langle T_1 z_1 + \alpha_1 J z_1, z_1 - \tilde{z}_1\rangle + \langle T_1 z_1 + \alpha_1 J z_1, \tilde{z}_1 - z_2\rangle \\ &\leq (\|T_1 z_1\|_* + \alpha_1 \|z_1\|)\sigma_0 \leq (M_2 + \alpha_1 M_1)\sigma_0. \end{aligned}$$

Analogously, one can obtain the estimate

$$\langle T_2 z_2 + \alpha_2 J z_2, z_2 - z_1\rangle \leq (\|T_2 z_2\|_* + \alpha_2 M_1)\sigma_0.$$

It is evident that

$$\|T_2 z_2\|_* \leq \|T_1 z_2\|_* + \|T_2 z_2 - T_1 z_2\|_* \leq M_2 + \omega\zeta(M_1).$$

And so it follows that

$$D \leq (2M_2 + \alpha_1 M_1 + \alpha_2 M_1 + \omega\zeta(M_1))\sigma_0 = c_2\sigma_0.$$

From the last inequality and from (36) we have

$$c_2\sigma_0 + \|z_1 - z_2\|(\omega\zeta(M_1) + M_1|\alpha_1 - \alpha_2|) \geq \alpha_1 (2L)^{-1}\delta_B(\|z_1 - z_2\|/c_1). \tag{37}$$

Only two cases are possible, either

$$(i) \qquad \|z_1 - z_2\| < c_1 g_B^{-1}\left(\sqrt{\frac{\sigma_0}{\alpha_1}} + c_4 \frac{|\alpha_1 - \alpha_2|}{\alpha_1} + c_5 \frac{\omega}{\alpha_1}\right) \tag{38}$$

or

$$(ii) \qquad \|z_1 - z_2\| \geq c_1 g_B^{-1}\left(\sqrt{\frac{\sigma_0}{\alpha_1}} + c_4 \frac{|\alpha_1 - \alpha_2|}{\alpha_1} + c_5 \frac{\omega}{\alpha_1}\right).$$

In the second case, since [22, 3]

$$\delta_B(\varepsilon) \leq \delta_H(\varepsilon) \leq \varepsilon^2/4,$$

we obtain

$$\frac{\|z_1 - z_2\|}{4c_1} \geq g_H\left(\frac{\|z_1 - z_2\|}{c_1}\right) \geq g_B\left(\frac{\|z_1 - z_2\|}{c_1}\right) \geq \sqrt{\frac{\sigma_0}{\alpha_1}}. \tag{39}$$

Consider now (37). We have

$$\frac{c_1\delta_B(\|z_1 - z_2\|/c_1)}{\|z_1 - z_2\|} \leq c_4\frac{|\alpha_1 - \alpha_2|}{\alpha_1} + \frac{2Lc_1c_2}{\|z_1 - z_2\|}\frac{\sigma_0}{\alpha_1} + c_5\frac{\omega}{\alpha_1}.$$

Using (39) , we get

$$\|z_1 - z_2\| \leq c_1 g_B^{-1}\left(\frac{Lc_2}{2}\sqrt{\frac{\sigma_0}{\alpha_1}} + c_4\frac{|\alpha_1 - \alpha_2|}{\alpha_1} + c_5\frac{\omega}{\alpha_1}\right).$$

Comparing the last expression with (38), we get (35). The proof of the lemma is completed. □

Remark 1. If $\sigma_0 = 0$, i.e. Ω_1 coincides to Ω_2 and $\omega = 0$, i.e. T_1 coincides with T_2 (at least on the elements of the sequence $\{z_2\}$) then

$$\|z_1 - z_2\| \leq c_1 g_B^{-1}\left(c_4\frac{|\alpha_1 - \alpha_2|}{\alpha_1}\right). \tag{40}$$

This estimate coincides with those obtained before (see [1, 6]). Proximity estimates of solutions to variational inequalities with perturbed operators and perturbed sets of constraints have not exist before.

Remark 2. In the case of multi-valued operators, by T_1 and T_2 we mean the single-valued sections of the corresponding operators.

3. Proof of Theorem 1

Consider the auxiliary inequality (33). Earlier, in [1], it was proved that the relation $\lim_{n\to\infty}\|z_n - \tilde{x}^*\| = 0$ holds in B while $\sigma/\alpha \to 0$ and $\|z_n\| \leq M_1$, $M_1 = const$. From (24) it follows that it is necessary to prove only (25) to obtain strong convergence and stability of the method (18)-(19).

