SINGULAR NONLINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$

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Abstract. This paper deals with existence, uniqueness and regularity of positive generalized solutions of singular nonlinear equations of the form $-\Delta u + a(x)u = h(x)u^{-\gamma}$ in $\mathbb{R}^N$ where $a, h$ are given, not necessarily continuous functions, and $\gamma$ is a positive number. We explore both situations where $a, h$ are radial functions, with $a$ being eventually identically zero, and cases where no symmetry is required from either $a$ or $h$. Schauder’s fixed point theorem, combined with penalty arguments, is exploited.

1. Introduction

This paper addresses existence, uniqueness and regularity questions on generalized solutions of the singular nonlinear elliptic problem

\[
\begin{cases}
-\Delta u + a(x)u = h(x)u^{-\gamma} & \text{in } \mathbb{R}^N \\
u > 0 & \text{in } \mathbb{R}^N
\end{cases}
\]

where $a, h$ are nonnegative $L^\infty_{loc}$ functions, $h \not\equiv 0$, (eventually we consider $a \equiv 0$), $\gamma > 0$ and $N \geq 3$. We point out that the search of positive solutions for the Dirichlet problem for the equation

\[-\Delta u + a(x)u = h(x)u^{-\gamma} \text{ in } \Omega\]

where $\Omega$ is a smooth bounded domain has deserved the attention of many authors. Nowosad [1] studied a related Hammerstein equation, namely

\[u(x) = \int_0^1 K(x, y)(u(x))^{-\gamma} dy,\]

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where \( \gamma = 1 \), \( \int_{0}^{1} K(x,y)dy \geq \delta > 0 \) and \( K(x,y) \) is positive semidefinite. Nowosad’s work was extended by Karlin and Nirenberg [2] where more general Hammerstein equations were considered including the case \( \gamma > 0 \) in the equation above. Crandall-Rabinowitz and Tartar [3] studied the Dirichlet problem

\[
Lu = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]

where \( L \) is a linear second order elliptic operator and \( f : \Omega \times (0, +\infty) \to \mathbb{R} \) is singular in the sense that \( f(x,r) \to \infty \) as \( r \to 0^+ \). Examples such as \( f(x,r) = r^{\gamma} \) with \( \gamma > 1 \), \( \gamma < 1 \) or \( \gamma = 1 \) were covered.

There is by now an extensive literature on singular elliptic problems. With respect to the case of bounded domains \( \Omega \subset \mathbb{R}^N \) we would like to further mention Gomes [4], Lazer and McKenna [5], Cac and Hernandez [8], Chen [9], Lair and Shaker [10], Shangbin [13] while for the case \( \Omega = \mathbb{R}^N \) we recall Kuzano and and Swanson [11], Lair and Shaker [12,14]. This reference list is far from complete. In the earlier papers concerning \( \Omega = \mathbb{R}^N \), \( h(x) \) is assumed at least continuous and several techniques are developed such as the method of lower and upper solutions. In this paper we assume \( h(x) \) only integrable and use the Schauder fixed point theorem and elliptic estimates. Singular equations appear in the theory of heat conduction in electrically conducting materials, (Fulks and Maybee [6]), in binary communications by signals (Nowosad [1]) and in the theory of pseudoplastic fluids (Nachman and Callegari [7]).

The following condition on \( a \) will be required in the first one of our main results stated below:

\( (a)_{R} \quad a(x) \geq a_0 \text{ for } |x| \geq R \text{ for some } a_0, R > 0. \)

In what follows we take \( \gamma, \alpha \in (0, 1) \) and \( h \in L^\theta \cap L^2 \) where \( \theta \equiv \frac{2}{2-(1-\gamma)}. \)

**Theorem 1.** Assume \( (a)_{R} \). Then (*) has a unique solution \( u \in D^{1,2} \cap W_{loc}^{2,p} \) where \( 1 < p < \infty \) with \( \int a(x)u^2 < \infty \). If \( a, h \) are radial functions the solution is radial, as well, and in fact, \( u(x) \to 0 \) as \( |x| \to \infty \). Moreover if \( a, h \in C_{loc}^\alpha \) then \( u \in C_{loc}^{2,\alpha} \).

