EXISTENCE OF ENTROPY SOLUTIONS FOR SOME NONLINEAR PROBLEMS IN ORLICZ SPACES

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We study in the framework of Orlicz Sobolev spaces $W_0^1 L_M(\Omega)$, the existence of entropic solutions to the nonlinear elliptic problems: $-\operatorname{div} a(x, u, \nabla u) + \operatorname{div} \phi(u) = f$ in Ω , for the case where the second member of the equation $f \in L^1(\Omega)$, and $\phi \in (C^0(\mathbb{R}))^N$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $A(u) = -\operatorname{div} a(x, u, \nabla u)$ be a Leray-Lions operator defined on $W_0^{1,p}(\Omega)$, $1 < p < \infty$.

We consider the nonlinear elliptic problem

$$\begin{aligned} -\operatorname{div} a(x, u, \nabla u) &= f - \operatorname{div} \phi(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where

$$f \in L^1(\Omega), \quad \phi \in (C^0(\mathbb{R}))^N. \tag{1.2}$$

Note that no growth hypothesis is assumed on the function ϕ , which implies that the term $\operatorname{div} \phi(u)$ may be meaningless, even as a distribution. The notion of entropy solution, used in [8], allows us to give a meaning to a possible solution of (1.1).

In fact Boccardo proved in [8], for p such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution u of problem (1.1), that is,

$$\begin{aligned} u &\in W_0^{1,q}(\Omega), && q < \tilde{p} = \frac{(p-1)N}{N-1}, \\ T_k(u) &\in W_0^{1,p}(\Omega), && \forall k > 0, \end{aligned}$$

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi] dx \leq \int_{\Omega} f T_k[u - \varphi] dx + \int_{\Omega} \phi(u) \nabla T_k[u - \varphi] dx$$

$$\forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad (1.3)$$

where

$$T_k(s) = s \quad \text{if } |s| \leq k \quad T_k(s) = k \frac{s}{|s|} \quad \text{if } |s| > k. \quad (1.4)$$

For the case $\phi = 0$ and f is a bounded measure, Benilan et al. proved in [3] the existence and uniqueness of entropy solutions.

We mention as a parallel development, the work of Lions and Murat [14] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [10] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution.

The functional setting in these works is that of the usual Sobolev space $W^{1,p}$. Accordingly, the function a is supposed to satisfy polynomial growth conditions with respect to u and its derivatives ∇u . When trying to generalize the growth condition on a , one is led to replace $W^{1,p}$ by a Sobolev space $W^1 L_M$ built from an Orlicz space L_M instead of L^p . Here the N -function M which defines L_M is related to the actual growth of the function a .

It is our purpose, in this paper, to prove the existence of entropy solution for problem (1.1) in the setting of the Orlicz Sobolev space $W_0^1 L_M(\Omega)$. Our result, Theorem 3.5, generalizes [8, Theorem 2.1] and gives in particular a refinement of his result (see Remark 3.6).

For some existence results for strongly nonlinear elliptic equations in Orlicz spaces [4, 5, 6].

2. Preliminaries

2.1. Let $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N -function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \rightarrow 0$ as $t \rightarrow 0$ and $M(t)/t \rightarrow \infty$ as $t \rightarrow \infty$.

Equivalently, M admits the representation $M(t) = \int_0^t a(\tau) d\tau$, where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \rightarrow \infty$ as $t \rightarrow \infty$.

In the following, we assume, for convenience, that all N -functions are twice continuously differentiable, see Benkirane and Gossez [7].

The N -function $\bar{M}$ conjugate to M is defined by $\bar{M}(t) = \int_0^t \bar{a}(\tau) d\tau$, where $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$, see [1, 13].

The N -function M is said to satisfy the Δ_2 -condition (resp., near infinity) if for some k and for every $t \geq 0$,

$$M(2t) \leq kM(t) \quad (\text{resp., for } t \geq \text{some } t_0). \quad (2.1)$$

Let M and P be two N -functions. The notation $P \ll M$ means that P grows essentially less rapidly than M , that is, for each $\epsilon > 0$, $P(t)/M(\epsilon t) \rightarrow 0$ as $t \rightarrow \infty$. This is the case if and only if $\lim_{t \rightarrow \infty} M^{-1}(t)/P^{-1}(t) = 0$. We will extend all N -functions into even functions on all $\mathbb{R}$.

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_{\Omega} M(u(x)) dx < \infty \quad (2.2)$$

(resp., $\int_{\Omega} M(u(x)/\lambda) dx < \infty$ for some $\lambda > 0$). The space $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) dx \leq 1 \right\} \quad (2.3)$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinity measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x) dx$, and the dual norm on $L_{\bar{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\bar{M}}$. We say that u_n converges to u for the modular convergence in $L_M(\Omega)$ if for some $\lambda > 0$

$$\int_{\Omega} M\left(\frac{|u_n - u|}{\lambda}\right) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.4)$$

If M satisfies the Δ_2 -condition, then the modular convergence coincide with the norm convergence.

2.3. The Orlicz Sobolev space $W^1 L_M(\Omega)$ (resp., $W^1 E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order one lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^{\alpha} u\|_M. \quad (2.5)$$

Thus, $W^1 L_M(\Omega)$ and $W^1 E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\bar{M}})$ and $\sigma(\prod L_M, \prod L_{\bar{M}})$.

The space $W_0^1 E_M(\Omega)$ is defined as the norm closure of $\mathcal{D}(\Omega)$ in $W^1 E_M(\Omega)$ and the space $W_0^1 L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\bar{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1 L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1 L_M(\Omega)$ if for some $\lambda > 0$

$$\int_{\Omega} M\left(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\right) dx \longrightarrow 0 \quad \forall |\alpha| \leq 1. \quad (2.6)$$

This implies the convergence $\sigma(\prod L_M, \prod L_{\tilde{M}})$.

