DOMAIN PERTURBATION METHOD AND LOCAL MINIMIZERS TO GINZBURG-LANDAU FUNCTIONAL WITH MAGNETIC EFFECT

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We prove the existence of vortex local minimizers to Ginzburg-Landau functional with a global magnetic effect. A domain perturbating method is developed, which allows us to extend a local minimizer on a nonsimply connected superconducting material to the local minimizer with vortex on a simply connected material.

1. Introduction

In this paper, we study the existence of nontrivial stable state solutions of Ginzburg-Landau equation with magnetic effect by the method of singular perturbation of domains. Particularly, we construct various topology classes of stable state solutions with vortex. The Ginzburg-Landau equation is deduced from the energy functional

$$\mathcal{H}_\lambda(\Phi, A) = \int_\Omega \left(\frac{1}{2}|(\nabla - iA)\Phi|^2 + \frac{\lambda}{4}\left(1-|\Phi|^2\right)^2\right)dx + \int_{\mathbb{R}^3} \frac{1}{2}|\operatorname{rot} A|^2 dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, Φ is a $\mathbb{C}$-valued function in Ω, A is an $\mathbb{R}^3$-valued function in $\mathbb{R}^3$ and $\lambda > 0$ is a parameter. Corresponding to a low temperature superconducting steady state in a superconductor Ω without external applied magnetic field, a local minimizer (Φ, A) of $\mathcal{H}_\lambda$ is a solution of the Ginzburg-Landau equation

$$\begin{gathered}
(\nabla - iA)^2\Phi + \lambda\left(1-|\Phi|^2\right)\Phi = 0 \quad \text{in } \Omega, \\
\frac{\partial \Phi}{\partial \nu} - i\langle A\cdot\nu\rangle\Phi = 0 \quad \text{on } \partial\Omega, \\
\operatorname{rot}\operatorname{rot} A + \left(\frac{i\left(\overline{\Phi}\nabla\Phi - \Phi\nabla\overline{\Phi}\right)}{2} + |\Phi|^2 A\right)\Lambda_\Omega = 0 \quad \text{in } \mathbb{R}^3,
\end{gathered} \tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard inner product of vectors in $\mathbb{R}^3$ and Λ_Ω is the characteristic function of Ω, that is, $\Lambda_\Omega(x) = 1$ in Ω and $\Lambda_\Omega(x) = 0$ in $\mathbb{R}^3 \setminus \Omega$. It is very interesting in physics to construct stable superconducting steady states.

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In [16], we proved the existence of local minimizers of (1.1) for large $\lambda > 0$ provided that $\Omega \subset \mathbb{R}^3$ is nonsimply connected (see also the work of Rubinstein and Sternberg in [19]). Let $\mathcal{M}$ denote the set of all continuous maps from $\overline{\Omega}$ into $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. We prove that for any $\theta_0 \in \mathcal{M}$, there exists a local minimizer $(\Phi_\lambda, A_\lambda)$ of $\mathcal{H}_\lambda$ for large λ such that Φ_λ does not vanish in $\overline{\Omega}$ and the continuous map

$$\overline{\Omega} \ni x \longmapsto \frac{\Phi_\lambda(x)}{\left|\Phi_\lambda(x)\right|} \in S^1 \subset \mathbb{C} \tag{1.3}$$

is homotopic to θ_0. We also prove an estimate from below for the second variation of (1.1) which guarantees the stability of $(\Phi_\lambda, A_\lambda)$.

In this paper, we consider the existence of local minimizers to (1.1) when the nonsimply connected domain Ω is replaced by a perturbated domain $\Omega(\zeta) : \Omega(\zeta) \to \Omega$ as $\zeta \to 0$, here the convergence means that

$$\mathrm{Vol}\left(\Omega(\zeta) \setminus \Omega\right) \longrightarrow 0 \quad \text{as } \zeta \longrightarrow 0. \tag{1.4}$$

The convergence is so weak that we can take $\Omega(\zeta)$ in various topological types. Note that $\Omega(\zeta)$ may be simply connected. We obtain a local minimizer $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ which has the same topological type as $(\Phi_\lambda, A_\lambda)$ on Ω (see Theorem 3.1). The point is that when $\Omega(\zeta)$ is simply connected, the solution $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ must take zero value somewhere. That is, the solution has vortices which is an important physical feature of type II superconductors.

The main result of this paper is Theorem 3.1 which is proved in Section 4. We first give the definition of notation used in this paper and some results in the case of nonsimply connected domain in Section 2. The nondegeneracy inequality for the energy functional (1.1) at the minimizers $(\Phi_\lambda, A_\lambda)$ of Proposition 2.5 is a key step to obtain the local minimizers to the Ginzburg-Landau energy functional on a perturbating domain $\Omega(\zeta)$.

Closely related to our paper [16], the main references in this subject are the works of Rubinstein and Sternberg in [19], Jimbo and Morita in [14], and Almeida in [2, 3]. These works establish results which, although closely related, present different points of view and very distinct techniques in their proofs.

