EXISTENCE THEOREMS FOR ELLIPTIC HEMIVARIATIONAL INEQUALITIES INVOLVING THE p-LAPLACIAN

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We study quasilinear hemivariational inequalities involving the p-Laplacian. We prove two existence theorems. In the first, we allow "crossing" of the principal eigenvalue by the generalized potential, while in the second, we incorporate problems at resonance. Our approach is based on the nonsmooth critical point theory for locally Lipschitz energy functionals.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with C^1 boundary Γ. In this paper, we study the following quasilinear hemivariational inequality:

$$-\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) \in \partial j(z,x(z)) \quad \text{a.e. on } Z, \qquad x|_\Gamma = 0. \tag{1.1}$$

Here, $2 \le p < \infty$, $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function which is measurable in $z \in Z$ and locally Lipschitz in $x \in \mathbb{R}$ and $\partial j(z,x)$ is the Clarke subdifferential of $j(z,\cdot)$. If $f : Z \times \mathbb{R} \to \mathbb{R}$ is a measurable function which is in general discontinuous in the $x \in \mathbb{R}$ variable, for almost all $z \in Z$, all $M > 0$, and all $|x| \le M$, we have $|f(z,x)| \le a_M(z)$ with $a_M \in L^1(Z)$ and we set $j(z,x) = \int_0^x f(z,r)\,dr$, then $j(z,x)$ is measurable in $z \in Z$, locally Lipschitz in $x \in \mathbb{R}$ and $\partial j(z,x) \subseteq [f_1(z,x), f_2(z,x)]$ where $f_1(z,x) = \liminf_{x' \to x} f(z,x')$ and $f_2(z,x) = \limsup_{x' \to x} f(z,x')$ (see Chang [5] and Clarke [6]). So problem (1.1) incorporates as a special case quasilinear elliptic problems with a discontinuous right-hand side which were studied by Chang [5]. Hemivariational inequalities are new type of inequality problems, which arise in mechanics and engineering when we wish to consider more realistic laws of nonmonotone and multivalued nature. This leads to energy functionals which are nonsmooth and nonconvex, and so the tools of differential calculus and convex analysis (which are used in the study of variational inequalities) are no longer suitable and new techniques based on the nonconvex and the

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nonsmooth analysis have to be developed. Concrete applications of hemivariational inequalities in mechanics and engineering can be found in the books of Naniewicz and Panagiotopoulos [15] and Panagiotopoulos [16]. Recently, semilinear (i.e., $p = 2$) hemivariational inequalities were studied by Goeleven et al. [10] and Gasiński and Papageorgiou [9]. Quasilinear problems with the p-Laplacian and a C^1 potential function were studied by Arcoya and Orsina [2], Boccardo et al. [3], Costa and Magalhães [7], and Hachimi and Gossez [8].

We prove two existence theorems for problem (1.1), which extend in different ways the above-mentioned smooth quasilinear works. Our approach is based on the nonsmooth critical point theory as developed by Chang [5] and extended by Kourogenis and Papageorgiou [13]. For the convenience of the reader in the next section, we recall the basics from the nonsmooth critical point theory.

2. Preliminaries

Let X be a Banach space and X^* its topological dual. A function $f \to X \to \mathbb{R}$ is said to be "locally Lipschitz," if for every $x \in X$ there exists a neighbourhood U of x and a constant $k_U > 0$ such that $|f(y) - f(x)| \leq k_U \|y - x\|$ for all $z, y \in U$. It is well known from the convex analysis that a proper, convex, and lower semicontinuous function $g : x \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is locally Lipschitz in the interior of its effective domain $\operatorname{dom} g = \{x \in X : g(x) < \infty\}$. In analogy with the directional derivative of a convex function, we define the "generalized directional derivative" of a locally Lipschitz function f at $x \in X$ in the direction $h \in X$, by

$$f^0(x;h) = \overline{\lim_{\substack{x' \to x \\ \lambda \downarrow 0}}} \frac{f(x' + \lambda h) - f(x')}{\lambda}. \tag{2.1}$$

It is easy to check that $h \to f^0(x;h)$ is sublinear, continuous and $|f^0(x;h)| \leq k_U \|h\|$. Therefore, by the Hahn-Banach theorem, $f^0(x;\cdot)$ is the support of a nonempty, convex, and w^*-compact set

$$\partial f(x) = \{x^* \in X^* : (x^*, h) \leq f^0(x;h) \ \forall h \in X\}. \tag{2.2}$$

The set $\partial f(x)$ is usually called the "generalized subdifferential" of f at x. Clearly for every $x^* \in \partial f(x)$, we have $\|x^*\|_* \leq k_U$. Also if $f, g : x \to \mathbb{R}$ are locally Lipschitz functions, then $\partial(f + g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\partial(\lambda f)(x) = \lambda \partial f(x)$ for all $\lambda \in \mathbb{R}$. Moreover, if in addition f is convex, then as we already said it is locally Lipschitz and the generalized subdifferential and the subdifferential in the sense of convex analysis coincide. Furthermore, if f is continuously differentiable (i.e., $f \in C^1(X)$), then we know that it is locally Lipschitz and $\partial f(x) = \{f'(x)\}$. For further details, we refer to Clarke [6].

Given a locally Lipschitz function $f : X \to \mathbb{R}$, a point $x \in X$ is said to be a "critical point" of f, if $0 \in \partial f(x)$. The value $c = f(x)$ is called "critical value." In the smooth critical point theory a compactness-type condition, known as the

"Palais-Smale condition" (PS-condition for short) plays a central role. In the present nonsmooth setting, this condition takes the following form: "A locally Lipschitz function $f : X \to \mathbb{R}$ satisfies the "nonsmooth PS-condition," if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ along which $\{f(x_n)\}_{n\geq 1}$ is bounded and $m(x_n) = \inf[\|x^*\|_* : x^* \in \partial f(x_n)] \to 0$ as $n \to \infty$, has a strongly convergent subsequence." If $f \in C^1(X)$, then $\partial f(x_n) = \{f'(x_n)\}$ and so the nonsmooth PS-condition coincides with the classical (smooth) PS-condition (see Rabinowitz [18]). In the smooth case, a generalization of the PS-condition was introduced by Cerami [4] and Bartolo et al. [3] showed that this more general condition suffices to prove a deformation theorem and then using it to prove minimax theorems locating critical points of C^1-energy functionals. In the context of the nonsmooth theory this was done by Kourogenis and Papageorgiou [13] who introduced the so-called "nonsmooth C-condition" which says the following: "A locally Lipschitz function $f : X \to \mathbb{R}$ satisfies the "nonsmooth C-condition," if any sequence $\{x_n\}_{n\geq 1} \subseteq X$ along which $\{f(x_n)\}_{n\geq 1}$ is bounded and $(1 + \|x_n\|)m(x_n) \to 0$ as $n \to \infty$ (here $m(x_n)$ is as before), has a strongly convergent subsequence." Using this condition, Kourogenis and Papageorgiou [13] proved the following generalization of the classical "Mountain-Pass theorem" (see Rabinowitz [17]).

