GENERIC UNIQUENESS OF A MINIMAL SOLUTION
FOR VARIATIONAL PROBLEMS ON A TORUS

ALEXANDER J. ZASLAVSKI

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We study minimal solutions for one-dimensional variational problems on a torus. We show that, for a generic integrand and any rational number $\alpha$, there exists a unique (up to translations) periodic minimal solution with rotation number $\alpha$.

1. Introduction

In this paper, we consider functionals of the form

$$I^f(a, b, x) = \int_a^b f(t, x(t), x'(t)) \, dt,$$

where $a$ and $b$ are arbitrary real numbers satisfying $a < b$, $x \in W^{1,1}(a, b)$ and $f$ belongs to a space of functions described below. By an appropriate choice of representatives, $W^{1,1}(a, b)$ can be identified with the set of absolutely continuous functions $x : [a, b] \to \mathbb{R}$, and henceforth we will assume that this has been done.

Denote by $\mathbb{M}$ the set of integrands $f = f(t, x, p) : \mathbb{R}^3 \to \mathbb{R}$ which satisfy the following assumptions:

(A1) $f \in C^3$ and $f(t, x, p)$ has period 1 in $t, x$;
(A2) $\delta f \leq f_{pp}(t, x, p) \leq \delta f^{-1}$ for every $(t, x, p) \in \mathbb{R}^3$;
(A3) $|f_{xp}| + |f_{tp}| \leq c_f(1 + |p|), |f_{xx}| + |f_{xt}| \leq c_f(1 + p^2),$

with some constants $\delta_f \in (0, 1)$, $c_f > 0$.

Clearly, these assumptions imply that

$$\tilde{\delta}_f p^2 - \tilde{c}_f \leq f(t, x, p) \leq \tilde{\delta}_f^{-1} p^2 + \tilde{c}_f$$

for every $(t, x, p) \in \mathbb{R}^3$ for some constants $\tilde{c}_f > 0$ and $0 < \tilde{\delta}_f < \delta_f$.

In this paper, we analyse extremals of variational problems with integrands $f \in \mathbb{M}$. The following optimality criterion was introduced by Aubry and Le
Uniqueness of a minimal solution


Let \( f \in \mathcal{M} \). A function \( x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) is called an \((f)\)-minimal solution if

\[
I^f(a, b, y) \geq I^f(a, b, x)
\]

for each pair of numbers \( a < b \) and each \( y \in W^{1,1}(a, b) \) which satisfies \( y(a) = x(a) \) and \( y(b) = x(b) \) (see [2, 9, 10, 12]).

Our work follows Moser [9, 10], who studied the existence and structure of minimal solutions in the spirit of Aubry-Mather theory [2, 7].

Consider any \( f \in \mathcal{M} \). It was shown in [9, 10] that \((f)\)-minimal solutions possess numerous remarkable properties. Thus, for every \((f)\)-minimal solution \( x(\cdot) \), there is a real number \( \alpha \) satisfying

\[
\sup \{ |x(t) - \alpha t| : t \in \mathbb{R}^1 \} < \infty
\]

which is called the rotation number of \( x(\cdot) \), and given any real \( \alpha \) there exists an \((f)\)-minimal solution with rotation number \( \alpha \). Senn [11] established the existence of a strictly convex function \( E_f : \mathbb{R}^1 \to \mathbb{R}^1 \), which is called the minimal average action of \( f \) such that, for each real \( \alpha \) and each \((f)\)-minimal solution \( x(\cdot) \) with rotation number \( \alpha \),

\[
(T_2 - T_1)^{-1} I^f(T_1, T_2, x) \to E_f(\alpha) \quad \text{as} \quad T_2 - T_1 \to \infty.
\]

This result is an analogue of Mather’s theorem about the average energy function for Aubry-Mather sets generated by a diffeomorphism of the infinite cylinder [8].

In this paper, we show that for a generic integrand \( f \) and any rational \( \alpha \), there exists a unique (up to translations) \((f)\)-minimal periodic solution with rotation number \( \alpha \).