Related topics on the Ginzburg-Landau energy functional can be found in [4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 19, 20, 21] and the references therein. Berger and Chen, Du, Gunzberger, and Peterson, Yang, Rubinstein and Sternberg have obtained many interesting results in this area (cf. [7, 9, 10, 19, 20, 21] and the references therein). For a cylindrical superconducting material with an applied magnetic field, Bauman, Phillips, and Tang in [6] considered the existence of local minimizers to GL functional with the vortex only in the axis of the cylinder. Bethuel, Brezis, and Hélein in [8] have given a full detail of the vortices problem of the Ginzburg-Landau equation with first kind boundary condition. Andre and Shafrir in [4, 5] considered the asymptotic behavior of the minimizers as $\lambda \to \infty$.

2. Preliminaries

We consider the variational problem of the energy functional (1.1) in the space

$$D(\Omega) = H^1(\Omega;\mathbb{C}) \times Z, \tag{2.1}$$

where

$$Z = \left\{A \in L^6\left(\mathbb{R}^3;\mathbb{R}^3\right) \mid \nabla A \in L^2\left(\mathbb{R}^3;\mathbb{R}^{3\times 3}\right)\right\}. \tag{2.2}$$

Note that the functional (1.1) as well as the equation (1.2) are invariant under the gauge transformation

$$(\Phi, A) \longmapsto \left(\Phi', A'\right), \quad \Phi' = e^{i\rho}\Phi,\ A' = A + \nabla\rho \tag{2.3}$$

for $\rho : \mathbb{R}^3 \to \mathbb{R}$.

Therefore, corresponding to a solution (Φ, A), there is a solution set $S(\Phi, A)$ generated from (Φ, A) by the gauge transformation. $\mathscr{H}_\lambda$ is constant on $S(\Phi, A)$ and hence the second variation degenerates in the tangential direction of $S(\Phi, A)$.

The tangent space $T(\Phi, A)$ of $S(\Phi, A)$ at (Φ, A) is obtained by direct calculation

$$T(\Phi, A) = \left\{\left((-v + iu)\xi, \nabla\xi\right) \in D(\Omega) \mid \xi \in L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right),\ \nabla\xi \in Z\right\}, \tag{2.4}$$

where $\Phi = u + vi$. We only need to consider the variation of $\mathscr{H}_\lambda$ in the transversal direction $N(\Phi, A)$ of $S(\Phi, A)$ at (Φ, A). To obtain the subspace $N(\Phi, A)$, we use the Helmholtz decomposition of L^6 (cf. [17])

$$L^6\left(\mathbb{R}^3;\mathbb{R}^3\right) = Y_1 \oplus Y_2, \tag{2.5}$$

where

$$\begin{aligned} Y_1 &= \left\{\nabla\xi \mid \xi \in L^2_{\mathrm{loc}}\left(\mathbb{R}^3\right),\ \nabla\xi \in L^6\left(\mathbb{R}^3;\mathbb{R}^3\right)\right\}, \\ Y_2 &= \left\{B \in L^6\left(\mathbb{R}^3;\mathbb{R}^3\right) \mid \operatorname{div} B = 0\right\}. \end{aligned} \tag{2.6}$$

Let P be the projector from $L^6(\mathbb{R}^3;\mathbb{R}^3)$ onto the space Y_2. Then

$$N(\Phi, A) := \left\{(\Psi, B) \in H^1(\Omega;\mathbb{C}) \times Z \mid \int_\Omega \operatorname{Im}\left(\Psi\overline{\Phi}\right) dx = 0,\ PB = B\right\} \tag{2.7}$$

is transversal to $T(\Phi, A)$.

Proposition 2.1 (see [16]). *For any C^1 function $(\Phi, A) \in H^1(\Omega;\mathbb{C}) \times Z$,*

$$H^1(\Omega;\mathbb{C}) \times Z = T(\Phi, A) \oplus N(\Phi, A). \tag{2.8}$$

Formula of the second variation of $\mathcal{H}_\lambda$

$$
\begin{aligned}
&\mathcal{L}_\lambda(\Phi, A, \Psi, B)\\
&\quad= \frac{d^2}{d\epsilon^2}\mathcal{H}_\lambda(\Phi+\epsilon\Psi, A+\epsilon B)_{|\epsilon=0}\\
&\quad= \int_\Omega \left\{|\nabla\phi+\psi A|^2+|\nabla\psi-\phi A|^2-\lambda\left(1-u^2-v^2\right)\left(\phi^2+\psi^2\right)+2\lambda(u\phi+v\psi)^2\right\}dx\\
&\qquad+\int_{\mathbb{R}^3}|\operatorname{rot} B|^2dx+\int_\Omega\left(u^2+v^2\right)B^2dx+4\int_\Omega\langle A, B\rangle(u\phi+v\psi)\,dx\\
&\qquad-2\int_\Omega\left\{\phi\langle\nabla v, B\rangle-\psi\langle\nabla u, B\rangle+u\langle\nabla\psi, B\rangle-v\langle\nabla\phi, B\rangle\right\}dx,
\end{aligned}
\tag{2.9}
$$

where we put $\Phi = u+vi$ and $\Psi = \phi+\psi i$.

