NULL CONTROLLABILITY OF A THERMOELASTIC PLATE

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Received 1 November 2001

Thermoelastic plate model with a control term in the thermal equation is considered. The main result in this paper is that with thermal control, locally distributed within the interior and square integrable in time and space, any finite energy solution can be driven to zero at the control time T.

1. Introduction

In this paper, we investigate the null controllability of thermoelastic plates when the control (heat source) acts in the thermal equation. In general, these models consist of an elastic motion equation and a heat equation, which are coupled in such a way that the energy transfer between them is taken into account.

The plate, we consider here, is derived in the light of [18]. Transverse shear effects are neglected (Euler-Bernoulli model), and the plate is hinged on its edge. In addition to internal and external heat source, the temperature dynamics are driven by internal frictional forces caused by the motion of the plate. The latter connection is expressed by the second law of thermodynamics for irreversible processes, which relates the entropy to the elastic strains. Accounting for thermal effects, we assume that the heat flux law involves only the temperature gradient by the Fourier law.

Let Ω be a bounded, open, connected subset of $\mathbb{R}^2$, with a C^∞ boundary and ω any open subset of Ω. Let $T > 0$ and set

$$Q := (0,T) \times \Omega, \qquad \Sigma := (0,T) \times \partial\Omega. \tag{1.1}$$

We consider a model which describes the small vibrations of a homogeneous, elastically and thermally isotropic Kirchhoff plate, under the influence of a control function $f \in L^2((0,T) \times \omega)$. In absence of exterior forces, and with

Abstract and Applied Analysis 7:11 (2002) 585–599
2000 Mathematics Subject Classification: 35B37, 74B05, 93B05
URL: http://dx.doi.org/10.1155/S108533750220408X

hinged mechanical and Dirichlet thermal boundary conditions, the system we are going to study is the following one:

$$\begin{aligned} u_{tt} + \Delta^2 u + \Delta\theta &= 0 \quad \text{in } Q, \\ \theta_t - \Delta\theta - \Delta u_t &= f \quad \text{in } Q, \\ u = \Delta u &= 0 \quad \text{on } \Sigma, \\ \theta &= 0 \quad \text{on } \Sigma, \\ u(0) = u^0, \quad u_t(0) &= u^1, \quad \theta(0) = \theta^0 \quad \text{on } \Omega. \end{aligned} \tag{1.2}$$

Here, u is the *vertical deflection* of the plate and θ is the *variation of temperature* of the plate with respect to its reference temperature. The subscript $\cdot_t$ denotes time derivative, and u^0, u^1, θ^0 are initial data in a suitable space.

We recall (cf. [20, 26]) that a system is *exactly controllable* at given time $T > 0$ if it can be driven from any state to any state belonging to the same space of states where the system evolves. A system is *null controllable* at time $T > 0$ if an arbitrary state can be transferred to 0 in time T, or equivalently, any state can be joined to any trajectory (e.g., attainability of the trajectories). The null controllability does not yield the exact controllability of the system (cf. the heat equation with distributed control in the domain Ω [26]).

In recent years, many efforts have been devoted to studying the controllability of thermoelastic systems, under varying boundary conditions, and with different choices of control on the boundary or in the control domain. In the classical literature some controllability results are established. Haraux [15] studies the internal controllability of a rectangular plate Ω in $\mathbb{R}^2$. By denoting with ω the set $\Omega \cap B \neq \emptyset$, where B is an open strip parallel to one side of Ω, he proves that for any $T > 0$ and any $(y_0, y_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega)$, there exists a control $h \in L^2((0,T) \times \Omega)$ with $\operatorname{supp} h \subset (0,T) \times \omega$ such that the solution of

$$\begin{aligned} y_{tt} + \Delta^2 y &= h \quad \text{in } Q, \\ y = \Delta y &= 0 \quad \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) &= y_1 \quad \text{on } \Omega \end{aligned} \tag{1.3}$$

satisfies $(y(T), y_t(T)) = (0,0)$ (or equivalently, the null controllability at any time $T > 0$).

Lasiecka and Triggiani [21] show the null controllability at any time $T > 0$ of the thermoelastic plate equation with hinged mechanical and Dirichlet thermal boundary conditions, under the influence of either mechanical or thermal

control on the *whole domain*, namely

$$\begin{gathered} u_{tt} + A^2 u - A\theta = f_1 \quad \text{in } Q, \\ \theta_t + A\theta + Au_t = f_2 \quad \text{in } Q, \\ u = Au = 0 \quad \text{on } \Sigma, \\ \theta = 0 \quad \text{on } \Sigma, \\ u(0) = u^0, \quad u_t(0) = u^1, \quad \theta(0) = \theta^0 \quad \text{on } \Omega, \end{gathered} \tag{1.4}$$

where A is a strictly positive, selfadjoint partial differential operator with compact resolvent, and $(f_1, f_2) = (0, h)$ or $(f_1, f_2) = (k, 0)$, with $h, k \in L^2((0,T) \times \Omega)$ and $h, k \not\equiv 0$.