Let \( k \geq 3 \) be an integer. Set \( \mathcal{M}_k = \mathcal{M} \cap C^k(\mathbb{R}^3) \). For \( f \in \mathcal{M}_k \) and \( q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3 \) satisfying \( q_1 + q_2 + q_3 \leq k \), we set

\[
|q| = q_1 + q_2 + q_3, \quad D^q f = \frac{\partial |q| f}{\partial t^{q_1} \partial x^{q_2} \partial p^{q_3}}.
\]

For \( N, \varepsilon > 0 \) we set

\[
E_k(N, \varepsilon) = \{(f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)| \\
\leq \varepsilon + \varepsilon \max \{|D^q f(t, x, p)|, |D^q g(t, x, p)|\} \\
\forall q \in \{0, 1, 2\}^3 \text{ satisfying } |q| \in \{0, 2\}, \forall (t, x, p) \in \mathbb{R}^3 \} \cap \{(f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)| \leq \varepsilon \\
\forall q \in \{0, \ldots, k\}^3 \text{ satisfying } |q| \leq k, \forall (t, x, p) \in \mathbb{R}^3 \} \quad \text{such that } |p| \leq N.
\]

(1.7)
2. Properties of minimal solutions

Consider any \( f \in \mathcal{M} \). We note that, for each pair of integers \( j \) and \( k \) the translations \( (t, x) \rightarrow (t + j, x + k) \) leave the variational problem invariant. Therefore, if \( x(\cdot) \) is an \((f)\)-minimal solution, so is \( x(\cdot) + j + k \). Of course, on the torus, this represents the same curve as does \( x(\cdot) \). This motivates the following terminology [9, 10].

We say that a function \( x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) has no self-intersections if for all pairs of integers \( j, k \) the function \( t \rightarrow x(t) + j + k - x(t) \) is either always positive, or always negative, or identically zero.

Denote by \( \mathbb{Z} \) the set of all integers. We have the following result (see [6, Proposition 3.2] and [9, 10]).

**Proposition 2.1.** (i) Let \( f \in \mathcal{M} \). Given any real \( \alpha \) there exists a nonself-intersecting \((f)\)-minimal solution with rotation number \( \alpha \).

(ii) For any \( f \in \mathcal{M} \) and any \((f)\)-minimal solution \( x \), there is the rotation number of \( x \).

For each \( f \in \mathcal{M} \), each rational number \( \alpha \), and each natural number \( q \) satisfying \( qa \in \mathbb{Z} \), we define

\[
\mathcal{N}(\alpha, q) = \{ x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) : x(t + q) = x(t) + aq, \ t \in \mathbb{R}^1 \},
\]

\[
\mathcal{M}_f(\alpha, q) = \{ x(\cdot) \in \mathcal{N}(\alpha, q) : I_f'(0, q, x) \leq I_f'(0, q, y) \ \forall y \in \mathcal{N}(\alpha, q) \}. \tag{2.1}
\]

We have the following result [9, Theorems 5.1, 5.2, 5.4, and Corollaries 5.3 and 5.5].

**Proposition 2.2.** Let \( f \in \mathcal{M} \), let \( \alpha \) be a rational number, and let \( p, q \geq 1 \) be integers satisfying \( pa, qa \in \mathbb{Z} \). Then \( \mathcal{M}_f(\alpha, q) = \mathcal{M}_f(\alpha, p) \neq \emptyset \), each \( x \in \mathcal{M}_f(\alpha, q) \) is a nonself-intersecting \((f)\)-minimal solution with rotation number \( \alpha \) and the set \( \mathcal{M}_f(\alpha, q) \) is totally ordered, that is, if \( x, y \in \mathcal{M}_f(\alpha, q) \), then either \( x(t) < y(t) \) for all \( t \), or \( x(t) > y(t) \) for all \( t \), or \( x(t) = y(t) \) identically.

For any \( f \in \mathcal{M} \) and any rational number \( \alpha \) we set \( \mathcal{M}^\text{per}_f(\alpha) = \mathcal{M}_f(\alpha, q) \), where \( q \) is a natural number satisfying \( qa \in \mathbb{Z} \).

We have the following result (see [6, Theorem 1.1]).

**Proposition 2.3.** Let \( f \in \mathcal{M} \). Then there exist a strictly convex function \( E_f : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) satisfying \( E_f(\alpha) \rightarrow \infty \) as \( |\alpha| \rightarrow \infty \) and a monotonically increasing function \( \Gamma_f : (0, \infty) \rightarrow [0, \infty) \) such that for each real \( \alpha \), each \((f)\)-minimal solution \( x \) with
rotation number $\alpha$ and each pair of real numbers $S$ and $T$,\[ |I^f(S, S + T, x) - E_f(\alpha)T| \leq \Gamma_f(|\alpha|). \] (2.2)

By Proposition 2.3 for each $f \in M$ there exists a unique number $\alpha(f)$ such that\[ E_f(\alpha(f)) = \min \{ E_f(\beta) : \beta \in \mathbb{R}^1 \}. \] (2.3)

Note that assumptions (A1), (A2), and (A3) play an important role in the proofs of Propositions 2.1, 2.2, and 2.3 (see [9, 10]).

