We establish a new form of the 3G theorem for polyharmonic Green function on the unit ball of \( \mathbb{R}^n \) \((n \geq 2)\) corresponding to zero Dirichlet boundary conditions. This enables us to introduce a new class of functions \( K_{m,n} \) containing properly the classical Kato class \( K_n \). We exploit properties of functions belonging to \( K_{m,n} \) to prove an infinite existence result of singular positive solutions for nonlinear elliptic equation of order \( 2m \).

1. Introduction

In [2], Boggio gave an explicit expression for the Green function \( G_{m,n} \) of \((-\Delta)^m\) on the unit ball \( B \) of \( \mathbb{R}^n \) \((n \geq 2)\) with Dirichlet boundary conditions

\[
    u = \frac{\partial}{\partial \nu} u = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0 \quad \text{on } \partial B,
\]

where \( \partial/\partial \nu \) is the outward normal derivative and \( m \) is a positive integer.

In fact, he proved that for each \( x, y \) in \( B \), we have

\[
    G_{m,n}(x, y) = k_{m,n} |x - y|^{2m-n} \int_1^{[x,y]/|x-y|} \frac{(v^2 - 1)^{m-1}}{v^{n-1}} dv,
\]

where \( k_{m,n} \) is a positive constant and \( [x, y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2) \), for each \( x, y \) in \( B \).

Hence, from its expression, it is clear that \( G_{m,n} \) is positive in \( B^2 \), which does not hold for the Green function for the biharmonic or \( m \)-polyharmonic operator for an arbitrary bounded domain (see, e.g., [5]). Only for the case \( m = 1 \), we do not have this restriction.

In [7], using the Boggio formula (1.2), Grunau and Sweers have established some interesting estimates for the Green function \( G_{m,n} \) in \( B \). In particular, they
obtained the following inequality called 3G theorem: there exists a constant $a_{m,n} > 0$ such that for each $x, y, z \in B$,

$$
\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq a_{m,n} \begin{cases}
|x-z|^{2m-n} + |y-z|^{2m-n}, & \text{for } 2m < n, \\
\log\left(\frac{3}{|x-z|}\right) + \log\left(\frac{3}{|z-y|}\right), & \text{for } 2m = n, \\
1, & \text{for } 2m > n.
\end{cases}
$$

(1.3)

The Green function for the Laplacian ($m = 1$) satisfies the above inequality in an arbitrary bounded $C^{1,1}$ domain $\Omega$ in $\mathbb{R}^n$. In fact, for the case $n \geq 3$, Zhao proved in [19] the existence of a positive constant $C_n$ such that for each $x, y, z$ in $\Omega$,

$$
\frac{G_{1,n}(x,z)G_{1,n}(z,y)}{G_{1,n}(x,y)} \leq C_n \left( \frac{1}{|x-z|^{n-2}} + \frac{1}{|y-z|^{n-2}} \right).
$$

(1.4)

Moreover, for the case $n = 2$, Chung and Zhao showed in [3] the existence of a positive constant $C_2$ such that for each $x, y, z$ in $\Omega$,

$$
\frac{G_{1,2}(x,z)G_{1,2}(z,y)}{G_{1,2}(x,y)} \leq C_2 \left[ \max\left(1, \log\left(\frac{1}{|x-z|}\right)\right) + \max\left(1, \log\left(\frac{1}{|y-z|}\right)\right) \right].
$$

(1.5)

The 3G theorem related to $G_{1,n}$ has been exploited in the study of functions belonging to the Kato class $K_n(\Omega)$ (see Definition 1.1), which was widely used in the study of some nonlinear differential equations (see [15, 18]).

More properties pertaining to this class can be found in [1, 3].

