EXISTENCE RESULTS FOR GENERAL INEQUALITY PROBLEMS WITH CONSTRAINTS

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Received 25 February 2002

To Professor Jean Mawhin on occasion of his 60th birthday

This paper is concerned with existence results for inequality problems of type
\[ F^0(u;v) + \Psi'(u;v) \geq 0, \quad \forall v \in X, \]
where \( X \) is a Banach space, \( F: X \rightarrow \mathbb{R} \) is locally Lipschitz, and \( \Psi: X \rightarrow (-\infty, +\infty] \) is proper, convex, and lower semicontinuous. Here \( F^0 \) stands for the generalized directional derivative of \( F \) and \( \Psi' \) denotes the directional derivative of \( \Psi \). The applications we consider focus on the variational-hemivariational inequalities involving the \( p \)-Laplacian operator.

1. Introduction

The paper deals with nonlinear inequality problems of type
\[ F^0(u;v - u) + h(v) - h(u) \geq 0, \quad \forall v \in C, \]  
(1.1)
where \( F^0 \) stands for the generalized directional derivative of a locally Lipschitz functional \( F \) (in the sense of Clarke [5]), \( h \) is a convex, lower semicontinuous (in short, l.s.c.), and proper function, and \( C \) is a nonempty, closed, and convex subset of a Banach space \( X \). It is clear that in problem (1.1) we can put \( h + I_C \) in place of \( h \), where \( I_C \) denotes the indicator function of the set \( C \), to give the formulation with \( v \) arbitrary in \( X \). However, we keep the statement (1.1) for allowing various possible choices separately on the data \( h \) and \( C \).

The type of problem stated in (1.1) fits in the framework of the nonsmooth critical point theory developed by Motreanu and Panagiotopoulos [9], which is constructed for the nonsmooth functionals having the form
\[ \Phi = \Psi + F \]  
(1.2)
with \( \Psi \) convex, l.s.c., and proper, and \( F \) locally Lipschitz. Namely, a solution of
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(1.1) means, in fact, a critical point of the associated nonsmooth functional (1.2) with $\Psi = h + I_C$.

The existence results in the present paper extend different theorems in the smooth and nonsmooth variational analyses (see, for comparison, Ambrosetti and Rabinowitz [2], Chang [4], Dincă et al. [8], Motreanu and Panagiotopoulos [9], Rabinowitz [10], and Szulkin [11]). In this respect, we solve problems of type

$$F^0(u;v) + \Psi'(u;v) \geq 0, \quad \forall v \in X,$$

(1.3)

where $\Psi'$ stands for the directional derivative of a convex, proper, l.s.c. functional $\Psi$. Consequently, we are able to handle the abstract hemivariational inequality problem

$$F^0(u;v - u) + \langle d\varphi(u), v - u \rangle \geq 0, \quad \forall v \in C,$$

(1.4)

where $\varphi$ is a convex, Gâteaux differentiable functional and $d\varphi$ is its differential. In particular, this contains the differential inclusion problem

$$d\varphi(u) \in \partial(-F)(u)$$

(1.5)

which we considered in our previous paper [8].

The rest of the paper is organized as follows. In Section 2, we briefly recall several elements of nonsmooth critical point theory developed by Motreanu and Panagiotopoulos [9]. In Section 3, we study some general inequality problems in relation with the nonsmooth critical point theory. Section 4 presents applications for different discontinuous boundary value problems with $p$-Laplacian.

2. Notions and preliminary results

Let $X$ be a real Banach space and $X^*$ its dual. The generalized directional derivative of a locally Lipschitz function $F : X \to \mathbb{R}$ at $u \in X$ in the direction $v \in X$ is defined by

$$F^0(u;v) = \limsup_{w \to u, t \to 0} \frac{F(w + tv) - F(w)}{t}.$$

(2.1)

The generalized gradient (in the sense of Clarke [5]) of $F$ at $u \in X$ is defined to be the subset of $X^*$ given by

$$\partial F(u) = \{ \eta \in X^* : F^0(u;\eta) \geq \langle \eta, v \rangle, \forall v \in X \},$$

(2.2)

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $X^*$ and $X$.

Let $\Psi : X \to (-\infty, +\infty]$ be a proper (i.e., $D(\Psi) := \{ u \in X : \Psi(u) < +\infty \} \neq \emptyset$), convex, and l.s.c. function and let $F : X \to \mathbb{R}$ be locally Lipschitz.

We define the functional $\Phi : X \to (-\infty, +\infty]$ by $\Phi = \Psi + F$. 
Definition 2.1 Motreanu and Panagiotopoulos [9]. An element \( u \in X \) is called a critical point of the functional \( \Phi \) if this inequality holds

\[
F^0(u; v - u) + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in X. \tag{2.3}
\]

Definition 2.2 Motreanu and Panagiotopoulos [9]. The functional \( \Phi \) is said to satisfy the Palais-Smale condition if every sequence \( \{u_n\} \subset X \) for which \( \Phi(u_n) \) is bounded and

\[
F^0(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in X, \tag{2.4}
\]

for a sequence \( \{\varepsilon_n\} \subset \mathbb{R}^+ \) with \( \varepsilon_n \to 0 \), contains a strongly convergent subsequence in \( X \).

For the proof of the next theorem, we refer the reader to [8, Proposition 2.1] and [9, Corollary 3.2] (also see [8, Theorem 2.2]).

Theorem 2.3. (i) If \( u \in X \) is a local minimum for \( \Phi \), then \( u \) is a critical point of \( \Phi \).

(ii) If \( \Phi \) satisfies the Palais-Smale condition and there exist a number \( \rho > 0 \) and a point \( e \in X \) with \( \|e\| > \rho \) such that

\[
\inf_{\|v\| = \rho} \Phi(v) > \Phi(0) \geq \Phi(e), \tag{2.5}
\]

then \( \Phi \) has a nontrivial critical point.

Remark 2.4. Definitions 2.1 and 2.2 recover and unify the nonsmooth critical point theories (and a fortiori the smooth critical point theory, see, e.g., Ambrosetti and Rabinowitz [2] and Rabinowitz [10]) due to Chang [4] and Szulkin [11]. Precisely, if \( \Psi = 0 \), Definitions 2.1 and 2.2 reduce to the corresponding definitions of Chang [4], while if \( F \in C^1(X, \mathbb{R}) \), then Definitions 2.1 and 2.2 coincide with those in Szulkin [11].