Proposition 2.2 (see [16]). *Let $(\Phi, A) \in D(\Omega)$ be a C^1-solution of (1.2). Then*

$$
\mathcal{L}_\lambda(\Phi, A, \Psi, B) = \mathcal{L}_\lambda\left(\Phi, A, \Psi', B'\right) \tag{2.10}
$$

provided that (Ψ, B), $(\Psi', B') \in H^1(\Omega;\mathbb{C})\times Z$ and $(\Psi-\Psi', B-B') \in T(\Phi, A)$.

In [16], we have proved the existence of local minimizers to (1.2) when Ω is non-simply connected (see also [19]). That is the following theorem holds.

Theorem 2.3. *For any $\theta_0 \in \mathcal{M}$, there exists a $\lambda_0 > 0$ such that (1.2) has a solution $(\Phi_\lambda, A_\lambda) \in D(\Omega)\cap(C^{2+\alpha}(\overline{\Omega};\mathbb{C})\times C^{1+\alpha}(\mathbb{R}^3;\mathbb{R}^3))$ for any $\lambda \geqq \lambda_0$ and $\alpha \in (0,1)$ with the following properties:*

$$
\Phi_\lambda(x) = W_\lambda(x)e^{i\theta_\lambda(x)}, \qquad \lim_{\lambda\to\infty}\sup_{x\in\Omega}\left|W_\lambda(x)-1\right| = 0, \tag{2.11}
$$

and the map $\theta_\lambda : \overline{\Omega} \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is homotopic to θ_0. Moreover, it is stable in the sense that there exists a constant $c > 0$ such that

$$
\mathcal{L}_\lambda\left(\Phi_\lambda, A_\lambda, \Psi, B\right) \geqq c\|(\Psi, B)\|^2_{D(\Omega)}, \tag{2.12}
$$

where

$$
\|(\Psi, B)\|_{D(\Omega)} \equiv \left(\|\Psi\|^2_{H^1(\Omega;\mathbb{C})}+\|B\|^2_{L^2(\Omega;\mathbb{R}^3)}+\|\nabla B\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})}\right)^{1/2} \tag{2.13}
$$

for $(\Psi, B) \in N(\Phi_\lambda, A_\lambda)$ and $\lambda \geqq \lambda_0$.

Remark 2.4. Concerning the regularity of the solution $(\Phi_\lambda, A_\lambda)$, it was proved that Φ_λ is bounded in $C^{2+\alpha}(\overline{\Omega};\mathbb{C})$ and A_λ is bounded in $C^{1+\alpha}(V;\mathbb{R}^3)$ for large $\lambda > 0$, where V is any fixed bounded set in $\mathbb{R}^3$ and $\alpha \in (0,1)$. Hence, the topology of the convergence of W_λ in (2.11) can be taken in the C^2 sense.

By the estimate (2.12), we get an infinitesimal stability property. A more comprehensive stability inequality (2.14) is proved in Section 5. The stability property is

obtained by directly comparing the values of the energy functional $\mathcal{H}_\lambda$ near a critical point $(\Phi_\lambda, A_\lambda)$.

PROPOSITION 2.5. *There exist constants $c' > 0$, $c'' > 0$ such that*

$$\mathcal{H}_\lambda\big(\Phi_\lambda + \Psi, A_\lambda + B\big) - \mathcal{H}_\lambda\big(\Phi_\lambda, A_\lambda\big) \geqq c'\|(\Psi, B)\|^2_{D(\Omega)} - c''\|(\Psi, B)\|^4_{D(\Omega)} \tag{2.14}$$

for $(\Psi, B) \in N(\Phi_\lambda, A_\lambda)$ and $\lambda \geqq \lambda_0$.

3. Main result

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^3 boundary. Assume that Ω is not simply connected. We consider a family of bounded domains $\{\Omega(\zeta)\}_{\zeta>0}$ with C^3 boundaries in $\mathbb{R}^3$, satisfying the following conditions:

(i) $\Omega(\zeta_1) \supset \Omega(\zeta_2) \supset \Omega$ for $\zeta_1 > \zeta_2 > 0$.

(ii) $\lim_{\zeta\to 0} \mathrm{Vol}(\Omega(\zeta) \setminus \Omega) = 0$.

$\mathrm{Vol}(X)$ is the 3-dim Lebesgue volume of $X \subset \mathbb{R}^3$. Note that the convergence in (ii) is very weak so that we can take $\Omega(\zeta)$ in any complicated topological type. Hereafter we put $Q(\zeta) = \Omega(\zeta) \setminus \Omega$ $(\zeta > 0)$.