In our second result we take \( a \equiv 0 \) and \( h \) radially symmetric that is, we study the problem

\[
(*)_0 \quad \begin{cases} -\Delta u = h(|x|)u^{-\gamma} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases}
\]

This problem shall be treated by first perturbing the equation by a radially symmetric term, then using the earlier result in the case \( a, h \) are radial functions and finally taking limits.

**Theorem 2.** Let \( a \equiv 0 \) and let \( h \) be radially symmetric. Then \( (*)_0 \) has a unique radially symmetric solution \( u \in D^{1,2} \cap W_{loc}^{2,p} \), \( 1 < p < \infty \) and \( u(x) \to 0 \) as \( |x| \to \infty \). Moreover, if \( h \in C_{loc}^\alpha \) then \( u \in C_{loc}^{2,\alpha} \).
2. Preliminaries

The main goal in this section is to prove theorem 1. For that purpose let \( \epsilon > 0 \) and consider the problem

\[
\begin{cases}
-\Delta u + a(x)u = \frac{h(x)}{(u+\epsilon)^\gamma} & \text{in } \mathbb{R}^N \\
u > 0 & \text{in } \mathbb{R}^N.
\end{cases}
\]

We are going to show by applying the Schauder fixed point theorem that (2.1) has a solution \( u \in W^{2,p}_{\text{loc}}, \ 1 < p < \infty \), and then by passing to the limit as \( \epsilon \to 0 \) we arrive at a solution of (\( * \)).

In order to deal with a first step namely, existence of a solution of (2.1), let \( f \in L^2 \) and consider the linear equation

\[
-\Delta u + a(x)u = f(x) \quad \text{in } \mathbb{R}^N.
\]

Recalling that the Hilbert space \( \mathcal{D}^{1,2} \) is defined as the closure of \( C_0^\infty \) with respect to the gradient norm \( \| \varphi \|^2 = \int |\nabla \varphi|^2 \) we introduce the space

\[
E \equiv \left\{ u \in \mathcal{D}^{1,2} \mid \int au^2 < \infty \right\}
\]

which endowed with the inner product and norm given respectively by

\[
\langle u, v \rangle = \int (\nabla u \cdot \nabla v + auv) \quad \text{and} \quad \| u \|^2 = \langle u, u \rangle
\]

is itself a Hilbert space. Under condition \((a)_R\) it follows that \( u \in E \) iff \( u \in W^{1,2}(\mathbb{R}^N) \).

Yet if \( f \in L^2 \) it follows by minimizing over \( E \) the energy functional associated with (2.2),

\[
I(u) = \frac{1}{2} \| u \|^2 - \int fu
\]

that (2.2) has a weak solution \( u \in E \), that is,

\[
\int (\nabla u \nabla \varphi + au \varphi) = \int f(x) \varphi, \quad \varphi \in E.
\]

The solution \( u \) is, in fact, unique. Letting \( S : L^2 \to E \) be the solution operator associated to (2.2) that is \( Sf = u \) for \( f \in L^2 \) it follows that \( S \) is linear and moreover

\[
\| Sf \| \leq C|f|_2, \quad f \in L^2
\]

for some \( C > 0 \). In addition, splitting \( u \) into \( u^+ - u^- \) where \( u^\pm \) are respectively the positive and negative parts of \( u \), taking \( \varphi = -u^- \) above and noticing that \( u^- \in E \) we infer that

\[
Sf \geq 0 \quad \text{whenever } \ f \geq 0.
\]

Now let \( u \in L^2 \) with \( u \geq 0 \). Since

\[
0 \leq \frac{h(x)}{(u+\epsilon)^\gamma} \leq \frac{h(x)}{\epsilon^\gamma}
\]
and $\frac{h(x)}{e^\gamma} \in L^2$ the operator

$$ Tu \equiv S \left[ \frac{h(x)}{(u + e)^\gamma} \right] $$

is continuous in $L^2$, and as a matter of fact, letting $w \equiv T(0)$ we have

$$ w = S \left[ \frac{h(x)}{e^\gamma} \right]. $$

Considering

$$ K \equiv \{ v \in L^2 \mid 0 \leq v \leq w \text{ a.e. in } \mathbb{R}^N \} $$

we shall prove that the following result holds true.