3. Critical points as solutions of inequality problems

Throughout this section, \( (X, \| \cdot \|_X) \) is a real reflexive Banach space, compactly embedded in the real Banach space \( (Z, \| \cdot \|_Z) \). Let \( \mathcal{F} : Z \to \mathbb{R} \) be a locally Lipschitz function and let \( \Psi : X \to (-\infty, +\infty] \) be convex, l.s.c., and proper.

We consider the inequality problem:

Find \( u \in D(\Psi) \) such that \( (\mathcal{F}\|_X)_0^0(u; v) + \Psi'(u; v) \geq 0, \quad \forall v \in X, \tag{3.1} \)

where \( (\mathcal{F}\|_X)_0^0 \) denotes the generalized directional derivative of the restriction \( \mathcal{F}\|_X \) while \( \Psi'(u; v) \) is the directional derivative of the convex function \( \Psi \) at \( u \) in the direction \( v \) (which is known to exist). Note that if the Gâteaux differential \( d\Psi(u) \) of \( \Psi \) at \( u \in D(\Psi) \) exists, then \( \langle d\Psi(u), v \rangle = \Psi'(u; v), \) for all \( v \in X. \)
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Proposition 3.1. Each solution of problem (3.1) solves the problem:

Find \( u \in D(\Psi) \) such that \( \mathcal{F}^0(u;v) + \Psi'(u;v) \geq 0, \forall v \in X \). (3.2)

If, in addition to our assumptions, \( X \) is densely embedded in \( Z \), then problems (3.1) and (3.2) are equivalent.

Proof. For \( u, v \in X \), the inequality below holds

\[
(\mathcal{F}|_X)^0(u;v) \leq \mathcal{F}^0(u;v).
\] (3.3)

This becomes an equality if \( X \) is continuously and densely embedded in \( Z \) (see [5, pages 46–47] and [9, pages 10–12]). □

Our approach for studying problem (3.1) is variational and relies on the use of the functional

\[ \Phi = \Psi + (\mathcal{F}|_X) : X \to (-\infty, +\infty) \] (3.4)

which is clearly of the form required in the previous section with \( F = \mathcal{F}|_X \).

The next result points out the relationship between the critical points of the functional \( \Phi \) in (3.4) and the solutions of problem (3.1).

Proposition 3.2. (i) If \( u \in X \) is a critical point of the functional \( \Phi \) in (3.4), that is,

\[
(\mathcal{F}|_X)^0(u;v - u) + \Psi(v) - \Psi(u) \geq 0, \forall v \in X,
\] (3.5)

then \( u \) is a solution of problem (3.1).

(ii) Conversely, assume that \( u \in X \) is a solution of problem (3.1). If either \( \Psi \) is Gâteaux differentiable at \( u \) or \( \Psi \) is continuous at \( u \), then \( u \) is a critical point of \( \Phi \), that is, relation (3.5) holds.

Proof. (i) As \( \Psi \) is proper, (3.5) obviously implies that \( u \in D(\Psi) \). For an arbitrary \( w \in X \), we set \( v = u + tw, t > 0 \), in (3.5). Dividing by \( t \) and then letting \( t \to 0^+ \), we arrive at the conclusion that \( u \) solves problem (3.1).

(ii) Let \( u \in D(\Psi) \) be a solution of problem (3.1). If \( \Psi \) is Gâteaux differentiable at \( u \), then

\[
\Psi(v) - \Psi(u) \geq \langle d\Psi(u), v - u \rangle = \Psi'(u;v - u), \forall v \in X
\] (3.6)

which leads to (3.5).

If \( \Psi \) is continuous at \( u \), then a standard result of convex analysis (see Barbu and Precupanu [3, page 106]) allows to write

\[
\Psi'(u;v) = \max \{ \langle x^*, v \rangle : x^* \in \partial\Psi(u) \}, \forall v \in X.
\] (3.7)

Using the definition of the subdifferential \( \partial\Psi(u) \), we obtain (3.5). □
Remark 3.3. In view of Proposition 3.2(i), each result stating the existence of critical points for $\Phi$ in (3.4) asserts a fortiori existence of solutions to problem (3.1).

**Theorem 3.4.** If $\Phi$ is coercive on $X$, that is,

$$\Phi(u) \rightarrow +\infty \quad \text{as} \quad \|u\|_X \rightarrow +\infty,$$

then $\Phi$ has a critical point.

**Proof.** The compact embedding of $X$ into $Z$ implies that $\mathcal{F}|_X$ is weakly continuous. We infer that $\Phi$ is sequentially weakly l.s.c. on $X$. Then, by standard theory, $\Phi$ is bounded from below and attains its infimum at some $u \in X$. From Theorem 2.3(i), $u$ is a critical point of $\Phi$. □

Towards the application of Theorem 2.3(ii) to the functional $\Phi$, we have to know when $\Phi$ satisfies the Palais-Smale condition. The following lemma provides a useful sufficient condition that improves the usual results based on the celebrated hypothesis $(p_5)$ in [2] or $(p_4)$ in [10].

**Lemma 3.5.** Assume, in addition, that $\Psi$ and $\mathcal{F}$, entering the expression of $\Phi$ in (3.4), satisfy the following hypotheses:

(H1) $D(\Psi)$ is a cone and there exist constants $a_0, a_1, b_0, b_1 \geq 0$, $\alpha > 0$, and $\sigma \geq 1$ such that

$$\Psi(u) - \alpha \Psi'(u; u) \geq a_0 \|u\|_X^\sigma - a_1, \quad \forall u \in D(\Psi),$$

$$\mathcal{F}(u) - \alpha (\mathcal{F}|_X)^0(u; u) \geq -b_0 \|u\|_X^\sigma - b_1, \quad \forall u \in D(\Psi),$$

$$a_0 > b_0 + \alpha \quad \text{if} \quad \sigma = 1, \quad a_0 > b_0 \quad \text{if} \quad \sigma > 1; \quad (3.11)$$

(H2) the following condition of $(S_\epsilon)$ type is satisfied: if $\{u_n\}$ is a sequence in $D(\Psi)$ provided $u_n \rightharpoonup u$ weakly in $X$ and $\limsup_{n \to \infty} (-\Psi'(u_n; u - u_n)) \leq 0$, then $u_n \to u$ strongly in $X$.

