We study a multiplicity result for the perturbed $p$-Laplacian equation

$$\Delta_p u - \lambda g(x) |u|^{p-2} u = f(x, u) + h(x) \quad \text{in} \quad \mathbb{R}^N,$$

where $1 < p < N$ and $\lambda$ is near $\lambda_1$, the principal eigenvalue of the weighted eigenvalue problem

$$\Delta_p u = \lambda g |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N.$$

Depending on which side $\lambda$ is from $\lambda_1$, we prove the existence of one or three solutions. This kind of results was firstly obtained by Mawhin and Schmitt (1990) for a semilinear two-point boundary value problem.

1. Introduction

In this paper, we study a class of $p$-Laplacian equations of the form

$$-\Delta_p u = \lambda g(x) |u|^{p-2} u + f(x, u) + h(x) \quad \text{in} \quad D^{1,p}(\mathbb{R}^N),$$

where $\Delta_p u = \text{div}(\nabla |u|^{p-2} \nabla u)$, $1 < p < N$, and $g \geq 0$ is a weight function. Here, $D^{1,p}(\mathbb{R}^N)$ is the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,p}} = \left( \int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p}. \quad (1.2)$$

This space, which is motivated by the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$, where $p^* = Np/(N - p)$, is in fact a reflexive Banach space characterized by

$$D^{1,p}(\mathbb{R}^N) = \left\{ u \in L^{p^*}(\mathbb{R}^N); \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N), \ 1 \leq i \leq N \right\}. \quad (1.3)$$

We refer the reader to Ben-Naoum et al. [5] for a quite complete discussion on the space $D^{1,p}(\mathbb{R}^N)$. 
Our study is based on a bifurcation result by Mawhin and Schmitt [15], related to the two-point boundary value problem,

\[-u'' - \lambda u = f(x, u) + h, \quad u(0) = u(\pi) = 0. \tag{1.4}\]

By assuming that \( f \) is bounded and satisfying a sign condition, they obtained the following result. If \( \lambda \) is sufficiently near to \( \lambda_1 \) from left, where \( \lambda_1 = 1 \) is the first eigenvalue of the corresponding linear problem, then (1.4) has at least three solutions. If \( 1 \leq \lambda < 4 \), then problem (1.4) has at least one solution. Some extensions and variations of their result were considered by other authors (cf. Badiale and Lupo [3], Chiappinelli et al. [7], Sanchez [18], and Ma et al. [13]). In [14], the multiplicity part of that result was extended to the \( p \)-Laplacian operator in bounded domains, using critical point theory. Our objective is to extend this problem to the \( p \)-Laplacian in \( \mathbb{R}^N \), with \( \lambda \) approaching to \( \lambda_1 \) from left and from right.

In order to state the Mawhin-Schmitt problem in the context of \( D^{1,p}(\mathbb{R}^N) \), we recall some facts about the eigenvalue problem for the weighted \( p \)-Laplacian in \( \mathbb{R}^N \)

\[-\Delta_p u = \lambda g|u|^{p-2}u \quad \text{in} \quad D^{1,p}(\mathbb{R}^N), \tag{1.5}\]

where \( g \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N) \) is a locally Hölder continuous weight function. It is known that for \( g \geq 0 \), there exists a first eigenvalue \( \lambda_1 = \lambda_1(g) \), characterized by

\[
\lambda_1 = \inf \left\{ \|u\|_{D^{1,p}}^p; \ u \in D^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} g|u|^p \, dx = 1 \right\}, \tag{1.6}
\]

which is simple and positive. This implies that

\[
\int_{\mathbb{R}^N} |\nabla u|^p \, dx \geq \lambda_1 \int_{\mathbb{R}^N} g|u|^p \, dx \quad \forall \, u \in D^{1,p}(\mathbb{R}^N). \tag{1.7}
\]

