This article investigates the existence of positive periodic solutions for a first-order functional differential equations of the form

\[ y'(t) = -a(t)y(t) + \lambda h(t)f(y(t - \tau(t))), \]

(1)

where \( a = a(t) \), \( h = h(t) \), and \( \tau = \tau(t) \) are continuous \( T \)-periodic functions. We will also assume that \( T > 0, \lambda > 0, f = f(t) \) as well as \( h = h(t) \) are positive, \( \int_0^T a(t) \, dt > 0 \).

Functional differential equations with periodic delays appear in a number of ecological models. In particular, our equation can be interpreted as the standard Malthus population model \( y' = -a(t)y \) subject to perturbation with periodical delay. One important question is whether these equations can support positive periodic solutions. Such questions have been studied extensively by a number of authors (cf. [1, 2, 3, 4, 6, 7] and the references therein). In this paper, we are concerned with the existence and nonexistence of periodic solutions when the parameter \( \lambda \) varies. For this purpose, we call a continuously differentiable and \( T \)-periodic function a periodic solution of (1) associated with \( \lambda^* \) if it satisfies (1) when \( \lambda = \lambda^* \). We show that there exists \( \lambda^* > 0 \) such that (1) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^* \] and does not have any \( T \)-periodic positive solutions for \( \lambda > \lambda^* \). Our technique is based on the well-known upper and lower solutions method (cf. [5]).

We proceed from (1) and obtain

\[ \left[ y(t) \exp \left( \int_0^t a(s) \, ds \right) \right]' = \lambda \exp \left( \int_0^t a(s) \, ds \right) h(t)f(y(t - \tau(t))). \]

(2)
After integration from \( t \) to \( t + T \), we obtain

\[
y(t) = \lambda \int_t^{t+T} G(t,s) h(s) f(y(s - \tau(s))) \, ds,
\]

where

\[
G(t,s) = \frac{\exp \left( \int_t^s a(u) \, du \right)}{\exp \left( \int_0^T a(u) \, du \right) - 1}.
\]

Note that the denominator in \( G(t,s) \) is not zero since we have assumed that \( \int_0^T a(t) \, dt > 0 \).

It is not difficult to check that any \( T \)-periodic function \( y(t) \) that satisfies (3) is also a \( T \)-periodic solution of (1). Note further that

\[
0 < N \equiv \min_{0 \leq s \leq T} G(t,s) \leq G(t,s) \leq \max_{0 \leq s \leq T} G(t,s) \equiv M, \quad t \leq s \leq t + T,
\]

\[
1 \geq \frac{G(t,s)}{\max_{0 \leq s \leq T} G(t,s)} \geq \frac{\min_{0 \leq s \leq T} G(t,s)}{\max_{0 \leq s \leq T} G(t,s)} = \frac{N}{M} > 0.
\]

Now let \( X \) be the set of all real \( T \)-periodic continuous functions, endowed with the usual linear structure as well as the norm

\[
\|y\| = \sup_{0 \leq t \leq T} |y(t)|.
\]

Then \( X \) is a Banach space with cones

\[
\Phi = \{ y(t) \in X : y(t) \geq 0 \},
\]

\[
\Omega = \{ y(t) : y(t) \geq \sigma \|y\|, \ t \in R \},
\]

where \( \sigma = N/M \).

Define a mapping \( F : X \to X \) by

\[
(Fy)(t) = \lambda \int_t^{t+T} G(t,s) h(s) f(y(s - \tau(s))) \, ds.
\]

Then it is easily seen that \( F \) is completely continuous on bounded subsets of \( \Omega \) and for \( y \in \Phi \),

\[
(Fy)(t) \leq \lambda M \int_0^T h(s) f(y(s - \tau(s))) \, ds
\]

so that

\[
(Fy)(t) \geq \lambda N \int_0^T h(s) f(y(s - \tau(s))) \, ds \geq \sigma \|Fy\|.
\]
That is, $F\Phi$ is contained in $\Omega$.

