EXISTENCE OF ENTROPY SOLUTIONS FOR SOME NONLINEAR PROBLEMS IN ORLICZ SPACES

A. BENKIRANE AND J. BENNOUNA

Received 20 October 2001

We study in the framework of Orlicz Sobolev spaces $W^{1,0}_{LM}(\Omega)$, the existence of entropic solutions to the nonlinear elliptic problems: $-\text{div} \, a(x, u, \nabla u) + \text{div} \phi(u) = f$ in $\Omega$, for the case where the second member of the equation $f \in L^1(\Omega)$, and $\phi \in (C^0(\mathbb{R}))^N$.

1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and let $A(u) = -\text{div} \, a(x, u, \nabla u)$ be a Leray-Lions operator defined on $W^{1,p}_0(\Omega)$, $1 < p < \infty$.

We consider the nonlinear elliptic problem

$$
-\text{div} \, a(x, u, \nabla u) = f - \text{div} \phi(u) \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega,
$$

where

$$f \in L^1(\Omega), \quad \phi \in (C^0(\mathbb{R}))^N. \quad (1.2)$$

Note that no growth hypothesis is assumed on the function $\phi$, which implies that the term $\text{div} \phi(u)$ may be meaningless, even as a distribution. The notion of entropy solution, used in [8], allows us to give a meaning to a possible solution of (1.1).

In fact Boccardo proved in [8], for $p$ such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution $u$ of problem (1.1), that is,

$$u \in W^{1,q}_0(\Omega), \quad q < \tilde{p} = \frac{(p-1)N}{N-1},$$

$$T_k(u) \in W^{1,p}_0(\Omega), \quad \forall k > 0,$$
\[ \int_{\Omega} a(x,u,\nabla u)\nabla T_k[u-\varphi] \, dx \leq \int_{\Omega} f T_k[u-\varphi] \, dx + \int_{\Omega} \phi(u)\nabla T_k[u-\varphi] \, dx \]
\[ \forall \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \]  
(1.3)

where
\[ T_k(s) = s \quad \text{if } |s| \leq k \]
\[ T_k(s) = k \frac{s}{|s|} \quad \text{if } |s| > k. \]  
(1.4)

For the case \( \phi = 0 \) and \( f \) is a bounded measure, Bénilan et al. proved in [3] the existence and uniqueness of entropy solutions.

We mention as a parallel development, the work of Lions and Murat [14] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [10] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution.

The functional setting in these works is that of the usual Sobolev space \( W^{1,p} \). Accordingly, the function \( a \) is supposed to satisfy polynomial growth conditions with respect to \( u \) and its derivatives \( \nabla u \). When trying to generalize the growth condition on \( a \), one is led to replace \( W^{1,p} \) by a Sobolev space \( W^{1}L_M \) built from an Orlicz space \( L_M \) instead of \( L^p \). Here the \( N \)-function \( M \) which defines \( L_M \) is related to the actual growth of the function \( a \).

It is our purpose, in this paper, to prove the existence of entropy solution for problem (1.1) in the setting of the Orlicz Sobolev space \( W^{1,0}L_M(\Omega) \). Our result, Theorem 3.5, generalizes [8, Theorem 2.1] and gives in particular a refinement of his result (see Remark 3.6).

For some existence results for strongly nonlinear elliptic equations in Orlicz spaces [4, 5, 6].

2. Preliminaries

2.1. Let \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) be an \( N \)-function, that is, \( M \) is continuous, convex, with \( M(t) > 0 \) for \( t > 0 \), \( M(t)/t \to 0 \) as \( t \to 0 \) and \( M(t)/t \to \infty \) as \( t \to \infty \).

Equivalently, \( M \) admits the representation \( M(t) = \int_0^t a(\tau) \, d\tau \), where \( a : \mathbb{R}^+ \to \mathbb{R}^+ \) is nondecreasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \).

In the following, we assume, for convenience, that all \( N \)-functions are twice continuously differentiable, see Benkirane and Gossez [7].

The \( N \)-function \( \bar{M} \) conjugate to \( M \) is defined by \( \bar{M}(t) = \int_0^t \bar{a}(\tau) \, d\tau \), where \( \bar{a} : \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \bar{a}(t) = \sup \{ s : a(s) \leq t \} \), see [1, 13].

The \( N \)-function \( M \) is said to satisfy the \( \Delta_2 \)-condition (resp., near infinity) if for some \( k \) and for every \( t \geq 0 \),
\[ M(2t) \leq kM(t) \quad (\text{resp., for } t \geq \text{some } t_0). \]  
(2.1)
Let $M$ and $P$ be two $N$-functions. The notation $P \ll M$ means that $P$ grows essentially less rapidly than $M$, that is, for each $\epsilon > 0$, $P(t) / M(\epsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t \to \infty} M^{-1}(t)/P^{-1}(t) = 0$. We will extend all $N$-functions into even functions on all $\mathbb{R}$.

2.2. Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions $u$ on $\Omega$ such that

$$\int_{\Omega} M(u(x)) \, dx < \infty$$

(resp., $\int_{\Omega} M(u(x)/\lambda) \, dx < \infty$ for some $\lambda > 0$). The space $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx \leq 1 \right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M$ satisfies the $\Delta_2$ condition, for all $t$ or for $t$ large according to whether $\Omega$ has infinity measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\bar{M}}(\Omega)$ by means of the pairing

$$\int_{\Omega} u(x)v(x) \, dx,$$

and the dual norm on $L_{\bar{M}}(\Omega)$ is equivalent to $\| \cdot \|_{\bar{M}}$. We say that $u_n$ converges to $u$ for the modular convergence in $L_M(\Omega)$ if for some $\lambda > 0$

$$\int_{\Omega} M\left(\frac{|u_n - u|}{\lambda}\right) \, dx \to 0 \quad \text{as} \quad n \to \infty.$$  

(2.4)

If $M$ satisfies the $\Delta_2$-condition, then the modular convergence coincide with the norm convergence.