Besides, the corresponding eigenfunction \( \varphi_1 \) belongs to \( D^{1,p}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N) \) and may be taken positive (see a complete proof in [12]). Putting

\[
W = \left\{ w \in D^{1,p}(\mathbb{R}^N); \int_{\mathbb{R}^N} g|\varphi_1|^{p-2}\varphi_1 w \, dx = 0 \right\} \tag{1.8}
\]

and \( V = \text{span}\{\varphi_1\} \), we have from the simplicity of \( \lambda_1 \),

\[
D^{1,p}(\mathbb{R}^N) = V \oplus W. \tag{1.9}
\]

Then, since \( \lambda_1 \) is also isolated (see [11]), we have

\[
\lambda_2 := \inf \left\{ \|w\|_{D^{1,p}}^p; \ w \in W, \int_{\mathbb{R}^N} g|w|^p \, dx = 1 \right\}. \tag{1.10}
\]
which satisfies $\lambda_1 < \lambda_2$. In addition,

$$\int_{\mathbb{R}^N} |\nabla w|^p \, dx \geq \lambda_2 \int_{\mathbb{R}^N} g|w|^p \, dx \quad \forall w \in W. \quad (1.11)$$

Next, we make some basic assumptions on the function $f$. We assume that $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|f(x,u)| \leq a(x)|u|^\sigma + b(x), \quad (1.12)$$

with $1 < \sigma < p$, $a \geq 0$, $a \in L^\infty(\mathbb{R}^N) \cap L^{(p^*/\sigma)'}(\mathbb{R}^N)$, and $b \in L^{p^*}(\mathbb{R}^N)$. Some of our hypotheses are given upon the primitive $F(x,u) = \int_0^u f(x,s) \, ds$, namely, there exists $\gamma \in L^1(\mathbb{R}^N)$ such that

$$F(x,u) \geq \gamma(x) \quad \text{a.e. in } \mathbb{R}^N, \quad \forall u \in \mathbb{R}. \quad (1.13)$$

We also consider the following: there exist $\alpha \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^{(p^*/\mu)'}(\mathbb{R}^N)$ and $\beta \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^{p^*}(\mathbb{R}^N)$ satisfying

$$pF(x,u) - f(x,u)u \geq \alpha(x)|u|^\mu + \beta(x) \quad \text{a.e. in } \mathbb{R}^N, \quad \forall u \in \mathbb{R}, \quad (1.14)$$

and $1 < \mu \leq \sigma < p$.

Now we are in a position to state our results.

**Theorem 1.1.** Assume that (1.12) and (1.13) hold. If in addition

$$\lim_{|u| \to \infty} F(x,u) = +\infty \quad \text{a.e. in } \mathbb{R}^N, \quad (1.15)$$

then for any $h \in L^{p^*}(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} h(x)\varphi_1(x) \, dx = 0, \quad (1.16)$$

problem (1.1) has at least three solutions when $\lambda$ is sufficiently close to $\lambda_1$ from left.

**Theorem 1.2.** Assume that (1.12) and (1.14) hold with $\alpha \geq \max\{a,g\}$. Assume further that $\lambda_1 \leq \lambda < \lambda_2$. Then for any $h \in L^{p^*}(\mathbb{R}^N)$ satisfying $|h| \leq \alpha$, problem (1.1) has at least one solution.

Since (1.14) implies (1.15), under the hypotheses of Theorem 1.2, we get an extension of the original work of Mawhin and Schmitt [15] to the $p$-Laplacian in $\mathbb{R}^N$. We note that our results do not assume $f$ bounded nor satisfying a sign condition. Theorem 1.2 is related to a class of double resonance problems.
introduced in [6] for semilinear elliptic equations. It was not considered for the $p$-Laplacian, even in bounded domains. Condition (1.14) was early used in [1, 8, 10] for example, as an Ambrosetti-Rabinowitz type condition [2]. A simple example of $g$ satisfying all the hypotheses of both theorems is

$$f(x,u) = \sigma a(x) |u|^{\sigma - 2}u, \quad (1.17)$$

where $a \in L^{\infty}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \cap L^{(p^{*}/\sigma)'}(\mathbb{R}^N)$ and $1 < \mu = \sigma < p - 1$.

The proofs of the theorems are given in Section 3. In Section 2, we present some preliminary results on the variational setting of the $p$-Laplacian equations in $D^{1,p}(\mathbb{R}^N)$ and the related Palais-Smale compactness.