**Definition 1.1** (see [1, 3]). A Borel measurable function $\phi$ in $\Omega$ belongs to the Kato class $K_n(\Omega)$ if $\phi$ satisfies the following conditions:

$$
\lim_{\alpha \to 0} \left( \sup_{x \in \Omega} \int_{\Omega \cap B(x,\alpha)} \frac{|\phi(y)|}{|x-y|^{n-2}} dy \right) = 0, \quad \text{if } n \geq 3,
$$

$$
\lim_{\alpha \to 0} \left( \sup_{x \in \Omega} \log\left(\frac{1}{|x-y|}\right) |\phi(y)| dy \right) = 0, \quad \text{if } n = 2.
$$

(1.6)

The purpose of this paper is two-folded. One is to give a new form of the 3G theorem to the Green function $G_{m,n}$ in $B^2$ which improves (1.3) and enables us to introduce a new Kato class $K_{m,n} := K_{m,n}(B)$ in the sense of Definition 1.2. The
second purpose is to investigate the existence of infinitely many singular positive solutions for the following nonlinear elliptic problem:

\[
\Delta^m u = (-1)^m f(\cdot, u) \quad \text{in } B \setminus \{0\} \text{ (in the sense of distributions),}
\]

\[
u = \frac{\partial}{\partial y} u = \cdots = \frac{\partial^{m-1}}{\partial y^{m-1}} u = 0 \quad \text{on } \partial B,
\]

\[u(x) \sim c\rho(x), \quad \text{near } x = 0, \text{ for any sufficiently small } c > 0,
\]

where

\[
\rho(x) = \begin{cases} 
\frac{1}{|x|^{n-2m}}, & \text{for } 2m < n, \\
\log \left( \frac{1}{|x|} \right), & \text{for } 2m = n, \\
1, & \text{for } 2m > n,
\end{cases}
\]

and \( f \) is required to satisfy suitable assumptions related to the class \( K_{m,n} \) which will be specified later.

The existence of infinitely many singular positive solutions for problem (1.7) in the case \( m = 1 \), for an arbitrary bounded \( C^{1,1} \) domain \( \Omega \) in \( \mathbb{R}^n (n \geq 3) \), has been established by Zhang and Zhao in [18] for the special nonlinearity

\[
f(x, t) = p(x)t^\mu, \quad \mu > 1,
\]

where the function \( p \) satisfies

\[x \longrightarrow \frac{p(x)}{|x|^{(n-2)(\mu-1)}} \in K_n(\Omega).
\]

This result has been recently extended by Mâagli and Zribi in [14], where \( f \) satisfies some appropriate conditions related to the class \( K_{1,n}(\Omega) \).

Here we extend these results to the high order.

The outline of the paper is as follows. In Section 2, we find again by a simpler argument some estimates on the Green function \( G_{m,n} \) given by Grunau and Sweers in [7] and we give further ones, including the following:

\[
\left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) \leq C \begin{cases} 
\frac{1}{|x - y|^{n-2m}}, & \text{for } 2m < n, \\
\log \left( \frac{3}{|x - y|} \right), & \text{for } 2m = n, \\
1, & \text{for } 2m > n.
\end{cases}
\]
Next, we establish the $3G$ theorem in this form: there exists $C_{m,n} > 0$ such that for each $x, y, z \in B,$

$$\frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} \leq C_{m,n} \left[ \left( \frac{\delta(z)}{\delta(x)} \right)^m G_{m,n}(x,z) + \left( \frac{\delta(z)}{\delta(y)} \right)^m G_{m,n}(y,z) \right],$$  \hspace{1cm} (1.12)

which improves (1.3). We note that, for $m = 1,$ (1.12) holds for an arbitrary bounded domain $\Omega$ in $\mathbb{R}^n.$ This was proved by Kalton and Verbitsky in [10] for $n \geq 3$ and by Selmi in [16] for the case $n = 2.$

In Section 3, we define and study some properties of functions belonging to the class $K_{m,n}.$

**Definition 1.2.** A Borel measurable function $\varphi$ in $B$ belongs to the class $K_{m,n}$ if $\varphi$ satisfies the following condition:

$$\lim_{\alpha \to 0} \left( \sup_{x \in B} \int_{B \cap B(x,\alpha)} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |\varphi(y)| \, dy \right) = 0. \hspace{1cm} (1.13)$$

In particular, we show that $K_{m,n}$ contains properly $K_{j,n},$ for $1 \leq j \leq m - 1,$ which contains properly $K_n(B).$ We close this section by giving a characterization of the radial functions belonging to the class $K_{m,n}.$