Then the functional $\Phi$ satisfies the Palais-Smale condition in the sense of Definition 2.2.

**Proof.** Let $\{u_n\}$ be a sequence in $X$ for which there is a constant $M > 0$ with

$$|\Phi(u_n)| \leq M, \quad \forall n \geq 1,$$

and inequality (2.4) holds for $F = \mathcal{F}|_X$ and a sequence $\epsilon_n \to 0^+$. By (3.12), each $u_n$ is in $D(\Psi)$. For $t > 0$, set $v = (1 + t)u_n$ in (2.4) with $F = \mathcal{F}|_X$. Dividing by $t$ and then letting $t \downarrow 0$, one obtains that

$$\Psi'(u_n; u_n) + (\mathcal{F}|_X)^0(u_n; u_n) \geq -\epsilon_n \|u_n\|_X, \quad \forall n \geq 1. \quad (3.13)$$
Inequalities (3.12) and (3.13) ensure that for \( n \) sufficiently large, one has
\[
M + \alpha \| u_n \|_X \geq \Psi(u_n) + \mathcal{F}(u_n) + \alpha \epsilon_n \| u_n \|_X \\
\geq \Psi(u_n) - \alpha \Psi'(u_n; u_n) + \left[ \mathcal{F}(u_n) - \alpha \mathcal{F}_X^0(u_n; u_n) \right].
\] (3.14)

Using (3.9) and (3.10), we find that
\[
M + \alpha \| u_n \|_X \geq (a_0 - b_0) \| u_n \|_X^\sigma - a_1 - b_1.
\] (3.15)

Then (3.11) and (3.15) show that \( \{ u_n \} \) is bounded in \( X \). By the compactness of the embedding of \( X \) into \( Z \), the sequence \( \{ u_n \} \) contains a subsequence, again denoted by \( \{ u_n \} \) such that
\[
u_n \rightharpoonup u \quad \text{weakly in } X,
\]
\[
u_n \rightarrow u \quad \text{strongly in } Z,
\] (3.16) (3.17)

for some \( u \in X \). Now put \( v = u_n + t(u - u_n), t > 0, \) in (2.4) with \( F = \mathcal{F}_X \). Similar to (3.13), we derive that
\[
\Psi'(u_n; u - u_n) + \mathcal{F}_X^0(u_n; u - u_n) \geq -\epsilon_n \| u - u_n \|_X, \quad \forall n \geq 1.
\] (3.18)

This implies
\[
\Psi'(u_n; u - u_n) + \mathcal{F}_X^0(u_n; u - u_n) \geq -\epsilon_n \| u - u_n \|_X, \quad \forall n \geq 1.
\] (3.19)

As \( \{ u_n \} \) is bounded in \( X \), we infer from (3.17) and the upper semicontinuity of \( \mathcal{F}_X^0 \) that
\[
\liminf_{n \to \infty} \Psi'(u_n; u - u_n) \geq 0.
\] (3.20)

Taking into account (3.16) and (3.20), assumption (H2) completes the proof. \( \square \)

**Remark 3.6.** If \( \Psi'(u; \cdot) \) is homogeneous, for all \( u \in D(\Psi) \), then (H2) becomes the usual form of the \((S)\) condition: if \( \{ u_n \} \) is a sequence in \( D(\Psi) \) provided \( u_n \rightharpoonup u \) weakly in \( X \) and \( \limsup_{n \to \infty} \Psi'(u_n; u_n - u) \leq 0, \) then \( u_n \rightarrow u \) strongly in \( X \).

We can now state the following result.

**Theorem 3.7.** Let \( \Phi \) be defined in (3.4) and assume Lemma 3.5(H1) and (H2) together with the following hypotheses.
(H3) There exists an element $u \in D(\Psi)$ such that
\[
a_1 + b_1 \leq (a_0 - b_0) \|\overline{u}\|_X, \tag{3.21}
\]
and
\[
\Phi(\overline{u}) < 0. \tag{3.22}
\]

(H4) There exists a constant $\rho > 0$ such that
\[
\inf_{\|v\|_X = \rho} \Phi(v) > \Phi(0). \tag{3.23}
\]

Then $\Phi$ has a nontrivial critical point $u \in X$. In particular, problem (3.1) has a nontrivial solution.

Proof. We apply Theorem 2.3(ii) to the functional $\Phi$ in (3.4). Lemma 3.5 guarantees that $\Phi$ satisfies the Palais-Smale condition. It remains to check that $\Phi$ verifies condition (2.5) with $\|e\|_X > \rho$. To this end, we prove that one can choose $e = t\overline{u}$ (with $\overline{u}$ entering (H3)) if $t > 0$ is sufficiently large.

First, note that $u \neq 0$. Indeed, from (3.9), (3.10), and (3.21), we have
\[
\Phi(u) - \alpha \left[ \Psi'(u; u) + (\overline{\mathcal{F}}|_X)^0(u; u) \right] \geq 0, \tag{3.24}
\]
which leads to a contradiction with (3.22) if $\overline{u} = 0$.

