EIGENVALUES AND RANGES FOR PERTURBATIONS OF NONLINEAR ACCRETIVE AND MONOTONE OPERATORS IN BANACH SPACES

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ABSTRACT. Various eigenvalue and range results are given for perturbations of m -accretive and maximal monotone operators. The eigenvalue results improve and extend some recent results by Guan and Kartsatos, while the range theorem gives an affirmative answer to a recent problem of Kartsatos.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real Banach space with norm $\|\cdot\|$ and normalized duality mapping J . An operator $T : X \supset D(T) \rightarrow 2^X$ is said to be “accretive” if for every $x, y \in D(T)$ and every $u \in Tx$, $v \in Ty$ there exists $j \in J(x - y)$ such that

$$(1) \quad \langle u - v, j \rangle \geq 0$$

An accretive operator T is “strongly accretive” if there exists a positive constant c such that the inequality (1) holds with 0 replaced by $c\|x - y\|^2$. An accretive operator T is “ m -accretive” if $R(T + \lambda I) = X$ for some $\lambda > 0$, (or, equivalently, for all $\lambda > 0$). For an m -accretive operator T , the resolvents $J_\lambda : X \rightarrow D(T)$ of T are defined by $J_\lambda = (I + \lambda T)^{-1}$ for all $\lambda > 0$. The Yosida approximants $T_\lambda : X \rightarrow X$ of T are defined by $T_\lambda = \frac{1}{\lambda}(I - J_\lambda)$. For the main properties of J_λ and T_λ the reader is referred to Barbu [1], Browder[2] and Lakshmikantham and Leela [9].

In what follows, the symbols ∂D , $\overline{D}$ denote the strong boundary and closure of the set D , respectively. We denote by $B_r(0)$ the open ball of X with center at zero and radius $r > 0$. The symbol $\rightarrow$ ($\rightharpoonup$) means strong

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(weak) convergence. We denote by N , R_+ the set of positive integers and the set $[0, \infty)$, respectively.

An operator $T : X \supset D(T) \rightarrow 2^X$ is called “bounded” if $T(A) = \cup\{Tx : x \in A\}$ is bounded for any bounded set $A \subset D(T)$. The operator T is said to be “ ϕ -expansive” on $E \subset D(T)$ if there exists a continuous strictly increasing function $\phi : R_+ \rightarrow R_+$ such that $\phi(0) = 0$ and

$$(2) \quad \|u - v\| \geq \phi(\|x - y\|)$$

for every $x, y \in E$ and all $u \in Tx, v \in Ty$. T is called “compact” if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets. It is “demicontinuous” (“completely continuous”) if it is strong-weak (weak-strong) continuous on $D(T)$. A linear operator in a reflexive Banach space is completely continuous if and only if it is compact.

By a “cone” we mean a closed and convex subset K of X which is closed under multiplication by nonnegative scalars and such that $K \cap (-K) = \{0\}$.

Recently, Guan and Kartsatos [4] established several results concerning the eigenvalue problem for a pair of operators T and C , where T is at least accretive or monotone while C is at least compact or bounded and continuous.

One of our objectives in this paper is to complement and improve the above results by using a well known theorem of Guo and Leray-Schauder degree theory. Our second objective is to provide an affirmative answer to a problem of Kartsatos [8] concerning ranges of perturbed maximal monotone operators (Theorem 2.4).

Before we state and prove our main results, we need some auxiliary results which follow. Unless otherwise stated, the symbol $d(\cdot, \cdot, \cdot)$ denotes the Leray-Schauder degree.

The following two lemmas can be found in [4].