Consider the GL functional in $D(\Omega(\zeta))$.

$$\mathcal{H}_{\lambda,\zeta}(\Phi, A) = \int_{\Omega(\zeta)} \left(\frac{1}{2}|(\nabla - iA)\Phi|^2 + \frac{\lambda}{4}\big(1 - |\Phi|^2\big)^2\right) dx + \int_{\mathbb{R}^3} \frac{1}{2}|\operatorname{rot} A|^2 dx. \tag{3.1}$$

The main result of this paper is the following theorem.

THEOREM 3.1. *Let $(\Phi_\lambda, A_\lambda)$ be the solution of (1.2) obtained in Theorem 2.3 with respect to the nonsimply connected domain Ω. For any $\lambda > \lambda_0$, there exists $\zeta_0 = \zeta_0(\lambda) > 0$ such that the functional $\mathcal{H}_{\lambda,\zeta}$ defined by (3.1) has a local minimizer $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta}) \in D(\Omega(\zeta))$ for $0 < \zeta < \zeta_0$ with*

$$\lim_{\zeta\to 0} \big\|\big(\Phi_{\lambda,\zeta|\Omega} - \Phi_\lambda, A_{\lambda,\zeta} - A_\lambda\big)\big\|_{D(\Omega)} = 0. \tag{3.2}$$

Here $\Phi_{\lambda,\zeta|\Omega}$ denotes the restriction of $\Phi_{\lambda,\zeta}$ in Ω.

Remark 3.2. By a compactness argument and the Schauder estimates for second order elliptic boundary value problem, we can choose a solution satisfying the following (3.3), (3.4) in addition to (3.2)

$$\lim_{\zeta\to 0} \Phi_{\lambda,\zeta} = \Phi_\lambda \quad \text{in } C^2\big(\overline{\Sigma(\eta)}\big), \tag{3.3}$$

$$\lim_{\zeta\to 0} A_{\lambda,\zeta} = A_\lambda \quad \text{in } C^1\big(\Gamma(\kappa)\big), \tag{3.4}$$

for any $\eta > 0$, $\kappa > 0$, where

$$\Sigma(\eta) = \big\{x \in \Omega \mid \mathrm{dist}\big(x, Q(\eta)\big) \geqq \eta\big\}, \qquad \Gamma(\kappa) = \big\{x \in \mathbb{R}^3 \mid |x| \leqq \kappa\big\}. \tag{3.5}$$

4. Proof of Theorem 3.1

Let $\Phi_\lambda^* \in C^2(\mathbb{R}^3;\mathbb{C})$ be an extension of the function Φ_λ, that is, $\Phi_\lambda^*(x) = \Phi_\lambda(x)$ for $x \in \Omega$. We first prepare an inequality derived immediately from Proposition 2.5.

LEMMA 4.1. *There exist constants $c' > 0$, $c'' > 0$, and $c''' = c'''(\Phi_\lambda^*, A_\lambda) > 0$ such that*

$$\begin{aligned}\mathcal{H}_{\lambda,\zeta}\big(\Phi_\lambda^*+\Psi, A_\lambda+B\big)-\mathcal{H}_{\lambda,\zeta}\big(\Phi_\lambda^*, A_\lambda\big)\\ \geqq c'\big\|\big(\Psi_{|\Omega}, B\big)\big\|_{D(\Omega)}^2 - c''\big\|\big(\Psi_{|\Omega}, B\big)\big\|_{D(\Omega)}^4 - c'''\operatorname{Vol}\big(Q(\zeta)\big)\end{aligned} \tag{4.1}$$

for $(\Psi, B) \in D(\Omega(\zeta))$ satisfying $(\Psi_{|\Omega}, B) \in N(\Phi_\lambda, A_\lambda)$, provided that $\lambda \geqq \lambda_0$ and $0 < \zeta < \zeta_0$.

Proof. From (3.1), we have

$$\begin{aligned}&\mathcal{H}_{\lambda,\zeta}\big(\Phi_\lambda^*+\Psi, A_\lambda+B\big)-\mathcal{H}_{\lambda,\zeta}\big(\Phi_\lambda^*, A_\lambda\big)\\ &\quad= \mathcal{H}_\lambda\big(\Phi_\lambda+\Psi_{|\Omega}, A_\lambda+B\big)-\mathcal{H}_\lambda\big(\Phi_\lambda, A_\lambda\big)\\ &\qquad+\int_{Q(\zeta)}\left(\frac{1}{2}\big|\big(\nabla-i\big(A_\lambda+B\big)\big)\big(\Phi_\lambda^*+\Psi\big)\big|^2+\frac{\lambda}{4}\Big(1-\big|\Phi_\lambda^*+\Psi\big|^2\Big)^2\right)dx\\ &\qquad-\int_{Q(\zeta)}\left(\frac{1}{2}\big|\big(\nabla-iA_\lambda\big)\Phi_\lambda^*\big|^2+\frac{\lambda}{4}\Big(1-\big|\Phi_\lambda^*\big|^2\Big)^2\right)dx\\ &\quad\geqq \mathcal{H}_\lambda\big(\Phi_\lambda+\Psi_{|\Omega}, A_\lambda+B\big)-\mathcal{H}_\lambda\big(\Phi_\lambda, A_\lambda\big)\\ &\qquad-\int_{Q(\zeta)}\left(\frac{1}{2}\big|\big(\nabla-iA_\lambda\big)\Phi_\lambda^*\big|^2+\frac{\lambda}{4}\Big(1-\big|\Phi_\lambda^*\big|^2\Big)^2\right)dx.\end{aligned} \tag{4.2}$$