Theorem 2.1. *If X is a reflexive Banach space, $\phi : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth C-condition, and for some $r > 0$ and $y \in X$ with $\|y\| > r$,*

$$\max\{\phi(0), \phi(y)\} \leq \inf[\phi(x) : \|x\| = r], \tag{2.3}$$

then ϕ has a nontrivial critical point $x \in X$ such that the critical value $c = f(x)$ is characterized by the following minimax principle

$$c = \inf_{\gamma\in\Gamma_0} \max_{0\leq t\leq 1} \phi(\gamma(t)), \tag{2.4}$$

where $\Gamma_0 = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = y\}$.

Given $\theta \in L^{N/p}(Z)_+$ (for $N > p$) and $\theta \in L^1(Z)_+$ (for $N \leq p$) with $\theta(z) > 0$ on a set of positive Lebesgue measure, we consider the following eigenvalue problem:

$$-\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = \lambda\theta(z)|x(z)|^{p-2}x(z) \quad \text{a.e. on } Z, \qquad x|_\Gamma = 0. \tag{2.5}$$

From Allegretto and Huang [1], we know that (2.5) has a first eigenvalue $\lambda_1(\theta)$ which is positive, isolated, simple (it is a principal eigenvalue) and admits the following variational characterization (Rayleigh quotient):

$$\lambda_1(\theta) = \inf\left[\frac{\|Dx\|_p^p}{\int_Z \theta(z)|x(z)|^p\,dz} : x \in W_0^{1,p}(Z),\ x \neq 0\right]. \tag{2.6}$$

Note that from (2.6) it follows easily that $\lambda_1(\theta) > 0$. Indeed, using Hölder's inequality we have for $N > p$, $\int_Z \theta(z)|x(z)|^p\,dz \le \|\theta\|_{N/p}\|x\|_{p^*}^p \le \gamma_1\|\theta\|_{N/p}\|Dx\|_p^p$ for some $\gamma_1 > 0$ (to obtain the second inequality, we used Poincaré's inequality). Hence, we have

$$\begin{aligned}\lambda_1(\theta) &= \frac{\|Dx\|_p^p}{\displaystyle\int_Z \theta(z)\left|x(z)\right|^p dz} \ge \frac{\|Dx\|_p^p}{\gamma_1\|\theta\|_{N/p}\|Dx\|_p^p} \\ &= \frac{1}{\gamma_1\|\theta\|_{N/p}} > 0 \quad \forall x \in W_0^{1,p}(Z),\ x \ne 0.\end{aligned} \tag{2.7}$$

Similarly if $N \le p$, in which case $p^* = +\infty$. Recall that

$$p^* = \begin{cases} \dfrac{Np}{N-p}, & \text{if } p < N \\ +\infty, & \text{if } p \ge N \end{cases} \quad \text{(critical Sobolev exponent).} \tag{2.8}$$

Note that if $\theta = 1$ (in which case we write $\lambda_1(\theta) = \lambda_1$), then we recover the properties of the principal eigenvalue of the negative p-Laplacian with Dirichlet boundary conditions (see Lindqvist [14]).

3. Existence theorems

In this section, we prove two existence theorems for problem (1.1). For the first existence theorem we need the following hypotheses on the generalized potential $j(z,x)$:

$\mathbf{H}(j)_1$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ locally Lipschitz;

(iii) for almost all $z \in Z$, all $z \in \mathbb{R}$, and all $u \in \partial j(z,x)$, $|u| \le a_1(z) + c_1|x|^{r-1}$, $1 \le r < p^*$, $a_1 \in L^\infty(Z)$, $c_1 > 0$, $j(\cdot,0) \in L^\infty(Z)$, and $\int_Z j(z,0)\,dz \ge 0$;

(iv) there exists a function $a \in L^\infty(Z)$ with $a(z) > 0$ a.e. on Z and $0 < \mu < p^*$ such that

$$\liminf_{|x|\to\infty} \frac{ux - pj(z,x)}{|x|^\mu} = a(z) \tag{3.1}$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$ and also there exists $s > 0$ such that $\max\{p,\mu\} < s < p(\max(N,p)+\mu)/\max(N,p)$ and $\limsup_{|x|\to\infty}(pj(z,x)/a(z)|x|^s) \le \gamma < +\infty$ uniformly for almost all $z \in Z$;

(v) there exists a function $\beta \in L^{N/p}(Z)_+$ if $N > p$ and $\beta \in L^1(Z)_+$ if $N \le p$ with $\beta(z) > 0$ for all z on a subset of positive Lebesgue measure such that $\lambda_1(\beta) > 1$ and

$$\limsup_{x\to 0} \frac{pj(z,x)}{|x|^p} \le \beta(z) \quad \text{uniformly for almost all } z \in Z; \tag{3.2}$$

(vi) there exists a function $\xi \in L^{N/p}(Z)_+$ if $N > p$ and $\xi \in L^1(Z)_+$ if $N \leq p$ with $\xi(z) > 0$ for all z on a subset of positive Lebesgue measure such that $\lambda_1(\xi) < 1$ and

$$\liminf_{|x|\to\infty} \frac{pj(z,x)}{|x|^p} \geq \xi(z) \quad \text{uniformly for almost all } z \in Z. \tag{3.3}$$

Remark 3.1. A more restrictive version of hypothesis $\mathbf{H}(j)_1$(iv) was first employed by Costa and Magalhães [7] for a smooth potential function. In their formulation, they assumed that $a(z) > a_0 > 0$ a.e. on Z and there are additional restrictions on the variation of μ. This hypothesis is a generalization of the well-known Ambrosetti-Rabinowitz condition for smooth potential functions (cf. Rabinowitz [17, Hypothesis p_4, page 9]). If $\beta(z) = \beta > \lambda_1 > \xi(z) = \xi > 0$ for all $z \in Z$ where λ_1 is the principal eigenvalue of the negative p-Laplacian with Dirichlet boundary conditions (i.e., of $(-\Delta_p, W_0^{1,p}(Z))$), then hypotheses $\mathbf{H}(j)_1$(v) and (vi) imply that the potential j "crosses" the principal eigenvalue λ_1. An example of a nonsmooth potential function $j(z,x)$ which satisfies hypotheses $\mathbf{H}(j)_1$ is the following:

$$j(z,x) = \begin{cases} \dfrac{\beta(z)}{p}|x|^p - x\ln|x|, & \text{if } |x| \leq 1, \\ \dfrac{\xi_1}{p}|x|^p - a(z)|x| + \dfrac{\beta(z)}{p}, & \text{if } |x| > 1, \end{cases} \tag{3.4}$$