Lemma 1.1. *Let $C : \overline{D} \rightarrow X$ be compact, where $D \subset X$ is open and bounded. Assume that there exists $y_0 \in X$ such that $y_0 \neq 0$ and $(I - C)x \neq \lambda y_0$, $\lambda \geq 0$, $x \in \partial D$. Then $d(I - C, D, 0) = 0$.*

Lemma 1.2. *Let X be a real infinite dimensional Banach space and $K \subset X$ be a compact set. Assume that there exists a positive constant α such that $\|x\| \geq \alpha$, for every $x \in K$. Then there exists $y_0 \in X$ with $\|y_0\| = 1$ and $\{ty_0 : t \geq \alpha\} \cap K = \emptyset$.*

We can now use the above Lemmas to deduce the following known result due to Guo [5]. Its proof is given for completeness.

Theorem 1.1. *Let X be a real infinite dimensional Banach space and let $D \subset X$ be open and bounded. Let $C : \overline{D} \rightarrow X$ be compact and assume that there exists a positive constant α such that $\|Cx\| \geq \alpha$, $x \in \partial D$. Assume, further, that $Cx = \mu x$, $x \in \partial D$ imply $\mu \notin (0, 1]$. Then $d(I - C, D, 0) = 0$.*

Proof. Choose y_0 as in Lemma 1.2 with $K = \overline{-C(\partial D)}$. Then $\{ty_0 : t \geq \alpha\} \cap K = \emptyset$.

We shall show that there exists $\lambda^* \geq 1$ such that

$$(3) \quad (I - \lambda^* C)x \neq \eta y_0, \quad \eta \geq 0, \quad x \in \partial D.$$

Indeed, assume (3) is false. Then there exist $x_n \in \partial D$, $\lambda_n, \eta_n \in (0, \infty)$ such that $\lambda_n \rightarrow \infty$ and

$$(4) \quad (I - \lambda_n C)x_n = \eta_n y_0.$$

Notice that $\{x_n\}$ is bounded. We know that $x_n/\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Hence,

$$(5) \quad \left\| Cx_n + \left(\frac{\eta_n}{\lambda_n} \right) y_0 \right\| \rightarrow 0$$

as $n \rightarrow \infty$. Since $\{Cx_n\}$ is bounded, we may assume that $\eta_n/\lambda_n \rightarrow \mu$ as $n \rightarrow \infty$. Then

$$(6) \quad \|Cx_n + \mu y_0\| \rightarrow 0$$

as $n \rightarrow \infty$, which implies $\mu \geq \alpha$. This contradicts the fact $\{ty_0 : t \geq \alpha\} \cap K = \emptyset$. Using Lemma 1.1, we see that $d(I - \lambda^* C, D, 0) = 0$.

Now, we consider the following compact homotopy

$$H(t, x) \equiv x - [t\lambda^* + (1-t)]Cx, \quad t \in [0, 1], \quad x \in \overline{D}.$$

By our assumption, $H(t, x) \neq 0$, $t \in [0, 1]$, $x \in \partial D$. By the Leray-Schauder degree theory, we have $d(I - C, D, 0) = d(I - \lambda^* C, D, 0) = 0$. The proof of Theorem 1.1 is complete. ■

Theorem 1.2. *Let X be a real Banach space and $K \subset X$ be a cone. Let $D \subset X$ be an open, bounded set. Assume that $C : \overline{D} \rightarrow K$ is a compact operator satisfying the following conditions:*

- (i) $\inf\{\|Cx\| : x \in \partial D\} > 0$;
- (ii) $Cx = \mu x$, $x \in \partial D$ imply $\mu \notin (0, 1]$.

Then $d(I - C, D, 0) = 0$.

Proof. Let $M = \sup\{\|x\| : x \in \overline{D}\}$ and $d = \inf\{\|Cx\| : x \in \partial D\} > 0$. Letting

$$\tau_0 > \max \left\{ \frac{2M}{d}, 1 \right\},$$

we have $x \neq \tau_0 Cx$ for all $x \in \partial D$. We shall show that there exists $\tau > \tau_0$ and some $u_0 \in X$, with $u_0 \neq 0$, such that