Let z_m and z_{m+1} be solutions of the variational inequalities

$$\langle Az_m + \alpha_m Jz_m - f, x - z_m\rangle \geq 0, \quad \forall x \in \Omega_{\sigma_m}, \quad m = n + n_0$$

and

$$\langle Az_{m+1} + \alpha_{m+1}Jz_{m+1} - f, x - z_{m+1}\rangle \geq 0, \quad \forall x \in \Omega_{\sigma_{m+1}}$$

respectively. Consider the Lyapunov functional

$$V(Jx, z) = (\|x\|^2 - 2\langle Jx, z\rangle + \|z\|^2)/2,$$

where x and z are arbitrary points from B. Let us state the properties of this functional (see [3]):

1. $V(Jx, z)$ is a convex, continuous and differentiable with respect to each argument $\varphi = Jx$ and z;
2. $\mathrm{grad}_\varphi V(Jx, z) = x - z$, if z is fixed;
3. $\mathrm{grad}_z V(Jx, z) = Jz - Jx$, if x is fixed;
4. $V(Jx, z) \geq 0$ $\forall x, z \in B$ and $V(Jx, z) = 0$, only if $x = z$;
5. $2^{-1}(\|x\| - \|z\|)^2 \leq V(Jx, z) \leq 2^{-1}(\|x\| + \|z\|)^2$;
6. $V(Jx, z) \to \infty$, if $\|x\| \to \infty$ and (or) $\|z\| \to \infty$.

From the convexity of $V(Jx, z)$ in z we have

$$\begin{aligned} V(Jx_{n+1}, z_{m+1}) &- V(Jx_{n+1}, z_m) \\ &\le \langle Jz_{m+1} - Jx_{n+1}, z_{m+1} - z_m \rangle \\ &\le \|Jz_{m+1} - Jx_{n+1}\|_* \|z_{m+1} - z_m\|. \end{aligned} \tag{41}$$

Now from the convexity of $V(Jx, z)$ in $\varphi = Jx$, we have

$$\begin{aligned} V(Jx_{n+1}, z_m) &- V(Jx_n, z_m) \\ &\le \langle Jx_{n+1} - Jx_n, x_{n+1} - z_m \rangle \\ &= \langle Jx_{n+1} - Jx_n, x_n - z_m \rangle + \langle Jx_{n+1} - Jx_n, x_{n+1} - x_n \rangle. \end{aligned} \tag{42}$$

(41) together with (42) give the following inequality

$$\begin{aligned} V(Jx_{n+1}, z_{m+1}) &- V(Jx_n, z_m) \\ &\le \|Jz_{m+1} - Jx_{n+1}\|_* \|z_{m+1} - z_m\| \\ &\quad + \langle Jx_{n+1} - Jx_n, x_{n+1} - x_n \rangle + \langle Jx_{n+1} - Jx_n, \bar{x}_n - z_m \rangle \\ &\quad + \langle Jx_{n+1} - Jx_n, x_n - \bar{x}_n \rangle. \end{aligned} \tag{43}$$

Let us estimate each of the four terms in the RHS of (43).

The first term. Using the estimate for the modulus of continuity of the duality mapping [9, 10], we get

$$\begin{aligned} \|Jx_{n+1} - Jz_{m+1}\|_* &\le c_6 g_{B^*}^{-1}(2c_6 L \|x_{n+1} - z_{m+1}\|) \\ &\le c_6 g_{B^*}^{-1}(2c_6 L(\|x_{n+1}\| + \|z_{m+1}\|)), \end{aligned}$$

where

$$c_6 = 2\max\{1, \|x_{n+1}\|, \|z_{m+1}\|\}. \tag{44}$$

Then, from Lemma 1 it follows that

$$\|z_{m+1} - z_m\| \le c_1 g_B^{-1}\left(c_4 \frac{|\alpha_m - \alpha_{m+1}|}{\alpha_m} + 2c_3\sqrt{\frac{\sigma_m}{\alpha_m}}\right).$$

We have used here the inequality:

$$\mathcal{H}_B(\Omega_{\sigma_m}, \Omega_{\sigma_{m+1}}) \le \mathcal{H}_B(\Omega, \Omega_{\sigma_m}) + \mathcal{H}_B(\Omega, \Omega_{\sigma_{m+1}}) \le \sigma_m + \sigma_{m+1} \le 2\sigma_m.$$

The second term. First consider the following estimate (see [9, 10])

$$\langle Jx_{n+1} - Jx_n, x_{n+1} - x_n \rangle \le 8\|Jx_{n+1} - Jx_n\|_*^2 + c_7 \rho_{B^*}(\|Jx_{n+1} - Jx_n\|_*),$$

where

$$c_7 = 8\max\{L, \|x_{n+1}\|, \|x_n\|\}. \tag{45}$$

It is evident (see e.g. (22)), that

$$\|f^{\omega_n}\|_* \le \|f\|_* + \omega_n.$$

By virtue of (21) and (32) we can write for all $y_n \in Ax_n$

$$\|y^{h_n}\|_* \le \|y^{h_n} - y_n\|_* + \|y_n\|_* \le h_n \zeta(\|x_n\|) + \varphi(\|x_n\|).$$