3. The main results

Theorem 3.1. Let $k \geq 3$ be an integer and $\alpha$ be a rational number. Then there exists a set $\mathcal{F}_k \subset \mathcal{M}_k$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_k$ such that for each $f \in \mathcal{M}_k$ the following assertions hold:

(1) If $x, y \in \mathcal{M}_f^{\text{per}}(\alpha)$, then there are integers $p, q$ such that $y(t) = x(t + p) - q$ for all $t \in \mathbb{R}^1$.

(2) Let $x \in \mathcal{M}_f^{\text{per}}(\alpha)$ and $\epsilon > 0$. Then there exists a neighborhood $\mathcal{V}$ of $f$ in $\mathcal{M}_k$ such that for each $g \in \mathcal{V}$ and each $y \in \mathcal{M}_g^{\text{per}}(\alpha)$ there are integers $p, q$ such that $|y(t) - x(t + p) + q| \leq \epsilon$ for all $t \in \mathbb{R}^1$.

It is not difficult to see that Theorem 3.1 implies the following result.

Theorem 3.2. Let $k \geq 3$ be an integer. Then there exists a set $\mathcal{F}_k \subset \mathcal{M}_k$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_k$ such that, for each $f \in \mathcal{M}_k$ and each rational number $\alpha$ the assertions (1) and (2) of Theorem 3.1 hold.

Note that minimal solutions with irrational rotation numbers were studied in [2, 7, 9, 10, 12].

4. An auxiliary result

Let $k \geq 3$ be an integer and $\beta \in \mathbb{R}^1$. For each $f \in \mathcal{M}_k$, define $\mathcal{A}f \in C^3(\mathbb{R}^3)$ by\[ (\mathcal{A}f)(t, x, u) = f(t, x, u) - \beta u, \quad (t, x, u) \in \mathbb{R}^3. \] (4.1)

Clearly $\mathcal{A}f \in \mathcal{M}_k$ for each $f \in \mathcal{M}_k$.

Proposition 4.1. The mapping $\mathcal{A} : \mathcal{M}_k \to \mathcal{M}_k$ is continuous.

Proof. Let $f \in \mathcal{M}_k$ and let $N, \epsilon > 0$. In order to prove the proposition, it is sufficient to show that there exists $\epsilon_0 \in (0, \epsilon)$ such that\[ \mathcal{A}(\{ g \in \mathcal{M}_k : (f, g) \in E_k(N, \epsilon_0) \}) \subset \{ h \in \mathcal{M}_k : (h, \mathcal{A}f) \in E_k(N, \epsilon) \}. \] (4.2)

Set\[ \Delta_0 = 2(|\beta| + 1). \] (4.3)
Equation (1.2) implies that there exists $c_0 > 0$ such that
\[ \Delta_0|u| - c_0 \leq f(t, x, u) \quad \forall(t, x, u) \in \mathbb{R}^3. \] (4.4)

Choose a number $\epsilon_0$ such that
\[ 0 < \epsilon_0 < \min\{1, \epsilon\}, \quad 4\epsilon_0 + 4\epsilon_0(1 - \epsilon_0)^{-1}(4 + c_0) < \epsilon. \] (4.5)

It follows from (4.3) and (4.4) that for each $(t, x, u) \in \mathbb{R}^3$,
\begin{align*}
|f(t, x, u) - \beta u| &\geq |f(t, x, u)| - |\beta u| \geq |f(t, x, u)| - |\beta|\Delta_0^{-1}(f(t, x, u) + c_0) \\
&\geq |f(t, x, u)|(1 - |\beta|\Delta_0^{-1}) - |\beta|\Delta_0^{-1}c_0 \\
&\geq 2^{-1}|f(t, x, u)| - 2^{-1}c_0.
\end{align*} (4.6)

Assume that
\[ g \in \mathcal{M}_k, \quad (f, g) \in E_k(N, \epsilon_0). \] (4.7)

