ON MODULI OF k -CONVEXITY

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We establish the continuity of some moduli of k -convexity. Let X be a Banach space. We denote by X^* the dual space of X and by B_X the unit ball of X . Several moduli of convexity for the norm of X have been defined; the last two definitions in the following are valid for spaces having dimension $\geq k$:

$$\begin{aligned}\delta_X(\epsilon) &= \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in B_X, \|x-y\| \geq \epsilon \right\} \quad (\text{see [2]}), \\ \delta_X^{(k)}(\epsilon) &= \inf \left\{ 1 - \frac{\|x_1 + \dots + x_{k+1}\|}{k+1} : x_1, \dots, x_{k+1} \in B_X, A(x_1, \dots, x_{k+1}) \geq \epsilon \right\} \quad (\text{see [10]}), \\ \Delta_X^{(k)}(\epsilon) &= \inf_{\|x\|=1} \inf_{\substack{Y \subset X \\ \dim(Y)=k}} \sup_{\substack{\|y\|=1 \\ y \in Y}} \{ \|x + \epsilon y\| - 1 \} \quad (\text{see [9]}),\end{aligned}\tag{1}$$

where

$$A(x_1, \dots, x_{k+1}) = \frac{1}{k!} \sup \left\{ \begin{vmatrix} 1 & \dots & 1 \\ f_1(x_1) & \dots & f_1(x_{k+1}) \\ \vdots & \dots & \vdots \\ f_k(x_1) & \dots & f_k(x_{k+1}) \end{vmatrix} : f_1, \dots, f_k \in B_{X^*} \right\}.\tag{2}$$

Evidently, by subtracting the first column from the other columns, the determinant can be replaced by

$$\begin{vmatrix} f_1(x_2 - x_1) & \dots & f_1(x_{k+1} - x_1) \\ \vdots & \dots & \vdots \\ f_k(x_2 - x_1) & \dots & f_k(x_{k+1} - x_1) \end{vmatrix}.\tag{3}$$

Also $A(x_1, \dots, x_{k+1})$ can be thought of as the “volume” of the convex hull of $x_1, \dots, x_{k+1}$ since that is the case in Euclidean spaces.

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X is called uniformly convex if $\delta_X(\epsilon) > 0$ for $\epsilon > 0$ and k -uniformly convex if $\delta_X^{(k)}(\epsilon) > 0$ for $\epsilon > 0$. Note that $\delta_X(\epsilon) = \delta_X^{(1)}(\epsilon)$; so 1-uniform convexity coincides with uniform convexity. Lin [8] proved that $\Delta_X^{(k)}(\epsilon) > 0$ for $\epsilon > 0$ is equivalent to k -uniform convexity. Gurarii [5] proved that $\delta_X(\epsilon)$ is continuous on $[0, 2)$ and there exist spaces of which $\delta_X(\epsilon) = 0$ for $0 \leq \epsilon < 2$ and $\delta_X(2) = 1$. The continuity problem of $\delta_X^{(k)}$ was mentioned in Kirk [6]. Let $\mu_X^{(k)} = \sup\{A(x_1, \dots, x_{k+1}) : x_1, \dots, x_{k+1} \in B_X\}$. Note that $\mu_X^{(1)} = 2$. In this paper, we prove that $\delta_X^{(k)}(\epsilon)$ is continuous on $[0, \mu_X^{(k)})$. It is quite evident that $\Delta_X^{(k)}(\epsilon)$ satisfy the Lipschitz condition with constant 1.

Definition 1. Let $k \geq 1$ and $0 \leq a < b \leq \infty$. A function $f(\epsilon)$ on (a, b) is called k -convex if

$$f\left(\left(\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k}\right)^k\right) \leq \lambda f(\epsilon_2) + (1-\lambda)f(\epsilon_1) \quad (4)$$

for every $\epsilon_1, \epsilon_2 \in (a, b)$, $0 \leq \lambda \leq 1$.

Obviously 1-convexity is simply the ordinary convexity.

LEMMA 2. Let $0 \leq a < b \leq \infty$ and let f be a nondecreasing k -convex function on (a, b) with $M = \sup_{a < x < y < b} (f(y) - f(x)) < \infty$. Let $\epsilon_1 < \epsilon_2$, $\epsilon_1, \epsilon_2 \in (a, b)$. Then

$$\frac{f(c) - f(\epsilon_1)}{c - \epsilon_1} \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}} \quad (5)$$

for every $\epsilon_1 < c < \epsilon_2$.

