LOCAL EXACT CONTROLLABILITY OF THE DIFFUSION EQUATION IN ONE DIMENSION

MARIUS BECEANU

Received 21 December 2002

This paper establishes the local exact null controllability of the diffusion equation in one dimension using distributed controls in the case of the Dirichlet boundary value problem. Most of the techniques used in the course of the proof are borrowed from Barbu (2002).

1. Formulation of the problem

Consider the diffusion equation

$$\begin{gathered} y_t - \big(a(y)\big)_{xx} = mu, \quad \forall (x,t) \in Q = I \times (0,T), \\ y(x,t) = 0, \quad \forall (x,t) \in \Sigma = \partial I \times (0,T), \\ y(x,0) = y_0, \quad x \in I, \end{gathered} \tag{1.1}$$

where $I = (i_1, i_2)$ is a bounded real interval and m is the characteristic function of an open subset ω of I.

The ellipticity of this operator is insured by the condition $0 < \mu \le a'(x)$, for all $x \in \mathbb{R}$. We need $|a'(x)|$, $|a''(x)|$, and $|a'''(x)| \le M < \infty$ as well in order to establish the desired controllability result.

Equation (1.1) is said to be exactly null controllable if there is $u \in L^2(Q)$ such that $y(T) \equiv 0$, where $y \in C([0,T]; L^2(I))$ is the solution to (1.1).

We will first establish some controllability results for the linearized equation

$$\begin{gathered} y_t - \big(a'(\tilde{y}) y_x\big)_x = mu, \quad \forall (x,t) \in Q, \\ y(x,t) = 0, \quad \forall (x,t) \in \Sigma, \\ y(x,0) = y_0, \quad x \in I, \end{gathered} \tag{1.2}$$

where $\tilde{y}$ is some function with $\tilde{y}_x$, $\sqrt{t}\tilde{y}_t \in L^\infty(Q)$, $\tilde{y}_{xt} \in L^2(Q)$, and $\tilde{y}|_\Sigma = 0$. Henceforth, $a'(\tilde{y})$ will be denoted by b.

Abstract and Applied Analysis 2003:14 (2003) 793–811
2000 Mathematics Subject Classification: 93B05, 93C10, 93C35
URL: http://dx.doi.org/10.1155/S1085337503303033

In order to establish the controllability of (1.2), it is sufficient to prove a Carleman-type inequality concerning the dual system

$$\begin{aligned} p_t + (bp_x)_x &= g, \quad \forall (x,t) \in Q, \\ p(x,t) &= 0, \quad \forall (x,t) \in \Sigma, \\ p(x,T) &= p_T, \quad x \in I. \end{aligned} \tag{1.3}$$

However, this only solves the problem for the linearized equation. Our purpose will be to eventually show that we can make $y = \tilde{y}$ in (1.2), since then the controllability of this equation would also imply that of the diffusion equation (1.1).

We will arrive at the desired conclusion by means of the Kakutani fixed-point theorem and Pontryagin's principle. Indeed, under strong regularity conditions for the initial data ($(a(y_0))_{xx}$, $(y_0)_x$, and y_0 belonging to $L^2(I)$ and of sufficiently small norm), we will see that the multifunction $y = \Phi(\tilde{y})$, where y is any solution to (1.2), takes an $L^2(Q)$-compact set into itself. The other conditions for applying Kakutani's theorem are easily met.

The final step will be to prove that we can dispense with at least some of the conditions on the initial data, due to the regularizing properties of the diffusion equation. We are going to show that, for every $y_0 \in H^1(I)$ of sufficiently small norm, the diffusion equation is null controllable.

2. The Carleman inequality

In the following, we will assume the function a to be fixed, and therefore we will not mention the constants μ and M explicitly. However, we will keep track of $\tilde{y}$ and its derivatives.

In the course of the proof, we are going to need the following lemma.

Lemma 2.1. *There exists a function $\psi \in C^2(\bar{I})$ such that $\psi(x) > 0$ for all $x \in I$, $\psi(x) = 0$ on ∂I, and $|\psi_x(x)| > 0$ for all $x \in \bar{I} \setminus \omega_0$, where ω_0 is an open set such that $\overline{\omega_0} \subset I$.*

The proof is obvious (see [1] for a more general case).

Throughout the paper, we will use a fixed ψ and a fixed $\overline{\omega_0} \subset \omega$.

In addition, we define, for any $\lambda > 0$, the functions α and $\phi : Q \to \mathbb{R}$ by

$$\alpha(x,t) = \frac{e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\bar{I})}}}{t(T-t)}, \qquad \phi(x,t) = \frac{e^{\lambda\psi(x)}}{t(T-t)}. \tag{2.1}$$

Note that $\phi(x,t) \geq c > 0$ for all $(x,t) \in Q$, and $e^{\delta\alpha}\phi^k \leq C < \infty$ for all $\delta > 0$, $k \in \mathbb{R}$.

We are now ready to formulate the Carleman inequality concerning (1.3). The proof below is, by many ways, identical to the one given by Imanuvilov in a different case (see [2]).

Theorem 2.2. *For any solution p of the dual system (1.3), with $\|b_x\|_{L^\infty(Q)}$, $\|\sqrt{t}b_t\|_{L^\infty(Q)} \le \rho$ and for every $\lambda \ge \lambda_0(\rho)$, $s \ge s_0(\lambda)$, the following inequality holds:*

$$\begin{aligned}\int_Q e^{2s\alpha}\left(s^3\phi^3p^2 + s\phi p_x^2 + s^{-1}\phi^{-1}\left(p_t^2 + p_{xx}^2\right)\right)\\ \le C(\lambda,\rho)\left(\int_{Q_\omega} e^{2s\alpha}s^3\phi^3p^2 + \int_Q e^{2s\alpha}g^2\right),\end{aligned} \tag{2.2}$$

where $C(\lambda,\rho)$ is a constant independent of p, g, and s, but which may depend on ψ, λ, and ρ, and $Q_\omega = \omega \times (0,T)$.

Proof. This proof follows step by step the one given for [1, Theorem 1.2.1] (the Carleman inequality concerning the heat equation [1, pages 145–152]).

By taking $z = e^{s\alpha}p$ in the dual system (1.3), we obtain

$$\begin{aligned}&z_t - s\alpha_t z + (bz_x)_x - 2s\lambda\phi\psi_x\cdot bz_x + \left(s^2\lambda^2\phi^2\psi_x^2 - s\lambda^2\phi\psi_x^2 - s\lambda\phi\psi_{xx}\right)bz\\ &\qquad - s\lambda\phi\psi_x b_x z = e^{s\alpha}g,\\ &z(0) = z(T) = 0 \quad \text{on } I,\\ &z|_\Sigma = 0.\end{aligned} \tag{2.3}$$

