# THE INTEGRAL WAVELET TRANSFORM IN WEIGHTED SOBOLEV SPACES

NGUYEN MINH CHUONG AND TA NGOC TRI

*Received 27 September 2001*

The integral wavelet transform is defined in weighted Sobolev spaces, in which some properties of the transform as well as its asymptotical behaviour for small dilation parameter are studied.

## 1. Preliminaries and notations

It is well known that the integral wavelet transform is a very powerful tool to study sciences and technology. In [5] wavelet theory has been investigated in very much functional spaces, even in BMO, VMO (for further details of those spaces, please refer to [5] and the references therein) even for pseudodifferential operators, however, the integral wavelet transform in weighted Sobolev spaces has not been studied yet neither in [5] nor in any other work.

The aim of this paper is to study this unsolved problem.

Let $\omega_\mu(x) \in L^\infty(\mathbb{R}^n)$, $\omega_\mu(x) > 0$, for almost all $x \in \mathbb{R}^n$ and for each $x$,

$$\omega_\mu(x+y) \le C_{1,\mu}\omega_\mu(x), \tag{1.1}$$

for almost all $y \in \mathbb{R}^n$, where $\mu$ is a multi-index.

We use the Sobolev space with weighted norm defined as follows:

$$W_\omega^{m,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) \mid \partial^k f \in L^p(\mathbb{R}^n),\ |k| \le m\} \tag{1.2}$$

equipped with the norm

$$\|f\|_{m,p,\omega} = \sum_{|\mu|\le m} \left( \int_{\mathbb{R}^n} \omega_\mu(x) \left|\partial^\mu f(x)\right|^p dx \right)^{1/p} < \infty, \tag{1.3}$$

where $\mu = (\mu_1, \dots, \mu_n)$, $|\mu| = \mu_1 + \cdots + \mu_n$, $\mu_i \ge 0$.


Abstract and Applied Analysis 7:3 (2002) 135–142
2000 Mathematics Subject Classification: 42Cxx, 42C40, 65Txx, 65T60
URL: http://dx.doi.org/10.1155/S1085337502000775

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all differentiable functions $\varphi$ on $\mathbb{R}^n$ such that for all multi-indices $\alpha$ and $\beta$

$$\sup_{x\in\mathbb{R}^n} \left|x^\alpha\left(D^\beta\varphi(x)\right)\right| < \infty. \tag{1.4}$$

The Fourier transform $\mathcal{F} : f \mapsto \hat{f}$ is given by

$$\hat{f}(y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-i(x,y)}\,dx, \tag{1.5}$$

where $(x,y) = x_1y_1+\cdots+x_ny_n$, $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ (see [1, 2, 4]).

As traditionally, it is not difficult to prove that $\mathcal{S}(\mathbb{R}^n)$ is dense in $W_\omega^{m,p}(\mathbb{R}^n)$. Now we recall that a basic wavelet is a nontrivial function $\psi\in L^1(\mathbb{R}^n)$ such that its integral on $\mathbb{R}^n$ is 0 and its Fourier transform $\hat{\psi}(\xi)$ satisfies the condition

$$(2\pi)^n\int_0^\infty \frac{\left|\hat{\psi}(a\xi)\right|^2}{a}\,da, \tag{1.6}$$

denoted by $C_\psi$ which is a constant for every $\xi\neq 0$ and $C_\psi\neq 0$.

With a basic wavelet $\psi$ and a function $f\in\mathcal{S}(\mathbb{R}^n)$, we define the following integral:

$$\left(L_\psi f\right)(b,a) = \frac{1}{\sqrt[2n]{C_\psi}}\frac{1}{\sqrt{|a|^n}}\int_{\mathbb{R}^n}\bar{\psi}\left(\frac{t-b}{a}\right)f(t)\,dt, \tag{1.7}$$

where $b\in\mathbb{R}^n$ and $a\in\mathbb{R}\setminus\{0\}$ (see [3, 5, 6]).

## 2. Some properties

Proposition 2.1. *If $f\in\mathcal{S}(\mathbb{R}^n)$ then*

$$\left\|\left(L_\psi f\right)(\cdot,a)\right\|_{m,p,\omega} \le C\|f\|_{m,p,\omega}, \tag{2.1}$$

*where $a\in\mathbb{R}$, $a\neq 0$ and fixed, $C$ is a constant independent of $f$.*

*Proof.* Obviously we have

$$\left(L_\psi f\right)(\cdot,a) = \frac{1}{\sqrt[2n]{C_\psi}}\left(f * D^{-a}\bar{\psi}\right)(\cdot), \tag{2.2}$$

where $D^a : L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ and $(D^a\psi)(a) = |a|^{-n/2}\psi(x/a)$; $a\neq 0$.

