1. Introduction

We consider the quasilinear elliptic boundary value problem,

\[-\Delta_p u = \alpha_+(x)(u^+)^{p-1} - \alpha_-(x)(u^-)^{p-1} + f(x, u), \quad u \in W^{1,p}_0(\Omega),\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n, n \geq 1\), \(\Delta_p u = \text{div}(\nabla |u|^{p-2} \nabla u)\) is the \(p\)-Laplacian, \(1 < p < \infty\), \(u^\pm = \max\{\pm u, 0\}\), \(\alpha_\pm \in L^{\infty}(\Omega)\), and \(f\) is a Carathéodory function on \(\Omega \times \mathbb{R}\) satisfying a growth condition,

\[|f(x, t)| \leq q V(x)^{p-q}|t|^{q-1} + W(x)^{p-1},\]

with \(1 \leq q < p\) and \(V, W \in L^p(\Omega)\). We assume that (1.1) is resonant from one side in the sense that either

\[\lambda_{l} \leq \alpha_\pm(x) \leq \lambda_{l+1} - \varepsilon\]

or

\[\lambda_{l} + \varepsilon \leq \alpha_\pm(x) \leq \lambda_{l+1},\]

for two consecutive variational eigenvalues, \(\lambda_l < \lambda_{l+1}\) of \(-\Delta_p\) on \(W^{1,p}_0(\Omega)\), and some \(\varepsilon > 0\) (see Section 2 for the definition of the variational spectrum).

The special case where \(\alpha_+(x) = \alpha_-(x) = \lambda_l\) and \(q = 1\) was recently studied by Arcoya and Orsina [1], Bouchala and Drábek [3], and Drábek and Robinson [8] (see also Cuesta et al. [6] and Dancer and Perera [7]). In the present paper, we prove a single existence theorem for the general case that includes all their results and much more.
Denote by $N$ the set of nontrivial solutions of the asymptotic problem

$$-\Delta_p u = \alpha_+(x)(u^+)^{p-1} - \alpha_-(x)(u^-)^{p-1}, \quad u \in W^{1,p}_0(\Omega), \quad (1.5)$$

and set

$$F(x, t) := \int_0^t f(x, s) \, ds, \quad H(x, t) := pF(x, t) - tf(x, t). \quad (1.6)$$

Our main result is the following theorem.

**Theorem 1.1.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds and $\int_\Omega H(x, u_j) \to +\infty$, 

(ii) equation (1.4) holds and $\int_\Omega H(x, u_j) \to -\infty$ for every sequence $(u_j)$ in $W^{1,p}_0(\Omega)$ such that $\|u_j\| \to \infty$ and $u_j/\|u_j\|$ converges to some element of $N$. In particular, (1.1) is solvable when (1.3) or (1.4) holds and $N$ is empty.

As is usually the case in resonance problems, the main difficulty here is the lack of compactness of the associated variational functional, which we will overcome by constructing a sequence of approximating nonresonance problems, finding approximate solutions for them using linking and min-max type arguments, and passing to the limit (see Rabinowitz [10] for standard details of the variational theory). But first we give some corollaries and deduce the results of [1, 3, 8]. In what follows, $(u_j)$ is as in the theorem, that is, $\rho_j := \|u_j\| \to \infty$ and $v_j := u_j/\rho_j \to v \in N$.

First, we give simple pointwise assumptions on $H$ that imply the limits in the theorem.

**Corollary 1.2.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds, $H(x, t) \to +\infty$ a.e. as $|t| \to \infty$, and $H(x, t) \geq -C(x)$, 

(ii) equation (1.4) holds, $H(x, t) \to -\infty$ a.e. as $|t| \to \infty$, and $H(x, t) \leq C(x)$ for some $C \in L^1(\Omega)$.

Note that this corollary makes no reference to $N$.