If we set

$$\begin{aligned}B(t)z &= -(bz_x)_x - \left(s^2\lambda^2\phi^2\psi_x^2 + s\lambda^2\phi\psi_x^2\right)bz + s\alpha_t z,\\ X(t)z &= -2s\lambda^2\phi\psi_x^2 bz - 2s\lambda\phi\psi_x\cdot bz_x,\\ Z(t)z &= s\lambda\phi\psi_x b_x z + s\lambda\phi\psi_{xx}bz,\end{aligned} \tag{2.4}$$

the equation can be rewritten as $z_t - B(t)z + X(t)z = Z(t)z + e^{s\alpha}g$. Then, starting with the relation

$$\begin{aligned}\frac{d}{dt}\int_I B(t)z\cdot z\,dx &= \int_I B(t)z_t\cdot z + B(t)z\cdot z_t + B_t(t)z\cdot z\,dx\\ &= 2\int_I B(t)z\left(B(t)z - X(t)z + Z(t)z + e^{s\alpha}g\right)dx + \int_I B_t(t)z\cdot z\,dx\end{aligned} \tag{2.5}$$

and integrating it on $(0,T)$, we obtain

$$2\int_Q (B(t)z)^2 + 2Y = -2\int_Q B(t)z\left(Z(t)z + e^{s\alpha}g\right) - \int_Q B_t(t)z\cdot z, \tag{2.6}$$

where $Y = -\int_Q B\cdot X$, that is,

$$Y = -\int_Q \left(2s\lambda\phi\psi_x\cdot bz_x + 2s\lambda^2\phi\psi_x^2 bz\right)\cdot\left((bz_x)_x + \left(s^2\lambda^2\phi^2\psi_x^2 + s\lambda^2\phi\psi_x^2\right)bz - s\alpha_t z\right). \tag{2.7}$$

Then, we evaluate

$$\int_Q B_t(t)z\cdot z = \int_Q b_t z_x^2 - \int_Q \left(s^2\lambda^2\phi^2\psi_x^2 b + s\lambda^2\phi\psi_x^2 b\right)_t z^2 + s\alpha_{tt}z^2. \tag{2.8}$$

Set $\gamma(\lambda) = e^{2\lambda\|\psi\|_{C(\overline{T})}}$ and we take $s \geq \gamma(\lambda)$ and $\lambda \geq \lambda_0$. In addition, put

$$D(s,\lambda,z) = \int_Q s^3\lambda^3\phi^3 z^2 + s\lambda\phi z_x^2. \tag{2.9}$$

We eventually obtain that

$$\left|\int_Q B_t(t)z\cdot z\right| \leq C\left(1+\|\phi^{-1}b_t\|_{L^\infty(Q)}\right)D(s,\lambda,z). \tag{2.10}$$

Furthermore,

$$\begin{aligned} &2\left|\int_Q B(t)z\left(Z(t)z+e^{s\alpha}g\right)\right| \\ &\qquad \leq \int_Q \left(B(t)z\right)^2 + C\left(1+\|\phi^{-1}b_x^2\|_{L^\infty(Q)}\right)D(s,\lambda,z) + C\int_Q e^{2s\alpha}g^2. \end{aligned} \tag{2.11}$$

From (2.6), (2.10), and (2.11), we obtain

$$Y \leq C\left(1+\|\sqrt{t}b_t\|_{L^\infty(Q)} + \|b_x\|^2_{L^\infty(Q)}\right)D(s,\lambda,z) + C\int_Q e^{2s\alpha}g^2. \tag{2.12}$$

On the other hand, we have the following inequalities that will give lower estimates:

$$\left|\int_Q \left(2s\lambda\phi\psi_x\cdot bz_x + s\lambda\phi\psi_x b_x z\right)s\alpha_t z\right| \leq CD(s,\lambda,z) \tag{2.13}$$

for $s \geq \gamma(\lambda)$; then

$$\begin{aligned} &\left|\int_Q s\lambda\phi\psi_x b_x z\cdot s\alpha_t z\right| \leq C\|b_x\|_{L^\infty(Q)}\cdot D(s,\lambda,z), \\ &\left|\int_Q \left(s\lambda^2\phi\psi_x^2 bz\right)s\alpha_t z\right| \leq CD(s,\lambda,z). \end{aligned} \tag{2.14}$$

Moreover,

$$\begin{aligned} -\int_Q s\lambda^2\phi\psi_x^2 bz\cdot\left(bz_x\right)_x &= \int_Q s\lambda^2\left(\phi\psi_x^2 bz\right)_x\cdot bz_x \\ &\geq \int_Q s\lambda^2\phi\psi_x^2 b^2 z_x^2 - C\left(1+\|b_x\|^2_{L^\infty(Q)}\right)D(s,\lambda,z). \end{aligned} \tag{2.15}$$

Furthermore,

$$\begin{aligned}
&-\int_Q (2s\lambda\phi\psi_x \cdot bz_x)(s^2\lambda^2\phi^2\psi_x^2 + s\lambda^2\phi\psi_x^2)bz \\
&\quad = \int_Q (s^3\lambda^3\phi^3\psi_x^3 b^2 + s^2\lambda^3\phi^2\psi_x^3 b^2)_x \cdot z^2 \\
&\quad \geq \int_Q (3s^3\lambda^4\phi^3\psi_x^4 + 2s^2\lambda^4\phi^2\psi_x^4) \cdot b^2 z^2 - C\Big(1 + \|b_x\|_{L^\infty(Q)}\Big)D(s,\lambda,z).
\end{aligned} \tag{2.16}$$

Finally,

$$-\int_Q 2s\lambda\phi\psi_x \cdot bz_x(bz_x)_x \geq -\int_\Sigma s\lambda\phi\frac{d\psi}{d\nu} \cdot b^2 z_x^2 d\sigma - CD(s,\lambda,\phi). \tag{2.17}$$

Since $\psi > 0$ on I and $\psi = 0$ on ∂I, we have $d\psi/d\nu \leq 0$, and therefore

$$-\int_\Sigma s\lambda\phi\frac{d\psi}{d\nu} \cdot b^2 z_x^2 d\sigma \geq 0. \tag{2.18}$$

Combining inequalities (2.13), (2.14), (2.15), (2.16), and (2.17), we obtain

$$Y \geq \int_Q s^3\lambda^4\phi^3\psi_x^4 b^2 z^2 + s\lambda^2\phi\psi_x^2 b^2 z_x^2 - C\Big(1 + \|b_x\|_{L^\infty(Q)}^2\Big)D(s,\lambda,z). \tag{2.19}$$

From (2.19) and (2.12), we get

$$\begin{aligned}
&\int_Q s^3\lambda^4\phi^3\psi_x^4 b^2 z^2 + s\lambda^2\phi\psi_x^2 b^2 z_x^2 \\
&\quad \leq C\Big(1 + \|\sqrt{t}b_t\|_{L^\infty(Q)} + \|b_x\|_{L^\infty(Q)}^2\Big)D(s,\lambda,z) + C\int_Q e^{2s\alpha}g^2 \\
&\quad \leq C(\rho)D(s,\lambda,z) + C\int_Q e^{2s\alpha}g^2.
\end{aligned} \tag{2.20}$$

Because $b \geq \mu$ and $|\psi_x| \geq c$ on $I \setminus \omega_0$, by making λ sufficiently large, it follows that

$$\begin{aligned}
&\int_Q s^3\lambda^4\phi^3 z^2 + s\lambda^2\phi z_x^2 \\
&\quad \leq C(\rho)\int_{Q_{\omega_0}} s^3\lambda^4\phi^3 z^2 + s\lambda^2\phi z_x^2 + C\int_Q e^{2s\alpha}g^2.
\end{aligned} \tag{2.21}$$

Hence, we can obtain, by exactly the methods used in [1, page 151], that

$$\begin{aligned}
&\int_Q e^{2s\alpha}s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi p_x^2 \\
&\quad \leq C(\rho)\int_{Q_{\omega_0}} e^{2s\alpha}s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi p_x^2 + C\int_Q e^{2s\alpha}g^2.
\end{aligned} \tag{2.22}$$