2.4. Let $W^{-1} L_{\tilde{M}}(\Omega)$ (resp., $W^{-1} E_{\tilde{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\tilde{M}}(\Omega)$ (resp., $E_{\tilde{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\prod L_M, \prod L_{\tilde{M}})$. Consequently, the action of a distribution in $W^{-1} L_{\tilde{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined.

2.5. We recall the following lemmas.

LEMMA 2.1 (see [5]). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P , and Q be N -functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^N$ be a Carathéodory function such that*

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|) \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}, \quad (2.7)$$

where $k_1, k_2 \in \mathbb{R}_+$, $c(x) \in E_Q(\Omega)$. Let N_f be the Nemytskii operator defined from $P(E_M(\Omega), 1/k_2) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2\}$ to $(E_Q(\Omega))^N$ by $N_f(u)(x) = f(x, u(x))$. Then N_f is strongly continuous.

LEMMA 2.2 (see [5]). *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N -function and let $u \in W_0^1 L_M(\Omega)$ (resp., $W_0^1 E_M(\Omega)$). Then $F(u) \in W_0^1 L_M(\Omega)$ (resp., $W_0^1 E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases} \quad (2.8)$$

Then $F : W_0^1 L_M(\Omega) \rightarrow W_0^1 L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\tilde{M}})$.

LEMMA 2.3 (see [11]). *Let Ω have the segment property. Then for each $v \in W_0^1 L_M(\Omega)$, there exists a sequence $v_n \in \mathcal{D}(\Omega)$ such that v_n converges to v for the modular convergence in $W_0^1 L_M(\Omega)$. Furthermore, if $v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ then*

$$\|v_n\|_{L^\infty(\Omega)} \leq (N+1)\|v\|_{L^\infty(\Omega)}. \quad (2.9)$$

2.6. We introduce the following notation, see [2, 15].

Definition 2.4. Let M be an N -function, and define the following set:

$$\mathcal{A}_M = \left\{ Q : Q \text{ is an } N\text{-function such that } \frac{Q''}{Q'} \leq \frac{M''}{M'}, \right. \\ \left. \int_0^1 Q \circ H^{-1} \left(\frac{1}{r^{1-1/N}} \right) dr < \infty \text{ where } H(r) = \frac{M(r)}{r} \right\}. \quad (2.10)$$

Remark 2.5. Let $M(t) = t^p$ and $Q(t) = t^q$, then the condition $Q \in \mathcal{A}_M$ is equivalent to the following conditions:

- (i) $2 - 1/N < p < N$
- (ii) $q < \tilde{p} = (p-1)N/(N-1)$, see (1).

Remark 2.6. We can give some examples of N -functions M for which the set $\mathcal{A}_M$ is not empty. Here, the N -functions M are defined only at infinity.

(1) For $M(t) = t^2 \log t$ and $Q(t) = t \log t$, we have $H(t) = t \log t$ and $H^{-1}(t) = t(\log t)^{-1}$ at infinity, see [13]. Then the N -function Q belongs to $\mathcal{A}_M$.

(2) For $M(t) = t^2 \log^2 t$ at infinity and $Q(t) = t \log^2 t$, we have $H(t) = t \log^2 t$ and $H^{-1}(t) = t(\log t)^{-2}$ at infinity, see [13]. Then the N -function Q belongs to $\mathcal{A}_M$.

3. Definition and existence of entropy solutions

Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. Let M, P be two N -functions such that $P \ll M$.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \rightarrow W^{-1} L_{\tilde{M}}(\Omega)$ be a mapping (not defined everywhere) given by $A(u) = -\operatorname{div} a(x, u, \nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, $\xi, \tilde{\xi}$ with $\xi \neq \tilde{\xi}$,

$$|a(x, t, \xi)| \leq d(x) + k_1 \bar{P}^{-1} M(k_2 |t|) + k_3 \bar{M}^{-1} M(k_4 |\xi|), \quad (3.1)$$

$$[a(x, t, \xi) - a(x, t, \tilde{\xi})] [\xi - \tilde{\xi}] > 0, \quad (3.2)$$

$$a(x, t, \xi) \xi \geq \alpha M \left(\frac{|\xi|}{\lambda} \right), \quad (3.3)$$

where $d(x) \in E_{\tilde{M}}(\Omega)$, $d \geq 0$, $\alpha, \lambda \in \mathbb{R}_+^*$, $k_1, k_2, k_3, k_4 \in \mathbb{R}_+$.

Consider the nonlinear elliptic problem (1.1) where

$$f \in L^1(\Omega) \quad (3.4)$$

and $\phi = (\phi_1, \dots, \phi_N)$ satisfies

$$\phi \in (C^0(\mathbb{R}))^N. \quad (3.5)$$

As in [8], we define the following notion of an entropy solution, which gives a meaning to a possible solution of (1.1).

Definition 3.1. Assume that (3.1), (3.2), (3.3), (3.4), and (3.5) hold true and $\mathcal{A}_M \neq \emptyset$. A function u is an entropy solution of problem (1.1) if

$$\begin{aligned} u &\in W_0^1 L_Q(\Omega) \quad \forall Q \in \mathcal{A}_M, \\ T_k(u) &\in W_0^1 L_M(\Omega) \quad \forall k > 0, \\ \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi] dx &\leq \int_{\Omega} f T_k[u - \varphi] dx + \int_{\Omega} \phi(u) \nabla T_k[u - \varphi] dx \\ &\quad \forall \varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega). \end{aligned} \quad (3.6)$$

We cannot use the solution u as a test function in (1.1), because u does not belong to $W_0^1 L_M(\Omega)$. An important role is played by $T_k(u)$ and the test functions

$$T_k[u - \varphi], \quad \varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega) \quad (3.7)$$

because both belong to $W_0^1 L_M(\Omega)$.