**Proof.** If (i) holds, then $H(x, u_j(x)) = H(x, \rho_j v_j(x)) \to +\infty$ for a.e. $x$ such that $v(x) \neq 0$ and $H(x, u_j(x)) \geq -C(x)$, so

$$\int_\Omega H(x, u_j) \geq \int_{v \neq 0} H(x, u_j) - \int_{v = 0} C(x) \to +\infty \quad (1.7)$$

by Fatou’s lemma. Similarly, $\int_\Omega H(x, u_j) \to -\infty$ if (ii) holds. \qed

Note that the above argument goes through as long as the limits in (i) and (ii) hold on subsets of $\{x \in \Omega : v(x) \neq 0\}$ with positive measure. Now, taking $w = v^\pm$ in

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla w = \int_\Omega \left[ \alpha_+ (x)^{p-1} - \alpha_- (x)(v^-)^{p-1} \right] w \quad (1.8)$$
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gives
\[
\|v^+\|^P = \int_{\Omega_+} \alpha^+(x) (v^+)^P \leq \|\alpha^+\|_{\infty} \|v^+\|^P p_n(\Omega_+)^{P/n} \\
\leq \|\alpha^+\|_{\infty} S^{-1} \|v^+\|^P p_n(\Omega_+)^{P/n},
\]
(1.9)
where \(\Omega_+ = \{ x \in \Omega : v(x) \geq 0 \} \), \(p^* = np/(n - p)\) is the critical Sobolev exponent, \(S\) is the best constant for the embedding \(W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)\), and \(\mu\) is the Lebesgue measure in \(\mathbb{R}^n\). So
\[
\mu(\Omega_+) \geq \left( S\|\alpha^+\|_{\infty}^{-1} \right)^{n/p},
\]
(1.10)
and hence
\[
\mu(\{ x \in \Omega : v(x) = 0 \}) \leq \mu(\Omega) - S^{n/p} (\|\alpha^+\|_{\infty}^{-n/p} + \|\alpha^-\|_{\infty}^{-n/p}).
\]
(1.11)
Thus, we have the following corollary.

**Corollary 1.3.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds, \(H(x,t) \to +\infty\) in \(\Omega'\) as \(|t| \to \infty\), and \(H(x,t) \geq -C(x)\),

(ii) equation (1.4) holds, \(H(x,t) \to -\infty\) in \(\Omega'\) as \(|t| \to \infty\), and \(H(x,t) \leq C(x)\)
for some \(\Omega' \subset \Omega\) with \(\mu(\Omega') > \mu(\Omega) - S^{n/p} (\|\alpha^+\|_{\infty}^{-n/p} + \|\alpha^-\|_{\infty}^{-n/p})\) and \(C \in L^1(\Omega)\).

Similar conditions on \(H\) were recently used by Furtado and Silva [9] in the semilinear case \(p = 2\).

Next, note that
\[
\frac{H^+(x)(v^+(x))^q + H^-(x)(v^-(x))^q}{\rho^q_j} \\
\leq \liminf_{t \to \pm\infty} \frac{H(x,u_j(x))}{\rho^q_j} \leq \limsup_{t \to \pm\infty} \frac{H(x,u_j(x))}{\rho^q_j} \\
\leq H^+(x)(v^+(x))^q + H^-(x)(v^-(x))^q,
\]
(1.12)
where
\[
H^+_\pm(x) = \liminf_{t \to \pm\infty} \frac{H(x,t)}{|t|^q}, \quad \overline{H}^+_\pm(x) = \limsup_{t \to \pm\infty} \frac{H(x,t)}{|t|^q}.
\]
(1.13)
Moreover,
\[
\frac{|H(x,u_j(x))|}{\rho^q_j} \leq (p + q)V(x)^{p-q}|v_j(x)|^q + \frac{(p + 1)W(x)^{p-1}|v_j(x)|^q}{\rho^{q-1}_j}
\]
(1.14)
by (1.2), so it follows that
\[
\int_\Omega \frac{H(x, u_j)}{\rho_j^q} \leq \limsup \int_\Omega \frac{H(x, u_j)}{\rho_j^q} \leq \int_\Omega \frac{H(x, u_j)}{\rho_j^q} \leq \liminf \int_\Omega \frac{H(x, u_j)}{\rho_j^q} \leq \int_\Omega H_+(v^+)q + H_-(v^-)q.
\]
(1.15)

Thus we have the following corollary.