with $a, \beta \in L^\infty(Z)_+$, $a(z) > 0$ a.e. on Z, $\beta(z) > 0$ for all z on a subset of positive Lebesgue measure, $\lambda_1(\beta) > 1$, and $0 < \lambda_1 \leq \xi_1$. Also if $a \in L^\infty(Z)$ with $a(z) \geq a_0 > 0$ a.e. on Z and $j(z,x) = (a(z)/p)|x|^p \ln|x| - |x|$, then we can check that it satisfies hypotheses $\mathbf{H}(j)_1$ with $\mu = p$, $\gamma = 0$, $\limsup_{|x|\to 0}(pj(z,x)/|x|^p) = -\infty$, and $\liminf_{|x|\to\infty}(pj(z,x)/|x|^p) = +\infty$. In this case, the classical Ambrosetti-Rabinowitz condition does not hold. Note that in this occasion $\partial j(z,x) = a(z)|x|^{p-2}x\ln|x| + (a(z)/p)|x|^p(\operatorname{sgn} x/|x|) - \operatorname{sgn} x$ where

$$\operatorname{sgn} x = \begin{cases} 1, & \text{if } x > 0, \\ [-1,1], & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases} \tag{3.5}$$

So $x\partial j(z,x) = a(z)|x|^p \ln|x| + (a(z)/p)|x|^p - |x|$.

Let $\phi : W_0^{1,p}(Z) \to \mathbb{R}$ be defined by

$$\phi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j\big(z, x(z)\big)\, dz. \tag{3.6}$$

By virtue of hypothesis $\mathbf{H}(j)_1$(iii) and the Lebourg mean value theorem (see Clarke [6, page 41]), for almost all $z \in Z$ and all $z \in \mathbb{R}$, we have

$$|j(z,x)| \leq a_1'(z) + c_1'|x|^r \tag{3.7}$$

with $a_1' \in L^\infty(Z)$ and $c_1' > 0$. Then the integral functional $J : L^p(Z) \to \mathbb{R}$ defined by $J(x) = \int_Z j(z,x(z))\,dz$ is locally Lipschitz (see Hu and Papageorgiou [12, page 313]). In particular then, since $W_0^{1,p}(Z)$ is embedded continuously (in fact compactly) in $L^r(Z)$ (recall that $r < p^*$), it follows that $\hat{J} = J|_{W_0^{1,p}(Z)}$ is locally Lipschitz too. Therefore, ϕ is a locally Lipschitz functional and we can use the nonsmooth critical point theory.

Proposition 3.2. *If hypotheses $\mathbf{H}(j)_1$ hold then ϕ satisfies the nonsmooth C-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$\begin{gathered} |\phi(x_n)| \leq M_1 \quad \forall n \geq 1 \text{ and some } M_1 > 0, \\ (1 + \|x_n\|)m(x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{gathered} \tag{3.8}$$

Choose $x_n^* \in \partial\phi(x_n)$ such that $m(x_n) = \|x_n\|$, $n \geq 1$. That such elements exist follows from the weak compactness of $\partial\phi(x_n)$, $n \geq 1$, and the weak lower semicontinuity of the norm in a Banach space (Weierstrass theorem). Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\,dz \quad \forall x, y \in W_0^{1,p}(Z). \tag{3.9}$$

Here and in what follows, we denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$ (where $1/p + 1/q = 1$). It is easy to check (see also Hu and Papageorgiou [12, page 323]) that A is continuous monotone, hence maximal monotone (see Hu and Papageorgiou [11, page 309]). Moreover, because $\hat{J} = J|_{W_0^{1,p}(Z)}$ is locally Lipschitz on the Sobolev space $W_0^{1,p}(Z)$ which is embedded continuously and densely in $L^r(Z)$, from [5, Theorem 2.2] and from Clarke [6, page 83], it follows that $\partial\hat{J}(x) \subseteq L^{r'}(Z)$ and if $u \in \partial\hat{J}(x)$, then $u(z) \in \partial j(z,x(z))$ a.e. on Z. For every $n \geq 1$, we have

$$x_n^* = A(x_n) - u_n, \quad \text{with } u_n \in \partial\hat{J}(x_n),\ n \geq 1. \tag{3.10}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$, we have

$$\begin{gathered} p\phi(x_n) = \|Dx_n\|_p^p - \int_Z pj(z,x_n(z))\,dz \leq pM_1, \\ -\varepsilon_n \leq -\langle A(x_n), x_n\rangle + \int_Z u_n(z)x_n(z)\,dz \leq \varepsilon_n \quad \text{with } \varepsilon_n \downarrow 0. \end{gathered} \tag{3.11}$$

We add the two inequalities above. Because $\langle A(x_n), x_n\rangle = \|Dx_n\|_p^p$, we obtain

$$\int_Z \big(u_n(z)x_n(z) - pj\big(z, x_n(z)\big)\big)\,dz \le M_2 \quad \forall n \ge 1 \text{ and some } M_2 > 0. \tag{3.12}$$

By virtue of hypothesis $\mathbf{H}(j)_1$(iv), we can find $k_1 > 0$ such that for almost all $z \in Z$, all $|x| \ge k_1$, and all $u \in \partial j(z,x)$, we have

$$\frac{a(z)}{2}|x|^\mu \le ux - pj(z,x). \tag{3.13}$$

Recalling that $|j(z,x)| \le a_1'(z) + c_1'|x|^r$ a.e. on Z and using hypothesis $\mathbf{H}(j)_1$ (iii) we infer that there exists $a_2 \in L^1(Z)_+$ such that for almost all $z \in Z$, all $|x| < k_1$, and all $u \in \partial j(z,x)$, we have

$$-a_2(z) \le ux - pj(z,x). \tag{3.14}$$

From (3.13) and (3.14) and with $a_3(z) = \|a\|_\infty/2 - a_2(z)$, $a_3 \in L^1(Z)$, for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$\frac{a(z)}{2}|x|^\mu - a_3(z) \le ux - pj(z,x). \tag{3.15}$$

Using this estimate in (3.12), we obtain

$$\frac{1}{2}\int_Z a(z)\big|x(z)\big|^\mu\,dz \le M_2 + \|a_3\|_1 = M_3. \tag{3.16}$$

From this we deduce that $\{x_n\}_{n\ge\infty}$ is bounded in the weighted Lebesgue space $L_a^\mu(Z)$, that is, $L_a^\mu(Z)$ is the Banach space of all equivalent classes (for the equivalence relation of a.e. equality on Z) of measurable functions $y : Z \to \mathbb{R}$ such that $\int_Z a(z)|y(z)|^\mu\,dz = \|y\|_{L_a^\mu(Z)}^\mu < \infty$.