Now let us estimate $\|Jx_{n+1} - Jx_n\|_*$:

$$\begin{aligned}\|Jx_{n+1} - Jx_n\|_* &= \varepsilon_m \|\bar{y}^{h_m} - f^{\omega_m} + \alpha_m J\bar{x}_n + q_m\|_* \\ &\le \varepsilon_m (2\|\bar{y}^{h_m}\|_* + 2\|f^{\omega_m}\|_* + 2\alpha_m \|\bar{x}_n\| + \alpha_m) \\ &\le \varepsilon_m c_8,\end{aligned}$$

where

$$c_8 = 2(\varphi(\|\bar{x}_n\|) + h_m \zeta(\|\bar{x}_n\|) + \alpha_m \|\bar{x}_n\| + \|f\|_* + \omega_m) + \alpha_m. \tag{46}$$

Therefore

$$\langle Jx_{n+1} - Jx_n, x_{n+1} - x_n\rangle \le 8c_8^2 \varepsilon_m^2 + c_7 \rho_{B^*}(c_8 \varepsilon_m).$$

The third term. For all $\bar{y}_n \in A\bar{x}_n$, $w_m \in Az_m$:

$$\begin{aligned}&\langle Jx_{n+1} - Jx_n, \bar{x}_n - z_m\rangle \\ &\quad = -\varepsilon_m \langle \bar{y}^{h_m} + \alpha_m J\bar{x}_n - f^{\omega_n} + q_m, \bar{x}_n - z_m\rangle \\ &\quad = -\varepsilon_m \langle \bar{y}^{h_m} - \bar{y}_n, \bar{x}_n - z_m\rangle - \varepsilon_m \alpha_m \langle J\bar{x}_n - Jz_m, \bar{x}_n - z_m\rangle \\ &\qquad - \varepsilon_m \langle \bar{y}_n - w_m, \bar{x}_n - z_m\rangle \\ &\qquad - \varepsilon_m \langle w_m + \alpha_m Jz_m \quad - f, \bar{x}_n - z_m\rangle \\ &\qquad - \varepsilon_m \langle f - f^{\omega_m}, \bar{x}_n - z_m\rangle - \langle q_m, \bar{x}_n - z_m\rangle.\end{aligned}$$

Taking into account that

$$\langle \bar{y}_n - w_m, \bar{x}_n - z_m\rangle \ge 0, \qquad \langle w_m + \alpha_m Jz_m - f, \bar{x}_n - z_m\rangle \ge 0$$

and

$$\langle q_m, \bar{x}_n - z_m\rangle \ge 0,$$

we have

$$\begin{aligned}\langle Jx_{n+1} - Jx_n, \bar{x}_n - z_m\rangle \le{}& (h_m \zeta(\|\bar{x}_n\|) + \omega_m)\varepsilon_m \|\bar{x}_n - z_m\| \\ &- \varepsilon_m \alpha_m (2L)^{-1} \delta_B(\|\bar{x}_n - z_m\|/c_9),\end{aligned}$$

where

$$c_9 = 2\max\{1, \|\bar{x}_n\|, \|z_m\|\}. \tag{47}$$

And, finally, *the fourth term.*

$$\begin{aligned}\langle Jx_{n+1} - Jx_n, x_n - \bar{x}_n\rangle &= -\varepsilon_m \alpha_m \langle J\bar{x}_n, x_n - \bar{x}_n\rangle - \varepsilon_m \langle q_m, x_n - \bar{x}_n\rangle \\ &\quad - \varepsilon_m \langle \bar{y}^{h_m} - f^{\omega_m}, x_n - \bar{x}_n\rangle \\ &\le (\varepsilon_m \alpha_m \|\bar{x}_n\| - \varepsilon_m \alpha_m - \varepsilon_m \alpha_m \|\bar{x}_n\| \\ &\quad - \varepsilon_m \|\bar{y}^{h_m} - f^{\omega_m}\|_* + \varepsilon_m \|\bar{y}^{h_m} - f^{\omega_m}\|_*)\|x_n - \bar{x}_n\| \\ &= -\varepsilon_m \alpha_m \|x_n - \bar{x}_n\|.\end{aligned}$$