Choose $\chi \in C_0^\infty(\omega)$ such that $\chi = 1$ in ω_0. If we multiply (1.3) by $e^{2s\alpha}\chi\phi p$ and integrate on Q, we obtain that

$$\begin{aligned}\int_Q e^{2s\alpha}\chi\phi b p_x^2 &= -\int_Q (e^{2s\alpha}\chi\phi)_x \cdot b p p_x + \int_Q e^{2s\alpha}\chi\phi p g + \int_Q \chi(e^{2s\alpha}\phi)_t p^2 \\ &\le C\Big(1 + \|b_x\|_{L^\infty(Q)}\Big)\int_{Q_\omega} e^{2s\alpha}s^2\lambda^2\phi^3 z^2 + C\int_Q e^{2s\alpha}g^2.\end{aligned} \tag{2.23}$$

Consequently, we get

$$\int_Q e^{2s\alpha}s^3\lambda^4\phi^3 p^2 + s\lambda^2\phi p_x^2 \le C(\rho)\int_{Q_\omega} e^{2s\alpha}s^3\lambda^4\phi^3 p^2 + C\int_Q e^{2s\alpha}g^2. \tag{2.24}$$

Equivalently, we may say that, for $\lambda \ge \lambda_0(\rho)$ and $s \ge s_0(\lambda)$,

$$\int_Q e^{2s\alpha}s^3\phi^3 p^2 + s\phi p_x^2 \le C(\rho,\lambda)\int_{Q_\omega} e^{2s\alpha}s^3\phi^3 p^2 + C\int_Q e^{2s\alpha}g^2. \tag{2.25}$$

Next, by squaring (1.3), multiplying it with $e^{2s\alpha}s^{-1}\phi^{-1}$, and integrating on Q, we obtain

$$\int_Q e^{2s\alpha}s^{-1}\phi^{-1}\Big(p_t + (bp_x)_x\Big)^2 \le C\int_Q e^{2s\alpha}g^2. \tag{2.26}$$

At the same time,

$$\begin{aligned}&\int_Q e^{2s\alpha}s^{-1}\phi^{-1}p_t \cdot (bp_x)_x \\ &\quad= -\int_Q (e^{2s\alpha}\phi^{-1}p_t)_x \cdot bp_x \\ &\quad\ge \frac{1}{2}\int_Q (e^{2s\alpha}\phi^{-1})_t \cdot p_x^2 - \frac{1}{2}\int_Q e^{2s\alpha}s^{-1}\phi^{-1}p_t^2 - C\int_Q s\lambda^2\phi p_x^2 \\ &\quad\ge -\frac{1}{2}\int_Q e^{2s\alpha}s^{-1}\phi^{-1}p_t^2 - C\int_Q s\lambda^2\phi p_x^2.\end{aligned} \tag{2.27}$$

By (2.26) and (2.27), we obtain

$$\int_Q e^{2s\alpha}s^{-1}\phi^{-1}\Big(p_t^2 + (bp_x)_x^2\Big) \le C\int_Q e^{2s\alpha}g^2 + C\int_Q s\lambda^2\phi p_x^2. \tag{2.28}$$

Combining inequalities (2.25) and (2.28), we obtain the final result (2.2). □

Proceeding as in [1, Corollary 1.2.1, page 145], one may obtain the following corollary.

Corollary 2.3. *Under the assumptions of Theorem 2.2, the following inequality holds:*

$$\int_I p^2(0)dx \le C(s,\lambda,\rho)\left(\int_{Q_\omega} e^{2s\alpha}s^3\phi^3p^2 + \int_Q e^{2s\alpha}g^2\right), \tag{2.29}$$

where $C(s,\lambda,\rho)$ is a constant that does not depend on p or g.

3. Estimates for y

First, we denote $Q_t = I\times(0,t)$ and $f = mu$. Thus, system (1.2) can be rewritten as

$$\begin{aligned} y_t - (by_x)_x &= f, \quad \forall (x,t)\in Q,\\ y(x,t) &= 0, \quad \forall (x,t)\in \Sigma,\\ y(x,0) &= y_0, \quad x\in I. \end{aligned} \tag{3.1}$$

We will prove a few estimates concerning the solution y of system (3.1).

Multiplying (3.1) by y_t and integrating on Q_t, we get

$$\int_{Q_t} y_t^2 + \int_{Q_t}\frac{d}{dt}by_x^2(t) \le C\left(\int_{Q_t} f^2 + \int_{Q_t} b_t y_x^2\right), \tag{3.2}$$

so

$$\begin{aligned} &\int_Q y_t^2 + \sup_{t\in[0,T]}\int_I y_x^2(t)dx\\ &\quad\le C\left(\int_I y_x^2(0)dx + \int_Q f^2 + \int_0^T \frac{dt}{\sqrt{t}}\cdot \|\sqrt{t}b_t\|_{L^\infty(Q)}\cdot \sup_{t\in[0,T]}\int_I y_x^2(t)dx\right)\\ &\quad\le C_0\left(\int_I y_x^2(0)dx + \int_Q f^2 + \|\sqrt{t}b_t\|_{L^\infty(Q)}\cdot \sup_{t\in[0,T]}\int_I y_x^2(t)dx\right). \end{aligned} \tag{3.3}$$

Making $\|\sqrt{t}b_t\|_{L^\infty(Q)}$ sufficiently small ($\|\sqrt{t}b_t\|_{L^\infty(Q)} \le 1/2C_0$), from (3.3) we infer that $y\in L^\infty((0,T);H_0^1(I))\subset L^\infty(Q)$ and

$$\|y\|^2_{L^\infty(Q)} \le C\sup_{t\in[0,T]}\int_I y_x^2(t)dx \le C\left(\int_I y_x^2(0)dx + \int_Q f^2\right). \tag{3.4}$$

As a consequence of the fact that $(by_x)_x = y_t - f$, we also obtain from (3.3), when $\|tb_t\|_{L^\infty(Q)}$ is sufficiently small, that

$$\int_Q (by_x)_x^2 \le C\left(\int_I y_x^2(0)dx + \int_Q f^2\right). \tag{3.5}$$

Multiplying (3.1) by $(by_x)_{xt}$ and integrating on Q_t, we get

$$\int_{Q_t} y_t(by_x)_{xt} - \int_{Q_t} (by_x)_x(by_x)_{xt} = \int_{Q_t} f(by_x)_{xt}, \tag{3.6}$$

and, by Green's formula,

$$\int_{Q_t} (by_{xt}^2 + b_t y_x y_{xt}) + \frac{1}{2}\int_{Q_t} \frac{d}{dt}(by_x)_x^2 = -\int_{Q_t} f(by_x)_{xt}. \tag{3.7}$$

Through integration by parts, we obtain

$$\begin{aligned}
&-\int_{Q_t} f(by_x)_{xt} \\
&\quad = -\int_I f(by_x)_x dx\Big|_0^t + \int_{Q_t} f_t(by_x)_x \\
&\quad \le \int_I (f^2(0) + f^2(t))dx + \frac{1}{4}\int_I \Big((by_x)_x^2(0) + (by_x)_x^2(t)\Big)dx + \int_{Q_t} (by_x)_x^2 + \int_{Q_t} f_t^2 \\
&\quad \le \frac{1}{4}\int_I \Big((by_x)_x^2(0) + (by_x)_x^2(t)\Big)dx + \int_{Q_t} (by_x)_x^2 + C\int_{Q_t} (f_t^2 + f^2).
\end{aligned} \tag{3.8}$$