**Lemma 3.** The set $K \subset L^2$ is closed, convex and bounded and moreover $T(K) \subset K$ and $T(K)$ is a compact subset of $L^2$.

Using lemma 3 and the Schauder fixed point theorem there is some $u_\epsilon \in K$ satisfying

$$ u_\epsilon = S \left[ \frac{h(x)}{(u_\epsilon + e)^\gamma} \right] $$

that is

$$ \begin{cases} f(\nabla u_\epsilon \nabla \varphi + a u_\epsilon \varphi) = f \left( \frac{h(x) \varphi}{(u_\epsilon + e)^\gamma} \right), \quad \varphi \in E \\ u_\epsilon \geq 0 \text{ a.e. in } \mathbb{R}^N, \ u_\epsilon \in E. \end{cases} $$

Now since by (2.3)

$$ \frac{h(x)}{(u_\epsilon + e)^\gamma} \in L^\infty_{loc} $$

it follows by the elliptic regularity theory that $u_\epsilon \in W^{1,p}_{loc}, 1 < p < \infty$, and further if $B \subset \mathbb{R}^N$ is a ball, then

$$ -\Delta u_\epsilon + a(x)u_\epsilon = \frac{h(x)}{(u_\epsilon + e)^\gamma} \text{ a.e. in } B. $$

In fact, it follows by the maximum principle that $u_\epsilon > 0$ in $B$ and so

$$ \begin{cases} -\Delta u_\epsilon + a(x)u_\epsilon = \frac{h(x)}{(u_\epsilon + e)^\gamma} \text{ a.e. in } \mathbb{R}^N \\ u_\epsilon > 0 \text{ in } \mathbb{R}^N. \end{cases} $$

On the other hand, if $f \in L^2_{rad}$ we get by minimizing the functional $I$ above over the space

$$ E_{rad} \equiv \left\{ u \in W^{1,2}_{rad} \mid \int a(r)u^2 < \infty \right\} $$

which endowed with the inner product and norm given above is also a Hilbert space, a weak solution $u \in E_{rad}$ of (2.2) that is

$$ \int (\nabla u \nabla \varphi + a u \varphi) = \int f(x) \varphi, \ \varphi \in E_{rad}. $$

The solution is also unique and as before the solution operator associated to (2.2), namely $S : L^2_{rad} \to E_{rad}$ satisfies

$$ \|Sf\| \leq C\|f\|_2 $$
for $f \in L^2_{rad}$ and further
\[ Sf \geq 0 \text{ whenever } f \geq 0. \]

Letting
\[ K \equiv \left\{ v \in L^2_{rad} \mid 0 \leq v \leq w \text{ a.e. in } \mathbb{R}^N \right\} \]
we have a corresponding symmetric variant of lemma 3 and so there is some $u_\epsilon \in E_{rad}$ with
\[ \int (\nabla u_\epsilon \nabla \varphi + a(r)u_\epsilon \varphi) = \int \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \varphi, \quad \varphi \in E_{rad}. \]

**Proof of Lemma 3.**

It is easy to show that $K$ is convex, closed and bounded. So we will only show that $T(K) \subset K$ and $T(K)$ is compact in $L^2$. If $v \in K$ then
\[ T(0) - T(v) = S \left[ h \left( \frac{1}{\epsilon^\gamma} - \frac{1}{(v + \epsilon)^\gamma} \right) \right] \geq 0 \]
that is $T(v) \leq w$ and hence $T(K) \subset K$.

In order to show that $\overline{T(K)} \subset L^2$ is compact let $v_n$ be a sequence in $T(K)$ say $v_n = T(u_n)$ for some $u_n \in K$. By (2.3)
\[ \frac{h(x)}{(u_n + \epsilon)^\gamma} \text{ is bounded in } L^2 \]
so that
\[ T(u_n) = S \left[ \frac{h(x)}{(u_n + \epsilon)^\gamma} \right] \text{ is bounded in } E. \]
Thus, passing to subsequences,
\[ T(u_n) \rightharpoonup v \text{ for some } v \in E \]
and
\[ T(u_n) \rightarrow v \text{ a.e. in } \mathbb{R}^N. \]

On the other hand, since $0 \leq T(u_n) \leq w$ it follows by Lebesgue’s theorem that
\[ T(u_n) \rightarrow v \text{ in } L^2. \]
showing that $\overline{T(K)}$ is compact in $L^2$, ending the proof of lemma 3. The radial case is handled similarly.

