ON THE LOCATION OF THE PEAKS OF LEAST-ENERGY SOLUTIONS TO SEMILINEAR DIRICHLET PROBLEMS WITH CRITICAL GROWTH

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We study the location of the peaks of solution for the critical growth problem $-\varepsilon^2 \Delta u + u = f(u) + u^{2^*-1}$, $u > 0$ in Ω , $u = 0$ on $\partial\Omega$, where Ω is a bounded domain; $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent and f has a behavior like u^p , $1 < p < 2^* - 1$.

1. Introduction

In this paper, we will study the location of the peaks of *least-energy* solution for the problem

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= f(u) + u^{2^*-1} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\varepsilon > 0$, and f is a function satisfying some subcritical conditions. Here $2^* = 2N/(N-2)$, $N \geq 3$, is the critical Sobolev exponent.

By *least-energy* solution for problem (1.1) we mean a critical point at the *Mountain-Pass* level of the associated *energy* functional

$$J_\varepsilon(u) = \frac{1}{2} \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) dz - \int_\Omega \left[F(u) + \frac{1}{2^*} (u^+)^{2^*} \right] dz, \quad (1.2)$$

(where $u^+ = \max\{u, 0\}$), defined on the Hilbert space $H_0^1(\Omega)$ endowed with the norm

$$\|u\|_\varepsilon^2 = \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) dz. \quad (1.3)$$

The Mountain-Pass level of J_ε is defined by

$$c_\varepsilon = \inf_{g \in \Gamma} \max_{0 \leq t \leq 1} J_\varepsilon(g(t)), \quad (1.4)$$

where Γ is the set of all continuous paths joining the origin and a fixed nonzero element e in $H_o^1(\Omega)$, such that $e \neq 0$ and $J_\varepsilon(e) \leq 0$. Under suitable hypothesis (e.g., (f_1) , (f_4) , (f_5) below), it is not hard to check that $c_\varepsilon > 0$ does not depend on the element $0 \neq v \in H_o^1(\Omega)$ and u is a *least-energy* solution if and only if $J_\varepsilon(u) = c$ and $J'_\varepsilon(u) = 0$, and $J_\varepsilon(u) \leq J_\varepsilon(v)$ for all $v \neq 0$ such that $J'_\varepsilon(v) = 0$.

The existence of least-energy solution of problem (1.1) was given in Brézis and Nirenberg in [3, Theorem 2.1] (see Lemma 2.4 in this paper).

In this paper, we will study some properties of the least-energy solution u_ε of problem (1.1) when ε is small. In order to describe these properties, we introduce the hypotheses on the function f .

Suppose that $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a $C^{1,\alpha}$ function such that

- (f₁) $f(0) = f'(0) = 0$;
- (f₂) there is $q_1 \in (1, (N+2)/(N-2))$ such that

$$\lim_{s \rightarrow \infty} \frac{f(s)}{s^{q_1}} = 0; \quad (1.5)$$

- (f₃) there are $q_2 \in (1, (N+2)/(N-2))$ and $\lambda > 0$ such that

$$f(s) \geq \lambda s^{q_2}, \quad \forall s > 0 \quad (1.6)$$

(when $N = 3$, we need $q_2 > 2$, otherwise we require a sufficiently large λ);

- (f₄) if $F(s) = \int_0^s f(t) dt$, for some $\theta \in (2, q_1 + 1)$ we have

$$0 < \theta F(s) \leq f(s)s, \quad \forall s > 0; \quad (1.7)$$

- (f₅) the function $f(s)/s$ is increasing for $s > 0$.

Since our interest is on positive solutions we define $f(s) = 0$, in $s \leq 0$.

Now we will state our main result.

THEOREM 1.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$; f satisfies (f_1) , (f_2) , (f_3) , (f_4) , (f_5) ; and let u_ε be the least-energy solution of (1.1). Then, there is a $\varepsilon_o > 0$ such that*

- (i) u_ε attains only one local maximum at some $z_\varepsilon \in \Omega$ (hence global maximum), for all $\varepsilon \in (0, \varepsilon_o]$;
- (ii) u_ε converges uniformly to zero over compact subsets of $\Omega \setminus \{z_\varepsilon\}$ as $\varepsilon \rightarrow 0$;
- (iii) $\text{dist}(z_\varepsilon, \partial\Omega) \rightarrow \max_{z \in \Omega} \text{dist}(z, \partial\Omega)$.

This statement is analogous to the one given by Ni and Wei in [8], in the subcritical case

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= h(u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \quad (1.8)$$

where h satisfies the following hypothesis:

- (i) (f_1) , (f_2) , (f_4) , and (f_5) hold;
- (ii) the global problem

$$-\Delta u + u = h(u), \quad \text{in } \mathbb{R}^N \quad (1.9)$$

has a unique positive solution in $H^1(\mathbb{R}^N)$;

- (iii) this solution is *nondegenerate* in the sense that

$$-\Delta v + v = h'(u)v, \quad \text{in } \mathbb{R}^N \quad (1.10)$$

has no nontrivial spherically symmetric solution in $L^2(\mathbb{R}^N)$.