On the other hand, by considering in addition the rotatory inertia in the plate motion equation, Avalos [5] complements the previous result. He proves the exact controllability at any time $T > 0$ for thermoelastic system

$$\begin{gathered} u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha \Delta \theta = f_1 \quad \text{in } Q, \\ \theta_t - \Delta\theta + \sigma\theta - \alpha \Delta u_t = f_2 \quad \text{in } Q, \\ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Sigma, \\ \theta = 0 \quad \text{on } \Sigma, \\ u(0) = u^0, \quad u_t(0) = u^1, \quad \theta(0) = \theta^0 \quad \text{on } \Omega \end{gathered} \tag{1.5}$$

in absence of control forces ($f_1 \equiv 0$), with a control $f_2 \in L^2(0,T;H^{-1}(\Omega))$ in the whole Ω. The choice of this control space yields the result of exact controllability.

De Teresa and Zuazua [10] consider the thermoelastic plate system (1.5) in presence of a control function $f_1 \in L^1(0,T;H^{-1}(\Omega))$, with $\operatorname{supp} f_1(\cdot,t) \subset \omega \subset \Omega$, in absence of heat sources ($f_2 \equiv 0$), and with $\sigma \equiv 0$ and $\gamma \neq 0$. Clamped boundary conditions are imposed on u. By using a decoupling result (see [16]) for three-dimensional thermoelasticity, a variational approach to controllability (see [13]), and some observability inequalities for the system of thermoelastic plate, a result of *exact-approximate controllability* is obtained. More precisely, they prove the exact controllability of the displacement and the approximate controllability of the temperature, when the control time T is large enough and the support ω of the control satisfies the geometric control conditions introduced in [8]. In other words, they find sufficient conditions on control time T and control region ω such that for every initial and final data (u^0, u^1, θ^0), (v^0, v^1, ξ^0), belonging to the space of states where system (1.5) evolves, and for every $\varepsilon > 0$ there exists a control function f_1 such that the solution of (1.5) satisfies

$$u(T) = v^0, \qquad u_t(T) = v^1, \qquad \|\theta(T) - \xi^0\|_{L^2(\Omega)} \leq \varepsilon. \tag{1.6}$$

Finally, we want to recall an important contribution by Lebeau and Zuazua [24] to the controllability theory of thermoelastic systems. By using the spectral decomposition of operators generated by the coupled wave and heat equations state variables, Lebeau and Zuazua study the null controllability at any time $T > 0$, when the control acts in the wave equation part as distributed control. They suppose the geometric control condition for the wave equation in the domain. With the same assumptions they also prove that the null controllability at any time $T > 0$ holds when the control acts on the heat equation.

Instead of an interior control implemented in the motion or in the heat equation, if the control time T is sufficiently large, Lagnese [19] gives the exact controllability of the displacement, for a boundary controlled thermoelastic system. In this work, free boundary conditions are imposed, instead of the clamped ones in (1.5). Moreover, constant γ is positive and coupling constant α is small enough.

By considering the same previous thermoelastic system, and inserting additional thermal control on an arbitrarily small subset of the boundary, Avalos and Lasiecka [6, 7] dispense with the smallness assumption on α, and they tackle the exact-approximate controllability of thermoelastic plates with variable coupling coefficient of thermal expansion α, respectively. Finally, we recall a recent paper of Eller et al. [11] where they study the exact-approximate boundary controllability of thermoelastic plates with thermal coefficient variable in space.

The null controllability problem for the heat equation has been developed recently by Carleman estimates [12, 14]. This result enables to study the nonlinear case with variable coefficients, and to apply Carleman estimates to thermoelastic plate. In particular, if T is sufficiently large, Albano and Tataru [2] prove some Carleman estimates for a coupled parabolic-hyperbolic system. They obtain a boundary observability estimate which, by duality, implies the null controllability for the adjoint system. Instead of the wave operator, Albano [1] gets a similar result in the case of the plate operator.

In this paper, we present the null controllability at any time $T > 0$ for thermoelastic problem (1.2). We consider an interior control applied in the heat equation, and supported in a subset ω of the domain Ω. Two results are obtained. Firstly, we study the case when $\omega \equiv \Omega$ and we find the null controllability at any time $T > 0$, as Lasiecka and Triggiani [21]. Our procedure is supported by introducing a quadratic function depending on the time (see [4]). Multipliers method is applied to construct this function [3, 5, 6]. Then, we consider the case when $\omega \Subset \Omega$, and the closure of ω does not intersect the boundary of Ω. By applying an iterative method and the observability estimates on the eigenfunctions of the Laplacian operator due to Lebeau and Robbiano [23] (see also [24]), we show that system (1.2) is null controllable at any time $T > 0$. In our proof, the analyticity property of semigroup associated to the thermoelastic system (recall $\gamma = 0$, see Lasiecka and Triggiani [22]), and the commutative property of the operators, which comes from the hinged boundary conditions, are crucial.