By (1.7) and (4.7) for each $(t, x, u) \in \mathbb{R}^3$,
\begin{align*}
|f(t, x, u) - g(t, x, u)| &\leq \epsilon_0 + \epsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\}, \\
\max\{|f(t, x, u)|, |g(t, x, u)|\} - \min\{|f(t, x, u)|, |g(t, x, u)|\} &\leq \epsilon_0 + \epsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\}, \\
(1 - \epsilon_0) \max\{|f(t, x, u)|, |g(t, x, u)|\} - \min\{|f(t, x, u)|, |g(t, x, u)|\} &\leq \epsilon_0 + \epsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\}.
\end{align*} (4.8)

We show that $(\mathcal{A}f, \mathcal{A}g) \in E_k(N, \epsilon)$. It follows from (1.7), (4.1), (4.5), and (4.7) that, for each $q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3$ satisfying $|q| \leq k$ and each $(t, x, p) \in \mathbb{R}^3$ satisfying $|p| \leq N$,
\[ |D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)| \leq \epsilon_0 < \epsilon. \] (4.9)

Let $q \in \{0, 1, 2\}^3$, $|q| \in \{0, 2\}$, and $(t, x, p) \in \mathbb{R}^3$. Equation (4.1) implies that
\[ |D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)|. \] (4.10)

If $|q| = 2$, then by (1.7), (4.1), (4.5), (4.7), and (4.10),
\begin{align*}
|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| &\leq \epsilon_0 + \epsilon_0 \max\{|D^q f(t, x, p)|, |D^q g(t, x, p)|\} \\
&< \epsilon + \epsilon \max\{|D^q(\mathcal{A}f)(t, x, p)|, |D^q(\mathcal{A}g)(t, x, p)|\}.
\end{align*} (4.11)
Assume that \( q = 0 \). By (1.7), (4.1), (4.5), (4.6), (4.7), and (4.8),

\[
\begin{align*}
|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| & \\
& = |f(t, x, p) - g(t, x, p)| \\
& \leq \varepsilon_0 + \varepsilon_0 \max \{|f(t, x, p)|, |g(t, x, p)|\} \\
& \leq \varepsilon_0 + \varepsilon_0 \max \{|f(t, x, p)|, (1 - \varepsilon_0)^{-1}|f(t, x, p)| + (1 - \varepsilon_0)^{-1} \varepsilon_0\} \\
& = \varepsilon_0 + \varepsilon_0 (1 - \varepsilon_0)^{-1} |f(t, x, p)| + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} \\
& \leq \varepsilon_0 + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} + \varepsilon_0 (1 - \varepsilon_0)^{-1} [2|f(t, x, p)| - \beta p] + 2\varepsilon_0 \\
& \leq \varepsilon_0 + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} + 2\varepsilon_0 (1 - \varepsilon_0)^{-1}c_0 + 2\varepsilon_0 (1 - \varepsilon_0)^{-1}|f(t, x, p) - \beta p| \\
& \leq 2\varepsilon_0 (1 - \varepsilon_0)^{-1}|(\mathcal{A}f)(t, x, p)| + \varepsilon \leq \varepsilon + \varepsilon |(\mathcal{A}f)(t, x, p)|.
\end{align*}
\]

Equations (4.9), (4.11), and (4.12) imply that \((\mathcal{A}f, \mathcal{A}g) \in E_k(N, \varepsilon)\). Proposition 4.1 is proved.

Let \(-\infty < T_1 < T_2 < \infty\) and \(x \in W^{1,1}(T_1, T_2)\). By (4.1) we have

\[
I^{\mathcal{A}f}(T_1, T_2, x) = \int_{T_1}^{T_2} (f(t, x(t), x'(t)) - \beta x'(t)) \, dt \\
= I^f(T_1, T_2, x) - \beta x(T_2) + \beta x(T_1).
\]

Therefore, each \(x \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)\) is an \((\mathcal{A}f)\)-minimal solution if and only if \(x(\cdot)\) is an \((f)\)-minimal solution.