In [8], Ni and Wei also have described the asymptotic profile (in ε) of u_ε , giving a detailed description for ε small. Here in the critical case, the solutions have the same profile.

In this work we will show that a ground state solution of the critical problem (1.1) is also solution of a subcritical problem (1.8) by showing that for small ε we have a uniform bound for the L^∞ norm of u_ε .

The difficulty here lies in finding an upper bound for $\|u_\varepsilon\|_{L^\infty(\Omega)}$ by obtaining a bound for u_ε in $L^p(\Omega)$ norm, for all $p \geq 2$. In the subcritical case this boundness is obtained since the family u_ε is bounded in $H^1(\Omega)$ but this argument does not work in the critical case. Here, we obtain an L^∞ -bound for u_ε through the estimate below, which is based on Moser's iteration technique (see [11]) and is essentially due to Brézis and Kato [2].

PROPOSITION 1.2. *Let Λ be an open subset and $q \in L^{N/2}(\Lambda)$. Suppose that $g : \Lambda \times \mathbb{R} \rightarrow \mathbb{R}$ is a Caratheodory function satisfying*

$$|g(x, s)| \leq (q(x) + C_g)|s|, \quad \forall s \in \mathbb{R}, x \in \Lambda \text{ and for some } C_g > 0. \quad (1.11)$$

Then, if $v \in H_0^1(\Lambda)$ is such that

$$-\Delta v = g(x, v), \quad \text{in } \Lambda \quad (1.12)$$

we have $v \in L^p(\Lambda)$ for all $2 \leq p < \infty$. Moreover, there is a positive constant $C_p = C(p, C_g, q)$ such that

$$\|v\|_{L^{2^*(p+1)}(\Lambda)} \leq C_p \|v\|_{L^{2(p+1)}(\Lambda)}. \quad (1.13)$$

Remark 1.3. The dependence on q of C_p can be given uniformly on a family of functions $\{q_\varepsilon\}_{\varepsilon>0}$ such that q_ε converges in $L^{N/2}$ (see the appendix).

We have organized this paper as follows: the next section contains the proof of [Theorem 1.1](#). This proof consists in a series of lemmas which show the L^∞ -bound for u_ε , where these functions are solutions of a class of subcritical problems (1.8). The third section is an appendix proving [Proposition 1.2](#), for the sake of completeness.

2. Proof of [Theorem 1.1](#)

Before proving [Theorem 1.1](#), let us fix some notation and preliminaries.

Remark 2.1. Throughout this section, we use the equivalent characterization of c_ε , which is more adequate to our purposes, given by

$$c_\varepsilon = \inf_{v \in H_0^1(\Omega) \setminus \{0\}} \max_{t \geq 0} J_\varepsilon(tv). \quad (2.1)$$

(see Willem [13, Theorem 4.2]).

We denote by $J : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ the functional given by

$$J(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} \left[F(u) + \frac{1}{2^*} (u_+)^{2^*} \right] dx, \quad (2.2)$$

where

$$\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) dx, \quad (2.3)$$

associated with the problem

$$-\Delta u + u = f(u) + |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N. \quad (2.4)$$

It is known that under assumptions (f_1) , (f_2) , (f_3) , (f_4) , (f_5) , and (2.4) possesses a *ground state* solution ω in the level

$$c = J(\omega) = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \geq 0} J(tv), \quad (2.5)$$

(see [1]).

Remark 2.2. It is easy to check that for each nonzero v in $H^1(\mathbb{R}^N)$, there is a unique $t_o = t(v)$ such that

$$J(t_o v) = \max_{t \geq 0} J(tv). \quad (2.6)$$

Indeed, since

$$J(tv) = \frac{t^2}{2} \|v\|^2 - \int_{\mathbb{R}^N} \left[F(tv) - \frac{t^{2^*}}{2^*} (v^+)^{2^*} \right] dx, \quad \text{for } t \geq 0, \quad (2.7)$$

the maximum point t_o of $J(tv)$ is given by

$$\|v\|^2 = \int_{\mathbb{R}^N} \left[t_o^{-1} v f(t_o v) + t_o^{2^*-2} (v^+)^{2^*} \right] dx. \quad (2.8)$$

We assume, without loss of generality that $0 \in \Omega$. Set $\Omega_\varepsilon = \{x \in \mathbb{R}^N; \varepsilon x \in \Omega\}$.