The next result states that the family $u_\epsilon$ increases when $\epsilon$ decreases.

**Lemma 4.** If $0 < \epsilon < \epsilon'$ then $u_{\epsilon'} \leq u_\epsilon$ in $\mathbb{R}^N$.

**Proof of Lemma 4.**

Letting $\omega \equiv u_{\epsilon'} - u_\epsilon$ we get
\[ -\Delta \omega + a(x)\omega = h(x) \left[ \frac{1}{(u_{\epsilon'} + \epsilon')^\gamma} - \frac{1}{(u_\epsilon + \epsilon)^\gamma} \right] \text{ a.e. in } \mathbb{R}^N \]
which gives
\[ \int |\nabla \omega^+|^2 + a(x)\omega^+ = \int h(x) \left[ \frac{1}{(u_\epsilon + \epsilon)^\gamma} - \frac{1}{(u_\epsilon + \epsilon)^\gamma} \right] \omega^+ \leq 0 \]
showing that \( \omega^+ = 0 \) and thus \( u_\epsilon \leq u_\epsilon \) a.e. in \( \mathbb{R}^N \), finishing the proof of lemma 4. □

3. Proofs of Main Results

Proof of Theorem 1.

Step 1 (the non-symmetric case).

Let \( \epsilon_n > 0 \) be a decreasing sequence converging to 0 and set \( u_n = u_{\epsilon_n} \). We claim that 
\[ \|u_n\| \text{ is bounded.} \]

Indeed,
\[
(3.1) \int (|\nabla u_n|^2 + a|u_n|^2) = \int \frac{h(x)u_n}{(u_n + \epsilon_n)^\gamma} \leq \int h(x)u_{1-\gamma} \leq C |h|_\|u_n\|^{1-\gamma}
\]
for some \( C > 0 \), showing that \( u_n \) is bounded in \( E \). Hence, passing to subsequences, we have
\[ u_n \to u \text{ in } E, \text{ and } u_n \to u \text{ a.e. in } \mathbb{R}^N. \]

Moreover since by lemma 4 \( 0 < u_1 \leq u_n \) in \( \mathbb{R}^N \) we infer that if \( \varphi \in E \) has compact support then \( \text{supp}(\varphi) \subset B \) for some ball \( B \subset \mathbb{R}^N \) and
\[ \frac{|h(x)\varphi|}{(u_n + \epsilon_n)^\gamma} \leq H(x) \text{ for some } H \in L^1 \]
which gives, by applying Lebesgue’s theorem to
\[ \int (\nabla u_n \nabla \varphi + au_n \varphi) = \int \frac{h(x)\varphi}{(u_n + \epsilon_n)^\gamma} \]
that
\[
\left\{ \begin{array}{l}
\int (\nabla u \nabla \varphi + au \varphi) = \int \frac{h(x)\varphi}{u^\gamma} \\
u \geq u_1 > 0 \text{ in } \mathbb{R}^N.
\end{array} \right.
\]
Using the regularity theory again we arrive at
\[
\left\{ \begin{array}{l}
-\Delta u + a(x)u = h(x)u^{-\gamma} \text{ a.e. in } \mathbb{R}^N \\
u \in W^{2,p}_{\text{loc}}, \ 1 < p < \infty \\
u > 0 \text{ in } \mathbb{R}^N.
\end{array} \right.
\]
In order to prove uniqueness let \( M \in C_0^\infty \) such that
\[
M(x) = 1 \text{ if } |x| \leq 1, \ M(x) = 0 \text{ if } |x| \geq 2 \text{ and } 0 \leq M \leq 1.
\]
Given \( \varphi \in E \), an integer \( j \geq 1 \) and letting
\[ \varphi_j(x) \equiv M(x)\varphi(x), \ x \in \mathbb{R}^N \]
it follows that $\varphi^j \in E$ and $\text{supp}(\varphi^j)$ is compact. Moreover as we will show in the Appendix

\[(3.2) \quad \varphi^j \to \varphi \quad \text{in} \quad E.\]