### 2. Preliminaries

We begin with some standard facts upon the variational formulation of problem (1.1). Let $J_\lambda : D^{1,p}(\mathbb{R}^N) \rightarrow \mathbb{R}$ be the functional defined by

$$J_\lambda(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{p} |\nabla u(x)|^p - \frac{\lambda}{p} g(x) |u(x)|^p - F(x,u(x)) - h(x)u(x) \right] dx. \quad (2.1)$$

It is proved in do Ó [10], that $J_\lambda$ is of class $C^1(\mathbb{R}^N)$ and

$$\langle J_\lambda'(u), \varphi \rangle = \int_{\mathbb{R}^N} |\nabla u|^p - 2 \nabla u \cdot \nabla \varphi dx - \lambda \int_{\mathbb{R}^N} g |u|^{p-2}u \varphi dx$$

$$- \int_{\mathbb{R}^N} f(x,u) \varphi dx - \int_{\mathbb{R}^N} h \varphi dx, \quad (2.2)$$

for all $\varphi \in D^{1,p}(\mathbb{R}^N)$. In addition, the critical points of $J_\lambda$ are precisely the weak solutions of (1.1).

Next we recall a compactness result which is proved in [5].

**Lemma 2.1 (see [5]).** The functional

$$u \mapsto \int_{\mathbb{R}^N} m(x) |u(x)|^q dx \quad (2.3)$$

is well defined and weakly continuous in $D^{1,p}(\mathbb{R}^N)$, for $1 \leq q < p^*$ and $m \in L^{(p^*/q)'}(\mathbb{R}^N)$.

As a consequence, under the conditions of the lemma, there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} m(x) |u(x)|^q dx \leq \|m\|_{L^{(p^*/q)'}(\mathbb{R}^N)} \|u\|_{L^{p^*}}^q \leq C \|u\|_{D^{1,p}}^q. \quad (2.4)$$
Lemma 2.2. Assume that (1.12) holds. Then the Nemytskii mapping,

$$ u \mapsto f(x,u) $$

(2.5)

is compact from $D^{1,p}(\mathbb{R}^N)$ to $L^{p^*}(\mathbb{R}^N)$.

Proof. Put $r = p^{\sigma'}$ and $q = (\sigma - 1)r$ so that $q < p^*$ and

$$ \left( \frac{p^*}{\sigma} \right) \left( \frac{p^*}{q} \right)' = r \left( \frac{p^*}{q} \right)' . $$

(2.6)

Then we get from (1.12) that $a^r \in L^{(p^*/q)'}(\mathbb{R}^N)$. Now let $(u_n)$ be a sequence such that $u_n \rightharpoonup u$ weakly for some $u \in D^{1,p}(\mathbb{R}^N)$. Then from Lemma 2.1 we have $a^{r/q} u_n \rightharpoonup a^{r/q} u$ strongly in $L^q(\mathbb{R}^N)$. It follows that

$$ a^{r/q} u_n \rightharpoonup a^{r/q} u, \quad |a^{r/q} u_n| \leq k \quad \text{a.e. in } \mathbb{R}^N, $$

(2.7)

for some $k \in L^q(\mathbb{R}^N)$. Hence, for all $n$ and a.e. $x \in \mathbb{R}^N$,

$$ |f(x,u_n(x))| \leq 2^r \left( |a(x)|^{(\sigma-1)r} + |b(x)|^r \right). $$

(2.8)

Since the last term is an integrable function, from Lebesgue theorem, we infer that $f(x,u_n) \rightharpoonup f(x,u)$ strongly in $L^r(\mathbb{R}^N)$.