Let \(x \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)\) be an \((f)\)-minimal solution with rotation number \(r\). By Proposition 2.1 there exists \(c_1 > 0\) such that for all \(s, t \in \mathbb{R}^1\),

\[
|x(t + s) - x(t) - rs| \leq c_1.
\]

Proposition 2.3 implies that there exists a constant \(c_2 > 0\) such that for each \(s \in \mathbb{R}^1\) and each \(t > 0\),

\[
|I^f(s, s + t, x) - E_f(r)t| \leq c_2, \quad I^{\mathcal{A}f}(s, s + t, x) - E_{\mathcal{A}f}(r)t| \leq c_2.
\]

It follows from (4.13), (4.14), (4.15), and (4.16) that, for each \(s \in \mathbb{R}^1\) and each \(t > 0\),

\[
|E_{\mathcal{A}f}(r)t + \beta tr - E_f(r)t| \\
\leq |E_{\mathcal{A}f}(r)t - I^{\mathcal{A}f}(s, s + t, x)| + |I^f(s, s + t, x) + \beta tr - I^f(s, s + t, x)| \\
\quad + |I^f(s, s + t, x) - E_f(r)t| \\
\leq c_2 + |\beta tr - \beta[x(t + s) - x(s)]| + c_2 \leq 2c_2 + |\beta|c_1.
\]

These inequalities imply that

\[
E_{\mathcal{A}f}(r) = E_f(r) - \beta r \quad \forall r \in \mathbb{R}^1.
\]
5. Proof of Theorem 3.1

Let $g \in \mathcal{M}$. We define

$$
\mu(g) = \inf \left\{ \liminf_{T \to \infty} T^{-1} \mathcal{I}^g(0, T, x) : x(\cdot) \in W^{1,1}_{\text{loc}}([0, \infty)) \right\}. \tag{5.1}
$$

In [13, Section 5] we showed that the number $\mu(g)$ is well defined and proved the following result [13, Theorem 5.1].

**Proposition 5.1.** Let $f \in \mathcal{M}$. Then there exists a constant $M_0 > 0$ such that:

(i) $I^f(0, T, x) - \mu(f) T \geq -M_0$ for each $x \in W^{1,1}_{\text{loc}}([0, \infty))$ and each $T > 0$.

(ii) For each $a \in \mathbb{R}^1$ there exists $x \in W^{1,1}_{\text{loc}}([0, \infty))$ such that $x(0) = a$ and

$$
\left| I^f(0, T, x) - \mu(f) T \right| \leq 4M_0 \quad \forall T > 0. \tag{5.2}
$$

Note that assertion (ii) of **Proposition 5.1** holds by the periodicity of $f$ in $x$.

Let $f \in \mathcal{M}$. A function $x \in W^{1,1}_{\text{loc}}([0, \infty))$ is called $(f)$-good (see [5]) if

$$
\sup \left\{ \left| I^f(0, T, x) - \mu(f) T \right| : T \in (0, \infty) \right\} < \infty. \tag{5.3}
$$

By [6, Theorem 4.1],

$$
E_f(\alpha(f)) = \mu(f) \quad \forall f \in \mathcal{M}. \tag{5.4}
$$

For $f \in \mathcal{M}$, $x, y, T_1 \in \mathbb{R}^1$, and $T_2 > T_1$ we set

$$
U^f(T_1, T_2, x, y) = \inf \{ I^f(T_1, T_2, v) : v \in W^{1,1}(T_1, T_2), v(T_1) = x, v(T_2) = y \}. \tag{5.5}
$$

It is not difficult to see that for each $x, y, T_1 \in \mathbb{R}^1$, $T_2 > T_1$,

$$
U^f(T_1, T_2, x+1, y+1) = U^f(T_1, T_2, x, y),
$$

$$
U^f(T_1+1, T_2+1, x, y) = U^f(T_1, T_2, x, y), \quad -\infty < U^f(T_1, T_2, x, y) < \infty,
$$

$$
\inf \{ U^f(T_1, T_2, a, b) : a, b \in \mathbb{R}^1 \} > -\infty. \tag{5.6}
$$

Denote by $\mathcal{M}_{\text{per}}$ the set of all $f \in \mathcal{M}$ such that $\alpha(f)$ is rational and denote by $\mathcal{M}_{\text{per}}^0$ the set of all $g \in \mathcal{M}_{\text{per}}$ for which there exist an $(g)$-minimal solution $w \in C^2(\mathbb{R}^1)$, a continuous function $\pi : \mathbb{R}^1 \to \mathbb{R}^1$, and integers $m, n$ such that the following properties hold:

(P1) $\pi(x+1) = \pi(x), x \in \mathbb{R}^1$;

(P2) $n \geq 1$ and $\alpha(g) = mn^{-1}$ is an irreducible fraction;