The paper is organized as follows. Section 2 is devoted to functional setting and notation. In Section 3, we study the case $\omega \equiv \Omega$. In particular, we prove the null controllability of system (1.2) by a control supported in the whole domain. Finally, Section 4 contains the main result of this paper.

2. Functional setting and notation

We introduce the Hilbert space

$$H := \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times L^2(\Omega) \times L^2(\Omega) \tag{2.1}$$

equipped with the inner product

$$\langle z_1, z_2 \rangle_H = \int_\Omega \left(\Delta u_1 \cdot \Delta u_2 + v_1 \cdot v_2 + \theta_1 \cdot \theta_2\right) dx, \tag{2.2}$$

where

$$z_i = \begin{bmatrix} u_i \\ v_i \\ \theta_i \end{bmatrix}, \quad i = 1, 2. \tag{2.3}$$

The induced norm is denoted by $\|\cdot\|_H$. Putting $v = u_t$ and

$$z(t) = \begin{bmatrix} u(t) \\ v(t) \\ \theta(t) \end{bmatrix}, \qquad z^0 = \begin{bmatrix} u^0 \\ v^0 \\ \theta^0 \end{bmatrix}, \tag{2.4}$$

problem (1.2) can be rewritten as an abstract linear evolution equation in H of the form

$$z_t = Az + Bf, \qquad z(0) = z^0 \in H, \tag{2.5}$$

where we set the operator $A : D(A) \to H$ by

$$A = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{bmatrix} \tag{2.6}$$

with domain

$$D(A) = \{z \in H : \Delta u, v, \theta \in H^2(\Omega) \cap H_0^1(\Omega)\}, \tag{2.7}$$

and the control operator $B : L^2(\omega) \to H$ by

$$Bf = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}. \tag{2.8}$$

Given $T > 0$, the problem of the null controllability of system (2.5) consists in to prove that, for any $z^0 \in H$, there exists a control $f \in L^2((0,T)\times\omega)$ such that the solution $z(t;z^0,f)$ of (2.5) satisfies $z(T;z^0,f) = 0$. This property is equivalent to (cf. [26, Theorem 2.6, page 213]): there exists a positive constant C_T such that

$$\|e^{A^*T}y^0\|_H^2 \le C_T \int_0^T \|B^* e^{A^*t} y^0\|_{L^2(\omega)}^2 \, dt, \quad \forall y^0 \in H. \tag{2.9}$$

We compute

$$A^* = \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & 0 & \Delta \\ 0 & -\Delta & \Delta \end{bmatrix} \tag{2.10}$$

with domain $D(A^*) = D(A)$, and

$$B^* = [0 \quad 0 \quad I]. \tag{2.11}$$

The adjoint system with respect to (1.2) is

$$\begin{gathered} \varphi_{tt} + \Delta^2\varphi + \Delta w = 0 \quad \text{in } Q, \\ w_t - \Delta w - \Delta\varphi_t = 0 \quad \text{in } Q, \\ \varphi = \Delta\varphi = 0 \quad \text{on } \Sigma, \\ w = 0 \quad \text{on } \Sigma, \\ \varphi(0) = \varphi^0, \quad \varphi_t(0) = \varphi^1, \quad w(0) = w^0 \quad \text{on } \Omega. \end{gathered} \tag{2.12}$$

Its solution can be written as

$$\begin{bmatrix} \varphi(t) \\ \varphi_t(t) \\ w(t) \end{bmatrix} = e^{A^*t} \begin{bmatrix} \varphi^0 \\ \varphi^1 \\ w^0 \end{bmatrix}, \tag{2.13}$$

$$B^* e^{A^*t} \begin{bmatrix} \varphi^0 \\ \varphi^1 \\ w^0 \end{bmatrix} = w(t). \tag{2.14}$$

Then, condition (2.9) is equivalent to require that there exists a positive constant C_T such that

$$\|\Delta\varphi(T)\|_{L^2(\Omega)}^2 + \|\varphi_t(T)\|_{L^2(\Omega)}^2 + \|w(T)\|_{L^2(\Omega)}^2 \le C_T \int_0^T \|w(t)\|_{L^2(\omega)}^2 \, dt \tag{2.15}$$

for any solution (2.13) of system (2.12).

3. Null controllability when $\omega \equiv \Omega$

We have the following result.

