NONLINEAR ERGODIC THEOREMS FOR A SEMITOPOLOGICAL SEMIGROUP OF NON-LIPSCHITZIAN MAPPINGS WITHOUT CONVEXITY

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Let G be a semitopological semigroup, C a nonempty subset of a real Hilbert space H , and $\mathfrak{S} = \{T_t : t \in G\}$ a representation of G as asymptotically nonexpansive type mappings of C into itself. Let $L(x) = \{z \in H : \inf_{s \in G} \sup_{t \in G} \|T_{ts}x - z\| = \inf_{t \in G} \|T_tx - z\|\}$ for each $x \in C$ and $L(\mathfrak{S}) = \bigcap_{x \in C} L(x)$. In this paper, we prove that $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(\mathfrak{S})$ is nonempty for each $x \in C$ if and only if there exists a unique nonexpansive retraction P of C into $L(\mathfrak{S})$ such that $PT_s = P$ for all $s \in G$ and $P(x) \in \overline{\text{conv}}\{T_sx : s \in G\}$ for every $x \in C$. Moreover, we prove the ergodic convergence theorem for a semitopological semigroup of non-Lipschitzian mappings without convexity.

1. Introduction and preliminaries

Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. Let G be a semitopological semigroup, that is, a semigroup with a Hausdorff topology such that for each $s \in G$ the mappings $s \mapsto s \cdot t$ and $s \mapsto t \cdot s$ of G into itself are continuous. Let C be a nonempty subset of H and let $\mathfrak{S} = \{T_t : t \in G\}$ be a semigroup on C , that is, $T_{st}(x) = T_s T_t(x)$ for all $s, t \in G$ and $x \in C$. Recall that a semigroup $\mathfrak{S}$ is said to be

- (a) nonexpansive if $\|T_tx - T_ty\| \leq \|x - y\|$ for $x, y \in C$ and $t \in G$.
- (b) asymptotically nonexpansive [6] if there exists a function $k : G \mapsto [0, \infty)$ with $\inf_{s \in G} \sup_{t \in G} k_{ts} \leq 1$ such that $\|T_tx - T_ty\| \leq k_t \|x - y\|$ for $x, y \in C$ and $t \in G$.
- (c) of asymptotically nonexpansive type [6] if for each x in C , there is a function $r(\cdot, x) : G \mapsto [0, \infty)$ with $\inf_{s \in G} \sup_{t \in G} r(ts, x) = 0$ such that $\|T_tx - T_ty\| \leq \|x - y\| + r(t, x)$ for all $y \in C$ and $t \in G$.

It is easily seen that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and that both the inclusions are proper (cf. [6, page 112]).

Baillon [1] proved the first nonlinear mean ergodic theorem for nonexpansive mappings in a Hilbert space: let C be a nonempty closed convex subset of a Hilbert space H and T a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T

is nonempty, then for each $x \in C$, the Cesáro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x \quad (1.1)$$

converge weakly as $n \rightarrow \infty$ to a point of $F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{\text{conv}}\{T^n x : n = 0, 1, 2, \dots\}$ for each $x \in C$, where $\overline{\text{conv}}A$ is the closure of the convex hull of A . The analogous results are given for nonexpansive semigroups on C by Baillon [2] and Brézis-Browder [3]. In [10], Mizoguchi-Takahashi proved a nonlinear ergodic retraction theorem for Lipschitzian semigroups by using the notion of submean. Recently, Li and Ma [8, 9] proved the nonlinear ergodic retraction theorems for non-Lipschitzian semigroups in a Banach space without using the notion of submean. Also, in 1992, Takahashi [13] proved the ergodic theorem for nonexpansive semigroups on condition that $\bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$.

