A BOUNDARY VALUE PROBLEM IN THE HYPERBOLIC SPACE

P. AMSTER, G. KEILHAUER, AND M. C. MARIANI

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We consider a nonlinear problem for the mean curvature equation in the hyperbolic space with a Dirichlet boundary data g . We find solutions in a Sobolev space under appropriate conditions on g .

1. Introduction

Let M be the open unit ball in $\mathbb{R}^3$ of center 0 and let

$$g_{ij}(x) = \frac{4\delta_{ij}}{(1 - |x|^2)^2} \quad (1.1)$$

be the hyperbolic metric on M . Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega \in C^{1,1}$, and let (u, v) be the variables in $\mathbb{R}^2$. We consider in this paper the Dirichlet problem for a function $X : \overline{\Omega} \rightarrow M$ which satisfies the equation of prescribed mean curvature

$$\begin{aligned} \nabla_{X_u} X_u + \nabla_{X_v} X_v &= -2H(X)X_u \wedge X_v \quad \text{in } \Omega, \\ X &= g \quad \text{on } \partial\Omega, \end{aligned} \quad (1.2)$$

where $H : M \rightarrow \mathbb{R}$ is a given continuous function, and $g \in W^{2,p}(\Omega, \mathbb{R}^3)$ for $1 < p < \infty$, with $\|g\|_\infty < 1$.

In the above equation X_u , X_v , and $X_u \wedge X_v : \Omega \rightarrow TM$ are the vector fields given by

$$\begin{aligned} X_u(u, v) &= \sum_{k=1}^3 \frac{\partial X_k}{\partial u} \Big|_{(u,v)} \frac{\partial}{\partial x_k} \Big|_{X(u,v)}, & X_v(u, v) &= \sum_{k=1}^3 \frac{\partial X_k}{\partial v} \Big|_{(u,v)} \frac{\partial}{\partial x_k} \Big|_{X(u,v)}, \\ X_u \wedge X_v(u, v) &= \sum_{k=1}^3 (X_u \wedge X_v)^k(u, v) \frac{\partial}{\partial x_k} \Big|_{X(u,v)}, \end{aligned} \quad (1.3)$$

where

$$\begin{aligned}(X_u \wedge X_v)^1(u, v) &= \varphi^{1/2}(X(u, v)) \left(\frac{\partial X_2}{\partial u} \Big|_{(u,v)} \frac{\partial X_3}{\partial v} \Big|_{(u,v)} - \frac{\partial X_3}{\partial u} \Big|_{(u,v)} \frac{\partial X_2}{\partial v} \Big|_{(u,v)} \right), \\(X_u \wedge X_v)^2(u, v) &= \varphi^{1/2}(X(u, v)) \left(\frac{\partial X_3}{\partial u} \Big|_{(u,v)} \frac{\partial X_1}{\partial v} \Big|_{(u,v)} - \frac{\partial X_1}{\partial u} \Big|_{(u,v)} \frac{\partial X_3}{\partial v} \Big|_{(u,v)} \right), \\(X_u \wedge X_v)^3(u, v) &= \varphi^{1/2}(X(u, v)) \left(\frac{\partial X_1}{\partial u} \Big|_{(u,v)} \frac{\partial X_2}{\partial v} \Big|_{(u,v)} - \frac{\partial X_2}{\partial u} \Big|_{(u,v)} \frac{\partial X_1}{\partial v} \Big|_{(u,v)} \right),\end{aligned}\tag{1.4}$$

for $\varphi(x) = 4/(1 - |x|^2)^2$.