In Theorem 3.5, we prove the existence of solution of problem (1.1), in the framework of entropy solutions.

LEMMA 3.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. If $u \in (W_0^1 L_M(\Omega))^N$ then $\int_{\Omega} \operatorname{div} u dx = 0$.*

Proof of Lemma 3.2. It is sufficient to use an approximation of u . □

We recall the following lemma (see [15, Lemma 2]).

LEMMA 3.3. *Let M be an N -function, $u \in W^1 L_M(\Omega)$ such that $\int_{\Omega} M(|\nabla u|) dx < \infty$, then*

$$\begin{aligned} -\mu'(t) &\geq N C_N^{1/N} \mu^{1-1/N}(t) \\ &\quad \times C \left(\frac{-1}{N C_N^{1/N} \mu^{1-1/N}(t)} \frac{d}{dt} \int_{\{|u|>t\}} M(|\nabla u|) dx \right) \quad \forall t > 0, \end{aligned} \quad (3.8)$$

where C is the function defined as

$$C(t) = \frac{1}{\sup \{r \geq 0, H(r) \leq t\}}, \quad H(r) = \frac{M(r)}{r}. \quad (3.9)$$

The function C_N is the measure of the unit ball of $\mathbb{R}^N$, and $\mu(t) = \operatorname{meas}\{|u| > t\}$.

LEMMA 3.4. *Let (X, τ, μ) be a measurable set such that $\mu(X) < \infty$. Let γ be a measurable function $\gamma : X \rightarrow [0, \infty)$ such that*

$$\mu(\{x \in X : \gamma(x) = 0\}) = 0, \quad (3.10)$$

then for each $\epsilon > 0$, there exists $\delta > 0$ such that $\int_A \gamma(x) dx < \delta$ implies

$$\mu(A) \leq \epsilon. \quad (3.11)$$

THEOREM 3.5. *Under assumptions (3.1), (3.2), (3.3), (3.4), and (3.5), with $\mathcal{A}_M \neq \emptyset$, there exists an entropy solution u of problem (1.1) (in the sense of Definition 3.1).*

Remark 3.6. In the case $M(t) = t^p$, Theorem 3.5 gives a refinement of the regularity result (1) (i.e., $u \in W_0^{1,q}(\Omega)$, $q < \tilde{p} = ((p-1)N/(N-1))$). In fact, by Theorem 3.5, we have $u \in W_0^1 L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$ (for example for $Q(t) = t^{\tilde{p}}/\log^\alpha(e+t)$, $\alpha > 1$).

Proof of Theorem 3.5

Step 1. Define, for each $n > 0$, the approximations

$$\phi_n(s) = \phi(T_n(s)), \quad f_n(s) = T_n[f(s)]. \quad (3.12)$$

Consider the nonlinear elliptic problem

$$u_n \in W_0^1 L_M(\Omega), \quad -\operatorname{div} a(x, u_n, \nabla u_n) = f_n - \operatorname{div} \phi_n(u_n) \quad \text{in } \Omega. \quad (3.13)$$

From Gossez and Mustonen [12, Proposition 1, Remark 2], problem (3.13) has at least one solution.

Step 2. We will prove that (u_n) is bounded in $W_0^1 L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Let φ be the truncation defined, for each $t, h > 0$, by

$$\varphi(\xi) = \begin{cases} 0 & \text{if } 0 \leq \xi \leq t, \\ \frac{1}{h}(\xi - t) & \text{if } t < \xi < t+h, \\ 1 & \text{if } \xi \geq t+h, \\ -\varphi(-\xi) & \text{if } \xi < 0. \end{cases} \quad (3.14)$$

Using the test function $v = \varphi(u_n)$ in (3.13) ($v \in W_0^1 L_M(\Omega)$ by Lemma 2.2), we have

$$\int_{\Omega} a(x, u_n, \nabla u_n) \varphi'(u_n) \nabla u_n dx = \int_{\Omega} f_n \varphi(u_n) dx + \int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) dx. \quad (3.15)$$

We claim now that

$$\int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) dx = 0. \quad (3.16)$$

Indeed,

$$\nabla \varphi(u_n) = \varphi'(u_n) \nabla u_n, \quad (3.17)$$

where

$$\varphi'(\xi) = \begin{cases} \frac{1}{h} & \text{if } t < |\xi| < t+h, \\ 0 & \text{otherwise,} \end{cases} \quad (3.18)$$

define $\theta(s) = \phi_n(s)(1/h)\chi_{\{t < |s| < t+h\}}$, and $\tilde{\theta}(s) = \int_0^s \theta(\tau) d\tau$, we have by [Lemma 2.2](#), $\tilde{\theta}(u_n) \in (W_0^1 L_M(\Omega))^N$, which implies

$$\begin{aligned} \int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) dx &= \int_{\Omega} \phi_n(u_n) \frac{1}{h} \chi_{\{t < |u_n| < t+h\}} \nabla u_n dx = \int_{\Omega} \theta(u_n) \nabla u_n dx \\ &= \int_{\Omega} \operatorname{div}(\tilde{\theta}(u_n)) dx = 0 \quad (\text{see } \text{Lemma 3.2}). \end{aligned} \quad (3.19)$$

This proves (3.16). By (3.3) and (3.15), we have (where we can suppose without loss of generality that $\lambda = 1$, since one can take $u'_n = u_n/\lambda$)

$$\frac{\alpha}{h} \int_{t < |u_n| < t+h} M(|\nabla u_n|) dx \leq \|f\|_{1,\Omega}. \quad (3.20)$$