Since $f\in\mathcal{S}(\mathbb{R}^n)$ the differentiation and integration can be interchanged

$$\partial^\mu\left(L_\psi f\right)(\cdot,a) = \frac{1}{\sqrt[2n]{C_\psi}}\left(D^{-a}\bar{\psi} * \partial^\mu f\right)(\cdot). \tag{2.3}$$

It is not difficult to see that

$$\begin{aligned}
&\|(L_\psi f)(\cdot,a)\|_{m,p,\omega} \\
&\quad= \sum_{|\mu|\le m}\left(\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\partial^\mu(L_\psi f)(\cdot,a)\right|^p d(\cdot)\right)^{1/p} \\
&\quad= \sum_{|\mu|\le m}\frac{1}{\sqrt[2n]{C_\psi}}\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|\int_{\mathbb{R}^n}(\partial^\mu f)(x-y)(D^{-a}\bar{\psi})(y)\,dy\right|^p dx\right)^{1/p} \\
&\quad\le \frac{1}{\sqrt[2n]{C_\psi}}\sum_{|\mu|\le m}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|(\partial^\mu f)(x-y)\right|^p\left|(D^{-a}\bar{\psi})(y)\right|^p\omega_\mu(x)\,dx\right)^{1/p}dy \\
&\quad\le \frac{1}{\sqrt[2n]{C_\psi}}\sum_{|\mu|\le m}\int_{\mathbb{R}^n}\left|D^{-a}\bar{\psi}(y)\right|\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x-y)\right|^p dx\right)^{1/p}dy.
\end{aligned} \tag{2.4}$$

However,

$$\begin{aligned}
\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x-y)\right|^p dx &= \int_{\mathbb{R}^n}\omega_\mu(u+y)\left|(\partial^\mu f)(u)\right|^p du \\
&\le C_{1,\mu}\int_{\mathbb{R}^n}\omega_\mu(u)\left|(\partial^\mu f)(u)\right|^p du.
\end{aligned} \tag{2.5}$$

Consequently,

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\partial^\mu(L_\psi f)(\cdot,a)\right|^p d(\cdot)\right)^{1/p} \\
&\quad\le \frac{(C_{1,\mu})^{1/p}}{\sqrt[2n]{C_\psi}}\int_{\mathbb{R}^n}\left|D^{-a}\bar{\psi}(y)\right|dy\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x)\right|^p dx\right)^{1/p} \\
&\quad\le C_\mu\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x)\right|^p dx\right)^{1/p},
\end{aligned} \tag{2.6}$$

where

$$C_\mu = \frac{(C_{1,\mu})^{1/p}}{\sqrt[2n]{C_\psi}}|a|^{n/2}\|\psi\|_1. \tag{2.7}$$

Therefore,

$$\|(L_\psi f)(\cdot,a)\|_{m,p,\omega} \le C\sum_{|\mu|\le m}\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x)\right|^p dx\right)^{1/p}, \tag{2.8}$$

where

$$C = \max_{|\mu|\le m} C_\mu, \tag{2.9}$$

that is,

$$\left\|(L_\psi f)(\cdot,a)\right\|_{m,p,\omega} \le C\|f\|_{m,p,\omega}. \tag{2.10}$$

□

By Proposition 2.1, we extend $(L_\psi f(\cdot,a))$ for fixed $a$ to a continuous mapping from $W_\omega^{m,p}(\mathbb{R}^n)$ to itself. It is called the *integral wavelet transform in weighted Sobolev space.*

Theorem 2.2. *If $\psi$ and $\varphi$ are basic wavelets and $f,g$ belong to $W_\omega^{m,p}(\mathbb{R}^n)$, then the following estimate holds true:*

$$\begin{aligned}&\left\|(L_\psi f)(\cdot,a)-(L_\varphi g)(\cdot,a)\right\|_{m,p,\omega}\\ &\quad\le C^1|a|^{n/2}\left(\left\|\frac{\psi}{\sqrt[2n]{C_\psi}}-\frac{\varphi}{\sqrt[2n]{C_\varphi}}\right\|_1\|f\|_{m,p,\omega}+\frac{\|\varphi\|_1}{\sqrt[2n]{C_\varphi}}\|f-g\|_{m,p,\omega}\right),\end{aligned} \tag{2.11}$$

*where $C^1$ is a constant independent of $f$ and $g$.*

*Proof.* It is sufficient to prove the case $f,g\in\mathcal{S}(\mathbb{R}^n)$.