2.3. The Orlicz Sobolev space $W^{1}L_M(\Omega)$ (resp., $W^{1}E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order one lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M.$$  

(2.5)

Thus, $W^{1}L_M(\Omega)$ and $W^{1}E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_M)$ and $\sigma(\prod L_M, \prod \bar{L}_M)$.

The space $W^{1}_0E_M(\Omega)$ is defined as the norm closure of $\mathcal{D}(\Omega)$ in $W^{1}E_M(\Omega)$ and the space $W^{1}_0L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_M)$ closure of $\mathcal{D}(\Omega)$ in $W^{1}L_M(\Omega)$.
We say that \( u_n \) converges to \( u \) for the modular convergence in \( W^1 L_M(\Omega) \) if for some \( \lambda > 0 \)
\[
\int_\Omega M\left(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\right) \, dx \to 0 \quad \forall |\alpha| \leq 1. \tag{2.6}
\]
This implies the convergence \( \sigma(\prod L_M, \prod L_M) \).

2.4. Let \( W^{-1} L_M(\Omega) \) (resp., \( W^{-1} E_M(\Omega) \)) denote the space of distributions on \( \Omega \) which can be written as sums of derivatives of order \( \leq 1 \) of functions in \( L_M(\Omega) \) (resp., \( E_M(\Omega) \)). It is a Banach space under the usual quotient norm.

If the open set \( \Omega \) has the segment property, then the space \( \mathcal{D}(\Omega) \) is dense in \( W^1_0 L_M(\Omega) \) for the modular convergence and thus for the topology \( \sigma(\prod L_M, \prod L_M) \). Consequently, the action of a distribution in \( W^{-1} L_M(\Omega) \) on an element of \( W^1_0 L_M(\Omega) \) is well defined.

2.5. We recall the following lemmas.

**Lemma 2.1** (see [5]). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with finite measure. Let \( M, P, \) and \( Q \) be \( N \)-functions such that \( Q \ll P \), and let \( f : \Omega \times \mathbb{R} \to \mathbb{R}^N \) be a Carathéodory function such that
\[
|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|) \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R}, \tag{2.7}
\]
where \( k_1, k_2 \in \mathbb{R}_+ \), \( c(x) \in E_Q(\Omega) \). Let \( N_f \) be the Nemitskii operator defined from \( P(E_M(\Omega), 1/k_2) = \{ u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2 \} \) to \( (E_Q(\Omega))^N \) by \( N_f(u)(x) = f(x, u(x)) \). Then \( N_f \) is strongly continuous.

**Lemma 2.2** (see [5]). Let \( F : \mathbb{R} \to \mathbb{R} \) be uniformly Lipschitzian, with \( F(0) = 0 \). Let \( M \) be an \( N \)-function and let \( u \in W^1_0 L_M(\Omega) \) (resp., \( W^1_0 E_M(\Omega) \)). Then \( F(u) \in W^1_0 L_M(\Omega) \) (resp., \( W^1_0 E_M(\Omega) \)). Moreover, if the set \( D \) of discontinuity points of \( F' \) is finite, then
\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \}, \\ 0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}. \end{cases} \tag{2.8}
\]
Then \( F : W^1_0 L_M(\Omega) \to W^1_0 L_M(\Omega) \) is sequentially continuous with respect to the weak* topology \( \sigma(\prod L_M, \prod E_M) \).

**Lemma 2.3** (see [11]). Let \( \Omega \) have the segment property. Then for each \( \nu \in W^1_0 L_M(\Omega) \), there exists a sequence \( \nu_n \in \mathcal{D}(\Omega) \) such that \( \nu_n \) converges to \( \nu \) for the modular convergence in \( W^1_0 L_M(\Omega) \). Furthermore, if \( \nu \in W^1_0 L_M(\Omega) \cap L^\infty(\Omega) \) then
\[
\|\nu_n\|_{L^\infty(\Omega)} \leq (N + 1) \|\nu\|_{L^\infty(\Omega)}. \tag{2.9}
\]
2.6. We introduce the following notation, see [2, 15].

**Definition 2.4.** Let $M$ be an $N$-function, and define the following set:

$$
\mathcal{A}_M = \left\{ Q : Q \text{ is an } N \text{-function such that } \frac{Q''}{Q'} \leq \frac{M''}{M'}, \right. \\
\left. \int_0^1 Q \circ H^{-1}\left(\frac{1}{r^{1-1/N}}\right) dr < \infty \right\}, \quad (2.10)
$$

**Remark 2.5.** Let $M(t) = t^p$ and $Q(t) = t^q$, then the condition $Q \in \mathcal{A}_M$ is equivalent to the following conditions:

(i) $2 - 1/N < p < N$

(ii) $q < \tilde{p} = (p - 1)N/(N - 1)$, see (1).

**Remark 2.6.** We can give some examples of $N$-functions $M$ for which the set $\mathcal{A}_M$ is not empty. Here, the $N$-functions $M$ are defined only at infinity.

(1) For $M(t) = t^2 \log t$ and $Q(t) = t \log t$, we have $H(t) = t \log t$ and $H^{-1}(t) = t/(\log t)$ at infinity, see [13]. Then the $N$-function $Q$ belongs to $\mathcal{A}_M$.

(2) For $M(t) = t^2 \log^2 t$ at infinity and $Q(t) = t \log^2 t$, we have $H(t) = t \log^2 t$ and $H^{-1}(t) = t/(\log t)^2$ at infinity, see [13]. Then the $N$-function $Q$ belongs to $\mathcal{A}_M$.

3. Definition and existence of entropy solutions

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with the segment property. Let $M, P$ be two $N$-functions such that $P \ll M$.