(P3) $w(t+n) = w(t) + m$ for all $t \in \mathbb{R}^1$;

(P4) $U^g(0, 1, x, y) - \mu(g) - \pi(x) + \pi(y) \geq 0$ for each $x, y \in \mathbb{R}^1$;

(P5) for any $u \in W^{1,1}(0, n)$, the equality

$$
I^g(0, n, u) = n\mu(g) + \pi(u(0)) - \pi(u(n)) \tag{5.7}
$$

holds if and only if there are integers $i, j$ such that $u(t) = w(t+i) - j$ for all $t \in [0, n]$.
Consider the manifold $(\mathbb{R}^1/\mathbb{Z})^2$ and the canonical mapping $P : \mathbb{R}^2 \rightarrow (\mathbb{R}^1/\mathbb{Z})^2$. We have the following result [13, Proposition 6.2].

**Proposition 5.2.** Let $\Omega$ be a closed subset of $(\mathbb{R}^1/\mathbb{Z})^2$. Then there exists a bounded nonnegative function $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ such that

$$\Omega = \{ x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0 \}. \quad (5.8)$$

**Proposition 5.2** is proved by using [1, Chapter 2, Section 3, Theorem 1] and the partition of unity (see [4, Appendix 1]).

We also have the following result (see [13, Proposition 6.3]).

**Proposition 5.3.** Suppose that $f \in M_{\text{per}}$, $\alpha(f) = mn^{-1}$ is an irreducible fraction ($m, n$ are integers, $n \geq 1$) and $w \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ is an $(f)$-minimal solution satisfying $w(t + n) = w(t) + m$ for all $t \in \mathbb{R}^1$. Let $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ be as guaranteed in Proposition 5.2 with

$$\Omega = \{ P(t, w(t)) : t \in [0, n] \}, \quad (5.9)$$

and let

$$g(t, x, p) = f(t, x, p) + \phi(P(t, x)), \quad (t, x, p) \in \mathbb{R}^3. \quad (5.10)$$

Then $g \in M^0_{\text{per}}$ and there is a continuous function $\pi : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ such that the properties (P1), (P2), (P3), (P4), and (P5) hold with $g, w, \pi, m, n$ and $\alpha(g) = \alpha(f)$.

In the sequel we need the following two lemmas proved in [13].

**Lemma 5.4** [13, Lemma 6.6]. Assume that $k \geq 3$ is an integer, $g \in M^0_{\text{per}} \cap M_k$, and properties (P1), (P2), (P3), (P4), and (P5) hold with a $g$-minimal solution $w(\cdot) \in C^2(\mathbb{R}^1)$, a continuous function $\pi : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ and integers $m, n$. Then for each $e \in (0, 1)$, there exists a neighborhood $\mathcal{U}$ of $g$ in $M_k$ such that for each $h \in \mathcal{U}$ and each $(h)$-good function $v \in W^{1,1}_{\text{loc}}([0, \infty))$ there are integers $p, q$ such that

$$|v(t) - w(t + p) - q| \leq e \quad \text{for all large enough } t. \quad (5.11)$$

**Lemma 5.5** [13, Corollary 6.1]. Assume that $k \geq 3$ is an integer, $g \in M^0_{\text{per}} \cap M_k$, and properties (P1), (P2), (P3), (P4), and (P5) hold with a $g$-minimal solution $w(\cdot) \in C^2(\mathbb{R}^1)$, a continuous function $\pi : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ and integers $m, n$. Then there exist a neighborhood $\mathcal{U}$ of $g$ in $M_k$ and a number $L > 0$ such that for each $h \in \mathcal{U}$ and each $(h)$-good function $v \in W^{1,1}_{\text{loc}}([0, \infty))$, the following property holds.