For the case $m = 1,$ this class has been extensively studied for an arbitrary bounded $C^{1,1}$ domain in $\mathbb{R}^n,$ in [14], for $n \geq 3,$ and in [12, 17] for the case $n = 2.$ To study problem (1.7) in Section 4, we assume that $f$ satisfies the following hypotheses:

(H1) $f$ is a Borel measurable function on $B \times (0, \infty),$ continuous with respect to the second variable;

(H2) $|f(x,t)| \leq tq(x,t),$ where $q$ is a nonnegative Borel measurable function in $B \times (0, \infty),$ such that the function $t \mapsto q(x,t)$ is nondecreasing on $(0, \infty)$ and $\lim_{t \to 0} q(x,t) = 0;$

(H3) the function $g,$ defined on $B$ by $g(x) = q(x, G_{m,n}(x,0)),$ belongs to the class $K_{m,n}.$

We point out that in the case $m = 1$ and $f(x,t) = p(x)t^\mu,$ the assumption (1.10) implies (H3).

In order to simplify our statements, we define some convenient notation.

**Notation.** (i) We denote $B = \{x \in \mathbb{R}^n; |x| < 1\}$ with $n \geq 2.$

(ii) We denote $s \wedge t = \min(s,t)$ and $s \vee t = \max(s,t)$ for $s, t \in \mathbb{R}.$

(iii) For $x, y \in B,$

$$[x,y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2),$$

$$\delta(x) = 1 - |x|,$$

$$\theta(x,y) = [x,y]^2 - |x - y|^2 = (1 - |x|^2)(1 - |y|^2).$$  \hspace{1cm} (1.14)
Note that \([x, y]^2 \geq 1 + |x|^2|y|^2 - 2|x||y| = (1 - |x||y|)^2\). So we have

\[
\delta(x) \leq [x, y], \quad \delta(y) \leq [x, y].
\] (1.15)

(iv) Let \(f\) and \(g\) be positive functions on a set \(S\).
We call \(f \sim g\) if there is \(c > 0\) such that

\[
\frac{1}{c}g(x) \leq f(x) \leq cg(x) \quad \forall x \in S.
\] (1.16)

We call \(f \preceq g\) if there is \(c > 0\) such that

\[
f(x) \leq cg(x) \quad \forall x \in S.
\] (1.17)

The following properties will be used several times:
(i) for \(s, t \geq 0\), we have

\[
s \wedge t \sim \frac{st}{s + t},
\] (1.18)

\[
(s + t)^p \sim s^p + t^p, \quad p \in \mathbb{R}^+; \quad (1.19)
\]

(ii) let \(\lambda, \mu > 0\) and \(0 < \gamma \leq 1\), then we have

\[
1 - t^\lambda \sim 1 - t^\mu \quad \text{for } t \in [0, 1],
\] (1.20)

\[
\log(1 + t) \leq t^\gamma \quad \text{for } t \geq 0,
\] (1.21)

\[
\log(1 + \lambda t) \sim \log(1 + \mu t) \quad \text{for } t \geq 0,
\] (1.22)

\[
\log(1 + t^3) \sim t^3 \log(2 + t) \quad \text{for } t \in [0, 1];
\] (1.23)

(iii) on \(B^2\) (i.e., \((x, y) \in B^2\)), we have

\[
\theta(x, y) \sim \delta(x)\delta(y),
\] (1.24)

\[
[x, y]^2 \sim |x - y|^2 + \delta(x)\delta(y).
\] (1.25)

2. Inequalities for the Green function

We first find another expression of \(G_{m,n}\) given by Hayman and Korenblum in [8], which will be used later.

**Proposition 2.1.** The Green function \(G_{m,n}\) satisfies

\[
G_{m,n}(x, y) = \alpha_{m,n} \sum_{k=0}^{\infty} \frac{\Gamma(n/2 + k)(\theta(x, y))^{m+k}}{(k+m)!|x, y|^{n+2k}},
\] (2.1)

where \(\alpha_{m,n}\) is some fixed positive constant.
Proof. Using the transformation \( v^2 = 1 + (\theta(x, y)/|x - y|^2)(1 - t) \) in (1.2), \( G_{m,n} \) becomes

\[
G_{m,n}(x, y) = \frac{k_{m,n}}{2} \frac{(\theta(x, y))^m}{[x, y]^n} \int_0^1 \frac{(1 - t)^{n-1}}{(1 - t(\theta(x, y)/[x, y]^2))^{n/2}} \, dt. \tag{2.2}
\]

