AN INVERSE PROBLEM FOR EVOLUTION INCLUSIONS

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An inverse problem, the determination of the shape and a convective coefficient on a part of the boundary from partial measurements of the solution, is studied using 2-person optimal control techniques.

1. Introduction

Let $H, \mathcal{H}_j, \mathcal{U}_j$; $j = 1, \dots, N$ be Hilbert spaces and let φ be a lower semi-continuous (l.s.c.) function from $H \times \prod_{j=1}^N \mathcal{U}_j$ into $\mathbb{R}^+$ with $\varphi(\cdot; u)$ convex on H .

Consider the initial-value problem

$$y' + \partial\varphi(y; u) + f(t, y; u) \ni 0 \quad \text{on } (0, T), \quad y(0) = y_0. \quad (1.1)$$

With some conditions on φ and on f , the set $\mathcal{R}(u)$ of all “strong” solutions of (1.1) is nonempty. Let f_j be mappings of $L^2(0, T; \mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$ and associate with (1.1) the cost functionals

$$J_j(y; u) = \int_0^T f_j(y(s); u) ds, \quad j = 1, \dots, N, \quad (1.2)$$

with $D(\varphi(\cdot, u)) \subset \mathcal{H}_j$ for all $u \in \mathcal{U} = \prod_{j=1}^N \mathcal{U}_j$.

The existence of an open loop of (1.1), (1.2) with φ independent of the control u , has been established in Ton [7]. With optimal shape design and with inverse problems in mind, we will consider the case when φ depends on the control u as it appears in the top order term of the partial differential operators involved in the problems.

Optimal design of domains has been investigated by Barbu and Friedman [1], Canadas et al. [2], Gunzburger and Kim [3], Pironneau [6], and others. Inverse

problems have been studied by Canadas et al. [2], Lenhart et al. [4], Lenhart and Wilson [5], and others.

In contrast with all the cited works where a single cost functional is involved, we will consider the N -person optimal control approach. It is well known that for N -control, open and closed loops are two different notions. In this paper, the existence of an open loop of (1.1), (1.2) is established in Section 3, that is, there exists $\tilde{u} \in U$ such that

$$J_j(\tilde{y}; \pi_j \tilde{u}, \tilde{u}_j) \leq J_j(y; \pi_j \tilde{u}, v_j), \quad \forall y \in \mathcal{R}(\pi_j \tilde{u}, v_j), \quad \forall v_j \in U_j; \quad j = 1, \dots, N, \quad (1.3)$$

where U_j are given compact convex subsets of the control spaces $\mathcal{U}_j$ and π_j is the projection of $\mathcal{U}$ onto $\prod_{k \neq j}^N \mathcal{U}_k$.

With a cost functional f_j defined by

$$f_j(y) = \|y(\cdot, t) - h(\cdot, t)\|_{L^2(0, T; L^2(\Omega))}^2, \quad (1.4)$$

where Ω is a proper subset of the domain and where h is a measurement of the solution y of (1.1) in the subdomain Ω , then (1.1), (1.2) becomes an inverse problem.

Applications to parabolic inequalities are carried out in Section 4 and the notation and the main assumptions of the paper are given in Section 2.

2. Notation and assumptions

Let $H, \mathcal{H}_j, \mathcal{U}_j; j = 1, \dots, N$ be Hilbert spaces. The norm in H is denoted by $\|\cdot\|$ and $(\cdot, \cdot)$ is the inner product in the space. Throughout, U is a given compact convex subset of the control space $\mathcal{U} = \prod_{j=1}^N \mathcal{U}_j$.

Assumption 2.1. Let φ be a mapping of $H \times \mathcal{U}$ into $\mathbb{R}^+$. We assume that

- (1) for each $u \in \mathcal{U}$, $D(\varphi(\cdot; u))$ is dense in H ;
- (2) $\varphi(y; u)$ is an l.s.c. function from $H \times \mathcal{U}$ into $\mathbb{R}^+$ and is convex on H for each given $u \in \mathcal{U}$;
- (3) there exists a positive constant c such that

$$c\|y\|^2 \leq \varphi(y; u), \quad \forall y \in D(\varphi(y; u)), \quad \forall u \in \mathcal{U}; \quad (2.1)$$

- (4) for each positive constant C ,

$$\{y : \varphi(y; u) \leq C\} \quad (2.2)$$

is a compact convex subset of H for each given $u \in \mathcal{U}$;

- (5) if $u_n \rightarrow u$ in $\mathcal{U}$, then

$$\int_0^T \varphi(y(s); u) ds = \lim_n \int_0^T \varphi(y(s); u_n) ds, \quad \forall y \in \bigcap_{u_n \in \mathcal{U}} D(\varphi(\cdot; u_n)) \cap L^2(0, T; H). \quad (2.3)$$

The subdifferential of $\varphi(y; u)$ at y is the set

$$\partial\varphi(y; u) = \{g : g \in H, \varphi(x; u) - \varphi(y; u) \geq (g, x - y), \forall x \in D(\varphi(\cdot; u))\}. \quad (2.4)$$

It is known that $A(y; u) = \partial\varphi(y; u)$ is maximal monotone in H . The images of $A(y; u)$ are closed, convex subsets of H .

Let $f(y; u)$ be a mapping of $L^2(0, T; H) \times \mathcal{U}$ into $L^2(0, T; H)$ satisfying the following assumption.

Assumption 2.2. We assume that there exists a constant C such that

$$\|f(y; u)\|_H^2 \leq C\{1 + \|u\|_{\mathcal{U}}^2 + \varphi(y; u)\} \quad (2.5)$$

for all $y \in D(\varphi(\cdot; u))$, all $u \in \mathcal{U}$.

Throughout, the set of solutions of (1.1) is denoted by $\mathcal{R}(u)$.

Assumption 2.3. Let f_j be mappings of $L^2(0, T; \mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$. We assume that

- (1) $D(\varphi(\cdot; u)) \subset \mathcal{H}_j$ for all $u \in \mathcal{U}$;
- (2) suppose that

$$\begin{aligned} \varphi(y^n; u^n) + \|(y^n)'\|_{L^2(0, T; H)} &\leq C, \\ u^n \in U, \{y^n, u^n\} &\longrightarrow \{y, u\} \quad \text{in } L^2(0, T; H) \times \mathcal{U}, \end{aligned} \quad (2.6)$$

then

$$\int_0^T f_j(y; u) dt = \lim_{n \rightarrow \infty} \int_0^T f_j(y^n; u^n) dt. \quad (2.7)$$

3. Open loop control

The main result of this section is the following theorem.