Let $A : D(A) \subset W^{1}_0 L_M(\Omega) \rightarrow W^{-1} L_M(\Omega)$ be a mapping (not defined everywhere) given by $A(u) = -\text{div} a(x, u, \nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, $\xi, \tilde{\xi}$ with $\xi \neq \tilde{\xi}$,

$$
|a(x, t, \xi)| \leq d(x) + k_1 \tilde{p}^{-1} M(k_2 |t|) + k_3 \tilde{M}^{-1} M(k_4 |\xi|), \quad (3.1)
$$

$$
[a(x, t, \xi) - a(x, t, \tilde{\xi})] [\xi - \tilde{\xi}] > 0, \quad (3.2)
$$

$$
a(x, t, \xi) \xi \geq aM\left(\frac{|\xi|}{\lambda}\right), \quad (3.3)
$$

where $d(x) \in E_M(\Omega)$, $d \geq 0$, $\alpha, \lambda \in \mathbb{R}^*_+$, $k_1, k_2, k_3, k_4 \in \mathbb{R}^*_+$.

Consider the nonlinear elliptic problem (1.1) where

$$
f \in L^1(\Omega) \quad (3.4)
$$

and $\phi = (\phi_1, \ldots, \phi_N)$ satisfies

$$
\phi \in (C^0(\mathbb{R}))^N. \quad (3.5)
$$

As in [8], we define the following notion of an entropy solution, which gives a meaning to a possible solution of (1.1).
Definition 3.1. Assume that (3.1), (3.2), (3.3), (3.4), and (3.5) hold true and \( \mathcal{A}_M \neq \emptyset \). A function \( u \) is an entropy solution of problem (1.1) if

\[
\begin{align*}
 u & \in W^1_0 \mathcal{L}_Q(\Omega), & \forall Q \in \mathcal{A}_M, \\
 T_k(u) & \in W^1_0 \mathcal{L}_M(\Omega), & \forall k > 0,
\end{align*}
\]

\[
\int_\Omega a(x, u, \nabla u) \nabla T_k[u - \varphi] \, dx \leq \int_\Omega f T_k[u - \varphi] \, dx + \int_\Omega \phi(u) \nabla T_k[u - \varphi] \, dx
\]

\[\forall \varphi \in W^1_0 \mathcal{L}_M(\Omega) \cap L^\infty(\Omega).\] (3.6)

We cannot use the solution \( u \) as a test function in (1.1), because \( u \) does not belong to \( W^1_0 \mathcal{L}_M(\Omega) \). An important role is played by \( T_k(u) \) and the test functions

\[
T_k[u - \varphi], \quad \varphi \in W^1_0 \mathcal{L}_M(\Omega) \cap L^\infty(\Omega)
\] (3.7)

because both belong to \( W^1_0 \mathcal{L}_M(\Omega) \).

In Theorem 3.5, we prove the existence of solution of problem (1.1), in the framework of entropy solutions.

Lemma 3.2. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with the segment property. If \( u \in (W^1_0 \mathcal{L}_M(\Omega))^N \) then \( \int_\Omega \text{div} \ u \, dx = 0 \).

Proof of Lemma 3.2. It is sufficient to use an approximation of \( u \). \( \Box \)

We recall the following lemma (see [15, Lemma 2]).

Lemma 3.3. Let \( M \) be an \( N \)-function, \( u \in W^1 \mathcal{L}_M(\Omega) \) such that \( \int_\Omega M(|\nabla u|) \, dx < \infty \), then

\[
-\mu'(t) \geq N C^1/N \mu^{1-1/N}(t)
\]

\[
\times C \left( \frac{-1}{N C^1/N \mu^{1-1/N}(t)} \frac{d}{dt} \int_{|u|>r} M(|\nabla u|) \, dx \right) \quad \forall t > 0,
\] (3.8)

where \( C \) is the function defined as

\[
C(t) = \frac{1}{\sup \{ r \geq 0, H(r) \leq t \}}, \quad H(r) = \frac{M(r)}{r}.
\] (3.9)

The function \( C_N \) is the measure of the unit ball of \( \mathbb{R}^N \), and \( \mu(t) = \text{meas}(|u| > t) \).

Lemma 3.4. Let \((X, \tau, \mu)\) be a measurable set such that \( \mu(X) < \infty \). Let \( \gamma \) be a measurable function \( \gamma : X \to [0, \infty) \) such that

\[
\mu(\{ x \in X : \gamma(x) = 0 \}) = 0,
\] (3.10)

then for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \int_A \gamma(x) \, dx < \delta \) implies

\[
\mu(A) \leq \epsilon.
\] (3.11)
Theorem 3.5. Under assumptions (3.1), (3.2), (3.3), (3.4), and (3.5), with $\mathcal{A}_M \neq \emptyset$, there exists an entropy solution $u$ of problem (1.1) (in the sense of Definition 3.1).

Remark 3.6. In the case $M(t) = t^p$, Theorem 3.5 gives a refinement of the regularity result (1) (i.e., $u \in W_0^{1,q}(\Omega), q < \tilde{p} = ((p-1)N/N - 1)$). In fact, by Theorem 3.5, we have $u \in W_0^1L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$ (for example for $Q(t) = t^{\tilde{p}}/\log^\alpha(e + t), \alpha > 1$).

Proof of Theorem 3.5
Step 1. Define, for each $n > 0$, the approximations
\[ \phi_n(s) = \phi(T_n(s)), \quad f_n(s) = T_n[f(s)]. \] (3.12)
Consider the nonlinear elliptic problem
\[ u_n \in W_0^1L_M(\Omega), \quad -\text{div} a(x, u_n, \nabla u_n) = f_n - \text{div} \phi_n(u_n) \quad \text{in } \Omega. \] (3.13)
From Gossez and Mustonen [12, Proposition 1, Remark 2], problem (3.13) has at least one solution.
Step 2. We will prove that $(u_n)$ is bounded in $W_0^1L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Let $\varphi$ be the truncation defined, for each $t, h > 0$, by
\[ \varphi(\xi) = \begin{cases} 0 & \text{if } 0 \leq \xi \leq t, \\ \frac{1}{h}(\xi - t) & \text{if } t < \xi < t + h, \\ 1 & \text{if } \xi \geq t + h, \\ -\varphi(-\xi) & \text{if } \xi < 0. \end{cases} \] (3.14)
Using the test function $v = \varphi(u_n)$ in (3.13) ($v \in W_0^1L_M(\Omega)$ by Lemma 2.2), we have
\[ \int_{\Omega} a(x, u_n, \nabla u_n) \varphi'(u_n) \nabla u_n \, dx = \int_{\Omega} f_n \varphi(u_n) \, dx + \int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) \, dx. \] (3.15)
We claim now that
\[ \int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) \, dx = 0. \] (3.16)
Indeed,
\[ \nabla \varphi(u_n) = \varphi'(u_n) \nabla u_n, \] (3.17)
where
\[ \varphi'(\xi) = \begin{cases} \frac{1}{h} & \text{if } t < |\xi| < t + h, \\ 0 & \text{otherwise,} \end{cases} \] (3.18)
92 Entropy solutions in Orlicz spaces