THEOREM 3.1. *For any $T > 0$ and for any $z^0 \in H$, there exists a control function $f \in L^2((0,T)\times\Omega)$ such that the solution $z(T;z^0,f)$ of (2.5) satisfies $z(T;z^0,f) = 0$. That is to say, problem (2.5) is null controllable in arbitrary time $T > 0$ on the space H within the class of $L^2((0,T)\times\Omega)$-controls.*

Remark 3.2. By a different method, Lasiecka and Triggiani [21] show the previous result.

Proof. In Section 2 we recalled that the request of null controllability is equivalent to: there exists a positive constant C_T such that

$$\|\Delta\varphi(T)\|^2_{L^2(\Omega)} + \|\varphi_t(T)\|^2_{L^2(\Omega)} + \|w(T)\|^2_{L^2(\Omega)} \le C_T \int_0^T \|w(t)\|^2_{L^2(\Omega)}\,dt \tag{3.1}$$

for any solution of system (2.12). Let $\begin{bmatrix}\varphi(t)\\ \varphi_t(t)\\ w(t)\end{bmatrix}$ be a solution of system (2.12) corresponding to an initial data $\begin{bmatrix}\varphi^0\\ \varphi^1\\ w^0\end{bmatrix} \in D(A^*) = D(A)$. We introduce the function

$$\begin{aligned}\mathcal{G}(t) = {} & \frac{t^5}{2}\Big[\|\Delta\varphi(t)\|^2_{L^2(\Omega)} + \|\varphi_t(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{L^2(\Omega)}\Big]\\ & + \alpha t^3\langle w(t),(-\Delta)^{-1}\varphi_t(t)\rangle_{L^2(\Omega)} + \beta t^4\langle\varphi(t),\varphi_t(t)\rangle_{L^2(\Omega)}\\ & + 4\beta t^3\langle(-\Delta)^{-1}w(t),\varphi(t)\rangle_{L^2(\Omega)} + \gamma t^3\|(-\Delta)^{-1/2}w(t)\|^2_{L^2(\Omega)}\\ & + \delta t^4\|(-\Delta)^{-2}w_t(t)\|^2_{L^2(\Omega)} - 2\delta t^4\langle(-\Delta)^{-1}\varphi(t),w(t)\rangle_{L^2(\Omega)}\\ & + \delta t^4\|(-\Delta)^{-1}w(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{3.2}$$

where α, β, γ, and δ are positive constants that will be chosen later. By a differentiation of $\mathcal{G}(t)$ with respect to t and by application of Young inequality, we find

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) \le {} & -\left(\alpha t^3 - \frac{5}{2}t^4 - \beta t^4 - 8\delta t^3\|(-\Delta)^{-1}\|^2 - \varepsilon t^3\right)\|\varphi_t(t)\|^2_{L^2(\Omega)}\\ & - \left(\beta t^4 - \frac{5}{2}t^4 - \varepsilon t^4\right)\|\Delta\varphi(t)\|^2_{L^2(\Omega)} + P(t)\|w(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{3.3}$$

where $\|\cdot\|$ denotes the norm in $\mathcal{L}(L^2(\Omega))$, $\varepsilon > 0$ and $P(t)$ is a polynomial of fifth order on time. We can choose α, β, γ, δ, ε in order to obtain

$$\frac{d}{dt}\mathcal{G}(t) \le P(t)\|w(t)\|^2_{L^2(\Omega)}, \tag{3.4}$$

and, for any $T > 0$,

$$\mathcal{G}(T) \geq \frac{T^5}{4}\left[\|\Delta\varphi(T)\|^2_{L^2(\Omega)} + \|\varphi_t(T)\|^2_{L^2(\Omega)} + \|w(T)\|^2_{L^2(\Omega)}\right]. \tag{3.5}$$

Integrating (3.4) on $[0,T]$ and considering (3.5), observability inequality (3.1) is obtained and the observability constant C_T is $\mathcal{O}(T^{-10})$. □

Remark 3.3. The controllability time estimate is surely not optimal, but it is sufficient to prove the results in Section 4.