Next, we do some remarks about the Palais-Smale condition for $J_\lambda$. We recall that $J_\lambda$ is said to satisfy the Palais-Smale condition at level $c$, (PS)$_c$, if every sequence for which

$$ J(u_n) \to c, \quad ||J'(u_n)||_{(D^{1,p})^*} \to 0 $$

(2.9)

possesses a convergent subsequence. When $J$ satisfies (PS)$_c$ for all $c \in \mathbb{R}^N$, we simply say that $J$ satisfies the (PS) condition. In Theorem 1.2, we use a weaker version of the (PS) condition due to Cerami (cf. [4]). We say that $J$ satisfies the Palais-Smale-Cerami condition, (PSC), if every sequence, for which

$$ J(u_n) \text{ is bounded,} \quad \left( 1 + ||u_n||_{D^{1,p}} \right)||J'(u_n)||_{(D^{1,p})^*} \to 0, $$

(2.10)

possesses a convergent subsequence.

Lemma 2.3. Assume that condition (1.12) holds. Then any bounded sequence satisfying (2.9) or (2.10) possesses a convergent subsequence.

Proof. Let $(u_n)$ be a bounded sequence satisfying (2.9). Then, passing to a subsequence if necessary, there exists $u \in D^{1,p}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $D^{1,p}(\mathbb{R}^N)$ and also in $L^{p^*}(\mathbb{R}^N)$. Consequently,

$$ \lim_{n \to \infty} \langle J'_\lambda(u_n), u_n - u \rangle = 0. $$

(2.11)
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On the other hand, from Lemma 2.2, we know that $f(x, u_n) \to f(x, u)$ strongly in $L^{p'}(\mathbb{R}^N)$ and therefore,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (f(x, u_n) + h)(u_n - u) \, dx = 0. \quad (2.12)$$

Noting that

$$\int_{\mathbb{R}^N} g \, |u_n|^{p-1} |u_n - u| \, dx \leq \left( \int_{\mathbb{R}^N} g \, |u_n|^p \, dx \right)^{1/p'} \left( \int_{\mathbb{R}^N} g \, |u_n - u|^p \, dx \right)^{1/p}, \quad (2.13)$$

and since $g \in L^{(p'/p)'}$, Lemma 2.1 implies that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} g \, |u_n|^{p-2} u_n (u_n - u) \, dx = 0. \quad (2.14)$$

Combining (2.11) with (2.12) and (2.14), we get

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla (u_n - u) \, dx = 0. \quad (2.15)$$

But since we also have

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla (u_n - u) \, dx = 0, \quad (2.16)$$

it follows that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u \right) \cdot \nabla (u_n - u) \, dx = 0. \quad (2.17)$$

Then from a well-known argument based on the Clarkson inequality (cf. Tolksdorf [19]), we conclude that

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^p \, dx = 0. \quad (2.18)$$

This completes the proof since (2.10) implies (2.9).

\[ \square \]

3. Proofs of Theorems 1.1 and 1.2

The proof of Theorem 1.1 is based on the Ekeland’s variational principle and the Ambrosetti-Rabinowitz Mountain-Pass theorem [2]. Theorem 1.2 is proved using the saddle point theorem of Rabinowitz [17].

Proof of Theorem 1.1. We divide the proof in several steps.

Step 1 (the coerciveness of $J_\lambda$). Since $\lambda < \lambda_1$ and (1.12) holds, from (1.7) and (2.4) we get

$$J_\lambda(u) \geq \left( \frac{\lambda_1 - \lambda}{p \lambda_1} \right) \|u\|^p_{D^{1,p}} - C \|u\|^p_{D^{1,p}} - C \|u\|_{D^{1,p}}, \quad (3.1)$$
where $C > 0$ denotes several constants. Then $J_\lambda$ is coercive as a consequence of the assumption that $1 < a < p$. This implies that any sequence satisfying (2.9) must be bounded, and therefore Lemma 2.2 implies that $J_\lambda$ satisfies the (PS)$_c$ for all $c \in \mathbb{R}$. Similarly, from (1.11),

$$J_{\lambda_1}(w) \geq \left( \frac{\lambda_2 - \lambda_1}{p\lambda_2} \right) \|w\|_{D^{1,p}}^p - C\|w\|_{D^{1,p}}^q - C\|w\|_{D^{1,p}},$$  \hspace{1cm} (3.2)

which shows that $J_{\lambda_1}$ is coercive in $W$. Noting that $J_{\lambda_1} \leq J_\lambda$ for all $\lambda < \lambda_1$, we have that

$$m = \inf_W J_{\lambda_1} \leq \inf_W J_\lambda.$$  \hspace{1cm} (3.3)