Now we can rewrite (43) in the following form:

$$\begin{aligned}V(Jx_{n+1}, z_{m+1}) \le V(Jx_n, z_m) - \alpha_m \varepsilon_m ((2L)^{-1} \delta_B(\|\bar{x}_n - z_m\|/c_9)) \\ + \|x_n - \bar{x}_n\|) + \gamma_m,\end{aligned} \tag{48}$$

where

$$\gamma_m = \varepsilon_m(h_m\zeta(\|\bar{x}_n\|) + \omega_m)\|\bar{x}_n - z_m\| + 8\varepsilon_m^2 c_8^2 + c_7\rho_{B^*}(c_8\varepsilon_m) + c_1c_6c_{10}g_B^{-1}\left(2c_3\sqrt{\frac{\sigma_m}{\alpha_m}} + c_4\frac{|\alpha_m - \alpha_{m+1}|}{\alpha_m}\right),$$

and

$$c_{10} = g_{B^*}^{-1}(2c_6L(\|x_{n+1}\| + \|z_{m+1}\|)). \tag{49}$$

Assume that $V(Jx_n, z_m) \le R_0$. Then $\|x_n\| \le \|z_m\| + \sqrt{2R_0} \le M_1 + \sqrt{2R_0} = K_1$. It follows from the 5-th property of the functional $V(Jx, z)$. It is easy to see that

$$\|\bar{x}_n\| \le \|\bar{x}_n - z_m\| + \|z_m\| \le 2\|x_n - z_m\| + \|z_m\| \le 2\|x_n\| + 3\|z_m\| \le 2K_1 + 3M_1 = K_2.$$

If $\|x_n\| \le K_1$ and $\|\bar{x}_n\| \le K_2$, then it is possible to derive from the expressions (44), (45), (46), (47), (49) the following inequality:

$$c_8 \le 2(\varphi(K_2) + \bar{h}\zeta(K_2) + \bar{\alpha}K_2 + \|f\|_* + \overline{\omega}) + \bar{\alpha} = \bar{c}_8, \tag{50}$$

Since

$$\|x_{n+1}\| \le \|x_n\| + \|Jx_{n+1} - Jx_n\|_* \le K_1 + \varepsilon_m c_8 \le K_1 + \bar{\varepsilon}\bar{c}_8,$$

then

$$c_9 \le 2\max\{1, K_2, M_1\} = \bar{c}_9, \tag{51}$$

$$c_6 \le 2\max\{1, K_1 + \bar{\varepsilon}\bar{c}_8, M_1\} = \bar{c}_6, \tag{52}$$

$$c_7 \le 8\max\{L, K_1 + \bar{\varepsilon}\bar{c}_8\} = \bar{c}_7, \tag{53}$$

and

$$c_{10} \le g_B^{-1}(2\bar{c}_6L(K_1 + \bar{\varepsilon}\bar{c}_8 + M_1) = \bar{c}_{10}, \tag{54}$$

where $\bar{c}_6$, $\bar{c}_7$, $\bar{c}_8$, $\bar{c}_9$, $\bar{c}_{10}$ are absolute constants. Getting back to (48), we obtain

$$\begin{aligned} V(Jx_{n+1}, z_{m+1}) &\le V(Jx_n, z_m) \\ &\quad - \varepsilon_m\alpha_m[(2L)^{-1}\delta_B(\|\bar{x}_n - z_m\|/\bar{c}_9) + \|x_n - \bar{x}_n\|] + \bar{\gamma}_m \\ &\le V(Jx_n, z_m) - \varepsilon_m\alpha_m[(2L)^{-1}\tilde{\delta}_B\|\bar{x}_n - z_m\|/\bar{c}_9) \\ &\quad + \|x_n - \bar{x}_n\|] + \bar{\gamma}_m, \end{aligned} \tag{55}$$

where

$$\bar{\gamma}_m = \varepsilon_m(h_m\zeta(K_2) + \omega_m)(K_2 + \bar{c}_8) + 8\varepsilon_m^2\bar{c}_8^2 + \bar{c}_7\rho_{B^*}(\varepsilon_m\bar{c}_8) + \bar{c}_6\,\bar{c}_{10}\,\bar{c}_1\,g_B^{-1}\left(2c_3\sqrt{\frac{\sigma_m}{\alpha_m}} + c_4\frac{|\alpha_m - \alpha_{m+1}|}{\alpha_m}\right). \tag{56}$$

It is evident that

$$\|x_n - \bar{x}_n\| = \frac{\delta(\|x_n - \bar{x}_n\|/\bar{c}_9)\bar{c}_9}{g_B(\|x_n - \bar{x}_n\|/\bar{c}_9)}. \tag{57}$$

At the same time, since $\|x_n - \bar{x}_n\| \le K_1 + K_2$,

$$g_B(\|x_n - \bar{x}_n\|/\bar{c}_g) \le g_B((K_1 + K_2)/\bar{c}_g) = c_{11}. \tag{58}$$