Now assume $u, v$ are two solutions of (*) and let $w_j \equiv u^j - v^j$. Then

\[
\langle u - v, u^j - v^j \rangle = \int (\nabla (u - v) \nabla w_j + a(x)(u - v)w_j) = \int h(x) \left( \frac{1}{u^j} - \frac{1}{v^j} \right) w_j.
\]

Assuming, by contradiction, that $u \neq v$ and once

\[
\langle u - v, u^j - v^j \rangle \to \|u - v\|^2
\]

we have

\[
\int h(x) \left( \frac{1}{u^j} - \frac{1}{v^j} \right) w_j > 0
\]

for large values of $j$. On the other hand,

\[
\int h(x) \left( \frac{1}{u^j} - \frac{1}{v^j} \right) w_j \leq \int_{\Omega_j} h(x)u^{1-\gamma} + \int_{\Omega_j} h(x)v^{1-\gamma}
\]

where $\Omega_j \equiv B_{2j} \setminus B_j$. Therefore, passing to the limit as $j \to \infty$ and noticing that the two integrals in the right hand side tend to zero we get a contradiction, that is $u = v$.

Assume now, $h \in C^{\alpha}_{loc}$. Then by the elliptic regularity theory more precisely, interior elliptic estimates, we get $u \in C^{2,\alpha}_{loc}$. This proves theorem 1 (in the case of Step 1).

**Step 2** (the symmetric case: $a, h$ are radial).

From section 2 we have found by Schauder’s Theorem some radial function $u_\epsilon \in K$, $u_\epsilon \neq 0$ satisfying $u_\epsilon = Tu_\epsilon$, which means

\[(3.3) \quad \int (\nabla u_\epsilon \nabla v + a(r)u_\epsilon v) = \int \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} v, \quad v \in E_{rad}.
\]

We will show next that $u_\epsilon \in W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ for $1 < p < \infty$, and

\[-\Delta u_\epsilon + a(r)u_\epsilon = \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} \quad \text{a.e. in} \quad \mathbb{R}^N \setminus \{0\}.
\]

Indeed, changing variables we get from (3.3)

\[
\int_S \int_0^{\infty} (u_\epsilon' v' + a(r)u_\epsilon v) r^{N-1} dr dS = \int_S \int_0^{\infty} \frac{h(r)}{(u_\epsilon + \epsilon)^\gamma} vr^{N-1} dr dS
\]

