SOLUTIONS TO $H$-SYSTEMS BY TOPOLOGICAL AND ITERATIVE METHODS

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Received 2 May 2002

We study $H$-systems with a Dirichlet boundary data $g$. Under some conditions, we show that if the problem admits a solution for some $(H_0, g_0)$, then it can be solved for any $(H, g)$ close enough to $(H_0, g_0)$. Moreover, we construct a solution of the problem applying a Newton iteration.

1. Introduction

We consider the Dirichlet problem in a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^2$ for a vector function $X : \overline{\Omega} \to \mathbb{R}^3$ which satisfies the equation of prescribed mean curvature

$$
\Delta X = 2H(u, v, X)X_u \wedge X_v \quad \text{in}\, \Omega,
$$

$$
X = g \quad \text{on}\, \partial\Omega,
$$

(1.1)

where $\wedge$ denotes the exterior product in $\mathbb{R}^3$, $H : \overline{\Omega} \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, and the boundary data $g$ is smooth. Problem (1.1) above arises in the Plateau and Dirichlet problems for the prescribed mean curvature equation that has been studied, for example, in [1, 2, 3, 4, 5].

In Section 2, we prove the following theorem.

**Theorem 1.1.** Let $X_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ be a solution of (1.1) for some $(H_0, g_0)$ with $g_0 \in W^{2,p}(\Omega, \mathbb{R}^3)$ ($2 < p < \infty$) and $H_0$ continuously differentiable with respect to $X$ over the graph of $X_0$. Set

$$
k = -2 \inf_{(u,v,Y) \in \Omega \times \mathbb{R}^3, |Y| = 1} \left( \frac{\partial H_0}{\partial X}(u,v,X_0)Y \right) \left( (X_0_u \wedge X_0_v)Y \right)
$$

(1.2)

and assume that

$$
k + 2\sqrt{\lambda_1} \left\| H_0(\cdot, X_0) \right\|_{\infty} \left\| \nabla X_0 \right\|_{\infty} < \lambda_1,
$$

(1.3)
where \( \lambda_1 \) is the first eigenvalue of \(-\Delta\). Then there exists a neighborhood \( \mathcal{B} \) of \((H_0, g_0)\) in the space \( C(\overline{\Omega} \times \mathbb{R}^3, \mathbb{R}) \times W^{2,p}(\Omega, \mathbb{R}^3) \) such that (1.1) is solvable for any \((H, g) \in \mathcal{B}\).

**Remark 1.2.** It is clear that

\[
0 \leq -2 \inf_{(u,v) \in \Omega} \frac{\partial H_0}{\partial X}(u, v, X_0)(X_{0_n} \wedge X_{0}) \leq k \leq 2 \left\| \frac{\partial H_0}{\partial X}(\cdot, X_0) \right\|_\infty \left\| X_{0_n} \wedge X_{0} \right\|_\infty.
\]

(1.4)

Moreover, a simple computation shows that \( k = 0 \) if and only if \((\partial H_0/\partial X)(\cdot, X_0)\) and \(X_{0_n} \wedge X_{0}\) are linearly dependent, with \((\partial H_0/\partial X)(u, v, X_0)(X_{0_n} \wedge X_{0}) \geq 0\) for every \((u, v) \in \Omega\).

In Section 3, we show that the solution provided by Theorem 1.1 can be obtained by a Newton iteration. For simplicity, we consider the case where \( H \) does not depend on \( X \) and prove the following theorem.

**Theorem 1.3.** Let \( X_0 \in W^{2,p}(\Omega, \mathbb{R}^3) \) be a solution of (1.1) for some \((H_0, g_0)\) with \( g_0 \in W^{2,p}(\Omega, \mathbb{R}^3) \) \((2 < p < \infty)\) and \( H_0 \) continuous, and assume that

\[
2 \left\| H_0 \right\|_\infty \left\| \nabla X_0 \right\|_\infty < \sqrt{\lambda_1}.
\]

(1.5)

Then, if \( H \) and \( g \) are close enough to \( H_0 \) and \( g_0 \), respectively, the sequence given by

\[
\Delta X_{n+1} = 2H \left[ (X_{n+1} - X_n)_u + (X_{n+1} - X_n)_v (X_{n+1} - X_n) - X_{n+1} \wedge X_{n} \right],
\]

\[
X_{n+1}|_{\partial \Omega} = g
\]

(1.6)

is well defined and converges in \( W^{2,p}(\Omega, \mathbb{R}^3) \) to a solution of (1.1).

2. Proof of Theorem 1.1

First we will prove a slight extension of a well-known result for linear elliptic second-order operators.