4. Null controllability when $\omega \Subset \Omega$

Theorem 4.1. *Let ω be an open subset of Ω and the closure of ω does not intersect the boundary of Ω. For any $T > 0$ and for any $z^0 \in H$, there exists a control function $f \in L^2((0,T)\times\omega)$ such that the solution $z(T;z^0,f)$ of (2.5) satisfies $z(T;z^0,f) = 0$. That is to say, problem (2.5) is null controllable at any time $T > 0$ on the space H within the class of $L^2((0,T)\times\omega)$-controls.*

The proof of this theorem follows the procedure developed by Lebeau and Robbiano in [23] (see also [24, 25]). We decompose the time interval $[0,T]$ as follows. We fix $\delta \in (0,T/2)$ and $\rho \in (0,1/n)$, $n = 2$ being the space dimension. For $l \geq 1$, we set

$$\sigma_l := 2^l, \qquad T_l := k2^{-\rho l} = k\sigma_l^{-\rho}, \tag{4.1}$$

where $k > 0$ is chosen such that $2\sum_{l\geq 1} T_l = T - 2\delta$. We introduce the operator

$$L_{t',t}(z,f) = e^{A(t-t')}z + \int_{t'}^{t} e^{A(t-t'-s)}Bf(s)\,ds, \tag{4.2}$$

with $0 \leq t' < t$. In particular, when $t' = 0$ we have

$$L_t(z,f) = e^{At}z + \int_0^t e^{A(t-s)}Bf(s)\,ds. \tag{4.3}$$

We observe that $z(t) = L_t(z^0,f)$ is the unique solution of system (2.5). We also set the sequence $(a_l)_{l\geq 1}$ by

$$a_0 = \delta, \qquad a_l = a_{l-1} + 2T_l, \quad l \geq 1. \tag{4.4}$$

We observe that $a_l \to T - \delta$ as $l \to \infty$. For any initial data $z \in H$, we denote by

$$g_l := K_{T_l,\sigma_l}(z) \tag{4.5}$$

a control function acting in the time interval $[a_{l-1}, a_{l-1} + T_l]$ in such a way as to drive to zero the projection of the solution $L_{a_{l-1},a_{l-1}+T_l}(z,g_l)$ on the subspace

$$H_{\sigma_l} = \operatorname{span}\left\{\Phi_k^j,\ 1 \leq k \leq \sigma_l,\ j = 1,2,3\right\}, \tag{4.6}$$

namely

$$\pi_{\sigma_l}\left(L_{a_{l-1},a_{l-1}+T_l}(z,g_l)\right)=0, \tag{4.7}$$

where π_{σ_l} is the projection operator on the subspace H_{σ_l}. We can construct a sequence of states

$$\begin{gathered} z^0\in H, \qquad z^1=e^{A\delta}z^0, \\ z^{l+1}=e^{AT_l}y^l, \quad y^l=L_{a_{l-1},a_{l-1}+T_l}(z^l,K_{T_l,\sigma_l}(z^l)). \end{gathered} \tag{4.8}$$

In the time interval $[0,\delta]$ we let the system to evolve freely without control. The second time interval $[a_0,a_1]=[\delta,\delta+2T_1]$ is split into two parts. Firstly, we introduce a control $g_1:=K_{T_1,\sigma_1}(z^1)$ which drive z^1 to a function of $H_{\sigma_1}^{\perp}$ in time T_1 (or equivalently, in $[\delta,\delta+T_1]$ we control to zero the projection π_{σ_1} of the solution). Subsequently, in the time interval $[\delta+T_1,\delta+2T_1]$, we let the system to evolve freely. By repeating this procedure on $[a_{l-1},a_{l-1}+2T_l]$, for any $l\geq 2$, we obtain the sequence $(z^l)_{l\geq 1}$. The main result is to prove the existence of z^l and that $\lim_{l\to\infty}z^l=0$. Then, the assertion of Theorem 4.1 follows. Since system (2.5) is invariant in time, the existence of control g_l, for any $l\geq 1$ amounts to the existence of a control for (2.5) for any $T>0$. Moreover, because of $\lim_{l\to\infty}T_l=0$, we will need to work with small T.

We start to consider the following lemma.

Lemma 4.2. *Let ω be an open subset of Ω and the closure of ω does not intersect the boundary of Ω. Let $T>0$. For any $Y^0\in H$ and any $l\geq 1$, there exists at least a control $f_l(T,Y^0)\in L^2((0,T)\times\omega)$ such that*

$$\pi_{\sigma_l}\left(L_T(Y^0,f_l)\right)=0 \tag{4.9}$$

with

$$\|f_l(T,Y^0)\|^2_{L^2((0,T)\times\omega)}\leq C_T e^{C\sqrt{\mu_{\sigma_l}}}\|Y^0\|^2_H \tag{4.10}$$

with C_T, $C>0$. In particular, C_T is $\mathcal{O}(T^{-10})$.

Remark 4.3. This shows that the projection of solutions $Y(T)=L_T(Y^0,f_l)$ over H_{σ_l}, $l\geq 1$, can be controlled to zero with a control of size $\sqrt{C_T}e^{C\sqrt{\mu_{\sigma_l}}}$.