where $S \subset \mathbb{R}^N$ is the unit sphere. Making

\[
v \equiv r^{-(N-1)}\psi, \quad r > 0, \quad \psi \in C^\infty_0(0, \infty)
\]
we have
\[ \int_{0}^{\infty} \left[ \left( r^{(N-1)} u' \right) \left( r^{-\gamma} \psi \right)' + a u \psi \right] \, dr = \int_{0}^{\infty} \frac{h(r)}{(u + \epsilon)^{\gamma}} \psi(r) \, dr, \]
for \( \psi \in C_0^\infty(0, \infty) \), and labelling
\[ \frac{h(r)}{(u + \epsilon)^{\gamma}} - a(r)u \equiv \tilde{H}(r), \quad r > 0 \]
we get
\[ -\frac{1}{r^{N-1}} (r^{(N-1)} u'_\epsilon)' = \tilde{H}(r) \text{ in } (0, \infty) \]
in the distribution sense. But since \( a, h, u, \epsilon \in L^p_{\text{loc}}(0, \infty) \), \( 1 < p < \infty \) it follows that \( \tilde{H} \in L^p_{\text{loc}}(0, \infty) \) and using the regularity theory we infer that \( u \in W^{2,p}_{\text{loc}}(0, \infty) \) and
\[ -\frac{1}{r^{N-1}} (r^{(N-1)} u'_\epsilon)' = \tilde{H}(r) \text{ a.e. in } (0, \infty). \]
By the maximum principle,
\[ u_\epsilon > 0 \text{ in } (0, \infty). \]
Since \( u_\epsilon \in W^{2,p}_{\text{loc}}(\mathbb{R}^N \setminus \{0\}) \) and
\[ -\Delta u_\epsilon = -\frac{1}{r^{N-1}} (r^{(N-1)} u'_\epsilon)' \]
we also have
\[ -\Delta u_\epsilon + a(r)u_\epsilon = \frac{h(r)}{(u + \epsilon)^{\gamma}} \text{ a.e. in } \mathbb{R}^N \setminus \{0\}. \]
Now, let \( \epsilon_0 > 0 \) such that \( \epsilon_0 \to 0 \) and label \( u_{\epsilon_0} \equiv u_0 \). Following the proof of lemma 4 we have \( u_0 \geq u_1 > 0 \). On the other hand we claim that
\[ \|u_n\| \text{ is bounded.} \]
Indeed, as in (3.1) we have
\[ \int \left( |\nabla u_n|^2 + a|u_n|^2 \right) \leq C \|h\| \|u_n\|^{1-\gamma} \]
so that \( u_n \) is bounded in \( E_{\text{rad}} \). Passing to subsequences we have
\[ u_n \rightharpoonup u \text{ in } E_{\text{rad}}, \text{ and } u_n \to u \text{ a.e. in } \mathbb{R}^N. \]
On the other hand, if \( v \in E_{\text{rad}} \) has compact support then, as in section 1, applying Lebesgue’s Theorem to
\[ \int (\nabla u_n \nabla v + a(r)u_nv) = \int \frac{h(r)}{(u_n + \epsilon_n)^{\gamma}} v, \]
gives
\[ \int (\nabla u \nabla v + a(r)uv) = \int \frac{h(r)}{u^{\gamma}} v. \]
Now changing variables, making again $v \equiv r^{-(N-1)} \psi$ where $r > 0$ and $\psi \in C_0^\infty(0, \infty)$ and arguing as above we obtain $u \in W^{2,p}_{loc}(\mathbb{R}^N \setminus \{0\})$ and

$$-\frac{1}{r^{N-1}r^{(N-1)}u'} + a(r)u = \frac{h(r)}{u^\gamma} \text{ a.e. in } (0, \infty)$$

and in addition,

$$-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \text{ a.e. in } \mathbb{R}^N \setminus \{0\}.$$ 

So, if $\varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ then

$$\int (\nabla u \nabla \varphi + a(r)u \varphi) = \int \frac{h(r)}{u^\gamma} \varphi$$

that is

$$-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \text{ in } \mathbb{R}^N \setminus \{0\}$$

in the distribution sense. Next we show that $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ and

$$\int (\nabla u \nabla \varphi + a(r)u \varphi) = \int \frac{h(r)}{u^\gamma} \varphi, \quad \varphi \in C_0^\infty(\mathbb{R}^N).$$

Indeed, let $\eta \in C^\infty(\mathbb{R}^N)$ such that

$$\eta(x) = 0 \text{ for } |x| \leq 1,\text{ and } \eta(x) = 1 \text{ for } |x| \geq 2$$

and let

$$\psi_\epsilon(x) \equiv \eta(\frac{x}{\epsilon}), \quad \epsilon > 0.$$ 

If $\varphi \in C_0^\infty(\mathbb{R}^N)$ then $\psi_\epsilon \varphi \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$ and from above

$$\int (\nabla u \nabla (\psi_\epsilon \varphi) + a(r)u(\psi_\epsilon \varphi)) = \int \frac{h(r)}{u^\gamma} (\psi_\epsilon \varphi)$$

so that

$$\int (\psi_\epsilon \nabla u \nabla \varphi + \varphi \nabla u \nabla \psi_\epsilon + a(r)u \psi_\epsilon \varphi) = \int \frac{h(r)}{u^\gamma} \psi_\epsilon \varphi.$$ 

Making $\epsilon \to 0$ and using Lebesgue’s Theorem we infer that

$$\int \psi_\epsilon \nabla u \nabla \varphi \to \int \nabla u \nabla \varphi,$$

$$\int a(r)u \psi_\epsilon \varphi \to \int a(r)u \varphi$$

and

$$\int \frac{h(r)}{u^\gamma} \psi_\epsilon \varphi \to \int \frac{h(r)}{u^\gamma} \varphi.$$ 