Obviously

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\partial^\mu\left[L_\psi f-L_\varphi g\right](\cdot,a)\right|^p d(\cdot)\right)^{1/p}\\ &\quad=\left\{\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\partial^\mu f*\left(\frac{D^{-a}\bar\psi}{\sqrt[2n]{C_\psi}}-\frac{D^{-a}\bar\varphi}{\sqrt[2n]{C_\varphi}}\right)(\cdot)\right|^p d(\cdot)\right\}^{1/p}\\ &\quad=\left\{\int_{\mathbb{R}^n}\omega_\mu(x)\left|\int_{\mathbb{R}^n}(\partial^\mu f)(x-y)\left(\frac{D^{-a}\bar\psi}{\sqrt[2n]{C_\psi}}-\frac{D^{-a}\bar\varphi}{\sqrt[2n]{C_\varphi}}\right)(y)\,dy\right|^p dx\right\}^{1/p}\\ &\quad\le\int_{\mathbb{R}^n}\left|\left(\frac{D^{-a}\bar\psi}{\sqrt[2n]{C_\psi}}-\frac{D^{-a}\bar\varphi}{\sqrt[2n]{C_\varphi}}\right)(y)\right|\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x-y)\right|^p dx\right)^{1/p}dy\\ &\quad\le C_{2,\mu}\left(\int_{\mathbb{R}^n}\omega_\mu(x)\left|(\partial^\mu f)(x)\right|^p dx\right)^{1/p},\end{aligned} \tag{2.12}$$

where

$$C_{2,\mu}=|a|^{n/2}\left\|\frac{\psi}{\sqrt[2n]{C_\psi}}-\frac{\varphi}{\sqrt[2n]{C_\varphi}}\right\|_1(C_{1,\mu})^{1/p}. \tag{2.13}$$

So

$$\left\|(L_\psi f)(\cdot,a)-(L_\varphi f)(\cdot,a)\right\|_{m,p,\omega} \le C^1|a|^{n/2}\left\|\frac{\psi}{\sqrt[2n]{C_\psi}}-\frac{\varphi}{\sqrt[2n]{C_\varphi}}\right\|_1 \|f\|_{m,p,\omega}, \tag{2.14}$$

where

$$C^1=\max_{|\mu|\le m}\left(C_{1,\mu}\right)^{1/p}. \tag{2.15}$$

Similarly we obtain

$$\left\|(L_\varphi f)(\cdot,a)-(L_\varphi g)(\cdot,a)\right\|_{m,p,\omega} \le C^1|a|^{n/2}\frac{\|\varphi\|_1}{\sqrt[2n]{C_\varphi}}\|f-g\|_{m,p,\omega}. \tag{2.16}$$

By the triangle inequality we get

$$\begin{aligned}&\left\|(L_\psi f)(\cdot,a)-(L_\varphi g)(\cdot,a)\right\|_{m,p,\omega}\\ &\quad\le C^1|a|^{n/2}\left(\left\|\frac{\psi}{\sqrt[2n]{C_\psi}}-\frac{\varphi}{\sqrt[2n]{C_\varphi}}\right\|_1\|f\|_{m,p,\omega}+\frac{\|\varphi\|_1}{\sqrt[2n]{C_\varphi}}\|f-g\|_{m,p,\omega}\right).\end{aligned} \tag{2.17}$$

□

## 3. Symptotical behaviour for small dilation parameter

From Theorem 2.2 the following proposition follows immediately.

Proposition 3.1. *If $\psi$ is a basic wavelet and $f\in W_\omega^{m,p}(\mathbb{R}^n)$, then*

$$\left\|(L_\psi f)(\cdot,a)\right\|_{m,p,\omega}=O\left(|a|^{n/2}\right). \tag{3.1}$$

Now consider the operator

$$(\Lambda_\psi f)(b,a)=(\psi_a * f)(b)=\frac{1}{a^n}\int_{\mathbb{R}^n} f(t)\psi\left(\frac{b-t}{a}\right)dt, \tag{3.2}$$

where $\psi\in L^1(\mathbb{R}^n)$, $f\in L^p(\mathbb{R}^n)$, $1\le p<\infty$, and

$$\psi_a(x)=\frac{1}{a^n}\psi\left(\frac{x}{a}\right),\quad a\ne 0. \tag{3.3}$$

In the sequel, the following lemma is needed.