Proof. Condition (4.9) is equivalent to require that for any $l\geq 1$ and any $Y^0\in H$, there exists at least a control $f_l\in L^2((0,T)\times\omega)$ such that

$$\pi_{\sigma_l}\left(e^{AT}Y^0+\int_0^T e^{A(T-s)}Bf_l(s)\,ds\right)=0, \tag{4.11}$$

or equivalently,

$$e^{AT}(\pi_{\sigma_l}Y^0)+\pi_{\sigma_l}\left(\int_0^T e^{A(T-s)}Bf_l(s)\,ds\right)=0. \tag{4.12}$$

We introduce the operator

$$\mathcal{L}_T : L^2((0,T)\times\omega) \longrightarrow H, \qquad f \longmapsto \int_0^T e^{A(T-s)} B f(s)\, ds. \tag{4.13}$$

Then (4.11) is equivalent to

$$\operatorname{Im}(\pi_{\sigma_l} \circ e^{AT}) \subset \operatorname{Im}(\pi_{\sigma_l} \circ \mathcal{L}_T). \tag{4.14}$$

Because $(\mathcal{L}_T^* h)(t) = B^* e^{A^*(T-t)} h$, for any $h \in H$, condition (4.14) can be rewritten as (cf. [26])

$$\|\Delta X_1(T)\|^2_{L^2(\Omega)} + \|X_2(T)\|^2_{L^2(\Omega)} + \|X_3(T)\|^2_{L^2(\Omega)} \le C_T \int_0^T \|X_3(t)\|^2_{L^2(\omega)}\, dt \tag{4.15}$$

for any solution $X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}$ of system

$$\begin{aligned} X_t &= A^* X \quad \text{in } (0,T)\times\Omega, \\ X(0) &= X^0 \quad \text{on } \Omega \end{aligned} \tag{4.16}$$

with $X^0 = \begin{bmatrix} X_1^0 \\ X_2^0 \\ X_3^0 \end{bmatrix} \in H_{\sigma_l}$. In this case, by inequality (3.1) we have

$$\|\Delta X_1(T)\|^2_{L^2(\Omega)} + \|X_2(T)\|^2_{L^2(\Omega)} + \|X_3(T)\|^2_{L^2(\Omega)} \le C_T \int_0^T \|X_3(t)\|^2_{L^2(\Omega)}\, dt \tag{4.17}$$

and C_T is $\mathcal{O}(T^{-10})$. Since the eigenspaces of A^* are invariant for $e^{A^* t}$ and $X^0 \in H_{\sigma_l}$, we have that $X(t) \in H_{\sigma_l}$ for any $t \in (0,T]$. The restriction A_l^* of the operator A^* on H_{σ_l} has the following representation:

$$A_l^* = \sum_{k=1}^{\sigma_l} \sum_{j=1}^{3} \overline{\lambda_k^j} \langle \cdot, \Phi_{k,j} \rangle_H \Phi_{k,j}^*, \tag{4.18}$$

where $(\overline{\lambda_k^j})$ are the eigenvalues of A^*, $(\Phi_{k,j}^*)$ are the corresponding eigenvectors, and $(\Phi_{k,j})$ are the eigenvectors of A such that $\langle \Phi_{k,j}, \Phi_{h,i}^* \rangle_H = \delta_{kh}\delta_{ji}$. Then, the associated semigroup is given by

$$e^{A_l^* t} \cdot = \sum_{k=1}^{\sigma_l} \sum_{j=1}^{3} e^{\overline{\lambda_k^j} t} \langle \cdot, \Phi_{k,j} \rangle_H \Phi_{k,j}^*. \tag{4.19}$$

This means that the solution $X(t)$ of system (4.16) with initial data $X^0 \in H_{\sigma_l}$, we get

$$X(t) = e^{A_l^* t} X^0 = \sum_{k=1}^{\sigma_l} \sum_{j=1}^{3} e^{\overline{\lambda_k^j} t} \langle X^0, \Phi_{k,j} \rangle_H \Phi_{k,j}^*. \tag{4.20}$$

Direct computations (see [3, 9]) give

$$\begin{aligned} X_3(t) &= \sum_{k=1}^{\sigma_l} \sum_{j=1}^{3} e^{\overline{\lambda_k^j} t} \langle X^0, \Phi_{k,j} \rangle_H \frac{\left(\overline{\lambda_k^j}\right)^2 + \mu_k^2}{\mu_k} e_k \\ &:= \sum_{k=1}^{\sigma_l} a_k(t) e_k, \end{aligned} \tag{4.21}$$

where e_k is a normalized eigenfunction of the Dirichlet Laplacian operator. From the observability estimates on the eigenfunctions of the Laplacian operator (see [17, Theorem 14.6, page 230]), there exist positive constants C, depending on Ω and ω, such that

$$\sum_{k=1}^{l} |a_k(t)|^2 \le C e^{C\sqrt{\mu_l}} \int_\omega \left| \sum_{k=1}^{l} a_k(t) e_k(x) \right|^2 dx. \tag{4.22}$$