$$(7) \quad x - \tau Cx \neq t u_0,$$

for any $t \geq 0$, $x \in \partial D$. In fact, (7) holds for $t = 0$ and any $\tau > \tau_0$, $u_0 \in X$. Thus, we only need to show (7) for $t > 0$. If this is not the case, then for any $u \in K$, with $\|u\| = 1$, there exist $x_n \in \partial D$, $\tau_n > \tau_0$, $t_n \in (0, \infty)$ such that $\tau_n \rightarrow \infty$ and

$$(8) \quad x_n - \tau_n Cx_n = t_n u.$$

Notice that

$$\left\| \frac{x_n}{\tau_n} - Cx_n \right\| = \left\| \frac{t_n}{\tau_n} u \right\| = \frac{t_n}{\tau_n}$$

and

$$(9) \quad \frac{\frac{x_n}{\tau_n} - Cx_n}{\left\| \frac{x_n}{\tau_n} - Cx_n \right\|} = \frac{t_n u}{\tau_n \left\| \frac{x_n}{\tau_n} - Cx_n \right\|} = u.$$

Since $\{x_n\}$ is bounded and C is compact, we may assume that $Cx_n \rightarrow y$ as $n \rightarrow \infty$. From (9) we know that $-y/\|y\| = u \in K$. On the other hand, since $C : \overline{D} \rightarrow K$, we have $y \in K$ and hence $y/\|y\| = -u \in K$. Thus, $u = 0$, which is a contradiction to $\|u\| = 1$. By Lemma 1.1,

$$(10) \quad d(I - \tau C, D, 0) = 0.$$

Now, we consider the compact homotopy

$$(11) \quad H(t, x) \equiv x - (t\tau + 1 - t)Cx, t \in [0, 1], x \in \overline{D}.$$

We have $H(t, x) \neq 0$, for any $t \in [0, 1]$, $x \in \partial D$. By the Leray-Schauder degree theory we have $d(I - C, D, 0) = d(I - \tau C, D, 0) = 0$, which completes the proof of Theorem 1.2. ■

From Theorem 1.2 we have the following two results. Since their proofs are straightforward, we omit them.

Theorem 1.3. *Let X be a real Banach space and K a cone in X . Let $D \subset X$ be an open, bounded and such that $0 \in D$. Assume that $C : \overline{D} \rightarrow K$ is compact and such that $\inf\{\|Cx\| : x \in \partial D\} > 0$. Then there exists $(\lambda, x) \in (0, \infty) \times (K \cap \partial D)$ such that $Cx = \lambda x$.*

Theorem 1.4. *Let X , D , K be as in Theorem 1.3. Assume that $C : \partial D \cap K \rightarrow K$ is compact and that there exists a constant $\alpha > 0$ such that $\inf\{\|Cx\| : x \in \partial D \cap K\} > \alpha$. Then the conclusion of Theorem 1.3 is true.*

2. MAIN RESULTS

We start with a theorem which improves the corresponding result (Theorem 2.5) in Guan and Kartsatos [4]. We do not assume that X and X^* are uniformly convex. We also assume that C compact, but not necessarily completely continuous. Finally, we assume only the boundedness of T on the intersection of a certain ball and its domain of definition.

Theorem 2.1. *Let X be a real infinite dimensional Banach space and let $D \subset X$ be open and bounded. Let $T : X \supset D(T) \rightarrow 2^X$ be an m -accretive operator with $0 \in T(0)$, $0 \in D \cap D(T)$ and $D \subset \overline{D(T)}$. Assume that $T(\overline{B_r(0)} \cap D(T))$ is bounded, where $r = \sup\{\|x\| : x \in \partial D\}$. Let $C : \overline{D(T)} \rightarrow X$ be compact and let there exist $\alpha > 0$ such that $\|Cx\| \geq \alpha$, for all $x \in \partial D$. Then (i) for every $c > 0$ there exists (λ_c, x_c) such that*

$$0 \in Tx_c + cx_c - \lambda_c Cx_c;$$

(ii) if, moreover, $0 \notin T(D(T) \cap \partial D)$ and T is ϕ -expansive on $D(T) \cap \partial D$, then there exists $(\lambda_0, x_0) \in (0, \infty) \times \partial D$ such that $0 \in Tx_0 - \lambda_0 Cx_0$.