We set $\eta = \min\{p^*, p(\max(N,p)+\mu)/\max(N,p)\}$. If $N \le p$, then $p^* = Np/N - p = p((N+p^*)/N) > p((N+\mu)/N)$ and so $\eta = p((N+\mu)/N) > p$. In addition since $\mu < p^* = Np/(N-p)$, we have $\mu < p((N+\mu)/N)$. Therefore, in both cases we have

$$\max\{p,\mu\} < \eta = \frac{\max(N,p)+\mu}{\max(N,p)}. \tag{3.17}$$

If $s > 0$ is as in the second part of hypothesis $\mathbf{H}(j)_1$(iv), we choose $t > 0$ such that $\max\{p,\mu,s\} < t < \eta \le p^*$ and for some $a_4 \in L^{r'}(Z)$, $a_5 > 0$ and for almost all $z \in Z$ and all $z \in \mathbb{R}$, we have

$$j(z,x) \le a_4(z) + a_5|x|^t. \tag{3.18}$$

Let

$$\theta = \begin{cases} \dfrac{p^*(t-\mu)}{t(p^*-\mu)}, & \text{if } N > p, \\ 1 - \dfrac{\mu}{t}, & \text{if } N \le p. \end{cases} \tag{3.19}$$

We have $0 < \theta < 1$ and $1/p = (1-\theta)/\mu + \theta/p^*$. From the interpolation inequality (see Showalter [18, page 45]) we have

$$\begin{aligned} \|x_n\|_{L_a^t} &\le \|x_n\|_{L_a^\mu}^{1-\theta} \|x_n\|_{L_a^{p^*}}^{\theta} \le M_4 \|x_n\|_{L_a^{p^*}}^{\theta} \\ &\le M_5 \|Dx_n\|_p^{\theta} \quad \text{for some } M_4, M_5 > 0. \end{aligned} \tag{3.20}$$

In (3.20) above, the last inequality follows from the Sobolev embedding theorem and the Poincaré inequality.

By virtue of hypothesis $\mathbf{H}(j)_1$(v), given $\varepsilon > 0$ we can find $0 < \delta \le 1$ such that for almost all $z \in Z$ and all $|x| \le \delta$, we have

$$pj(z,x) \le (\beta(z) + \varepsilon)|x|^p. \tag{3.21}$$

Also because of hypothesis $\mathbf{H}(j)_1$(iv) we can find $k_2 \ge 1$ such that for almost all $z \in Z$ and all $|x| \ge k_2$, we have

$$pj(z,x) \le (\gamma+1)a(z)|x|^t \quad \text{recall that } s < t. \tag{3.22}$$

From (3.18), (3.21), and (3.22) we deduce that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$pj(z,x) \le (\beta(z)+\varepsilon)|x|^p + (\gamma+1)a(z)|x|^t + a_6(z) \quad \text{with } a_6 \in L^1(Z). \tag{3.23}$$

From the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ we have

$$\begin{aligned} &\|Dx_n\|_p^p - \int_Z pj(z,x_n(z))\,dz \le pM_1 \\ &\quad \Longrightarrow \|Dx_n\|_p^p \le pM_1 + \int_Z (\beta(z)+\varepsilon)|x_n(z)|^p\,dz \\ &\qquad\qquad + (\gamma+1)\int_Z a(z)|x_n(z)|^t\,dz + \|a_6\|_1 \quad (\text{using } (3.23)) \\ &\quad \Longrightarrow \|Dx_n\|_p^p - \int_Z \beta(z)|x_n(z)|^p\,dz - \varepsilon\|x_n\|_p^p \le (\gamma+1)\|x_n\|_{L_a^t}^t + \|a_6\|_1 \\ &\quad \Longrightarrow \left(1 - \frac{1}{\lambda_1(\beta)} - \frac{\varepsilon}{\lambda_1}\right)\|Dx_n\|_p^p \le M_7\|Dx_n\|_p^{\theta t} + M_8 \\ &\qquad\qquad \text{for some } M_7, M_8 > 0 \text{ (see (2.6) and (3.20))}. \end{aligned} \tag{3.24}$$

Since $\lambda_1(\beta) > 1$, we can choose $\varepsilon > 0$ small so that $(1 - (1/\lambda_1(\beta)) - \varepsilon/\lambda_1) = \xi_0 > 0$. Moreover, by a simple calculation we can check that $\theta t < p$. Therefore from (3.24) and Poincaré's inequality it follows that $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded. Hence by passing to a subsequence if necessary, we may assume that

$$\begin{gathered} x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z), \quad x_n \longrightarrow x \text{ in } L^p(Z), \quad x_n(z) \longrightarrow x(z) \quad \text{a.e. on } Z, \\ |x_n(z)| \leq k(z) \quad \text{a.e. on } Z\ \forall n \geq 1, \text{ with } k \in L^p(Z). \end{gathered} \tag{3.25}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ we have

$$\begin{aligned} \langle x_n^*, x_n - x\rangle &= \langle A(x_n), x_n - x\rangle - \langle u_n, x_n - x\rangle \leq \varepsilon_n \|x_n - x\| \\ &\Longrightarrow \langle A(x_n), x_n - x\rangle \leq \varepsilon_n \|x_n - x\| + \int_Z u_n(z)(x_n - x)(z)\,dz \\ &\Longrightarrow \limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0. \end{aligned} \tag{3.26}$$

But recall that A is maximal monotone, hence generalized pseudomonotone (see Hu and Papageorgiou [11, page 365]). So we have $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \Rightarrow \|Dx_n\|_p \to \|Dx\|_p$. Because $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and the latter space has the Kadec-Klee property (because it is uniformly convex, see Hu and Papageorgiou [11, page 28]), we infer that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$ and so $x_n \to x$ in $W_0^{1,p}(Z)$. This proves that ϕ exhibits the nonsmooth PS-condition. □

Proposition 3.3. *If hypotheses* $\mathbf{H}(j)_1$ *hold, then there exists* $\rho > 0$ *such that for all* $x \in W_0^{1,p}(Z)$ *with* $\|x\| = \rho$ *we have* $\phi(x) \geq \xi_1 > 0$.