Recalling that

$$||X_3(t)||^2_{L^2(\Omega)} = \sum_{k=1}^{\sigma_l} |a_k(t)|^2, \tag{4.23}$$

by (4.21) and (4.22) we obtain

$$\int_\Omega |X_3(t,x)|^2 dx \le C e^{C\sqrt{\mu_{\sigma_l}}} \int_\omega |X_3(t,x)|^2 dx. \tag{4.24}$$

Applying estimates (4.24) in (4.17), we get

$$\begin{aligned} &||\Delta X_1(T)||^2_{L^2(\Omega)} + ||X_2(T)||^2_{L^2(\Omega)} + ||X_3(T)||^2_{L^2(\Omega)} \\ &\quad \le C_T \int_0^T ||X_3(t)||^2_{L^2(\Omega)} dt \\ &\quad \le C_T e^{C\sqrt{\mu_{\sigma_l}}} \int_0^T ||X_3(t)||^2_{L^2(\omega)} dt \end{aligned} \tag{4.25}$$

with C_T is $\mathbb{O}(T^{-10})$. Then, the existence of the required control follows. The constant controlling observability inequality is the one which gives the size of the norm of the control (see [20]), then estimate (4.10) follows. □

Corollary 4.4. *Let ω be an open subset of Ω and the closure of ω does not intersect the boundary of Ω. For any $l \ge 1$ and any $z^l \in H$, there exists at least a control $g_l \in L^2((a_{l-1}, a_{l-1} + T_l) \times \omega)$ such that*

$$\pi_{\sigma_l}(L_{a_{l-1}, a_{l-1}+T_l}(z^l, g_l)) = 0 \tag{4.26}$$

with

$$\|g_l\|^2_{L^2((a_{l-1},a_{l-1}+T_l)\times\omega)} \le C_{T_l} e^{C\sqrt{\mu_{\sigma_l}}} \|z^l\|^2_H \tag{4.27}$$

with $C_{T_l}, C > 0$. *In particular,* C_{T_l} *is* $\mathbb{O}(T_l^{-10})$.

Proof. By application of Lemma 4.2 to every interval $(a_{l-1}, a_{l-1}+T_l)$, $l \ge 1$, with $Y^0 = z^l$, $T = T_l$, and $g_l(t) = f_l(t - a_{l-1})$, for any $t \in (a_{l-1}, a_{l-1}+T_l)$, our conclusion follows. □

Proof of Theorem 4.1. Recalling (4.8) and $y^l \in H^{\perp}_{\sigma_l}$, we have

$$\begin{aligned} \|z^{l+1}\|_H = \|e^{AT_l} y^l\|_H &= \left\| \sum_{k\ge\sigma_l+1} \sum_{j=1}^{3} e^{\lambda_k^j T_l} \langle y^l, \Phi^*_{k,j}\rangle_H \Phi_{k,j} \right\|_H \\ &\le \left(\sum_{k\ge\sigma_l+1} \sum_{j=1}^{3} e^{2\operatorname{Re}\lambda_k^j T_l} \left| \langle y^l, \Phi^*_{k,j}\rangle_H \right|^2 \right)^{1/2} \le e^{\operatorname{Re}\lambda^2_{\sigma_l+1} T_l} \|y^l\|_H. \end{aligned} \tag{4.28}$$

Since e^{At} is a semigroup of contractions, from (4.8) and (4.27), we find

$$\begin{aligned} \|y^l\|_H &= \|L_{a_{l-1},a_{l-1}+T_l}(z^l, K_{T_l,\sigma_l}(z^l))\|_H \\ &\le \|e^{AT_l} z^l\|_H + \left\| \int_{a_{l-1}}^{a_{l-1}+T_l} e^{A(T_l-s)} B g_l(s)\, ds \right\|_H \\ &\le \|z^l\|_H + \|g_l\|_{L^2((a_{l-1},a_{l-1}+T_l)\times\omega)} \\ &\le \left(1 + \sqrt{C_{T_l}} e^{C\sqrt{\mu_{\sigma_l}}}\right) \|z^l\|_H. \end{aligned} \tag{4.29}$$

From (4.28) and (4.29) we have

$$\|z^{l+1}\|_H \le e^{\operatorname{Re}\lambda^2_{\sigma_l+1} T_l} \left(1 + \sqrt{C_{T_l}} e^{C\sqrt{\mu_{\sigma_l}}}\right) \|z^l\|_H. \tag{4.30}$$