**Claim.**

$$\int \varphi \nabla u \nabla \psi_\epsilon \to 0.$$ 

Assuming the Claim has been proved we have

$$\int (\nabla u \nabla \varphi + a(r)u \varphi) = \int \frac{h(r)}{u^\gamma} \varphi$$

where

$$v \equiv r^{-(N-1)} \psi \text{ where } r > 0 \text{ and } \psi \in C_0^\infty(0, \infty).$$
and since $a, h \in L^\infty_{\text{loc}}$ we get by the regularity theory that $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for $1 < p < \infty$ and

$$-\Delta u + a(r)u = \frac{h(r)}{u^\gamma} \quad \text{a.e. in } \mathbb{R}^N$$

and if in addition $a, h \in C^\alpha_{\text{loc}}$ then $u \in C^{2,\alpha}_{\text{loc}}$ by the interior Schauder estimates.

**Verification of the Claim.**

Using Schwarz inequality we have

$$| \int \varphi \nabla u \nabla \psi | \leq | \varphi |_{\infty} \left( \int_{|x| \leq 2\epsilon} |\nabla u|^2 \right)^{\frac{1}{2}} \left( \int_{|x| \leq 2\epsilon} |\nabla \psi|^2 \right)^{\frac{1}{2}}$$

$$\leq | \varphi |_{\infty} | \nabla \eta |_2 \left( \int_{|x| \leq 2\epsilon} |\nabla u|^2 \right)^{\frac{1}{2}} \epsilon^{\frac{N-2}{2}}$$

where $N \geq 3$. Letting $\epsilon \to 0$ shows the Claim.

As for the uniqueness the argument in the proof of theorem 1 (Step 1) applies ending the proof of theorem 1 (in case of Step 2). The proof of theorem 1 is finished.

**Proof of Theorem 2.**

In order to solve $(\ast)_0$ we consider the family of problems

$$(3.4) \quad \begin{cases}
-\Delta u + \frac{1}{k} u = h(|x|) u^{-\gamma} \quad \text{in } \mathbb{R}^N \\
u > 0 \quad \text{in } \mathbb{R}^N.
\end{cases}$$

where $k \geq 1$ is an integer. Making $a(x) \equiv \frac{1}{k}$ in theorem 1 (radial case), it follows that $(3.4)$ has a solution $u_k \in H^1_{\text{rad}} \cap W^{2,p}_{\text{loc}}, \ 1 < p < \infty$ satisfying

$$\int |\nabla u_k|^2 + \frac{1}{k} u_k^2 = \int h(r) u_k^{-\gamma}.$$

Using both Hölder’s inequality and the continuous embedding $D^{1,2} \to L^{2^*}$ in the equality above we infer that

$$(3.5) \quad \int |\nabla u_k|^2 \leq C_1 \text{ for some } C_1 > 0.$$ 

By a well known property of radial functions $u \in D^{1,2}$, namely

$$|u(x)| \leq \frac{C_2}{|x|^{\frac{N-2}{2}}} \|u\|_{D^{1,2}}, \ x \neq 0 \text{ for some } C_2 > 0$$

we get

$$(3.6) \quad 0 \leq u_k(x) \leq \frac{C}{|x|^{\frac{N-2}{2}}}, \ x \neq 0 \text{ for some } C > 0.$$ 

We shall need the following result which says that the sequence $u_k$ increases with $k$.

**Lemma 5.** If $k < k'$ then $u_k \leq u_{k'}$, a.e. in $\mathbb{R}^N$. 

By the boundedness of \( u_k \) in \( D^{1,2} \) and lemma 5 there is some radial function \( u \in D^{1,2} \) such that

\[
u_k \rightarrow u \text{ in } D^{1,2}, \quad u_k \rightarrow u \text{ a.e. in } \mathbb{R}^N\]

and

\[
u_1 \leq \nu_2 \leq \ldots \leq \nu_k \leq \ldots \leq u \text{ a.e. in } \mathbb{R}^N.
\]

Now if \( \varphi \in C_0^\infty(\mathbb{R}^N) \) then

\[
(3.7) \quad \int \left( \nabla u_k \nabla \varphi + \frac{1}{k} u_k \varphi \right) = \int h u_k^{-\gamma} \varphi.
\]

