RELAXED SUBMONOTONE MAPPINGS

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The notions of *relaxed submonotone* and *relaxed monotone* mappings in Banach spaces are introduced and many of their properties are investigated. For example, the Clarke subdifferential of a locally Lipschitz function in a separable Banach space is relaxed submonotone on a residual subset. For example, it is shown that this property need not be valid on the whole space. We prove, under certain hypotheses, the surjectivity of the relaxed monotone mappings.

1. Preliminaries

The notions *submonotone* and *strictly submonotone* mappings in $\mathbb{R}^n$ were introduced by Spingarn in 1981 (see [11]) as a local version of monotone operators. In the recent papers [5, 6], these notions were extended in arbitrary Banach spaces and many of their properties were considered. It was shown in particular in [6] that the subdifferential of Pshenichnyi (see [10]) is almost strictly submonotone almost everywhere in separable Banach spaces.

In this paper, we extend these notions to the so-called *relaxed submonotone mappings*. We prove that some of the main properties of the submonotone mappings are valid also for the relaxed submonotone ones. We show that, in separable Banach spaces, the Clarke subdifferential of every locally Lipschitz real valued functions is almost everywhere (in Baire sense) relaxed submonotone. We also give an example of a Lipschitz function from $\mathbb{R}$ into $\mathbb{R}$ which Clarke subdifferential is not relaxed submonotone on a dense set.

Firstly, we recall some definitions and notations.

We will use the following abbreviations: USC—upper semicontinuous, UHC—upper hemicontinuous, RM—relaxed monotone, RSM—relaxed submonotone, and SRSM—strictly relaxed submonotone.

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Let $(E, \|\cdot\|)$ be a Banach space with dual E^* and $P(E^*)$ the set of all nonempty and bounded subsets of E^*. If $A \in P(E)$ and $x \in E^*$, we denote by $\sigma(x, A) = \sup_{a\in A}\langle x, a\rangle$—the support function of A. Here, $\langle\cdot,\cdot\rangle$ is the dual brackets between E and E^*.

The multifunction $F : E \supset D(F) \to P(E^*)$ is called locally bounded, if for every $x \in D(F)$ ($D(F)$ is called domain of F), there exists $\delta > 0$ such that F is bounded on the set $U(x, \delta) := (x + \delta B) \cap D(F)$ (B is the unit ball centered in the origin). That is, $\|F(y)\| := \sup_{e\in F(y)} \|e\|$ is bounded on $y \in U(x, \delta)$. If for every $x_0 \in D(F)$ and every $l \in S$ (where S is the unit sphere in E), there exists $\delta > 0$ such that F is bounded on the set

$$U(x_0, e, \delta) := \left\{x \in E : x \neq x_0,\ \|x - x_0\| < \delta,\ \left\|\frac{x - x_0}{\|x - x_0\|} - e\right\| < \delta\right\} \cap D(F), \tag{1.1}$$

then F is called directionally locally bounded. If $y \neq x \to y$ and $(x - y)/\|x - y\| \to e$, we will write $x \to_e y$.

The multifunction $F : X \to Y$, where X and Y are topological spaces, is called USC at x, when for every open set $V \supset F(x)$ there exists an open set $U \ni x$ such that $F(y) \subset V$ for every $y \in U$. It is called USC when it is USC at every point of $D(F) = \{x \in X : F(x) \neq \emptyset\}$.

The function $f^0(x; h) = \limsup_{\substack{z\to x\\ t\downarrow 0}}(f(z + th) - f(z))/t$ is said to be *Clarke's derivative* at the point $x \in E$ in the direction $h \in E$ for a locally Lipschitz function $f : E \to \mathbb{R}$ (see [2]).

The function $f'(x; h) = \lim_{t\downarrow 0}(f(x + th) - f(x))/t$ (if it exists) is said to be the *directional derivative* at the point x (in the direction h). The set

$$\partial f(x) = \{x^* \in E^* : \langle h, x^*\rangle \leq f^0(x; h),\ \forall h \in E\} \tag{1.2}$$

is said to be *Clarke's subdifferential* at x for f (see [2]).

Definition 1.1. The mapping $F : D(F) \to P(E^*)$ is said to be RSM at $x \in E$ when the following two conditions hold:

$$\begin{aligned} &\liminf_{x\neq y\to_e x} \frac{\sigma(y - x, F(y)) - \sigma(y - x, F(x))}{\|y - x\|} \geq 0 \quad \forall e \in S, \\ &\liminf_{x\neq y\to_e x} \frac{\sigma(x - y, F(x)) - \sigma(x - y, F(y))}{\|y - x\|} \geq 0 \quad \forall e \in S. \end{aligned} \tag{1.3}$$

The mapping F is said to be SRSM when

$$\liminf_{\substack{x\leftarrow z\neq y\to x\\ y-z\to_e 0}} \frac{\sigma(y - z, F(y)) - \sigma(y - z, F(z))}{\|y - z\|} \geq 0 \tag{1.4}$$

for every $e \in S$.