We observe that, due to the fact that $\overline{u} \in D(\Psi)$ and since $D(\Psi)$ is a cone, the convex function $s \mapsto \Psi(s\overline{u})$ is locally Lipschitz on $(0, +\infty)$. A straightforward computation shows that
\[
\partial_s(s^{-1/\alpha}\Phi(s\overline{u})) = \partial_s(s^{-1/\alpha}\Psi(s\overline{u}) + s^{-1/\alpha}\overline{\mathcal{F}}|_X(s\overline{u}))
\]
\[
\subseteq - \frac{1}{\alpha} s^{-1/\alpha - 1}\Psi(s\overline{u}) + s^{-1/\alpha} \partial_s(\Psi(s\overline{u}))
\]
\[
+ \left( - \frac{1}{\alpha} s^{-1/\alpha - 1}\overline{\mathcal{F}}(s\overline{u}) + s^{-1/\alpha} \left( \partial(\overline{\mathcal{F}}|_X)(s\overline{u}) , \overline{u} \right) \right), \quad \forall s > 0, \tag{3.25}
\]
where the notation $\partial_s$ stands for the generalized gradient with respect to $s$. For an arbitrary $t > 1$, Lebourg’s mean value theorem yields some $\tau = \tau(t) \in (1, t)$ such that
\[
t^{-1/\alpha}\Phi(t\overline{u}) - \Phi(\overline{u}) = \xi(t - 1), \tag{3.26}
\]
where $\xi \in \partial_s(s^{-1/\alpha}\Phi(s\overline{u}))|_{s = \tau}$. This implies
\[
t^{-1/\alpha}\Phi(t\overline{u}) - \Phi(\overline{u}) = \frac{1}{\alpha} (t - 1)^{-1/\alpha - 1} \left[ (\alpha \tau \partial_s(\Psi(s\overline{u}))|_{s = \tau} - \Psi(\tau\overline{u}))
\right.
\]
\[
+ \left( - \overline{\mathcal{F}}(\tau\overline{u}) + \alpha \left( \partial(\overline{\mathcal{F}}|_X)(\tau\overline{u}), \tau\overline{u} \right) \right)]. \tag{3.27}
\]
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Then, taking into account the convexity of $s \mapsto \Psi(su)$, the regularity property of a convex function (see Clarke [5, pages 39–40]) and relations (3.9) and (3.10), we get that

$$\Phi(tu) \leq t^{1/\alpha} \Phi(u) + \frac{1}{\alpha} t^{1/\alpha} (t - 1) \tau^{-1/\alpha - 1} \left[ (\alpha \Psi'(\tau u; \tau u) - \Psi(\tau u)) + ( - \mathcal{F}(\tau u) + \alpha(\mathcal{F}|_X)^0(\tau u; \tau u) \right]$$

$$\leq t^{1/\alpha} \Phi(u) + \frac{1}{\alpha} t^{1/\alpha} (t - 1) \tau^{-1/\alpha - 1} \left[ - (a_0 - b_0) \tau^\alpha \|u\|_X^\alpha + a_1 + b_1 \right], \quad \forall t > 1.$$ (3.28)

By (3.21) and because $\tau > 1$, we derive that

$$\Phi(tu) \leq t^{1/\alpha} \Phi(u), \quad \forall t > 1.$$ (3.29)

Then (3.29) and assumption (3.22) imply

$$\lim_{t \to +\infty} \Phi(tu) = -\infty.$$ (3.30)

Now, by means of (3.30), we can choose $\bar{t} > 0$ sufficiently large to satisfy

$$\bar{t}\|u\|_X > \rho, \quad \Phi(\bar{t}u) \leq \Phi(0),$$ (3.31)

for $\rho > 0$ entering (H4). If we compare (3.23) and (3.31), it is seen that the requirement in (2.5) is achieved for $e = \bar{t}u$. Theorem 2.3(ii) assures that $\Phi$ in (3.4) has a nontrivial critical point $u \in X$. Furthermore, Remark 3.3 shows that $u$ is a (nontrivial) solution of problem (3.1). The proof of Theorem 3.7 is thus complete. □

In the final part of this section, we are concerned with the case when

$$\Psi = \Psi_C := \varphi + I_C,$$ (3.32)

where $C$ is a nonempty, closed, and convex subset of $X$, $I_C$ denotes the indicator function of $C$, and $\varphi : X \to \mathbb{R}$ is a convex, Gâteaux differentiable functional. Note that $\Psi_C$ is convex, l.s.c., and proper and $D(\Psi_C) = C$. Therefore, the functional

$$\Phi = \Psi_C + \mathcal{F}|_X,$$ (3.33)

with $\mathcal{F}$ as at the beginning of this section, has the form required in (3.4).

Consider the following problem of variational-hemivariational inequality type:

Find $u \in C$ such that $(\mathcal{F}|_X)^0(u; \nu - u) + \langle d\varphi(u), \nu - u \rangle \geq 0, \quad \forall \nu \in C.$ (3.34)
Remark 3.8. (i) Taking into account that, for $u \in C$,

$$\Psi'_C(u; v) = \begin{cases} 
\langle d\varphi(u), v \rangle & \text{if } u + tv \in C \text{ for some } t \in (0, 1], \\
+\infty & \text{otherwise},
\end{cases}$$

(3.35)
a straightforward computation shows that problem (3.34) is equivalent to the following problem of type (3.1):

Find $u \in D(\Psi_C) = C$ such that

$$
\langle H5106(\bar{u}), v \rangle + \Psi'_C(u; v) \geq 0, \quad \forall v \in X. 
$$

(3.36)

(ii) If $C$ is a nonempty, closed, and convex cone, then each solution of problem (3.34) solves also the problem:

Find $u \in C$ such that

$$
\langle H5106(\bar{u}), v \rangle + \langle d\varphi(u), v \rangle \geq 0, \quad \forall v \in C.
$$

(3.37)

Proposition 3.9. If $u \in X$ is a critical point of $\Phi$ in (3.33) and (3.32), then $u$ is a solution of problem (3.34).

Proof. Viewing Remark 3.8(i), the conclusion follows from Proposition 3.2(i). □

Theorem 3.10. If the functional $\Phi$ in (3.33) and (3.32) is coercive on $X$, then problem (3.34) has a solution.

Proof. It is a direct consequence of Theorem 3.4 and Proposition 3.9. □

Theorem 3.11. For the defining $\Phi$ data entering (3.33) and (3.32), we assume the following.

(H1’) The set $C$ is a nonempty, closed, and convex cone in $X$ and there exist constants $a_0, a_1, b_0, b_1 \geq 0$, $\alpha > 0$, and $\sigma \geq 1$ such that one has (3.11),

$$
\varphi(u) - \alpha(d\varphi(u), u) \geq a_0\|u\|^\sigma_X - a_1, \quad \forall u \in C,
$$

(3.38)

$$
\mathcal{F}(u) - \alpha(\mathcal{F}|_X)^0(u; u) \geq -b_0\|u\|^\sigma_X - b_1, \quad \forall u \in C.
$$

(3.39)

(H2’) The following condition of $(S_\ast)$ type is satisfied: if $\{u_n\}$ is a sequence in $C$ provided $u_n \rightharpoonup u$ weakly in $X$ and $\limsup_{n \to \infty} (d\varphi(u_n), u_n - u) \leq 0$, then $u_n \rightharpoonup u$ strongly in $X$.