In this paper, without using the concept of submean, we prove nonlinear ergodic theorem for semitopological semigroup of non-Lipschitzian mappings without convexity in a Hilbert space. We first prove that if C is a nonempty subset of a Hilbert space H , G a semitopological semigroup, and $\mathfrak{S} = \{T_t : t \in G\}$ a representation of G as asymptotically nonexpansive type mappings of C into itself, then $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(\mathfrak{S})$ is nonempty for each $x \in C$ if and only if there exists a unique nonexpansive retraction P of C into $L(\mathfrak{S})$ such that $PT_s = P$ for all $s \in G$ and Px is in the closed convex hull of $\{T_s x : s \in G\}$, where $L(x) = \{z : \inf_{s \in G} \sup_{t \in G} \|T_{ts}x - z\| = \inf_{t \in G} \|T_t x - z\|\}$ and $L(\mathfrak{S}) = \bigcap_{x \in C} L(x)$. By using this result, we also prove the ergodic convergence theorem for semitopological semigroup of non-Lipschitzian mapping without convexity. Our results are generalizations and improvements of the previously known results of Brézis-Browder [3], Hirano-Takahashi [4], Mizoguchi-Takahashi [10], Takahashi-Zhang [14], and Takahashi [11, 12, 13] in many directions. Further, it is safe to say that in the results [1, 2, 3, 4, 5, 7, 10, 11, 12, 13, 14], many key conditions are not necessary.

2. Ergodic convergence theorems

Throughout this paper, we assume that C is a nonempty subset of a real Hilbert space H , G a semitopological semigroup, and $\mathfrak{S} = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C . For each $x \in C$, define $L(x)$ and $L(\mathfrak{S})$ by

$$L(x) = \left\{ z : \inf_{s \in G} \sup_{t \in G} \|T_{ts}x - z\| = \inf_{t \in G} \|T_t x - z\| \right\}, \quad L(\mathfrak{S}) = \bigcap_{x \in C} L(x), \quad (2.1)$$

respectively. We denote $F(\mathfrak{S})$ by the set $\{x \in C : T_s(x) = x \text{ for all } s \in G\}$ of common fixed point of $\mathfrak{S}$. We begin with the following lemma.

LEMMA 2.1. *Let C be a nonempty subset of a Hilbert space H and $\mathfrak{S} = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C . Then $F(\mathfrak{S}) \subset L(\mathfrak{S})$.*

Proof. Let $x \in C$ and $f \in F(\mathfrak{S})$. Since $\mathfrak{S}$ is asymptotically nonexpansive type, for an arbitrary $\varepsilon > 0$, there exists $s_0 \in G$ such that for all $t \in G$

$$r(ts_0, f) < \varepsilon. \quad (2.2)$$

Hence, for each $a \in G$,

$$\begin{aligned} \inf_{s \in G} \sup_{t \in G} \|T_{ts}x - f\| &\leq \sup_{t \in G} \|T_{ts_0a}x - f\| \leq \sup_{t \in G} (\|T_ax - f\| + r(ts_0, f)) \\ &\leq \|T_ax - f\| + \varepsilon. \end{aligned} \quad (2.3)$$

Since $\varepsilon > 0$ is arbitrary, we have $\inf_{s \in G} \sup_{t \in G} \|T_{ts}x - f\| \leq \inf_{t \in G} \|T_tx - f\|$. Therefore, $f \in L(x)$. This completes the proof. $\square$

Remark 2.2. It is not easy to prove that $F(\mathfrak{S})$ is nonempty when C is not a convex subset. However, we can show that $L(\mathfrak{S})$ is nonempty under some conditions and it is important for the ergodic convergence theorem.

The following proposition plays a crucial role in the proof of our main theorems in this paper.

PROPOSITION 2.3. *Let G be a semitopological semigroup, C a nonempty subset of a Hilbert space H , and $\mathfrak{S} = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C . Then, for every $x \in C$, the set*

$$\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(x), \quad (2.4)$$

consists of at most one point.