**Corollary 1.4.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds and \(\int_\Omega H_+(v^+)q + H_-(v^-)q > 0\) for all \(v \in \mathbb{N}\),

(ii) equation (1.4) holds and \(\int_\Omega H_+(v^+)q + H_-(v^-)q < 0\) for all \(v \in \mathbb{N}\).

When \(\alpha_+(x) = \alpha_-(x) \equiv \lambda_1\) and \(q = 1\) this reduces to the result of Bouchala and Drábek [3].

Finally, we note that if
\[
\frac{tf(x, t)}{|t|^q} \to f_\pm(x) \quad \text{a.e. as } t \to \pm\infty,
\]
(1.16)
then
\[
\frac{F(x, t)}{|t|^q} = \frac{1}{|t|^q} \int_0^t \left[ \frac{sf'(x, s)}{|s|^q} - f_\pm(x) \right] |s|^{q-2} s ds + \frac{f_\pm(x)}{q} \to \frac{f_\pm(x)}{q}
\]
(1.17)
and hence
\[
\frac{H(x, t)}{|t|^q} \to \left( \frac{p}{q} - 1 \right) f_\pm(x),
\]
(1.18)
so Corollary 1.4 implies the following corollary.

**Corollary 1.5.** Problem (1.1) has a solution in the following cases:

(i) equation (1.3) holds and \(\int_\Omega f_+(v^+)q + f_-(v^-)q > 0\) for all \(v \in \mathbb{N}\),

(ii) equation (1.4) holds and \(\int_\Omega f_+(v^+)q + f_-(v^-)q < 0\) for all \(v \in \mathbb{N}\).

This was proved in Arcoya and Orsina [1] and Drábek and Robinson [8] for the special case \(\alpha_+(x) = \alpha_-(x) \equiv \lambda_1\) and \(q = 1\).

2. **Proof of Theorem 1.1**

First we recall some facts about the variational spectrum of the \(p\)-Laplacian. It is easily seen from the Lagrange multiplier rule that the eigenvalues of \(-\Delta_p\) on \(W^{1,p}_0(\Omega)\) correspond to the critical values of
\[
J(u) = \int_\Omega |\nabla u|^p, \quad u \in S := \{ u \in W^{1,p}_0(\Omega) : \|u\|_p = 1 \}.
\]
(2.1)
Moreover, $J$ satisfies the Palais-Smale condition (cf. Drábek and Robinson [8]). Thus, we can define an unbounded sequence of min-max eigenvalues by

$$\lambda_l := \inf_{A \in \mathcal{F}_l} \max_{u \in A} J(u), \quad l \in \mathbb{N},$$  \hspace{1cm} (2.2)

where

$$\mathcal{F}_l = \{ A \subset S : \exists \text{ a continuous odd surjection } h : S^{l-1} \rightarrow A \}$$  \hspace{1cm} (2.3)

and $S^{l-1}$ is the unit sphere in $\mathbb{R}^l$.

**Lemma 2.1.** $\lambda_l$ is an eigenvalue of $-\Delta_p$ and $\lambda_l \rightarrow \infty$.

**Proof.** If $\lambda_l$ is a regular value of $J$, then there is an $\varepsilon > 0$ and an odd homeomorphism $\eta : S \rightarrow S$ such that $\eta(J^{\lambda_l+\varepsilon}) \subset J^{\lambda_l-\varepsilon}$ by [2, Theorem 2.5] (the standard first deformation lemma is not sufficient because the manifold $S$ is not of class $C^{1,1}$ when $p < 2$). But then taking $A \in \mathcal{F}_l$ with $\max A \leq \lambda_l + \varepsilon$ and setting $\tilde{A} = \eta(A)$, we get a set in $\mathcal{F}_l$ for which $\max \tilde{A} \leq \lambda_l - \varepsilon$, contradicting the definition of $\lambda_l$.