There is a number $T_0 > 0$ such that

$$|v(t_2) - v(t_1) - \alpha(g)(t_2 - t_1)| \leq L \quad (5.12)$$

for each $t_1 \geq T_0$ and each $t_2 > t_1$. 
Completion of the proof of Theorem 3.1. Let $k \geq 3$ be an integer and let $\alpha = mn^{-1}$ be an irreducible fraction ($n \geq 1$ and $m$ are integers). Let $f \in \mathcal{M}_k$. By Proposition 2.2 there exists an $(f)$-minimal solution $w_f(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ such that
\[ w_f(t+n) = w_f(t) + m \quad \forall t \in \mathbb{R}^1. \tag{5.13} \]
Choose
\[ \beta \in \partial E_f(\alpha). \tag{5.14} \]
Consider a mapping $\mathcal{A} : \mathcal{M}_k \to \mathcal{M}_k$ defined by (4.1). By Proposition 4.1 the mapping $\mathcal{A}$ is continuous. Clearly there exists a continuous $\mathcal{A}^{-1} : \mathcal{M}_k \to \mathcal{M}_k$. Equations (5.14) and (4.18) imply that
\[ 0 \in \partial E_{\mathcal{A}f}(\alpha), \quad E_{\mathcal{A}f}(\alpha) = \min \{ E_{\mathcal{A}f}(r) : r \in \mathbb{R}^1 \} = \mu(\mathcal{A}f) \tag{5.15} \]
and that $\mathcal{A}f \in \mathcal{M}_{\text{per}}$. It follows from Proposition 5.2 that there exists a bounded nonnegative function $\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)$ such that
\[ \{ x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0 \} = \{ P(t, w_f(t)) : t \in [0, n] \}. \tag{5.16} \]
Set $f(\beta) = \mathcal{A}f$ and for each $\gamma \in (0, 1)$ define
\[ f_\gamma(t, x, u) = f(t, x, u) + \gamma \phi(P(t, x)), \quad (t, x, u) \in \mathbb{R}^3, \quad f_\gamma(\beta) = \mathcal{A}(f_\gamma). \tag{5.17} \]
Proposition 5.3 implies that for each $\gamma \in (0, 1)$,
\[ f_\gamma(\beta) \in \mathcal{M}_{\text{per}}^0 \cap \mathcal{M}_k, \]
\[ f_\gamma \to f \quad \text{as} \quad \gamma \to 0^+, \quad f_\gamma(\beta) \to f(\beta) \quad \text{as} \quad \gamma \to 0^+ \text{ in } \mathcal{M}_k. \tag{5.18} \]
Fix $\gamma \in (0, 1)$ and an integer $n \geq 1$. By Proposition 5.3 the properties (P1), (P2), (P3), (P4), and (P5) hold with $g = f_\gamma(\beta)$, $\alpha(g) = \alpha$ and $w(\cdot) = w_f$. By Lemmas 5.4 and 5.5, there exists an open neighborhood $V(f, \gamma, n)$ of $f_\gamma(\beta)$ in $\mathcal{M}_\gamma$ and a number $L(f, \gamma, n) > 0$ such that the following properties hold:
(i) for each $h \in V(f, \gamma, n)$ and each $(h)$-good function $v \in W^{1,1}_{\text{loc}}([0, \infty))$, there are integers $p, q$ such that
\[ |v(t) - w_f(t + p) - q| \leq \frac{1}{n} \tag{5.19} \]
for all large enough $t$;
(ii) for each $h \in V(f, \gamma, n)$ and each $(h)$-good function $v \in W^{1,1}_{\text{loc}}([0, \infty))$, there is a number $T_0$ such that
\[ |v(t_2) - v(t_1) - \alpha(f_\gamma(\beta))(t_2 - t_1)| \leq L \tag{5.20} \]
for each $t_1 \geq T_0$ and each $t_2 > t_1$. 
Let \( h \in V(f, \gamma, n) \) and let \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) be an \((h)\)-minimal solution with rotation number \( \alpha(h) \). Then by Proposition 2.3, (2.3), (5.4), and property (ii), \( v\big|_{[0, \infty)} \) is an \((h)\)-good function and there is \( T_0 \) such that (5.20) holds for each \( t_1 \geq T_0 \) and each \( t_2 > t_1 \). Since \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) has rotation number \( \alpha(h) \) it follows from Proposition 2.1 that there exists \( c_1 > 0 \) such that

\[
|v(t+s) - v(t) - \alpha(h)s| \leq c_1 \quad \forall s, t \in \mathbb{R}. \tag{5.21}
\]

Equations (5.15), (5.17), (5.20), and (5.21) imply that

\[
\alpha(h) = \alpha(f^{(\beta)}) = \alpha(f^{(\iota)}) = \alpha. \tag{5.22}
\]

Thus we have shown that

\[
\alpha(h) = \alpha \quad \forall h \in V(f, \gamma, n). \tag{5.23}
\]