Proof. Let $u, v \in \bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(x)$, without loss of generality, we assume that

$$\inf_{t \in G} \|T_tx - u\|^2 \leq \inf_{t \in G} \|T_tx - v\|^2. \quad (2.5)$$

Now, for each $t, s \in G$, since

$$\|u - v\|^2 + 2(T_{ts}x - u, u - v) = \|T_{ts}x - v\|^2 - \|T_{ts}x - u\|^2, \quad (2.6)$$

we have

$$\begin{aligned} \|u - v\|^2 + 2 \inf_{t \in G} (T_{ts}x - u, u - v) &\geq \inf_{t \in G} \|T_{ts}x - v\|^2 - \sup_{t \in G} \|T_{ts}x - u\|^2 \\ &\geq \inf_{t \in G} \|T_tx - v\|^2 - \sup_{t \in G} \|T_{ts}x - u\|^2. \end{aligned} \quad (2.7)$$

From $u \in L(x)$, we have

$$\begin{aligned} \|u - v\|^2 + 2 \sup_{s \in G} \inf_{t \in G} (T_{ts}x - u, u - v) &\geq \inf_{t \in G} \|T_tx - v\|^2 - \inf_{s \in G} \sup_{t \in G} \|T_{ts}x - u\|^2 \\ &= \inf_{t \in G} \|T_tx - v\|^2 - \inf_{t \in G} \|T_tx - u\|^2 \geq 0. \end{aligned} \quad (2.8)$$

Therefore, for $\varepsilon > 0$ there is an $s_1 \in G$ such that

$$\|u - v\|^2 + 2(T_{ts_1}x - u, u - v) > -\varepsilon \quad \forall t \in G. \quad (2.9)$$

From $v \in \overline{\text{conv}}\{T_{ts_1}x : t \in G\}$, we have

$$\|u - v\|^2 + 2(v - u, u - v) \geq -\varepsilon. \quad (2.10)$$

This inequality implies that $\|u - v\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $u = v$. This completes the proof. $\square$

Remark 2.4. In the Takahashi-Zhang's result [14], it is assumed that C is a closed convex subset, G a reversible semigroup, and $\mathfrak{S}$ an asymptotically nonexpansive semigroup. Proposition 2.3 shows those key conditions are not necessary.

Let $m(G)$ be the Banach space of all bounded real-valued functions on a semi-topological semigroup G with the supremum norm and let X be a subspace of $m(G)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a mean on X if $\|\mu\| = \mu(1) = 1$. Let μ be a mean on X and $f \in X$. Then, according to time and circumstances, we use $\mu_t(f(t))$ instead of $\mu(f)$. For each $s \in G$ and $f \in m(G)$, we define elements $l_s f$ and $r_s f$ in $m(G)$ given by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in G$, respectively.

Throughout the rest of this section, let X be a subspace of $m(G)$ containing constants invariant under l_s and r_s for each $s \in G$. Furthermore, suppose that for each $x \in C$ and $y \in H$, a function $f(t) = \|T_t x - y\|^2$ is in X . For $\mu \in X^*$, we define the value $\mu_t(T_t x, y)$ of μ at this function. By Riesz theorem, there exists a unique element $\mathfrak{S}_\mu x$ in X such that

$$\mu_t(T_t x, y) = (\mathfrak{S}_\mu x, y) \quad \forall y \in H. \quad (2.11)$$

LEMMA 2.5. Suppose that X has an invariant mean μ . Then we have

$$\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(x) = \{\mathfrak{S}_\mu x\} \quad \text{for every } x \in C. \quad (2.12)$$

Further, if T_t is continuous for each $t \in G$ and $\bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$, then $\mathfrak{S}_\mu x \in F(\mathfrak{S})$.

Proof. Since μ is an invariant mean, it is easy to show that $\mathfrak{S}_\mu x \in \bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\}$ for each $x \in C$. By Proposition 2.3, it is enough to prove that $\mathfrak{S}_\mu x \in L(x)$ for each $x \in C$. To this end, let $\varepsilon > 0$, since $\mathfrak{S}$ is an asymptotically nonexpansive type semigroup, for each $t \in G$ there is an $h_t \in G$ such that for each $h \in G$,

$$r(hh_t, T_t x) < \varepsilon. \quad (2.13)$$

Put $M = \sup_{t, s \in G} \|T_t x - T_s x\|$, then we have

$$\begin{aligned} \|T_{hh_t}x - \mathfrak{S}_\mu x\|^2 - \|T_t x - \mathfrak{S}_\mu x\|^2 &= \mu_s \left(\|T_{hh_t}x - T_s x\|^2 - \|T_t x - T_s x\|^2 \right) \\ &= \mu_s \left(\|T_{hh_t}x - T_{hh_t}x\|^2 - \|T_t x - T_s x\|^2 \right) \\ &\leq 2M\varepsilon \quad \text{for each } h \in G. \end{aligned} \quad (2.14)$$