Since \( 0 < \theta(x, y)/|x, y|^2 \leq 1 \), and for each \( t \in [0, 1[, \) we have

\[
(1 - t)^{-n/2} = \sum_{k=0}^{\infty} \frac{\Gamma(n/2 + k)}{k! \Gamma(n/2)} t^k; \tag{2.3}
\]

it follows that

\[
G_{m,n}(x, y) = \frac{k_{m,n}}{2} \sum_{k=0}^{\infty} \frac{\Gamma(n/2 + k)}{k! \Gamma(n/2)} \frac{(\theta(x, y))^{m+k}}{[x, y]^{n+2k}} B(k+1, m), \tag{2.4}
\]

where \( B(k+1, m) := \int_0^1 t^k (1 - t)^{m-1} \, dt = k! (m - 1)! / (k + m)! \).

That is,

\[
G_{m,n}(x, y) = \alpha_{m,n} \sum_{k=0}^{\infty} \frac{\Gamma((n/2) + k)}{(k + m)!} \frac{(\theta(x, y))^{m+k}}{[x, y]^{n+2k}}, \tag{2.5}
\]

with \( \alpha_{m,n} > 0 \). \( \square \)

Moreover, from formula (1.2), we may prove, by simpler argument, the following estimates on \( G_{m,n} \) given in [7].

**Proposition 2.2.** On \( B^2 \), the following estimates hold:

(i) for \( 2m < n \),

\[
G_{m,n}(x, y) \sim |x - y|^{2m-n} \left( 1 \wedge \frac{(\delta(x)\delta(y))^m}{|x - y|^{2m}} \right); \tag{2.6}
\]

(ii) for \( 2m = n \),

\[
G_{m,n}(x, y) \sim \log \left( 1 + \frac{(\delta(x)\delta(y))^m}{|x - y|^{2m}} \right); \tag{2.7}
\]

(iii) for \( 2m > n \),

\[
G_{m,n}(x, y) \sim (\delta(x)\delta(y))^{m-n/2} \left( 1 \wedge \frac{(\delta(x)\delta(y))^{n/2}}{|x - y|^{n}} \right). \tag{2.8}
\]

**Proof.** Using in (1.2) the transformation \( t = (v^2 - 1)^m \), we obtain the following expression for \( G_{m,n} \):

\[
G_{m,n}(x, y) = C|x - y|^{2m-n} \int_0^1 \frac{(\theta(x, y)/|x - y|^2)^m}{(t^{1/m} + 1)^{n/2}} \, dt. \tag{2.9}
\]
Now, from (1.19) we have

\[ G_{m,n}(x, y) \sim |x - y|^{2m-n} \int_0^{(\theta(x,y)/|x-y|)^{2m}/|x-y|^{2m}} dt \frac{dt}{(m/2m + 1)}. \] (2.10)

Next, we distinguish the following cases.

**Case 1** \((2m = n)\). It follows from (2.10), (1.22), and (1.24) that

\[ G_{m,n}(x, y) \sim \log \left(1 + \frac{(\theta(x,y))^{m}}{|x-y|^{2m}}\right) \sim \log \left(1 + \frac{(\delta(x)\delta(y))^{m}}{|x-y|^{2m}}\right). \] (2.11)

**Case 2** \((2m < n)\). Using the fact that for each \(a > 0\) and \(\lambda > 1\), we have

\[ \int_a^1 \frac{1}{t^\lambda + 1} dt \sim 1 \wedge a, \] (2.12)

hence, we deduce from (2.10) and (1.24) that

\[ G_{m,n}(x, y) \sim |x - y|^{2m-n} \left(1 \wedge \frac{(\theta(x,y))^{m}}{|x-y|^{2m}}\right) \sim |x - y|^{2m-n} \left(1 \wedge \frac{(\delta(x)\delta(y))^{m}}{|x-y|^{2m}}\right). \] (2.13)