Let $h \rightarrow 0$, then

$$-\frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) dx \leq C \quad \text{with } C = \frac{\|f\|_{1,\Omega}}{\alpha}. \quad (3.21)$$

We prove the following inequality, which allows us to obtain the boundedness of (u_n) in $W_0^1 L_Q(\Omega)$,

$$\begin{aligned} &-\frac{d}{dt} \int_{|u_n| > t} Q(|\nabla u_n|) dx \\ &\leq -\mu'_n(t) Q \circ H^{-1} \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) dx \right). \end{aligned} \quad (3.22)$$

Indeed, let $C(s) = 1/H^{-1}(s)$, where $H(r) = M(r)/r$ and $H^{-1}(s) = \sup\{r \geq 0, H(r) \leq s\}$. Then

$$C(s) = \frac{s}{M \circ H^{-1}(s)}. \quad (3.23)$$

By [Lemma 3.3](#) we have, with $\mu_n(t) = \operatorname{meas}\{|u_n| > t\}$,

$$\begin{aligned} -\mu'_n(t) &\geq NC_N^{1/N} \mu_n(t)^{1-1/N} \\ &\times C \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n| > t} M(|\nabla u_n|) dx \right), \end{aligned} \quad (3.24)$$

then

$$\begin{aligned} &-\mu'_n(t) \cdot M \circ H^{-1} \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n| > t} M(|\nabla u_n|) dx \right) \\ &\geq NC_N^{1/N} \mu_n(t)^{1-1/N} \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n| > t} M(|\nabla u_n|) dx \right), \end{aligned} \quad (3.25)$$

and also

$$\begin{aligned} & \frac{1}{\mu'_n(t)} \frac{d}{dt} \int_{\{|u_n|>t\}} M(|\nabla u_n|) dx \\ & \leq M \circ H^{-1} \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n|>t\}} M(|\nabla u_n|) dx \right) \end{aligned} \quad (3.26)$$

which gives

$$\begin{aligned} & M^{-1} \left(\frac{1}{\mu'_n(t)} \frac{d}{dt} \int_{\{|u_n|>t\}} M(|\nabla u_n|) dx \right) \\ & \leq H^{-1} \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n|>t\}} M(|\nabla u_n|) dx \right). \end{aligned} \quad (3.27)$$

Let $Q \in \mathcal{A}_M$ and let $D(s) = M(Q^{-1}(s))$, D is then convex, and the Jensen's inequality gives

$$D \left(\frac{\int_{\{t < |u_n| < t+h\}} Q(|\nabla u_n|) dx}{-\mu_n(t+h) + \mu_n(t)} \right) \leq \frac{\int_{\{t < |u_n| < t+h\}} M(|\nabla u_n|) dx}{-\mu_n(t+h) + \mu_n(t)}, \quad (3.28)$$

then

$$\begin{aligned} & Q^{-1} \left(\frac{1}{\mu'_n(t)} \frac{d}{dt} \int_{\{|u_n|>t\}} Q(|\nabla u_n|) dx \right) \\ & \leq M^{-1} \left(\frac{1}{\mu'_n(t)} \frac{d}{dt} \int_{\{|u_n|>t\}} M(|\nabla u_n|) dx \right) \\ & \leq H^{-1} \left(-\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n|>t\}} M(|\nabla u_n|) dx \right) \end{aligned} \quad (3.29)$$

which gives (3.22). By (3.21) and (3.22) and since the function

$$t \longrightarrow \int_{\{|u_n|>t\}} Q(|\nabla u_n|) dx \quad (3.30)$$

is absolutely continuous (see [15]), we have

$$\begin{aligned} \int_{\Omega} Q(|\nabla u_n|) dx &= \int_0^\infty \left(-\frac{d}{dt} \int_{\{|u_n|>t\}} Q(|\nabla u_n|) dx \right) dt \\ &\leq \int_0^\infty -\mu'_n(t) Q \circ H^{-1} \left(\frac{C}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \right) dt \\ &\leq \frac{1}{C'} \int_0^{C' \cdot \text{meas}(\Omega)} Q \circ H^{-1} \left(\frac{1}{r^{1-1/N}} \right) dr < \infty \end{aligned} \quad (3.31)$$

which implies that (∇u_n) is bounded in $L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Then u_n is bounded in $W_0^1 L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Passing to a subsequence if necessary, we can assume that

$$u_n \rightharpoonup u \quad \text{weakly in } W_0^1 L_Q(\Omega) \text{ for } \sigma\left(\prod L_Q, \prod E_{\bar{Q}}\right), \text{ a.e. in } \Omega. \quad (3.32)$$

Step 3. We prove that $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^1 L_M(\Omega)$ for all $k > 0$. Using the test function $T_k(u_n)$ in (3.13), we obtain

$$\int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n) dx = \int_{\Omega} f_n T_k(u_n) dx + \int_{\Omega} \phi_n(u_n) \nabla T_k(u_n) dx. \quad (3.33)$$

We claim that

$$\int_{\Omega} \phi_n(u_n) \nabla T_k(u_n) dx = 0. \quad (3.34)$$

Indeed, $\nabla T_k(u_n) = \nabla u_n \chi_{\{|u_n| \leq k\}}$, define $\theta(t) = \phi_n(t) \chi_{\{|t| \leq k\}}$, and $\tilde{\theta}(t) = \int_0^t \theta(\tau) d\tau$, we have by Lemma 2.2, $\tilde{\theta}(u_n) \in (W_0^1 L_M(\Omega))^N$,

$$\begin{aligned} \int_{\Omega} \phi_n(u_n) \nabla T_k(u_n) dx &= \int_{\Omega} \phi_n(u_n) \chi_{\{|u_n| \leq k\}} \nabla u_n dx \\ &= \int_{\Omega} \theta(u_n) \nabla u_n dx \\ &= \int_{\Omega} \operatorname{div}(\tilde{\theta}(u_n)) dx = 0 \quad (\text{by Lemma 3.2}) \end{aligned} \quad (3.35)$$

which proves the claim.