THEOREM 3.1. *Let φ, f be as in Assumptions 2.1 and 2.2, and let f_j be continuous mappings of $L^2(0, T; \mathcal{H}_j) \times \mathcal{U}$ into $\mathbb{R}^+$. Suppose that $y_0 \in D(\varphi(\cdot; u))$ for all $u \in U$. Then there exists $\{\tilde{y}, \tilde{u}\} \in \{L^2(0, T; H) \cap \mathcal{R}(\tilde{u})\} \times U$ such that*

$$J_j(\tilde{y}; \pi_j \tilde{u}, \tilde{u}_j) \leq J_j(y; \pi_j \tilde{u}, v_j), \quad \forall y \in \mathcal{R}(\pi_j \tilde{u}, v_j), \quad \forall v_j \in U_j, j = 1, \dots, N. \quad (3.1)$$

Moreover, there exists a positive constant C , independent of u such that

$$\begin{aligned} \text{ess sup } \varphi(\tilde{y}(t); \tilde{u}) + \|\tilde{y}'\|_{L^2(0, T; H)}^2 + \|A(\tilde{y}; \tilde{u})\|_{L^2(0, T; H)}^2 \\ \leq C \left\{ 1 + \sup_{u \in U} \varphi(y_0; u) \right\}, \end{aligned} \quad (3.2)$$

where $A(\tilde{y}; \tilde{u})$ is an element of the set $\partial\varphi(\tilde{y}; \tilde{u})$.

First, we will show that the set $\mathcal{R}(u)$ is nonempty.

THEOREM 3.2. *Suppose all the hypotheses of [Theorem 3.1](#) are satisfied. Then for each given $u \in U$, there exists a solution y of (1.1) with*

$$\|y'\|_{L^2(0,T;H)}^2 + \|A(y;u)\|_{L^2(0,T;H)}^2 + \text{ess sup } \varphi(y(t);u) \leq C\{1 + \|u\|_{\mathcal{Q}_U}^2\}. \quad (3.3)$$

The constant C is independent of u and $A(y;u)$ is an element of $\partial\varphi(y;u)$.

Proof. For a given $u \in U$, the existence of a solution y of (1.1) with

$$\{y, y', A(y;u)\} \in L^\infty(0, T; H) \times (L^2(0, T; H))^2 \quad (3.4)$$

is known (cf. Yamada [8]).

We will now establish the estimate of [Theorem 3.2](#). We have

$$(y', \partial\varphi(y;u)) + \|\partial\varphi(y;u)\|^2 + (f(y;u), \partial\varphi(y;u)) = 0. \quad (3.5)$$

With our hypotheses on f , we get

$$\frac{d}{dt} \varphi(y;u) + \|\partial\varphi(y;u)\|^2 \leq C\{1 + \|u\|_{\mathcal{Q}_U}^2 + \varphi(y(t);u)\}. \quad (3.6)$$

It follows from the Gronwall lemma that

$$\text{ess sup}_{t \in [0, T]} \varphi(y(t);u) + \|\partial\varphi(y;u)\|_{L^2(0,T;H)}^2 \leq C\{1 + \|u\|_{\mathcal{Q}_U}^2\}. \quad (3.7)$$

The different constants C are all independent of u .

With the estimate (2.1), we deduce from (1.1) and from [Assumption 2.2](#) that

$$\|y'\|_{L^2(0,T;H)}^2 \leq C\{1 + \|u\|_{\mathcal{Q}_U}^2\}. \quad (3.8)$$

The theorem is thus proved. $\square$

Set

$$\mathcal{B}_C = \left\{ y : \|y'\|_{L^2(0,T;H)} + \sup_{u \in U} \text{ess sup } \varphi(y;u) \leq C \left(1 + \sup_{u \in U} \|u\|_{\mathcal{Q}_U} \right) \right\}. \quad (3.9)$$

Consider the evolution inclusion

$$y' + \partial\varphi(y;u) + f(x;u) \ni 0 \quad \text{on } (0, T), \quad y(0) = y_0 \quad (3.10)$$

with $x \in \mathcal{B}_C$.

In view of [Theorem 3.2](#), inclusion (3.10) has a unique solution which we will write as $y = R(x;u)$.

Denote by

$$J_j(x; y; u) = \int_0^T f_j(y(s); u) ds, \quad j = 1, \dots, N, \quad (3.11)$$

the cost functionals associated with (3.10) and where $y = R(x; u)$ is the unique solution of (3.10).

Let

$$\Psi(x; u, v) = \sum_{j=1}^N J_j(x; y_j; \pi_j u, v_j), \quad (3.12)$$

where $y_j = R(x; \pi_j u, v_j)$.

LEMMA 3.3. Suppose all the hypotheses of Theorem 3.1 are satisfied. Then for each given $\{x, u\} \in \mathcal{B}_C \times U$, there exists $v^* \in U$ such that

$$\Psi(x; u, v^*) = d(x; u) = \inf \{ \Psi(x; u, v) : v \in U \}. \quad (3.13)$$

Proof. Let $\{v^n\}$ be a minimizing sequence of (3.13) with

$$d(x; u) \leq \Psi(x; u, v^n) \leq d(x; u) + n^{-1}. \quad (3.14)$$

Since $v^n \in U$ and U is a compact subset of $\mathcal{O}_U$, we obtain by taking subsequences that $v^{n_k} \rightarrow v^*$ in $\mathcal{O}_U$. Let $y_j^n = R(x; \pi_j u, v_j^n)$, then from the estimates of Theorem 3.2 we obtain, by taking subsequences, that

$$\begin{aligned} & \{y_j^{n_k}, (y_j^{n_k})', A(y_j^{n_k}; \pi_j u, v_j^{n_k})\} \\ & \longrightarrow \{y_j^*, (y_j^*)', \chi_j\} \quad \text{in } L^2(0, T; H) \times (L^2(0, T; H))_{\text{weak}}^2. \end{aligned} \quad (3.15)$$

From the definition of subdifferential, we have

$$\begin{aligned} & \int_0^T \varphi(z(t); \pi_j u, v_j^{n_k}) dt - \int_0^T \varphi(y_j^{n_k}(t); \pi_j u, v_j^{n_k}) dt \\ & \geq \int_0^T (A(y_j^{n_k}(t); \pi_j u, v_j^{n_k}), z - y_j^{n_k}) dt, \end{aligned} \quad (3.16)$$

for all $z \in L^2(0, T; H)$.