define \( \theta(s) = \phi_n(s)(1/h)\chi_{|t|<|s|<t+h}, \) and \( \tilde{\theta}(s) = \int_0^s \theta(\tau) \, d\tau, \) we have by Lemma 2.2, \( \tilde{\theta}(u_n) \in (W_0^1L_M(\Omega))^N, \) which implies

\[
\int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) \, dx = \int_{\Omega} \phi_n(u_n) \frac{1}{h} \chi_{|t|<|u_n|<t+h} \nabla u_n \, dx = \int_{\Omega} \theta(u_n) \nabla u_n \, dx \\
= \int_{\Omega} \text{div}(\tilde{\theta}(u_n)) \, dx = 0 \quad \text{(see Lemma 3.2).} \tag{3.19}
\]

This proves (3.16). By (3.3) and (3.15), we have (where we can suppose without loss of generality that \( \lambda = 1, \) since one can take \( u'_n = u_n/\lambda \))

\[
\alpha \frac{1}{h} \int_{|u_n|<t} M(|\nabla u_n|) \, dx \leq \|f\|_{1,\Omega}. \tag{3.20}
\]

Let \( h \to 0, \) then

\[
-\frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \leq C \quad \text{with} \quad C = \frac{\|f\|_{1,\Omega}}{\alpha}. \tag{3.21}
\]

We prove the following inequality, which allows us to obtain the boundedness of \((u_n)\) in \( W_0^1L_Q(\Omega), \)

\[
-\frac{d}{dt} \int_{|u_n|>t} Q(|\nabla u_n|) \, dx \\
\leq -\mu'_n(t)Q \circ H^{-1}\left( -\frac{1}{NC_N^{1/N}} \frac{1}{\mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right). \tag{3.22}
\]

Indeed, let \( C(s) = 1/H^{-1}(s), \) where \( H(r) = M(r)/r \) and \( H^{-1}(s) = \sup\{r \geq 0, H(r) \leq s\}. \) Then

\[
C(s) = \frac{s}{M \circ H^{-1}(s)}. \tag{3.23}
\]

By Lemma 3.3 we have, with \( \mu_n(t) = \text{meas}\{|u_n|>t\}, \)

\[
-\mu'_n(t) \geq NC_N^{1/N} \mu_n(t)^{1-1/N} \\
\times C\left( -\frac{1}{NC_N^{1/N}} \mu_n(t)^{1-1/N} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right), \tag{3.24}
\]

then

\[
-\mu'_n(t) \cdot M \circ H^{-1}\left( -\frac{1}{NC_N^{1/N}} \mu_n(t)^{1-1/N} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right) \\
\geq NC_N^{1/N} \mu_n(t)^{1-1/N} \left( -\frac{1}{NC_N^{1/N}} \mu_n(t)^{1-1/N} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right), \tag{3.25}
\]
and also

\[
\frac{1}{\mu(t)} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) \, dx \\
\leq M \circ H^{-1}\left( -\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) \, dx \right)
\]

(3.26)

which gives

\[
M^{-1}\left( \frac{1}{\mu_n(t)} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) \, dx \right) \\
\leq H^{-1}\left( -\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) \, dx \right).
\]

(3.27)

Let \( Q \in \mathcal{D}_M \) and let \( D(s) = M(Q^{-1}(s)) \), \( D \) is then convex, and the Jensen's
inequality gives

\[
D\left( \int_{\{t < |u_n| < t+h\}} \frac{Q(|\nabla u_n|)}{-\mu_n(t+h) + \mu_n(t)} \, dx \right) \\
\leq \int_{\{t < |u_n| < t+h\}} \frac{M(|\nabla u_n|)}{-\mu_n(t+h) + \mu_n(t)} \, dx,
\]

(3.28)

then

\[
Q^{-1}\left( \frac{1}{\mu_n(t)} \frac{d}{dt} \int_{\{|u_n| > t\}} Q(|\nabla u_n|) \, dx \right) \\
\leq M^{-1}\left( \frac{1}{\mu_n(t)} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) \, dx \right) \\
\leq H^{-1}\left( -\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{\{|u_n| > t\}} M(|\nabla u_n|) \, dx \right)
\]

(3.29)

which gives (3.22). By (3.21) and (3.22) and since the function

\[
t \rightarrow \int_{\{|u_n| > t\}} Q(|\nabla u_n|) \, dx
\]

(3.30)

is absolutely continuous (see [15]), we have

\[
\int_{\Omega} Q(|\nabla u_n|) \, dx = \int_{0}^{\infty} \left( -\frac{d}{dt} \int_{\{|u_n| > t\}} Q(|\nabla u_n|) \right) \, dt \\
\leq \int_{0}^{\infty} -\mu_n(t) Q \circ H^{-1}\left( \frac{C}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \right) \, dt
\]

(3.31)

\[
\leq \frac{1}{C} \int_{C^{\text{meas}(\Omega)}} Q \circ H^{-1}\left( \frac{1}{r^{1-1/N}} \right) dr < \infty
\]
which implies that \((\nabla u_n)\) is bounded in \(L_Q(\Omega)\) for each \(Q \in \mathcal{A}_M\). Then \(u_n\) is bounded in \(W^1_0L_Q(\Omega)\) for each \(Q \in \mathcal{A}_M\). Passing to a subsequence if necessary, we can assume that

\[
u_n \rightharpoonup u \quad \text{weakly in } W^1_0L_Q(\Omega) \text{ for } \sigma(\prod L_Q, \prod E_Q), \quad \text{a.e. in } \Omega. \tag{3.32}
\]