Note that the last term does not depend on Ψ and B. By using Proposition 2.5, we obtain the desired inequality. □

We define a set F in $D(\Omega(\zeta))$ in which we seek for a local minimizer of $\mathcal{H}_{\lambda,\zeta}$.

$$\begin{aligned}F(\delta,\epsilon,\zeta) := \Big\{(\Phi, A) \in D\big(\Omega(\zeta)\big) \mid \big(\Phi_{|\Omega}-\Phi_\lambda, A-A_\lambda\big) \in N\big(\Phi_\lambda, A_\lambda\big),\\ \big\|\big(\Phi_{|\Omega}-\Phi_\lambda, A-A_\lambda\big)\big\|_{D(\Omega)} \leqq \delta,\ \mathcal{H}_{\lambda,\zeta}(\Phi, A)-\mathcal{H}_{\lambda,\zeta}\big(\Phi_\lambda^*, A_\lambda\big) \leqq \epsilon\Big\}.\end{aligned} \tag{4.3}$$

We prove the existence of the (global) minimizer of $\mathcal{H}_{\lambda,\zeta}$ in $F(\delta,\epsilon,\zeta)$ of $D(\Omega(\zeta))$ by taking $\delta > 0$ and $\epsilon > 0$ adequately for small $\zeta > 0$. Put

$$\epsilon(\zeta) = c'''\operatorname{Vol}\big(Q(\zeta)\big), \qquad \delta(\zeta) = \left(\frac{16c'''\operatorname{Vol}\big(Q(\zeta)\big)}{c'}\right)^{1/2}, \tag{4.4}$$

where c', c'', and c''' are positive constants given in Lemma 4.1. Then we have the following lemma.

LEMMA 4.2. *There exists $\zeta_0 > 0$ such that for $\zeta \in (0, \zeta_0)$,*

$$\left\| \left(\Phi_{|\Omega} - \Phi_\lambda, A - A_\lambda\right) \right\|_{D(\Omega)} \leqq \frac{\delta(\zeta)}{2} \quad \textit{for } (\Phi, A) \in F\big(\delta(\zeta), \epsilon(\zeta), \zeta\big). \tag{4.5}$$

Proof. Put $\Psi = \Phi - \Phi_\lambda^*$, $B = A - A_\lambda$. Take $\zeta_0 > 0$ small so that $0 < \delta(\zeta) < (c'/2c'')^{1/2}$ for $\zeta \in (0, \zeta_0)$. Then from the definition of $F(\delta(\zeta), \epsilon(\zeta), \zeta)$ and (4.4),

$$\left\| \left(\Psi_{|\Omega}, B\right) \right\|_{D(\Omega)} \leqq \delta(\zeta) < \left(\frac{c'}{2c''}\right)^{1/2}, \quad \forall (\Phi, A) \in F\big(\delta(\zeta), \epsilon(\zeta), \zeta\big), \tag{4.6}$$

and consequently, we have

$$c' \left\| \left(\Psi_{|\Omega}, B\right) \right\|_{D(\Omega)}^2 - c'' \left\| \left(\Psi_{|\Omega}, B\right) \right\|_{D(\Omega)}^4 \geqq \frac{c'}{2} \left\| \left(\Psi_{|\Omega}, B\right) \right\|_{D(\Omega)}^2. \tag{4.7}$$

Using Lemma 4.1, we obtain

$$\begin{aligned} \frac{c'}{2} \left\| \left(\Psi_{|\Omega}, B\right) \right\|_{D(\Omega)}^2 &\leqq \mathcal{H}_{\lambda,\zeta}(\Phi, A) - \mathcal{H}_{\lambda,\zeta}\left(\Phi_\lambda^*, A_\lambda\right) + c''' \operatorname{Vol}\big(Q(\zeta)\big) \\ &\leqq \epsilon(\zeta) + c''' \operatorname{Vol}\big(Q(\zeta)\big) \leqq 2c''' \operatorname{Vol}\big(Q(\zeta)\big), \end{aligned} \tag{4.8}$$

then $\|(\Psi_{|\Omega}, B)\|_{D(\Omega)}^2 \leqq (4c'''/c') \operatorname{Vol}(Q(\zeta)) = \delta(\zeta)^2/4$. □

Proof of Theorem 3.1. We consider the minimizing problem of $\mathcal{H}_{\zeta,\lambda}(\Phi, A)$ on $F(\delta(\zeta), \epsilon(\zeta), \zeta)$.