If F satisfies Definition 1.1 with "$\geq$" replaced by "$\leq$," then F will be called relaxed subdissipative at x. Let $F : D(F) \to P(E^*)$. For given $l \in S$ and $x \in \operatorname{int}(D(F))$, we let $\Phi_{x,l}(t) = \sigma(l, F(x+tl))$.

Proposition 1.2. *If F is RSM, then $\Phi_{x,l}$ is submonotone for every $l \in S$ and $x \in \operatorname{int}(D(F))$, that is,*

$$\limsup_{s\uparrow t} \Phi_{x,l}(s) \leq \Phi_{x,l}(t) \leq \liminf_{s\downarrow t} \Phi_{x,l}(s) \quad \forall t \in \mathbb{R}. \tag{1.5}$$

Proof. Let F be RSM. If $y = x + tl$, $z = y + sl$ and $s \geq 0$, then

$$\liminf_{s\downarrow 0} \frac{(\sigma(z-y, F(z)) - \sigma(z-y, F(y)))}{\|z-y\|} \geq 0, \tag{1.6}$$

that is,

$$\liminf_{s\downarrow 0} \frac{(s\sigma(l, F(z)) - s\sigma(l, F(y)))}{s\|l\|} \geq 0. \tag{1.7}$$

Thus, $\liminf_{s\downarrow 0}(\sigma(l, F(x+(t+s)l)) - \sigma(l, F(x+tl))) \geq 0$. Hence, $\liminf_{s\downarrow 0} \Phi_{x,l}(t+s) \geq \Phi_{x,l}(t)$. If $z = y - sl$, then

$$\liminf_{s\downarrow 0} \frac{(\sigma(y-z, F(y)) - \sigma(y-z, F(z)))}{\|z-y\|} \geq 0, \tag{1.8}$$

that is,

$$\liminf_{s\downarrow 0} (\sigma(l, F(x+tl)) - \sigma(l, F(x+(t-s)l))) \geq 0. \tag{1.9}$$

That is, $\limsup_{s\downarrow 0} \Phi_{x,l}(t-s) \leq \Phi_{x,l}(t)$. □

Remark 1.3. Let the map $f(\cdot, v)$ be RSM at x_0 for every $v \in V$ where v is a parameter. If lim inf in Definition 1.1 is uniform, with respect to $v \in V$, we will say that the mappings $\{f(\cdot, v) : v \in V\}$ are equi-RSM. The equi-SRSM multifunctions are defined analogously.

The following property of RSM (SRSM) mappings is obvious.

Proposition 1.4. *If the mappings $\{f(\cdot, v) : v \in V\}$ are equi-RSM (resp., equi-SRSM), then the mapping $F(x) = \overline{\operatorname{co}}\, f(x, V)$ is RSM (resp., SRSM).*

We finish this section with an example, showing that the class of RSM mappings is substantially different from the class of continuous plus submonotone mappings.

Example 1.5. Define the mapping $F : R \to P(R)$ as follows:

$$F(x) = \left[\min_{a \in \{f(x), g(x)\}} a, \max_{b \in \{f(x), g(x)\}} b \right], \tag{1.10}$$

where

$$\begin{aligned} f(x) &= \begin{cases} [0,1] & \text{if } x = k \ (k \text{ is integer number}), \\ \sqrt[3]{k+1-x} & \text{if } x \in (k, k+1), \end{cases} \\ g(x) &= \begin{cases} \left[\sqrt[3]{-\frac{1}{2}}, \sqrt[3]{\frac{1}{2}} \right] & \text{if } x = k + \frac{1}{2}, \\ \sqrt[3]{k+1-x} & \text{if } x \in \left(k + \frac{1}{2}, k + 1 + \frac{1}{2} \right). \end{cases} \end{aligned} \tag{1.11}$$

It is easily seen that F is RSM (even SRSM) and cannot be represented as a sum of continuous and submonotone mappings.

2. Relaxed submonotone operators

In this section, we study the main properties of RSM.

Proposition 2.1. *Let $F : E \to P(E^*)$ be directionally locally bounded at $x_0 \in E$. Then, F is RSM at x_0 if and only if, for every $e \in S$ and every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\sigma(e, F(x)) \geq \sigma(e, F(x_0)) - \varepsilon, \qquad \sigma(-e, F(x_0)) \geq \sigma(-e, F(x)) - \varepsilon \tag{2.1}$$

for every $x \in U(x_0, e, \delta)$.

Proof. Let F be RSM at x_0. For every $\varepsilon > 0$ and every $e \in S$, there exists $\delta > 0$ such that

$$\begin{aligned} \frac{\sigma(x - x_0, F(x)) - \sigma(x - x_0, F(x_0))}{\|x - x_0\|} &\geq -\frac{\varepsilon}{3}, \\ \frac{\sigma(x_0 - x, F(x_0)) - \sigma(x_0 - x, F(x))}{\|x - x_0\|} &\geq -\frac{\varepsilon}{3}, \end{aligned} \tag{2.2}$$

for $x \in U(x_0, e, \delta)$. Since $F(\cdot)$ is directionally locally bounded at x_0, there exists $\delta_1 \in (0, \delta)$ such that F is bounded on $U(x_0, e, \delta_1)$, say, by b. Then for