**Lemma 2.1.** Let \( L : W^{2,p}(\Omega, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3) \) be the linear elliptic operator given by

\[
LX = \Delta X + AX_u + BX_v + CX
\]

with \( A, B, C \in L^\infty(\Omega, \mathbb{R}^{3 \times 3}) \) \((2 < p < \infty)\), and assume that \( r := ((\|A\|^2 + \|B\|^2)/\lambda_1)^{1/2} < 1 \) and that \( CY \cdot Y \leq \kappa \|Y\|^2 \) for every \( Y \in \mathbb{R}^3 \) with \( \kappa < \lambda_1(1 - r) \). Then \( L|_{W^{2,p}_0 \cap W^{1,p}_0(\Omega, \mathbb{R}^3)} : W^{2,p}_0 \cap W^{1,p}_0(\Omega, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3) \) is an isomorphism.
Proof. Let $Z_n \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)$ be a sequence such that $\|LZ_n\|_p \to 0$. Then $\|LZ_n\|_2 \to 0$, and from the inequalities

$$-\int LZ_nZ_n \geq \frac{1}{\lambda_1}(\|\nabla Z_n\|_2^p)^{1/2}\|\nabla Z_n\|_2 - \int CZ_nZ_n \geq \left(1 - r - \frac{\kappa}{\lambda_1}\right)\|\nabla Z_n\|_2^2,$$  

we deduce that $\|\nabla Z_n\|_2 \to 0$. Thus, $\|Z_n\|_2 \to 0$ and hence $\|\Delta Z_n\|_p \to 0$. From the invertibility of $\Delta$, there exists a subsequence (still denoted $Z_n$) such that $\|Z_n\|_{1,p} \to 0$. By Sobolev imbedding, $\|Z_n\|_{1,p} \to 0$ and we conclude that $\|\Delta Z_n\|_p \to 0$.

In order to prove that $L$ is onto, it suffices to consider for any $\phi \in L^p(\Omega)$, the homotopy

$$\Delta X = \sigma(\phi - AX_u - BX_v - CX)$$  

and apply a Leray-Schauder argument. □

Now we are able to prove Theorem 1.1. Consider a pair $(H, g)$ with $\|g - g_0\|_{2,p} < \delta$ and $\|(H - H_0)\|_{\infty} < \epsilon$ for some compact $K$ containing a neighborhood of the graph of $X_0$. Setting $Y = X - X_0$, equation (1.1) is equivalent to the problem

$$LY = F(u, v, Y, Y_u, Y_v) \quad \text{in } \Omega,$$

$$Y = g - g_0 \quad \text{on } \partial \Omega,$$  

(2.3)

where $L$ is the linear operator given by

$$LY = \Delta Y - 2H_0(u, v, X_0)\left[X_0_u \wedge Y_v + Y_u \wedge X_0_v\right] - 2\left(\frac{\partial H_0}{\partial X} (u, v, X_0) Y\right) X_0_u \wedge X_0_v,$$  

(2.4)

and

$$F(u, v, Y, Y_u, Y_v) := 2\left(H(u, v, X_0 + Y) Y_u \wedge Y_v \right.$$

$$+ [H(u, v, X_0 + Y) - H_0(u, v, X_0)](X_0_u \wedge Y_v + Y_u \wedge X_0_v)$$

$$+ \left[H(u, v, X_0 + Y) - H_0(u, v, X_0) \frac{\partial H_0}{\partial X} (u, v, X_0) Y\right] X_0_u \wedge X_0_v\Big)$$

(2.5)

We define an operator $T : C^1(\bar{\Omega}, \mathbb{R}^3) \to C^1(\bar{\Omega}, \mathbb{R}^3)$ given by $T(\bar{Y}) = Y$ where $Y$ is the unique solution of the linear problem

$$LY = F(u, v, \bar{Y}, \bar{Y}_u, \bar{Y}_v) \quad \text{in } \Omega,$$

$$Y = g - g_0 \quad \text{on } \partial \Omega.$$  

(2.6)
As \( L \) satisfies the hypothesis of Lemma 2.1, it is immediate to prove that \( T \) is well defined and continuous. Furthermore, the range of a bounded set is bounded with \( \| \cdot \|_{2,p} \), and by Sobolev imbedding, we conclude that \( T \) is compact.