The restriction of J to $H_o^1(\Omega_\varepsilon)$ is the energy functional,

$$J(u) = \frac{1}{2} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2) dx - \int_{\Omega_\varepsilon} \left[F(u_+) + \frac{1}{2^*} u_+^{2^*} \right] dx, \quad u \in H_o^1(\Omega_\varepsilon), \quad (2.9)$$

associated with the problem

$$\begin{aligned} -\Delta u + u &= f(u) + u^{2^*-1} && \text{in } \Omega_\varepsilon, \\ u &= 0 && \text{on } \partial\Omega_\varepsilon. \end{aligned} \quad (2.10)$$

If u_ε is a critical point of J_ε , the family

$$v_\varepsilon(x) = u_\varepsilon(z) = u_\varepsilon(\varepsilon x), \quad z = \varepsilon x \quad (2.11)$$

is such that each v_ε is a critical point of functional J restricted to $H_o^1(\Omega_\varepsilon)$ at the level

$$b_\varepsilon = J(v_\varepsilon) = \inf_{v \in H_o^1(\Omega_\varepsilon) \setminus \{0\}} \max_{t \geq 0} J(tv). \quad (2.12)$$

It is easy to check that $b_\varepsilon = \varepsilon^{-N} c_\varepsilon$ and from the definition of c it follows that $b_\varepsilon \geq c$ for all $\varepsilon > 0$.

We will start with the following property of $\{b_\varepsilon\}_{\varepsilon>0}$.

LEMMA 2.3. *For $\{b_\varepsilon\}_{\varepsilon>0}$, $\lim_{\varepsilon \rightarrow 0} b_\varepsilon = c$.*

Proof. Fix ω a ground state solution of problem (2.4) and let $\psi_\varepsilon(x) = \varphi(\varepsilon x)\omega(x)$, where φ is a C^1 -function such that

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in B_1, \\ 0 & \text{if } x \notin B_2, \end{cases} \quad (2.13)$$

$B_1 = B_\rho(0)$, $B_2 = B_{2\rho}(0) \subset \Omega$. Observe that $\psi_\varepsilon \rightarrow \omega$ in $H^1(\mathbb{R}^N)$ and the support of ψ_ε is in Ω_ε . By definition of b_ε , we have $t_\varepsilon > 0$ such that

$$b_\varepsilon \leq \max_{t>0} J(t\psi_\varepsilon) = J(t_\varepsilon \psi_\varepsilon). \quad (2.14)$$

From (2.8) and condition (f₃) it follows that

$$\begin{aligned} \|\psi_\varepsilon\|^2 &= \int_{\mathbb{R}^N} \left[t_\varepsilon^{-1} \psi_\varepsilon f(t_\varepsilon \psi_\varepsilon) + t_\varepsilon^{2^*-2} \psi_\varepsilon^{2^*} \right] dx \\ &\geq \int_{\mathbb{R}^N} \left[\lambda t_\varepsilon^{q_2-1} \psi_\varepsilon^{q_2+1} + t_\varepsilon^{2^*-2} \psi_\varepsilon^{2^*} \right] dx, \end{aligned} \quad (2.15)$$

so that, t_ε is bounded. Equality (2.15) and Remark 2.2 show that $t_\varepsilon \rightarrow t(\omega) = 1$, as $\varepsilon \rightarrow 0$. Then we have $t_\varepsilon \psi_\varepsilon \rightarrow \omega$ in $H^1(\mathbb{R}^N)$ and

$$\lim_{\varepsilon \rightarrow 0} J(t_\varepsilon \psi_\varepsilon) = J(\omega) = c. \quad (2.16)$$

Combining (2.14), (2.16), and the inequality $b_\varepsilon \geq c$, for all $\varepsilon > 0$, we have proved this lemma. $\square$

LEMMA 2.4. *The inequality $c < (1/N)S^{N/2}$ holds, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.*

Proof. For each $h > 0$, consider the function

$$\phi_h(x) = \frac{[N(N-2)h]^{(N-2)/4}}{(h + |x|^2)^{(N-2)/2}}. \quad (2.17)$$

We recall that ϕ_h satisfies the problem

$$\begin{aligned} -\Delta u &= u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u(x) &> 0, \quad \int_{\mathbb{R}^N} |\nabla u|^2 dx < \infty, \\ \int_{\mathbb{R}^N} |\nabla \phi_h|^2 dx &= \int_{\mathbb{R}^N} \phi_h^{2^*} dx = S^{N/2} \quad (\text{see Talenti [12]}). \end{aligned} \quad (2.18)$$

Now, consider $\psi_h(x) = \varphi \phi_h(x) / \|\varphi \phi_h\|_{L^{2^*}(\mathbb{R}^N)}$, where φ is the function defined in the proof of Lemma 2.3. From condition (f₃) we have

$$J(t\psi_h) \leq \frac{t^2}{2} \int_{B_2} (|\nabla \psi_h|^2 + \psi_h^2) dx - \frac{\lambda t^{q_2+1}}{q_2+1} \int_{B_2} \psi_h^{q_2+1} dx - \frac{t^{2^*}}{2^*}. \quad (2.19)$$

Using arguments as in [7], there exists $h > 0$ such that

$$\max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{B_2} (|\nabla \psi_h|^2 + \psi_h^2) dx - \frac{\lambda t^{q_2+1}}{q_2+1} \int_{B_2} \psi_h^{q_2+1} dx - \frac{t^{2^*}}{2^*} \right\} < \frac{1}{N} S^{N/2}. \quad (2.20)$$

Therefore, from (2.19) and (2.20) we have that

$$\max_{t \geq 0} J(t\psi_h) < \frac{1}{N} S^{N/2}, \quad (2.21)$$

and the proof of the lemma is completed. $\square$

Notice that the same proof of Lemma 2.4 can be used to show that $b_\varepsilon < (1/N)S^{N/2}$, for all $\varepsilon > 0$. Using [3, Theorem 2.1], this inequality implies the existence of v_ε and then the existence of u_ε .