We remark that if X_u and X_v are linearly independent, then $X(\Omega) \subset M$ is an imbedded submanifold and $X_u \wedge X_v(u, v)$ is the only vector orthogonal to $X(\Omega)$ at $X(u, v)$ that satisfies, for any $z = \sum_{k=1}^3 z^k (\partial/\partial x_k)|_{X(u,v)}$

$$\langle z, X_u \wedge X_v(u, v) \rangle = \omega(X(u, v))(z, X_u(u, v), X_v(u, v)),\tag{1.5}$$

where ω is the volume element of $(M, \langle \cdot, \cdot \rangle)$, namely

$$\omega = \sqrt{\det(g_{ij})} dx_1 \wedge dx_2 \wedge dx_3 = \varphi^{3/2} dx_1 \wedge dx_2 \wedge dx_3.\tag{1.6}$$

If ∇ is the Levi-Civita connection associated to $\langle \cdot, \cdot \rangle$ and $\Gamma_{ij}^k : M \rightarrow \mathbb{R}$ are the Christoffel symbols

$$\Gamma_{ij}^k = \sum_{r=1}^3 \frac{g^{rk}}{2} \left(\frac{\partial g_{rj}}{\partial x_i} + \frac{\partial g_{ri}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_r} \right)\tag{1.7}$$

with $(g^{ij}) = (g_{ij})^{-1}$, then a simple computation shows that

$$\Gamma_{ij}^i(x) = \Gamma_{ji}^i(x) = \frac{2x_j}{1 - |x|^2}, \quad \Gamma_{ii}^k(x) = \begin{cases} -\frac{2x_k}{1 - |x|^2} & \text{if } k \neq i, \\ 0 & \text{otherwise.} \end{cases}\tag{1.8}$$

Let $E, F, G : \Omega \rightarrow \mathbb{R}$ be the coefficients of the first fundamental form, and the unit normal $N : \Omega \rightarrow TM$ be given by

$$N = \frac{1}{\sqrt{EG - F^2}} X_u \wedge X_v\tag{1.9}$$

which is orthogonal to the tangent space $\{X(\Omega)\}_x$ for any $x = X(u, v)$. Then, if $H : \Omega \rightarrow \mathbb{R}$ is the mean curvature of $X(\Omega)$ we obtain

$$\left\langle N, \frac{G}{EG - F^2} \nabla_{X_u} X_u + \frac{E}{EG - F^2} \nabla_{X_v} X_v - 2 \frac{F}{EG - F^2} \nabla_{X_u} X_v \right\rangle = -2H.\tag{1.10}$$

In particular, if X is isothermal, that is, $E = G$, $F = 0$, then $\langle \nabla_{X_u} X_u + \nabla_{X_v} X_v, X_u \rangle = 0 = \langle \nabla_{X_u} X_u + \nabla_{X_v} X_v, X_v \rangle$ and consequently

$$\nabla_{X_u} X_u + \nabla_{X_v} X_v = -2H X_u \wedge X_v.\tag{1.11}$$

Thus, (1.11) is the equation of prescribed mean curvature for an imbedded submanifold of M .

2. A Dirichlet problem for (1.11)

With the notations of the previous section, we consider the Dirichlet problem (1.2). The equation of prescribed mean curvature for a surface in $\mathbb{R}^3$ has been studied for constant H in [3, 5], and for H nonconstant in [1, 2].

Without loss of generality, we may assume that g is harmonic in Ω . Our existence result reads as follows.

THEOREM 2.1. *Let c_0 and c_1 be some positive constants to be specified. Then (1.2) is solvable for any $g \in W^{2,p}(\Omega, \mathbb{R}^3)$ harmonic such that*

$$\|g\|_\infty + 2\left(c_1 + \sqrt{c_1(c_1 + c_0)}\right) \|\text{grad}(g)\|_{2p} \leq 1. \quad (2.1)$$