Recalling that

$$\sigma_l = 2^l, \quad T_l = k2^{-\rho l} = k\sigma_l^{-\rho}, \quad C_{T_l} = \frac{C}{T_l^{10}} \quad \text{for } l \longrightarrow +\infty, \tag{4.31}$$

and by Weyl's formula, $\mu_{\sigma_l} \sim C(\Omega)(\sigma_l)^{2/n} = C(\Omega)\sigma_l$, we deduce that

$$\operatorname{Re}\lambda^2_{\sigma_l+1} \sim -0.2151\mu_{\sigma_l+1} \sim -C\sigma_l, \tag{4.32}$$

for suitable positive constant C. Then

$$\operatorname{Re}\lambda^2_{\sigma_l+1} T_l \sim -C(\sigma_l)^{1-\rho} \tag{4.33}$$

and we find

$$\frac{C\sqrt{\mu_{\sigma_l}}}{\operatorname{Re}\lambda_{\sigma_l+1}^2 T_l} \sim -C\frac{(\sigma_l)^{1/2}}{(\sigma_l)^{1-\rho}} = -C\frac{1}{(\sigma_l)^{1/2-\rho}}. \tag{4.34}$$

Because of $\rho \in (0, 1/2)$, we obtain

$$\sqrt{C_{T_l}}e^{\operatorname{Re}\lambda_{\sigma_l+1}^2 T_l + C\sqrt{\mu_{\sigma_l}}} \le \frac{C}{T_l^5}e^{-C(\sigma_l)^{1-\rho}} = C\sigma_l^{5\rho}e^{-C(\sigma_l)^{1-\rho}}. \tag{4.35}$$

Applying (4.35) to (4.30), we get

$$\begin{aligned}
\|z^{l+1}\|_H &\le C\sigma_l^{5\rho}e^{-C(\sigma_l)^{1-\rho}}\|z^l\|_H \\
&= C2^{5\rho l}e^{-C2^{(1-\rho)l}}\|z^l\|_H \\
&\le C^2 2^{5\rho l}2^{5\rho(l-1)}e^{-C2^{(1-\rho)l}}e^{-C2^{(1-\rho)(l-1)}}\|z^{l-1}\|_H \\
&\le C^l 2^{5\rho(l+1)l/2}e^{-C2^{(1-\rho)l}[1+2^{-(1-\rho)}+\cdots+2^{-(1-\rho)(l-1)}]}\|z^1\|_H \\
&\le C^l 2^{5\rho(l+1)l/2}e^{-C2^{(1-\rho)l}}\|z^0\|_H.
\end{aligned} \tag{4.36}$$

We find that

$$\lim_{l\to+\infty} C^l 2^{5\rho(l+1)l/2}e^{-C2^{(1-\rho)l}} = \lim_{l\to+\infty} e^{l\ln C+((l+1)l/2)\ln 2^{5\rho}-C2^{(1-\rho)l}} = 0, \tag{4.37}$$

since $0 < \rho < 1/2$ and

$$\begin{aligned}
&\lim_{l\to+\infty}\left[l\ln C + \frac{(l+1)l}{2}\ln 2^{5\rho} - C2^{(1-\rho)l}\right] \\
&\quad = \lim_{l\to+\infty} 2^{(1-\rho)l}\left\{\left[l\ln C + \frac{(l+1)l}{2}\ln 2^{5\rho}\right]2^{-(1-\rho)l} - C\right\} = -\infty.
\end{aligned} \tag{4.38}$$

Then, (4.36) shows that

$$\lim_{l\to+\infty}\|z^{l+1}\|_H = 0. \tag{4.39}$$

Finally, denoting by

$$f(t,\cdot) = \begin{cases} 0 & \text{if } 0 \le t < a_0 = \delta \\ g_l(t,\cdot) & \text{if } a_{l-1} \le t < a_{l-1} + T_l,\ l \ge 1 \\ 0 & \text{if } a_{l-1} + T_l \le t < a_{l-1} + 2T_l = a_l,\ l \ge 1 \\ 0 & \text{if } T - \delta \le t \le T, \end{cases} \tag{4.40}$$

the control function, by (4.27) and (4.36) it can be estimated as

$$\begin{aligned}\|f\|_{L^2((0,T)\times\omega)} &\le \sum_{l=1}^{\infty} \|g_l\|_{L^2((a_{l-1},a_{l-1}+T_l)\times\omega)} \\ &\le C\sum_{l=1}^{\infty} T_l^{-5} e^{C\sqrt{\mu_{\sigma_l}}} \|z^l\|_H \\ &\le C\left\{\sum_{l=1}^{\infty} C^{l-1} 2^{5\rho(l+1)l/2} e^{C[2^{l/2}-2^{(1-\rho)(l-1)}]}\right\} \|z^0\|_H.\end{aligned} \tag{4.41}$$

Since the series on the right-hand side converges in view of $0<\rho<1/2$, the control $f\in L^2((0,T)\times\omega)$. □

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