Thus, from (57), it follows that

$$\|x_n - \bar{x}_n\| \ge \bar{c}_9 c_{11}^{-1} \delta_B(\|x_n - \bar{x}_n\|/\bar{c}_9) \ge \bar{c}_9 c_{11}^{-1} \tilde{\delta}_B(\|x_n - \bar{x}_n\|/\bar{c}_9).$$

By our assumption, $\tilde{\delta}_B(\varepsilon)$ is a convex function. Therefore,

$$\begin{aligned}
&(2L)^{-1}\tilde{\delta}_B(\|\bar{x}_n - z_m\|/\bar{c}_9) + \|\bar{x}_n - x_n\| \\
&\ge (2L)^{-1}\tilde{\delta}_B(\|\bar{x}_n - z_m\|/\bar{c}_9) + \bar{c}_9 c_{11}^{-1}\tilde{\delta}_B(\|\bar{x}_n - x_n\|/\bar{c}_9) \\
&\ge \frac{1}{2} c_{12}(\tilde{\delta}_B(\|\bar{x}_n - z_m\|/\bar{c}_9) + \tilde{\delta}_B(\|\bar{x}_n - x_n\|/\bar{c}_9) \\
&\ge c_{12}\tilde{\delta}_B((2\bar{c}_9)^{-1}(\|\bar{x}_n - z_m\| + \|x_n - \bar{x}_n\|)) \\
&\ge c_{12}\tilde{\delta}_B(\|x_n - z_m\|/2\bar{c}_9).
\end{aligned} \tag{59}$$

Here,

$$c_{12} = 2\min\{(2L)^{-1}, \bar{c}_9\, c_{11}^{-1}\}. \tag{60}$$

Taking all this into consideration, along with (55), we have the following numerical inequality:

$$V(Jx_{n+1}, z_{m+1}) \le V(Jx_n, z_m) - \varepsilon_m \alpha_m c_{12}\tilde{\delta}_B(\|x_n - z_m\|/2\bar{c}_9) + \bar{\gamma}_m. \tag{61}$$

Recall that $m = n + n_0$. Choose n_0 according to the formula:

$$n_0 = \min\{k : 2\bar{c}_9\tilde{\delta}_B^{-1}(\bar{\gamma}_k/c_{12}\varepsilon_k\alpha_k) \le \mu^{-1}(R_0)\}, \tag{62}$$

where

$$\mu(t) = 8t^2 + C\rho_B(t),$$
$$C = 8\max\{L, M_1 + \sqrt{2R_1}\},$$
$$R_1 = R_0 + \bar{r}$$

and

$$\bar{r} = \bar{\varepsilon}\bar{c}_8(K_1 + M_1 + \bar{\varepsilon}\bar{c}_8) + c_1(K_1 + M_1)g_B^{-1}(2c_3\sqrt{a} + c_4 b).$$

Let us calculate the Lyapunov functional $V(Jx, z)$ at the point (x_{n+1}, z_{m+1}).

$$\begin{aligned}
V(Jx_{n+1}, z_{m+1}) \le\ & 2^{-1}\|Jx_n\|^2 - \langle Jx_n, z_m\rangle + 2^{-1}\|z_m\|^2 \\
&+ \varepsilon_m\|A^{h_m}\bar{x}_n + \alpha_m J\bar{x}_n - f^{\omega_m} + q_m\|(\|x_n\| + \|z_{m+1}\|) \\
&+ 2^{-1}\varepsilon_m^2\|A^{h_m}\bar{x}_n + \alpha_m J\bar{x}_n - f^{\omega_m} + q_m\|^2 \\
&+ \langle Jx_n, z_m - z_{m+1}\rangle + 2^{-1}(\|z_{m+1}\|^2 - \|z_m\|^2).
\end{aligned}$$

Let us estimate the last two terms.

$$\begin{aligned}
&\langle Jx_n, z_m - z_{m+1}\rangle + 2^{-1}(\|z_{m+1}\|^2 - \|z_m\|^2) \\
&\qquad \le (\|x_n\| + 2^{-1}(\|z_{m+1}\| + \|z_m\|))\|z_m - z_{m+1}\| \\
&\qquad \le (K_1 + M_1)c_1 g_B^{-1}\left(2c_3\sqrt{\frac{\sigma_m}{\alpha_m}} + c_4\frac{|\alpha_m - \alpha_{m+1}|}{\alpha_m}\right).
\end{aligned}$$