Proof. (i) Let $c > 0$ be given. Since $0 \in D$, we know that $r > 0$. Set $M = \sup\{\|y\| : y \in Tx, x \in \overline{B_r(0)} \cap D(T)\}$ and choose

$$\lambda^* > \frac{(M + cr)}{\alpha}.$$

Since $T : D(T) \rightarrow 2^X$ is m-accretive and $C : \overline{D(T)} \rightarrow X$ is compact, we see that $A = (T + cI)^{-1}\lambda^*C : \overline{D} \rightarrow D(T)$ is compact.

Now we verify that the operator A satisfies all conditions in Theorem 1.1.

(I) $\inf\{\|Ax\| : x \in \partial D\} > 0$. If this is not the case, there exists $\{x_n\} \subset \partial D$ such that $Ax_n \rightarrow 0$ as $n \rightarrow \infty$. Let $u_n = Ax_n$. Then

$$(12) \quad \lambda^*Cx_n \in Tu_n + cu_n, \quad n \in N.$$

Thus, there exist $v_n \in Tu_n$ such that

$$(13) \quad v_n + cu_n = \lambda^*Cx_n.$$

Since $u_n \rightarrow 0$ as $n \rightarrow \infty$, we have $u_n \in B_r(0) \cap D(T)$ for all large $n \in N$. Thus, $\|v_n\| \leq M$ for all large n . From (13) we have

$$\lambda^* = \frac{\|v_n + cu_n\|}{\|Cx_n\|} \leq \frac{M + cr}{\alpha},$$

which contradicts the choice on λ^* .

(II) If $Ax = \mu x$, $x \in \partial D$, then $\mu \notin (0, 1]$. If this is not the case, there exist $x_0 \in \partial D$ and $\mu_0 \in (0, 1]$ such that $Ax_0 = \mu_0 x_0$. Then $\mu_0 x_0 \in \overline{B_r(0)} \cap D(T)$ and $\lambda^*Cx_0 \in T(\mu_0 x_0) + c\mu_0 x_0$. Thus we can find $v_0 \in T(\mu_0 x_0)$ such that

$$(14) \quad v_0 + c\mu_0 x_0 = \lambda^*Cx_0.$$

Consequently,

$$\lambda^* = \frac{\|v_0 + c\mu_0 x_0\|}{\|Cx_0\|} \leq \frac{M + cr}{\alpha},$$

i.e., a contradiction. By Theorem 1.1, we have $d(I - A, D, 0) = 0$, i.e.,

$$(15) \quad d(I - (T + cI)^{-1}\lambda^*C, D, 0) = 0.$$

Now we construct a compact homotopy as follows:

$$H(t, x) \equiv x - (T + cI)^{-1}\lambda^*tCx, \quad t \in [0, 1], \quad x \in \overline{D}.$$

Indeed, for every $t \in [0, 1]$, $(T + cI)^{-1}\lambda^*tC : \overline{D} \rightarrow D(T)$ is relatively compact. On the other hand, for every $\{t_n\} \subset [0, 1]$, with $t_n \rightarrow t$ as $n \rightarrow \infty$, and all $x \in \overline{D}$, we have

$$\begin{aligned} & \|(T + cI)^{-1}\lambda^*t_nCx - (T + cI)^{-1}\lambda^*tCx\| \\ & \leq \frac{\lambda^*}{c} \sup\{\|Cx\| : x \in \overline{D}\} |t_n - t| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Hence, $(T + cI)^{-1}\lambda^*tC : [0, 1] \times \overline{D} \rightarrow D(T)$ is a compact operator. Notice that $H(0, \cdot) = I$, $0 \in D$. We have $d(H(0, \cdot), D, 0) = 1$. By the Leray-Schauder degree theory, we must have $(t_c, x_c) \in (0, 1] \times \partial D$ such that $x_c = (T + cI)^{-1}\lambda^*Cx_c$. This says that $0 \in Tx_c + cx_c$, where $\lambda_c = \lambda^*t_c \in (0, \infty)$.