Proof. From hypothesis $\mathbf{H}(j)_1$(v), we know that given $\varepsilon > 0$ we can find $0 < \delta \leq 1$ such that for almost all $z \in Z$ and all $|x| \leq \delta$ we have

$$j(z,x) \leq \frac{1}{p}(\beta(z) + \varepsilon)|x|^p. \tag{3.27}$$

On the other hand, recall that $|j(z,x)| \leq a_1'(z) + c_1'|x|^r$ with $a_1' \in L^\infty(Z)$, $c_1' > 0$. So for $\gamma_1 > 0$ large enough we can write that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$j(z,x) \leq \frac{1}{p}(\beta(z) + \varepsilon)|x|^p + \gamma_1 |x|^{p^*}. \tag{3.28}$$

Hence, we have

$$\begin{aligned} \phi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z))\,dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \beta(z)|x(z)|^p\,dz - \frac{\varepsilon}{p}\|x\|_p^p - \gamma_1\|x\|_{p^*}^{p^*} \\ &\geq \frac{1}{p}\left(1 - \frac{1}{\lambda_1(\beta)} - \varepsilon\right)\|Dx\|_p^p - \gamma_2\|Dx\|_p^{p^*} \quad \text{for some } \gamma_2 > 0. \end{aligned} \tag{3.29}$$

Because $\lambda_1(\beta) > 1$, we can choose $\varepsilon > 0$ small, so that $\sigma = 1 - 1/\lambda_1(\beta) - \varepsilon > 0$. Therefore,

$$\phi(x) \geq \frac{\sigma}{p}\|Dx\|_p^p - \gamma_2\|Dx\|_p^{p^*}. \tag{3.30}$$

Since $p < p^*$ we can find $\rho_1 > 0$ small so that $\inf[\phi(x) : \|Dx\|_p = \rho_1] = \xi_1 > 0$ which completes the proof since $\|Dx\|_p$ is an equivalent norm in $W_0^{1,p}(Z)$. □

Proposition 3.4. *If hypotheses* $\mathbf{H}(j)_1$ *hold then there exists* $\eta \in C_0^\infty(Z)$ *such that* $\lim_{|\tau|\to\infty}\phi(\tau\eta) = -\infty$.

Proof. From the definition of $\lambda_1(\xi)$ (see (2.6)) and the density of $C_0^\infty(Z)$ in the Sobolev space $W_0^{1,p}(Z)$, we see that given $\varepsilon > 0$ we can find $\eta \in C_0^\infty(Z)$ such that

$$\|D\eta\|_p^p \leq (\lambda_1(\xi) + \varepsilon)\int_Z \xi(z)|\eta(z)|^p\,dz. \tag{3.31}$$

Also from hypothesis $\mathbf{H}(j)_1$(vi), we can find $M_9 > 0$ such that for almost all $z \in Z$ and all $|x| \geq M_9$, we have

$$\frac{1}{p}(\xi(z) - \varepsilon)|x|^p \leq j(z,x). \tag{3.32}$$

Thus for $\tau \in \mathbb{R}$, we can write that

$$\begin{aligned}\int_Z j(z,\tau\eta(z))\,dz &= \int_{\{|\tau\eta|\geq M_9\}\cap\operatorname{supp}\eta} j(z,\tau\eta(z))\,dz \\ &\quad + \int_{\{|\tau\eta|<M_9\}\cap\operatorname{supp}\eta} j(z,\tau\eta(z))\,dz \\ &\geq \frac{\tau^p}{p}\int_{\{|\tau\eta|\geq M_9\}\cap\operatorname{supp}\eta}(\xi(z)-\varepsilon)|\eta(z)|^p\,dz - M_{10} \\ &\qquad\qquad \text{for some } M_{10} > 0.\end{aligned} \tag{3.33}$$

Using this estimate in the definition of ϕ, we obtain

$$\begin{aligned}\phi(\tau\eta) &\leq \frac{|\tau|^p}{p}\|D\eta\|_p^p - \frac{|\tau|^p}{p}\int_{\{|\tau\eta|\geq M_9\}\cap\operatorname{supp}\eta}(\xi(z)-\varepsilon)|\eta(z)|^p\,dz + M_{10} \\ &= \frac{|\tau|^p}{p}\|D\eta\|_p^p - \frac{|\tau|^p}{p}\int_Z(\xi(z)-\varepsilon)|\eta(z)|^p\,dz \\ &\quad + \frac{|\tau|^p}{p}\int_{\{|\tau\eta|<M_9\}\cap\operatorname{supp}\eta}(\xi(z)-\varepsilon)|\eta(z)|^p\,dz + M_{10} \\ &\leq \frac{|\tau|^p}{p}\left(1 - \frac{1}{\lambda_1(\xi)} + \frac{\varepsilon}{\lambda_1}\right)\|D\eta\|_p^p \\ &\quad + \frac{|\tau|^p}{p}\int_{\{|\tau\eta|<M_9\}\cap\operatorname{supp}\eta}(\xi(z)-\varepsilon)|\eta(z)|^p\,dz + M_{10}.\end{aligned} \tag{3.34}$$

Recall that $\lambda_1(\xi) < 1$. So we can choose $\varepsilon > 0$ small such that $\sigma_1 = 1 - 1/\lambda_1(\xi) + \varepsilon/\lambda_1 < 0$. Also note that $\int_{\{|\tau\eta|<M_9\}\cap \operatorname{supp}\eta}(\xi(z)-\varepsilon)|\eta(z)|^p\,dz \to 0$ as $|\tau| \to \infty$. Thus we can find $\rho_1 > 0$ such that for $|\tau| > \rho_1$ we have

$$\sigma_2(\tau) = \sigma_1\|D\eta\|_p^p + \int_{\{|\tau\eta|<M_9\}\cap \operatorname{supp}\eta}\big(\xi(z)-\varepsilon\big)\big|\eta(z)\big|^p\,dz < 0. \tag{3.35}$$

Therefore, it follows that

$$\phi(\tau\eta) \le \frac{|\tau|^p}{p}\sigma_2(\tau) \longrightarrow -\infty \quad \text{as } |\tau| \longrightarrow \infty. \tag{3.36}$$

□

Propositions 3.2, 3.3, and 3.4 lead to the first existence theorem for problem (1.1).

Theorem 3.5. *If hypotheses* $\mathbf{H}(j)_1$ *hold, then problem (1.1) has a nontrivial solution* $x \in W_0^{1,p}(Z)$.

Proof. Propositions 3.2, 3.3, 3.4 and since $\int_Z j(z,0)\,dz \ge 0$ (hypothesis $\mathbf{H}(j_1)$(iii)) permit the use of Theorem 2.1. So we obtain $x \in W_0^{1,p}(Z)$ such that

$$\phi(z) \ge \xi_1 > 0 \ge \phi(0), \qquad 0 \in \partial\phi(x). \tag{3.37}$$

So $x \neq 0$. Moreover, the inclusion $0 \in \partial\phi(x)$ implies that

$$\begin{aligned} A(x) = u \quad & \text{for some } u \in L^q(Z)\left(\frac{1}{p}+\frac{1}{q}=1\right), \\ & u \in \partial J(x)\ \big(\text{hence } u(z) \in \partial j\big(z,x(z)\big) \text{ a.e.}\big) \\ \Longrightarrow \langle A(x),\psi\rangle = \langle u,\psi\rangle = (u,\psi)_{pq} &= \int_Z u(z)\psi(z)\,dz \quad \forall \psi \in C_0^\infty(Z) \\ \Longrightarrow \int_Z \|Dx(z)\|^{p-2}\big(Dx(z),D\psi(z)\big)_{\mathbb{R}^N}\,dz &= \int_Z u(z)\psi(z)\,dz. \end{aligned} \tag{3.38}$$