**Lemma 1.** The mapping $F$ maps $\Phi$ into $\Omega$.

**Lemma 2.** Suppose that

$$\lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \quad (11)$$

Let $I$ be a compact subset of $(0, +\infty)$. Then there exists a constant $b_I > 0$ such that $\|u\| < b_I$ for all $\lambda \in I$ and all possible $T$-periodic positive solutions $u$ of (1) associated with $\lambda$.

**Proof.** Suppose to the contrary that there is a sequence $\{u_n\}$ of $T$-periodic positive solutions of (1) associated with $\{\lambda_n\}$ such that $\lambda_n \in I$ for all $n$ and $\|u_n\| \to +\infty$ as $n \to \infty$. Since $u_n \in \Omega$, $\min_{0 \leq t \leq T} u_{n}\left(t\right) \geq \sigma \|u_n\|$.

By (11), we may choose $R_f > 0$ such that $f(u) \geq \eta u$ for all $u \geq R_f$, and there exists $n_0$ such that $\sigma \|u_{n_0}\| \geq R_f$, where $\eta$ satisfies

$$\sigma \eta N \lambda \int_0^T h(s) \, ds > 1. \quad (13)$$

Thus, we have

$$\|u_{n_0}\| \geq u_{n_0}\left(t\right) = \lambda \int_t^{t+T} G(t, s)h(s)f\left(u_{n_0}\left(s-\tau(s)\right)\right) \, ds \geq \sigma \eta N \lambda \int_0^T h(s)\|u_{n_0}\| \, ds > \|u_{n_0}\|. \quad (14)$$

This is a contradiction. The proof is complete. $\Box$

**Lemma 3.** Suppose that

$$f \text{ is nondecreasing on } [0, +\infty) \text{ and } f(0) > 0. \quad (15)$$

Let (1) have a $T$-periodic positive solution $y(t)$ associated with $\bar{\lambda} > 0$. Then (1) also has a positive $T$-periodic solution associated with $\lambda \in (0, \bar{\lambda})$.

**Proof.** In view of (3) and (15), we have

$$y(t) = \bar{\lambda} \int_t^{t+T} G(t, s)h(s)f\left(y(s-\tau(s))\right) \, ds \geq \lambda \int_t^{t+T} G(t, s)h(s)f\left(y(s-\tau(s))\right) \, ds, \quad 0 < \lambda \int_t^{t+T} G(t, s)h(s)f(0) \, ds. \quad (16)$$
Let $\bar{y}_0(t) = y(t)$,
\[
\bar{y}_{k+1}(t) = \lambda \int_t^{t+T} G(t,s) h(s) f\left(\bar{y}_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots, \quad (17)
\]
$y_0(t) = 0$, and
\[
y_{k+1}(t) = \lambda \int_t^{t+T} G(t,s) h(s) f\left(y_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots. \quad (18)
\]
Clearly, we have
\[
\bar{y}_0(t) \geq \bar{y}_1(t) \geq \cdots \geq \bar{y}_k(t) \geq \cdots \geq y_0(t). \quad (19)
\]
If we now let $y(t) = \lim_{k \to \infty} \bar{y}_k(t)$, then $y(t)$ satisfies (3). Clearly, we have
\[
y(t) \geq y(t) = \lambda \int_t^{t+T} G(t,s) h(s) f(0) ds > 0. \quad (20)
\]
This completes our proof. □

**Lemma 4.** Suppose that (11) and (15) hold. Then there exists $\lambda_* > 0$ such that (1) has a $T$-periodic positive solution.