**Step 2** (estimating $J_\lambda$ in $V$). From (1.16) we have for $t \in \mathbb{R}$,

$$J_\lambda(t\varphi_1) = \left( \frac{\lambda_1 - \lambda}{p} \right) \int_{\mathbb{R}^N} |t\varphi_1(x)|^p \, dx - \int_{\mathbb{R}^N} F(x, t\varphi_1(x)) \, dx.$$  \hspace{1cm} (3.4)

Now, from (1.13) there exist constants $R, C > 0$ such that

$$\int_{|x| > R} F(x, t\varphi_1(x)) \, dx \geq \int_{|x| > R} \gamma(x) \, dx \geq -C, \quad \forall t \in \mathbb{R}.$$  \hspace{1cm} (3.5)

Choosing $t^+ > 0$ sufficiently large, we get from (1.15) that

$$\int_{|x| \leq R} F(x, t^+\varphi_1(x)) \, dx > -m + C.$$  \hspace{1cm} (3.6)

Then we have

$$\int_{\mathbb{R}^N} F(x, t^+\varphi_1(x)) \, dx > -m,$$  \hspace{1cm} (3.7)

so that

$$J_{\lambda}(t^+\varphi_1) \leq \left( \frac{\lambda_1 - \lambda}{p} \right) \int_{\mathbb{R}^N} |t^+\varphi_1(x)|^p \, dx + m.$$  \hspace{1cm} (3.8)

Then for $\lambda$ sufficiently near to $\lambda_1$, $J_{\lambda}(t^+\varphi_1) < m$. The same conclusion holds for a $t^- < 0$. 

**Step 3** (the existence of the first two solutions). Put

$$C^\pm = \{ u \in D^{1,p}(\mathbb{R}^N); \ u = \pm t\varphi_1 + w \ \text{with} \ t > 0, \ w \in W \}.$$  \hspace{1cm} (3.9)

Then from Step 2, for $\lambda$ sufficiently near to $\lambda_1$,

$$-\infty < \inf_{C^\pm} I_\lambda < m.$$  \hspace{1cm} (3.10)
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Now let $u_n \in \mathcal{O}^+$ be a sequence satisfying (2.9) for $c < m$. Then from coerciveness of $J_\lambda$, $(u_n)$ has a convergent subsequence, say, $(u_n)$ itself. Noting that $W = \partial \mathcal{O}^+$ and $\inf_W J_\lambda \geq m$ (Step 1), we conclude that $(u_n)$ converges to an interior point $u \in \mathcal{O}^+$. This means that $J_\lambda$ satisfies the (PS)$_c$ condition inside $\mathcal{O}^+$ for all $c < m$.

Then applying the Ekeland variational principle in $\mathcal{O}^+$, we set that $J_\lambda$ has a critical point $u^+$ as a local minimum in $\mathcal{O}^+$. Similarly, we obtain a critical point $u^-$ of $J_\lambda$ in $\mathcal{O}^-$. Taking into account that $\mathcal{O}^- \cap \mathcal{O}^+ = \emptyset$, the existence of two weak solutions of (1.1) is proved.

**Step 4** (the third solution). To fix ideas, suppose that $J_\lambda(u^+) \leq J_\lambda(u^-)$. If $u^-$ is not an isolated critical point, then $J_\lambda$ has at least three solutions. Otherwise, putting

$$I(u) = J_\lambda(u + u^-) - J_\lambda(u^-), \quad e = u^+ - u^-,$$

we have that $I(0) = 0, I(e) \leq 0$, and there exist $r, \rho > 0$ such that $I(u) \geq \rho$ if $\|u\|_{D^{1,p}} = r$. Then, since $I' = J'_\lambda$ and $I$ also satisfies the (PS) condition, from the Mountain-Pass theorem, the number

$$c = \inf_{y \in \Gamma} \max_{t \in [0,1]} J_\lambda(y(t)),$$

where

$$\Gamma = \{ y \in C([0,1], D^{1,p}(\mathbb{R}^N)) ; y(0) = u^-, y(1) = u^+ \}$$

is a critical value of $J_\lambda$. Noting that all paths joining $u^-$ to $u^+$ pass through $W$, we have $c \geq m$. Therefore we have obtained a third critical point of $J_\lambda$. The proof is now complete.