$\delta_2 = \min\{\varepsilon/(3b), \delta_1\}$ and $x \in U(x_0, \varepsilon, \delta_2)$, we have

$$\begin{aligned}\sigma(e, F(x_0)) &\le \frac{\sigma(x - x_0, F(x_0))}{||x - x_0||} + \delta_2 b \\ &\le \frac{\sigma(x - x_0, F(x))}{||x - x_0||} + \frac{2\varepsilon}{3} \\ &\le \sigma(e, F(x)) + \varepsilon.\end{aligned} \tag{2.3}$$

The proofs of the second inequality and the opposite direction of the proposition are similar. □

The following lemma is a modification of a Kenderov's lemma [8, Lemma 1.7] for monotone mappings. Here the proof is analogous.

Lemma 2.2. *Let $T : E \to P(E^*)$ be a RSM mapping, let $A \subset E^*$ be a w^*-compact and convex subset of E^*, and let $T_{-1}(A) := \{x \in E : T(x) \subset A\}$ be dense in some open set $U \subset E$. Then $T(x) \subset A$ for every $x \in U$.*

Proof. (a) Assume that $T(x_0) \not\subset A$ for some $x_0 \subset U$. Choose $x_0^* \in T(x_0) \setminus A$. Then, by the separation theorem, there exists $e_0 \in S_1$ and $\varepsilon > 0$ such that $\langle e, x_0^* \rangle > \sigma(e, A) + \varepsilon$, $\forall e \in B(e_0, \varepsilon)$. Since T is RSM at x_0, there exists $\delta \in (0, \varepsilon)$ such that

$$\sigma\left(\frac{x - x_0}{||x - x_0||}, T(x)\right) > \sigma\left(\frac{x - x_0}{||x - x_0||}, T(x_0)\right) - \varepsilon \quad \forall x \in U(x_0, e_0, \delta). \tag{2.4}$$

Hence, for $x_1 \in T_{-1}(A) \cap U(x_0, e_0, \delta)$, $e_1 := (x_1 - x_0)/\|x_1 - x_0\|$, we have

$$\sigma(e_1, A) \ge \sigma(e_1, T(x_1)) \ge \langle e_1, x_0^* \rangle - \varepsilon > \sigma(e_1, A), \tag{2.5}$$

a contradiction. □

Recall that a subset X_1 of a topological space X is said to be residual, if $X \setminus X_1$ is of first Baire category.

Lemma 2.3. *Let X be a topological space and let $P \subset X$ be a subset. Then the set $X(P) \subset X$, defined by the properties: for every $x_0 \in X(P)$, either*

(1) *$x_0 \in P$, or*
(2) *there exists an open set $U \ni x_0$ and a dense subset $U' \subset U$ such that $U' \cap P = \emptyset$,*

is residual in X.

Proof. We have

$$X \setminus X(P) = \{x \in X : x \notin P,\ \forall U \ni x \text{ open } \forall U' \subset U \text{ dense in } U, U' \cap P \neq \emptyset\}. \tag{2.6}$$

Assume that $X \setminus X(P)$ is dense in some open U. Then, by definition of $X \setminus X(P)$, we have simultaneously $(X \setminus X(P)) \cap P = \emptyset$ and $(X \setminus X(P)) \cap P \neq \emptyset$, a contradiction. □

Theorem 2.4. *Every RSM mapping $T : D(T) \to P(E^*)$ is locally bounded at the points of some dense and open (hence residual) subset of* $\operatorname{int} D(T)$ *(assume that* $\operatorname{int} D(T) \neq \emptyset$*).*

Proof. The assertion follows from Lemmas 2.2 and 2.3, applied to the sets $P_n = \{x \in E : T(x) \not\subset n\bar{B}^*\}$, where $n\bar{B}^* := \{x^* \in E^* : \|x^*\| \leq n\}$. Indeed, by Lemma 2.3, applied to P_n, the set $E(P_n)$ (defined by (1) and (2) in Lemma 2.3, when X is replaced by E) is residual in E. Define $R = \bigcap_{n=1}^{\infty} E(P_n)$ and let $x_0 \in R$. Since $T(x_0)$ is bounded, there exists n_0 such that $T(x_0) \subset n_0\bar{B}^*$, that is, $x_0 \notin P_{n_0}$. By (2) of Lemma 2.3, there exists an open set $U_{x_0} \ni x_0$ and a dense subset $U' \subset U_{x_0}$ such that $x \notin P_{n_0}$ for every $x \in U'$, that is, $T(x) \subset n_0\bar{B}^*$ for every $x \in U'$. Now we apply Lemma 2.2 (since $n_0\bar{B}^*$ is w^*-compact) and obtain that $T(x) \subset n_0\bar{B}^*$ for every $x \in U_{x_0}$. Obviously, the set $R_0 = \bigcup\{U_x : x \in R\}$ is open and dense in E and T is locally bounded at every $x \in R_0$. □

Now we can prove the main result in this section.

Theorem 2.5. *Every SRSM $T : D(T) \to P(E^*)$ is locally bounded at every point of* $\operatorname{int} D(T)$ *(assume that* $\operatorname{int} D(T) \neq \emptyset$*).*

Proof. We follow [6] where the local boundedness of strictly submonotone operators is proved.