(H3’) There exists an element $\bar{u} \in C$ such that (3.21) holds with $a_0, a_1, b_0$, and $b_1$ from (H1’) together with

$$
\mathcal{F}(\bar{u}) + \varphi(\bar{u}) < 0.
$$

(3.40)

(H4’) There exists a constant $\rho > 0$ such that

$$
\inf_{\|v\|_X = \rho} (\mathcal{F}(v) + \varphi(v)) > \mathcal{F}(0) + \varphi(0).
$$

(3.41)
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Then $\Phi$ in (3.33) and (3.32) has a nontrivial critical point $u \in C$. In particular, problem (3.34) has a nontrivial solution.

Proof. Note that assumptions (H1'), (H2'), (H3'), and (H4') are just (H1), (H2), (H3), and (H4), respectively, in the case where $D(\Psi) = C$ is a closed convex cone and $\Psi$ is given by (3.32). Thus it suffices to apply Theorem 3.7 and Proposition 3.9 to the functional $\Phi$ in (3.33) and (3.32). □

Remark 3.12. It is worth pointing out that if we take $C = X$, then problem (3.34) becomes

$$\text{Find } u \in X, \text{ such that } d\phi(u) \in \partial(-\mathcal{F}|_X)(u).$$

Thus, [8, Theorems 3.2 and 3.4] are immediate consequences of Theorems 3.10 and 3.11, respectively.

4. Applications to nonsmooth boundary value problems

In order to illustrate how the abstract results of Section 3 can be applied, we consider a concrete problem of type (3.34). To this end, let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with Lipschitz-continuous boundary $\Gamma = \partial\Omega$ and let $\omega \subset \overline{\Omega}$ be a measurable set. Given $p \in (1, \infty)$, the Sobolev space $W^{1,p}(\Omega)$ is endowed with its usual norm (see [1, page 44]).

We denote

$$W_0 = \{ v \in W^{1,p}(\Omega) : v|_\Gamma = 0 \},$$

$$W_1 = \{ v \in W^{1,p}(\Omega) : \int_\Omega v = 0 \},$$

$$W_2 = \{ v \in W_1 : v|_\Gamma = \text{constant} \}.$$ (4.1)

In the sequel, $W$ will stand for any of the above (closed) subspaces $W_0$, $W_1$, and $W_2$ of $W^{1,p}(\Omega)$. By the Poincaré-Wirtinger inequality, the functional

$$W \ni v \mapsto \|v\|_{1,p} := \left( \int_\Omega |\nabla v|^p \right)^{1/p}$$ (4.2)

is a norm on $W$, equivalent to the induced norm from $W^{1,p}(\Omega)$. The dual space $W^*$ is considered endowed with the dual norm of $\| \cdot \|_{1,p}$.

Now, we define the $p$-Laplacian operator $-\Delta_p : W \to W^*$ by

$$\langle -\Delta_p u, v \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v, \quad \forall u, v \in W.$$ (4.3)

Arguments similar to those in [7] show that the convex functional $\phi : W \to \mathbb{R}$ defined by

$$\phi(u) = \frac{1}{p} \|u\|^p_{1,p}, \quad \forall u \in W,$$ (4.4)
is continuously differentiable on $W$ and its differential is $-\Delta_p$, that is,

$$\langle d\varphi(u), v \rangle = \langle -\Delta_p u, v \rangle, \quad \forall u, v \in W. \quad (4.5)$$

Moreover, as $d\varphi$ is the duality mapping on $W$, corresponding to the gauge function $t \mapsto t^{p-1}$ and because $W$ is uniformly convex, $d\varphi$ satisfies condition $(S_+)$ (see Remark 3.6).

If $p^*$ stands for the Sobolev critical exponent, that is,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases} \quad (4.6)$$

then, for any fixed $q \in (1, p^*)$, by the Rellich-Kondrachov theorem, the embedding $W \hookrightarrow L^q(\Omega)$ is compact (the space $L^q(\Omega)$ is understood with its usual norm $\| \cdot \|_{0,q}$).

The results in Section 3 will be applied by taking $X = W$, $Z = L^q(\Omega)$, and $\varphi$ defined in (4.4).

Further, to complete the setting, let a function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be measurable and satisfy the growth condition

$$|g(x,s)| \leq c_1|s|^{q-1} + c_2 \quad \text{for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \quad (4.7)$$

where $c_1, c_2 \geq 0$ are constants. For a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, we put

$$g(x, s) = \lim_{\delta \to 0^+} \inf_{|t-s| < \delta} g(x, t),$$
$$\overline{g}(x, s) = \lim_{\delta \to 0^+} \sup_{|t-s| < \delta} g(x, t). \quad (4.8)$$

The following condition will be invoked below:

$$g \text{ and } \overline{g} \text{ are } N\text{-measurable} \quad (4.9)$$

(recall that a function $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is called $N\text{-measurable}$ if $h(\cdot, u(\cdot)) : \Omega \to \mathbb{R}$ is measurable whenever $u : \Omega \to \mathbb{R}$ is measurable).

By (4.7), the primitive $G : \Omega \times \mathbb{R} \to \mathbb{R}$ of function $g$:

$$G(x, s) = \int_0^s g(x, t)dt \quad \text{for a.e. } x \in \Omega, \forall s \in \mathbb{R}, \quad (4.10)$$

satisfies

$$|G(x,s)| \leq \frac{c_1}{q}|s|^q + c_2|s| \quad \text{for a.e. } x \in \Omega, \forall s \in \mathbb{R}. \quad (4.11)$$
existence results for inequality problems

Taking into account (4.11), we define the functional $\mathcal{G} : L^q(\Omega) \to \mathbb{R}$ by putting

$$\mathcal{G}(u) = -\int_{\Omega} G(x,u), \quad \forall u \in L^q(\Omega).$$

It is known (see, e.g., Chang [4]) that $\mathcal{G}$ is Lipschitz continuous on the bounded subsets of $L^q(\Omega)$. At this stage, we introduce the closed convex cone $K$ in $W$:

$$K = \{ u \in W : u(x) \geq 0 \text{ for a.e. } x \in \omega \}$$

and we formulate the problem:

