ON MULTIPLE POSITIVE SOLUTIONS OF POSITONE AND NON-POSITONE PROBLEMS

F. J. S. A. CORRÊA

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1. Introduction

In this paper, we consider the following problem:

$$-\triangle u = f(u) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is the ball $B_R = \{x \in \mathbb{R}^N; |x| < R\}$, $|\cdot|$ is the Euclidean norm in $\mathbb{R}^N$, and $f : \mathbb{R}^+ \to \mathbb{R}$ is a locally Lipschitzian continuous function. We are concerned with two classes of problems, namely,

(i) the positone problem: $f(0) \geq 0$;
(ii) the non-positone problem: $f(0) < 0$.

The study of positone problems was initiated by Keller and Cohen [14], see also [15], motivated by problems arose from the theory of nonlinear heat generation. In the past twenty five years there has been considerable interest in this class of semilinear elliptic boundary value problems and there is a wide literature on this subject. The reader may consult the survey by Lions [17], and the references therein, where many interesting questions are studied under a different point of view.

In the positone case we consider the following assumptions:

(f1) there exist $0 < a_1 < a_2 < a_3$ so that $f(a_1) = f(a_2) = 0$ and $F(a_3) > F(a_1)$, where $F(t) = \int_0^t f$;
(f2) $|f(t)| \leq \alpha$, for all $t \in \mathbb{R}^+$, for some constant $\alpha > 0$;
(P) $f(0) > 0$ or, if $f(0) = 0$, then $f'_+(0) > 0$, where $f'_+(0) = \lim_{t\to 0^+}(f(t)/t)$ and prove the result below.

THEOREM 1.1. *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies (f1), (f2), and (P). If R is sufficiently large, problem (1.1) possesses at least three radial positive solutions u_1, u_2, and u_3 such that $\partial u_i/\partial r < 0$, for $i = 1, 2, 3$ and $0 < r < R$. In particular*

$$0 < |u_1|_\infty < a_1 < a_2 < |u_2|_\infty, \tag{1.2}$$

where $|\cdot|_\infty$ is the sup norm.

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Remark 1.2. Some results like the one above have been obtained by several authors when the area condition $F(a_3) > F(a_1)$ has been considered. This condition was first used by Brown and Budin [3] who proved a result similar to Theorem 1.1 by using a combination of variational and monotone iteration methods. Later Hess [13] studied this problem by using variational methods and degree theory. In [9], the above condition is also used and, by using solely variational methods, the existence of three ordered positive solutions is shown. These authors studied the case in which f has a third root, that is, $f(a_3) = 0$. In Theorem 1.1 this assumption is not made and we use only variational methods. Also, Theorem 1.1 is used as an essential tool in the study of non-positone case.

The non-positone case has been studied in recent years mainly by Brown, Castro, Shivaji, Arcoya, and Calahorrano, among others. See, for example, [2, 4, 5]. In this case the following assumption is posed:

$$f(0) < 0. \tag{1.3}$$

Motivated by the study of discontinuous nonlinear problems we consider, as in [2], the following multivalued problem:

$$-\triangle u(x) \in \hat{f}\big(u(x)\big) \quad \text{a.e in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $\hat{f}$ is the multivalued function defined by

$$\hat{f}(t) = \begin{cases} 0, & \text{if } t < 0 \\ [f(0), 0], & \text{if } t = 0 \\ f(t), & \text{if } t > 0. \end{cases} \tag{1.5}$$

By a solution of (1.4), we mean a function $u \in C^1(\overline{\Omega}) \cap C^2(\Omega^*(u))$ with $\Omega^*(u) = \{x \in \Omega; u(x) \neq 0\}$ and verifying (1.4).

Remark 1.3. Some authors (cf. Chang [6]) have treated discontinuous problems by using a direct variational approach. In the present case, we consider the multivalued problem (1.4) as the limit of smooth approximating problems in order to use Theorem 1.1. Thus two nonnegative solutions u_0 and v_0, in the sense of (1.4), are obtained as limits of smooth approximating solutions. Then we use the symmetry results in [11] and the maximum principle to show that u_0 and v_0 are positive classical solutions of (1.1). For this a crucial step is to show that the set $\Omega_0(u) = \Omega / \Omega^*(u)$ has null Lebesgue measure for $u = u_0, v_0$.

The following assumptions on f are considered:

(f3) there exists $\theta > 0$ such that $f(t) < 0$ if $0 \leq t < \theta$ and $f(\theta) = 0$,

(f4) there is $a > \theta$ satisfying $F(a) > 0$.

We are now ready to state the following theorem.