Hence, we have

$$\inf_{s \in G} \sup_{h \in G} \|T_{hs}x - \mathfrak{S}_\mu x\|^2 \leq \|T_t x - \mathfrak{S}_\mu x\|^2 + 2M\varepsilon \quad \forall t \in G. \quad (2.15)$$

Since $\varepsilon > 0$ is arbitrary, we have $\mathfrak{S}_\mu x \in L(x)$. Finally, suppose that $\bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ and each T_t is continuous from C into itself. Then, we can easily prove that $\mathfrak{S}_\mu x \in \bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\}$ and hence we have $\mathfrak{S}_\mu x \in C$. For each $h \in G$ and $\varepsilon \in (0, 1)$, there exists $0 < \delta < \varepsilon$ such that $\|T_h y - T_h \mathfrak{S}_\mu x\| < \varepsilon$ whenever $y \in C$ and $\|y - \mathfrak{S}_\mu x\| \leq \delta$. Since $\mathfrak{S}$ is an asymptotically nonexpansive type semigroup, there is $s_0 \in G$ such that

$$r(ts_0, \mathfrak{S}_\mu x) < \frac{1}{2(M_1 + 1)}\delta^2 \quad \forall t \in G, \quad (2.16)$$

where $M_1 = \sup_{t \in G} \|T_t x - \mathfrak{S}_\mu x\|$. Then for each $t, s \in G$, we have

$$\begin{aligned} & \|T_{ss_0} \mathfrak{S}_\mu x - \mathfrak{S}_\mu x\|^2 + 2(T_t x - \mathfrak{S}_\mu x, \mathfrak{S}_\mu x - T_{ss_0} \mathfrak{S}_\mu x) \\ &= \|T_t x - T_{ss_0} \mathfrak{S}_\mu x\|^2 - \|T_t x - \mathfrak{S}_\mu x\|^2 \\ &= \|T_{ss_0} T_t x - T_{ss_0} \mathfrak{S}_\mu x\|^2 - \|T_t x - \mathfrak{S}_\mu x\|^2 - \|T_{ss_0} T_t x - T_{ss_0} \mathfrak{S}_\mu x\|^2 + \|T_t x - T_{ss_0} \mathfrak{S}_\mu x\|^2 \\ &\leq \delta^2 - \|T_{ss_0} T_t x - T_{ss_0} \mathfrak{S}_\mu x\|^2 + \|T_t x - T_{ss_0} \mathfrak{S}_\mu x\|^2. \end{aligned} \quad (2.17)$$

It follows that

$$\|T_{ss_0} \mathfrak{S}_\mu x - \mathfrak{S}_\mu x\| \leq \delta \quad \forall s \in G. \quad (2.18)$$

This implies that

$$\|T_h \mathfrak{S}_\mu x - \mathfrak{S}_\mu x\| \leq \|T_h \mathfrak{S}_\mu x - T_h T_{ss_0} \mathfrak{S}_\mu x\| + \|T_{hs_0} \mathfrak{S}_\mu x - \mathfrak{S}_\mu x\| < 2\varepsilon. \quad (2.19)$$

Since $\varepsilon > 0$ is arbitrary, we have $T_h \mathfrak{S}_\mu x = \mathfrak{S}_\mu x$. This completes the proof. $\square$

Now, we prove a nonlinear ergodic theorem for asymptotically nonexpansive type semigroups without convexity. Before doing this, we give a definition concerning means. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on X , where A is a directed set. Then $\{\mu_\alpha : \alpha \in A\}$ is said to be asymptotically invariant if for each $f \in X$ and $s \in G$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \longrightarrow 0, \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \longrightarrow 0. \quad (2.20)$$