LEMMA 2.5. *There are $\varepsilon_o > 0$; a family $\{y_\varepsilon\}_{\{0 < \varepsilon \leq \varepsilon_o\}} \subset \mathbb{R}^N$, $y_\varepsilon \in \Omega_\varepsilon$; constants $R > 0$ and $\beta > 0$ such that*

$$\int_{B_R(y_\varepsilon)} v_\varepsilon^2 dx \geq \beta > 0, \quad \forall 0 < \varepsilon \leq \varepsilon_o, \quad (2.22)$$

$$\lim_{\varepsilon \rightarrow 0} d(y_\varepsilon, \partial\Omega_\varepsilon) = \infty. \quad (2.23)$$

Proof. Start by showing that there is a family satisfying inequality (2.22). Arguing to the contrary, there is $\varepsilon_n \searrow 0$ such that for all $R > 0$

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^N} \int_{B_R(x)} v_{\varepsilon_n}^2 dx = 0. \quad (2.24)$$

Using (Lions [6, Lemma I.1]) we have

$$\int_{\mathbb{R}^N} v_{\varepsilon_n}^q dx = o_n(1), \quad \text{as } n \rightarrow \infty, \quad \forall 2 < q < 2^*, \quad (2.25)$$

and, from (f₁) and (f₂),

$$\int_{\mathbb{R}^N} F(v_{\varepsilon_n}) dx = \int_{\mathbb{R}^N} v_{\varepsilon_n} f(v_{\varepsilon_n}) dx = o_n(1). \quad (2.26)$$

Since $J'(v_{\varepsilon_n}) \cdot v_{\varepsilon_n} = 0$, we conclude from (2.26) that

$$\|v_{\varepsilon_n}\|^2 = \int_{\mathbb{R}^N} v_{\varepsilon_n}^{2^*} dx + o_n(1). \quad (2.27)$$

Let $\ell \geq 0$ be such that $\|v_{\varepsilon_n}\|^2 \rightarrow \ell$. Passing to the limit in $J(v_{\varepsilon_n}) = b_{\varepsilon_n}$ and using (2.26) we have

$$\ell = Nc \quad (2.28)$$

and hence $\ell > 0$. Now, using the definition of the constant S , we have

$$\|v_{\varepsilon_n}\|^2 \geq S \left(\int_{\mathbb{R}^N} v_{\varepsilon_n}^{2^*} dx \right)^{2/2^*}. \quad (2.29)$$

Taking the limit in the above inequalities, as $n \rightarrow \infty$, we achieve that

$$\ell \geq S\ell^{2/2^*}, \quad (2.30)$$

and by (2.28), that

$$c \geq \frac{1}{N} S^{N/2} \quad (2.31)$$

which contradicts Lemma 2.4 and then (2.22) holds.

Finally, to establish (2.23), suppose the contrary. That is, there exist $\varepsilon_n \rightarrow 0$ and $R > 0$ such that $\text{dist}(y_{\varepsilon_n}, \partial\Omega_{\varepsilon_n}) \leq R$, hence $\text{dist}(\varepsilon_n y_{\varepsilon_n}, \partial\Omega) \leq \varepsilon_n R$. Without loss of generality, we have $\varepsilon_n y_{\varepsilon_n} \rightarrow y_o$ for some $y_o \in \partial\Omega$. The arguments that follow can be found in [8].

Let ν be the unit interior normal to $\partial\Omega$ at y_o , and $\delta > 0$ such that $B_\delta(y_o + \delta\nu) \subset \Omega$ and $B_\delta(y_o - \delta\nu) \cap \Omega = \emptyset$. Let $\Omega_n = \{x \in \mathbb{R}^N : y_o + \varepsilon_n x \in \Omega\}$ and $w_n(x) = u_{\varepsilon_n}(y_o + \varepsilon_n x)$. This sequence w_n is bounded in $H^1(\mathbb{R}^N)$, $-\Delta w_n + w_n = f(w_n) + w_n^{2^*-1}$ in Ω_n ,

$$\int_{B_{2R}(0)} w_n^2 dx \geq \int_{B_R(y_{\varepsilon_n})} v_{\varepsilon_n}^2 dx \geq \beta > 0, \quad \forall n, \quad (2.32)$$

and we have that w_n converges weakly to some w in $H^1(\mathbb{R}^N)$.