THEOREM 1.4. *Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ is a bounded C^1-function satisfying (f3), (f4), and (1.3). Then problem (1.4) has at least two radial positive solutions u_0 and v_0 if R*

is sufficiently large. In fact $\Omega_0(u_0) = \Omega_0(v_0) = \phi$, *which implies* u_0 *and* v_0 *are radial classical positive solutions of (1.1) such that* $\partial u/\partial r < 0$, *for* $0 < r < R$ *and* $u = v_0, u_0$.

Remark 1.5. It is well known in the positone case, see, for example, Cohen and Laetsch [7], that if f is concave, problem (1.1) has at most one positive solution, but may have multiple solutions when f is convex. In contrast with this case Theorem 1.4 shows that in the non-positone problem we may obtain multiplicity of positive solutions with f concave. The function $f(t) = \alpha - e^{-t}$, $0 < \alpha < 1$ and $t \geq 0$, is a simple example of a bounded concave function satisfying (f3), (f4), and (1.3).

2. Proof of Theorem 1.1

First solution. Let us consider the following function:

$$f_1(t) = \begin{cases} f(0), & \text{if } t \leq 0; \\ f(t), & \text{if } 0 \leq t \leq a_1; \\ 0, & \text{if } a_1 \leq t; \end{cases} \tag{2.1}$$

and the functional $I_1 : E \to \mathbb{R}, E := H_0^1(\Omega)$ with the usual norm $\|u\|^2 = \int |\nabla u|^2$, where the integrals are taken over all Ω, unless we state the contrary, defined by

$$I_1(u) = \frac{1}{2}\int |\nabla u|^2 - \int F_1(u), \quad F_1(u) = \int_0^u f_1. \tag{2.2}$$

Note that I_1 is the Euler-Lagrange functional associated to the problem

$$-\triangle u = f_1(u) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Since I_1 is coercive and weakly lower semi-continuous (w.l.s.c.), see [10], it achieves its minimum at some point $u_1 \in E$, which is a weak solution of (2.3) and a bootstrap argument shows that u_1 is a classical solution of (2.3). Assumption (P) implies that $\liminf_{t\to 0^+}(f(t)/t) > \lambda_1 := \lambda_1(R)$, if R is sufficiently large, where λ_1 is the first eigenvalue of $(-\triangle, H_0^1(\Omega))$. In particular, this yields $I_1(u_1) < 0$, that is, $u_1 \not\equiv 0$ in Ω. So the maximum principle provides $0 < u_1(x) < a_1$ in Ω and then u_1 satisfies (1.1). Moreover, it is easy to show that

$$I_1\big(u_1\big) > -K_N F\big(a_1\big) R^N, \tag{2.4}$$

where K_N is a positive constant depending only on N. Actually (2.4) is valid for all $u \in E$ satisfying $0 < u(x) < a_1$ a.e. in Ω.

Second solution. We now consider the function $I : E \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int |\nabla u|^2 - \int F(u), \quad F(u) = \int_0^u f, \tag{2.5}$$

where we still denote by f the extension of the former function f and defined by

$$f(t) = \begin{cases} f(0), & \text{if } t \leq 0; \\ f(t), & \text{if } 0 < t. \end{cases} \tag{2.6}$$

A standard calculation shows that I is coercive and w.l.s.c. Hence I attains its minimum at $u_2 \in E$. We may not guarantee, up to now, that $u_1 \neq u_2$. For this we consider the following function used by Klaasen and Mitidieri [16]:

$$u_R(x) = \begin{cases} a_{3,} & \text{if } |x| \leq R-1; \\ (R-|x|)a_3, & \text{if } R-1 < |x| < R. \end{cases} \tag{2.7}$$

Setting $C_R = \{x \in \mathbb{R}^N; R-1 < |x| < R\}$ we obtain

$$\begin{aligned} I(u_R) &\leq \frac{1}{2}\int_{C_R} |\nabla u_R|^2 - \int_{B_{R-1}} F(a_3) - \int_{C_R} F(u_R) \\ &\leq \frac{a_3^2}{2}|C_R| - F(a_3)|B_{R-1}| + C_1|C_R| \\ &\leq \frac{a_3^2}{2} h_N R^{N-1} - F(a_3) K_N (R-1)^N + C_1 h_N R^{N-1}. \end{aligned} \tag{2.8}$$

Thus

$$I(u_R) \leq C_N R^{N-1} - K_N F(a_3) R^N, \tag{2.9}$$

where the constants C_1, K_N, h_N, and C_N do not depend on R. Since $F(a_3) - F(a_1) > 0$ one has, for R sufficiently large,

$$K_N\big(F(a_3) - F(a_1)\big)R^N > C_N R^{N-1} \tag{2.10}$$

which implies

$$I(u_R) < -K_N F(a_1) R^N < I_1(u_1) = I(u_1). \tag{2.11}$$

Then the minimum u_2 of I satisfies

$$I(u_2) \leq I(u_R) < I(u_1). \tag{2.12}$$

This shows that $u_1 \neq u_2$. A bootstrap argument guarantees that $u_2 \in C^{2,\alpha}(\overline{\Omega})$ and the maximum principle implies that $0 < u_2(x)$, for all $x \in \Omega$, and $a_2 < |u_2|_\infty$, because $I(u_2) < -K_N F(a_1) R^N < I(u)$, for all u so that $0 < u(x) < a_1$ in Ω.