It follows from Assumption 2.1 that

$$\int_0^T \varphi(z(t); \pi_j u, v_j^*) dt - \int_0^T \varphi(y_j^*(t); \pi_j u, v_j^*) dt \geq \int_0^T (\chi_j, z - y_j^*(t)) dt. \quad (3.17)$$

Hence

$$\chi_j = A(y_j^*; \pi_j u, v_j^*). \quad (3.18)$$

It is clear that $y_j^* = R(x; \pi_j u, v_j^*)$ and thus,

$$d(x; u) = \Psi(x; u, v^*) = \sum_{j=1}^N J_j(x; y_j, \pi_j u, v_j^*), \quad (3.19)$$

where $y_j = R(x; \pi_j u, v_j^*)$.

The lemma is proved. $\square$

Let

$$X(x; u) = \{v^* : \Psi(x; u, v^*) \leq \Psi(x; u, v), \forall v \in U\}. \quad (3.20)$$

LEMMA 3.4. *Let g_j be a continuous mapping of U_j into $\mathbb{R}^+$ and suppose that g_j is 1-1. Then there exists a unique $\hat{v} \in X(x; u)$ such that*

$$g_j(\hat{v}_j) = \inf \{g_j(v_j^*) : v^* \in X(x; u)\}. \quad (3.21)$$

Proof. The set $X(x; u)$ is nonempty and with our hypothesis on g_j , it is clear that

$$d_j(x; u) = \inf \{g_j(v_j^*) : v^* \in X(x; u)\} \quad (3.22)$$

exists.

Let v_j^n be a minimizing sequence of the optimization problem (3.22) with

$$d_j(x; u) \leq g_j(v_j^n) \leq d_j(x; u) + n^{-1}, \quad j = 1, \dots, N, \quad (3.23)$$

and $v^n \in X(x, u)$.

Let $y_j^n = R(x; \pi_j u, v_j^n)$ be the unique solution of (3.10) with controls $\{\pi_j u, v_j^n\}$ and $f(x; \pi_j u, v_j^n)$. Then from the estimates of Theorem 3.2, we obtain, by taking subsequences, that

$$\{y_j^n, (y_j^n)', A(y_j^n; \pi_j u, v_j^n)\} \longrightarrow \{\hat{y}_j, \hat{y}_j', \chi_j\} \quad \text{in } L^2(0, T; H) \times (L^2(0, T; H))_{\text{weak}}^2. \quad (3.24)$$

Since $v^n \in U$, we get by taking subsequences that $v^n \rightarrow \hat{v}$ in $\mathcal{U}$.

A proof, as in that of Lemma 3.3, shows that

$$\chi_j = A(\hat{y}_j; \pi_j u, \hat{v}_j), \quad \hat{y}_j = R(x; \pi_j u, \hat{v}_j). \quad (3.25)$$

Hence $\hat{v} \in X(x; u)$. We now have

$$g_j(\hat{v}_j) = d_j(x; u) = \inf \{g_j(v_j^*) : v^* \in X(x; u)\}. \quad (3.26)$$

Since g_j is 1-1, $\hat{v}$ is unique. The lemma is proved. $\square$

Let $\mathcal{L}$ be the nonlinear mapping of $\mathcal{B}_C \times U$ into $\mathcal{B}_C \times U$, defined by

$$\mathcal{L}(x, u) = \{\hat{y}, \hat{v}\}, \quad (3.27)$$

where $\hat{v}$ is the element of U given by Lemma 3.4 and $\hat{y} = R(x; \pi_j u, \hat{v}_j)$ is the unique solution of (3.10) with control $\{\pi_j u, \hat{v}_j\}$ and $f(x; \pi_j u, \hat{v}_j)$.

LEMMA 3.5. *Suppose all the hypotheses of Theorem 3.1 are satisfied. Then $\mathcal{L}$, defined by (3.27), has a fixed point, that is, there exists $\{\tilde{y}, \tilde{u}\} \in \mathcal{B}_C \times U$ such that $\mathcal{L}(\tilde{y}, \tilde{u}) = \{\tilde{y}, \tilde{u}\}$.*

Proof. (1) We now show that $\mathcal{L}$ has a fixed point by applying Schauder's theorem. Since $\mathcal{B}_C \times U$ is a compact convex subset of $L^2(0, T; H) \times \mathcal{U}$ and since $\mathcal{L}$ takes $\mathcal{B}_C \times U$ into itself, it suffices to show that $\mathcal{L}$ is continuous.

(2) Let $\{x^n, u^n\}$ be in $\mathcal{B}_C \times U$ and let

$$y_j^n = R(x^n; \pi_j u^n, \hat{v}_j^n), \quad \hat{v}^n \text{ as in Lemma 3.4.} \quad (3.28)$$

Since $\{x^n u^n\} \in \mathcal{B}_C \times U$ and $\mathcal{B}_C \times U$ is a compact subset of $L^2(0, T; H) \times \mathcal{U}$, there exists a subsequence such that

$$\{x^n, u^n, \hat{v}^n\} \longrightarrow \{x^*, u^*, \hat{v}\} \quad \text{in } L^2(0, T; H) \times \mathcal{U} \times \mathcal{U}. \quad (3.29)$$

From the estimates of Theorem 3.2, we get

$$\{y_j^n, (y_j^n)', A(y_j^n; u^n)\} \longrightarrow \{y_j^*, (y_j^*)', \chi_j\} \quad \text{in } L^2(0, T; H) \times (L^2(0, T; H))^2_{\text{weak}}. \quad (3.30)$$

A proof, as in that of Lemma 3.3, shows that

$$\chi_j = A(y_j^*; u^*), \quad y_j^* = R(x^*; \pi_j u^*, \hat{v}_j). \quad (3.31)$$