THEOREM 2.6. *Let C be a nonempty subset of a Hilbert space H , X an invariant subspace of $m(G)$ containing constants, and $\mathfrak{S} = \{T_t : t \in G\}$ an asymptotically nonexpansive type semigroup on C . If for each $x \in C$ and $y \in H$, the function f on G defined by $f(t) = \|T_t x - y\|^2$ belong to X , then for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ on X , the net $\{\mathfrak{S}_{\mu_\alpha} x\}_{\alpha \in A}$ converges weakly to an element $x_0 \in L(x)$.*

Further, if T_t is continuous for each $t \in G$ and $\bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$, then $x_0 \in F(\mathfrak{S})$.

Proof. Let W be the set of all weak limit points of subnet of the net $\{\mathfrak{S}_{\mu_\alpha}x : \alpha \in A\}$. By Proposition 2.3, it is enough to prove that

$$W \subset \bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(x). \quad (2.21)$$

To show this, let $z \in W$ and let $\{\mathfrak{S}_{\mu_{\alpha_\beta}}x\}$ be a subnet of $\{\mathfrak{S}_{\mu_\alpha}x\}$ such that $\{\mathfrak{S}_{\mu_{\alpha_\beta}}x\}$ converges weakly to z . Now, without loss of generality, we can suppose that $\{\mathfrak{S}_{\mu_{\alpha_\beta}}x\}$ converges weakly* to $\mu \in X^*$. It is easily seen that μ is an invariant mean on X and then Lemma 2.5 implies that $z = \mathfrak{S}_\mu x \in \bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(x)$. This completes the proof. $\square$

Let $C(G)$ be the Banach space of all bounded continuous real-valued functions on G and let $RUC(G)$ be the space of all bounded right uniformly continuous functions on G , that is, all $f \in C(G)$ such that the mapping $s \mapsto r_s f$ is continuous. Then $RUC(G)$ is a closed subalgebra of $C(G)$ containing constants and invariant under l_s and r_s .

As a direct consequence of Theorem 2.6, we obtain the following corollary.

COROLLARY 2.7 (see [13]). *Let C be a nonempty subset of a Hilbert space H and let G be a semitopological semigroup such that $RUC(G)$ has an invariant mean. Let $\mathfrak{S} = \{T_t : t \in G\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in G\}$ is bounded and $\bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$ for some $x \in C$. Then, $F(\mathfrak{S}) \neq \emptyset$. Further, for an asymptotically invariant net $\{\mu_\alpha\}_{\alpha \in A}$ of means on $RUC(G)$, the net $\{\mathfrak{S}_{\mu_\alpha}\}_{\alpha \in A}$ converges weakly to an element $x_0 \in F(\mathfrak{S})$.*

Remark 2.8. For the proof of Corollary 2.7, Takahashi [13] used the condition $\bigcap_{s \in G} \overline{\text{conv}}\{T_{st}x : t \in G\} \subset C$. But, from Theorem 2.6, we can prove the result without this condition except proving the fact that the weak limit of $\{\mathfrak{S}_{\mu_\alpha}x\}$ is in $F(\mathfrak{S})$.

3. Nonexpansive retractions

In this section, we prove an ergodic retraction theorem for a semitopological semigroup of asymptotically nonexpansive type mappings without convexity.

THEOREM 3.1. *Let C be a nonempty subset of a Hilbert space H and let $\mathfrak{S} = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $L(\mathfrak{S}) \neq \emptyset$. Then the following statements are equivalent:*

- (a) $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \bigcap L(\mathfrak{S}) \neq \emptyset$ for each $x \in C$.
- (b) There is a unique nonexpansive retraction P of C into $L(\mathfrak{S})$ such that $PT_t = P$ for every $t \in G$ and $Px \in \overline{\text{conv}}\{T_t x : t \in G\}$ for every $x \in C$.

Proof. (b) $\Rightarrow$ (a). Let $x \in C$, then $Px \in L(\mathfrak{S})$. Also $Px \in \bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\}$. In fact, for each $s \in G$, $Px = PT_s x \in \overline{\text{conv}}\{T_t T_s x : t \in G\} = \overline{\text{conv}}\{T_{ts}x : t \in G\}$.