Let $\{(\Phi_m, A_m)\}_{m=1}^\infty \subset F(\delta(\zeta), \epsilon(\zeta), \zeta)$ be a minimizing sequence of $\mathcal{H}_{\zeta,\lambda}$ for a fixed $\zeta \in (0, \zeta_0)$.

Note the identity

$$\|\nabla G\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times 3})}^2 = \|\operatorname{div} G\|_{L^2(\mathbb{R}^3)}^2 + \|\operatorname{rot} G\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^2, \tag{4.9}$$

for any vector valued function G with $\nabla G \in L^2(\mathbb{R}^3; \mathbb{R}^{3\times 3})$. From the boundedness of $\mathcal{H}_{\zeta,\lambda}(\Phi_m, A_m)$ $(m \geqq 1)$, the definitions (2.7), (4.3), and $\operatorname{div} A_m = 0$ in $\mathbb{R}^3$, we see that

$$\int_{\Omega(\zeta)} \left|\left(\nabla - iA_m\right)\Phi_m\right|^2 dx \quad (m \geqq 1), \tag{4.10}$$

$$\int_{\Omega(\zeta)} \left|\Phi_m\right|^4 dx \quad (m \geqq 1), \tag{4.11}$$

$$\int_{\mathbb{R}^3} \left|\nabla A_m\right|^2 dx \quad (m \geqq 1), \tag{4.12}$$

are bounded. By the Sobolev inequality (cf. [11, Chapter 7]), (4.12) and $A_m \in L^6(\mathbb{R}^3; \mathbb{R}^3)$, we get the boundedness of

$$\int_{\mathbb{R}^3} \left|A_m\right|^6 dx \quad (m \geqq 1). \tag{4.13}$$

Relations (4.11) and (4.13) yield that $\{\Phi_m A_m\} \subset L^2(\Omega(\zeta);\mathbb{C}^3)$ is bounded and the boundedness relations of $\int_{\Omega(\zeta)} |\nabla\Phi_m|^2\,dx$ $(m \geqq 1)$ follows from (4.10).

By the usual weak convergence argument in the reflexive Banach space with the aid of the Sobolev imbedding theorem and the Kondrachov compact imbedding theorem (cf. [1, Chapter 7]), there exist subsequences $\{\Phi_{m'}\} \subset \{\Phi_m\}$, $\{A_{m'}\} \subset \{A_m\}$ and their limits $\Phi' \in H^1(\Omega(\zeta);\mathbb{C})$, $A' \in Z$ such that

$$\begin{aligned} \nabla\Phi_{m'} &\longrightarrow \nabla\Phi \quad \text{weakly in } L^2\big(\Omega(\zeta);\mathbb{C}^3\big), \\ \Phi_{m'} &\longrightarrow \Phi \quad \text{strongly in } L^4\big(\Omega(\zeta);\mathbb{C}\big), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \nabla A_{m'} &\longrightarrow \nabla A' \quad \text{weakly in } L^2\big(\mathbb{R}^3;\mathbb{R}^{3\times 3}\big), \\ A_{m'} &\longrightarrow A' \quad \text{strongly in } L^4\big(\Omega(\zeta);\mathbb{R}^3\big). \end{aligned} \tag{4.15}$$

Relations (4.14) and (4.15) yield

$$\liminf_{m\to\infty} \mathscr{H}_{\zeta,\lambda}\big(\Phi_{m'}, A_{m'}\big) \geqq \mathscr{H}_{\zeta,\lambda}\big(\Phi', A'\big) \tag{4.16}$$

and $(\Phi', A') \in F(\delta(\zeta), \epsilon(\zeta), \zeta)$.

Therefore, (Φ', A') is a (global) minimizer of $\mathscr{H}_{\zeta,\lambda}$ in $F(\delta(\zeta), \epsilon(\zeta), \zeta)$.

Hereafter, we denote (Φ', A') by $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$. It is easy to see from Lemma 4.2 that $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ is the interior point of $F(\delta(\zeta), \epsilon(\zeta), \zeta)$ in the relative topology of $N(\Phi_\lambda, A_\lambda)$.