Finally, denoting by $\mu_l \rightarrow \infty$ the usual Ljusternik-Schnirelmann eigenvalues, we have $\lambda_l \geq \mu_l$ since the genus of each $A$ in $\mathcal{F}_l$ is $l$, so $\lambda_l \rightarrow \infty$. \hfill $\square$

It is not known whether this is a complete list of eigenvalues when $p \neq 2$ and $n \geq 2$. However, the variational structure provided by this portion of the spectrum is sufficient to show that the associated functional admits a linking geometry in the nonresonant case. We only consider (i) as the proof for (ii) is similar. Let

$$\alpha^j_\pm(x) = \begin{cases} 
\alpha_\pm(x), & \text{if } \alpha_\pm(x) \geq \lambda_l + \frac{1}{j}, \\
\lambda_l + \frac{1}{j}, & \text{if } \alpha_\pm(x) < \lambda_l + \frac{1}{j},
\end{cases}$$  \hspace{1cm} (2.4)

so that

$$\lambda_l + \frac{1}{j} \leq \alpha^j_\pm(x) \leq \lambda_{l+1} - \varepsilon, \quad |\alpha^j_\pm(x) - \alpha_\pm(x)| \leq \frac{1}{j},$$  \hspace{1cm} (2.5)

and let

$$\Phi_j(u) = \int_{\Omega} |\nabla u|^p - \alpha_+(x)(u_+)^p - \alpha_-(x)(u_-)^p - p F(x,u), \quad u \in W_0^{1,p}(\Omega).$$  \hspace{1cm} (2.6)

First, we show that there is a $u_j \in W_0^{1,p}(\Omega)$ such that

$$\|u_j\| \|\Phi_j'(u_j)\| \rightarrow 0, \quad \inf \Phi_j(u_j) > -\infty.$$  \hspace{1cm} (2.7)

By (2.2), there is an $A \in \mathcal{F}_l$ such that

$$J(u) \leq \lambda_l + \frac{1}{2j}, \quad u \in A.$$  \hspace{1cm} (2.8)
For \( u \in A \) and \( R > 0 \),
\[
\Phi_j(Ru) = R^p \left[ J(u) - \int_{\Omega} \alpha_+^j(x)(u^+)^p + \alpha_-^j(x)(u^-)^p \right] - \int_{\Omega} pF(x, Ru)
\]
\[
\leq - \frac{R^p}{2j} + p \left( \| V \|_p^{p-q} R^q + \| W \|_p^{p-1} R \right)
\]
\[
(2.9)
\]
by (1.2), (2.5), and (2.8), so
\[
\max_{u \in A} \Phi_j(Ru) \to -\infty \quad \text{as} \quad R \to \infty.
\]
(2.10)

Next, let
\[
\mathcal{F} = \left\{ u \in W_0^{1,p}(\Omega) : \int_{\Omega} |\nabla u|^p \geq \lambda_{l+1} \int_{\Omega} |u|^p \right\}.
\]
(2.11)

For \( u \in \mathcal{F} \),
\[
\Phi_j(u) \geq \varepsilon \| u \|_p^p - p \left( \| V \|_p^{p-q} \| u \|_p^q + \| W \|_p^{p-1} \| u \|_p \right),
\]
so
\[
\inf_{u \in \mathcal{F}} \Phi_j(u) \geq C := \min_{r \geq \theta} \left[ \varepsilon r^p - p \left( \| V \|_p^{p-q} r^q + \| W \|_p^{p-1} r \right) \right] > -\infty.
\]
(2.12)

Now use (2.10) to fix \( R > 0 \) so large that
\[
\max_{u \in A} \Phi_j(RA) < C,
\]
(2.14)

where \( RA = \{ Ru : u \in A \} \).