**Step 3.** We prove that \(T_k(u_n) \rightharpoonup T_k(u)\) weakly in \(W^1_0L_M(\Omega)\) for all \(k > 0\). Using the test function \(T_k(u_n)\) in (3.13), we obtain

\[
\int \Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n) \, dx = \int \Omega f_n T_k(u_n) \, dx + \int \Omega \phi_n(u_n) \nabla T_k(u_n) \, dx. \tag{3.33}
\]

We claim that

\[
\int \Omega \phi_n(u_n) \nabla T_k(u_n) \, dx = 0. \tag{3.34}
\]

Indeed, \(\nabla T_k(u_n) = \nabla u_n \chi_{|u_n| \leq k}\), define \(\tilde{\theta}(t) = \phi_n(t)\chi_{|t| \leq k}\), and \(\tilde{\theta}(t) = \int_0^t \tilde{\theta}(r) \, dr\), we have by Lemma 2.2, \(\tilde{\theta}(u_n) \in (W^1_0L_M(\Omega))^N\),

\[
\int \Omega \tilde{\theta}(u_n) \nabla u_n \, dx = \int \Omega \phi_n(u_n) \chi_{|u_n| \leq k} \nabla u_n \, dx
\]

\[
= \int \Omega \theta(u_n) \nabla u_n \, dx
\]

\[
= \int \Omega \text{div}(\tilde{\theta}(u_n)) \, dx = 0 \quad \text{(by Lemma 3.2)}
\]

which proves the claim.

On the other hand, (3.33) can be written as

\[
\int \Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n) \, dx = \int \Omega a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u_n) \, dx
\]

\[
= \int \Omega f_n T_k(u_n) \, dx,
\]

which implies, with (3.3), that \(\nabla T_k(u_n)\) is bounded in \((L_M(\Omega))^N\), and \(T_k(u_n)\) is bounded in \((W^1_0L_M(\Omega))^N\). Since \(u_n \rightharpoonup u\) a.e. in \(\Omega\) then \(T_k(u_n) \rightharpoonup T_k(u)\) a.e. in \(\Omega\). Then

\[
T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W^1_0L_M(\Omega) \text{ for } \sigma(\prod L_M, \prod E_M). \tag{3.37}
\]

**Step 4.** We will prove that \(\nabla u_n \rightharpoonup \nabla u\) a.e. in \(\Omega\). Let \(\lambda > 0, \epsilon > 0\). For \(B > 1, k > 0\), we consider as in [9] for \(n, m \in \mathbb{N}\),

\[
E_1 = \{ |\nabla u_n| > B \} \cup \{ |\nabla u_m| > B \} \cup \{ |u_n| > B \} \cup \{ |u_m| > B \},
\]

\[
E_2 = \{ |u_n - u_m| > k \},
\]

\[
E_3 = \{ |u_n - u_m| \leq k, |u_n| \leq B, |u_m| \leq B, |\nabla u_n| \leq B, |\nabla u_m| \leq B, |\nabla u_n - \nabla u_m| \geq \lambda \},
\]

we have \(||\nabla u_n - \nabla u_m| \geq \lambda| \subset E_1 \cup E_2 \cup E_3\).
Since \((u_n)\) and \((\nabla u_n)\) are bounded in \(L^1(\Omega)\) (since \(u_n\) is bounded in \(W_0^1L_Q(\Omega)\)), we have

\[
2B\mu(E_1) < \int_{E_1} |\nabla u_n| + |u_n| \, dx < \int_{\Omega} |\nabla u_n| + |u_n| \, dx < C. \tag{3.39}
\]

Then \(\text{meas} E_1 \leq \epsilon\) for \(B\) sufficiently large enough, independently of \(n, m\). Thus we fix \(B\) in order to have

\[
\text{meas} E_1 \leq \epsilon. \tag{3.40}
\]

Now we claim that \(\text{meas} E_3 \leq \epsilon\) for \(n\) and \(m\) large. Let \(C_1\) be such that \(\|u_n\|_1 \leq C_1\) and \(\|\nabla u_n\|_1 \leq C_1\). As in [9], the assumption (3.2) gives the existence of a measurable function \(\gamma(x)\) such that

\[
\text{meas} \left( \{ x \in \Omega : \gamma(x) = 0 \} \right) = 0,
\]

\[
[a(x, t, \xi) - a(x, t, \tilde{\xi})] [\xi - \tilde{\xi}] \geq \gamma(x) > 0, \tag{3.41}
\]

for all \(t \in \mathbb{R}, \xi, \tilde{\xi} \in \mathbb{R}^N, |t|, |\xi|, |\tilde{\xi}| \leq B, |\xi - \tilde{\xi}| \geq \lambda\) a.e. in \(\Omega\). We have

\[
\int_{E_3} \gamma(x) \, dx \leq \int_{E_3} \left[ a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m) \right] \left[ \nabla u_n - \nabla u_m \right] \, dx
\]

\[
\leq \int_{E_3} \left[ a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_n) \right] \left[ \nabla u_n - \nabla u_m \right] \, dx \tag{3.42}
\]

\[
+ \int_{E_3} \left[ a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m) \right] \left[ \nabla u_n - \nabla u_m \right] \, dx.
\]