Lemma 3.2. *Let $f\in W_\omega^{m,p}(\mathbb{R}^n)$, $1\le p<\infty$, $\psi\in L^1(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, with*

$$\int_{\mathbb{R}^n}\psi(t)\,dt=1,\qquad \frac{1}{p}+\frac{1}{q}=1. \tag{3.4}$$

*Then*

(i) $(\Lambda_\psi f)(\cdot,a) \to f(\cdot)$ *in* $W_\omega^{m,p}(\mathbb{R}^n)$ *as* $a \to 0^+$;

(ii) $[\partial^\mu(\Lambda_\psi f)](\cdot,a) = [\Lambda_\psi(\partial^\mu f)](\cdot,a) = a^{-|\mu|}(\Lambda_{\partial^\mu\psi} f)(\cdot,a)$ *if* $\partial^\mu\psi \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $|\mu| < m$, $a > 0$.

*Proof.* Since $(\Lambda_\psi f)(\cdot,a) = (\psi_a * f)(\cdot)$, and for each multi-index $\alpha$, $\sum_{i=1}^n \alpha_i \le m$ such that $\partial^\mu f \in L^p(\mathbb{R}^n)$, $\psi \in L^q(\mathbb{R}^n)$, we obtain

$$\begin{aligned}
&\left[\partial^\mu(\psi_a * f)\right](\cdot) = (\psi_a * \partial^\mu f)(\cdot),\\
&\left\|(\Lambda_\psi f)(\cdot,a) - f(\cdot)\right\|_{m,p,\omega}\\
&\quad = \sum_{|\mu|\le m}\left(\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\partial^\mu\left[(\Lambda_\psi f)(\cdot,a) - f(\cdot)\right]\right|^p d(\cdot)\right)^{1/p}\\
&\quad = \sum_{|\mu|\le m}\left(\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\Lambda_\psi(\partial^\mu f)(\cdot,a) - (\partial^\mu f)(\cdot)\right|^p d(\cdot)\right)^{1/p}.
\end{aligned} \tag{3.5}$$

Taking into account that $\partial^\mu f \in L^p(\mathbb{R}^n)$, it is easy to see that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\omega_\mu(\cdot)\left|\Lambda_\psi(\partial^\mu f)(\cdot,a) - (\partial^\mu f)(\cdot)\right|^p d(\cdot)\\
&\quad \le \|\omega_\mu\|_\infty \int_{\mathbb{R}^n}\left|\Lambda(\partial^\mu f)(\cdot,a) - (\partial^\mu f)(\cdot)\right|^p d(\cdot) \longrightarrow 0 \quad \text{as } a \longrightarrow 0^+.
\end{aligned} \tag{3.6}$$

Consequently, $(\Lambda_\psi f)(\cdot,a) \to f(\cdot)$ in $W_\omega^{m,p}(\mathbb{R}^n)$ as $a \to 0^+$, that is, (i) is proved.

To check (ii) take $f_r \in \mathcal{S}(\mathbb{R}^n)$, $f_r \to f$ in $W_\omega^{m,p}(\mathbb{R}^n)$. It is obvious that in $W_\omega^{m-|\mu|,p}(\mathbb{R}^n)$

$$\left[\partial^\mu(\Lambda_\psi f_r)\right](\cdot,a) = \left[\Lambda_\psi(\partial^\mu f_r)\right](\cdot,a) = a^{-|\mu|}(\Lambda_{\partial^\mu\psi} f_r)(\cdot,a). \tag{3.7}$$

By the continuity of the operators

$$\begin{aligned}
\Lambda_\psi &: W_\omega^{m,p}(\mathbb{R}^n) \longrightarrow W_\omega^{m,p}(\mathbb{R}^n),\\
\partial^\mu &: W_\omega^{m,p}(\mathbb{R}^n) \longrightarrow W_\omega^{m-|\mu|,p}(\mathbb{R}^n),
\end{aligned} \tag{3.8}$$

for $\psi \in L^q(\mathbb{R}^n)$, $|\mu| < m$, letting $r \to \infty$ we get (ii). □