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain such that \( \text{supp}(\varphi) \subset \Omega \). Then

\[
|h u_k^{-\gamma} \varphi| \leq h u_1^{-\gamma} |\varphi| \in L^p(\Omega), \quad 1 \leq p < \infty
\]

and

\[
\int h u_k^{-\gamma} \varphi \rightarrow \int h u^{-\gamma} \varphi.
\]

On the other hand, using (3.6) we get

\[
\frac{1}{k} \int u_k \varphi \rightarrow 0.
\]

Passing to the limit in (3.7) gives

\[
\int \nabla u \nabla \varphi = \int h u^{-\gamma} \varphi.
\]

Since \( 0 < u_1 \leq u \) and \( u_1 \in W^{2,p}_{\text{loc}}(\mathbb{R}^N) \) it follows that \( h u^{-\gamma} \in L^p_{\text{loc}}(\mathbb{R}^N) \) and by the regularity theory \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N) \). In addition \( u \in C^{2,\alpha}_{\text{loc}} \) when \( h \in C^\alpha_{\text{loc}} \).

This proves Theorem 2. \( \blacksquare \)

**Proof of Lemma 5.**

Letting \( \omega = u_k - u_{k'} \) we have

\[
\int |\nabla \omega^+|^2 + \frac{1}{k'}(\omega^+)^2 \leq \int \nabla \omega \nabla \omega^+ + \frac{1}{k} \omega \omega^+
\]

\[
\leq \int h \left( \frac{1}{u_k} - \frac{1}{u_{k'}} \right) \omega^+
\]

showing that \( \omega^+ = 0 \) and so \( \omega \leq 0 \), ending the proof of lemma 5. \( \blacksquare \)

**Verification of (3.2).**

Indeed,

\[
a |\varphi^j - \varphi|^2 \leq 4a \varphi^2 \in L^1 \quad \text{and} \quad a |\varphi^j - \varphi|^2 \rightarrow 0 \text{ a.e. in } \mathbb{R}^N
\]

so that by Lebesgue’s theorem

\[
\int a |\varphi^j - \varphi|^2 \rightarrow 0.
\]
Now
\[ \frac{\partial \varphi^j}{\partial x_i} = \frac{1}{j} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi + M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i}. \]

Hence
\[ \int \left| \frac{\partial \varphi^j}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \right|^2 = \int \left| \frac{1}{j} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi + M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \right|^2 \]
\[ \leq C \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 + \int \left| M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \right|^2. \]

Arguing as above we infer that
\[ M \left( \frac{x}{j} \right) \frac{\partial \varphi}{\partial x_i} \rightarrow \frac{\partial \varphi}{\partial x_i} \text{ in } L^2. \]

It remains to show that
\[ \int \left| \frac{1}{j} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \rightarrow 0. \]

At first we remark that
\[ \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 = \int_{B_{1j} \setminus B_j} \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \]
\[ \leq C \int_{B_{1j} \setminus B_j} \varphi^2. \]

Now using Hölder inequality with exponents \( \frac{N}{N-2} \) and \( \frac{N}{2} \) in the last integral we obtain
\[ \int \left| \frac{1}{j^2} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \leq C \left( \int_{B_{2j} \setminus B_j} 1 \right)^{\frac{N}{N-2}} \left( \int_{B_{2j} \setminus B_j} |\varphi|^{2^*} \right)^{\frac{1}{2^*}} \]
\[ \leq C \left( \int_{B_{2j}} 1 \right)^{\frac{N}{N-2}} \left( \int_{B_{2j}} |\varphi|^{2^*} \right)^{\frac{1}{2^*}} \]
\[ \leq C \omega_N \frac{N}{j^2} \left( \int_{B_{2j}} |\varphi|^{2^*} \right)^{\frac{1}{2^*}} \]

where \( \omega_N \) denotes the volume of the unit sphere of \( \mathbb{R}^N \).

Next passing to the limit we get
\[ \int \left| \frac{1}{j} \frac{\partial}{\partial x_i} M \left( \frac{x}{j} \right) \varphi \right|^2 \rightarrow 0. \]

This shows that \( \varphi^j \rightarrow \varphi \) in \( E \) proving (3.2).

\textbf{References}


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