Let $\mathbb{R}_{+, \nu}^N$ be the half space $\{x \in \mathbb{R}^N : x \cdot \nu > 0\}$. Notice that $B_{\varepsilon_n^{-1}\delta}(\varepsilon_n^{-1}\delta\nu) \subset \Omega_n$ and $B_{\varepsilon_n^{-1}\delta}(-\varepsilon_n^{-1}\delta\nu) \cap \Omega_n = \emptyset$ and then we can prove that for all compacts $K_+ \subset \mathbb{R}_{+, \nu}^N$ and $K_- \subset \mathbb{R}_{-, \nu}^N = \mathbb{R}^N \setminus \overline{\mathbb{R}_{+, \nu}^N}$, we have $K_+ \subset \Omega_n$ and $K_- \cap \Omega_n = \emptyset$, for n large.

Then for each $\phi \in C_0^\infty(\mathbb{R}_{+, \nu}^N)$ such that $\text{supp } \phi \subset \Omega_n$, we have

$$\int_{\mathbb{R}_{+, \nu}^N} (\nabla w_n \nabla \phi + w_n \phi) dx = \int_{\mathbb{R}_{+, \nu}^N} (f(w_n) + w_n^{2^*-1}) \phi dx. \quad (2.33)$$

From (2.33), usual arguments show that $w \in H^1(\mathbb{R}^N) \cap C^2(\mathbb{R}_+^N)$ and satisfies $-\Delta w + w = f(w) + w^{2^*-1}$, in $\mathbb{R}_{+, \nu}^N$, and $w \equiv 0$ in $\mathbb{R}_{-, \nu}^N$. Theorem I.1, due to Esteban and Lions in [4], shows that $w \equiv 0$ which contradicts

$$\int_{B_{2R}(0) \cap \mathbb{R}_{+, \nu}^N} w^2 dx \geq \beta > 0. \quad (2.34)$$

This completes the proof of the lemma. $\square$

Now we will consider the translation of v_ε , defined by $\omega_\varepsilon(x) = v_\varepsilon(x + y_\varepsilon) = u_\varepsilon(\varepsilon y_\varepsilon + \varepsilon x)$ in $\tilde{\Omega}_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon y_\varepsilon + \varepsilon x \in \Omega\}$ and $\omega_\varepsilon = 0$ outside $\tilde{\Omega}_\varepsilon$. From (2.23), any compact subset of $\mathbb{R}^N$ is contained in $\tilde{\Omega}_\varepsilon$, for ε sufficiently small.

From Lemma 2.5,

$$\int_{B_R(0)} \omega_\varepsilon^2 dx \geq \beta > 0, \quad \forall 0 < \varepsilon \leq \varepsilon_o. \quad (2.35)$$

Consider a sequence $\varepsilon_n \searrow 0$ and set $\tilde{\Omega}_n = \tilde{\Omega}_{\varepsilon_n}$, $\omega_n = \omega_{\varepsilon_n}$, $v_n = v_{\varepsilon_n}$, $y_\varepsilon = y_{\varepsilon_n}$.

We will prove that ω_n is bounded in the L^∞ norm. In that case, u_ε is also bounded in $L^\infty(\Omega)$ norm and the proof of [Theorem 1.1](#) follows from the subcritical case, as [Lemma 2.8](#) will show.

Since the sequence ω_n a translation of v_n , we have a uniform bound for $\|\omega_n\|$ and there is a $\omega_o \in H^1(\mathbb{R}^N)$ which is weak limit of ω_n in $H^1(\mathbb{R}^N)$. From [\(2.35\)](#) we have $\omega_o \neq 0$. We can write limit [\(2.23\)](#) in the following form

$$\lim_{n \rightarrow \infty} d(0, \partial \tilde{\Omega}_n) = \infty. \quad (2.36)$$

Then for each $\phi \in C_0^\infty(\mathbb{R}^N)$ and large n such that $\text{supp } \phi \subset \tilde{\Omega}_n$, we have

$$\int_{\mathbb{R}^N} (\nabla \omega_n \nabla \phi + \omega_n \phi) dx = \int_{\mathbb{R}^N} (f(\omega_n) + \omega_n^{2^*-1}) \phi dx, \quad \forall n. \quad (2.37)$$

From [\(2.37\)](#), usual arguments show that ω_o is a solution of problem [\(2.4\)](#), hence a critical point of J , and $J(\omega_o) \geq c$.

LEMMA 2.6. *The sequence ω_n converges to ω_o in $H^1(\mathbb{R}^N)$ and $J(\omega_o) = c$.*

Proof. This fact comes from [Lemma 2.5](#) and Fatou's lemma applied in the positive sequence $\omega_n f(\omega_n) - \theta F(\omega_n)$. Observe that

$$\begin{aligned} b_{\varepsilon_n} &= J(v_n) - \frac{1}{\theta} J'(v_n) v_n \\ &= \left(\frac{\theta - 2}{2\theta} \right) \|v_n\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^N} [v_n f(v_n) - \theta F(v_n)] + \left(\frac{2^* - \theta}{2^* \theta} \right) \int_{\mathbb{R}^N} v_n^{2^*} \\ &= \left(\frac{\theta - 2}{2\theta} \right) \|\omega_n\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^N} [\omega_n f(\omega_n) - \theta F(\omega_n)] + \left(\frac{2^* - \theta}{2^* \theta} \right) \int_{\mathbb{R}^N} \omega_n^{2^*}. \end{aligned} \quad (2.38)$$