An elementary computation also proves that

$$\begin{aligned}
\int_{Q_t} y_{xt} b_t y_x &\le \frac{1}{4c}\int_{Q_t} y_{xt}^2 + c\int_0^t \|b_t(s)\|_{L^\infty(I)}^2 ds \cdot \sup_{s\in[0,t]} \int_I y_x^2(s)dx \\
&\le \frac{1}{4c}\int_{Q_t} y_{xt}^2 + C\|b_{xt}\|_{L^2(Q)}^2 \cdot \sup_{s\in[0,T]} \int_I y_x^2(s)dx.
\end{aligned} \tag{3.9}$$

Applying the preceding estimates into the equality (3.7) gives us

$$\begin{aligned}
&\int_{Q_t} by_{xt}^2 + \int_I (by_x)_x^2(t)dx \\
&\le C\left(\int_I (by_x)_x^2(0)dx + \int_{Q_t} (by_x)_x^2 + \int_{Q_t} (f_t^2 + f^2) + \|b_{xt}\|_{L^2(Q)}^2 \cdot \sup_{s\in[0,t]} \int_I y_x^2(s)dx\right).
\end{aligned} \tag{3.10}$$

Taking into account (3.4) and (3.5), for any $t \in [0,T]$, it is true that

$$\begin{aligned}\int_Q y_{xt}^2 + \int_I (by_x)_x^2(t)dx \\ \leq C\left(\int_I (by_x)_x^2(0)dx + \int_Q f_t^2 + \left(1+\|b_{xt}\|_{L^2(Q)}^2\right)\left(\int_I y_x^2(0)dx + \int_Q f^2\right)\right).\end{aligned} \tag{3.11}$$

One consequence of this relation and of (3.4) is that $by_x \in L^\infty((0,T);H^1(I)) \subset L^\infty(Q)$, and we get the estimate

$$\begin{aligned}\|y_x\|_{L^\infty(Q)}^2 \\ \leq C\left(\int_I (by_x)_x^2(0)dx + \int_Q f_t^2 + \left(1+\|b_{xt}\|_{L^2(Q)}^2\right)\left(\int_I y_x^2(0)dx + \int_Q f^2\right)\right).\end{aligned} \tag{3.12}$$

Taking the derivative of (3.1) with respect to t, we obtain

$$y_{tt} - (by_{tx})_x - (b_t y_x)_x = f_t. \tag{3.13}$$

Multiplying this with ty_{tt} and integrating on Q_t, we have

$$\int_{Q_t} ty_{tt}^2 + \int_{Q_t} tby_{xt}y_{xtt} = \int_{Q_t} t((b_t y_x)_x + f_t)y_{tt}. \tag{3.14}$$

It follows that

$$\int_{Q_t} ty_{tt}^2 + \frac{1}{2}\int_{Q_t} \frac{d}{dt}(tby_{xt}^2) \leq C\left(\int_{Q_t} tf_t^2 + t(b_t y_x)_x^2 + \int_{Q_t} (tb_t + b)y_{xt}^2\right). \tag{3.15}$$

Then, since $(b_t y_x)_x = b_{tx}y_x + b_t y_{xx}$,

$$\begin{aligned}\int_{Q_t} t(b_t y_x)_x^2 &\leq 2\left(\int_Q tb_{tx}^2 y_x^2 + \int_Q tb_t^2 y_{xx}^2\right) \\ &\leq C\|b_{xt}\|_{L^2(Q)}^2 \cdot \|y_x\|_{L^\infty(Q)}^2 + \|\sqrt{t}b_t\|_{L^\infty(Q)}^2 \cdot \int_Q y_{xx}^2.\end{aligned} \tag{3.16}$$

Furthermore, it is clearly true that

$$\begin{aligned}\int_{Q_t} tb_t y_{xt}^2 &\leq \int_0^t \frac{ds}{\sqrt{s}} \cdot \|\sqrt{t}b_t\|_{L^\infty(Q)} \cdot \sup_{s\in[0,t]} \int_I sy_{xt}^2(s)dx \\ &\leq C\|\sqrt{t}b_t\|_{L^\infty(Q)} \cdot \sup_{t\in[0,T]} \int_I ty_{xt}^2(t)dx.\end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned}
&\int_Q y_{tt}^2 + \sup_{t\in[0,T]} \int_I t y_{xt}^2(t)dx \\
&\quad \le C_1 \Bigg(\int_Q f_t^2 + \|y_x\|_{L^\infty(Q)}^2 \cdot \|b_{xt}\|_{L^2(Q)}^2 \\
&\qquad + \|\sqrt{t} b_t\|_{L^\infty(Q)}^2 \int_Q y_{xx}^2 + \int_Q y_{xt}^2 + \|\sqrt{t} b_t\|_{L^\infty(Q)} \cdot \sup_{t\in[0,T]} \int_I t y_{xt}^2(t)dx \Bigg).
\end{aligned} \tag{3.18}$$

Since $by_{xx} = (by_x)_x - b_x y_x$, from (3.4) and (3.5) we can infer that

$$\int_Q y_{xx}^2 \le C\Big(1 + \|b_x\|_{L^\infty(Q)}^2\Big)\Bigg(\int_I y_x^2(0)dx + \int_Q f^2\Bigg). \tag{3.19}$$

Making this substitution, as well as those possible in virtue of (3.12) and (3.11), into (3.18), we get (for every b with $\|\sqrt{t} b_t\|_{L^\infty(Q)} \le 1/2C_1$)

$$\begin{aligned}
&\int_Q y_{tt}^2 + \sup_{t\in[0,T]} \int_I t y_{xt}^2(t)dx \\
&\le C\Bigg(\int_Q f_t^2 + \Big(\|b_{xt}\|_{L^2(Q)}^2 + \|\sqrt{t} b_t\|_{L^\infty(Q)}^2 \Big(1 + \|b_x\|_{L^\infty(Q)}^2\Big)\Big)\Bigg(\int_I y_x^2(0)dx + \int_Q f^2\Bigg) \\
&\qquad + \int_I (by_x)_x^2(0)dx + \int_Q f_t^2 + \Big(1 + \|b_{xt}\|_{L^2(Q)}^2\Big)\Bigg(\int_I y_x^2(0)dx + \int_Q f^2\Bigg)\Bigg),
\end{aligned} \tag{3.20}$$

which becomes (after sorting out the terms)

$$\begin{aligned}
&\int_Q y_{tt}^2 + \sup_{t\in[0,T]} \int_I t y_{xt}^2(t)dx \\
&\le C\Bigg(\int_Q f_t^2 + \int_I (by_x)_x^2(0)dx \\
&\qquad + \Big(1 + \|b_{xt}\|_{L^2(Q)}^2 + \|\sqrt{t} b_t\|_{L^\infty(Q)}^2 \Big(1 + \|b_x\|_{L^\infty(Q)}^2\Big)\Big)\Bigg(\int_I y_x^2(0)dx + \int_Q f^2\Bigg)\Bigg).
\end{aligned} \tag{3.21}$$

As a consequence, we see that $\sqrt{t} y_t \in L^\infty((0,T); H_0^1(I)) \subset L^\infty(Q)$ (with $\|\sqrt{t} y_t\|_{L^\infty(Q)} \le C\|\sqrt{t} y_{xt}\|_{L^\infty((0,T);L^2(I))}$).