(3) We now show that $u^* \in X(x^*, \hat{v})$. Since

$$\mathcal{L}\{u^n, x^n\} = \{v^n, y^n\}, \quad (3.32)$$

it follows from the definition of $\mathcal{L}$ that

$$\begin{aligned} \Psi(x^n; u^n, v^n) &\leq \Psi(x^n; u^n, v), \quad \forall v \in U, \\ \sum_{j=1}^N J_j(x^n; y_j^n; \pi_j u^n, v_j^n) &\leq \sum_{j=1}^N J_j(x^n; z_j^n; \pi_j u^n, v_j), \quad \forall v \in U, \end{aligned} \quad (3.33)$$

where $z_j^n = R(x^n; \pi_j u^n, v_j)$ is the unique solution of (3.10) with controls $\{\pi_j u^n, v_j\}$ and $f(x^n; \pi_j u^n, v_j)$.

Again from the estimates of Theorem 3.2, we deduce as above that

$$\{z_j^n, (z_j^n)', A(z_j^n; u^n)\} \longrightarrow \{z_j, z_j', A(z_j; u^*)\} \quad \text{in } L^2(0, T; H) \times (L^2(0, T; H))^2_{\text{weak}}. \quad (3.34)$$

It then follows from (3.33) that

$$\sum_{j=1}^N J_j(x^*; y_j^*; \pi_j u^*, \hat{v}_j) \leq \sum_{j=1}^N J_j(x^*; z_j; \pi_j u^*, v_j), \quad \forall v \in U, \quad (3.35)$$

that is,

$$\Psi(x^*; u^*, \hat{v}) \leq \Psi(x^*; u^*, v), \quad \forall v \in U. \quad (3.36)$$

Hence

$$d(x^*, u^*) = \Psi(x^*; u^*, \hat{v}) = \inf \{ \Psi(x^*; u^*, v) : v \in U \}. \quad (3.37)$$

Moreover, we have

$$\lim_n g_j(v_j^n) = g_j(\hat{v}_j), \quad j = 1, \dots, N. \quad (3.38)$$

By hypothesis, g_j is 1-1 and so $\hat{v}$, the unique element of $X(x^*; u^*)$, with

$$g_j(\hat{v}_j) = \inf \{ g_j(v_j) : v \in X(x^*; u^*) \}, \quad (3.39)$$

is in $X(x^*; u^*)$. It follows that $\mathcal{L}\{x^*, u^*\} = \{y^*, \hat{v}\}$.

The operator $\mathcal{L}$ is continuous and thus, it has a fixed point by Schauder's theorem. The lemma is thus proved. $\square$

Proof of Theorem 3.1. Let $\mathcal{L}$ be as in (3.33). Then it follows from Lemma 3.5 that $\mathcal{L}$ has a fixed point, that is, there exists $\{\tilde{y}, \tilde{u}\}$ with

$$\mathcal{L}\{\tilde{y}, \tilde{u}\} = \{\tilde{y}, \tilde{u}\}. \quad (3.40)$$

Thus,

$$\tilde{y}' + A(\tilde{y}; \tilde{u}) + f(\tilde{y}; \tilde{u}) = 0 \quad \text{on } (0, T); \quad \tilde{y}(0) = y_0. \quad (3.41)$$

Moreover,

$$\sum_{j=1}^N J_j(\tilde{y}; \pi_j \tilde{u}, \tilde{u}_j) \leq \sum_{j=1}^N J_j(y_j; \pi_j \tilde{u}, v_j), \quad \forall y_j \in \mathcal{R}(\pi_j \tilde{u}, v_j), \quad \forall v \in U. \quad (3.42)$$

Take $v = (\pi_j \tilde{u}, v_j)$ and we obtain from (3.42) that

$$J_j(\tilde{y}; \pi_j \tilde{u}, \tilde{u}_j) \leq J_j(y_j; \pi_j \tilde{u}, v_j), \quad \forall y_j \in \mathcal{R}(\pi_j \tilde{u}, v_j). \quad (3.43)$$

Repeating the process N times we get the theorem. $\square$

4. Applications

In this section, we give some applications of Theorem 3.1 to parabolic initial boundary value problems. For simplicity, we take $N = 2$.

Let G be a bounded open subset of $\mathbb{R}^2$ with a smooth boundary and let

$$\begin{aligned} Q &= G \times (0, 2), & \Gamma &= G \times \{2\}, \\ Q(u_1) &= \{(\xi, \eta) : \xi \in G, 0 < \eta < u_1(\xi)\}, \end{aligned} \quad (4.1)$$

where u_1 is a continuous function of G into $[1, 2]$. The top of the cylinders $Q(u_1)$, Q are

$$\Gamma(u_1) = \{(\xi, u_1(\xi)) : \xi \in G\}, \quad \Gamma. \quad (4.2)$$

Make the change of variable $\zeta = 2\eta/u_1$ and set

$$y(\xi, \eta) = y\left(\xi, \frac{u_1 \zeta}{2}\right) = Y(\xi, \zeta). \quad (4.3)$$

As done in great details in [4, pages 946–948], we get

$$\nabla^2 y = \nabla_{\xi, \zeta} F(\xi, \zeta; u_1) \nabla_{\xi, \zeta} Y(\xi, \zeta) + u_1^{-1} F \nabla Y \cdot \nabla u_1, \quad (4.4)$$

where $F(\xi, \zeta; u_1)$ is the matrix

$$\begin{pmatrix} 1 & 0 & -\zeta(\partial_{\xi_1} u_1)u_1^{-1} \\ 0 & 1 & -\zeta(\partial_{\xi_2} u_1)u_1^{-1} \\ -\zeta(\partial_{\xi_2} u_1)u_1^{-1} & -\zeta(\partial_{\xi_1} u_1)u_1^{-1} & \zeta^2 |\nabla u_1|^2 u_1^{-2} + 4u_1^{-2} \end{pmatrix}. \quad (4.5)$$

Set

$$\mu(u_1) = 2u_1^{-1} \sqrt{1 + |\nabla u_1|^2}. \quad (4.6)$$

4.1. An inverse problem for a nonlinear heat equation. Consider the initial boundary value problem

$$\begin{aligned} y' - \Delta y &= \tilde{f}(y) && \text{on } Q(u_1) \times (0, T), \\ y &= 0 && \text{on } \partial Q(u_1)/\Gamma \times (0, T), \\ -\frac{\partial y}{\partial n} &\in u_2 \beta(y) && \text{on } \Gamma(u_1) \times (0, T), \\ y(\cdot, 0) &= y_0 && \text{on } Q(u_1), \end{aligned} \quad (4.7)$$

where $\beta \in \partial j(r)$ and $j(r)$ is an l.s.c. convex function from $\mathbb{R}^+$ to $[0, \infty]$.