(ii) We take $c = \frac{1}{n}$. Then there exists $(\lambda_n, x_n) \in (0, \infty) \times \partial D$ such that

$$(16) \quad 0 \in Tx_n + \frac{1}{n}x_n - \lambda_n Cx_n.$$

Thus, $v_n + \frac{1}{n}x_n = \lambda_n Cx_n$, where $v_n \in Tx_n$. Since $\{x_n\} \subset \overline{B_r}(0) \cap D(T)$, we have $\|v_n\| \leq M$. Hence $\{\lambda_n\}$ must be bounded. We may assume that $\lambda_n \rightarrow \lambda_0$, $Cx_n \rightarrow y$ as $n \rightarrow \infty$. Then $v_n \rightarrow \lambda_0 y$ as $n \rightarrow \infty$. Since T is ϕ -expansive on $D(T) \cap \partial D$, we know that $x_n \rightarrow x \in D(T)$ and $\lambda_0 y \in Tx$. Hence $Cx_n \rightarrow Cx$ as $n \rightarrow \infty$. So, $y = Cx$. Thus, we have $\lambda_0 Cx \in Tx$, i.e., $0 \in Tx - \lambda_0 Cx$, where $(\lambda_0, x_0) \in (0, \infty) \times \partial D$. The proof of Theorem 2.1 is finished. ■

For cones of Banach spaces we have the following two theorems.

Theorem 2.2. *Let X be a real Banach space with a cone K . Let $T : X \supset D(T) \rightarrow 2^X$ have a continuous single-valued inverse $T^{-1} : X \rightarrow D(T)$. Assume that $C : \overline{D(T)} \rightarrow K$ is compact and there exist an open bounded set G and a constant $\alpha > 0$ such that: $\overline{G} \subset D(T)$, $0 \in T(G)$, $T(G)$ is bounded and $\|Cx\| \geq \alpha$, $x \in \partial G$. Then there exists $(\lambda, x) \in (0, \infty) \times \partial G$ such that $0 \in \lambda Tx - Cx$.*

Proof. Since $T^{-1} : X \rightarrow D(T)$ is continuous, we know that $T(G)$ is open and $T(\overline{G})$ is closed, moreover, $\partial T(G) \subset T(\partial G)$. Notice that $CT^{-1} : \overline{T(G)} \rightarrow K$ is compact and $\|CT^{-1}u\| \geq \alpha$, $u \in \partial T(G)$. By Theorem 1.3, there exists $(\lambda, u) \in (0, \infty) \times K \cap \partial T(G)$ such that $CT^{-1}u = \lambda u$. Letting $x = T^{-1}u$, we have $0 \in \lambda Tx - Cx$, $x \in \partial G$. The proof of Theorem 2.2 is finished. ■

Theorem 2.3. *Let X be a real Banach space and $K \subset X$ be a cone. Let $T : D(T) \subset K \rightarrow X$ be an m -accretive operator with $T0 = 0$. Let D be a bounded open in X with $0 \in D(T) \cap D$ and $D \subset \overline{D(T)}$. Assume that $T(\overline{B_r}(0) \cap D(T))$ is bounded, where $r = \sup\{\|x\| : x \in \partial D\}$. Let $C : \overline{D(T)} \rightarrow X$ be compact and assume that there exists $\alpha > 0$ such that $\|Cx\| \geq \alpha$, $x \in \partial D$. Then*

(i) *For every $c > 0$, there exists $(\lambda_c, x_c) \in (0, \infty) \times \partial D \cap K$ such that $0 \in Tx_c + cx_c - \lambda_c Cx_c$;*

(ii) *if, moreover, $0 \notin T(D(T) \cap \partial D)$ and T is ϕ -expansive on $D(T) \cap \partial D$, then there exists $(\lambda_0, x_0) \in (0, \infty) \times \partial D \cap K$ such that $0 \in Tx_0 - \lambda_0 Cx_0$.*

Proof. The proof is similar to the proof of Theorem 2.1. It is therefore omitted. ■

We now turn our attention to an open problem stated by Kartsatos in [8].