Coming back to $\tilde{y}$ (recall that $b = a'(\tilde{y})$), we note that

$$\begin{aligned}
&\|b_x\|_{L^\infty(Q)} \le C\|\tilde{y}_x\|_{L^\infty(Q)}, \qquad \|\sqrt{t}b_t\|_{L^\infty(Q)} \le C\|\sqrt{t}\tilde{y}_t\|_{L^\infty(Q)},\\
&\|b_{xt}\|_{L^2(Q)} \le C\Big(\|\tilde{y}_x\|_{L^\infty(Q)} \cdot \|\tilde{y}_t\|_{L^2(Q)} + \|\tilde{y}_{xt}\|_{L^2(Q)}\Big)\\
&\qquad\qquad \le C\|\tilde{y}_x\|_{L^\infty(Q)}\Big(1 + \|\tilde{y}_{xt}\|_{L^2(Q)}\Big).
\end{aligned} \tag{3.22}$$

We also recall that $f = mu$.

Taking into account all these facts, we obtain

$$\begin{aligned}
\|y_x\|^2_{L^\infty(Q)} + \|y_{xt}\|^2_{L^2(Q)} &\le C\Big(1 + \|\tilde{y}_x\|^2_{L^\infty(Q)}\Big(1 + \|\tilde{y}_{xt}\|^2_{L^2(Q)}\Big)\Big)\\
&\quad \cdot \Big(\int_I (by_x)^2_x(0) + y_x^2(0)dx + \int_Q u_t^2 + u^2\Big),\\
\|\sqrt{t}y_t\|^2_{L^\infty(Q)} &\le C\Big(1 + \|\sqrt{t}\tilde{y}_t\|^2_{L^\infty(Q)}\Big)\Big(1 + \|\tilde{y}_x\|^2_{L^\infty(Q)}\Big(1 + \|\tilde{y}_{xt}\|^2_{L^2(Q)}\Big)\Big)\\
&\quad \cdot \Big(\int_I (by_x)^2_x(0) + y_x^2(0)dx + \int_Q u_t^2 + u^2\Big),
\end{aligned} \tag{3.23}$$

for all $\tilde{y}$ with $\|\sqrt{t}\tilde{y}_t\|_{L^\infty(Q)} \le C_2 = (1/\mu)\max(1/2C_0, 1/2C_1)$, where C_2 is a constant which does not depend on $\tilde{y}$, u, or y.

Further combining these results, we arrive at

$$\begin{aligned}
&\|y_x\|^2_{L^\infty(Q)} + \|\sqrt{t}y_t\|^2_{L^\infty(Q)} + \|y_{xt}\|^2_{L^2(Q)}\\
&\quad \le C\Big(1 + \|\sqrt{t}\tilde{y}_t\|^2_{L^\infty(Q)}\Big)\\
&\qquad \cdot \Big(1 + \|\tilde{y}_x\|^4_{L^\infty(Q)} + \|\tilde{y}_{xt}\|^4_{L^2(Q)}\Big)\Big(\int_I (by_x)^2_x(0) + y_x^2(0)dx + \int_Q u_t^2 + u^2\Big).
\end{aligned} \tag{3.24}$$

4. Optimization and the main result

Now, we are ready to establish the main theorem, first for y_0 in a narrower class of functions, then for $y_0 \in H^1(I)$.

Theorem 4.1. *For any $\delta > 0$, there exists $\eta > 0$ such that, for every $y_0 \in L^2(I)$ with $\|(a(y_0))_{xx}\|_{L^2(I)} + \|(y_0)_x\|_{L^2(I)} + \|y_0\|_{L^2(I)} \le \eta$, (1.1) is exactly null controllable, with a controller u satisfying*

$$\int_Q e^{2s(\delta-1)\alpha}(u^2 + u_t^2) \le C_3(\delta)\|y_0\|^2_{L^2}. \tag{4.1}$$

Proof. We define

$$K = \left\{ y \mid \|y_x\|_{L^\infty(Q)} \le \rho,\ \|\sqrt{t}\, y_t\|_{L^\infty(Q)} \le \rho,\ \|y_{xt}\|_{L^2(Q)} \le \rho,\ y(0) = y_0 \right\}. \tag{4.2}$$

Note that this set is compact in the $L^2(Q)$ topology. Indeed, it can be easily seen that the set is closed, and the fact that $\|y\|_{H^1(Q)}$ is bounded insures its precompactness.

Consider the linearized equation (1.2) for $\tilde{y} \in K$. By the definition of K, for all $\tilde{y} \in K$, we have a fixed ρ (which does not depend on $\tilde{y}$) in the Carleman inequality (Theorem 2.2). Setting a sufficiently small ρ ($\rho \le C_4$), we also find that we can apply the main result of Section 3, (3.24), for all the elements of K.

For brevity, in the following we will denote any constant which does not depend on p, y, $\tilde{y}$, g, and u by C (such a constant may, however, depend on s, λ, ρ, or δ).

Consider the optimal control problem: minimize

$$\int_Q e^{-2s\alpha}\phi^{-3}u^2 + \epsilon^{-1}\int_I y^2(T)\,dx \tag{4.3}$$

subject to (1.2). By the Pontryagin principle, this problem has a unique solution (u_ϵ, y_ϵ). We have $u_\epsilon = m p_\epsilon e^{2s\alpha}\phi^3$, where p_ϵ is a solution of the dual system

$$\begin{aligned} (p_\epsilon)_t + \left(b\cdot(p_\epsilon)_x\right)_x &= 0, \quad \forall (x,t) \in Q, \\ p_\epsilon(x,t) &= 0, \quad \forall (x,t) \in \Sigma, \\ p_\epsilon(x,T) &= -\frac{1}{\epsilon} y_\epsilon(x,T), \quad \forall x \in I. \end{aligned} \tag{4.4}$$

Multiplying (4.4) by y_ϵ and (1.2) by p_ϵ, adding the two equations together, and integrating on Q, we get

$$\begin{aligned} \int_{Q_\omega} e^{2s\alpha}\phi^3 p_\epsilon^2 + \epsilon^{-1}\int_I y_\epsilon^2(T)\,dx &= -\int_I y_0 p(0)\,dx \\ &\le \frac{1}{2k}\int_I y_0^2\,dx + \frac{k}{2}\int_I p(0)^2\,dx. \end{aligned} \tag{4.5}$$

Applying Corollary 2.3 and choosing a small enough k in (4.5), we further obtain

$$\int_{Q_\omega} e^{2s\alpha}\phi^3 p_\epsilon^2 + \epsilon^{-1}\int_I y_\epsilon^2(T)\,dx \le C\int_I y_0^2\,dx. \tag{4.6}$$