Using the test function \(T_k(u_n - u_m)\) in (3.13) and integrating on \(E_3\), we obtain

\[
\int_{E_3} \left[ a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m) \right] \nabla T_k(u_n - u_m) \, dx
\]

\[
= \int_{E_3} (f_n - f_m) T_k(u_n - u_m) \, dx \tag{3.43}
\]

\[
+ \int_{E_3} \left[ \phi_n(u_n) - \phi_m(u_m) \right] \nabla T_k(u_n - u_m) \, dx,
\]

with

\[
\int_{E_3} \left[ \phi_n(u_n) - \phi_m(u_m) \right] \nabla T_k(u_n - u_m) \, dx
\]

\[
\leq 2B \int_{E_3} |\phi_n(u_n) - \phi_m(u_m)| \, dx \tag{3.44}
\]

\[
\leq 2B \int_{E_3} \left[ |\phi(T_n(u_n)) - \phi(u_n)| + |\phi(u_n) - \phi(u_m)| + |\phi(u_m) - \phi(T_m(u_m))| \right] \, dx.
\]
Let $n_0 \geq B$, then for $n, m \geq n_0$ we have $T_n(u_n) = u_n$ and $T_m(u_m) = u_m$ on $E_3$, which implies that the first and the third integral of the last inequality vanish. The second integral of (3.42) is bounded for $n, m \geq n_0$ by

$$2k\|f\|_{1, \Omega} + 2B \int_{E_3} |\phi(u_n) - \phi(u_m)| \, dx. \quad (3.45)$$

For a.e. $x \in \Omega$ and $\epsilon_1 > 0$ there exist $\eta(x) \geq 0$ (meas$\{x \in \Omega : \eta(x) = 0\} = 0$) such that $|s - s'| \leq \eta(x)$, $|s|, |s'|, |\xi| \leq B$ implies

$$|a(x, s, \xi) - a(x, s', \xi)| \leq \epsilon_1. \quad (3.46)$$

We use now the continuity of $\phi$, to obtain for a.e. $x \in \Omega$ and $\epsilon_2 > 0$, $\eta_1(x) \geq 0$ (meas$\{x \in \Omega : \eta_1(x) = 0\} = 0$) such that $|s - s'| \leq \eta_1(x)$, $|s|, |s'| \leq B$ implies

$$|\phi(s) - \phi(s')| \leq \epsilon_2. \quad (3.47)$$

Then

$$\int_{E_3} y(x) \, dx \leq \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] \times [\nabla u_n - \nabla u_m] \, dx + \int_{E_3 \cap \{x \in \Omega : \eta(x) \geq k\}} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m)] \times [\nabla u_n - \nabla u_m] \, dx + 2k\|f\|_{1, \Omega} + 2B \int_{E_3 \cap \{x \in \Omega : \eta_1(x) < k\}} |\phi(u_n) - \phi(u_m)| \, dx + \int_{E_3 \cap \{x \in \Omega : \eta_1(x) \geq k\}} |\phi(u_n) - \phi(u_m)| \, dx. \quad (3.48)$$

By using for the first integral the definition of $E_3$ and condition (3.1), for the second integral the definition of $E_3$ and (3.46), for the fourth integral the definition of $E_3$ and $|\phi(u_n)| \leq C(B)$ (since $|u_n| \leq B$ and $\phi$ continuous), and for the last integral the definition of $E_3$ and (3.47), we obtain

$$\int_{E_3} y(x) \, dx \leq C(B) \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} [1 + d(x)] \, dx + 2C_1(B)\epsilon_1 + 2k\|f\|_{1, \Omega} + 2C(B) \text{meas}\{x \in \Omega : \eta_1(x) < k\} + C_2\epsilon_2. \quad (3.49)$$

We have meas$\{x \in \Omega : \eta(x) < k\} \to 0$ when $k \to 0$, and meas$\{x \in \Omega : \eta_1(x) < k\} \to 0$ when $k \to 0$. Let $\epsilon > 0$ and let $\delta$ be the real, in Lemma 3.4, corresponding to $\epsilon$, we choose $\epsilon_1, \epsilon_2$ such that

$$2C_1(B)\epsilon_1 \leq \frac{\delta}{5}, \quad C_2\epsilon_2 \leq \frac{\delta}{5}, \quad (3.50)$$
and $k$ such that
\[
C'(B) \int_{E_3 \cap \{x \in \Omega : \eta_1(x) < k\}} \left[1 + d(x)\right] \, dx < \frac{\delta}{5}, \quad 2k\|f\|_{L^1(\Omega)} \leq \frac{\delta}{5}.
\]
(3.51)

Then $\int_{E_3} y(x) \, dx < \delta$ and Lemma 3.4 implies that
\[
\text{meas } E_3 < \epsilon \quad \forall \, n, m \geq n_0.
\]
(3.52)

This completes the proof of the claim.

Let the last $k$ be fixed, $u_n$ a Cauchy sequence in measure, we choose $n_1$ such that
\[
\text{meas } E_2 \leq \epsilon \quad \forall \, n, m \geq n_1.
\]
(3.53)

Then
\[
\text{meas } \{x \in \Omega : |\nabla u_n - \nabla u_m| \geq \lambda\} \leq \epsilon \quad \forall \, n, m \geq \max(n_1, n_0)
\]
(3.54)

and $\nabla u_n \rightharpoonup \nabla u$ in measure, consequently
\[
\nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in } \Omega.
\]
(3.55)

Step 5. Let $\varphi \in W^1_1 L^1(\Omega) \cap L^\infty(\Omega)$. From Lemma 2.3, there exists a sequence $(\varphi_j) \in \mathcal{D}(\Omega)$ such that $\varphi_j$ converges to $\varphi$ for the modular convergence in $W^1_1 L_M(\Omega)$ with
\[
\|\varphi_j\|_{L^\infty(\Omega)} \leq (N + 1)\|\varphi\|_{L^\infty(\Omega)}.
\]
(3.56)