Let X be a real reflexive Banach space with norm $\|\cdot\|$ and normalized duality mapping J . As it is often assumed, the spaces X , X^* are locally uniformly convex. Thus, J is a bicontinuous mapping.

An operator $T : X \supset D(T) \rightarrow 2^{X^*}$ is “monotone” if for every $x, y \in D(T)$ and $u^* \in Tx$, $v^* \in Ty$ we have

$$(17) \quad \langle u^* - v^*, x - y \rangle \geq 0$$

A monotone operator T is “maximal monotone” if $T + \lambda J$ is surjective for all $\lambda > 0$. An operator $T : X \supset D(T) \rightarrow Y$, with Y another real Banach space, is “bounded” if it maps bounded subsets of $D(T)$ onto bounded sets. It is “compact” if it is continuous and maps bounded subsets of $D(T)$ onto relatively compact sets.

Recently, Kartsatos [8, Theorem 7] proved the following result.

Theorem K. *Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone and $C : D(T) \rightarrow X^*$. Let $(T + J)^{-1}$ be compact. Let $G \subset X$ be open, bounded and such that, for some $z \in D(T) \cap G$ and $v^* \in Tz$,*

$$0 \notin Tx - v^*, \quad \langle u^* + Cx, x - z \rangle > 0, \quad (x, u^*) \in (D(T) \cap \partial G) \times Tx.$$

Assume, further, that the operator $C(\lambda T + J)^{-1}$ is compact, where λ is a fixed constant, and the set $C(D(T) \cap \overline{G})$ is bounded. Then $0 \in (T + C)(D(T) \cap \overline{G})$.

Kartsatos asked in [8] the following question: is Theorem K true without the assumption that $0 \notin Tx - v^*$, for every $x \in D(T) \cap \partial G$?

We shall solve the above open problem, in the affirmative, by using Kartsatos’ degree theory from [7]. For this purpose, we shall first solve the perturbed problem:

$$0 \in Tx + Cx + \epsilon Jx.$$

By a limiting process, we can then pass to the solution of the original problem. The key step of the proof is to construct a homotopy equation:

$$u = H(t, u) \equiv -tC(T + \epsilon J)^{-1}u, \quad t \in [0, 1], \quad u \in X^*,$$

which satisfies the condition of Kartsatos [7, Theorem 1].

Theorem 2.4. *Let the assumptions of Theorem K be satisfied except, possibly, the one on $Tx - v^*$. Then $0 \in (T + C)(D(T) \cap \overline{G})$.*

Proof. By the proof of Kartsatos [8, Theorem 7], we see that it suffices to show Theorem 2.4 for $z = 0$ and $0 \in T0$. Otherwise, we reduce the problem to this case by a suitable transformation.

Let $\epsilon \in (0, 1)$ be given. Set $U = (T + \epsilon J)(D(T) \cap G)$, and $V = (T + \epsilon J)(D(T) \cap \overline{G})$.

Since $(T + \epsilon J)^{-1} : X^* \rightarrow D(T)$ is continuous, we know that $U \subset X^*$ is open and $V \subset X^*$ is closed. Thus, $\overline{U} \subset V$.