By Theorem 2.4, T is locally bounded at the points of some residual subset of $\operatorname{int} D(T)$. Let $x_0 \in \operatorname{int} D(T)$ be an arbitrary point and let $\varepsilon > 0$, $e \in S$. Since T is SRSM at x_0, it follows that there exists $\delta \in (0,1)$ such that the conditions

$$\begin{gathered} x_1 \neq x_2, \qquad \|x_i - x_0\| < \delta, \qquad y_i \in T(x_i), \\ \langle x_1 - x_2, y_1 \rangle = \max_{y \in T(x_1)} \langle x_1 - x_2, y \rangle, \end{gathered} \tag{2.7}$$

either

$$\left\| \frac{x_1 - x_2}{\|x_1 - x_2\|} - e \right\| < \delta \qquad \text{or} \qquad \left\| \frac{x_1 - x_2}{\|x_1 - x_2\|} + e \right\| < \delta, \tag{2.8}$$

imply

$$\frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{\|x_1 - x_2\|} > -\varepsilon. \tag{2.9}$$

Since $U(0, e, \delta/2)$ is open, there exists $y \in U(0, e, \delta/2)$ such that $y_1 := x_0 + y \in \operatorname{int} D(T)$, $y_2 := x_0 - y \in \operatorname{int} D(T)$ and T is locally bounded at y_i, $i = 1, 2$, that is, there exist $\varepsilon_1 > 0$ and $C > 0$ such that $y_i + \varepsilon_1 B \subset \operatorname{int} D(T)$ and $\|z^*\| < C$ for every $z^* \in T(y_i + \varepsilon_1 B)$, $i = 1, 2$. Let $\varepsilon_2 \in (0, \min\{\delta\|y\|/4, \delta/2, \varepsilon_1\})$. We will show

that the set $T(x_0+(\varepsilon_2/2)B)$ is bounded. Let $x\in x_0+\varepsilon_2 B/2$, $x^*\in T(x)$. Then $x=x_0+z$ for some $z\in\varepsilon_2 B/2$. Let $v\in B$. We have, as in [6, page 8],

$$\left\|\frac{(y+\varepsilon_2 v/2)}{\|y+\varepsilon_2 v/2\|}-e\right\|<\delta,\qquad \left\|y_1+\frac{\varepsilon_2 v}{2}+z-x_0\right\|\le\delta \tag{2.10}$$

and $\|x-x_0\|<\varepsilon_2/2<\delta/4$. For $x_1=y_1+\varepsilon_2 v/2+z$, $x_2=x$, $y_i^*\in T(x_i)$, $\langle x_1-x_2,y_1^*\rangle=\max_{y^*\in T(x_1)}\langle x_1-x_2,y^*\rangle$, we obtain $\langle y+\varepsilon_2 v/2,y_1^*-x^*\rangle/\|y+\varepsilon_2 v/2\|\ge -\varepsilon$. Hence, since $x_1-x_2=y+\varepsilon_2 v/2$,

$$\begin{aligned}\left\langle y+\frac{\varepsilon_2 v}{2},x^*\right\rangle &\le \left\langle y+\frac{\varepsilon_2 v}{2},y_1^*\right\rangle+\varepsilon\left\|y+\frac{\varepsilon_2 v}{2}\right\|\\ &\le C\left(\|y\|+\frac{\varepsilon_2}{2}\right)+\varepsilon\left(\|y\|+\frac{\varepsilon_2}{2}\right)\\ &<(C+\varepsilon)\left(\frac{\delta}{2}+\frac{\varepsilon_2}{2}\right).\end{aligned} \tag{2.11}$$

Analogously, we have $\|(-y+\varepsilon_2 v/2)/\|-y+\varepsilon_2 v/2\|+e\|<\delta$, $\|y+\varepsilon_2 v/2+z-x_0\|<\delta$ and as above, we obtain

$$\left\langle -y+\frac{\varepsilon_2 v}{2},x^*\right\rangle<(C+\varepsilon)\left(\frac{\delta}{2}+\frac{\varepsilon_2}{2}\right). \tag{2.12}$$

Adding (2.11) and (2.12), we get $\varepsilon_2\langle v,x^*\rangle<(C+\varepsilon)(\delta+\varepsilon_2)$ which is true for every $v\in B$. Hence, $\|x^*\|<(C+\varepsilon)((\delta+\varepsilon_2)/\varepsilon_2)$ which proves the locally boundedness of T at x_0. □

Theorem 2.6. *Let E be a Banach space such that E^* is separable and let $T:E\to P(E^*)$ be a RSM mapping with compact images on a residual subset E_1 of E. Then, T is norm to norm continuous on a residual subset of E.*