**Proof of Theorem 1.2.** The proof is based on the arguments from [8, 10].

**Step 1** (the growth of $F$). We prove that for some $C_1, C_2 > 0$,

$$\int_{\mathbb{R}^N} F(x, t\varphi_1) \, dx \geq C_1 \|t\varphi_1\|_{D^{1,p}}^\mu - C_2. \quad (3.14)$$

In fact, from (1.14) we have

$$\frac{d}{du} \left( \frac{F(x, u)}{|u|^p} \right) \leq -\alpha(x)|u|^{\mu-p-2}u - \beta(x)|u|^{-p-2}u \quad (u > 0). \quad (3.15)$$

Integrating from $u > 0$ to $+\infty$, and noting that $F(x, \theta)/(\theta^p) \to 0$ as $\theta \to \infty$, we get

$$F(x, u) \geq \frac{\alpha(x)}{p-\mu} |u|^{\mu} + \frac{\beta(x)}{p}. \quad (3.16)$$
Since this inequality holds for \( u < 0 \), we have
\[
\int_{\mathbb{R}^N} F(x, t\phi_1) \, dx \geq C|t|^\mu - C_2,
\]
and inequality (3.14) follows.

**Step 2** (the (PSC) condition). Let \((u_n)\) be a sequence satisfying (2.9). Then from Lemma 2.2, it suffices to prove that \((u_n)\) is bounded. In fact, first we note that \((u_n)\) satisfies
\[
\langle J'_\lambda(u_n), u_n \rangle - pJ_\lambda(u_n) = \int_{\mathbb{R}^N} \left[ pF(x, u_n) - f(x, u_n)u_n + (p - 1)hu_n \right] \, dx \\
\geq \int_{\mathbb{R}^N} \alpha|u|^\mu \, dx + \int_{\mathbb{R}^N} \beta \, dx + (p - 1) \int_{\mathbb{R}^N} hu_n \, dx.
\]
(3.18)

Now, since \(|h| \leq \alpha\),
\[
\left| \int_{\mathbb{R}^N} hu_n \, dx \right| \leq \int_{\mathbb{R}^N} \alpha|u_n| \, dx \leq \|\alpha\|_{L^1}^{1/\mu} \left( \int_{\mathbb{R}^N} \alpha|u_n|^{\mu} \, dx \right)^{1/\mu}.
\]
(3.19)

Then from the boundedness of \(\langle J'_\lambda(u_n), u_n \rangle - pJ_\lambda(u_n)\), we deduce that
\[
\int_{\mathbb{R}^N} \alpha|u_n|^{\mu} \, dx \leq C + C \left( \int_{\mathbb{R}^N} \alpha|u_n|^{\mu} \, dx \right)^{1/\mu},
\]
(3.20)
so that
\[
\int_{\mathbb{R}^N} \alpha|u_n|^{\mu} \, dx \leq C.
\]
(3.21)

Now we use an interpolation inequality. Since \(0 < \mu < p < p^*\), there exists \(t \in (0, 1)\) such that
\[
1 = \frac{p(1-t)}{\mu} + \frac{pt}{p^*}.
\]
(3.22)

Then from Hölder inequality,
\[
\int_{\mathbb{R}^N} \alpha(x)|u|^p \, dx = \int_{\mathbb{R}^N} (\alpha^{1/\mu}|u|)^{p(1-t)}(\alpha^{1/p^*}|u|)^{pt} \, dx \\
\leq \left( \int_{\mathbb{R}^N} \alpha|u|^{\mu} \, dx \right)^{p(1-t)/\mu} \left( \int_{\mathbb{R}^N} \alpha|u|^{p^*} \, dx \right)^{pt/p^*}.
\]
(3.23)