Let

$$\begin{aligned} J_1(y; u_1, u_2) &= \int_0^T \int_G |y(\xi, u_1(\xi))|^2 d\xi dt, \\ J_2(y; u_1, u_2) &= \int_0^T \int_\Omega |y - h(\xi, \eta)|^2 d\xi d\eta dt \end{aligned} \quad (4.8)$$

be the cost functionals associated with (4.7) and let h be the measurement of the solution y of (4.7) in the sub-region Ω .

We denote

$$U_j = \{u_j : \|u_j\|_{H^3(G)} \leq C, 1 \leq u_1(\xi) \leq 2, 0 \leq u_2(\xi) \leq C\} \quad (4.9)$$

and let $\mathcal{U}_j = L^2(G)$. It is clear that the U_j are compact convex subsets of the space of controls $\mathcal{U}_j$.

We will take

$$H = L^2(Q), \quad \mathcal{H}_1 = L^2(G), \quad \mathcal{H}_2 = L^2(\Omega), \quad \Omega \subset Q. \quad (4.10)$$

The main result of this subsection is the following theorem.

THEOREM 4.1. Let y_0 be in $H_0^1(Q)$ and let $\tilde{f}$ be a continuous function of y, u with

$$|\tilde{f}(y; u)| \leq C\{1 + |y| + |u|\}. \quad (4.11)$$

Let h be a given function in $L^2(0, T; L^2(\Omega))$ where Ω is a proper subset of Q and let $j(r)$ be an l.s.c. convex function on $\mathbb{R}$ with values in $[0, +\infty]$. Then there exists

$$\begin{aligned} \{\hat{y}, \hat{y}', \hat{u}\} &\in L^2(0, T; H^1(Q(\hat{u}_1))) \cap L^\infty(0, T; L^2(Q(\hat{u}_1))) \\ &\times L^2(0, T; L^2(Q(\hat{u}_1))) \times U \end{aligned} \quad (4.12)$$

such that $\hat{y}$ is a solution of the initial boundary value problem (4.7) in $Q(\hat{u}_1) \times (0, T)$; and

$$\begin{aligned} J_1(\hat{y}; \hat{u}_1, \hat{u}_2) &\leq J_1(y; \hat{u}_1, v_2), \quad \forall v_2 \in U_2, \\ J_2(\hat{y}; \hat{u}_1, \hat{u}_2) &\leq J_2(x; v_1, \hat{u}_2), \quad \forall v_1 \in U_1, \end{aligned} \quad (4.13)$$

where x, y are the solutions of (4.7) with controls $\{v_1, \hat{u}_2\}, \{\hat{u}_1, v_2\}$ in $Q(v_1) \times (0, T)$ and in $Q(\hat{u}_1) \times (0, T)$, respectively.

Problems of type (4.7) arise in the study of heat transfer between solids and gases under nonlinear boundary conditions.

As carried out in [4], we make the change of variable $\zeta = 2u_1^{-1}\eta$ and set $y(\xi, \eta) = Y(\xi, \zeta)$. Then (4.7) is transformed into the following problem:

$$\begin{aligned} Y' - \nabla(F(u_1) \cdot \nabla Y) + u_1^{-1} F \nabla Y \cdot \nabla u_1 &= \tilde{f}(Y, u) \quad \text{on } Q \times (0, T), \\ Y &= 0 \quad \text{on } \partial Q / \Gamma \times (0, T), \\ -\frac{\partial Y}{\partial n} &\in \mu(u_1) u_2 \beta(Y) \quad \text{on } \Gamma \times (0, T), \\ Y(\cdot, 0) &= y_0 \quad \text{on } Q \end{aligned} \quad (4.14)$$

with cost functionals

$$J_1(Y; u_1, u_2) = \int_0^T \int_G |Y(\xi, 2)|^2 d\xi dt, \quad (4.15)$$

$$J_2(Y; u_1, u_2) = \int_0^T \int_\Omega \left| Y\left(\xi, \frac{2\eta}{u_1}\right) - h(\xi, \eta; t) \right|^2 d\xi d\eta dt, \quad (4.16)$$

where μ is as in expression (4.6).

Our aim is to find the controls u_1, u_2 so that the solution y of (4.7), if it is unique, is as close to the measurement h in Ω as possible.

Let φ be the mapping of $H \times U_1 \times U_2$ into $\mathbb{R}^+$ given by

$$\varphi(Y; u_1, u_2) = \begin{cases} \frac{1}{2} \|F(u) \nabla Y\|_{L^2(Q)}^2 + \int_\Gamma \mu(u_1) u_2 j(Y) d\sigma, & j(Y) \in L^1(\Gamma), \\ +\infty, & \text{otherwise,} \end{cases} \quad (4.17)$$

where $j(r)$ is an l.s.c. convex function from $\mathbb{R}$ to $[0, +\infty]$ with $j(0) = 0$.

By abuse of notation, we will write y for $Y(\xi, \zeta, t)$ when there is no confusion possible.

LEMMA 4.2. *Let φ be as in (4.17). Then φ satisfies Assumption 2.1.*

Proof. (1) It is clear that $\varphi(y; u)$ is an l.s.c. function from $H \times U$ into $\mathbb{R}^+$ and that $C_0^\infty(Q) \subset D(\varphi(\cdot, u))$ for all $u \in U$.

(2) It was shown in [4, pages 949–952] that

$$\int_Q F(u) |\nabla y|^2 d\xi d\zeta \geq c \|y\|_{H^1(Q)}^2, \quad (4.18)$$

for all y with $F(u) \nabla y \in H$, $y = 0$ on $\partial Q/\Gamma$.