Proof. Since E^* is separable, E^* has a countable base $\alpha=\{V_i\}_{i=1}^{\infty}$ of open sets. Let $\mathcal{I}=\{I_n:n\in\mathbb{N}\}$ be the set of all finite subsets of the set of natural numbers $\mathbb{N}$. For any $I_n\in\mathcal{I}$, define

$$P_n=\{x\in E:\exists i\in I_n:T(x)\not\subset V_i\}. \tag{2.13}$$

By Lemma 2.3, the set $E_2:=E_1\cap(\bigcap_{n=1}^{\infty}X(P_n))$ is residual. Let $x\in E_2$ and let V be an open subset such that $T(x)\subset V$. Since $T(x)$ is compact, there exist a finite number of elements $V_{n_i}\in\alpha$, $i=1,\dots,m$, such that $T(x)\subset\bigcup_{i=1}^{m}V_{n_i}\subset V$. By Theorem 2.5 and Lemma 2.2, we obtain norm upper semicontinuity of T at x. Applying Fort's theorem [7, Theorem 2.95], we obtain the result. □

Theorem 2.7. *Let E be a separable Banach space and let $f : E \to \mathbb{R}$ be a locally Lipschitz function. Then, the Clarke subdifferential ∂f is RSM on a dense G_δ subset of E.*

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be a dense subset of E. Since $f^0(\cdot;e_n)$ is upper semicontinuous (see Clarke [2]), $f^0(\cdot;e_n)$ is lower semicontinuous on a dense G_δ subset E_n of E. Let E_0 be the intersection of all E_n, $n > 1$. Afterwards, we use the formula $f^0(x;v) = \max\{\langle x^*, v\rangle : x^* \in \partial f(x)\}$ (see [2, Proposition 2.1.2(b)]) and the fact that $f^0(x;\cdot)$ is Lipschitz with the same Lipschitz constant L as f in a neighborhood of x. Denoting $e_y = (y-x)/\|y-x\|$, we obtain

$$\begin{aligned}
&\liminf_{x\neq y\to_e x} \frac{\sigma(y-x,F(y)) - \sigma(y-x,F(x))}{\|y-x\|} \\
&\qquad = \liminf_{x\neq y\to_e x} \left(f^0(y;e_y) - f^0(x;e_y)\right) \\
&\qquad \geq \liminf_{x\neq y\to_e x} \left(f^0(y;e) - f^0(x;e) - 2L\|e-e_y\|\right) \\
&\qquad \geq 0 \quad \text{(using lower semicontinuity of } f^0(\cdot;e)\text{)}.
\end{aligned} \tag{2.14}$$

Analogously, we obtain

$$\begin{aligned}
&\liminf_{x\neq y\to_e x} \frac{\sigma(x-y,F(x)) - \sigma(x-y,F(y))}{\|x-y\|} \\
&\qquad = \liminf_{x\neq y\to_e x} \left(f^0(x;-e_y) - f^0(y;-e_y)\right) \\
&\qquad \geq \liminf_{x\neq y\to_e x} \left(f^0(x;-e) - f^0(y;-e) - 2L\|e-e_y\|\right) \\
&\qquad \geq 0 \quad \text{(using upper semicontinuity of } f^0(\cdot;-e)\text{)}.
\end{aligned} \tag{2.15}$$

□

The last theorem cannot be improved. That is, there exist (globally) Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$ whose subdifferential ∂f is not RSM at some points x.

Example 2.8. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} (x-k)\sqrt{3}, & x \in \left[k, k+\dfrac{1}{\sqrt{3}}\right], \\ 1 - \left(x-k-\dfrac{1}{\sqrt{3}}\right)\dfrac{\sqrt{3}}{\sqrt{3}-1}, & x \in \left[k+\dfrac{1}{\sqrt{3}}, k+1\right]. \end{cases} \tag{2.16}$$

Here $k = \pm 1, \pm 2, \ldots$. Its subdifferential (in the sense of Clarke) is $\partial f(x) = \overline{\text{co}}\, g(x)$, where

$$g(x) = \begin{cases} \sqrt{3}, & x \in \left[k, k+\dfrac{1}{\sqrt{3}}\right], \\ -\dfrac{\sqrt{3}}{\sqrt{3}-1}, & x \in \left[k+\dfrac{1}{\sqrt{3}}, k+1\right]. \end{cases} \tag{2.17}$$

Obviously, $\partial f(\cdot)$ is not RSM at any point $x = k + 1/\sqrt{3}$, $k = 0, \pm 1, \pm 2, \ldots$.

Moreover, let $\{e_n\}_{n=1}^{\infty}$ be a sequence consisting of all rational numbers in the interval $(0,1)$. Let $f_n(x)$ be defined by

$$f_n(x) = \begin{cases} (x-k-e_n)\sqrt{3}, & x \in \left[k+e_n, k+\dfrac{1}{\sqrt{3}}+e_n\right], \\ 1-\left(x-k-\dfrac{1}{\sqrt{3}}-e_n\right)\dfrac{\sqrt{3}}{\sqrt{3}-1}, & x \in \left[k+\dfrac{1}{\sqrt{3}}+e_n, k+1+e_n\right]. \end{cases} \tag{2.18}$$