Using (3.21) and (2.4),
\[
\int_{\mathbb{R}^N} \alpha(x)|u_n(x)|^p \, dx \leq C\|u_n\|_{D^{1,p}}^p.
\]
(3.24)
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Taking into account the boundedness of $J_\lambda(u_n)$,

$$\frac{1}{p} ||u_n||_{D^{1,p}}^p \leq C + \frac{\lambda}{p} \int_{\mathbb{R}^N} g |u_n|^p \, dx + \int_{\mathbb{R}^N} F(x,u_n) \, dx + \int_{\mathbb{R}^N} h u_n \, dx,$$  \hspace{1cm} (3.25)

and since $\alpha \geq \max\{a,g\}$,

$$\frac{1}{p} ||u_n||_{D^{1,p}}^p \leq C + \frac{\lambda}{p} \int_{\mathbb{R}^N} \alpha |u_n|^p \, dx + \frac{1}{\sigma} \int_{\mathbb{R}^N} \alpha |u_n|^{\sigma} \, dx$$
$$+ \int_{\mathbb{R}^N} b |u_n| \, dx + \int_{\mathbb{R}^N} h u_n \, dx.$$  \hspace{1cm} (3.26)

Consequently, if $\mu = \sigma$ we have, from (3.24),

$$||u_n||_{D^{1,p}}^p \leq C\left(1 + ||u_n||_{D^{1,p}}^p + ||u_n||_{D^{1,p}}^{s\sigma}\right).$$  \hspace{1cm} (3.27)

Otherwise, we have $\mu < \sigma < p^*$, and as before, we get $s \in (0,1)$ such that

$$\int_{\mathbb{R}^N} \alpha |u_n|^{\sigma} \, dx \leq C ||u_n||_{D^{1,p}}^{s\sigma}. \hspace{1cm} (3.28)$$

Then

$$||u_n||_{D^{1,p}}^p \leq C\left(1 + ||u_n||_{D^{1,p}}^p + ||u_n||_{D^{1,p}}^{s\sigma} + ||u_n||_{D^{1,p}}^{s\sigma}\right).$$  \hspace{1cm} (3.29)

In both cases, we see that $||u_n||_{D^{1,p}}$ is uniformly bounded.

*Step 3* (the saddle point theorem). It is well known that the (PS) condition can be replaced by the (PSC) condition in the saddle point theorem of Rabinowitz (see [4, 17]). Then to conclude that $J_\lambda$ has a critical point it suffices to show that

$$\lim_{||v||_{D^{1,p}} \to \infty} J_\lambda(v) = -\infty, \hspace{1cm} \lim_{||w||_{D^{1,p}} \to \infty} J_\lambda(w) = +\infty,$$  \hspace{1cm} (3.30)

where $v \in V$ and $w \in W$, as defined in (1.9). Now, from (3.14),

$$J_\lambda(t\varphi_1) \leq -\left(\frac{\lambda - \lambda_1}{p\lambda_1}\right)||t\varphi_1||_{D^{1,p}}^p - C_1 ||t\varphi_1||_{D^{1,p}}^{\mu} + C_2 ||t\varphi_1||_{D^{1,p}}^p + C_2.$$  \hspace{1cm} (3.31)

Since $\lambda \geq \lambda_1$, the first part of (3.30) holds. Finally, since $\lambda < \lambda_2$, the argument in *Step 1* of the proof of Theorem 1.1 implies the second statement of (3.30). The proof is now complete. \(\square\)

*Note 3.1.* Just before the completion of this paper, we noticed that P. De Nápóli and M. C. Mariani [9] studied problem (1.1) in the same framework of our Theorem 1.1. However, they considered only the case $\lambda \to \lambda_1$ from left. Our assumptions on $f$ are slightly more general.
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References


[9] P. De Nápoli and M. C. Mariani, Three solutions for quasilinear equations in $\mathbb{R}^n$ near resonance, Proceedings of the USA-Chile Workshop on Nonlinear Analysis (Viña del Mar-Valparaiso, 2000), Southwest Texas State University, Texas, 2001, pp. 131–140.


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