From Green's identity and since $-\operatorname{div}(\|Dx\|^{p-2}Dx) \in W^{-1,q}(Z) = W_0^{1,p}(Z)^*$ (cf. Hu and Papageorgiou [12, page 866]), we obtain

$$\big\langle -\operatorname{div}\big(\|Dx\|^{p-2}Dx\big),\psi\big\rangle = \langle u,\psi\rangle. \tag{3.39}$$

Because $\psi \in C_0^\infty(Z)$ is arbitrary and since $C_0^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we obtain

$$
\begin{aligned}
&-\operatorname{div}\left(\|Dx\|^{p-2}Dx\right) = u \in L^q(Z) \\
&\qquad\qquad \Longrightarrow \begin{cases} -\operatorname{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = u(z) & \text{a.e. on } Z, \\ x|_\Gamma = 0 \end{cases} \\
&\qquad\qquad \Longrightarrow x \in W_0^{1,p}(Z) \text{ is a nontrivial solution of (1.1).}
\end{aligned}
\tag{3.40}
$$

□

In the second existence theorem, we allow at $\pm\infty$ interaction with the first eigenvalue (problems at resonance) but we strengthen the growth condition on j and restrict the variation of μ. Moreover, the nonresonance condition at the origin (hypothesis $\mathbf{H}(j)_2$(v) is removed). Smooth problems at resonance were investigated by Arcoya and Orsina [2], Costa and Magalhães [7, Theorem 2], and Hachimi and Gossez [8]. Our hypotheses on the generalized potential $j(z,x)$ are now the following:

$\mathbf{H}(j)_2$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$, and all $u \in \partial j(z,x)$, we have $|u| \le a_1(z) + c_1|x|^{p-1}$ with $a_1 \in L^q(Z)$, $c_1 > 0$, $j(\cdot,0) \in L^1(Z)$, $\int_Z j(z,0)\,dz = 0$ and there exists $\varepsilon > 0$ such that $j(z,x) \ge c|x|^r$ for μ-almost all $z \in Z$, all $0 < x < \varepsilon$, $1 \le r < p$, and $c > \lambda_1/p$ if $r = p$;

(iv) there exists $a \in L^\infty(Z)$ with $a(z) \ge \eta > 0$ a.e. on Z and $0 < \mu < p$ such that

$$\liminf_{|x|\to\infty} \frac{ux - pj(z,x)}{|x|^\mu} = a(z) \tag{3.41}$$

uniformly for almost all $z \in Z$ and all $u \in \partial j(z,x)$;

(v) there exists $\xi \in L^{N/p}(Z)_+$ with $\xi(z) > 0$ for all z on a subset of positive Lebesgue measure such that $\lambda_1(\xi) \ge 1$ and

$$\limsup_{|x|\to\infty} \frac{pj(z,x)}{|x|^p} \le \xi(z) \tag{3.42}$$

uniformly for almost all $z \in Z$.

Remark 3.6. If $a \in L^\infty(Z)$ and $\xi \in L^{N/p}(Z)_+$ are as above, $0 < k < \lambda_1$ it is easy to see that the nonsmooth potential function

$$j(z,x) = \begin{cases} \dfrac{k}{p}|x|^{p-1}, & \text{if } |x| \le 1, \\ \dfrac{\xi(z)}{p}|x|^p - a(z)|x| + \left(a(z) - \dfrac{\xi(z)}{p} + 1\right), & \text{if } |x| > 1 \end{cases} \tag{3.43}$$

satisfies hypotheses $\mathbf{H}(j)_2$.

As before, we consider the locally Lipschitz energy functional defined by

$$\phi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))\,dz. \tag{3.44}$$

Proposition 3.7. *If hypotheses* $\mathbf{H}(j)_2$ *hold, then* ϕ *satisfies the nonsmooth C-condition.*

Proof. Let $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ be a sequence such that

$$|\phi(x_n)| \leq M_{11}, \quad (1+\|x_n\|)m(x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \ (M_{11} > 0). \tag{3.45}$$

Since the function $x \to |x|^r$ $(0 < r < \infty)$ is locally Lipschitz and $|x|^r \neq 0$ for $x \neq 0$, using Clarke [6, Proposition 2.3.14, page 48], for $x \neq 0$ we have

$$\begin{aligned}\partial\left(\frac{j(z,x)}{|x|^p}\right) &\subseteq \frac{|x|^p\partial j(z,x) - pj(z,x)|x|^{p-2}x}{|x|^{2p}} \\ &= \begin{cases} \dfrac{x\partial j(z,x) - pj(z,x)}{x^{p+1}}, & \text{if } x > 0, \\ \dfrac{-x\partial j(z,x) + pj(z,x)}{|x|^{p+1}}, & \text{if } x < 0. \end{cases}\end{aligned} \tag{3.46}$$

By virtue of hypothesis $\mathbf{H}(j)_2$(iv), given $\varepsilon > 0$ we can find $M_{12} > 0$ such that for almost all $z \in Z$, all $x \geq M_{12}$, and all $u \in \partial j(z,x)$ we have

$$\frac{ux - pj(z,x)}{x^\mu} \geq a(z) - \varepsilon \Longrightarrow \frac{ux - pj(z,x)}{x^{p+1}} \geq (a(z) - \varepsilon)x^{\mu-p-1}. \tag{3.47}$$

Therefore for almost all $z \in Z$, all $x \geq M_{12}$, and all $\theta(z,x) \in \partial(j(z,x)/x^p)$ we can write

$$\theta(z,x) \geq (a(z) - \varepsilon)x^{\mu-p-1}. \tag{3.48}$$

For $x \geq M_{12} > 0$ the function $x \to (j(z,x)/x^p)$ is locally Lipschitz and so for all $z \in \mathbb{R} \setminus D_1(z)$, $|D_1(z)| = 0$ (here by $|\cdot|$ we denote the Lebesgue measure on $\mathbb{R}$) is differentiable. Thus we may choose

$$\theta(z,x) = \begin{cases} \dfrac{d}{dx}\dfrac{j(z,x)}{x^p}, & \text{if } x \in \mathbb{R} \setminus D_1(z), \\ 0, & \text{if } x \in D_1(z). \end{cases} \tag{3.49}$$