The functions f_n (or $-fn$) are regular in the sense of Clarke (see [2, Definition 2.3.4]). By Corollary 3 to Proposition 2.3.3 and Remark 2.3.5 of [2], we have $\partial\sum_{n=1}^{k}(1/n^2)f_n(x) = \sum_{n=1}^{k}(1/n^2)\partial f_n(x)$ for every k. Obviously, every $f_n(\cdot)$ is Lipschitz with a constant $\sqrt{3}$. Hence, the functions $h_k(x)\sum_{n=k+1}^{\infty}(1/n^2)f_n(x)$ are Lipschitz with constants $\sqrt{3}\sum_{n=k+1}^{\infty}1/n^2 = L_k$. Thus,

$$\left|\partial h_k(x)\right| = \sup_{\alpha\in\partial h_k(x)}|\alpha| \le L_k. \tag{2.19}$$

Hence, $\partial h_0(x) = \sum_{n=1}^{k}(1/n^2)\partial f_n(x) + \partial h_k(x)$. Furthermore

$$\left|\sum_{n=k+1}\left(\frac{1}{n^2}\right)\partial f_n(x)\right| \le L_k. \tag{2.20}$$

Consequently, $D_H(\partial h_0(x), \sum_{n=1}^{\infty}(1/n)\partial f_n(x)) \le 2L_k$ for every k. Since $\lim_{k\to\infty}L_k = 0$, we have $\partial\sum_{n=1}^{\infty}(1/n^2)f_n(x) = \sum_{n=1}^{\infty}(1/n^2)\partial f_n(x)$. Therefore, the function $h_0(x)$ is Lipschitz and its subdifferential $\partial h_0(x)$ is not RSM at any point $t \in Q$ (the set of the rational numbers). However, $\partial h(x)$ is relaxed subdissipative at every such point.

Example 2.9. There exist Lipschitz function whose subdifferentials are neither RSM nor relaxed subdissipative at any point of a dense set. Indeed, consider the interval $[0,1]$. Let $0 < t_1 < t_2 < \cdots < T = 1/\sqrt{2} < \cdots < s_2 < s_1 < 1$. Here $t_i = T(1-1/2^i)$ and $s_i = T(1+1/2^i)$. Define the function f such that $f(x) = 0$ for $x = 0, 1$, $f(T) = 1$. Furthermore, $f(s_{2i}) = f(t_{2i}) = 1 - 1/2^i$ and $f(s_{2i+1}) = f(t_{2i+1}) = f(t_{2i}) - 1/2^{i+1}$. On $[t_i, t_{i+1}]$ and on $[s_{i+1}, s_i]$ $f(\cdot)$ is linear. Let $\{r_n\}_{n=1}^{\infty}$ be the set of all rational numbers and let $f_n(\cdot)$ be defined by $f_n(t) = f(t-r_n)$. Then, f_n are Lipschitz and regular in the sense of Clarke, the function $g(x) = \sum_{n=1}^{\infty}(1/n^2)f_n(x)$ is Lipschitz on $\mathbb{R}$ and $\partial g(x) = \sum_{n=1}^{\infty}(1/n^2)\partial f_n(x)$. However, $\partial g(x)$ is neither RSM nor relaxed subdissipative at any point of the form $x = T + r_n$.

Remark 2.10. Let $F:\mathbb{R}\to\mathbb{R}$ be discontinuous at x_0. Obviously at least one of $F(\cdot)$ or $-F(\cdot)$ is not submonotone at x_0. Furthermore, we can easily show that the multifunction $F:\mathbb{R}\to 2^{\mathbb{R}}$ defined by $F(x) = [\alpha(x), \beta(x)]$ is RSM if and only

if $\alpha(\cdot)$ and $\beta(\cdot)$ are submonotone. Using this fact and [1, Theorem 3.2, Example 3.4], we can see that there exist functions whose subdifferential in the sense of Clarke is not RSM on a subset of $\mathbb{R}$ with positive Lebesgue measure.

Remark 2.11. It is well known that there exist everywhere differentiable real functions whose derivatives are not (locally) Riemann integrable, since their derivatives have sets of discontinuity points with positive measures. Furthermore, we can find such functions whose derivatives are not RSM on sets with positive measures.

3. Relaxed monotone mappings

In this section, we introduce and study the main properties of RM mappings. The first author has studied similar properties of the so-called one sided Lipschitz maps (see [4]). Some of the results here are contained in [3]. We refer to [7, 9] for the corresponding properties of the monotone operators.

Definition 3.1. The multivalued map $F : D(F) \to P(E^*)$ is said to be RM when

$$\sigma(x-y, F(x)) - \sigma(x-y, F(y)) \geq 0 \quad \forall x, y \in D(F). \tag{3.1}$$

The next result may be proved in the same fashion as Proposition 1.2 was.

Proposition 3.2. *$F(\cdot)$ is RM if and only if $\Phi_{x,l}(\cdot)$ is monotone for all $l \in S$.*

Obviously every RM map is also SRSM. By Theorem 2.5, every RM map is locally bounded in the interior of its domain. However, we can prove a stronger result.