Clearly, considering the Carleman inequality (2.2),

$$\begin{aligned}
\int_Q e^{2s(\delta-1)\alpha}u_\epsilon^2 &= \int_{Q_\omega} e^{2s\alpha}\phi^3 p_\epsilon^2 \cdot e^{2s\delta\alpha}\phi^3 \le C\int_{Q_\omega} e^{2s\alpha}\phi^3 p_\epsilon^2,\\
\int_Q e^{2s(\delta-1)\alpha}(u_\epsilon)_t^2 &= \int_{Q_\omega} e^{2s(\delta-1)\alpha}\left(e^{2s\alpha}\phi^3(p_\epsilon)_t + 2se^{2s\alpha}\alpha_t\phi^3 p_\epsilon + 3e^{2s\alpha}\phi^2\phi_t p_\epsilon\right)^2\\
&\le C\left(\int_Q e^{2s\alpha}\phi^{-1}(p_\epsilon)_t^2 \cdot e^{2s\delta\alpha}\phi^7 + \int_{Q_\omega} e^{2s\alpha}\phi^3 p_\epsilon^2 \cdot e^{2s\delta\alpha}\phi^7\right)\\
&\le C\int_{Q_\omega} e^{2s\alpha}\phi^3 p_\epsilon^2.
\end{aligned} \tag{4.7}$$

Combining (4.7) and (4.6) gives

$$\int_Q e^{2s(\delta-1)\alpha}\left(u_\epsilon^2 + (u_\epsilon)_t^2\right) + \epsilon^{-1}\int_I y_\epsilon^2(T)dx \le C_3\int_I y_0^2 dx. \tag{4.8}$$

Taking $\delta = 1$ and applying (3.24) (and also considering that $\|\tilde{y}_x\|_{L^\infty(Q)} \le \rho$, $\|\sqrt{t}\tilde{y}_t\|_{L^\infty(Q)} \le \rho$, $\|\tilde{y}_{xt}\|_{L^2(Q)} \le \rho$) give

$$\begin{aligned}
&\left\|(y_\epsilon)_x\right\|_{L^\infty(Q)}^2 + \left\|\sqrt{t}(y_\epsilon)_t\right\|_{L^\infty(Q)}^2 + \left\|(y_\epsilon)_{xt}\right\|_{L^2(Q)}^2\\
&\quad\le C(1+\rho^2)(1+2\rho^4)\left(\int_I \left(b\cdot(y_0)_x\right)_x^2 + (y_0)_x^2 dx + \int_Q (u_\epsilon)_t^2 + u_\epsilon^2\right)\\
&\quad\le C_4(1+\rho^6)\int_I \left(b\cdot(y_0)_x\right)_x^2 + (y_0)_x^2 + y_0^2 dx.
\end{aligned} \tag{4.9}$$

Furthermore, since $b(0) = a'(\tilde{y}(0)) = a'(y_0)$, we obtain

$$\int_I \left(b\cdot(y_0)_x\right)_x^2 + (y_0)_x^2 + y_0^2 dx = \int_I (a(y_0))_{xx}^2 + (y_0)_x^2 + y_0^2 dx, \tag{4.10}$$

hence the condition imposed on y_0 in the statement of the theorem. Thus we get

$$\begin{aligned}
&\left\|(y_\epsilon)_x\right\|_{L^\infty(Q)}^2 + \left\|\sqrt{t}(y_\epsilon)_t\right\|_{L^\infty(Q)}^2 + \left\|(y_\epsilon)_{xt}\right\|_{L^2(Q)}^2\\
&\qquad\le C_4(1+\rho^6)\int_I (a(y_0))_{xx}^2(0) + (y_0)_x^2 + y_0^2 dx.
\end{aligned} \tag{4.11}$$

By making

$$\int_I (a(y_0))_{xx}^2 + (y_0)_x^2 + y_0^2 dx < \eta \tag{4.12}$$

for a sufficiently small η (such as $\eta = \rho/C_4(1+\rho^6)$), we obtain that $y_\epsilon \in K$.

In conclusion, by (4.8) with $\delta = 1$, u_ϵ and $(u_\epsilon)_t$ belong to a bounded set in $L^2(Q)$, the same for every ϵ, and we have also proved that y_ϵ belongs to K, a set that is compact in $L^2(Q)$.

Choose a sequence of (u_ϵ, y_ϵ), $\epsilon \to 0$, that achieves the optimum in expression (4.3). From the above, it follows that (on a subsequence)

$$\begin{aligned} u_\epsilon &\longrightarrow u \quad \text{weakly in } L^2(Q), \\ (u_\epsilon)_t &\longrightarrow v \quad \text{weakly in } L^2(Q), \\ y_\epsilon &\longrightarrow y \quad \text{strongly in } L^2(Q). \end{aligned} \tag{4.13}$$

Obviously, $v = u_t$. Other immediate consequences are that $(y_\epsilon)_t$ tends to y_t in $H^{-1}(Q)$ and that $(b(y_\epsilon)_x)_x$ tends to $(by_x)_x$ in $L^2((0,T);H^{-2}(I))$; then (u,y) satisfies the linearized equation (1.2). Furthermore, the strong convergence of y_ϵ implies that $y \in K$. Last but not least, by making $\epsilon \to 0$ in (4.6), we see that $y(T) \equiv 0$.

As for $\|e^{s(1-\delta)\alpha}u\|_{L^2(Q)}$ and $\|e^{s(1-\delta)\alpha}u_t\|_{L^2(Q)}$, we recall that, for any sequence $(f_n)_{n\in\mathbb{N}}$ that converges weakly to f in L^2, we have $\|f\|_{L^2(Q)} \le \liminf \|f_n\|_{L^2(Q)}$.

Thus, for any auxiliary function $\tilde{y} \in K$, we have obtained a pair (y,u) that satisfies system (1.2), with $y \in K$, $y(T) \equiv 0$, and

$$\int_Q e^{2s(\delta-1)\alpha}(u^2 + u_t^2) \le C_3 \|y_0\|_{L^2}^2. \tag{4.14}$$

We are now ready to apply Kakutani's theorem. Consider $\Phi : K \to 2^K$,

$$\begin{aligned} \Phi(\tilde{y}) = \Big\{ & y \mid y \in K,\ y(T) \equiv 0, \text{ and } \exists u \in H^1([0,T];L^2(I)), \\ & \text{with } \int_Q e^{2s(\delta-1)\alpha}(u^2 + u_t^2) \le C_3 \|y_0\|_{L^2}, \text{ such that } (u,y) \text{ satisfies (1.2)} \Big\}. \end{aligned} \tag{4.15}$$

Clearly Φ is well defined, takes nonempty values for every $\tilde{y} \in K$, and has convex values.

In order to prove that Φ has closed values, first we fix a $\tilde{y}$, consider a convergent sequence

$$y_n \longrightarrow y \quad \text{in the } L^2(Q) \text{ norm}, \tag{4.16}$$

with $y_n \in \Phi(\tilde{y})$, and choose the corresponding u_n as in the definition of Φ (4.15). Since $y_n \in K$, for all n, and K is compact, it follows that their limit y

is also in K. Furthermore,

$$\begin{aligned} ||y_n(T) - y(T)||_{L^2(I)} &\le ||y_n - y||_{C([0,T];L^2(I))} \\ &\le ||y_n - y||^{1/2}_{H^1([0,T];L^2(I))} ||y_n - y||^{1/2}_{L^2(Q)} \\ &\le C\rho^{1/2} \cdot ||y_n - y||^{1/2}_{L^2(Q)}, \end{aligned} \tag{4.17}$$

so $y(T) \equiv 0$.