(a) $\Rightarrow$ (b). Let $x \in C$. Then by Proposition 2.3, $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(\mathfrak{N})$ contains exactly one point Px . For each $a \in G$, we have

$$\begin{aligned} \{PT_a x\} &= \bigcap_{s \in G} \overline{\text{conv}}\{T_{tsa}x : t \in G\} \cap L(\mathfrak{N}) \\ &\supseteq \bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap L(\mathfrak{N}) = \{Px\} \end{aligned} \quad (3.1)$$

and hence we have $PT_a = P$ for every $a \in G$.

Finally, we have to show that P is nonexpansive. Let $x, y \in C$ and $0 < \lambda < 1$. Then for any $\varepsilon > 0$, there exists $s_1 \in G$ such that

$$\sup_{t \in G} \|T_{ts_1}x - Py\| \leq \inf_{t \in G} \|T_t x - Py\| + \varepsilon, \quad (3.2)$$

from $Py \in L(\mathfrak{N})$. Hence, we have

$$\begin{aligned} &\|\lambda T_{ts_1}x + (1-\lambda)Px - Py\|^2 \\ &= \|\lambda(T_{ts_1}x - Py) + (1-\lambda)(Px - Py)\|^2 \\ &= \lambda\|T_{ts_1}x - Py\|^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\|T_{ts_1}x - Px\|^2 \\ &\leq \lambda(\|T_{ab}x - Py\| + \varepsilon)^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\inf_{t \in G} \|T_t x - Px\|^2, \end{aligned} \quad (3.3)$$

for each $t, s, a, b \in G$. Since $\varepsilon > 0$ is arbitrary, this implies

$$\begin{aligned} &\inf_{s \in G} \sup_{t \in G} \|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2 \\ &\leq \lambda\|T_{ab}x - Py\|^2 + (1-\lambda)\|Px - Py\|^2 - \lambda(1-\lambda)\inf_{t \in G} \|T_t x - Px\|^2 \\ &= \|\lambda T_{ab}x + (1-\lambda)Px - Py\|^2 + \lambda(1-\lambda)\|T_{ab}x - Px\|^2 - \lambda(1-\lambda)\inf_{t \in G} \|T_t x - Px\|^2. \end{aligned} \quad (3.4)$$

Then it is easily seen that

$$\begin{aligned} &\inf_{s \in G} \sup_{t \in G} \|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2 - \lambda(1-\lambda)\inf_{b \in G} \sup_{a \in G} \|T_{ab}x - Px\|^2 \\ &\leq \sup_{b \in G} \inf_{a \in G} \|\lambda T_{ab}x + (1-\lambda)Px - Py\|^2 - \lambda(1-\lambda)\inf_{t \in G} \|T_t x - Px\|^2. \end{aligned} \quad (3.5)$$

Since $Px \in L(\mathfrak{N})$, we have

$$\inf_{s \in G} \sup_{t \in G} \|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2 \leq \sup_{s \in G} \inf_{t \in G} \|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2. \quad (3.6)$$

Let

$$h(\lambda) = \inf_{s \in G} \sup_{t \in G} \|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2. \quad (3.7)$$

Then for any $\varepsilon > 0$, there exists $s_2 \in G$ such that for all $t \in G$,

$$\|\lambda T_{ts_2}x + (1-\lambda)Px - Py\|^2 \leq h(\lambda) + \varepsilon \quad (3.8)$$

and hence

$$(\lambda T_{ts_2}x + (1-\lambda)Px - Py, Px - Py) \leq (h(\lambda) + \varepsilon)^{1/2} \|Px - Py\| \quad \forall t \in G. \quad (3.9)$$

From $Px \in \overline{\text{conv}}\{T_{ts_2}x : t \in G\}$, we have

$$(\lambda Px + (1-\lambda)Px - Py, Px - Py) \leq (h(\lambda) + \varepsilon)^{1/2} \|Px - Py\|. \quad (3.10)$$