**Proof.** Let
\[
\beta(t) = \int_t^{t+T} G(t,s) h(s) ds, \quad M_f = \max_{0 \leq t \leq T} f(\beta(t - \tau(t))), \quad \lambda_* = \frac{1}{M_f}. \quad (21)
\]
We have
\[
\beta(t) = \int_t^{t+T} G(t,s) h(s) ds \geq \lambda_* \int_t^{t+T} G(t,s) h(s) f(\beta(s - \tau(s))) ds,
\]
\[
0 < \lambda_* \int_t^{t+T} G(t,s) h(s) f(0) ds. \quad (22)
\]
Let $\bar{y}_0(t) = \beta(t)$,
\[
\bar{y}_{k+1}(t) = \lambda_* \int_t^{t+T} G(t,s) h(s) f\left(\bar{y}_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots, \quad (23)
\]
$y_0(t) = 0$, and
\[
y_{k+1}(t) = \lambda_* \int_t^{t+T} G(t,s) h(s) f\left(y_k(s - \tau(s))\right) ds, \quad k = 0, 1, 2, \ldots. \quad (24)$
Clearly, we have
\[
\overline{y}_0(t) \geq \overline{y}_1(t) \geq \cdots \geq \overline{y}_k(t) \geq \cdots \geq \overline{y}_1(t) \geq \overline{y}_0(t). \tag{25}
\]

If we now let \( y(t) = \lim_{k \to \infty} \overline{y}_k(t) \), then \( y(t) \) satisfies (3). Clearly, we have
\[
y(t) \geq y_1(t) = \lambda \int_t^{t+T} G(t,s) h(s) f(0) \, ds > 0. \tag{26}
\]

The proof is complete. \( \square \)

**Theorem 5.** Suppose that (11) and (15) hold. Then there exists \( \lambda^* > 0 \) such that (1) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*] \) and does not have any \( T \)-periodic positive solutions for \( \lambda > \lambda^* \).

**Proof.** Suppose to the contrary that there is a sequence \( \{u_n\} \) of \( T \)-periodic positive solutions of (1) associated with \( \{\lambda_n\} \) such that \( \lim_{n \to \infty} \lambda_n = \infty \). Then either we have \( \|u_n_j\| \to +\infty \) as \( j \to \infty \) or there is \( \tilde{M} > 0 \) such that \( \|u_n\| \leq \tilde{M} \). Assume the former case holds. Note that \( u_n \in \Omega \) and thus
\[
\min_{0 \leq t \leq T} u_n(t) \geq \sigma \|u_n\|. \tag{27}
\]

By (11), we may choose \( R_f > 0 \) and \( \eta_1 > 0 \) such that \( f(u) \geq \eta_1 u \) when \( \sigma u \geq R_f \). On the other hand, there exist \( \{t_n\} \subset [0,T] \) such that \( u_n_j(t_n) = \|u_n\| \) and \( u_n'(t_n) = 0 \) by the periodicity of \( \{u_n_j(t)\} \). In view of (1), we have
\[
a(t_n_j) \|u_n_j\| = a(t_n) u(t_n) = \lambda_n h(t_n) f(u_n_j(t_n - \tau(t_n))) \\
\geq \lambda_n \eta_1 \sigma h(t_n) \|u_n\| \tag{28}
\]

for all large \( j \). That is, we have \( \lambda_n \leq a(t_n)/\eta_1 \sigma h(t_n) \). Note that \( a(t)/h(t) \) is bounded. Thus, we obtain a contradiction.

Next, suppose that the latter case holds. In view of (15), there exists \( \eta_2 > 0 \) such that \( f(0) \geq \eta_2 \tilde{M} \). Then as above, we will obtain
\[
a(t_n) \|u_n\| = a(t_n) u(t_n) = \lambda_n h(t_n) f(u_n(t_n - \tau(t_n))) \\
\geq \lambda_n \eta_2 h(t_n) \tilde{M} \geq \lambda_n \eta_2 h(t_n) \|u_n\| \tag{29}
\]

for all \( n \). A contradiction will again be reached.
Thus, there exists $\lambda^* > 0$ such that (1) has at least one positive $T$-periodic solution for $\lambda \in (0, \lambda^*)$ and no $T$-periodic positive solutions for $\lambda > \lambda^*$.