Since $\int_Q e^{2s(\delta-1)\alpha}(u_n^2 + (u_n)_t^2) \le C_3 \| y_0 \|^2_{L^2(I)}$, we may select a weakly convergent subsequence which tends to a limit u with $\int_Q e^{2s(\delta-1)\alpha}(u^2 + u_t^2) \le C_3 \| y_0 \|^2_{L^2(I)}$ (the proof is as above). It can be seen that $(y_n)_t \to y_t$ strongly in the $H^{-1}(Q)$ norm and $(b \cdot (y_n)_x)_x \to (b y_x)_x$ strongly in the $L^2((0,T);H^{-2}(I))$ norm. By passing to the weak limit in (1.2), satisfied by (u_n, y_n), we obtain that (u, y) also satisfies the equation for the same $\tilde{y}$. Therefore, $y \in \Phi(\tilde{y})$, and thus $\Phi(\tilde{y})$ is closed; in fact, since $\Phi(\tilde{y}) \subset K$, it follows that $\Phi(\tilde{y})$ is compact for every $\tilde{y} \in K$.

In such a case, the lower semicontinuity of Φ can be obtained from the fact that it has a closed graph. Indeed, if $\tilde{y}_n \in K$, $\tilde{y}_n \to \tilde{y}$, and $y_n \in \Phi(\tilde{y}_n) \to y$ in the $L^2(Q)$ norm, consider the corresponding u_n, as above, and we obtain (on a subsequence) that

$$\begin{aligned} u_n &\longrightarrow u && \text{weakly in } L^2(Q), \\ (u_n)_t &\longrightarrow u_t && \text{weakly in } L^2(Q), \text{ with } \int_Q e^{2s(\delta-1)\alpha}(u^2 + u_t^2) \le C_3 ||y_0||_{L^2}; \\ (y_n)_t &\longrightarrow y_t && \text{strongly in } H^{-1}(Q); \\ (y_n)_x &\longrightarrow y_x && \text{strongly in } L^2((0,T);H^{-1}(I)); \\ \tilde{y}_n &\longrightarrow \tilde{y} && \text{strongly in } L^2(Q). \end{aligned} \tag{4.18}$$

Since $|a'(\tilde{y}_n) - a'(\tilde{y})| \le C|y_n - y|$, it also follows that $a'(\tilde{y}_n) \to a'(\tilde{y})$ strongly in $L^2(Q)$ (and weakly-star in $L^\infty(Q)$). Then $a'(\tilde{y}_n)(y_n)_x \to a'(\tilde{y})y_x$ weakly in $L^2((0,T);H^{-1}(I))$, or, equivalently, $(a'(\tilde{y}_n)(y_n)_x)_x \to (a'(\tilde{y})y_x)_x$ weakly in $L^2((0,T);H^{-2}(I))$. By going to the weak limit in (1.2), we obtain that the pair (u, y) also satisfies the linearized system, with $b = a'(\tilde{y})$. The other conditions ($\tilde{y}$, $y \in K$, $y(T) \equiv 0$) are obviously satisfied (see above the details of the proof), so $(\tilde{y}, y)$ belongs to the graph of Φ.

Thus we can apply Kakutani's theorem and obtain that there is $y \in K$ such that $y \in \Phi(y)$. Such a y is a solution of the diffusion equation (1.1) with $y(0) = y_0$ and $y(T) \equiv 0$. In addition, its controller u satisfies the required estimate. □

Now, we come back to the more general case, $y_0 \in H^1(I)$.

Theorem 4.2. *For any $\delta > 0$, there exists $\eta > 0$ such that, for every $y_0 \in H^1(I)$ with $\|y_0\|_{H^1(I)} \le \eta$, (1.1) is exactly null controllable, with a controller u satisfying*

$$\int_Q e^{2s(\delta-1)\alpha}(u^2 + u_t^2) \le C_3 \|y_0\|^2_{L^2(I)}. \tag{4.19}$$

Proof. We are going to divide the interval $[0, T]$ into two parts by choosing $0 < T_0 < T$. On the first part of the interval, we will make $u \equiv 0$; the equation becomes

$$\begin{aligned} y_t - (by_x)_x &= 0, \quad \forall (x,t) \in Q, \\ y(x,t) &= 0, \quad \forall (x,t) \in \Sigma, \\ y(x,0) &= y_0, \quad x \in I, \end{aligned} \tag{4.20}$$

where we have renamed $I \times [0, T_0] \equiv Q_{T_0} \equiv Q$ and $\partial I \times [0, T_0] \equiv \Sigma$ for greater convenience. We will use the regularizing properties of (4.20), eventually obtaining that

$$\int_I (a(y))^2_{xx}(T_0) + y_x^2(T_0) + y^2(T_0)\,dx \le C \int_I y_0^2 + y_x^2(0)\,dx. \tag{4.21}$$

Then, applying Theorem 4.1, we will be able to establish the null controllability of y on the interval $[T_0, T]$. Finally, we note that the function u defined by

$$u(x,t) = \begin{cases} 0, & \text{for } t \in [0, T_0], \\ u^*(x,t), & \text{for } t \in [T_0, T], \end{cases} \tag{4.22}$$

where u^* is a controller for y on $[T_0, T]$, still has all the desired properties (it belongs to $H^1([0,T]; L^2(I))$ and it satisfies (4.19)). Then the proof is complete.

Now, we establish estimate (4.21). For the beginning, we will only assume in (4.20) that $b_t \in L^2(Q)$ and $b_x \in L^\infty([0, T_0]; L^2(I))$.

Multiplying (4.20) by y and integrating on Q_t, we get

$$\int_I y^2(t)\,dx + 2\int_{Q_t} by_x^2 = \int_I y^2(0)\,dx. \tag{4.23}$$

Multiplying the same equation by y_t and integrating by parts for t, we get

$$\int_{Q_t} y_t^2 + \frac{1}{2}\int_I by_x^2(t) = \frac{1}{2}\int_I by_x^2(0) + \frac{1}{2}\int_{Q_t} b_t y_x^2. \tag{4.24}$$

Then, it is true that

$$\int_{Q_t} y_t^2 + \int_I y_x^2(t)\,dx \le C\left(\int_I y_x^2(0)\,dx + \left(\int_{Q_t} b_t^2\right)^{1/2} \cdot \left(\int_{Q_t} y_x^4\right)^{1/2}\right), \tag{4.25}$$

and, consequently, (since $0 \le t \le T_0$)

$$\int_Q y_t^2 + \sup_{t\in[0,T_0]} \int_I y_x^2(t)dx \le C\left(\int_I y_x^2(0)dx + ||b_t||_{L^2(Q)} \cdot ||y_x||_{L^2((0,T_0);L^\infty(I))} \cdot \sup_{t\in[0,T_0]} ||y_x(t)||_{L^2(I)}\right). \tag{4.26}$$

We note that $\|y_x\|_{L^2((0,T_0);L^\infty(I))} \le C\|(by_x)_x\|_{L^2(Q)} = C\|y_t\|_{L^2(Q)}$, so

$$\int_Q y_t^2 + \sup_{t\in[0,T_0]} \int_I y_x^2(t)dx \le C_5\left(\int_I y_x^2(0)dx + ||b_t||_{L^2(Q)}\left(\int_Q y_t^2 + \sup_{t\in[0,T_0]} \int_I y_x^2(t)dx\right)\right). \tag{4.27}$$

Making $\|b_t\|_{L^2(Q)}$ sufficiently small (e.g., $\|b_t\|_{L^2(Q)} \le 1/2C_5$), we obtain

$$\int_Q y_t^2 + \sup_{t\in[0,T_0]} \int_I y_x^2(t)dx \le C\int_I y_x^2(0)dx. \tag{4.28}$$

As an immediate consequence, we also get $\int_Q (by_x)_x^2 \le C\int_I y_x^2(0)$ (this will be useful later on).