Theorem 3.3. *If $A : E \to P(E^*)$ is an RM mapping, then A is locally bounded at every absorbing point of $D(A)$.*

Proof. Here we follow with modifications the method presented in [7] for monotone mappings. Let $x_0 \in D(A)$ be an absorbing point. Without loss of generality, we may assume that the images of A are w^*-closed (since the mapping $\overline{A}(x) := w^* - \mathrm{cl}A(x)$ is also RM), $x_0 = 0$, and $(0,0) \in \mathrm{graph}\,A$ (by choosing any $x_0^* \in A(x_0)$ and considering instead the operator $z \to A(x_0+z) - x_0^*$, which is also RM). We need to show that A is locally bounded at 0. Define the function $\varphi : E \to \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(x) = \sup\{\langle x-z, z^*\rangle : z \in D(A),\ \|z\| \leq 1,\ z^* \in A(z)\}. \tag{3.2}$$

Let $C = \{x \in E : \varphi(x) \leq 1 + \varphi(0)\}$. The function φ is convex and lower semicontinuous, $0 \leq \varphi(x)$ for all $x \in E$, that is, C is closed, convex, and $0 \in C$. We will prove that C is absorbing. Let $x \in E$. By hypothesis, $D(A)$ is absorbing, that is, there exists $\lambda > 0$ such that $A(\lambda x) \neq \varepsilon$. Let $M = \max_{x^* \in A(\lambda x)} |x^*|$, $z \in U \cap D(A)$, $z^* \in A(z)$ and $u^* \in A(\lambda x)$, $\langle \lambda x - z, u^*\rangle = \max_{x^* \in A(\lambda x)} \langle \lambda x - z, x^*\rangle$. Since A is

RM, we have

$$\langle \lambda x - z, z^* \rangle \le \langle \lambda x - z, u^* \rangle \le \|u^*\|(1+\lambda\|x\|) \le M(1+\lambda\|x\|). \tag{3.3}$$

Hence,

$$\varphi(\lambda x) \le M(1+\lambda\|x\|). \tag{3.4}$$

Let $\beta \in [0,1]$ such that $\beta\varphi(\lambda x) < 1$. Since φ is convex, we have $\varphi(\beta\lambda x) \le \beta\varphi(\lambda x) + (1-\beta)\varphi(0) < 1 + \varphi(0)$. Thus, $\beta\lambda x \in C$ and so C is absorbing. It follows that $0 \in \operatorname{int} C$. Then there exists $\delta > 0$ such that for $x \in 2\delta U$, we have $\varphi(x) < 1 + \varphi(0)$, that is, if $\|x\| < 2\delta$, then $\langle x, z^* \rangle < \langle z, z^* \rangle + 1 + \varphi(0)$ for all $(z, z^*) \in \operatorname{graph} A$ with $\|z\| < 1$. If $z \in (\delta B) \cap D(A)$ and $z^* \in A(z)$, we have

$$\begin{aligned} 2\delta\|z^*\| &= \sup\{\langle v, z^* \rangle : \|v\| \le 2\delta\} \\ &\le \|z^*\| \cdot \|z\| + 1 + \varphi(0) \\ &\le \delta\|z^*\| + 1 + \varphi(0), \end{aligned} \tag{3.5}$$

which implies $\|z^*\| \le (1+\varphi(0))/\delta$, that is, the set $\{A(z),\ z \in (\delta B) \cap D(A)\}$ is bounded. □

Definition 3.4. The multifunction $F(\cdot)$ is said to be UHC when it is USC on the finite dimensional subspaces of E.

Lemma 3.5 (lemma of Minty). *Let $\Omega \subset E$ be convex and let $A : \Omega \to P(E^*)$ be RM and UHC. Fix $u \in \Omega$ and $z \in E^*$. The following conditions are equivalent:*

(i) $\sigma(v-u, A(u)-z) \ge 0$ *for all* $v \in \Omega$,
(ii) $\sigma(v-u, A(v)-z) \ge 0$ *for all* $v \in \Omega$.

Proof. Obviously $\sigma(v-u, A(u)-z) = \sigma(v-u, A(u)) - \langle v-u, z \rangle$. Hence, $\sigma(v-u, A(v)-z) - \sigma(v-u, A(u)-z) \ge 0$, that is, (i) implies (ii).

Conversely, for $w \in \Omega$ and $t \in [0,1]$, we let $v = tu + (1-t)w$, that is, $v - u = (1-t)(w-u)$. If

$$\sigma(v-u, A(v)-z) = (1-t)\sigma(w-u, A(v)-z) \ge 0, \tag{3.6}$$

then

$$\sigma(w-u, A(v)-z) \ge 0. \tag{3.7}$$

Since A is UHC, there exists

$$0 \le \limsup_{t\to 1^+} \sigma(w-u, A(v)-z) \le \sigma(w-u, A(u)-z). \tag{3.8}$$

□

We will use the following lemma.