**Case 3** \((2m > n)\). We recall that \(0 < \theta(x, y)/[x, y]^2 \leq 1\), which yields

\[ \int_0^1 \frac{(1-t)^{m-1}}{(1-t(\theta(x,y)/[x,y]^2))^{n/2}} dt \sim 1. \] (2.14)

This implies, with (2.2), that

\[ G_{m,n}(x, y) \sim \frac{(\theta(x,y))^{m}}{|x,y|^n}, \] (2.15)

which, together with (1.24), (1.18), and (1.19), gives that

\[ G_{m,n}(x, y) \sim (\delta(x)\delta(y))^{m/2} \left(1 \wedge \frac{(\delta(x)\delta(y))^{n/2}}{|x-y|^n}\right). \] (2.16)

\[ \square \]

**Corollary 2.3.** On \(B^2\), the following estimates hold:
(i) if $2m < n$, 
\[
G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}(|x-y|^2 + \delta(x)\delta(y))^m} \\
\sim \frac{(\delta(x)\delta(y))^m}{|x-y|^{n-2m}m_{x,y}^2m} \\
\sim \frac{1}{|x-y|^{n-2m}} \left(\frac{1}{|x-y|^2 + (\delta(x)\delta(y))^{m/2}}\right)^{(n-2m)/2m};
\]
(2.17)

(ii) if $2m = n$, 
\[
G_{m,n}(x,y) \sim \left(1 \wedge \frac{(\delta(x)\delta(y))^m}{|x-y|^{2m}}\right) \log \left(2 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right) \\
\sim \frac{(\delta(x)\delta(y))^m}{(|x-y|^2 + \delta(x)\delta(y))^m} \log \left(2 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right) \\
\sim \frac{(\delta(x)\delta(y))^m}{|x,y|^2m} \log \left(1 + \frac{|x,y|^2}{|x-y|^2}\right);
\]
(2.18)

(iii) if $2m > n$, 
\[
G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{(|x-y|^2 + (\delta(x)\delta(y)))^{n/2}} \\
\sim \frac{(\delta(x)\delta(y))^m}{|x,y|^n}.
\]
(2.19)

Proof. The proof follows immediately from Proposition 2.2 and the statements (1.18), (1.19), (1.20), (1.22), (1.23), (1.24), and (1.25).

From the above estimates, we derive some inequalities for the Green function $G_{m,n}$ including (1.11), which will be done in the following corollaries.

**Corollary 2.4.** On $B^2$, the following estimates hold:
\[
\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \leq \begin{cases} 
\frac{1}{|x-y|^{n-2m}}, & \text{for } 2m < n, \\
\log \left(\frac{3}{|x-y|}\right), & \text{for } 2m = n, \\
1, & \text{for } 2m > n.
\end{cases}
\]
(2.20)

Proof. Using Corollary 2.3 and inequalities (1.15), we deduce that

(i) if $2m < n$, 
\[
\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x,y) \leq \frac{1}{|x-y|^{n-2m}} \left(\delta(y)^{2m}/|x,y|^{2m}\right) \leq \frac{1}{|x-y|^{n-2m}};
\]
(2.21)
(ii) if \(2m = n\),

\[
\left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) \leq \log \left( 1 + \frac{[x, y]^2}{|x - y|^2} \right) \frac{(\delta(y))^{2m}}{|x, y|^{2m}} \leq \log \left( \frac{3}{|x - y|} \right); \quad (2.22)
\]

(iii) if \(2m > n\),

\[
\left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) \leq \frac{(\delta(y))^{2m}}{|x, y|^n} \leq 1. \quad (2.23)
\]

\[
\square
\]

**Corollary 2.5.** For each \(x, y \in B\) such that \(|x - y| \geq r\),

\[
G_{m,n}(x, y) \leq \frac{(\delta(x)\delta(y))^m}{r^n}. \quad (2.24)
\]

Moreover, on \(B^2\), the following estimates hold:

\[
(\delta(x)\delta(y))^m \leq G_{m,n}(x, y), \quad (2.25)
\]

\[
G_{m,n}(x, y) \leq (\delta(x))^m \wedge (\delta(y))^m \quad \text{if } m \geq n, \quad (2.26)
\]

\[
G_{m,n}(x, y) \leq \frac{(\delta(x))^m \wedge (\delta(y))^m}{|x - y|^{n-m}} \quad \text{if } 1 \leq m < n. \quad (2.27)
\]

**Proof.** Assertions (2.24) and (2.25) are obviously obtained using the estimates in Corollary 2.3 and the fact that \(|x - y| \leq [x, y] \leq 1\).