Since $j(r)$ and μ are both positive functions, we get

$$c \|y\|_{H^1(Q)}^2 \leq \varphi(y; u), \quad \forall y \in D(\varphi). \quad (4.19)$$

(3) By the Sobolev imbedding theorem, the set

$$\{y : \varphi(y; u) \leq C\} \quad (4.20)$$

is a compact subset of $H = L^2(Q)$.

(4) Suppose that $u_1^n \rightarrow u_1$ in H with $u_1^n \in U_1$. Since u_1^n is in U_1 , it follows from the definition of U_1 and from the Sobolev imbedding theorem that there exists a subsequence such that $u_1^n \rightarrow u_1 \in H^2(G)$ and in $C^1(\bar{G})$.

With $F(u)$, $\mu(u)$ as above, it is trivial to check that we have

$$\lim_{n \rightarrow \infty} \int_0^T \varphi(y(s); u_1^n) ds = \int_0^T \lim_{n \rightarrow \infty} \varphi(y(s); u_1^n) ds. \quad (4.21)$$

□

LEMMA 4.3. *Let φ be as in (4.16). Then $\partial\varphi(y; u) = -\nabla \cdot (F(u) \nabla y) = A(y; u)$ with*

$$D(A(y; u)) = \left\{ y : \nabla \cdot (F(u) \nabla y) \in H, y = 0 \text{ on } \partial Q/\Gamma, \right. \\ \left. -\frac{\partial y}{\partial n} \in \mu(u_1) u_2 \beta(y) \text{ on } \Gamma \right\}. \quad (4.22)$$

Proof. For $y \in H^1(Q)$ with $\nabla \cdot F(u) \nabla y$ in $L^2(Q)$, we know that $F(u) \nabla y \cdot n \in H^{-1/2,2}(\partial Q)$.

Let $A(y; u) = -\nabla \cdot F(u) \nabla y$ with

$$D(A(y; u)) = \left\{ y : y \in H, \nabla \cdot (F(u) \nabla y) \in H, y = 0 \text{ on } \partial Q/\Gamma, \right. \\ \left. -\frac{\partial y}{\partial n} y \in \mu(u_1) u_2 y \text{ on } \Gamma \right\}. \quad (4.23)$$

We now show that A is maximal monotone on H and that $A \subset \partial\varphi(y; u)$.

(1) It is clear that $A(\cdot; u)$ is monotone in H . For $y \in D(A(\cdot; u))$ and $x \in D(\varphi(\cdot; u))$, we have

$$-(\nabla \cdot F(u) \nabla y, x - y) = (F(u) \nabla y, \nabla x - y) - \left\langle \frac{\partial y}{\partial n}, x - y \right\rangle, \quad (4.24)$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1/2,2}(\Gamma)$ and its dual.

It follows that

$$-(\nabla \cdot F(u) \nabla y, x - y) \leq \varphi(x; u) - \varphi(y; u). \quad (4.25)$$

Hence $A(y; u) \in \partial \varphi(y; u)$.

(2) To show that $A(y; u)$ is maximal monotone, it suffices to show that $I + A(\cdot; u)$ is onto.

Since $\beta(y) \in \partial j(y)$ is maximal monotone, its resolvent operator $(I + \lambda \beta)^{-1}$ is nonexpansive for all $\lambda > 0$.

Consider the elliptic boundary value problem

$$\begin{aligned} -\nabla \cdot (F(u) \nabla y_\lambda) &= f \quad \text{on } Q, \quad y_\lambda = 0 \quad \text{on } \partial Q / \Gamma, \\ \mu(u_1) u_2 y_\lambda + \lambda \frac{\partial}{\partial n} y_\lambda &= \mu(u_1) u_2 (I + \lambda \beta)^{-1} x \quad \text{on } \Gamma. \end{aligned} \quad (4.26)$$

For $(f, x) \in L^2(Q) \times L^2(\Gamma)$, there exists a unique solution y_λ of (4.17) with $y_\lambda \in H^1(Q)$. Let L be the mapping of $L^2(\Gamma)$ into itself given by

$$L\left(\sqrt{\mu(u_1) u_2} x\right) = \sqrt{\mu(u_1) u_2} y_\lambda|_\Gamma. \quad (4.27)$$

(3) We now show that L is a contraction. Let L be as above, then

$$\int_Q F(u) |\nabla (y_\lambda^1 - y_\lambda^2)|^2 - \left\langle \frac{\partial}{\partial n} (y_\lambda^1 - y_\lambda^2), y_\lambda^1 - y_\lambda^2 \right\rangle = 0. \quad (4.28)$$

As shown in [4, pages 949 and 952] we have

$$c \|y_\lambda^1 - y_\lambda^2\|_{H^1(Q)}^2 - \left\langle \frac{\partial}{\partial n} (y_\lambda^1 - y_\lambda^2), y_\lambda^1 - y_\lambda^2 \right\rangle \leq 0. \quad (4.29)$$

Thus,

$$\begin{aligned} c \|y_\lambda^1 - y_\lambda^2\|_{H^1(Q)}^2 + \lambda^{-1} \left\| \sqrt{\mu(u_1) u_2} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)}^2 \\ \leq \lambda^{-1} (\mu(u_1) u_2 \{ (I + \lambda \beta)^{-1} x^1 - (I + \lambda \beta)^{-1} x^2 \}, y_\lambda^1 - y_\lambda^2) \\ \leq \left\| \sqrt{\mu(u_1) u_2} \lambda^{-1} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)} \left\| \sqrt{\mu(u_1) u_2} (x^1 - x^2) \right\|_{L^2(\Gamma)}. \end{aligned} \quad (4.30)$$

We have used the nonexpansive property of $(I + \lambda \beta)^{-1}$ in the above estimate. We know that

$$a \|y_\lambda^1 - y_\lambda^2\|_{L^2(\Gamma)}^2 \leq \|y_\lambda^1 - y_\lambda^2\|_{H^1(Q)}^2, \quad (4.31)$$

where a is a positive constant.