Observing that $\overline{U} = U \cup \partial U$ and $V = U \cup (T + \epsilon J)(D(T) \cap \partial G)$, we have

$$(18) \quad \partial U \subset (T + \epsilon J)(D(T) \cap \partial G).$$

We solve first the perturbed problem

$$(19) \quad 0 \in Tx + Cx + \epsilon Jx.$$

Since $T : X \supset D(T) \rightarrow 2^{X^*}$ is maximal monotone, we see that (19) is equivalent to

$$(20) \quad u = -C(T + \epsilon J)^{-1}u.$$

We now consider the homotopy equation

$$(21) \quad u = H(t, u) \equiv -tC(T + \epsilon J)^{-1}u, \quad t \in [0, 1], \quad u \in X^*.$$

It is easy to see from the resolvent identity for maximal monotone operators (see, for example, [8]) that $C(T + \epsilon J)^{-1} : X^* \rightarrow X^*$ is compact. Notice that $H(t, \cdot)\bar{U} = -tC(T + \epsilon J)^{-1}\bar{U} \subset -tC(D(T) \cap \bar{G})$ is bounded. By Kartsatos' degree theory [7, Theorem 1], we only need to prove that the equation (21) has no solution in ∂U . Assume that the contrary is true. Then there exists $x_t \in \partial U$ such that

$$x_t = -tC(T + \epsilon J)^{-1}x_t.$$

Setting $u_t = (T + \epsilon J)^{-1}x_t$, we see that $0 \in Tu_t + \epsilon Ju_t + tCu_t$.

Noting that $x_t \in \partial U \subset V = (T + \epsilon J)(D(T) \cap \partial G)$, we have $u_t = (T + \epsilon J)^{-1}x_t \in D(T) \cap \partial G$. By our assumption, for $u^* \in Tu_t$,

$$\begin{aligned} 0 &= \langle u^* + tCu_t + \epsilon Ju_t, u_t \rangle \\ &= \langle (1-t)u^* + tu^* + tCu_t, u_t \rangle + \epsilon \|u_t\|^2 \\ &= (1-t) \langle u^*, u_t \rangle + t \langle u^* + Cu_t, u_t \rangle + \epsilon \|u_t\|^2 \\ &\geq \epsilon \|u_t\|^2 > 0, \end{aligned}$$

i.e., a contradiction. Hence

$$d(I + C(T + \epsilon J)^{-1}, U, 0) = d(I, U, 0) = 1.$$

This shows that the equation (20) is solvable in $u \in U$, and hence the equation (19) is solvable in $x \in D(T) \cap \bar{G}$.

At this point we can choose $\epsilon_n = \frac{1}{n}$. Then there exists $x_n \in D(T) \cap \bar{G}$ such that

$$0 \in Tx_n + Cx_n + \frac{1}{n}Jx_n,$$

which leads to

$$(22) \quad x_n = (T + J)^{-1}((1 - \frac{1}{n})Jx_n - Cx_n).$$

Since $\{Jx_n\}$, $\{Cx_n\}$ are bounded and $(T + J)^{-1}$ is compact, we see that $\{x_n\}$ lies in a compact set. Without loss of generality, we may assume that $x_n \rightarrow x$ as $n \rightarrow \infty$.

It follows from (22) that

$$(23) \quad Cx_n = C(T + J)^{-1}((1 - \frac{1}{n})Jx_n - Cx_n).$$

Since $\{Jx_n\}$, $\{Cx_n\}$ are bounded and $C(T + J)^{-1}$ is compact, we see that $\{Cx_n\}$ lies in a compact set. We may assume that $Cx_n \rightarrow y$ as $n \rightarrow \infty$. Consequently, taking limit on both sides of (23), we obtain

$$y = C(T + J)^{-1}(Jx - y).$$

Set $u = (T + J)^{-1}(Jx - y)$. Then $y = Cu$ and $Jx - y \in Tu + Ju$, which implies that $Jx \in Tu + Cu + Ju$.

On the other hand, by taking the limit on both sides of (22), we have

$$x = (T + J)^{-1}(Jx - y) = u,$$

and hence $Ju = Jx$, $0 \in Tu + Cu$, $u \in D(T) \cap \bar{G}$, which ends the proof of Theorem 2.4. ■

Remark 2.1. By Theorem 2.4, we know that Corollary 4 of Kartsatos [8] is true without the assumption that $0 \notin Tx$, for every $x \in D(T) \cap \partial G$.

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