On the other hand, (3.33) can be written as

$$\begin{aligned} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n) dx &= \int_{\Omega} a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u_n) dx \\ &= \int_{\Omega} f_n T_k(u_n) dx, \end{aligned} \quad (3.36)$$

which implies, with (3.3), that $\nabla T_k(u_n)$ is bounded in $(L_M(\Omega))^N$, and $T_k(u_n)$ is bounded in $(W_0^1 L_M(\Omega))^N$. Since $u_n \rightarrow u$ a.e. in Ω then $T_k(u_n) \rightarrow T_k(u)$ a.e. in Ω . Then

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma\left(\prod L_M, \prod E_{\bar{M}}\right). \quad (3.37)$$

Step 4. We will prove that $\nabla u_n \rightharpoonup \nabla u$ a.e. in Ω . Let $\lambda > 0$, $\epsilon > 0$. For $B > 1$, $k > 0$, we consider as in [9] for $n, m \in \mathbb{N}$,

$$\begin{aligned} E_1 &= \{|\nabla u_n| > B\} \cup \{|\nabla u_m| > B\} \cup \{|u_n| > B\} \cup \{|u_m| > B\}, \\ E_2 &= \{|u_n - u_m| > k\}, \\ E_3 &= \{|u_n - u_m| \leq k, |u_n| \leq B, |u_m| \leq B, |\nabla u_n| \leq B, \\ &\quad |\nabla u_m| \leq B, |\nabla u_n - \nabla u_m| \geq \lambda\}, \end{aligned} \quad (3.38)$$

we have $\{|\nabla u_n - \nabla u_m| \geq \lambda\} \subset E_1 \cup E_2 \cup E_3$.

Since (u_n) and (∇u_n) are bounded in $L^1(\Omega)$ (since u_n is bounded in $W_0^1 L_Q(\Omega)$), we have

$$2B\mu(E_1) < \int_{E_1} |\nabla u_n| + |u_n| dx < \int_{\Omega} |\nabla u_n| + |u_n| dx < C. \quad (3.39)$$

Then $\text{meas} E_1 \leq \epsilon$ for B sufficiently large enough, independently of n, m . Thus we fix B in order to have

$$\text{meas} E_1 \leq \epsilon. \quad (3.40)$$

Now we claim that $\text{meas} E_3 \leq \epsilon$ for n and m large. Let C_1 be such that $\|u_n\|_1 \leq C_1$ and $\|\nabla u_n\|_1 \leq C_1$. As in [9], the assumption (3.2) gives the existence of a measurable function $\gamma(x)$ such that

$$\begin{aligned} \text{meas}(\{x \in \Omega : \gamma(x) = 0\}) &= 0, \\ [a(x, t, \xi) - a(x, t, \bar{\xi})][\xi - \bar{\xi}] &\geq \gamma(x) > 0, \end{aligned} \quad (3.41)$$

for all $t \in \mathbb{R}$, $\xi, \bar{\xi} \in \mathbb{R}^N$, $|t|, |\xi|, |\bar{\xi}| \leq B$, $|\xi - \bar{\xi}| \geq \lambda$ a.e. in Ω . We have

$$\begin{aligned} \int_{E_3} \gamma(x) dx &\leq \int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u_m)] [\nabla u_n - \nabla u_m] dx \\ &\leq \int_{E_3} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] [\nabla u_n - \nabla u_m] dx \\ &\quad + \int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m)] [\nabla u_n - \nabla u_m] dx. \end{aligned} \quad (3.42)$$

Using the test function $T_k(u_n - u_m)$ in (3.13) and integrating on E_3 , we obtain

$$\begin{aligned} &\int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m)] \nabla T_k(u_n - u_m) dx \\ &= \int_{E_3} (f_n - f_m) T_k(u_n - u_m) dx \\ &\quad + \int_{E_3} [\phi_n(u_n) - \phi_m(u_m)] \nabla T_k(u_n - u_m) dx, \end{aligned} \quad (3.43)$$

with

$$\begin{aligned} &\int_{E_3} [\phi_n(u_n) - \phi_m(u_m)] \nabla T_k(u_n - u_m) dx \\ &\leq 2B \int_{E_3} |\phi_n(u_n) - \phi_m(u_m)| dx \\ &\leq 2B \int_{E_3} [|\phi(T_n(u_n)) - \phi(u_n)| + |\phi(u_n) - \phi(u_m)| \\ &\quad + |\phi(u_m) - \phi(T_m(u_m))|] dx. \end{aligned} \quad (3.44)$$