In the proof of Theorem 2.1, we ignore the canonical isomorphism $\partial/\partial x_k|_{X(u,v)} \rightarrow e_k$ (with $\{e_k\}$ the usual basis of $\mathbb{R}^3$), and considering $X_u, X_v \in \mathbb{R}^3$ we may write (1.2) as a system

$$\begin{aligned} -\Delta X_k &= \psi_k(X, X_u, X_v) \quad \text{in } \Omega, \\ X_k &= g_k \quad \text{on } \partial\Omega \end{aligned} \quad (2.2)$$

with $\psi_k(X, X_u, X_v) = 2H(X)(X_u \wedge X_v)^k + \sum_{i,j} \Gamma_{ij}^k(X) \text{grad}(X_i) \text{grad}(X_j)$, $1 \leq k \leq 3$. For fixed $\bar{X} \in W_0^{1,2p}(\Omega, \mathbb{R}^3)$ such that $\|g + \bar{X}\|_\infty < 1$, we define $X = T\bar{X}$ as the unique solution in $W^{2,p}(\Omega, \mathbb{R}^3) \hookrightarrow W^{1,2p}(\Omega, \mathbb{R}^3)$ of the linear problem

$$\begin{aligned} -\Delta X_k &= \psi_k(\bar{X} + g, (\bar{X} + g)_u, (\bar{X} + g)_v) \quad \text{in } \Omega, \\ X_k &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (2.3)$$

Then, for $B = \{X \in W_0^{1,2p}(\Omega, \mathbb{R}^3) \mid \|g + X\|_\infty < 1\}$ the operator $T : B \rightarrow W_0^{1,2p}(\Omega, \mathbb{R}^3)$ is well defined and a strong solution of (1.2) in $W^{2,p}$ can be regarded as $Y = g + X$, where X is a fixed point of T . By the usual a priori bounds for the Laplacian and the compactness of the imbedding $W^{2,p}(\Omega, \mathbb{R}^3) \hookrightarrow W_0^{1,2p}(\Omega, \mathbb{R}^3)$ we get the following lemma.

LEMMA 2.2. *$T : B \rightarrow W_0^{1,2p}(\Omega, \mathbb{R}^3)$ is continuous. Furthermore, if*

$$C_{R_1, R_2} = \{X \in W_0^{1,2p}(\Omega, \mathbb{R}^3) \mid \|g + X\|_\infty \leq R_1, \|\text{grad}(X)\|_{2p} \leq R_2\} \quad (2.4)$$

with $R_1 < 1$, then $T(C_{R_1, R_2})$ is precompact.

Proof. For $X = T(\bar{X})$, $Y = T(\bar{Y})$, as $X = Y$ on $\partial\Omega$ we obtain that

$$\begin{aligned} \|\text{grad}(X_k - Y_k)\|_{2p} &\leq c \|\Delta(X_k - Y_k)\|_p \\ &= c \left\| \psi_k(\bar{X} + g, (\bar{X} + g)_u, (\bar{X} + g)_v) - \psi_k(\bar{Y} + g, (\bar{Y} + g)_u, (\bar{Y} + g)_v) \right\|_p \end{aligned} \quad (2.5)$$

and the continuity of T follows. On the other hand, if $\bar{X} \in C_{R_1, R_2}$, then

$$\begin{aligned} \|\operatorname{grad}(X_k)\|_{2p} &\leq c \|\Delta X_k\|_p = c \left\| \psi_k \left(\bar{X} + g, (\bar{X} + g)_u, (\bar{X} + g)_v \right) \right\|_p \\ &\leq \bar{c} (R_2 + \|\operatorname{grad}(g)\|_{2p})^2 \end{aligned} \quad (2.6)$$

for some constant $\bar{c}$ and the result follows. $\square$

Remark 2.3. By definition of ψ_k , it is clear that $\bar{c} \leq c_1/(1 - R_1)$ for some constant c_1 .

Proof of Theorem 2.1. With the notation of the previous lemma, by Schauder fixed point theorem, it suffices to see that C_{R_1, R_2} is T -invariant for some R_1, R_2 . From the previous computations, we have

$$\|\operatorname{grad}(X)\|_{2p} \leq \frac{c_1}{1 - R_1} (R_2 + \|\operatorname{grad}(g)\|_{2p})^2. \quad (2.7)$$

Moreover, by Poincaré's inequality

$$\|g + X\|_\infty \leq \|g\|_\infty + c_0 \|\operatorname{grad}(X)\|_{2p}. \quad (2.8)$$