With this choice of θ we integrate (3.48) on the interval $[y,x]$, $M_{12} \le y < x$. We obtain

$$\begin{aligned}
\frac{j(z,x)}{x^p} - \frac{j(z,y)}{y^p} &\ge \int_y^x (a(z)-\varepsilon) r^{\mu-p-1}\,dr = \frac{a(z)-\varepsilon}{\mu-p}\left(x^{\mu-p} - y^{\mu-p}\right) \\
&\Longrightarrow \limsup_{x\to+\infty} \frac{j(z,x)}{x^p} - \frac{j(z,y)}{y^p} \ge -\frac{a(z)-\varepsilon}{\mu-p} y^{\mu-p} \text{ (recall } \mu < p) \\
&\Longrightarrow \frac{1}{p}\xi(z) - \frac{j(z,y)}{y^p} \\
&\ge -\frac{a(z)-\varepsilon}{\mu-p} y^{\mu-p} \quad \text{(see hypothesis } \mathbf{H}(j)_2(\mathrm{v})) \\
&\Longrightarrow \frac{j(z,y)}{y^p} \le \frac{a(z)-\varepsilon}{\mu-p} y^{\mu-p} + \frac{1}{p}\xi(z) = \frac{1}{p}\xi(z) - \frac{a(z)-\varepsilon}{p-\mu} y^{\mu-p} \\
&\Longrightarrow j(z,y) \le \frac{1}{p}\xi(z) y^p - \frac{a(z)-\varepsilon}{p-\mu} y^{\mu} \\
&\qquad\qquad \text{a.e. on } Z\ \forall y \ge M_{12} > 0.
\end{aligned} \tag{3.50}$$

In a similar fashion, we show that

$$j(z,y) \le \frac{1}{p}\xi(z)|y|^p - \frac{a(z)-\varepsilon}{p-\mu}|y|^{\mu} \quad \text{a.e. on } Z,\ \forall y \le -M_{12} < 0. \tag{3.51}$$

From (3.50) and (3.51), we infer that for almost all $z \in Z$ and all $|x| \ge M_{12}$ we have

$$j(z,x) \le \frac{1}{p}\xi(z)|x|^p - \frac{a(z)-\varepsilon}{p-\mu}|x|^{\mu}. \tag{3.52}$$

We will show that $\{x_n\}_{\ge 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose not. Then by passing to a subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Set $y_n = x_n/\|x_n\|$, $n \ge 1$. We may assume (at least for a subsequence) that

$$\begin{gathered}
y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z), \quad y_n \longrightarrow y \text{ in } L^p(Z), \quad y_n(z) \longrightarrow y(z) \quad \text{a.e. on } Z, \\
|y_n(z)| \le k(z) \quad \text{a.e. on } Z\ \forall n \ge 1, \text{ with } k \in L^p(Z).
\end{gathered} \tag{3.53}$$

From the choice of the sequence $\{x_n\}_{n\ge 1}$, we have that

$$\langle A(x_n), y_n - y\rangle - \langle u_n, y_n - y\rangle \le \varepsilon_n \|y_n - y\|. \tag{3.54}$$

Divide by $\|x_n\|^{p-1}$. We obtain

$$\langle A(y_n), y_n - y\rangle - \left\langle \frac{u_n}{\|x_n\|^{p-1}}, y_n - y \right\rangle \le \frac{\varepsilon_n}{\|x_n\|^{p-1}} \|y_n - y\|. \tag{3.55}$$

Note that since $u_n \in L^q(Z)$ and $y_n - y \in L^p(Z)$, we have

$$\begin{aligned}\left\langle \frac{u_n}{\|x_n\|^{p-1}}, y_n - y \right\rangle &= \int_Z \frac{u_n}{\|x_n\|^{p-1}} (y_n - y)(z)\, dz \\ &\le \int_Z \left(\frac{a_1(z)}{\|x_n\|^{p-1}} + c_1 |y_n(z)|^{p-1} \right) (y_n - y)(z)\, dz \longrightarrow 0 \\ &\qquad \text{as } n \longrightarrow \infty.\end{aligned} \tag{3.56}$$

Therefore passing to the limit in (3.55), we obtain

$$\lim_{n\to\infty} \sup \langle A(y_n), y_n - y \rangle \le 0. \tag{3.57}$$

Because A is maximal, it is generalized pseudomonotone and so the above inequality implies that $\langle A(y_n), y_n \rangle \to \langle A(y), y \rangle$, hence $\|Dy_n\|_p \to \|Dy\|_p$. As in the proof of Proposition 3.2, via Kadec-Klee property, we obtain that $y_n \to y$ in $W_0^{1,p}(Z)$ and since $\|y_n\| = 1$, $n \ge 1$, we have that $\|y\| = 1$ and so $y \ne 0$.

Recall that from the choice of the sequence $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$, we have that

$$\begin{aligned}&|\phi(x_n)| \le M_{11} \quad \forall n \ge 1 \\ &\quad \Longrightarrow \frac{1}{p}\|Dx_n\|^p - \int_Z j(z, x_n(z))\, dz \le M_{11} \\ &\quad \Longrightarrow \frac{1}{p}\|Dy_n\|_p^p - \int_Z \frac{j(z, x_n(z))}{\|x_n\|^p}\, dz \le \frac{M_{11}}{\|x_n\|^p}.\end{aligned} \tag{3.58}$$

Note that

$$\begin{aligned}\int_Z \frac{j(z, x_n(z))}{\|x_n\|^p}\, dz &= \int_{\{|x_n| \ge M_{12}\}} \frac{j(z, x_n(z))}{|x_n(z)|^p} |y_n(z)|^p\, dz \\ &\quad + \int_{\{|x_n| < M_{12}\}} \frac{j(z, x_n(z))}{|x_n(z)|^p} |y_n(z)|^p\, dz \\ &\le \int_Z \frac{1}{p} \xi(z) |y_n(z)|^p\, dz \\ &\quad - \frac{1}{\|x_n\|^p} \int_{\{|x_n| \ge M_{12}\}} \frac{a(z) - \varepsilon}{p - \mu} |x_n(z)|^\mu\, dz + \frac{M_{13}}{\|x_n\|^p}\end{aligned} \tag{3.59}$$

with $M_{13} > 0$ (see (3.52)).

Arguing as in the proof of Proposition 3.2, we can check that $\{x_n\}_{n\geq 1} \subseteq L^{\mu}_{a}(Z)$ is bounded. So we have

$$\begin{aligned}
&\frac{1}{\|x_n\|^p}\left|\int_{\{|x_n|\geq M_{12}\}} \frac{a(z)-\varepsilon}{p-\mu}|x_n(z)|^{\mu}\,dz\right| \\
&\quad\leq \frac{1}{\|x_n\|^p}\int_Z \frac{a(z)}{p-\mu}|x_n(z)|^{\mu}\,dz \\
&\quad= \frac{1}{\|x_n\|^p}\frac{\|x\|^{\mu}_{L^{\mu}_a}}{p-\mu} \longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{3.60}$$

Therefore returning to (3.58), we can write that

$$\frac{1}{p}\|Dy_n\|^p_p - \frac{1}{p}\int_Z \xi(z)|y_n(z)|^p\,dz - \delta_n \leq \frac{M_{11}}{\|x_n\|^p}, \quad \text{with } \delta_n \downarrow 0. \tag{3.61}$$