Let $n_0 \geq B$, then for $n, m \geq n_0$ we have $T_n(u_n) = u_n$ and $T_m(u_m) = u_m$ on E_3 , which implies that the first and the third integral of the last inequality vanish. The second integral of (3.42) is bounded for $n, m \geq n_0$ by

$$2k\|f\|_{1,\Omega} + 2B \int_{E_3} |\phi(u_n) - \phi(u_m)| dx. \quad (3.45)$$

For a.e. $x \in \Omega$ and $\epsilon_1 > 0$ there exist $\eta(x) \geq 0$ ($\text{meas}\{x \in \Omega : \eta(x) = 0\} = 0$) such that $|s - s'| \leq \eta(x)$, $|s|, |s'|, |\xi| \leq B$ implies

$$|a(x, s, \xi) - a(x, s', \xi)| \leq \epsilon_1. \quad (3.46)$$

We use now the continuity of ϕ , to obtain for a.e. $x \in \Omega$ and $\epsilon_2 > 0$, $\eta_1(x) \geq 0$ ($\text{meas}\{x \in \Omega : \eta_1(x) = 0\} = 0$) such that $|s - s'| \leq \eta_1(x)$, $|s|, |s'| \leq B$ implies

$$|\phi(s) - \phi(s')| \leq \epsilon_2. \quad (3.47)$$

Then

$$\begin{aligned} \int_{E_3} \gamma(x) dx &\leq \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] \\ &\quad \times [\nabla u_n - \nabla u_m] dx \\ &\quad + \int_{E_3 \cap \{x \in \Omega : \eta(x) \geq k\}} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] \\ &\quad \times [\nabla u_n - \nabla u_m] dx \\ &\quad + 2k\|f\|_{1,\Omega} + 2B \int_{E_3 \cap \{x \in \Omega : \eta_1(x) < k\}} |\phi(u_n) - \phi(u_m)| dx \\ &\quad + \int_{E_3 \cap \{x \in \Omega : \eta_1(x) \geq k\}} |\phi(u_n) - \phi(u_m)| dx. \end{aligned} \quad (3.48)$$

By using for the first integral the definition of E_3 and condition (3.1), for the second integral the definition of E_3 and (3.46), for the fourth integral the definition of E_3 and $|\phi(u_n)| \leq C(B)$ (since $|u_n| \leq B$ and ϕ continuous), and for the last integral the definition of E_3 and (3.47), we obtain

$$\begin{aligned} \int_{E_3} \gamma(x) dx &\leq C'(B) \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} [1 + d(x)] dx + 2C_1(B)\epsilon_1 \\ &\quad + 2k\|f\|_{1,\Omega} + 2C(B) \text{meas}\{x \in \Omega : \eta_1(x) < k\} + C_2\epsilon_2. \end{aligned} \quad (3.49)$$

We have $\text{meas}\{x \in \Omega : \eta(x) < k\} \rightarrow 0$ when $k \rightarrow 0$, and $\text{meas}\{x \in \Omega : \eta_1(x) < k\} \rightarrow 0$ when $k \rightarrow 0$. Let $\epsilon > 0$ and let δ be the real, in Lemma 3.4, corresponding to ϵ , we choose ϵ_1, ϵ_2 such that

$$2C_1(B)\epsilon_1 \leq \frac{\delta}{5}, \quad C_2\epsilon_2 \leq \frac{\delta}{5}, \quad (3.50)$$

and k such that

$$\begin{aligned} C'(B) \int_{E_3 \cap \{x \in \Omega : \eta_1(x) < k\}} [1 + d(x)] dx &< \frac{\delta}{5}, \quad 2k \|f\|_{1,\Omega} \leq \frac{\delta}{5}, \\ 2C(B) \text{meas} \{x \in \Omega : \eta_1(x) < k\} &< \frac{\delta}{5}. \end{aligned} \quad (3.51)$$

Then $\int_{E_3} \gamma(x) dx < \delta$ and [Lemma 3.4](#) implies that

$$\text{meas } E_3 < \epsilon \quad \forall n, m \geq n_0. \quad (3.52)$$

This completes the proof of the claim.

Let the last k be fixed, u_n a Cauchy sequence in measure, we choose n_1 such that

$$\text{meas } E_2 \leq \epsilon \quad \forall n, m \geq n_1. \quad (3.53)$$

Then

$$\text{meas} \{x \in \Omega : |\nabla u_n - \nabla u_m| \geq \lambda\} \leq \epsilon \quad \forall n, m \geq \max(n_1, n_0) \quad (3.54)$$

and $\nabla u_n \rightarrow \nabla u$ in measure, consequently

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega. \quad (3.55)$$

Step 5. Let $\varphi \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$. From [Lemma 2.3](#), there exists a sequence $(\varphi_j) \in \mathcal{D}(\Omega)$ such that φ_j converges to φ for the modular convergence in $W_0^1 L_M(\Omega)$ with

$$\|\varphi_j\|_{L^\infty(\Omega)} \leq (N+1)\|\varphi\|_{L^\infty(\Omega)}. \quad (3.56)$$

Using $T_k[u_n - \varphi_j]$ as a test function in [\(3.13\)](#) we obtain

$$\begin{aligned} &\int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] dx \\ &= \int_{\Omega} f_n T_k[u_n - \varphi_j] dx + \int_{\Omega} \phi_n(u_n) \nabla T_k[u_n - \varphi_j] dx \end{aligned} \quad (3.57)$$

which gives, if $n \rightarrow \infty$,

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] dx \\ &\geq \liminf_{n \rightarrow \infty} \int_{\Omega} [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla \varphi_j)] \nabla T_k[u_n - \varphi_j] dx \\ &\quad + \lim_{n \rightarrow \infty} \int_{\Omega} a(x, T_{k+\|\varphi_j\|_{L^\infty(\Omega)}}(u_n), \nabla \varphi_j) \nabla T_k[u_n - \varphi_j] dx \\ &\geq \int_{\Omega} [a(x, u, \nabla u) - a(x, u, \nabla \varphi_j)] \nabla T_k[u - \varphi_j] dx \\ &\quad + \int_{\Omega} a(x, u, \nabla \varphi_j) \nabla T_k[u - \varphi_j] dx, \end{aligned} \quad (3.58)$$

where we have used Fatou lemma for the first integral, and for the second the convergences $\nabla T_k[u_n - \varphi_j] \rightarrow \nabla T_k[u - \varphi_j]$ by (3.37) in $(L_M(\Omega))^N$ for $\sigma(\prod L_M, \prod E_{\tilde{M}})$ and $a(x, T_{k+\|\varphi_j\|_{L^\infty(\Omega)}}(u_n), \nabla \varphi_j) \rightarrow a(x, T_{k+\|\varphi_j\|_{L^\infty(\Omega)}}(u), \nabla \varphi_j)$ strongly in $(E_{\tilde{M}}(\Omega))^N$ by (3.1), which implies that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] dx \geq \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] dx. \quad (3.59)$$