Using $T_k[u_n - \varphi_j]$ as a test function in (3.13) we obtain
\[
\int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] \, dx
= \int_{\Omega} f_n T_k[u_n - \varphi_j] \, dx + \int_{\Omega} \phi_n(u_n) \nabla T_k[u_n - \varphi_j] \, dx
\]
(3.57)

which gives, if $n \to \infty$,

\[
\liminf_{n \to \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] \, dx
\geq \liminf_{n \to \infty} \int_{\Omega} \left[a(x, u_n, \nabla u_n) - a(x, \eta u_n, \nabla \varphi_j)\right] \nabla T_k[u_n - \varphi_j] \, dx
+ \lim_{n \to \infty} \int_{\Omega} a(x, T_k[u_n - \varphi_j], \nabla \varphi_j) \nabla T_k[u_n - \varphi_j] \, dx
\geq \int_{\Omega} \left[a(x, u, \nabla u) - a(x, \eta u, \nabla \varphi_j)\right] \nabla T_k[u - \varphi_j] \, dx
+ \int_{\Omega} a(x, u, \nabla \varphi_j) \nabla T_k[u - \varphi_j] \, dx.
\]
(3.58)
where we have used Fatou lemma for the first integral, and for the second the convergences \( \nabla T_k[u_n - \varphi_j] \to \nabla T_k[u - \varphi_j] \) by (3.37) in \((L_M(\Omega))^N\) for \( \sigma(\prod L_M, \prod E_M) \) and \( a(x, T_k + \|\varphi\|_{L^\infty(\Omega)}(u_n), \nabla \varphi_j) \to a(x, T_k + \|\varphi\|_{L^\infty(\Omega)}(u), \nabla \varphi_j) \) strongly in \((E_M(\Omega))^N\) by (3.1), which implies that

\[
\liminf_{n \to \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] \, dx \geq \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] \, dx. \tag{3.59}
\]

For \( n \geq k + (N + 1)\|\varphi\|_{L^\infty(\Omega)}, \)

\[
\int_{\Omega} \phi_n(u_n) \nabla T_k[u_n - \varphi_j] \, dx = \int_{\Omega} \phi(T_k + (N + 1)\|\varphi\|_{L^\infty(\Omega)}(u_n)) \nabla T_k[u_n - \varphi_j] \, dx
\]

\[
\quad \xrightarrow{n \to \infty} \int_{\Omega} \phi(T_k + (N + 1)\|\varphi\|_{L^\infty(\Omega)}(u)) \nabla T_k[u - \varphi_j] \, dx, \tag{3.60}
\]

we have used the convergences \( \nabla T_k[u_n - \varphi_j] \to \nabla T_k[u - \varphi_j] \) by (3.37) in \((L_M(\Omega))^N\) and \( \phi(T_k + (N + 1)\|\varphi\|_{L^\infty(\Omega)}(u_n)) \to \phi(T_k + (N + 1)\|\varphi\|_{L^\infty(\Omega)}(u)) \) strongly in \((E_M(\Omega))^N\) since \( \phi \) is continuous. On the other hand, since \( f_n \to f \) strongly in \( L^1(\Omega) \) and \( T_k[u_n - \varphi_j] \to T_k[u - \varphi_j] \) weakly* in \( L^\infty(\Omega) \), we have

\[
\int_{\Omega} f_n T_k[u_n - \varphi_j] \, dx \to \int_{\Omega} f T_k[u - \varphi_j] \, dx. \tag{3.61}
\]

Then

\[
\int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] \, dx \geq \int_{\Omega} \phi(T_k + (N + 1)\|\varphi\|_{L^\infty(\Omega)}(u)) \nabla T_k[u - \varphi_j] \, dx
\]

\[
\quad + \int_{\Omega} f T_k[u - \varphi_j] \, dx. \tag{3.62}
\]

Now, if \( j \to \infty \) in (3.62), we get

\[
\liminf_{j \to \infty} \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] \, dx
\]

\[
\geq \liminf_{j \to \infty} \int_{\Omega} \left[ a(x, u, \nabla u) - a(x, u, \nabla \varphi_j) \right] \nabla T_k[u - \varphi_j] \, dx
\]

\[
\quad + \lim_{j \to \infty} \int_{\Omega} a(x, u, \nabla \varphi_j) \nabla T_k[u - \varphi_j] \, dx \tag{3.63}
\]

\[
\geq \int_{\Omega} \left[ a(x, u, \nabla u) - a(x, u, \nabla \varphi) \right] \nabla T_k[u - \varphi] \, dx
\]

\[
\quad + \int_{\Omega} a(x, u, \nabla \varphi) \nabla T_k[u - \varphi] \, dx,
\]

where we have used Fatou lemma for the first integral, and for the second the convergences \( \nabla T_k[u - \varphi_j] \to \nabla T_k[u - \varphi] \) in \((L_M(\Omega))^N\) for the modular convergence and \( a(x, u, \nabla \varphi_j) \to a(x, u, \nabla \varphi) \) in \((L_M(\Omega))^N\) for the modular convergence,
which implies that
\[
\liminf_{j \to \infty} \int_{\Omega} a(x, u, \nabla u) \nabla T_k [u - \varphi_j] \, dx \geq \int_{\Omega} a(x, u, \nabla u) \nabla T_k [u - \varphi] \, dx. \tag{3.64}
\]

On the other hand, since \( \nabla T_k [u - \varphi_j] \to \nabla T_k [u - \varphi] \) in \((L_M(\Omega))^N\) for the modular convergence, then weakly for \( \sigma(\prod L_M, \prod L_M) \) and \( \phi(T_{k(N+1)}[\varphi]|_{L^\infty(\Omega)}(u)) \in (L_M(\Omega))^N \) we have
\[
\int_{\Omega} \phi(T_{k(N+1)}[\varphi]|_{L^\infty(\Omega)}(u)) \nabla T_k [u - \varphi_j] \, dx \xrightarrow{j \to \infty} \int_{\Omega} \phi(T_{k(N+1)}[\varphi]|_{L^\infty(\Omega)}(u)) \nabla T_k [u - \varphi] \, dx \tag{3.65}
\]
\[
= \int_{\Omega} \phi(u) \nabla T_k [u - \varphi] \, dx.
\]

Since \( f \in L^1(\Omega) \) and \( T_k [u - \varphi_j] \to T_k [u - \varphi] \) weakly* in \( L^\infty(\Omega) \), we have
\[
\int_{\Omega} f T_k [u - \varphi_j] \, dx \to \int_{\Omega} f T_k [u - \varphi] \, dx. \tag{3.66}
\]

Then
\[
\int_{\Omega} a(x, u, \nabla u) \nabla T_k [u - \varphi] \, dx \geq \int_{\Omega} \phi(u) \nabla T_k [u - \varphi] \, dx + \int_{\Omega} f T_k [u - \varphi] \, dx \tag{3.67}
\]
and \( u \) is an entropy solution of problem (1.1). \( \square \)

**Theorem 3.7.** Suppose, in Theorem 3.5, that the \( N \)-function \( M \) satisfies, furthermore, the \( \Delta_2 \)-condition and \( f \geq 0 \), then the entropy solution \( u \) of problem (1.1) satisfies \( u \geq 0 \).