Find $u \in K$ such that

$$(\mathcal{G}|_W)^0(u;v-u) + \langle -\Delta_p u, v-u \rangle \geq 0, \quad \forall v \in K.$$  (4.14)

Thus, the functional framework in Section 3 is now accomplished by taking $\mathcal{F} = \mathcal{G}$ and $C = K$. Clearly, problem (4.14) is of the same type as (3.34). Before passing on to obtaining existence results for problem (4.14), it should be noticed that the nonsmooth functional $\Phi = \Phi_K : W \to (-\infty, +\infty]$, defined by

$$\Phi_K = \mathcal{G}|_W + \varphi + I_K$$

with $\varphi$ in (4.4), $I_K$ the indicator function of the cone $K$ in (4.13), has the form required in (3.33) and (3.32).

We also need to invoke the following constant, depending on the cone $K$ in the Banach space $W$:

$$\lambda_1 = \lambda_{1,K} := \inf \left\{ \frac{\|v\|_{1,p}}{\|v\|_{0,p}} : v \in K \setminus \{0\} \right\}.$$

(4.16)

Note that

$$\|v\|_{0,p} \leq \lambda_1^{-1/p}\|v\|_{1,p}, \quad \forall v \in K.$$  (4.17)

**Theorem 4.1.** Assume (4.7) together with

(i) $\limsup_{s \to -\infty} pG(x,s)/|s|^p < \lambda_1$ uniformly for a.e. $x \in \Omega \setminus \omega$;

(ii) $\limsup_{s \to +\infty} pG(x,s)/s^p < \lambda_1$ uniformly for a.e. $x \in \Omega$.

Then problem (4.14) has a solution.

**Proof.** By Theorem 3.10, it suffices to show that the functional $\Phi_K$ in (4.15) is coercive on $W$.

From (i) and (ii), there are numbers $\varepsilon \in (0,\lambda_1)$ and $s_0 > 0$ such that

$$G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p} |s|^p \quad \text{for a.e. } x \in \Omega \setminus \omega, \quad \forall s < -s_0,$$

$$G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p} s^p \quad \text{for a.e. } x \in \Omega, \quad \forall s > s_0.$$  (4.18)
Using (4.11), we can find a positive constant $k = k(s_0)$ with
\[ |G(x, s)| \leq k \quad \text{for a.e. } x \in \Omega, \quad \forall s \in [-s_0, s_0]. \tag{4.20} \]

For $u \in K$, we put
\[ \Omega_- := \{ x \in \Omega : u < 0 \}, \quad \Omega_+ := \Omega \setminus \Omega_- . \tag{4.21} \]

Notice that by (4.13) we have $\Omega_- \subset \Omega \setminus \omega$. Then by (4.18) and (4.20), it follows that
\[
\int_{\Omega_-} G(x, u) = \int_{u < s_0} G(x, u) + \int_{-s_0 \leq u < 0} G(x, u) \leq \frac{\lambda_1 - \varepsilon}{p} \int_{\Omega_-} |u|^p + k|\Omega|. \tag{4.22}
\]

On the other hand, by (4.19) and (4.20), one sees that
\[
\int_{\Omega_+} G(x, u) = \int_{u > s_0} G(x, u) + \int_{0 \leq u \leq s_0} G(x, u) \leq \frac{\lambda_1 - \varepsilon}{p} \int_{\Omega_+} |u|^p + k|\Omega|. \tag{4.23}
\]

Combining (4.22) and (4.23), the following estimate holds:
\[
\int_{\Omega} G(x, u) \leq 2k|\Omega| + \frac{\lambda_1 - \varepsilon}{p} \|u\|_{0,p}^p, \quad \forall u \in K. \tag{4.24}
\]

Then, from (4.15), it follows that
\[
\Phi_K(u) = \frac{1}{p} \|u\|_{1,p}^p - \int_{\Omega} G(x, u) \geq \frac{1}{p} \|u\|_{1,p}^p - \frac{\lambda_1 - \varepsilon}{p} \|u\|_{0,p}^p - 2k|\Omega|, \quad \forall u \in W. \tag{4.25}
\]

By (4.17), we infer
\[
\Phi_K(u) \geq \frac{\varepsilon}{p\lambda_1} \|u\|_{1,p}^p - 2k|\Omega|, \quad \forall u \in W, \tag{4.26}
\]

showing that
\[
\lim_{\|u\|_{1,p} \to \infty} \Phi_K(u) = +\infty. \tag{4.27}
\]

**Theorem 4.2.** Assume (4.7), (4.9), and $\text{int}(\Omega \setminus \omega) \neq \emptyset$ if $W = W_1$ or $W = W_2$, together with

(i) $\limsup_{s \to 0} P G(x, s)/|s|^p < \lambda_1$ uniformly for a.e. $x \in \Omega \setminus \omega$;

(ii) $\limsup_{s \to 0} P G(x, s)/s^p < \lambda_1$ uniformly for a.e. $x \in \Omega$.

□
and there are numbers \( \theta > p \) and \( s_0 > 0 \) such that

(iii) \( 0 < \theta G(x,s) \leq sg(x,s) \) for a.e. \( x \in \Omega \setminus \omega, \forall s \leq -s_0, \)

(iv) \( 0 < \theta G(x,s) \leq sg(x,s) \) for a.e. \( x \in \Omega, \forall s \geq s_0. \)

Then problem (4.14) has a nontrivial solution.