Since $\varepsilon > 0$ is arbitrary, this yields that

$$\|Px - Py\|^2 \leq h(\lambda). \quad (3.11)$$

That is,

$$\|Px - Py\|^2 \leq \inf_{s \in G} \sup_{t \in G} \|\lambda T_{ts}x + (1-\lambda)Px - Py\|^2. \quad (3.12)$$

Now, one can choose an $s_3 \in G$ such that $\|T_{ts_3}x - Px\| \leq M$ for all $t \in G$, where $M = 1 + \inf_{t \in G} \|T_t x - Px\|$. Then, we have

$$\begin{aligned} & \|\lambda T_{ts_3}x + (1-\lambda)Px - Py\|^2 \\ &= \|\lambda(T_{ts_3}x - Px) + (Px - Py)\|^2 \\ &= \lambda^2 \|T_{ts_3}x - Px\|^2 + \|Px - Py\|^2 + 2\lambda(T_{ts_3}x - Px, Px - Py) \\ &\leq M^2\lambda^2 + \|Px - Py\|^2 + 2\lambda(T_{ts_3}x - Px, Px - Py). \end{aligned} \quad (3.13)$$

It then follows from (3.6) and (3.12) that

$$\begin{aligned} & 2\lambda \sup_{s \in G} \inf_{t \in G} (T_{ts}x - Px, Px - Py) \\ &\geq 2\lambda \sup_{s \in G} \inf_{t \in G} (T_{ts_3}x - Px, Px - Py) \\ &\geq \sup_{s \in G} \inf_{t \in G} \|\lambda T_{ts_3}x + (1-\lambda)Px - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2 \\ &= \sup_{s \in G} \inf_{t \in G} \|\lambda T_{ts}T_{s_3}x + (1-\lambda)PT_{s_3}x - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2 \\ &\geq \|PT_{s_3}x - Py\|^2 - \|Px - Py\|^2 - M^2\lambda^2 \\ &= -M^2\lambda^2. \end{aligned} \quad (3.14)$$

Hence, we have

$$\sup_{s \in G} \inf_{t \in G} (T_{ts}x - Px, Px - Py) \geq -\frac{1}{2}M^2\lambda. \quad (3.15)$$

Letting $\lambda \rightarrow 0$, then we have

$$\sup_{s \in G} \inf_{t \in G} (T_{ts}x - Px, Px - Py) \geq 0. \quad (3.16)$$

Let $\varepsilon > 0$, then there is $s_4 \in G$ such that

$$r(ts_4, x) < \varepsilon \quad \forall t \in G. \quad (3.17)$$

For such an $s_4 \in G$, from (3.16), we have

$$\sup_{s \in G} \inf_{t \in G} (T_{ts} T_{s_4} x - P T_{s_4} x, P T_{s_4} x - P y) \geq 0 \quad (3.18)$$

and hence there is $s_5 \in G$ such that

$$\inf_{t \in G} (T_{ts_5} T_{s_4} x - P T_{s_4} x, P T_{s_4} x - P y) > -\varepsilon. \quad (3.19)$$

Then, from $P T_{s_4} x = P x$, we have

$$\inf_{t \in G} (T_{ts_5 s_4} x - P x, P x - P y) > -\varepsilon. \quad (3.20)$$

Similarly, from (3.16), we also have

$$\sup_{s \in G} \inf_{t \in G} (T_{ts} T_{s_5 s_4} y - P T_{s_5 s_4} y, P T_{s_5 s_4} y - P x) \geq 0, \quad (3.21)$$

and there exists $s_6 \in G$ such that

$$\inf_{t \in G} (T_{ts_6 s_5 s_4} y - P T_{s_5 s_4} y, P T_{s_5 s_4} y - P x) \geq -\varepsilon, \quad (3.22)$$

that is,

$$\inf_{t \in G} (P y - T_{ts_6 s_5 s_4} y, P x - P y) \geq -\varepsilon. \quad (3.23)$$

On the other hand, from (3.20)

$$\inf_{t \in G} (T_{ts_6 s_5 s_4} x - P x, P x - P y) > -\varepsilon. \quad (3.24)$$