**Proof of Theorem 3.7.** Using \( \varphi = T_l(u^+) \) as test function in the definition of entropy solution, we obtain
\[
\int_{\Omega} a(x, u, \nabla u) \nabla T_k [u - T_l(u^+)] \, dx \leq \int_{\Omega} f T_k [u - T_l(u^+)] \, dx + \int_{\Omega} \phi(u) \nabla T_k [u - T_l(u^+)] \, dx. \tag{3.68}
\]

We have
\[
\int_{\Omega} f T_k [u - T_l(u^+)] \, dx \leq \int_{\{u \geq l\}} f T_k [u - T_l(u)] \, dx. \tag{3.69}
\]
100 Entropy solutions in Orlicz spaces

Indeed,

\[
\int_\Omega f T_k [u - T_l(u^+)] \, dx = \int_{u \geq l} f T_k [u - T_l(u^+)] \, dx \\
+ \int_{0 < u < l} f T_k [u - T_l(u^+)] \, dx \\
+ \int_{u \leq 0} f T_k [u - T_l(u^+)] \, dx.
\] (3.70)

If \( 0 < u < l \) then \( u - T_l(u^+) = 0 \) and \( \int_{0 < u < l} f T_k [u - T_l(u^+)] \, dx = 0 \). If \( u \leq 0 \) then \( u - T_l(u^+) = u \) and \( \int_{u \leq 0} f T_k [u - T_l(u^+)] \, dx \leq 0 \) since \( f \) is positive. If \( u \geq l \) then \( u^+ = u \) and \( \int_{u \geq l} f T_k [u - T_l(u^+)] \, dx \leq \int_{u \geq l} f T_k [u - T_l(u)] \, dx \).

On the other hand, we claim that

\[
\int_\Omega \phi(u) \nabla T_k [u - T_l(u^+)] \, dx = 0.
\] (3.71)

Indeed, if \( 0 < u < l \), then \( u - T_l(u^+) = 0 \), \( \int_{0 < u < l} \phi(u) \nabla T_k [u - T_l(u^+)] \, dx = 0 \). If \( u \leq 0 \), then \( u - T_l(u^+) = u \),

\[
\int_{u \leq 0} \phi(u) \nabla T_k [u - T_l(u^+)] \, dx = \int_{-k \leq u \leq 0} \phi(u) \nabla u \, dx
\] (3.72)

We verify that the third integral of the last inequality vanishes. For this, define \( \theta(t) = \phi(t) \chi_{[-k \leq t \leq 0]} \), and \( \bar{\theta}(t) = \int_0^t \theta(\tau) \, d\tau \) we have, by Lemma 2.2, \( \bar{\theta}(u) \in (W^1_{0,LM}(\Omega))^N \) which implies

\[
\int_\Omega \phi(u) \nabla u \chi_{[-k \leq u \leq 0]} \, dx = \int_\Omega \theta(u) \nabla u \, dx
\] (3.73)

If \( u \geq l \) then \( u^+ = u \) and

\[
\int_{u \geq l} \phi(u) \nabla T_k [u - T_l(u^+)] \, dx = \int_{l \leq u \leq l+k} \phi(u) \nabla u \, dx
\] (3.74)

Similarly, we verify that

\[
\int_\Omega \phi(u) \nabla u \chi_{[l \leq u \leq l+k]} \, dx = 0.
\] (3.75)
This completes the proof of the claim which implies that

\[ \int_{\Omega} a(x, u, \nabla u) \nabla T_k [u - T_l(u^+)] \, dx \leq \int_{u \geq l} f T_k [u - T_l(u)] \, dx \quad (3.76) \]

or

\[ \int_{\Omega} a(x, u, \nabla u) \nabla T_k [u - T_l(u^+)] \, dx \]

\[ = \int_{l \leq u \leq l+k} a(x, u, \nabla u) \nabla u \, dx + \int_{-k \leq u \leq 0} a(x, u, \nabla u) \nabla u \, dx \]

\[ \geq \int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx + \int_{-k \leq u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx, \]

which gives

\[ \int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx + \int_{-k \leq u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx \leq \int_{u \geq l} f T_k [u - T_l(u)] \, dx. \quad (3.78) \]

Letting \( l \to \infty \) in (3.78) we have

\[ \int_{u \geq l} f T_k [u - T_l(u)] \, dx \to 0 \quad \text{since} \quad f T_k [2u] \in L^1(\Omega), \]

\[ \int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx \geq \int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx \]

\[ = \int_{l \leq u} M \left( \frac{|\nabla T_k(u)|}{\lambda} \right) \, dx \]

\[ \to 0, \quad \text{when} \quad l \to \infty, \]

since \( M(|\nabla T_k(u)|/\lambda) \in L^1(\Omega) \) and \( M \) satisfies the \( \Delta_2 \)-condition. Then

\[ \int_{-k \leq u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx = 0 \quad \forall k, \quad (3.80) \]

which implies that,

\[ \int_{u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx = \int_{\Omega} M \left( \frac{|\nabla u^-|}{\lambda} \right) \, dx = 0, \quad (3.81) \]

\[ \nabla u^- = 0, \quad u^- = c \quad \text{a.e. in} \ \Omega. \]

Or \( u^- \in W_0^1 L^q(\Omega) \) then \( u^- = 0 \) a.e. in \( \Omega \) which proves that

\[ u \geq 0 \quad \text{a.e. in} \ \Omega. \quad (3.82) \]
References


A. Benkirane and J. Bennouna: Département de Mathématique, Faculté des Sciences Dhar Mezhraz, Université Sidi Mohamed Ben Abdallah, BP 1796 Atlas Fes, Morocco
Submit your manuscripts at http://www.hindawi.com