Theorem 3.3. *Let* $f \in W_\omega^{m,p}(\mathbb{R}^n)$, $1 \le p < \infty$, $\psi \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *such that*

$$\int_{\mathbb{R}^n}\psi(t)\,dt = 1. \tag{3.9}$$

*Moreover, assume that* $\partial^\mu\psi$ *is a basic wavelet for each multi-index* $\mu, |\mu| \in \{0,1,2,\dots,m\}$, *and* $\partial^\mu\psi \in L^q(\mathbb{R}^n)$, *with* $1/p + 1/q = 1$. *Suppose additionally that* $f$ *and* $\psi$ *are real-valued functions and* $a > 0$. *Then*

$$\lim_{a\to 0^+}\left\|\frac{1}{a^{|\mu|+n/2}}(L_{\partial^\mu\psi} f)(\cdot,-a) - \frac{1}{\sqrt[2n]{C_{\partial^\mu\psi}}}\partial^\mu f(\cdot)\right\|_{m-|\mu|,p,\omega} = 0. \tag{3.10}$$

*Proof.* Obviously

$$\begin{aligned}(L_{\partial^\mu\psi}f)(\cdot,-a) &= \frac{a^{n/2}}{\sqrt[2n]{C_{\partial^\mu\psi}}}\left[(\partial^\mu\psi)_a * f\right](\cdot)\\ &= \frac{a^{n/2}}{\sqrt[2n]{C_{\partial^\mu\psi}}}(\psi_a * \partial^\mu f)(\cdot) = \frac{a^{|\mu|+n/2}}{\sqrt[2n]{C_{\partial^\mu\psi}}}\left[\partial^\mu(\psi_a * f)\right](\cdot).\end{aligned} \tag{3.11}$$

Under the assumptions of the theorem, the differentiation and integration can be interchanged, and furthermore by the continuity of the operator

$$\partial^\mu : W_\omega{}^{m,p}(\mathbb{R}^n) \longrightarrow W_\omega^{m-|\mu|,p}(\mathbb{R}^n), \tag{3.12}$$

for $|\mu| < m$, we get

$$\begin{aligned}&\left\|\frac{1}{a^{|\mu|+n/2}}(L_{\partial^\mu\psi}f)(\cdot,-a) - \frac{1}{\sqrt[2n]{C_{\partial^\mu\psi}}}(\partial^\mu f)(\cdot)\right\|_{m-|\mu|,p,\omega}\\ &\quad\le \frac{1}{\sqrt[2n]{C_{\partial^\mu\psi}}}\left\|\left[\partial^\mu(\psi_a * f)\right](\cdot) - (\partial^\mu f)(\cdot)\right\|_{m-|\mu|,p,\omega}\\ &\quad\le \frac{C}{\sqrt[2n]{C_{\partial^\mu\psi}}}\left\|(\psi_a * f)(\cdot) - f(\cdot)\right\|_{m,p,\omega}.\end{aligned} \tag{3.13}$$

Lemma 3.2 implies now that the last term in (3.13) tends to 0 as $a \to 0^+$. □

## Acknowledgment

This paper was supported by the National Fundamental Research Fund Grant for Natural Science, Vietnam.

## References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York, 1975.

[2] N. M. Chuong, N. M. Tri, and L. Q. Trung, *Theory of Partial Differential Equations*, Science and Technology Publishing House, Hanoi, 1995.

[3] N. M. Chuong and T. N. Tri, *The integral wavelet transform in $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$*, Fract. Calc. Appl. Anal. **3** (2000), no. 2, 133–140.

[4] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I. Distribution Theory and Fourier Analysis*, Fundamental Principles of Mathematical Sciences, vol. 256, Springer-Verlag, Berlin, 1983.

[5] Y. Meyer, *Ondelettes et Opérateurs. I* [*Wavelets and Operators. I*], Hermann, Paris, 1990.

[6] A. Rieder, *The wavelet transform on Sobolev spaces and its approximation properties*, Numer. Math. **58** (1991), no. 8, 875–894.

Nguyen Minh Chuong: Institute of Mathematics, P.O. Box 631, Bo Ho, 10 000 Hanoi, Vietnam
*E-mail address*: nmchuong@thevinh.ncst.ac.vn

Ta Ngoc Tri: University of Pedagogy Hanoi II, Me Linh, Vinh Phu, Vietnam