Combining (3.23) and (3.24), we have

$$\begin{aligned} -2\varepsilon &< (T_{ts_6 s_5 s_4} x - T_{ts_6 s_5 s_4} y, P x - P y) - \|P x - P y\|^2 \\ &\leq \|T_{ts_6 s_5 s_4} x - T_{ts_6 s_5 s_4} y\| \cdot \|P x - P y\| - \|P x - P y\|^2 \\ &\leq (r(ts_6 s_5 s_4, x) + \|x - y\|) \cdot \|P x - P y\| - \|P x - P y\|^2 \\ &\leq (\varepsilon + \|x - y\|) \cdot \|P x - P y\| - \|P x - P y\|^2. \end{aligned} \quad (3.25)$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|P x - P y\| \leq \|x - y\|$. The proof is completed. $\square$

Using Lemma 2.1, we have the following ergodic retraction theorem for asymptotically nonexpansive type semigroups.

THEOREM 3.2. *Let C be a nonempty subset of a real Hilbert space H and let $\mathfrak{S} = \{T_t : t \in G\}$ be a semitopological semigroup of asymptotically nonexpansive type mappings on C such that $F(\mathfrak{S}) \neq \emptyset$. Then the following statements are equivalent:*

- (a) $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts} x : t \in G\} \cap F(\mathfrak{S}) \neq \emptyset$ for each $x \in C$.
- (b) There is a unique nonexpansive retraction P of C onto $F(\mathfrak{S})$ such that $P T_t = T_t P = P$ for every $t \in G$ and $P x \in \overline{\text{conv}}\{T_t x : t \in G\}$ for every $x \in C$.

We denote by $B(G)$ the Banach space of all bounded real-valued functions on G with supremum norm. Let X be a subspace of $B(G)$ containing constants. Then, according to Mizoguchi-Takahashi [10], a real-valued function μ on X is called a submean on X if the following conditions are satisfied:

- (1) $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
- (2) $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
- (3) for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
- (4) $\mu(c) = c$ for every constant c .

The following corollaries are immediately deduced from Theorem 3.2.

COROLLARY 3.3 (see [10]). *Let C be a closed convex subset of a Hilbert space H and let X be an r_s -invariant subspace of $B(G)$ containing constants which has a right invariant submean. Let $\mathfrak{S} = \{T_t : t \in G\}$ be a Lipschitzian semigroup on C with $\inf_s \sup_t k_{ts}^2 \leq 1$ and $F(\mathfrak{S}) \neq \emptyset$, where k_t is the Lipschitzian constants. If for each $x, y \in C$, the function f on G defined by*

$$f(t) = \|T_t x - y\|^2 \quad \forall t \in G \quad (3.26)$$

and the function g on G defined by

$$g(t) = k_t^2 \quad \forall t \in G \quad (3.27)$$

belong to X , then the following statements are equivalent:

- (a) $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap F(\mathfrak{S}) \neq \emptyset$ for each $x \in C$.
- (b) *There is a nonexpansive retraction P of C onto $F(\mathfrak{S})$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{\text{conv}}\{T_t x : t \in G\}$ for every $x \in C$.*

COROLLARY 3.4 (see [7]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathfrak{S} = \{T_t : t \in G\}$ be a continuous representation of a semitopological semigroup as nonexpansive mappings from C into itself. If for each $x \in C$, the set $\bigcap_{s \in G} \overline{\text{conv}}\{T_{ts}x : t \in G\} \cap F(\mathfrak{S}) \neq \emptyset$, then there exists a nonexpansive retraction P of C onto $F(\mathfrak{S})$ such that $PT_t = T_t P = P$ for every $t \in G$ and $Px \in \overline{\text{conv}}\{T_t x : t \in G\}$ for every $x \in C$.*

Remark 3.5. By Theorem 3.2, many key conditions, in Corollaries 3.3 and 3.4, such as C is convex closed subset and $\mathfrak{S}$ is continuous Lipschitzian semigroup, are not necessary.

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