Proof. We will apply Theorem 3.11. Without loss of generality, we may suppose in (4.7) that \( q \in (p, p^\ast). \) For \( u \in K \) (see (4.13)), the sets \( \Omega_{-} \) and \( \Omega_{+} \) will be considered as being defined by (4.21), and recall that \( \Omega_{-} \subset \Omega \setminus \omega. \)

First we check (H4'). By (i) and (ii), one can find numbers \( \varepsilon \in (0, \lambda_1) \) and \( \delta_0 > 0 \) such that

\[
G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p} |s|^p \quad \text{for a.e. } x \in \Omega \setminus \omega, \forall s \in [-\delta_0, 0),
\]

(4.28)

\[
G(x,s) \leq \frac{\lambda_1 - \varepsilon}{p} |s|^p \quad \text{for a.e. } x \in \Omega, \forall s \in (0, \delta_0].
\]

(4.29)

From (4.11), there exists a constant \( c = c(\delta_0) \) with

\[
G(x,s) \leq c|s|^q \quad \text{for a.e. } x \in \Omega, \forall |s| > \delta_0.
\]

(4.30)

For an arbitrary \( u \in K, \) by (4.28) and (4.30) we have

\[
\int_{\Omega_{-}} G(x,u) = \int_{\Omega_{-} \cap [-\delta_0 \leq u]} G(x,u) + \int_{|u|<\delta_0} G(x,u)
\leq \frac{\lambda_1 - \varepsilon}{p} \int_{\Omega_{-}} |u|^p + c \int_{\Omega_{-}} |u|^q.
\]

(4.31)

Similarly, (4.29) and (4.30) imply

\[
\int_{\Omega_{+}} G(x,u) = \int_{\Omega_{+} \cap [u\leq\delta_0]} G(x,u) + \int_{|u|>\delta_0} G(x,u)
\leq \frac{\lambda_1 - \varepsilon}{p} \int_{\Omega_{+}} |u|^p + c \int_{\Omega_{+}} |u|^q.
\]

(4.32)

Then, combining (4.31) and (4.32), we infer

\[
\int_{\Omega} G(x,u) \leq \frac{\lambda_1 - \varepsilon}{p} \|u\|^p_{1,p} + c \|u\|_{0,q}^q.
\]

(4.33)

Taking into account the continuity of the embedding \( W \hookrightarrow L^q(\Omega), \) from (4.33) and (4.17) we get, for a constant \( \tilde{c}, \) the relations

\[
\mathcal{G}(u) + \varphi(u) = -\int_{\Omega} G(x,u) + \frac{1}{p} \|u\|_{1,p}^p \geq \frac{\varepsilon}{\lambda_1 p} \|u\|_{1,p}^p - \tilde{c} \|u\|_{1,p}^q > 0
\]

(4.34)
provided \( u \in K \) and \( \|u\|_{1,p} = \rho > 0 \) is sufficiently small. Therefore, Theorem 3.11(H4') is satisfied.

To check hypothesis (H1'), we proceed as follows. From (iv), we have

\[
\frac{G(x,s)}{s} \leq \frac{1}{\theta} g(x,s) \text{ for a.e. } x \in \Omega, \forall s \geq s_0. \tag{4.35}
\]

For a.e. \( x \in \Omega \), the primitive \( G(x,s) \) as a function of \( s \) being continuous (even locally Lipschitz), (4.35) implies

\[
\frac{G(x,s)}{s} \leq \frac{1}{\theta} g(x,s) \text{ for a.e. } x \in \Omega, \forall s > s_0. \tag{4.36}
\]

Similarly, by (iii), we get

\[
G(x,s) \leq \frac{1}{\theta} s g(x,s) \text{ for a.e. } x \in \Omega \setminus \omega, \forall s < -s_0. \tag{4.37}
\]

Recall that under the assumptions (4.7) and (4.9), for \( u \in L^q(\Omega) \), the following inclusion holds (see [4, Theorem 2.1]):

\[
\partial(-\mathcal{G})(u) \subset [g(x,u),\overline{g}(x,u)] \text{ for a.e. } x \in \Omega. \tag{4.38}
\]

Then, from (4.20), (4.36), (4.37), (4.38), and (4.7), for an arbitrary \( u \in K \), we obtain

\[
-\mathcal{G}(u) = \int_{\Omega} G(x,u) = \int_{[u < -s_0]} G(x,u) + \int_{[u > s_0]} G(x,u) + \int_{[-s_0 \leq u \leq s_0]} G(x,u)
\leq \frac{1}{\theta} \left[ \int_{[u < -s_0]} u \overline{g}(x,u) + \int_{[u > s_0]} u g(x,u) \right] + k|\Omega|
\leq \frac{1}{\theta} \left[ \int_{[u < -s_0]} uw + \int_{[u > s_0]} uw \right] + k|\Omega|
= \frac{1}{\theta} \left[ \int_{\Omega} uw - \int_{|u| \leq s_0} uw \right] + k|\Omega|
\leq \frac{1}{\theta} \int_{\Omega} uw + k_0, \forall w \in \partial(-\mathcal{G})(u), \tag{4.39}
\]

for a constant \( k_0 > 0 \). As \( \partial(-\mathcal{G})(u) = -\partial \mathcal{G}(u) \), it follows that

\[
\mathcal{G}(u) \geq \frac{1}{\theta} \int_{\Omega} uw - k_0, \forall w \in \partial \mathcal{G}(u). \tag{4.40}
\]
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Taking the supremum over \( w \in \partial \mathcal{G}(u) \) in (4.40), we deduce

\[
\mathcal{G}(u) - \frac{1}{\theta} (\mathcal{G}|_{W})^0(u;u) \geq -k_0, \quad \forall u \in K. \tag{4.41}
\]

By virtue of (4.4) and (4.5), one has

\[
\varphi(u) - \frac{1}{\theta} (d\varphi(u), u) = \left( \frac{1}{p} - \frac{1}{\theta} \right) \|u\|_{1,p}^p \quad \forall u \in W. \tag{4.42}
\]

From (4.41) and (4.42), it turns out that Theorem 3.11(H1') is fulfilled with

\[
\alpha = \frac{1}{\theta}, \quad a_0 = \frac{1}{p} - \frac{1}{\theta}, \quad a_1 = 0, \quad \sigma = p, \quad b_0 = 0, \quad b_1 = k_0. \tag{4.43}
\]

To check condition Theorem 3.11(H3'), we first note that, on the basis of (i), (ii) and arguing as in the proof of [7, Proposition 7], one has

\[
G(x,t) \geq \gamma_1(x)t^\theta \quad \text{for a.e. } x \in \Omega, \quad \forall t > s_0, \tag{4.44}
\]

\[
G(x,t) \geq \gamma_2(x)|t|^\theta \quad \text{for a.e. } x \in \Omega \setminus \omega, \quad \forall t < -s_0, \tag{4.45}
\]

where \( \gamma_1, \gamma_2 \in L^\infty(\Omega), \gamma_1(x) > 0 \) for a.e. \( x \in \Omega \), and \( \gamma_2(x) > 0 \) for a.e. \( x \in \Omega \setminus \omega \).