Thus,

$$\begin{aligned} \lambda ac \|y_\lambda^1 - y_\lambda^2\|_{L^2(\Gamma)}^2 + \left\| \sqrt{\mu(u_1)u_2} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)} \\ \leq \left\| \sqrt{\mu(u_1)u_2} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)} \left\| \sqrt{\mu(u_1)u_2} (x^1 - x^2) \right\|_{L^2(\Gamma)}. \end{aligned} \quad (4.32)$$

It follows that

$$\left\| \sqrt{\mu(u_1)u_2} (y_\lambda^1 - y_\lambda^2) \right\|_{L^2(\Gamma)} \leq \gamma \left\| \sqrt{\mu(u_1)u_2} (x^1 - x^2) \right\|_{L^2(\Gamma)} \quad (4.33)$$

with

$$\gamma = \frac{\|\mu(u_1)u_2\|_{L^\infty(G)}}{\lambda ac + \|\mu(u_1)u_2\|_{L^\infty(G)}} < 1. \quad (4.34)$$

Thus, L is a contraction mapping. There exists a unique y_λ such that

$$\begin{aligned} -\nabla \cdot (F(u_1) \nabla y_\lambda) &= f \quad \text{on } Q, \\ y_\lambda &= 0 \quad \text{on } \partial Q/\Gamma, \end{aligned} \quad (4.35)$$

$$\mu(u_1)u_2 y_\lambda + \lambda \frac{\partial y_\lambda}{\partial n} = \mu(u_1)u_2 (I + \lambda\beta)^{-1} y_\lambda \quad \text{on } \Gamma.$$

(4) By a standard argument, we get from (4.35) the following estimate:

$$\|y_\lambda\|_{H^1(Q)}^2 \leq C \|f\|_{L^2(Q)}. \quad (4.36)$$

Let $\lambda \rightarrow 0^+$, and we get by taking subsequences that $y_\lambda \rightarrow y$ in $(H^1(Q))_{\text{weak}} \cap L^2(Q)$. It is clear that $y = 0$ on $\partial Q/\Gamma$. On the other hand,

$$-\frac{\partial y_\lambda}{\partial n} = \mu(u_1)u_2 \lambda^{-1} \{I - (I + \lambda\beta)^{-1}\} y_\lambda = \mu(u_1)u_2 \beta_\lambda(y_\lambda), \quad (4.37)$$

where β_λ is the Yosida approximation of β .

Since

$$\beta_\lambda(y_\lambda) \in \beta((I + \lambda\beta)^{-1} y_\lambda), \quad (I + \lambda\beta)^{-1} y_\lambda \rightarrow y \quad \text{in } L^2(\Gamma), \quad (4.38)$$

it follows from the maximal monotonicity of β that

$$-\frac{\partial}{\partial n} y \in \mu(u_1)u_2 \beta(y). \quad (4.39)$$

The lemma is proved. $\square$

Proof of Theorem 4.1. Consider the optimal control problem

$$\begin{aligned} Y' - \nabla \cdot (F(u) \nabla Y) + g(Y; u) &= 0 && \text{on } Q \times (0, T), \\ Y &= 0 && \text{on } (\partial Q/\Gamma) \times (0, T), \\ -\frac{\partial}{\partial n} Y &\in \mu(u_1)u_2 \beta(Y) && \text{on } \Gamma \times (0, T), \\ Y(\cdot, 0) &= y_0 && \text{on } Q \end{aligned} \quad (4.40)$$

with

$$g(Y; u) = -u_1^{-1} F(u_1) \nabla Y \cdot \nabla u_1 - \tilde{f}(Y, u) \quad (4.41)$$

and cost functionals

$$\begin{aligned} J_1(Y; u_1, u_2) &= \int_0^T \int_G |Y(\xi, 2; t)|^2 d\xi dt, \\ J_2(Y; u_1, u_2) &= \int_0^T \int_\Omega \left| Y\left(\xi, \frac{2\eta}{u_1}, t\right) - h(\xi, \eta, t) \right|^2 d\xi d\eta dt. \end{aligned} \quad (4.42)$$

It is easy to check that g and J_1, J_2 satisfy Assumptions 2.2 and 2.3, respectively. It follows from Lemmas 4.2 and 4.3 and from Theorem 3.1 that there exists an open loop control $\tilde{u}$ of (4.36) and (4.40), that is, we have

$$\begin{aligned} \tilde{Y} &\in L^2(0, T; H^1(Q)) \cap L^\infty(0, T; L^2(Q)), \\ \{\tilde{Y}', A(\tilde{Y}; \tilde{u})\} &\in (L^2(0, T; L^2(Q)))^2, \end{aligned} \quad (4.43)$$

solution of (4.36) with controls $\tilde{u}$. Moreover,

$$\begin{aligned} J_1(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) &\leq J_1(y; \tilde{u}_1, v_2), \\ J_2(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) &\leq J_2(x; u_1, \tilde{u}_2), \end{aligned} \quad (4.44)$$

for all $y \in \mathcal{R}(\tilde{u}_1, v_2)$, for all $v_2 \in U_2$, all $x \in \mathcal{R}(u_1, \tilde{u}_2)$, and all $u_1 \in U_1$.

Now set

$$\hat{y}(\xi, \eta) = \tilde{Y}(\xi, \zeta) = \tilde{Y}\left(\xi, \frac{2\eta}{\tilde{u}_1}\right) \quad (4.45)$$

and we get the stated result. $\square$

4.2. Parabolic variational inequalities. Consider the initial boundary value problem

$$\begin{aligned} y' - \Delta y &= \tilde{f}(y) && \text{on } Q(u_1) \times (0, T), \\ y &= 0 && \text{on } (\partial Q / \Gamma) \times (0, T), \\ y(\cdot, t) &\geq u_2(\xi) && \text{on } \Gamma \times (0, T), \\ y(\cdot, 0) &= y_0 && \text{on } Q \end{aligned} \quad (4.46)$$

with cost functionals

$$\begin{aligned} J_1(y; u_1, u_2) &= \int_0^T \int_G |y(\xi, u_1(\xi); t)|^2 d\xi, \\ J_2(y; u_1, u_2) &= \int_0^T \int_\Omega |y(\xi, \eta; t) - h(\xi, \eta)|^2 d\xi d\eta dt, \end{aligned} \quad (4.47)$$

where h is the partial measurement of the solution y of (4.46) in the subdomain $\Omega \times (0, T)$, U_1 is as before and

$$U_2 = \{v : \|v\|_{H^3(G)} \leq C, 0 \leq v \text{ on } G\}. \quad (4.48)$$

The main result of this subsection is the following theorem.