More precisely, for \( \| \mathcal{Y} \|_{1,\infty} \leq R \) we obtain

\[
\| T(\mathcal{Y}) \|_{1,\infty} \leq \| g - g_0 \|_{1,\infty} + c \| T(\mathcal{Y}) - (g - g_0) \|_{2,p} \\
\leq \| g - g_0 \|_{1,\infty} + c_1 \left( \| L(T(\mathcal{Y})) \|_p + \| L(g - g_0) \|_p \right) \\
\leq k_0 \delta + c_1 \| F(\cdot, \overline{\mathcal{Y}}, \overline{\mathcal{Y}}_u, \overline{\mathcal{Y}}_v) \|_p
\]

for some constants \( k_0 \) and \( c_1 \).

On the other hand, a simple computation shows that

\[
\| F(\cdot, \overline{\mathcal{Y}}, \overline{\mathcal{Y}}_u, \overline{\mathcal{Y}}_v) \|_p \leq k_1 R^2 + k_2 \varepsilon R + k_3 \varepsilon
\]

for some constants \( k_1, k_2, \) and \( k_3 \). Hence, if \( \delta \) and \( \varepsilon \) are small, it is possible to choose \( R \) such that \( T(B_R) \subset B_R \) and the result follows by Schauder’s Theorem.

### 3. A Newton iteration for problem (1.1)

In this section, we apply a Newton iteration to (1.1). For simplicity, we will assume that \( H \) does not depend on \( X \).

Let \( X_0 \) be a solution of (1.1) for some \( H_0 \) and \( g_0 \) with

\[
2\| H_0 \|_{\infty} \| \nabla X_0 \|_{\infty} < \sqrt{\lambda_1}.
\]

In order to define a sequence that converges to a solution of (1.1) for \((H, g)\) close to \((H_0, g_0)\), we consider the function \( F : g + (W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)) \to L^p(\Omega, \mathbb{R}^3) \) given by

\[
F(X) = \Delta X - 2HX_u \wedge X_v.
\]

Thus, the problem is equivalent to find a zero of \( F \). The well-known Newton method consists in defining a recursive sequence

\[
X_{n+1} = X_n - (DF(X_n))^{-1}(F(X_n))
\]

or equivalently

\[
DF(X_n)(X_{n+1} - X_n) = -F(X_n).
\]
A simple computation shows that in this case,

$$DF(X)(Y) = \Delta Y - 2H (X_u \wedge Y_v + Y_u \wedge X_v).$$  \hfill (3.5)

According to this, we start at $X_0$ and define the sequence $\{X_n\}$ from the following problem:

$$\Delta X_{n+1} - 2H (X_{n_u} \wedge (X_{n+1} - X_n)_v + (X_{n+1} - X_n)_u \wedge X_n_v) = 2HX_{n_u} \wedge X_{n_v}$$ \hfill (3.6)

with Dirichlet condition

$$X_{n+1}|_{\partial \Omega} = g.$$ \hfill (3.7)

We will prove that if $H$ and $g$ are close enough to $H_0$ and $g_0$, respectively, this sequence is well defined (i.e., $DF(X_n)$ is invertible for every $n$) and converges.

Fix a positive $R$ such that

$$R < \frac{\sqrt{\lambda_1}}{2\|H_0(\cdot, X_0)\|_\infty} - \|\nabla X_0\|_\infty$$ \hfill (3.8)

and set

$$\mathcal{C} = \left\{ X \in W^{2,p}(\Omega, \mathbb{R}^3) : X|_{\partial \Omega} = g, \|X - X_0\|_{2,p} \leq R \right\}.$$ \hfill (3.9)

We will assume that

$$\|H - H_0\|_\infty < \varepsilon, \quad \|g - g_0\|_{2,p} < \delta \leq R$$ \hfill (3.10)

with

$$\varepsilon < \frac{\sqrt{\lambda_1}}{2(\|\nabla X_0\|_\infty + R)} - \|H(\cdot, X_0)\|_\infty.$$ \hfill (3.11)

For each $X \in \mathcal{C}$, we define the linear operator $L_X$ given by

$$L_X Y = \Delta Y - 2H (X_u \wedge Y_v + Y_u \wedge X_v).$$ \hfill (3.12)