Since, by assumption, \( \text{int}(\Omega \setminus \omega) \neq \emptyset \) if \( W = W_1 \) or \( W = W_2 \), there is some \( \overline{\Omega} \in K \) such that \( |\Omega(\overline{\Omega})| > 0 \), where \( \Omega(\overline{\Omega}) = \{x \in \Omega : \overline{u} > s_0\} \). For \( t \geq 1 \), using (4.20), (4.44), (4.45), and the inclusion \( \{t\overline{u} < -s_0\} \subset \Omega \setminus \omega \), we estimate \(-\mathcal{G}\) as follows:

\[
-\mathcal{G}(t\overline{u}) = \int_{\{t\overline{u} > s_0\}} G(x,t\overline{u}) + \int_{\{t\overline{u} \leq s_0\}} G(x,t\overline{u}) \geq \int_{\{t\overline{u} > s_0\}} G(x,t\overline{u}) - k|\Omega| = \int_{\{\overline{u} > s_0\}} G(x,t\overline{u}) + \int_{\{\overline{u} < -s_0\}} G(x,t\overline{u}) - k|\Omega| \geq t^\theta \left[ \int_{\Omega(\overline{\Omega})} \gamma_1(x)\overline{u}^\theta + \int_{\Omega \setminus \omega} \gamma_2(x)|\overline{u}|^\theta \right] - k|\Omega| \geq t^\theta \int_{\Omega(\overline{\Omega})} \gamma_1(x)\overline{u}^\theta - k|\Omega|. \tag{4.46}
\]

Therefore,

\[
\mathcal{G}(t\overline{u}) + \varphi(t\overline{u}) \leq -t^\theta \int_{\Omega(\overline{\Omega})} \gamma_1(x)\overline{u}^\theta + \frac{t^p}{p} \|\overline{u}\|_{1,p}^p + k|\Omega|, \quad \forall t \geq 1. \tag{4.47}
\]

Taking into account \( \theta > p \), it follows that \( \Phi_K(t\overline{u}) \to -\infty \) as \( t \to +\infty \). This establishes (H3') with \( \overline{u} \) replaced by \( t\overline{u} \), for some \( t \geq 1 \) sufficiently large.
Finally, hypothesis (H2’) is also satisfied because, as we have already noted, the duality mapping $d\varphi$ verifies condition $(S_\star)$.

The application of Theorem 3.11 concludes the proof. \hfill\Box

**Remark 4.3.** If $\omega = \emptyset$, then $K = W$. Taking into account Remark 3.12, in this case, problem (4.14) becomes

\[
\text{Find } u \in W \text{ such that } -\Delta_p u \in \partial(-\mathcal{G}_{|W})(u). \tag{4.48}
\]

This means that for $u \in W$, it corresponds $h \in \partial(-\mathcal{G}_{|W})(u) \subset \partial(-\mathcal{G})(u) \subset L^{q'}(\Omega)$, with $1/q + 1/q' = 1$, such that $u$ satisfies the variational equality

\[
\int_{\Omega} \left( |\nabla u|^p - 2 \nabla u \nabla v + hv \right) = 0, \quad \forall v \in W. \tag{4.49}
\]

Assuming (4.7) and (4.9), inclusion (4.38) and equality (4.49) show that each solution of problem (4.48) for $W = W_0$ also solves the differential inclusion problem:

\[
\text{Find } u \in W_0 = W_0^{1,p}(\Omega) \text{ such that } -\Delta_p u \in \left[ g(x,u), \widehat{g}(x,u) \right] \quad \text{for a.e. } x \in \Omega. \tag{4.50}
\]

In the case $W = W_1$, denoting by $\hat{w} = \left(1/|\Omega|\right) \int_{\Omega} w$ the mean value of any $w \in L^1(\Omega)$, relation (4.49) is expressed as follows:

\[
\int_{\Omega} \left( |\nabla u|^p - 2 \nabla u \nabla w + h(w - \hat{w}) \right) = 0, \quad \forall w \in W^{1,p}(\Omega), \tag{4.51}
\]

or, equivalently,

\[
\int_{\Omega} \left[ |\nabla u|^p - 2 \nabla u \nabla w + (h - \hat{h})w \right] = 0, \quad \forall w \in W^{1,p}(\Omega). \tag{4.52}
\]

Thus, if $W = W_1$, with $u \in W$ in (4.48), the following problem is solved:

\[
\text{Find } u \in W_1 \text{ such that } -\Delta_p u \in \left[ g(x,u) - \widehat{g}(\cdot,u), \widehat{g}(x,u) - \widehat{g}(\cdot,u) \right] \quad \text{for a.e. } x \in \Omega. \tag{4.53}
\]

A problem similar to (4.53) is solved when $W = W_2$ in (4.48).

**Corollary 4.4 (see [8, Theorem 5.1]).** Assume (4.7), (4.9), and

\[
\limsup_{|s| \to +\infty} \frac{pG(x,s)}{|s|^p} < \lambda_{1,W_0} \quad \text{uniformly for a.e. } x \in \Omega. \tag{4.54}
\]

Then problem (4.50) has a solution.
Proof. Theorem 4.1 applies with $\omega = \emptyset$. □

Corollary 4.5 (see [6, Theorem 3.6] and [8, Theorem 5.2]). Assume (4.7) and (4.9) together with

$$\limsup_{s \to 0} \frac{pG(x,s)}{|s|^p} < \lambda_{1,W_0} \quad \text{uniformly for a.e. } x \in \Omega. \quad (4.55)$$

If there are numbers $\theta > p$ and $s_0 > 0$ such that

$$0 < \theta G(x,s) \leq sg(x,s) \quad \text{for a.e. } x \in \Omega, \quad \forall |s| \geq s_0, \quad (4.56)$$

then problem (4.50) has a nontrivial solution.

Proof. We apply Theorem 4.2 with $\omega = \emptyset$. □

Acknowledgment

The authors are grateful to the referee for valuable comments and suggestions.

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