Passing to the limit and since $Dy_n \to Dy$ in $L^p(Z)$, we obtain

$$\begin{aligned}
\frac{1}{p}\|Dy\|^p_p \leq \frac{1}{p}\int_Z \xi(z)|y(z)|^p\,dz &\Longrightarrow \frac{\lambda_1(\xi)}{p}\int_Z \xi(z)|y(z)|^p\,dz \\
&\leq \frac{1}{p}\int_Z \xi(z)|y(z)|^p\,dz \quad \text{since } y\neq 0 \text{ (see (2.6))}.
\end{aligned} \tag{3.62}$$

If $\lambda_1(\xi) > 1$, then we have a contradiction. If $\lambda_1(\xi) = 1$, then from (3.62) it follows that $y = \pm u_1(\xi)$ the eigenfunction corresponding to the principal eigenvalue $\lambda_1(\xi)$ (recall that $\lambda_1(\xi) > 0$ is simple). From Allegretto and Huang [1], we know that $u_1(\xi)(z) > 0$ for all $z \in \{z \in Z : \xi(z) > 0\}$. Also from the $L^{\mu}_a(Z)$-boundedness of the sequence $\{x_n\}_{n\geq 1}$, we have that

$$\begin{aligned}
&\|x_n\|^{\mu}_{L^{\mu}_a} \leq M_{14} \quad \text{for some } M_{14} > 0 \text{ and all } n\geq 1 \\
&\quad\Longrightarrow \int_Z a(z)|x_n(z)|^{\mu}\,dz \leq M_{14} \\
&\quad\Longrightarrow \int_Z a(z)|y_n(z)|^{\mu}\,dz \leq \frac{M_{14}}{\|x_n\|^{\mu}}.
\end{aligned} \tag{3.63}$$

Passing to the limit as $n\to\infty$ we obtain

$$\int_Z a(z)|u_1(\xi)(z)|^{\mu}\,dz \leq 0, \tag{3.64}$$

a contradiction. So $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_0(Z)$ is bounded and then arguing as in the last part of the proof of Proposition 3.2, we can show (at least for a subsequence) that $x_n \to x$ in $W^{1,p}_0(Z)$. Therefore, ϕ satisfies the nonsmooth C-condition. □

Using this proposition we can prove a second existence theorem for problem (1.1).

Theorem 3.8. *If hypotheses* $\mathbf{H}(j)_2$ *hold, then problem (1.1) has a nontrivial solution* $x \in W_0^{1,p}(Z)$.

Proof. From the proof of Proposition 3.7, we know that for all $x \in W_0^{1,p}(Z)$, $x \neq 0$, we have

$$\begin{aligned}\phi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))\,dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \xi(z)|x(z)|^p\,dz \\ &\quad + \int_{\{|x(z)|\geq M_{12}\}} \frac{a(z)-\varepsilon}{p-\mu}|x(z)|^\mu\,dz - M_{13} \\ &\geq \frac{1}{p}\left(1-\frac{\varepsilon}{\lambda_1(\xi)}\right)\|Dx\|_p^p + M_{14}\|x\|_{L_a^\mu}^\mu - M_{13}\end{aligned} \tag{3.65}$$

with $M_{14} = (\eta-\varepsilon)/(p-\mu) > 0$ if we choose $\varepsilon < \eta$. We infer that ϕ is bounded below. We can apply Zhong's [20] variational principle and obtain a sequence $\{x_n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ such that $\phi(x_n) \downarrow m_0 = \inf\phi$ and

$$\begin{aligned}&\phi(x_n) \leq \phi(u) + \frac{\varepsilon_n}{1+\|x_n\|}\|x_n-u\| \quad \forall u \in W_0^{1,p}(Z) \text{ with } \varepsilon_n \downarrow 0 \\ &\Longrightarrow -\frac{\varepsilon_n}{1+\|x_n\|}\|x_n-u\| \leq \phi(u)-\phi(x_n).\end{aligned} \tag{3.66}$$

Let $u = x_n + tw$ with $t > 0$ and $w \in W_0^{1,p}(Z)$. We obtain

$$-\frac{\varepsilon_n}{1+\|x_n\|}\|w\| \leq \frac{\phi(x_n+tw)-\phi(x_n)}{t} \Longrightarrow -\frac{\varepsilon_n}{1+\|x_n\|}\|w\| \leq \phi^0(x_n;w). \tag{3.67}$$

Let $\psi_n(w) = ((1+\|x_n\|)/\varepsilon_n)\phi^0(x_n;w)$. Evidently $\psi_n(0) = 0$ and $-\|w\| \leq \psi_n(w)$. So we can apply [19, Lemma 1.3] and obtain $y_n^* \in W^{-1,q}(Z)$, $\|y_n^*\|_* \leq 1$ such that $\langle y_n^*, w\rangle \leq \psi_n(w)$ for all $w \in W_0^{1,p}(Z)$. Setting $x_n^* = (\varepsilon_n/(1+\|x_n\|))y_n^*$ we have $\langle x_n^*, w\rangle \leq \phi^0(x_n;w)$ for all $w \in W_0^{1,p}(Z)$. Moreover, $(1+\|x_n\|)\|x_n^*\|_* \leq \varepsilon_n \to 0$, hence $(1+\|x_n\|)m(x_n) \to 0$. So by virtue of Proposition 3.7 and by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $W_0^{1,p}(Z)$. Then $\phi(x) = m_0 = \inf\phi$ and $0 \in \partial\phi(x)$. Arguing as in the proof of Theorem 3.5, we check that $x \in W_0^{1,p}(Z)$ solves (1.1).

Let now $u_1(\cdot)$ be the principal eigenfunction of problem (2.5) with $\theta = 1$. Then from the Rayleigh quotient we have that $\|Du_1\|^p = \lambda_1\|u_1\|^p$. Moreover, from hypothesis $\mathbf{H}(j)_2$(iii) and as $u_1 \in C^1(Z)$ we can find $0 < s < 1$ so small that

$0 < su_1(z) < \varepsilon$ for every $z \in Z$. Thus, if $r = p$

$$\begin{aligned}\phi(su_1) &= \frac{1}{p}\lambda_1 \|su_1\|_p^p - \int_Z j(z, su_1(z))\, dz \\ &\le s^p\left(\frac{\lambda_1}{p} - c\right)\|u_1\|_p^p < 0 = \phi(0).\end{aligned} \tag{3.68}$$

If $r < p$, then for some $c_1 > 0$

$$\phi(su_1) = \frac{\lambda_1 s^p}{p}\|u_1\|_p^p - c_1 s^r \|u_1\|_p^r. \tag{3.69}$$

Since $r < p$, if $0 < s < 1$ is small enough we have $\phi(su_1) < 0 = \phi(0)$. Therefore, $x \neq 0$. □

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