For $n \geq k + (N+1)\|\varphi\|_{L^\infty(\Omega)}$,

$$\begin{aligned} \int_{\Omega} \phi_n(u_n) \nabla T_k[u_n - \varphi_j] dx &= \int_{\Omega} \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u_n)) \nabla T_k[u_n - \varphi_j] dx \\ &\xrightarrow{n \rightarrow \infty} \int_{\Omega} \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)) \nabla T_k[u - \varphi_j] dx, \end{aligned} \quad (3.60)$$

we have used the convergences $\nabla T_k[u_n - \varphi_j] \rightarrow \nabla T_k[u - \varphi_j]$ by (3.37) in $(L_M(\Omega))^N$ and $\phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u_n)) \rightarrow \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u))$ strongly in $(E_{\tilde{M}}(\Omega))^N$ since ϕ is continuous. On the other hand, since $f_n \rightarrow f$ strongly in $L^1(\Omega)$ and $T_k[u_n - \varphi_j] \rightharpoonup T_k[u - \varphi_j]$ weakly* in $L^\infty(\Omega)$, we have

$$\int_{\Omega} f_n T_k[u_n - \varphi_j] dx \rightarrow \int_{\Omega} f T_k[u - \varphi_j] dx. \quad (3.61)$$

Then

$$\begin{aligned} \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] dx &\geq \int_{\Omega} \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)) \nabla T_k[u - \varphi_j] dx \\ &\quad + \int_{\Omega} f T_k[u - \varphi_j] dx. \end{aligned} \quad (3.62)$$

Now, if $j \rightarrow \infty$ in (3.62), we get

$$\begin{aligned} \liminf_{j \rightarrow \infty} \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] dx &\geq \liminf_{j \rightarrow \infty} \int_{\Omega} [a(x, u, \nabla u) - a(x, u, \nabla \varphi_j)] \nabla T_k[u - \varphi_j] dx \\ &\quad + \lim_{j \rightarrow \infty} \int_{\Omega} a(x, u, \nabla \varphi_j) \nabla T_k[u - \varphi_j] dx \\ &\geq \int_{\Omega} [a(x, u, \nabla u) - a(x, u, \nabla \varphi)] \nabla T_k[u - \varphi] dx \\ &\quad + \int_{\Omega} a(x, u, \nabla \varphi) \nabla T_k[u - \varphi] dx, \end{aligned} \quad (3.63)$$

where we have used Fatou lemma for the first integral, and for the second the convergences $\nabla T_k[u - \varphi_j] \rightharpoonup \nabla T_k[u - \varphi]$ in $(L_M(\Omega))^N$ for the modular convergence and $a(x, u, \nabla \varphi_j) \rightarrow a(x, u, \nabla \varphi)$ in $(L_{\tilde{M}}(\Omega))^N$ for the modular convergence,

which implies that

$$\liminf_{j \rightarrow \infty} \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] dx \geq \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi] dx. \quad (3.64)$$

On the other hand, since $\nabla T_k[u - \varphi_j] \rightarrow \nabla T_k[u - \varphi]$ in $(L_M(\Omega))^N$ for the modular convergence, then weakly for $\sigma(\prod L_M, \prod L_{\tilde{M}})$ and $\phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)) \in (L_{\tilde{M}}(\Omega))^N$ we have

$$\begin{aligned} & \int_{\Omega} \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)) \nabla T_k[u - \varphi_j] dx \\ & \xrightarrow{j \rightarrow \infty} \int_{\Omega} \phi(T_{k+(N+1)\|\varphi\|_{L^\infty(\Omega)}}(u)) \nabla T_k[u - \varphi] dx \\ & = \int_{\Omega} \phi(u) \nabla T_k[u - \varphi] dx. \end{aligned} \quad (3.65)$$

Since $f \in L^1(\Omega)$ and $T_k[u - \varphi_j] \rightharpoonup T_k[u - \varphi]$ weakly* in $L^\infty(\Omega)$, we have

$$\int_{\Omega} f T_k[u - \varphi_j] dx \longrightarrow \int_{\Omega} f T_k[u - \varphi] dx. \quad (3.66)$$

Then

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi] dx \geq \int_{\Omega} \phi(u) \nabla T_k[u - \varphi] dx + \int_{\Omega} f T_k[u - \varphi] dx \quad (3.67)$$

and u is an entropy solution of problem (1.1). $\square$

THEOREM 3.7. Suppose, in [Theorem 3.5](#), that the N -function M satisfies, furthermore, the Δ_2 -condition and $f \geq 0$, then the entropy solution u of problem (1.1) satisfies $u \geq 0$.