Now, if \(m \geq n\), then we deduce from Corollary 2.3 and (1.15) that

\[
G_{m,n}(x, y) \sim \frac{(\delta(x)\delta(y))^m}{|x, y|^n} \leq (\delta(x))^m \wedge (\delta(y))^m. \quad (2.28)
\]

Then (2.26) holds.

To prove (2.27), we suppose that \(1 \leq m < n\). So we obtain, from Corollary 2.3, inequalities (1.15), and \(|x - y| \leq [x, y] \leq 1\) that

(i) if \(2m < n\), then we have

\[
G_{m,n}(x, y) \sim \frac{(\delta(x)\delta(y))^m}{|x - y|^{n-2m}[x, y]^{2m}} \leq \frac{(\delta(x))^m}{|x - y|^{n-m}} \frac{(\delta(y))^m}{|x, y|^m} \leq \frac{(\delta(x))^m}{|x - y|^{n-m}}; \quad (2.29)
\]
(ii) if $2m = n$, then using further inequality (1.21), we deduce that

$$G_{m,n}(x,y) \sim \log \left(1 + \frac{|x,y|^2}{|x-y|^2}\right) \left(\delta(x)\delta(y)\right)^m \frac{\delta(x)\delta(y)^m}{|x,y|^{2m}} \leq \frac{|x,y|}{|x-y|} \frac{\delta(x)\delta(y)^m}{|x,y|^{2m}} \leq \frac{(\delta(x))^m (\delta(y))^m}{|x-y|^m |x,y|^m} \leq \frac{(\delta(x))^m}{|x-y|^m};$$

(2.30)

(iii) if $2m > n$, then we have

$$G_{m,n}(x,y) \sim \frac{(\delta(x)\delta(y))^m}{|x,y|^n} \leq \frac{(\delta(x))^m (\delta(y))^m}{|x-y|^{n-m} |x,y|^m} \leq \frac{(\delta(x))^m}{|x-y|^{n-m}}.$$  

(2.31)

Hence interchanging the roles of $x$ and $y$, (2.27) is proved. □

In the sequel, for a nonnegative measurable function $f$ on $B$, we put

$$V_{m,n}f(x) = \int_B G_{m,n}(x,y)f(y)dy \quad \text{for } x \in B.$$  

(2.32)

**Remark 2.6.** Let $m \geq n$. Then there exists a positive constant $C_1$ such that, for each $f \in L^1(B)$ and $x \in B$, we have

$$\frac{1}{C_1} \left(\int_B (\delta(y))^m f(y)dy\right) \left(\delta(x)\right)^m \leq V_{m,n}f(x) \leq C_1 \|f\|_1 \left(\delta(x)\right)^m.$$  

(2.33)

In particular, we have $V_{m,n}1(x) \sim (\delta(x))^m$.

Moreover, let $1 \leq m < n$. Then there exists a positive constant $C_2$ such that for each $f \in L^p_+(B)$ with $p > n/m$ and $x \in B$, we have

$$\frac{1}{C_2} \left(\int_B (\delta(y))^m f(y)dy\right) \left(\delta(x)\right)^m \leq V_{m,n}f(x) \leq C_2 \|f\|_p \left(\delta(x)\right)^m.$$  

(2.34)

Indeed, (2.33) holds by (2.25) and (2.26). To prove (2.34), we use (2.25) and (2.27) and we apply the Hölder inequality, so we obtain that, for $x \in B$,

$$\left(\int_B (\delta(y))^m f(y)dy\right) \left(\delta(x)\right)^m \leq V_{m,n}f(x) \leq \left(\delta(x)\right)^m \|f\|_p \left(\int_B \frac{dy}{|x-y|^{(n-m)p/(p-1)}}\right)^{(p-1)/p}. $$  

(2.35)
Now, for each \( x \in B \), we have

\[
\int_B \frac{dy}{|x - y|^{(n-m)p/(p-1)}} \leq \int_{B(0,2)} \frac{d\xi}{|\xi|^{(n-m)p/(p-1)}},
\]  

(2.36)

and this last integral is finite if and only if \( p > n/m \), which gives (2.34).