Rewriting (4.28) and recalling that, by definition, $b = a'(\tilde{y})$, we obtain

$$||y_t||_{L^2(Q)} + \sup_{t\in[0,T_0]} ||y_x(t)||_{L^2(I)} \le C_6||y_x(0)||_{L^2(I)} \tag{4.29}$$

if $\|\tilde{y}_t\|_{L^2(Q)} \le 1/2MC_5$.

Consider the set

$$K = \left\{y \mid ||y_t||_{L^2(Q)} + \sup_{t\in[0,T_0]} ||y_x(t)||_{L^2(I)} \le \frac{1}{2MC_5},\ y(0) = y_0\right\}, \tag{4.30}$$

which is clearly compact in $L^2(Q)$ (being closed in $L^2(Q)$ and bounded in $H^1(Q)$). Then, define $\Phi : K \to 2^K$ by

$$\Phi(\tilde{y}) = \{y \mid y \in K,\ y \text{ is a solution of (4.20)}\}. \tag{4.31}$$

For $\|y_x(0)\|_{L^2(I)} \le 1/(C_6 \cdot 2MC_5)$, the multifunction Φ has nonempty values everywhere. It can be easily checked that it also has convex values.

Consider a sequence of solutions $y_n \in \Phi(\tilde{y})$ which converges in $L^2(Q)$ to a limit y. Then, $(y_n)_t \to y_t$ in $H^{-1}(Q)$ and $(b \cdot (y_n)_x)_x \to (by_x)_x$ in $L^2((0,T_0); H^{-2}(I))$. By going to the limit in (4.20) for y_n, we obtain that $y_t - (by_x)_x = 0$ in the weak sense (in $H^{-2}(Q)$), and therefore $y \in \Phi(\tilde{y})$. Thus we have proved that Φ has closed values (compact, as a matter of fact).

All that is left is to show that Φ has a closed graph (and thus is upper semi-continuous). Indeed, if we take two sequences y_n and $\tilde{y}_n$, $y_n \in \Phi(\tilde{y}_n)$, converging to y and to $\tilde{y}$, respectively, we successively obtain, in the same manner as in the proof of Theorem 4.1, that

$$\begin{aligned}
&(y_n)_t \longrightarrow y_t && \text{in } H^{-1}(Q),\\
&(y_n)_x \longrightarrow y_x && \text{in } L^2((0,T_0); H^{-1}(I)),\\
&\tilde{y}_n \longrightarrow \tilde{y} && \text{in } L^2(Q),\\
&a'(\tilde{y}_n) \longrightarrow a'(\tilde{y}) && \text{weakly-star in } L^\infty(Q),\\
&a'(\tilde{y}_n) \cdot (y_n)_x \longrightarrow a'(\tilde{y}) y_x && \text{in } L^2((0,T_0); H^{-1}(I)),\\
&\left(a'(\tilde{y}_n) \cdot (y_n)_x\right)_x \longrightarrow (a'(\tilde{y}) y_x)_x && \text{in } L^2((0,T_0); H^{-2}(I)).
\end{aligned} \tag{4.32}$$

By going to the limit in (4.20), satisfied by all the functions y_n with $b = a'(\tilde{y}_n)$, we obtain that $y \in \Phi(\tilde{y})$, so the graph is closed.

Now, all the conditions needed to apply Kakutani's theorem are fulfilled and we obtain that Φ has a fixed point $y \in \Phi(y)$, and therefore the diffusion equation

$$\begin{gathered}
y_t - (a(y))_{xx} = 0, \quad \forall (x,t) \in Q = I \times (0,T_0),\\
y(x,t) = 0, \quad \forall (x,t) \in \Sigma = \partial I \times (0,T_0),\\
y(x,0) = y_0, \quad x \in I,
\end{gathered} \tag{4.33}$$

has a solution y with

$$\|y_t\|_{L^2(Q)} + \sup_{t\in[0,T_0]} \|y_x(t)\|_{L^2(I)} \le \underbrace{\frac{1}{2MC_5}}_{C} \|y_x(0)\|_{L^2(I)}, \tag{4.34}$$

for $\|y_x(0)\|_{L^2(I)} \le 1/(C_6 \cdot 2MC_5)$.

However, the solution of (4.33) has further regularity properties, of which we are going to employ only one. For convenience, we will keep the notation $b = a'(\tilde{y}) \equiv a'(y)$. By multiplying (3.1) by $(by_x)_{xt}$, we get

$$\int_{Q_t} (by_x)_x (by_x)_{xt} + \int_{Q_t} b y_{xt}^2 + \int_{Q_t} y_{xt} b_t y_x = 0, \tag{4.35}$$

hence

$$\frac{1}{2}\int_I t(by_x)_x^2(t)dx + \int_{Q_t} t b y_{xt}^2 + \int_{Q_t} t y_{xt} b_t y_x = \frac{1}{2}\int_{Q_t} (by_x)_x^2. \tag{4.36}$$

In this equation, we have

$$\left| \int_{Q_t} t y_{xt} b_t y_x \right| \leq \frac{\mu}{2} \int_{Q_t} t y_{xt}^2 + \frac{1}{2\mu} \int_{Q_t} t b_t^2 y_x^2, \tag{4.37}$$

where we may take μ to be the same constant as in the beginning of the paper ($\mu \leq a'(x)$, for all $x \in \mathbb{R}$, so that $\mu \leq a'(y) = b$). Then,

$$\int_{Q_t} t b_t^2 y_x^2 \leq \int_Q b_t^2 \cdot \|t b y_x^2\|_{L^\infty(Q)} \leq C \int_Q y_t^2 \cdot \sup_{t\in[0,T_0]} \int_I t (b y_x)_x^2(t) dx. \tag{4.38}$$

Consequently, we obtain (by using evaluation (4.34) for $\|y_t\|_{L^2(Q)}$) that

$$\begin{aligned} &\sup_{t\in[0,T_0]} \int_I t (b y_x)_x^2(t) dx + \int_Q t y_{xt}^2 \\ &\qquad \leq C_7 \left(1 + \sup_{t\in[0,T_0]} \int_I t (b y_x)_x^2(t) dx \right) \int_I y_x^2(0) dx, \quad \forall t \in [0, T_0]. \end{aligned} \tag{4.39}$$

Again, for sufficiently small $\|y_x(0)\|_{L^2(I)}$ (such as $\|y_x(0)\|_{L^2(I)}^2 \leq 1/2C_7$), we have proved that

$$\sup_{t\in[0,T_0]} \int_I t (b y_x)_x^2(t) dx + \int_Q t y_{xt}^2 \leq C \int_I y_x^2(0) dx. \tag{4.40}$$

By choosing $t = T_0$ in (4.23), (4.34), and (4.40), we obtain

$$\int_I (a(y))_{xx}^2(T_0) + y_x^2(T_0) + y^2(T_0) dx \leq C \left(\int_I y_0^2 dx + \int_I y_x^2(0) dx \right), \tag{4.41}$$

for any y_0 with $\|(y_0)_x\|_{L^2(I)}$ sufficiently small. The proof of Theorem 4.2 is thus concluded. □

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Marius Beceanu: Department of Mathematics, Princeton University, Princeton, NJ 0544, USA

E-mail address: mbeceanu@princeton.edu