Finally, we assert that (1) has at least one $T$-periodic positive solution for $\lambda = \lambda^*$. Indeed, let $\{\lambda_n\}$ satisfy $0 < \lambda_1 < \cdots < \lambda_k < \lambda^*$ and $\lim_{k \to \infty} \lambda_k = \lambda^*$. Since $u_n(t)$ is $T$-periodic positive solution of (1) associated with $\lambda_n$ and Lemma 2 implies that the set $\{u_n(t)\}$ of solutions is uniformly bounded in $\Omega$, the sequence $\{u_n(t)\}$ has a subsequence converging to $u(t) \in \Omega$. We can now apply the Lebesgue convergence theorem to show that $u(t)$ is a $T$-periodic positive solution of (1) associated with $\lambda = \lambda^*$. The proof is complete. \[\square\]

**Example 6.** Consider the equation

$$x'(t) + a(t)x(t) = \lambda h(t)\{x^\gamma(t - \tau(t)) + 1\}, \quad \gamma > 1, \quad (30)$$

where $a$, $h$, and $\tau$ satisfy the same assumptions stated for (1). In view of Theorem 5, there exists a $\lambda^* > 0$ such that (30) has at least one $T$-periodic positive solution for $\lambda \in (0, \lambda^*)$ and no $T$-periodic positive solution for $\lambda > \lambda^*$.

**Example 7.** Consider the equation

$$y'(t) = -ay(t) + \lambda b(y^2(t) + \epsilon), \quad (31)$$

where $a$, $b$, $\epsilon > 0$. Note that the function $f(x) = (x^2 + \epsilon)$ satisfies (11) and (15) in Theorem 5. Therefore Theorem 5 may be applied. However, we may give a direct proof that, for $\lambda > a/(2b\sqrt{\epsilon})$, this equation cannot have any positive $2\pi$-periodic solutions associated with $\lambda$. Indeed, assume to the contrary that $y(t)$ is such a solution. Then $y'(\xi) = 0$ for some $\xi \in [0, 2\pi]$. Hence

$$-ay(\xi) + \lambda b y^2(\xi) + \lambda b \epsilon = 0. \quad (32)$$

However, since the discriminant of the quadratic equation

$$\lambda bx^2 - ax + \lambda b \epsilon = 0 \quad (33)$$

satisfies

$$a^2 - 4\lambda^2 b^2 \epsilon < 0, \quad (34)$$

a contradiction is obtained. We remark that when $\epsilon = 0$, our equation reduces to the well-known logistic equation.

Similarly, we can consider the equation

$$x'(t) = a(t)x(t) - \lambda h(t)f(x(t - \tau(t))), \quad (35)$$
where \( a = a(t) \), \( h = h(t) \), and \( f = f(t) \) satisfy the same assumptions stated for (1).

By (35), we have

\[
  x(t) = \int_t^{t+T} H(t,s) h(s) f(x(s - \tau(s))) \, ds,
\]

where

\[
  H(t,s) = \frac{\exp \left( -\int_t^s a(u) \, du \right)}{1 - \exp \left( -\int_0^T a(u) \, du \right)} = \frac{\exp \left( \int_t^{t+T} a(u) \, du \right)}{\exp \left( \int_0^T a(u) \, du - 1 \right)}
\]

which satisfies

\[
  M \geq H(t,s) \geq N, \quad t \leq s \leq t + T,
\]

for some \( M \) and \( N > 0 \), and \( \sigma = N/M \leq 1 \).

**Theorem 8.** Suppose that (11) and (15) hold. Then there exists \( \lambda^* > 0 \) such that (35) has at least one positive \( T \)-periodic solution for \( \lambda \in (0, \lambda^*] \) and no \( T \)-periodic positive solution for \( \lambda > \lambda^* \).

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**References**


286 Bifurcation in functional differential equations


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