Proof. Let $z(x)$, $\epsilon_1 \leq x \leq \epsilon_2$ be the function whose graph is defined by

$$\begin{aligned} x &= \left(\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k}\right)^k \\ y &= \lambda f(\epsilon_2) + (1-\lambda)f(\epsilon_1) \end{aligned} \quad 0 \leq \lambda \leq 1. \quad (6)$$

By direct computations, we have

$$z'(x) = \frac{f(\epsilon_2) - f(\epsilon_1)}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})(\lambda\epsilon_2^{1/k} + (1-\lambda)\epsilon_1^{1/k})^{k-1}} \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}. \quad (7)$$

If $\epsilon_1 < c < \epsilon_2$, then by the k -convexity of f and the mean-value theorem,

$$\frac{f(c) - f(\epsilon_1)}{c - \epsilon_1} \leq \frac{z(c) - z(\epsilon_1)}{c - \epsilon_1} = z'(\psi) \leq \frac{M}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}. \quad (8)$$

□

The inequality in the following lemma is a consequence of a more general result proved in Bernal-Sullivan [1].

LEMMA 3. Let X be a Banach space and $x_1, \dots, x_{k+1} \in X$. Then

$$A(x_1, \dots, x_{k+1}) \leq \frac{1}{k!} k^{k/2} \|x_2 - x_1\| \cdots \|x_{k+1} - x_1\|. \quad (9)$$

Proof. Hadamard inequality says that if $r_1, r_2, \dots, r_k$ are the rows (or columns) of a $k \times k$ matrix, then

$$\det(r_1, r_2, \dots, r_k) \leq \|r_1\|_2 \|r_2\|_2 \cdots \|r_k\|_2. \quad (10)$$

Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^k$. Since the Euclidean norm of the j th column of the determinant in (3) is $\leq k^{1/2} \|x_{j-1} - x_1\|$, the inequality follows. $\square$

The inequality in the next theorem for the case $k = 1$ improves the one obtained in [5]. The general idea is similar to that in Goebel [3]. However, the reader should be aware that the assertion of Lemma 1 in that paper (that $\delta(\epsilon)$ is convex) is incorrect; a counterexample can be found in [7] or [4].

THEOREM 4. Let X be a Banach space. Then

$$\frac{\delta_X^{(k)}(c) - \delta_X^{(k)}(\epsilon_1)}{c - \epsilon_1} \leq \frac{1}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}} \quad (11)$$

for every $0 < \epsilon_1 < c < \epsilon_2 < \mu_X^{(k)}$.

Proof. For simplicity, in the following we will consider $k = 2$ and will indicate how to generalize to general k . Note that if $A(x_1, x_2, x_3) > 0$, then $x_2 - x_1$ and $x_3 - x_1$ are linearly independent.

For unit vectors u, u_{21}, u_{31} , and u_{32} in X , with $\{u_{21}, u_{31}\}$ linearly independent, consider the set

$$\begin{aligned} N(u, u_{21}, u_{31}, u_{32}; \epsilon) = \Big\{ (x_1, x_2, x_3) \in X^3 : & x_1 + x_2 + x_3 = \lambda u, x_2 - x_1 = \lambda_{21} u_{21}, \\ & x_3 - x_1 = \lambda_{31} u_{31}, x_3 - x_2 = \lambda_{32} u_{32} \\ & \text{for some } \lambda, \lambda_{21}, \lambda_{31}, \lambda_{32} \geq 0 \text{ and } A(x_1, x_2, x_3) \geq \epsilon \Big\}, \end{aligned} \quad (12)$$

and define

$$\delta(u, u_{21}, u_{31}, u_{32}; \epsilon) = \inf \left\{ 1 - \frac{\|x_1 + x_2 + x_3\|}{3} : (x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon) \right\}. \quad (13)$$

Obviously, $\delta(u, u_{21}, u_{31}, u_{32}; \epsilon)$ is nondecreasing and has values in $[0, 1]$.

If $(x_1, x_2, x_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_1)$, $(y_1, y_2, y_3) \in N(u, u_{21}, u_{31}, u_{32}; \epsilon_2)$, and

$$\begin{aligned} x_1 + x_2 + x_3 &= \lambda u, & x_2 - x_1 &= \lambda_{21} u_{21}, & x_3 - x_1 &= \lambda_{31} u_{31}, & x_3 - x_2 &= \lambda_{32} u_{32}, \\ y_1 + y_2 + y_3 &= \alpha u, & y_2 - y_1 &= \alpha_{21} u_{21}, & y_3 - y_1 &= \alpha_{31} u_{31}, & y_3 - y_2 &= \alpha_{32} u_{32} \end{aligned} \quad (14)$$

for some $\lambda, \lambda_{ij}, \alpha, \alpha_{ij} \geq 0$, then by linear independence of $\{u_{21}, u_{31}\}$, there exists $c \geq 0$ such that

$$\alpha_{21} = c\lambda_{21}, \quad \alpha_{31} = c\lambda_{31}, \quad \alpha_{32} = c\lambda_{32}. \quad (15)$$

Indeed, $\lambda_{32}u_{32} = x_3 - x_2 = (x_3 - x_1) - (x_2 - x_1) = \lambda_{31}u_{31} - \lambda_{21}u_{21}$ and $\alpha_{32}u_{32} = \alpha_{31}u_{31} - \alpha_{21}u_{21}$ imply

$$(\alpha_{32}\lambda_{31} - \lambda_{32}\alpha_{31})u_{31} - (\alpha_{32}\lambda_{21} - \lambda_{32}\alpha_{21})u_{21} = 0 \quad (16)$$

from which it follows that $\alpha_{31}/\lambda_{31} = \alpha_{32}/\lambda_{32} = \alpha_{21}/\lambda_{21}$.