It remains to show that there exists a neighbourhood of $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ in $D(\Omega(\zeta))$ such that $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ is also a minimizer in this neighbourhood. In Lemma 4.3, we prove that there is a neighbourhood of $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ in $D(\Omega(\zeta))$ which can be mapped into $F(\delta(\zeta), \epsilon(\zeta), \zeta)$ by the gauge transformation (2.3). Thus from the gauge invariance of the GL functional, we obtain that $(\Phi_{\zeta,\lambda}, A_{\zeta,\lambda})$ is also a local minimizer of $\mathscr{H}_{\zeta,\lambda}$ in $D(\Omega(\zeta))$. □

Lemma 4.3. *For $\zeta \in (0, \zeta_0)$, there exists $\eta = \eta(\zeta) > 0$ such that for any $(\Phi, A) \in D(\Omega(\zeta))$ with*

$$\Big\| \big((\Phi - \Phi_{\zeta,\lambda})_{|\Omega}, A - A_{\lambda,\zeta} \big) \Big\|_{D(\Omega)} \leqq \eta, \tag{4.17}$$

there exists a real-valued function ρ satisfying $\nabla\rho \in Z$ and $(\Phi e^{i\rho}, A + \nabla\rho) \in F(\delta(\zeta), \epsilon(\zeta), \zeta)$.

Proof. For a given (Φ, A), we consider the system of equations for ρ

$$\int_\Omega \operatorname{Im}\big((\Phi e^{i\rho} - \Phi_\lambda)\overline{\Phi}_\lambda \big)\,dx = 0, \tag{4.18}$$

$$\operatorname{div}(A + \nabla\rho) = 0 \quad \text{in } \mathbb{R}^3. \tag{4.19}$$

If ρ is a solution of (4.18) and (4.19), then

$$\big(\Phi e^{i\rho} - \Phi_\lambda, A + \nabla\rho - A_\lambda\big) \in N\big(\Phi_\lambda, A_\lambda\big) \tag{4.20}$$

provided that $\operatorname{div} A_\lambda = 0$. From the gauge invariance of the GL functional, we may assume that $\operatorname{div} A_\lambda = 0$ (cf. (2.3) and the discussion behind it).

By the operator $P : L^6(\mathbb{R}^3;\mathbb{R}^3) \to Y_2$ defined in Section 1, we have the decomposition

$$A = (I-P)A + PA \in Y_1 \oplus Y_2, \quad (I-P)A = -\nabla\rho, \tag{4.21}$$

for a real-valued function ρ satisfying (4.19). If we impose

$$\int_\Omega \rho\, dx = 0 \tag{4.22}$$

on ρ, it is determined uniquely by A. We denote this ρ by $\rho(A)$. Noting that $A_{\lambda,\zeta} = PA_{\lambda,\zeta}$, we obtain $-\nabla\rho(A) = (I-P)(A - A_{\lambda,\zeta})$. By the continuity of P, we get

$$\|\nabla\rho(A)\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} \leqq c_1 \|A - A_{\lambda,\zeta}\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)}. \tag{4.23}$$

Putting $G = A - A_{\lambda,\zeta} = P(A - A_{\lambda,\zeta}) - \nabla\rho(A)$ in (4.9), we get

$$\begin{aligned} \left\|\nabla\left(A - A_{\lambda,\zeta}\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})} &\geqq \left\|\operatorname{div}\left(-\nabla\rho(A)\right)\right\|_{L^2(\mathbb{R}^3)} \\ &= \left\|\nabla\left(-\nabla\rho(A)\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})}. \end{aligned} \tag{4.24}$$

Relations (4.22), (4.23), and (4.24) yield

$$\|\rho(A)\|_{H^2(\Omega)} \leqq c_2\Big(\left\|A - A_{\lambda,\zeta}\right\|_{L^6(\mathbb{R}^3;\mathbb{R}^3)} + \left\|\nabla\left(A - A_{\lambda,\zeta}\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})}\Big). \tag{4.25}$$

From the Sobolev imbedding theorem,

$$\|\rho(A)\|_{L^\infty(\Omega)} \leqq \left\|\nabla\left(A - A_{\lambda,\zeta}\right)\right\|_{L^2(\mathbb{R}^3;\mathbb{R}^{3\times3})}. \tag{4.26}$$

Note that $\rho = \rho(A) + \xi$ also satisfies (4.19) for any constant $\xi \in \mathbb{R}$. Equation (4.18) can be rewritten in terms of $\xi \in \mathbb{R}$.

$$\int_\Omega \operatorname{Im}\left(\left(\Phi - \Phi_\lambda\right)\overline{\Phi}_\lambda e^{(\rho(A)+\xi)i}\right) dx + \int_\Omega \left|\Phi_\lambda\right|^2 \sin\left(\rho(A)+\xi\right) dx = 0. \tag{4.27}$$

Applying the implicit function theorem to (4.27), we see that there exits a unique solution $\xi(A) \in (-\pi/4, \pi/4)$ if η in (4.17) is small.