Since \( A \in \mathcal{F}_l \), there is a continuous odd surjection \( h : S^{l-1} \to A \). Let
\[
\Gamma = \left\{ \varphi \in C\left( D^l, W_0^{1,p}(\Omega) \right) : \varphi|_{S^{l-1}} = Rh \right\},
\]
(2.15)

where \( D^l \) is the unit disk in \( \mathbb{R}^l \) with boundary \( S^{l-1} \). We claim that \( RA \) links \( \mathcal{F} \) with respect to \( \Gamma \), that is,
\[
\varphi(D^l) \cap \mathcal{F} \neq \emptyset \quad \forall \varphi \in \Gamma.
\]
(2.16)

To see this, first note that the proof is done if \( 0 \in \varphi(D^l) \). Otherwise, denoting by \( \pi \) the radial projection onto \( S, \tilde{A} := \pi(\varphi(D^l)) \subseteq \mathcal{F}_{l+1} \), and hence
\[
\max_{u \in \pi(\varphi(D^l))} J(u) = \max_{u \in \tilde{A}} J(u) \geq \lambda_{l+1},
\]
(2.17)
so \( \pi(\varphi(D^l)) \cap \mathcal{F} \neq \emptyset \), which implies that \( \varphi(D^l) \cap \mathcal{F} \neq \emptyset \).

Now it follows from a deformation argument of Cerami [5] that there is a \( u_j \) such that
\[
\| u_j \| \| \Phi_j'(u_j) \| \to 0, \quad | \Phi_j(u_j) - c_j | \to 0,
\]
(2.18)
where

\[ c_j := \inf_{\varphi \in \Gamma} \max_{u \in \varphi(D^p)} \Phi_j(u) \geq C, \quad (2.19) \]

from which (2.7) follows.

We complete the proof by showing that a subsequence of \((u_j)\) converges to a solution of (1.1). It is easy to see that this is the case if \((u_j)\) is bounded, so suppose that \(\rho_j := \|u_j\| \to \infty\). Setting \(v_j := u_j/\rho_j\) and passing to a subsequence, we may assume that \(v_j \to v\) weakly in \(W^{1,p}_0(\Omega)\), strongly in \(L^p(\Omega)\), and a.e. in \(\Omega\). Then

\[
\int_{\Omega} |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla (v_j - v) = \frac{(\Phi'_j(u_j), v_j - v)}{p \rho_j^{p-1}} + \int_{\Omega} \left[ \alpha_j^+(x)(v_j^+)^{p-1} - \alpha_j^-(x)(v_j^-)^{p-1} + \frac{f(x, u_j)}{\rho_j^{p-1}} \right] (v_j - v) \to 0,
\]

and we deduce that \(v_j \to v\) strongly in \(W^{1,p}_0(\Omega)\) (cf. Browder [4]). In particular, \(\|v\| = 1\), so \(v \neq 0\). Moreover, for each \(w \in W^{1,p}_0(\Omega)\), passing to the limit in

\[
\frac{(\Phi'_j(u_j), w)}{p \rho_j^{p-1}} = \int_{\Omega} |\nabla v_j|^{p-2} \nabla v_j \cdot \nabla w - \left[ \alpha_j^+(x)(v_j^+)^{p-1} - \alpha_j^-(x)(v_j^-)^{p-1} + \frac{f(x, u_j)}{\rho_j^{p-1}} \right] w
\]

and passing to the limit in

\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \cdot \nabla w - \left[ \alpha_+(x)(v^+)^{p-1} - \alpha_-(x)(v^-)^{p-1} \right] w = 0,
\]

so \(v \in N\). Thus,

\[
\frac{(\Phi'_j(u_j), u_j)}{p} - \Phi_j(u_j) = \int_{\Omega} H(x, u_j) \to +\infty,
\]

contradicting (2.7).

References

One-sided resonance


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