Let \( h \in V(f, \gamma, n) \) and let \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}) \) be an \((h)\)-minimal solution with rotation number \( \alpha \). It follows from Proposition 2.3, (2.3), and (5.4) that \( v\big|_{[0, \infty)} \) is an \((h)\)-good function. By property (i) there exist integers \( p, q \) such that

\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n} \quad \text{for all large enough } t. \tag{5.24}
\]

Therefore we proved the following property:

(iii) for each \( h \in V(f, \gamma, n) \) and each \((h)\)-minimal solution \( v \in \mathcal{M}^{\text{per}}_h(\alpha) \), there exist integers \( p, q \) such that

\[
|v(t) - w_f(t + p) - q| \leq \frac{1}{n} \quad \forall t \in \mathbb{R}. \tag{5.25}
\]

Define

\[
\mathcal{U}(f, \gamma, n) = \mathcal{A}^{-1}(V(f, \gamma, n)). \tag{5.26}
\]

Clearly \( \mathcal{U}(f, \gamma, n) \) is an open neighborhood of \( f \gamma \) in \( \mathcal{M}_k \). By property (iii) the following property holds:

(iv) for each \( \xi \in \mathcal{U}(f, \gamma, n) \) and each \((\xi)\)-minimal solution \( v \in \mathcal{M}^{\text{per}}_\xi(\alpha) \), there exist integers \( p, q \) such that (5.25) holds.

Define

\[
\mathcal{F}_{k\alpha} = \cap_{n=1}^{\infty} \cup \{ \mathcal{U}(f, \gamma, i) : f \in \mathcal{M}_k, \ \gamma \in (0, 1), \ i \geq n \}. \tag{5.27}
\]

It is not difficult to see that \( \mathcal{F}_{k\alpha} \) is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \).
Let \( g \in \mathcal{F}_{k_{\alpha}}, e \in (0, 1) \) and \( x, y \in \mathcal{M}^{(\text{per})}_{g}(\alpha) \). Choose a natural number \( n > 8e^{-1} \).

By (5.27) there exist \( f \in \mathcal{M}_{k}, \gamma \in (0, 1) \) and an integer \( i \geq n \) such that
\[
g \in \mathcal{U}(f, y, i).
\]
(5.28)

It follows from (5.28) and property (iv) that there exist integers \( p_{1}, q_{1}, p_{2}, q_{2} \) such that
\[
|x(t) - w_{f}(t + p_{1}) - q_{1}| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^{1},
\]
(5.29)
\[
|y(t) - w_{f}(t + p_{2}) - q_{2}| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^{1},
\]
(5.30)

where \( w_{f} \in \mathcal{M}^{(\text{per})}_{f}(\alpha) \).

It follows from (5.29) and (5.30) that for all \( t \in \mathbb{R}^{1} \),
\[
|x(t - p_{1}) - w_{f}(t) - q_{1}| \leq \frac{1}{i},
\]
\[
|y(t - p_{2}) - w_{f}(t) - q_{2}| \leq \frac{1}{i},
\]
\[
|x(t - p_{1} - q_{1}) - (y(t - p_{2}) - q_{2})| \leq \frac{2}{i},
\]
\[
|x(t + p_{2} - p_{1}) - y(t) - q_{1} + q_{2}| \leq \frac{2}{i} \leq \frac{2}{n} < \varepsilon.
\]
(5.31)

Since \( e \) is any number in \((0, 1)\), we conclude that there exist integers \( p, q \) such that
\[
x(t + p) - q = y(t) \quad \forall t \in \mathbb{R}^{1}.
\]
(5.32)

Assume that \( h \in \mathcal{U}(f, y, i) \) and \( z \in \mathcal{M}^{(\text{per})}_{h}(\alpha) \). By the property (iv) there exist integers \( p_{3}, q_{3} \) such that
\[
|z(t) - w_{f}(t + p_{3}) - q_{3}| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^{1}.
\]
(5.33)

Combined with (5.29) this inequality implies that
\[
|z(t - p_{3}) - q_{3} - x(t - p_{1}) + q_{1}| \leq \frac{2}{i} \leq \frac{2}{n} < \varepsilon
\]
(5.34)

for all \( t \in \mathbb{R}^{1} \). This completes the proof of Theorem 3.1.

References


Uniqueness of a minimal solution


Alexander J. Zaslavski: Department of Mathematics, Technion-Israel Institute of Technology, Haifa 32000, Israel

E-mail address: ajzasl@techunix.technion.ac.il