Let

$$C(u_{21}, u_{31}) = \sup \left\{ \left| \begin{matrix} f_1(u_{21}) & f_1(u_{31}) \\ f_2(u_{21}) & f_2(u_{31}) \end{matrix} \right| : f_1, f_2 \in B_{X^*} \right\}. \quad (17)$$

Then $A(x_1, x_2, x_3) = \lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \epsilon_1$ and $A(y_1, y_2, y_3) = c^2\lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \epsilon_2$.

For $0 \leq \zeta \leq 1$, let $z_i = \zeta x_i + (1 - \zeta)y_i$, $i = 1, 2, 3$. Then

$$\begin{aligned} z_2 - z_1 &= (\zeta\lambda_{21} + (1 - \zeta)c\lambda_{21})u_{21} = (\zeta + (1 - \zeta)c)\lambda_{21}u_{21}, \\ z_3 - z_1 &= (\zeta + (1 - \zeta)c)\lambda_{31}u_{31}, \\ z_3 - z_2 &= (\zeta + (1 - \zeta)c)\lambda_{32}u_{32}, \\ z_1 + z_2 + z_3 &= (\zeta\lambda + (1 - \zeta)\alpha)u, \end{aligned} \quad (18)$$

$$A(z_1, z_2, z_3) = (\zeta + (1 - \zeta)c)^2\lambda_{21}\lambda_{31}C(u_{21}, u_{31}) \geq \left(\zeta\epsilon_1^{1/2} + (1 - \zeta)\epsilon_2^{1/2}\right)^2,$$

$$\begin{aligned} 1 - \frac{\|z_1 + z_2 + z_3\|}{3} &= 1 - \frac{\|\zeta(x_1 + x_2 + x_3) + (1 - \zeta)(y_1 + y_2 + y_3)\|}{3} \\ &= 1 - \frac{\|\zeta\lambda u + (1 - \zeta)\alpha u\|}{3} \\ &= 1 - \frac{\zeta\lambda + (1 - \zeta)\alpha}{3} \\ &= \zeta\left(1 - \frac{\lambda}{3}\right) + (1 - \zeta)\left(1 - \frac{\alpha}{3}\right) \\ &= \zeta\left(1 - \frac{\|x_1 + x_2 + x_3\|}{3}\right) + (1 - \zeta)\left(1 - \frac{\|y_1 + y_2 + y_3\|}{3}\right). \end{aligned} \quad (19)$$

Hence

$$\begin{aligned} &\delta\left(u, u_{21}, u_{31}, u_{32}; \left(\zeta\epsilon_1^{1/2} + (1 - \zeta)\epsilon_2^{1/2}\right)^2\right) \\ &\leq \zeta\delta(u, u_{21}, u_{31}, u_{32}; \epsilon_1) + (1 - \zeta)\delta(u, u_{21}, u_{31}, u_{32}; \epsilon_2). \end{aligned} \quad (20)$$

Since

$$\begin{aligned} \delta_X^{(2)}(\epsilon) &= \inf \left\{ \delta(u, u_{21}, u_{31}, u_{32}; \epsilon) : \|u\| = \|u_{21}\| = \|u_{31}\| = \|u_{32}\| = 1, \right. \\ &\quad \left. \{u_{21}, u_{31}\} \text{ linearly independent} \right\}, \end{aligned} \quad (21)$$

and the inequality in Lemma 2 is preserved under passing to infimum, inequality (11) for $k = 2$ follows.

For general k , we have $\binom{k+1}{2} + 1$ unit vectors $u, u_{21}, \dots$ and the proof is similar to the one above. $\square$

COROLLARY 5. *Let X be a Banach space. Then $\delta_X^{(k)}(\epsilon)$ is continuous on $[0, \mu_X^{(k)})$.*

Proof. Take $\|x_1\| = 1$ and $x_2, \dots, x_{k+1}$ in a small ball centered at x_1 . Then, by Lemma 3, $A(x_1, \dots, x_{k+1})$ is small. Since $1 - \|x_1 + \dots + x_{k+1}\|/(k+1)$ is close to 0, we see that $\delta_X^{(k)}(\epsilon)$ is continuous at 0.

Continuity of $\delta_X^{(k)}(\epsilon)$ on $(0, \mu_X^{(k)})$ follows immediately from the inequality (11). $\square$

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