Therefore, according to our notation,

$$V(Jx_{n+1}, z_{m+1}) \leq R_0 + r_m, \tag{63}$$

where

$$r_m = \varepsilon_m c_8(K_1 + M_1 + \varepsilon_m c_8) + c_1(K_1 + M_1) g_B^{-1} \left(c_4 \frac{|\alpha_m - \alpha_{m+1}|}{\alpha_m} + 2c_3 \sqrt{\frac{\sigma_m}{\alpha_m}} \right).$$

Then it is clear that $r_m \leq \bar{r}$, since $\varepsilon_m \leq \bar{\varepsilon}$, $\sigma_m/\alpha_m \leq a$, $|\alpha_m - \alpha_{m+1}|/\alpha_m \leq b$ and

$$V(Jx_{n+1}, z_{m+1}) \leq R_0 + \bar{r} = R_1. \tag{64}$$

Let us show now that (64) holds for all $n \geq 1$. Let $V(Jx_0, z_{n_0}) \leq R_0$ and the minimal n be such that

$$R_0 < V(Jx_n, z_{n+n_0}) < R_0 + \bar{r}. \tag{65}$$

For this n we have the following alternatives:
(H_1) $\tilde{\delta}_B(\|x_n - z_m\|/2\bar{c}_9) > \bar{\gamma}_m/c_{12}\alpha_m\varepsilon_m$;
(H_2) $\tilde{\delta}_B(\|x_n - z_m\|/2\bar{c}_9 \leq \bar{\gamma}_m/c_{12}\alpha_m\varepsilon_m$.

From (H_1) and (61) it follows:

$$V(Jx_{n+1}, z_{m+1}) < R_0 + \bar{r}.$$

The inequality H_2 is impossible. Indeed, suppose that the contrary holds. Then we obtain from (62)

$$\|x_n - z_m\| \leq 2\bar{c}_9 \tilde{\delta}_B^{-1}(\bar{\gamma}_m/c_{12}\alpha_m\varepsilon_m) \leq \mu^{-1}(R_0). \tag{66}$$

Let us estimate $V(Jx_n, z_m)$ using $\|x_n - z_m\|$. As $V(Jx_n, x_n) = 0$,

$$V(Jx_n, z_m) \leq \langle Jx_n - Jz_m, x_n - z_m \rangle.$$

But in [9, 10] it is shown that

$$\langle Jx_n - Jz_m, x_n - z_m \rangle \leq 8\|x_n - z_m\|^2 + c_{13}\rho_B(\|x_n - z_m\|),$$

where

$$c_{13} = 8\max\{L, \|x_n\|, \|z_m\|\} \leq 8\max\{L, M_1 + \sqrt{2R_1}\} = \bar{c}_{13}.$$

Therefore,

$$V(Jx_n, z_m) \leq 8\|x_n - z_m\|^2 + \bar{c}_{13}\rho_B(\|x_n - z_m\|) = \mu(\|x_n - z_m\|). \tag{67}$$

With regard to (66) it brings

$$V(Jx_n, z_m) \leq R_0,$$

which contradicts (65). Thus, (64) takes place for all $n \geq 1$, i.e. $\|x_n\|$ are uniformly bounded. From (67)

$$\|x_n - z_m\| \geq \mu^{-1}(V(Jx_n, z_m)),$$

therefore, since $\tilde{\delta}_B(\varepsilon)$ is nondecreasing function, then

$$\tilde{\delta}_B(\|x_n - z_m\|/2\bar{c}_9) \geq \tilde{\delta}_B(\mu^{-1}(V(Jx_n, z_m))/2\bar{c}_9). \tag{68}$$

Denote $\psi(\lambda) = \tilde{\delta}_B(\mu^{-1}(\lambda/2\bar{c}_9))$. It is clear that $\psi(\lambda)$ is continuous and nondecreasing function, and that $\psi(0) = 0$. Then, finally, we obtain the following numerical inequality

$$V(Jx_{n+1}, z_{m+1}) \leq V(Jx_n, z_m) - \alpha_m \varepsilon_m c_{12} \psi(V(Jx_n, z_m)) + \bar{\gamma}_m,$$

or, letting $\lambda_n = V(Jx_n, z_m) \geq 0$,

$$\lambda_{n+1} \leq \lambda_n - \alpha_m \varepsilon_m c_{12} \psi(\lambda_n) + \bar{\gamma}_m. \tag{69}$$

Lemma 2. (see [7, 14]) *If the sequence of nonnegative numbers* $\{\lambda_n\}$ *satisfies the inequality (69),* $\varepsilon_m > 0$, $\alpha_m > 0$,

$$\sum_{m=1}^{\infty} \alpha_m \varepsilon_m = \infty, \qquad \lim_{m\to\infty} \frac{\bar{\gamma}_m}{\alpha_m \varepsilon_m} \to 0,$$

$\psi(\lambda)$ *is continuous, nondecreasing function and* $\psi(0) = 0$, *then* $\lim_{n\to\infty} \lambda_n = 0$.