From [\(2.38\)](#)

$$\begin{aligned} c &\leq J(\omega_o) = J(\omega_o) - \frac{1}{\theta} J'(\omega_o) \omega_o \\ &= \left(\frac{\theta - 2}{2\theta} \right) \|\omega_o\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^N} [\omega_o f(\omega_o) - \theta F(\omega_o)] + \left(\frac{2^* - \theta}{2^* \theta} \right) \int_{\mathbb{R}^N} \omega_o^{2^*} \\ &\leq \liminf \left(\frac{\theta - 2}{2\theta} \right) \|\omega_n\|^2 + \frac{1}{\theta} \int_{\mathbb{R}^N} [\omega_n f(\omega_n) - \theta F(\omega_n)] + \left(\frac{2^* - \theta}{2^* \theta} \right) \int_{\mathbb{R}^N} \omega_n^{2^*} \\ &= \lim_{n \rightarrow \infty} b_{\varepsilon_n} = c. \end{aligned} \quad (2.39)$$

We have proved that $J(\omega_o) = c$ and then [\(2.39\)](#) becomes an equality.

Combining (2.39) with the three following inequalities:

$$\begin{aligned} \|\omega_o\|^2 &\leq \liminf \|\omega_n\|^2, \\ \int_{\mathbb{R}^N} [\omega_o f(\omega_o) - \theta F(\omega_o)] dx &\leq \liminf \int_{\mathbb{R}^N} [\omega_n f(\omega_n) - \theta F(\omega_n)] dx, \\ \int_{\mathbb{R}^N} \omega_o^{2^*} dx &\leq \liminf \int_{\mathbb{R}^N} \omega_n^{2^*} dx, \end{aligned} \quad (2.40)$$

we conclude that $\|\omega_n\| \rightarrow \|\omega_o\|$ and then $\omega_n \rightarrow \omega_o$ in $H^1(\mathbb{R}^N)$. $\square$

We are ready to conclude the proof of our main result. From [Proposition 1.2](#) and [Remark 1.3](#) with $q(x) = \omega_n^{2^*-2} \in L^{N/2}$; $g(x, s) = f(s) + s^{2^*} - s$, we have $\omega_n \in L^t$ for all $t \geq 2$ and

$$\|\omega_n\|_{L^t} \leq C_t, \quad (2.41)$$

where C_t does not depend on n .

Now we will make use of a very particular version of [\[5, Theorem 8.17\]](#), due to Trudinger.

PROPOSITION 2.7. *Suppose that $t > N$, $g \in L^{t/2}(\Lambda)$, and $u \in H_0^1(\Lambda)$ satisfies (in the weak sense)*

$$-\Delta u + u \leq \tilde{g}(x), \quad (2.42)$$

where Λ is an open subset of $\mathbb{R}^N$. Then for any ball $B_{2R}(y) \subset \Lambda$,

$$\sup_{B_R(y)} u \leq C(\|u^+\|_{L^2(B_{2R}(y))} + \|g\|_{L^{t/2}(B_{2R}(y))}), \quad (2.43)$$

where C depends on N , t , and R .

We know that each ω_n satisfies

$$-\Delta \omega_n + \omega_n = \omega_n^{2^*-1} + f(\omega_n), \quad \text{in } \tilde{\Omega}_n \quad (2.44)$$

and this implies that

$$-\Delta \omega_n + \omega_n \leq g_n(x) = \omega_n^{2^*-1} + f(\omega_n), \quad \text{in } \mathbb{R}^N \quad (2.45)$$

in the weak sense.

Since (2.41) holds, $\|g_n\|_{L^t}$ is bounded from above for some $t > N$. Using [Proposition 2.7](#) in (2.45) we have

$$\sup_{B_1(y)} \omega_n \leq C(\|\omega_n\|_{L^2(B_{2R}(y))} + \|g_n\|_{L^t(B_{2R}(y))}) \quad (2.46)$$

for all $y \in \mathbb{R}^N$, which implies that there is a constant $a > 0$, independent of n , such that

$$\omega_n(x) \leq a, \quad \forall x \in \mathbb{R}^N. \quad (2.47)$$

It follows that there is a $\varepsilon_0 > 0$ such that

$$u_\varepsilon(z) \leq a, \quad \forall z \in \Omega, \quad \forall \varepsilon < \varepsilon_0. \quad (2.48)$$

To conclude the proof observe that u_ε becomes a solution of the subcritical case (1.8) with h given by

$$h(s) = \begin{cases} f(s) + s^{2^*-1}, & \text{if } s \leq a, \\ f(s) + \frac{(2^* - 1)}{(\theta - 1)} a^{2^* - \theta} s^{\theta-1} - \frac{(2^* - \theta)}{(\theta - 1)} a^{2^*-1}, & \text{if } s > a, \end{cases} \quad (2.49)$$

where $\theta > 2$ is that one fixed in condition (f_4) . It is easy to check that h is a $C^{1,\alpha}$ function, h and $H(s) = \int_0^s h(\tau) d\tau$ satisfy (f_1) , (f_2) , (f_3) , (f_4) , and (f_5) . Let $\tilde{J}_\varepsilon$ be the C^1 -functional on $H_0^1(\Omega)$ given by

$$\tilde{J}_\varepsilon(u) = \frac{1}{2} \int_\Omega (\varepsilon^2 |\nabla u|^2 + u^2) dz - \int_\Omega H(u) dz. \quad (2.50)$$