Third solution. Using an argument as in de Figueiredo [10] we may prove that u_1 is a local minimum of I. A straightforward computation shows that I satisfies the Palais-Smale condition, see [1]. So by the Ambrosetti and Rabinowitz's Mountain Pass Theorem, see [1], we find a third solution (positive) u_3 of (1.1) satisfying

$$I(u_3) \geq -K_N F(a_1) R^N. \tag{2.13}$$

Finally we observe, in view of symmetry results in Gidas, Ni, and Nirenberg [11], that $u_i,\ i = 1, 2, 3$, are radial and $\partial u_1/\partial r < 0,\ i = 1, 2, 3$, for all $0 < r < R$. This proves Theorem 1.1. □

Remark 2.1. If f is decreasing on $[0, a_1]$, there exists only the solution u_1 such that $0 < u_1(x) < a_1$. Hence, in this case, u_3 also satisfies $a_2 < |u_3|_\infty$. In fact the Cosner and Schmitt's result, see [8], implies that $|u_2|_\infty, |u_3|_\infty > a_3$.

3. Proof of Theorem 1.4

To prove Theorem 1.4 we use mainly Theorem 1.1.

For this, we first consider smooth approximations of $\hat{f}$ given by

$$f_n(t) = \begin{cases} f(t), & \text{if } \dfrac{1}{n} \le t; \\ f_n(t) \ge f(t), & \text{if } 0 \le t \le \dfrac{1}{n}; \\ f_n(t) = \dfrac{1}{n}, & \text{if } t \le 0; \end{cases} \tag{3.1}$$

f_n is C^1, there is $a_{1n} \in (0, 1/n)$ such that $F_n(a) > F_n(a_{1n})$ and $f_n(t) \to f(t)$, for all $t > 0$. As usual $F_n(t) = \int_0^t f_n$. Let us now consider the problem

$$-\triangle u = f_n(u) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{3.2}$$

In order to apply Theorem 1.1 to problem (3.2) we consider the functional I_n on E defined by

$$I_n(u) = \frac{1}{2}\int |\nabla u|^2 - \int F_n(u). \tag{3.3}$$

Since $f_n : \mathbb{R} \to \mathbb{R}$ satisfies the assumptions of Theorem 1.1, problem (3.2) possesses at least three positive solutions u_{1n}, u_{2n}, and u_{3n}. In particular

$$I_n\big(u_{2n}\big) \le C_N R^{N-1} - K_N F_n(a) R^N < -K_N F_n\big(a_{1n}\big) R^N \le I_n\big(u_{3n}\big). \tag{3.4}$$

Elliptic regularity yields $|u_{2n}|_{W^{2,p}}, |u_{3n}|_{W^{2,p}} \le C_p$, for all $p \ge 1$. For $p > N$ one has $u_{2n} \to u_0$ and $u_{3n} \to v_0$ in $\mathrm{C}^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$, eventually for subsequences. Since $F_n(a) \to F(a)$ and $F_n(a_{1n}) \to 0$ as $n \to \infty$, one has

$$\begin{aligned} &\int \big|\nabla u_{2n}\big|^2 \longrightarrow \int \big|\nabla u_0\big|^2, \qquad \int \big|\nabla u_{3n}\big|^2 \longrightarrow \int \big|\nabla v_0\big|^2, \\ &\int F\big(u_{2n}\big) \longrightarrow \int F\big(u_0\big), \qquad \int F\big(u_{3n}\big) \longrightarrow \int F\big(v_0\big). \end{aligned} \tag{3.5}$$

Taking limits in the inequalities in (3.4) we obtain

$$I\big(u_0\big) \le C_N R^{N-1} - K_N F(a) R^N \le I\big(v_0\big). \tag{3.6}$$

Noticing that $-K_N F_n(a_{1n})R^N \le I_n(u_{3n})$ and $F_n(a_{1n}) \to 0$ one has $0 \le I(v_0)$. Because $F(a) > 0$ we obtain $C_N R^{N-1} - K_N F(a) R^N < 0$ if R is sufficiently large.

Thus

$$I\big(u_0\big) \le C_N R^{N-1} - K_N F(a) R^N < 0 \le I\big(v_0\big) \tag{3.7}$$

and so $u_0 \ne v_0$.