From this lemma along with the conditions (34) it follows that

$$V(Jx_n, z_m) \to 0, n \to \infty. \tag{70}$$

And finally, analogously to [3], the following estimate has been obtained

$$\|x - y\| \leq 4R_1 \tilde{\delta}_B^{-1}(V(Jx,y)/4R_1^2). \tag{71}$$

Indeed, for arbitrary elements $x, y \in B$

$$\begin{aligned} \|\frac{x+y}{2}\|^2 - \|x\|^2 &\geq \langle Jx, y - x\rangle + 2R_1 \delta_B(\frac{\|x-y\|}{4R_1}) \\ &\geq \langle Jx, y - x\rangle + 2R_1 \tilde{\delta}_B(\frac{\|x-y\|}{4R_1}), \end{aligned} \tag{72}$$

where

$$R_1 = \sqrt{(\|x\|^2 + \|y\|^2)/2}.$$

Using the equality $\langle Jx, x\rangle = \|x\|^2$, we get

$$\langle Jx, y\rangle \leq \|(x+y)/2\|^2 - 4R_1 \tilde{\delta}_B(\|x-y\|/4R_1).$$

From the convexity of $\tilde{\delta}_B(\varepsilon)$, it follows that $\tilde{\delta}_B(t\varepsilon) \leq t\tilde{\delta}_B(\varepsilon)$ for each $t \in [0, 1]$. Particularly, for $t = 1/2$, we have $2\tilde{\delta}_B(\varepsilon/2) \leq \tilde{\delta}_B(\varepsilon)$. Hence,

$$\begin{aligned} V(x, y) &= 2^{-1}(\|x\|^2 - 2\langle Jx, y\rangle + \|y\|^2) \\ &\geq 2^{-1}(\|x\|^2 + \|y\|^2 - 2^{-1}\|x+y\|^2 + 4R_1^2 \tilde{\delta}_B(\|x-y\|/4R_1)) \\ &\geq R_1^2 \tilde{\delta}_B(\|x-y\|/2R_1) + 2R_1^2 \tilde{\delta}_B(\|x-y\|/4R_1) \\ &\geq 4R_1^2 \tilde{\delta}_B(\|x-y\|/4R_1). \end{aligned}$$

This gives (71). From (70) we obtain the final limit relation

$$\lim_{n\to\infty} \|x_n - z_{n+n_0}\| = 0,$$

since $\|x_n\|$ and $\|z_m\|$ are uniformly bounded.

The proof of the theorem is complete.

Remark 3. This scheme of the proof could be applied to the study of equations $Ax = f$ with accretive operators $A : B \to B$:

$$\langle J(x-y), Ax - Ay\rangle \geq 0, \quad \forall x, y \in B.$$

It gives the possibility to get rid of the assumption of the a priori boundedness of the sequence $\{x_n\}$ in the method of iterative regularization

$$x_{n+1} = x_n - \varepsilon_m(y^{h_m} x_n + \alpha_m x_n - f^{\omega_m}), \quad n = 1, 2, ...$$

(cf. [10, 34]).

Remark 4. In the proof of Theorem 1 the monotonicity of the operators A^{h_n} is used only in (8), to provide the existence of the maximal monotone extension. Thus, continuous A^{h_n} could be not necessary monotone.

References

[1] Y. I. Alber, The solution of nonlinear equations with monotone operators in Banach spaces, *Siberian Math. J.* **16** (1975), 1-8.

[2] Y. I. Alber, The regularization method for variational inequalities with nonsmooth unbounded operators in Banach space, *Appl. Math. Lett.* **6** (1993), 63-68.

[3] Y. I. Alber, Metric and generalized projection operators in Banach spaces: properties and applications, (to appear).

[4] Y. I. Alber, The monotonicity method and the approximate computation of the value of the nonlinear unbounded operator, *Siberian Math. J.* **19** (1978), 179-183.

[5] Y. I. Alber, A continuous regularization of linear operator equations in Hilbert spaces, *Math. Notes* **9** (1968), 42-54.

[6] Y. I. Alber, Iterative regularization in Banach spaces, *Soviet Math. (Izv. VUZ)* **30** (1986), 1-8.

[7] Y. I. Alber, The solution of equations and variational inequality with maximal monotone operators, *Soviet Math. Dokl.* **20** (1979) 871-876.

[8] Y. I. Alber and O. A. Liskovets, The principle of the smoothing functional for solution of equations of the first kind with monotone operators, *Differential Equations* **20** (1984), 603-608.

[9] Y. I. Alber and A. I. Notik, Parallelogram inequalities in Banach spaces and some properties of the duality mapping, *Ukrainian Math. J.* **40** (1988), 650-652.