Since $f(s) + s^{2^*-1} \geq h(s)$ for all $s > 0$, we have that

$$J_\varepsilon(u) \leq \tilde{J}_\varepsilon(u), \quad \forall u \in H_0^1(\Omega), \quad (2.51)$$

$J_\varepsilon(u_\varepsilon) = \tilde{J}_\varepsilon(u_\varepsilon)$, $J'_\varepsilon(u_\varepsilon) = \tilde{J}'_\varepsilon(u_\varepsilon) = 0$. We conclude that u_ε is a least-energy solution of the subcritical problem (1.8).

LEMMA 2.8. (i) If $\tilde{c}_\varepsilon$ is the minimax level of $\tilde{J}_\varepsilon$, then $\tilde{c}_\varepsilon = c_\varepsilon$;

(ii) each u_ε is a critical point of $\tilde{J}_\varepsilon$ in the minimax level and satisfies (1.8).

Since global problem (1.9) has a unique nondegenerate positive solution (cf. [9, 10]), Theorem 1.1 comes from [8, Theorem 2.2] applied to the functional $\tilde{J}_\varepsilon$, and the asymptotic profile comes from [8, Theorem 2.3].

Appendix

Let Λ be some general domain in $\mathbb{R}^N$ (bounded or unbounded). We will start with the following lemma due to Brézis and Kato [2].

LEMMA A.1. Let $q \in L^{N/2}(\Lambda)$ be a nonnegative function. Then, for every $\varepsilon > 0$, there is a constant $\sigma_\varepsilon = \sigma(\varepsilon, q) > 0$ such that

$$\int_\Lambda q(x) u^2 dx \leq \varepsilon \int_\Lambda |\nabla u|^2 dx + \sigma_\varepsilon \int_\Lambda u^2 dx, \quad \forall u \in H_0^1(\Lambda). \quad (A.1)$$

Remark A.2. If $q_k \rightarrow q$ in $L^{N/2}(\Lambda)$, we can choose a constant σ_ε independent of k . That is, $\sigma(\varepsilon, q_k) = \sigma_\varepsilon$ and

$$\int_{\Lambda} q_k(x) u^2 dx \leq \varepsilon \int_{\Lambda} |\nabla u|^2 dx + \sigma_\varepsilon \int_{\Lambda} u^2 dx, \quad \forall u \in H_o^1(\Lambda), k \in \mathbb{N}. \quad (\text{A.2})$$

Proof. Let $\sigma_\varepsilon = \sigma(\varepsilon, q) > 0$ be such that

$$\|q\|_{L^{N/2}(\{q \geq \sigma_\varepsilon\})} \leq \varepsilon S, \quad (\text{A.3})$$

where S is a best constant in the Sobolev immersion $H_o^1(\Lambda) \hookrightarrow L^{2^*}(\Lambda)$, where $2^* = 2N/(N-2)$. For all $u \in H_o^1(\Lambda)$, we have

$$\begin{aligned} \int_{\Lambda} q(x) u^2 dx &= \int_{\{q \geq \sigma_\varepsilon\}} q(x) u^2 dx + \int_{\{q \leq \sigma_\varepsilon\}} q(x) u^2 dx \\ &\leq \sigma_\varepsilon \int_{\{q \leq \sigma_\varepsilon\}} u^2 dx + \int_{\{q \geq \sigma_\varepsilon\}} q(x) u^2 dx \\ &\leq \sigma_\varepsilon \int_{\Lambda} u^2 dx + \|q\|_{L^{N/2}(\{q \geq \sigma_\varepsilon\})} \|u\|_{L^{2^*}(\{q \geq \sigma_\varepsilon\})}^2. \end{aligned} \quad (\text{A.4})$$

Inequality (A.1) follows from Sobolev estimate and the choice of σ_ε . $\square$

Remark 1.3 follows from the proof of [Lemma A.1](#) and the inequality

$$\int_{\Lambda} q_k(x) u^2 dx \leq \int_{\Lambda} q(x) u^2 dx + \|q_k - q\|_{L^{N/2}(\Lambda)} \|u\|_{L^{2^*}(\Lambda)}^2. \quad (\text{A.5})$$

Proof of Proposition 1.2. For any $n \in \mathbb{N}$ and $p > 0$, consider $A_n = \{x \in \Lambda : |v|^p \leq n\}$, $B_n = \Lambda \setminus A_n$, and define v_n by

$$v_n = v|v|^{2p} \quad \text{in } A_n, \quad v_n = n^2 v \quad \text{in } B_n. \quad (\text{A.6})$$