Lemma 3.6. *Let $D \subset \mathbb{R}^n$ be compact convex nonempty subset. Let $A : D \to P(R^n)$ be USC with convex compact images. Then, there exists $x_0 \in D$ such that $\sigma(y - x_0, Ax_0) \geq 0$ for all $y \in D$.*

Proof. By [7, Theorem 4.41], there exists a continuous function $g_n : \mathbb{R}^n \to \mathbb{R}^n$ with $\operatorname{graph}(g_n) \subset \operatorname{graph}(I_A) + (1/n)B$. By Brouwer's theorem (see [9] for instance) for the mapping $x \mapsto \operatorname{Proj}_D(g_n(x))$, there exists a fixed point x_n of it, that is, $x_n = \operatorname{Proj}_D(g_n(x_n))$. Therefore,

$$0 \leq -\langle y - x_n, g_n(x_n) - x_n \rangle. \tag{3.9}$$

Since D is compact, we may assume that $x_n \to x_0$. Passing to limits, we obtain $0 \leq \langle y - x_0, x_0^* \rangle$, where $x_0^* \in A(x_0)$, therefore $0 \leq \sigma(y - x_0, A(x_0))$. □

Now we are ready to prove the main result in this section.

Theorem 3.7. *Let E be a Banach space with dual E^*. Let $F : \bar{B}^* \to P(E)$ be UHC and RM mapping with convex strongly compact values. If $\sigma(-x, F(x)) < 0$ for all $x \in \partial B^*$, then there exists $x_0 \in \bar{B}^*$ with $F(x_0) \ni 0$.*

Proof. For every $y \in \bar{B}^*$ consider the set $S(y) = \{x \in \bar{B}^* : \sigma(y - x, F(y)) \geq 0\}$. Let $\{x_i\}_{i=1}^{\infty} \subset S(y)$ and let $x_i \to x_0$ weakly* in E^*. Let $y_i \in F(y)$ such that $\langle y - x_i, y_i \rangle = \sigma(y - x_i, F(y))$. Since $F(y)$ is strongly compact, passing to subsequences, we have $y_i \to y_0$ strongly in E. Therefore, $\langle y - x_i, y_i \rangle \to \langle y - x_0, y_0 \rangle$, that is, $\sigma(y - x_0, F(y)) \geq 0$. Consequently, $S(y)$ is weakly* closed. Since $S(y) \subset \bar{B}^*$, $S(y)$ is weakly* compact. By Lemmas 3.5 and 3.6, we obtain that the family $\{S(y) : y \in \bar{B}^*\}$ has finite intersection property. By compactness, there exists $x_0 \in \cap\{S(y) : y \in \bar{B}^*\}$. By Lemma 3.5, we obtain $\sigma(y - x_0, F(x_0)) \geq 0$ for every $y \in \bar{B}^*$. Taking $y = 0$, we conclude that x_0 cannot belong to the boundary of B^*, so $x_0 \in \operatorname{int} B^*$, which implies $0 \in F(x_0)$. □

Corollary 3.8. *Let $F : E^* \to P(E)$ be UHC and RM mapping with convex and strongly compact values. If $\lim_{\|x\| \to \infty} \sigma(-x, F(x))/\|x\| = -\infty$, that is, F is coercive, then F is surjective.*

Proof. Let $y \in E$. Consider the map $A(x) = F(x) - y$. Obviously A satisfies the assumptions of Theorem 3.7 for the set $U^* = r\bar{B}^*$ where r is sufficiently large. Therefore, there exists $x_0 \in E$ such that $A(x_0) \ni 0$, that is, $y \in F(x_0)$. □

Remark 3.9. Let E be reflexive. In this case $E = (E^*)^*$. Therefore, due to Corollary 3.8, if $F : E \to P(E^*)$ is RM and coercive, then F is surjective. That is, [3, Corollary 2] follows from Corollary 3.8. Furthermore, [3, Theorem 5] follows from Theorem 3.7.

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References

[1] J. M. Borwein and S. Fitzpatrick, *Characterization of Clarke subgradients among one-dimensional multifunctions*, Proceedings of the Optimization Miniconference II, University of Ballarat Press, Australia, 1995, CECM Research Report 94:006, pp. 61–73.

[2] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, 1983.

[3] T. Donchev, *Surjectivity and fixed points of relaxed dissipative multifunctions*, preprint, 2001.

[4] ———, *Properties of one-sided Lipschitz multivalued maps*, Nonlinear Anal. **49** (2002), no. 1, Ser. A: Theory Methods, 13–20.

[5] P. G. Georgiev, *Submonotone mappings in Banach spaces and differentiability of non-convex functions*, C. R. Acad. Bulgare Sci. **42** (1989), no. 6, 13–16.

[6] ———, *Submonotone mappings in Banach spaces and applications*, Set-Valued Anal. **5** (1997), no. 1, 1–35.

[7] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis. Vol. I. Theory*, Mathematics and Its Applications, vol. 419, Kluwer Academic Publishers, Dordrecht, 1997.

[8] P. S. Kenderov, *Continuity-like properties of set-valued mappings*, Serdica **9** (1983), no. 2, 149–160.

[9] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences New York University, New York, 1974.

[10] B. N. Pshenichnyi, *Necessary Conditions for an Extremum*, Pure and Applied Mathematics, vol. 4, Marcel Dekker, New York, 1971.

[11] J. E. Spingarn, *Submonotone subdifferentials of Lipschitz functions*, Trans. Amer. Math. Soc. **264** (1981), no. 1, 77–89.

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