[10] Y. I. Alber and A. I. Notik, Geometric properties of Banach spaces and approximate methods for solving nonlinear operator equations, *Soviet Math. Dokl.* **29** (1984), 611-615 .

[11] Y. I. Alber and A. I. Notik, Perturbed unstable variational inequalities with unbounded operator on approximately given sets, *Set-Valued Anal.* **1** (1993), 393-402.

[12] Y. I. Alber and A. I. Notik, Iterative processes in Orlicz spaces, *Methods of Optimization and Operational Research*, Irkutsk, 1984, 114-123 (in Russian).

[13] Y. I. Alber and A. I. Notik, On minimization of functionals and solution of variational inequalities in Banach spaces, *Soviet Math. Dokl.* **34** (1986), 296-300.

[14] Y. I. Alber and S. Reich, An iterative method for solving a class of nonlinear operator equations in Banach spaces, *Panamer. Math. J.* **4** (1994), 39-54.

[15] Y. I. Alber and I. P. Ryazantseva, The residual principle in nonlinear problems with discontinuous monotone mappings is a regularizing algorithm, *Soviet Math. Dokl.* **19** (1978), 437-440.

[16] Y. I. Alber and I. P. Ryazantseva, Variational inequalities with discontinuous monotone mappings, *Soviet Math. Dokl.* **25** (1982), 206-210.

[17] A. B. Bakushinskii, Methods for solution of monotone variational inequalities that are based on the principle of iterative regularization, *Zh. Vychisl. Mat. i Mat. Fiz.* **17** (1977), 1350-1362 (in Russian).

[18] A. B. Bakushinskii, On the principle of iterative regularization, *Comput. Math. Math. Phys.* **19** (1979), 256-260.
[19] F. E. Browder, Nonlinear maximal monotone operators in Banach space, *Math. Ann.* **175** (1968), 89-113.
[20] F. E. Browder, Existence and approximation of solution of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. USA* **56** (1966), 1080-1086.
[21] R. E. Bruck, Jr., A strongly convergent iterative solution of $0 \in Ux$ for a maximal monotone operator U in Hilbert space, *J. Math. Anal. Appl.* **48** (1974), 114–126.
[22] J. Diestel, *The geometry of Banach spaces*, Lecture Notes in Math. **485**, Springer, 1975.
[23] V.K. Ivanov, Approximate solution of operator equations of the first kind, *Zh. Vychisl. Mat. i Mat. Fiz.* **6** (1966), 1089-1094. (in Russian).
[24] A. Kartsatos, New results in the perturbation theory of maximal monotone and m-accretive operators in Banach spaces, Trans. Amer. Math. Soc. (to appear)
[25] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.
[26] R. Lattes and J.-L. Lions, *Méthode de quasiriversibilité et applications*, Dunod, Paris, 1967.
[27] M. M. Lavrentjev, *On some incorrect problems of mathematical physics*, Novosibirsk, 1962.
[28] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
[29] O. A. Liskovets, External approximations for regularization of monotone variational inequalities, *Soviet Math. Dokl.* **36** (1988), 220-224.
[30] O. A. Liskovets, Discrete convergence of elements and operators for ill-posed problems with a monotone operator, *Soviet Math. Dokl.* **31** (1985), 202-206.
[31] U. Mosco, Perturbation of variational inequalities, in *Nonlinear Functional Analysis*, Proc. Sympos. Pure. Math. **18**, Part 1, AMS, Providence, 1970, 182-184.
[32] V. A. Morozov, Linear and nonlinear ill-posed problems, *J. Soviet Math.* **4** (1975), 706-736.
[33] G. Pisier, Martingales with values in uniformly convex spaces, *Israel J. Math.* **20** (1975), 329-350.
[34] S. Reich, Constructive techniques for accretive and monotone operators, *Applied Nonlinear Analysis*, Academic Press, New York, 1979, 335-345.
[35] I. P. Ryazantseva, Solution of variational inequalities with monotone operators by the regularization method, *Comput. Math. Math. Phys.* **23** (1983), 145-148.
[36] I. P. Ryazantseva, Variational inequalities with monotone operators on sets which are specified approximately, *Comput. Math. Math.Phys.* **24** (1984), 194-197.
[37] G. M. Vainikko, The residual principle for one class of regularization methods, *Comput. Math. Math.Phys.* **22** (1982), 1-19.

Department of Mathematics, The Technion - Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: alberya@tx.technion.ac.il

Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA
E-mail address: hermes@gauss.math.usf.edu

Department of Theoretical Mathematics, Weizmann Institute of Science, 76100 Rehovot, Israel
E-mail address: elenal@wisdom.weizmann.ac.il