We show that u_0 and v_0 are nontrivial nonnegative solutions of (1.4). First we observe that f_n may be chosen decreasing in $(0, 1/n)$, for each $n = 1, 2, \ldots$. Thus problem (3.2) possesses a unique positive solution u_{1n} satisfying $0 < u_{1n}(x) < 1/n$, for each $n = 1, 2, \ldots$. Hence $|u_{2n}|_\infty, |u_{3n}|_\infty > \theta$, for all $n = 1, 2, \ldots$. In fact, the result in Cosner-Schmitt [8] implies that $|u_{2n}|_\infty, |u_{3n}|_\infty \ge a$ for all $n = 1, 2, \ldots$. Consequently, $|u_0|_\infty, |v_0|_\infty \ge a$ and u_0 and v_0 are not identically zero.

If $t_n > 0$ and $t_n \to t > 0$, then $f_n(t_n) \to f(t)$. From now on, we set $u = u_0$ or $u = v_0$. Thus if $x \in \Omega^*(u)$ one has $\lim_{n\to\infty} f_n(u_n(x)) = f(u(x))$, where $u_n = u_{2n}$ or $u_n = u_{3n}$. Taking $\varphi \in C_0^\infty(\Omega^*(u))$ and using the fact that

$$-\triangle u_n(x) = f_n\big(u_n(x)\big) \quad \text{in } \Omega,\ u_n = 0 \text{ on } \partial\Omega, \tag{3.8}$$

we obtain

$$\int_{\Omega^*} \nabla u_n \cdot \nabla\varphi = \int_{\Omega^*(u)} f_n\big(u_n\big)\varphi \Longrightarrow \int_{\Omega^*(u)} \nabla u \cdot \nabla\varphi = \int_{\Omega^*(u)} f(u)\varphi. \tag{3.9}$$

Consequently, $u \in C^2(\Omega^*(u))$ and $-\triangle u = f(u)$ in $\Omega^*(u)$. In $\Omega_0(u)$ we have, as a consequence of a well-known result by Stampacchia (cf. [15, Lemma A.4]), $-\triangle u(x) = 0 \in [f(0), 0]$ a.e., and so u_0 and v_0 are both nontrivial nonnegative solutions of (1.4). Let us now prove that if $u(x_0) = 0$ for some $|x_0| = r < R$, it follows that $u(x) = 0$ for all $r < |x| < R$. Actually, if $u(x) > 0$ for some $r < |x| < R$, then $(\partial u/\partial r)(y) > 0$, for some $r < |y| < R$, which is a contradiction in view of $(\partial u/\partial r)(x) \le 0$ if $0 < |x| < R$. Remember that $\partial u_n/\partial r < 0$ for $0 < r < R$ which implies that $\partial u/\partial r \le 0$ because $u_n \to u$ in $C^{1,\alpha}(\overline{\Omega})$. So $\Omega_0(u)$ is a set like $A_\rho = B_R - B_{R-\rho}$, for some $\rho \ge 0$. Suppose that $\rho > 0$. In view of $f(u) \in C^1(\bar{B}_{R-\rho})$ one has

$$-\triangle\left(\frac{\partial u}{\partial r}\right) = f'(u)\frac{\partial u}{\partial r} \quad \text{in } B_{R-\rho}. \tag{3.10}$$

Since f is C^1 there is $\mu > 0$ satisfying $f'(t) + \mu \ge 0$, for all $t \in [0, M]$, with $M = \max\{|u_0|_\infty, |v_0|_\infty\}$. Consequently,

$$-\triangle\left(-\frac{\partial u}{\partial r}\right) + \mu\left(-\frac{\partial u}{\partial r}\right) = \big(f'(u) + \mu\big)\left(-\frac{\partial u}{\partial r}\right) \quad \text{in } B_{R-\rho}. \tag{3.11}$$

Therefore, in view of $\partial u/\partial r = 0$ in A_ρ we have by applying the generalized Green's formula on $B_{R-\rho}$ that

$$-\triangle\left(-\frac{\partial u}{\partial r}\right) + \mu\left(-\frac{\partial u}{\partial r}\right) \ge 0 \quad \text{in } \Omega \tag{3.12}$$

in the weak sense and as $u > 0$ in $B_{R-\rho}$ we have $\partial u/\partial r \not\equiv 0$ and so, by using the generalized maximum principle, (cf. [12, Theorem 8.19]), $\partial u/\partial r < 0$ in Ω, which is a

contradiction with assumption $\rho > 0$. Hence, $\rho = 0$. This implies that $\Omega_0(u) = \phi$, then $u > 0$ in Ω and $\partial u/\partial r < 0$ if $0 < r < R$. It follows that u_0 and v_0 are classical solutions of (1.1). This concludes the proof of Theorem 1.4. □

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F. J. S. A. Corrêa: Departamento de Matemática-C.C.E.N, Universidade Federal do Pará, 66.075-900-Belém-PA, Brazil

E-mail address: fjulio@ufpa.br