By Lemma 2.1, $L_X|_{W_0^{1,p}(\Omega)}$ is invertible for any $X \in \mathcal{C}$. Furthermore, we claim that $\|L_X^{-1}\|$ is bounded over $\mathcal{C}$. Indeed, for $Z \in W^{2,p} \cap W_0^{1,p}(\Omega, \mathbb{R}^3)$ and $X, Y \in \mathcal{C}$, we have

$$\|L_Y Z\|_p \geq \|L_X Z\|_p - \|(L_X - L_Y) Z\|_p \geq \left( \frac{1}{\|L_X^{-1}\|} - 2\|H\|_\infty \|\nabla (X - Y)\|_\infty \right) \|Z\|_{2,p}.$$ \hfill (3.13)
Taking, for example, \( Y \) such that \( \| \nabla (Y - X) \|_\infty \leq 1/(4 \| H \|_\infty \| L^{-1}_X \|) := R_X \), we obtain
\[
\| L^{-1}_Y \| \leq 2 \| L^{-1}_X \|. \tag{3.14}
\]
By compactness, there exist \( X^1, \ldots, X^n \in \mathcal{C} \) such that
\[
\mathcal{C} \subset \bigcup_{i=1}^n \{ Y : \| \nabla (Y - X^i) \|_\infty \leq R_X \} \tag{3.15}
\]
and hence,
\[
\| L^{-1}_X \| \leq 2 \max_{1 \leq i \leq n} \| L^{-1}_X \|. \tag{3.16}
\]
Let \( Z_n = X_{n+1} - X_n \). For \( n = 0 \), we have
\[
\| Z_0 \|_{2,p} \leq \| g - g_0 \|_{2,p} + \| Z_0 - (g - g_0) \|_{2,p} \leq \| g - g_0 \|_{2,p} + c \left( \| L_X Z_0 \|_p + \| L_X (g - g_0) \|_p \right) \leq 2\delta (1 + \| H \|_\infty \| \nabla X_0 \|_\infty) + c \| L_X Z_0 \|_p. \tag{3.17}
\]
As
\[
\| L_X Z_0 \|_p = \| 2(H - H_0) X_0 \wedge X_0 \|_{2,p} \leq \varepsilon \| \nabla X_0 \|_p, \tag{3.18}
\]
we conclude that
\[
\| Z_0 \|_{2,p} \leq 2\delta (1 + (\| H_0 \|_\infty + \varepsilon) \| \nabla X_0 \|_\infty) + c\varepsilon \| \nabla X_0 \|_{2,p} \leq c(\delta, \varepsilon). \tag{3.19}
\]
Then we may establish a more precise version of Theorem 1.3.

**Theorem 3.1.** With the previous notations, assume that
\[
c(\delta, \varepsilon) \leq \frac{R}{1 + R c_0 (\| H_0 \|_\infty + \varepsilon)}, \tag{3.20}
\]
where \( c_0 \) is the constant of the imbedding \( W^{2,p}(\Omega, \mathbb{R}^3) \hookrightarrow C^1(\overline{\Omega}, \mathbb{R}^3) \). Then the sequence given by (1.6) is well defined and converges in \( W^{2,p}(\Omega, \mathbb{R}^3) \) to a solution of (1.1).

**Proof.** By (3.20), we have that \( \| Z_0 \|_{2,p} \leq c(\delta, \varepsilon) \leq R \), proving that \( X_1 \in \mathcal{C} \). For \( n > 0 \), we assume as inductive hypothesis that \( X_k \in \mathcal{C} \) for \( k \leq n \), and then
\[
\| Z_n \|_{2,p} \leq c \| L_X Z_n \|_p = 2c \| H Z_{n-1} \wedge Z_{n-1} \|_p \leq c \| H \|_\infty \| \nabla Z_{n-1} \|_\infty \| \nabla Z_{n-1} \|_p \leq c_0 c \| H \|_\infty \| Z_{n-1} \|_{2,p}^2. \tag{3.21}
\]
Inductively,

\[ \| Z_n \|_{2,p} \leq (c_0 \| H \|_{\infty})^{2^n-1} \| Z_0 \|_{2,p} = A^{2^n-1} \| Z_0 \|_{2,p}, \]  

(3.22)

where \( A = c_0 \| H \|_{\infty} \| Z_0 \|_{2,p} \). By hypothesis, it is immediate that \( A < 1 \), and hence

\[ \| X_{n+1} - X_0 \|_{2,p} \leq \sum_{j=0}^{n} \| Z_j \|_{2,p} \leq \| Z_0 \|_{2,p} \frac{1}{1 - A} \leq R. \]  

(3.23)

Thus, \( X_n \in C \) for every \( n \), and

\[ \| X_{n+k} - X_n \|_{2,p} \leq \frac{A^{2^n-1}}{1 - A} \]  

(3.24)

for every \( k \geq 0 \). Then \( X_n \) is a Cauchy sequence, and the result follows. \( \square \)

**Remark 3.2.** It is clear from definition that \( c(\delta, \varepsilon) \to 0 \) for \( (\delta, \varepsilon) \to (0, 0) \).

**Acknowledgment**

This work was partially supported by UBACYT TX45.

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