Thus, a sufficient condition for obtaining $T(C_{R_1, R_2}) \subset C_{R_1, R_2}$ is that

$$\frac{c_1}{1 - R_1} (R_2 + \|\operatorname{grad}(g)\|_{2p})^2 \leq R_2, \quad \|g\|_\infty + c_0 R_2 \leq R_1. \quad (2.9)$$

For R small enough we may fix $R_1 = \|g\|_\infty + c_0 R < 1$, and then the theorem is proved if

$$c_1 (R + \|\operatorname{grad}(g)\|_{2p})^2 \leq R(1 - \|g\|_\infty - c_0 R) \quad (2.10)$$

for some $R > 0$. As last condition is equivalent to our hypothesis, the result holds. $\square$

3. Regularity of the solutions of problem (1.2)

In this section, we state the following regularity result.

THEOREM 3.1. *Let $X \in W^{1,2p}(\Omega, \mathbb{R}^3)$ be a solution of (1.2). Then*

- (a) *if $g \in W^{2,q}(\Omega, \mathbb{R}^3)$ for some $q > 1$, then $X \in W^{2,q}(\Omega, \mathbb{R}^3)$,*
- (b) *if $\partial\Omega \in C^{k+2,\alpha}$, $H \in C^{k,\alpha}(\mathbb{R}^3, \mathbb{R})$, $g \in C^{k+2,\alpha}(\bar{\Omega}, \mathbb{R}^3)$ for some $0 < \alpha < 1$, $k \geq 0$, then $X \in C^{k+2,\alpha}(\bar{\Omega}, \mathbb{R}^3)$.*

Proof. (a) Let $\Delta X = f \in L^p$. If $p \geq q$, let Z be the unique solution in $W^{2,q}$ of the problem $\Delta Z = f$, $Z|_{\partial\Omega} = g$. As $\Delta(X - Z) = 0$ and $X = Z$ on $\partial\Omega$ the result follows. On the other hand, if $p < q$, we obtain in the same way that $X \in W^{2,p}$. For $2 \leq p < q$ this implies that $X \in W^{1,2q}$ and the result follows.

Now we consider the case $p < 2, q$. Let $p_0 = p$ and define

$$p_{n+1} = \begin{cases} \frac{p_n^*}{2} & \text{if } p_n < 2, q, \\ q & \text{otherwise,} \end{cases} \quad (3.1)$$

where p_n^* is the critical Sobolev exponent $2p_n/(2-p_n)$. Then $\{p_n\}$ is bounded, and $X \in W^{1,2p_n}$ for every n . If $p_n < 2, q$ for every n , then p_n is increasing and taking $r = \lim_{n \rightarrow \infty} p_n$, we obtain that $r/(2-r) = r$, a contradiction. Hence, $p_n \geq q$ or $q > p_n \geq 2$ for some n , and the proof is complete.

(b) Case $k = 0$: by part (a), choosing $q > 2/(1-\alpha)$ we obtain that $X \in W^{2,q} \hookrightarrow C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Then $\Delta X = f \in C^\alpha(\overline{\Omega}, \mathbb{R}^3)$. By [4, Theorem 6.14] the equation $\Delta Z = f$ in Ω , $Z = g$ in $\partial\Omega$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and the result follows from the uniqueness in [4, Theorem 9.15].

The general case is now immediate, from [4, Theorem 6.19]. $\square$

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P. AMSTER: DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES PABELLÓN I, CIUDAD UNIVERSITARIA, 1428, BUENOS AIRES, ARGENTINA

E-mail address: pamster@dm.uba.ar

G. KEILHAUER: DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES PABELLÓN I, CIUDAD UNIVERSITARIA, 1428, BUENOS AIRES, ARGENTINA

E-mail address: wkeilh@dm.uba.ar

M. C. MARIANI: DEPARTAMENTO DE MATEMÁTICA, FACULTAD DE CIENCIAS EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES PABELLÓN I, CIUDAD UNIVERSITARIA, 1428, BUENOS AIRES, ARGENTINA

E-mail address: mcmarian@dm.uba.ar