THEOREM 4.4. *Let y_0 be an element of $H^1(Q)$ with*

$$y_0 = 0 \quad \text{on } \frac{\partial Q}{\Gamma}, \quad y_0 \geq v \geq 0 \quad \text{on } \Gamma, \quad \forall v \in U_2. \quad (4.49)$$

Let $h \in L^2(0, T; L^2(\Omega))$ where Ω is a proper subset of $Q(u_1)$ for all $u_1 \in U_1$ and let $\tilde{f}$ be as in [Assumption 2.2](#). Then there exists

$$\begin{aligned} \{\hat{y}, \hat{y}', \hat{u}\} &\in L^2(0, T; H^1(Q(\hat{u}_1))) \cap L^\infty(0, T; L^2(Q(\hat{u}_1))) \\ &\times L^2(0, T; L^2(Q(\hat{u}_1))) \times U \end{aligned} \quad (4.50)$$

with

$$\begin{aligned} J_1(\hat{y}; \hat{u}_1, \hat{u}_2) &\leq J_1(y; \hat{u}_1, v_2), \\ J_2(\hat{y}; \hat{u}_1, \hat{u}_2) &\leq J_2(x; u_1, \hat{u}_2), \end{aligned} \quad (4.51)$$

for all solutions y of (4.46) with controls $\hat{u}_1, v_2$ all solutions x of (4.42) with controls $u_1, \hat{u}_2$ and all $\{u_1, v_2\} \in U_1 \times U_2$.

As before, we make the change of variables $\zeta = 2\eta/u_1$ and as in [Section 4.1](#), we transform (4.42) into a problem in a fixed domain

$$\begin{aligned} Y' - \nabla \cdot F((u_1) \nabla Y) &= \tilde{f}(Y, u) + u^{-1} F(u_1) \nabla Y \cdot \nabla u_1 \quad \text{on } Q \times (0, T), \\ Y &= 0 \quad \text{on } \partial Q / \Gamma \times (0, T), \\ Y &\geq u_2 \quad \text{a.e. on } \Gamma \times (0, T), \\ Y(\cdot, 0) &= y_0 \quad \text{on } Q. \end{aligned} \quad (4.52)$$

The cost functionals become

$$\begin{aligned} J_1(Y; u_1, u_2) &= \int_0^T \int_G |Y(\xi, 2; t)|^2 d\xi dt, \\ J_2(Y; u_1, u_2) &= \int_0^T \int_\Omega \left| Y\left(\xi, \frac{2\eta}{u_1}; t\right) - h(\xi, \eta; t) \right|^2 d\xi d\eta dt. \end{aligned} \quad (4.53)$$

Set

$$K(u_2) = \{y : y \in L^2(0, T; L^2(Q)), y \geq u_2 \text{ a.e. on } \Gamma \times (0, T)\}. \quad (4.54)$$

Then $K(u_2)$ is a closed convex subset of $L^2(0, T; H)$. Let

$$\varphi(y; u) = \frac{1}{2} \int_0^T \int_Q F(u) |\nabla y|^2 d\xi d\zeta dt + I_{K(u_2)}(y), \quad (4.55)$$

where $I_{K(u_2)}$ is the indicator function of the closed convex set $K(u_2)$ of $L^2(0, T; H)$ and

$$D(\varphi(y; u)) = \left\{ y : y \in L^2(0, T; H^1(Q)), y = 0 \text{ on } (\partial Q/\Gamma) \times (0, T), \right. \\ \left. y \geq u_2 \text{ on } \Gamma \times (0, T) \right\}. \quad (4.56)$$

LEMMA 4.5. *Let φ be as in (4.53). Then φ satisfies [Assumption 2.1](#).*

Proof. As in the proof of [Lemma 4.2](#), we have

$$\varphi(y; u) \geq c \|y\|_{H^1(Q)}^2, \quad \forall y \in D(\varphi(\cdot, u)). \quad (4.57)$$

It is clear that

$$\partial\varphi(y; u) = \nabla(F(u) \cdot \nabla y) + \partial I_{K(u_2)}(y). \quad (4.58)$$

All the other conditions of [Assumption 2.1](#) can be verified without any difficulty. $\square$

LEMMA 4.6. *Suppose all the hypotheses of [Theorem 4.4](#) are satisfied. Then there exists a solution $\tilde{Y}$ of*

$$\tilde{Y}' + \partial\varphi(\tilde{Y}; \tilde{u}) \ni \tilde{f}(\tilde{Y}, \tilde{u}) + \tilde{u}_1^{-1} F(\tilde{u}_1) \nabla \tilde{Y} \cdot \nabla \tilde{u}_1, \quad \tilde{Y}(\cdot, 0) = y_0, \quad (4.59)$$

$$\{\tilde{Y}, \tilde{Y}', \partial\varphi(\tilde{Y}; \tilde{u}), \cdot \tilde{u}\} \in (L^2(0, T; H^1(Q)) \cap L^\infty(0, T; L^2(Q))) \\ \times (L^2(0, T; L^2(Q)))^2 \times U. \quad (4.60)$$

Moreover,

$$J_1(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) \leq J_1(y; \tilde{u}_1, v_2), \\ J_2(\tilde{Y}; \tilde{u}_1, \tilde{u}_2) \leq J_2(x; u_1, \tilde{u}_2), \quad (4.61)$$

for all solutions y, x of (4.55) with controls $\{\tilde{u}_1, v_2\}$, $\{u_1, \tilde{u}_2\}$, respectively, and for all $\{u_1, v_2\}$ in $U_1 \times U_2$.

Proof. The proof is an immediate consequence of [Theorem 3.1](#) and [Lemma 4.5](#). $\square$

Proof of [Theorem 4.4](#). Let $\{\tilde{Y}, \tilde{u}\}$ be as in [Lemma 4.6](#) and set $\hat{y}(\xi, \eta; t) = \tilde{Y}(\xi, 2\eta/\tilde{u}_1)$. Then $\hat{y}, \tilde{u}$ is a solution of (4.52) and (4.53). The theorem is proved. $\square$

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