Proof of Theorem 3.7. Using $\varphi = T_l(u^+)$ as test function in the definition of entropy solution, we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - T_l(u^+)] dx \\ & \leq \int_{\Omega} f T_k[u - T_l(u^+)] dx + \int_{\Omega} \phi(u) \nabla T_k[u - T_l(u^+)] dx. \end{aligned} \quad (3.68)$$

We have

$$\int_{\Omega} f T_k[u - T_l(u^+)] dx \leq \int_{\{u \geq l\}} f T_k[u - T_l(u)] dx. \quad (3.69)$$

Indeed,

$$\begin{aligned} \int_{\Omega} f T_k[u - T_l(u^+)] dx &= \int_{u \geq l} f T_k[u - T_l(u^+)] dx \\ &+ \int_{0 < u < l} f T_k[u - T_l(u^+)] dx \\ &+ \int_{u \leq 0} f T_k[u - T_l(u^+)] dx. \end{aligned} \quad (3.70)$$

If $0 < u < l$ then $u - T_l(u^+) = 0$ and $\int_{0 < u < l} f T_k[u - T_l(u^+)] dx = 0$. If $u \leq 0$ then $u - T_l(u^+) = u$ and $\int_{u \leq 0} f T_k[u - T_l(u^+)] dx \leq 0$ since f is positive. If $u \geq l$ then $u^+ = u$ and $\int_{u \geq l} f T_k[u - T_l(u^+)] dx \leq \int_{u \geq l} f T_k[u - T_l(u)] dx$.

On the other hand, we claim that

$$\int_{\Omega} \phi(u) \nabla T_k[u - T_l(u^+)] dx = 0. \quad (3.71)$$

Indeed, if $0 < u < l$, then $u - T_l(u^+) = 0$, $\int_{0 < u < l} \phi(u) \nabla T_k[u - T_l(u^+)] dx = 0$. If $u \leq 0$, then $u - T_l(u^+) = u$,

$$\begin{aligned} \int_{u \leq 0} \phi(u) \nabla T_k[u - T_l(u^+)] dx &= \int_{-k \leq u \leq 0} \phi(u) \nabla u dx \\ &= \int_{\Omega} \phi(u) \nabla u \chi_{\{-k \leq u \leq 0\}} dx. \end{aligned} \quad (3.72)$$

We verify that the third integral of the last inequality vanishes. For this, define $\theta(t) = \phi(t) \chi_{\{-k \leq t \leq 0\}}$, and $\tilde{\theta}(t) = \int_0^t \theta(\tau) d\tau$ we have, by [Lemma 2.2](#), $\tilde{\theta}(u) \in (W_0^1 L_M(\Omega))^N$ which implies

$$\begin{aligned} \int_{\Omega} \phi(u) \nabla u \chi_{\{-k \leq u \leq 0\}} dx &= \int_{\Omega} \theta(u) \nabla u dx \\ &= \int_{\Omega} \operatorname{div}(\tilde{\theta}(u)) dx = 0 \quad (\text{by } \text{Lemma 3.2}). \end{aligned} \quad (3.73)$$

If $u \geq l$ then $u^+ = u$ and

$$\begin{aligned} \int_{\{u \geq l\}} \phi(u) \nabla T_k[u - T_l(u^+)] dx &= \int_{l \leq u \leq l+k} \phi(u) \nabla u dx \\ &= \int_{\Omega} \phi(u) \nabla u \chi_{\{l \leq u \leq l+k\}} dx. \end{aligned} \quad (3.74)$$

Similarly, we verify that

$$\int_{\Omega} \phi(u) \nabla u \chi_{\{l \leq u \leq l+k\}} dx = 0. \quad (3.75)$$

This completes the proof of the claim which implies that

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - T_l(u^+)] dx \leq \int_{u \geq l} f T_k[u - T_l(u)] dx \quad (3.76)$$

or

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - T_l(u^+)] dx \\ &= \int_{l \leq u \leq l+k} a(x, u, \nabla u) \nabla u dx + \int_{-k \leq u \leq 0} a(x, u, \nabla u) \nabla u dx \\ &\geq \int_{l \leq u \leq l+k} M\left(\frac{|\nabla u|}{\lambda}\right) dx + \int_{-k \leq u \leq 0} M\left(\frac{|\nabla u|}{\lambda}\right) dx, \end{aligned} \quad (3.77)$$

which gives

$$\int_{l \leq u \leq l+k} M\left(\frac{|\nabla u|}{\lambda}\right) dx + \int_{-k \leq u \leq 0} M\left(\frac{|\nabla u|}{\lambda}\right) dx \leq \int_{u \geq l} f T_k[u - T_l(u)] dx. \quad (3.78)$$

Letting $l \rightarrow \infty$ in (3.78) we have

$$\begin{aligned} & \int_{u \geq l} f T_k[u - T_l(u)] dx \rightarrow 0 \quad \text{since } f T_k[2u] \in L^1(\Omega), \\ & \int_{l \leq u \leq l+k} M\left(\frac{|\nabla u|}{\lambda}\right) dx \geq \int_{l \leq u \leq k} M\left(\frac{|\nabla u|}{\lambda}\right) dx \\ &= \int_{l \leq u} M\left(\frac{|\nabla T_k(u)|}{\lambda}\right) dx \\ &\rightarrow 0, \quad \text{when } l \rightarrow \infty, \end{aligned} \quad (3.79)$$

since $M(|\nabla T_k(u)|/\lambda) \in L^1(\Omega)$ and M satisfies the Δ_2 -condition. Then

$$\int_{-k \leq u \leq 0} M\left(\frac{|\nabla u|}{\lambda}\right) dx = 0 \quad \forall k, \quad (3.80)$$

which implies that,

$$\begin{aligned} & \int_{u \leq 0} M\left(\frac{|\nabla u|}{\lambda}\right) dx = \int_{\Omega} M\left(\frac{|\nabla u^-|}{\lambda}\right) dx = 0, \\ & \nabla u^- = 0, \quad u^- = c \quad \text{a.e. in } \Omega. \end{aligned} \quad (3.81)$$

Or $u^- \in W_0^1 L_Q(\Omega)$ then $u^- = 0$ a.e. in Ω which proves that

$$u \geq 0 \quad \text{a.e. in } \Omega. \quad (3.82)$$

□

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