Observe that $v_n \in H_o^1(\Lambda)$, $v_n \leq |v|^{2p+1}$ and

$$\nabla v_n = (2p+1)|v|^{2p} \nabla v \quad \text{in } A_n, \quad \nabla v_n = n^2 \nabla v \quad \text{in } B_n. \quad (\text{A.7})$$

So, using v_n as a test function

$$\int_{\Lambda} \nabla v \nabla v_n dx = \int_{\Lambda} g(x, v) v_n dx. \quad (\text{A.8})$$

Using (A.7), we have

$$\begin{aligned} (2p+1) \int_{A_n} |v|^{2p} |\nabla v|^2 dx + n^2 \int_{B_n} |\nabla v|^2 dx \\ \leq \int_{\Lambda} |g(x, v) v_n| dx \leq \int_{\Lambda} (q(x) + C_g) |v v_n| dx. \end{aligned} \quad (\text{A.9})$$

Now consider

$$\omega_n = v|v|^p \quad \text{in } A_n, \quad \omega_n = nv \quad \text{in } B_n. \quad (\text{A.10})$$

Notice that $\omega_n^2 = v\nu_n \leq |v|^{2(p+1)}$ and

$$\nabla \omega_n = (p+1)|v|^p \nabla v \quad \text{in } A_n, \quad \nabla \nu_n = n \nabla v \quad \text{in } B_n. \quad (\text{A.11})$$

Therefore,

$$\int_{\Lambda} |\nabla \omega_n|^2 dx = (p+1)^2 \int_{A_n} |v|^{2p} |\nabla v|^2 dx + n^2 \int_{B_n} |\nabla v|^2 dx. \quad (\text{A.12})$$

Combining (A.9) and (A.12), we obtain

$$\frac{2p+1}{(p+1)^2} \int_{\Lambda} |\nabla \omega_n|^2 dx \leq \int_{\Lambda} (q(x) + C_g) \omega_n^2 dx. \quad (\text{A.13})$$

Let σ_p be given by Lemma A.1 with $\varepsilon = (2p+1)/2(p+1)^2$. Then

$$\int_{\Lambda} |\nabla \omega_n|^2 dx \leq \tilde{C}_p \int_{\Lambda} \omega_n^2 dx, \quad (\text{A.14})$$

where $\tilde{C}_n = (2(p+1)^2/(2p+1))(C_g + \sigma_p)$. Suppose that $v \in L^{2(p+1)}(\Lambda)$ for some $p \geq 2$. Applying Sobolev immersion in inequality (A.14) we have

$$\left[\int_{A_n} \omega_n^{2^*} dx \right]^{2/2^*} \leq \left[\int_{\Lambda} \omega_n^{2^*} dx \right]^{2/2^*} \leq S \tilde{C}_p \int_{\Lambda} |v|^{2(p+1)} dx \quad (\text{A.15})$$

that is,

$$\left[\int_{A_n} |v|^{2^*(p+1)} dx \right]^{2/2^*} dx \leq C_p \int_{\Lambda} |v|^{2(p+1)} dx, \quad (\text{A.16})$$

where

$$C_p = \frac{2(p+1)^2}{2p+1} S(C_g + \sigma_p). \quad (\text{A.17})$$

Now, passing to the limit in (A.16) we have $v \in L^{2^*(p+1)}(\Lambda)$ and

$$\|v\|_{L^{2^*(p+1)}(\Lambda)} \leq C_p \|v\|_{L^{2(p+1)}(\Lambda)}. \quad (\text{A.18})$$

The proof follows from the following iteration argument: let p_1 a positive such that $2(p_1+1) = 2^*$. It is easy to see that $0 < p_1$ and $v \in L^{2(p_1+1)}(\Lambda)$. Using inequality (A.18) we have

$$v \in L^{2^*(p_1+1)}(\Lambda). \quad (\text{A.19})$$

Now choose p_2 such that $2(p_2 + 1) = 2^*(p_1 + 1)$. It is easy to see that $0 < p_1 < p_2$ and $v \in L^{p_2+1}(\Lambda)$. Using inequality (A.18) we have

$$v \in L^{2^*(p_2+1)}(\Lambda). \quad (\text{A.20})$$

Continuing with this iteration we obtain an increasing sequence p_k given by $2(p_{k+1} + 1) = 2^*(p_k + 1)$ such that $v \in L^{2(p_{k+1}+1)}(\Lambda)$ for all $k \in \mathbb{N}$. From

$$p_{k+1} + 1 = \frac{N}{N-2}(p_k + 1), \quad (\text{A.21})$$

it follows that

$$p_{k+1} + 1 = \left[\frac{N}{N-2} \right]^k 2^*. \quad (\text{A.22})$$

This shows that p_k goes to ∞ and therefore,

$$v \in L^p(\Lambda), \quad \forall p \geq 2. \quad (\text{A.23})$$

□

Remark A.3. [Proposition 1.2](#) is valid for positive subsolutions of problem (1.12) as we can check in its proof.

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