Thus we have proved that there is a ρ such that $\nabla\rho \in Z$ and

$$\left(\Phi e^{i\rho} - \Phi_\lambda, A + \nabla\rho - A_\lambda\right) \in N\left(\Phi_\lambda, A_\lambda\right). \tag{4.28}$$

The other conditions in (4.3) can be satisfied if we take η small enough. □

Remark 4.4. We have shown that $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ is a local minimizer of $\mathscr{H}_{\lambda,\zeta}$ in $D(\Omega(\zeta))$. Applying the regularity argument to the weak solution of the third kind elliptic boundary value problem (cf. [18]), we get $\Phi_{\lambda,\zeta} \in H^2(\Omega(\zeta))$. Note that $(\Phi_{\lambda,\zeta}, A_{\lambda,\zeta})$ satisfies the

variational equation (GL equation):

$$\begin{aligned}
&\left(\nabla - iA'\right)^2\Phi' + \lambda\left(1-\left|\Phi'\right|^2\right)\Phi' = 0 \quad \text{in } \Omega(\zeta),\\
&\frac{\partial\Phi'}{\partial\nu} - i\left\langle A'\cdot\nu\right\rangle\Phi' = 0 \quad \text{on } \partial\Omega(\zeta),\\
&\text{rot rot } A' + \left(\frac{i\left(\overline{\Phi}'\nabla\Phi' - \Phi'\nabla\overline{\Phi}'\right)}{2} + \left|\Phi'\right|^2 A'\right)\Lambda_{\Omega(\zeta)} = 0 \quad \text{in } \mathbb{R}^3,
\end{aligned} \tag{4.29}$$

in the distribution sense. Using div $A' = 0$ in $\mathbb{R}^3$, the second equation is written as

$$-\Delta A' + \left(\frac{i\left(\overline{\Phi}'\nabla\Phi' - \Phi'\nabla\overline{\Phi}'\right)}{2} + \left|\Phi'\right|^2 A\right)\Lambda_{\Omega(\zeta)} = 0 \quad \text{in } \mathbb{R}^3. \tag{4.30}$$

Applying the regularity argument and the Schauder estimate to the obtained elliptic system, we see that $\Phi_{\lambda,\zeta}$ is C^2 in $\Omega(\zeta)$ and $A_{\lambda,\zeta}$ is C^1 in $\mathbb{R}^3$.

5. Proof of Proposition 2.5

Let $(\Phi, A) = (\Phi_\lambda, A_\lambda) = (u + vi, A) \in D(\Omega) \cap C^1$ be the solution of (1.2) in Theorem 2.3. A direct calculation yields

$$\mathscr{H}_\lambda(\Phi+\Psi, A+B) - \mathscr{H}_\lambda(\Phi, A) = \frac{1}{2}\mathscr{L}_\lambda(\Phi, A, \Psi, B) + I_\lambda(\Phi, A, \Psi, B), \tag{5.1}$$

where

$$\begin{aligned}
I_\lambda(\Phi, A, \Psi, B) = \int_\Omega \Bigg(&-\lambda(u\phi + v\psi)\left(\phi^2+\psi^2\right) + \frac{\lambda\left(\phi^2+\psi^2\right)^2}{4} + (u\phi+v\psi)B^2\\
&+ \langle\nabla\phi + A\psi, B\psi\rangle + \langle\nabla\psi - A\phi, -B\phi\rangle + \left(\phi^2+\psi^2\right)B^2\Bigg)\,dx.
\end{aligned} \tag{5.2}$$

The first term in the right-hand side of (5.1) is the second variation at (Ψ, B) and its estimate is given by Theorem 2.3. Since (Φ, A) is a solution in Ω, the first variation vanishes and $\mathscr{L}_\lambda(\Phi, A; \Psi, B)$ becomes the leading term.

For any small δ, we estimate the right of I_λ by

$$\begin{aligned}
\int_\Omega |u\phi + v\psi|\left(\phi^2+\psi^2\right)dx &\leqq \int_\Omega \delta\left(u^2+v^2\right)\left(\phi^2+\psi^2\right) + C_\delta\left(\phi^4+\psi^4\right)dx\\
&\leqq \delta\max_\Omega\left(|u|^2+|v|^2\right)\int_\Omega\left(\phi^2+\psi^2\right)dx\\
&\quad + C\left(\|\phi\|^4_{H^1(\Omega)} + \|\psi\|^4_{H^1(\Omega)}\right),
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
\int_\Omega |\nabla\phi||B\psi|dx &\leqq \delta\int_\Omega |\nabla\phi|^2dx + C_\delta\int_\Omega\left(|B|^4+|\psi|^4\right)dx\\
&\leqq \delta\int_\Omega |\nabla\phi|^2dx + C\left(\|B\|^4_Z + \|\psi\|^4_{H^1(\Omega)}\right),
\end{aligned} \tag{5.4}$$

and similar inequalities for the remaining terms. Thus, we obtain

$$\left|I_\lambda(\Phi, A, \Psi, B)\right| \leqq \delta\|(\Psi, B)\|^2_{D(\Omega)} + C\|(\Psi, B)\|^4_{D(\Omega)}. \tag{5.5}$$

By taking δ small, the proposition is proved. □

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