Next, we aim to prove inequality (1.12). So, we need the following key lemma.

**Lemma 2.7** (see [11, 13]). Let \( x, y \in B \). Then the following properties are satisfied:

1. if \( \delta(x) \delta(y) \leq |x - y|^2 \), then \( (\delta(x) \vee \delta(y)) \leq ((\sqrt{5} + 1)/2)|x - y| \);
2. if \( |x - y|^2 \leq \delta(x) \delta(y) \), then \( ((3 - \sqrt{5})/2)\delta(x) \leq \delta(y) \leq ((3 + \sqrt{5})/2)\delta(x) \).

**Proof.** (1) We may assume that \( (\delta(x) \vee \delta(y)) = \delta(y) \). Then the inequalities \( \delta(y) \leq \delta(x) + |x - y| \) and \( \delta(x) \delta(y) \leq |x - y|^2 \) imply that

\[
(\delta(y))^2 - \delta(y)|x - y| - |x - y|^2 \leq 0,
\]  

(2.37)

that is,

\[
\left( \delta(y) + \frac{(\sqrt{5} - 1)}{2}|x - y| \right) \left( \delta(y) - \frac{(\sqrt{5} + 1)}{2}|x - y| \right) \leq 0. 
\]  

(2.38)

It follows that

\[
(\delta(x) \vee \delta(y)) \leq \frac{(\sqrt{5} + 1)}{2}|x - y|.
\]  

(2.39)

(2) For each \( z \in \partial B \), we have \( |y - z| \leq |x - y| + |x - z| \) and since \( |x - y|^2 \leq \delta(x) \delta(y) \), we obtain

\[
|y - z| \leq \sqrt{\delta(x)\delta(y)} + |x - z| \leq \sqrt{|x - z||y - z|} + |x - z|,
\]  

(2.40)

that is,

\[
\left( \sqrt{|y - z|} + \frac{(\sqrt{5} - 1)}{2}\sqrt{|x - z|} \right) \left( \sqrt{|y - z|} - \frac{(\sqrt{5} + 1)}{2}\sqrt{|x - z|} \right) \leq 0. 
\]  

(2.41)

It follows that

\[
|y - z| \leq \frac{(3 + \sqrt{5})}{2}|x - z|.
\]  

(2.42)

Thus, interchanging the roles of \( x \) and \( y \), we have

\[
\left( \frac{3 - \sqrt{5}}{2} \right)|x - z| \leq |y - z| \leq \left( \frac{3 + \sqrt{5}}{2} \right)|x - z|,
\]  

(2.43)
which gives
\[
\left(\frac{3 - \sqrt{5}}{2}\right) \delta(x) \leq \delta(y) \leq \left(\frac{3 + \sqrt{5}}{2}\right) \delta(x).
\] (2.44)

**Theorem 2.8 (3G theorem).** There exists a constant \(C_{m,n} > 0\) such that, for each \(x, y, z \in B\),
\[
\frac{G_{m,n}(x, z) G_{m,n}(z, y)}{G_{m,n}(x, y)} \leq C_{m,n} \left[\left(\frac{\delta(z)}{\delta(x)}\right)^m G_{m,n}(x, z) + \left(\frac{\delta(z)}{\delta(y)}\right)^m G_{m,n}(y, z)\right].
\] (2.45)

**Proof.** To prove the inequality, we denote \(A(x, y) := (\delta(x)\delta(y))^m/G_{m,n}(x, y)\) and we claim that \(A\) is a quasimetric, that is, for each \(x, y, z \in B\),
\[
A(x, y) \leq A(x, z) + A(y, z).
\] (2.46)

To show the claim, we separate the proof into three cases.

**Case 1.** For \(2m < n\), using Proposition 2.2, we have
\[
A(x, y) \sim |x - y|^{n - 2m}(|x - y|^2 \vee (\delta(x)\delta(y)))^m.
\] (2.47)

We distinguish the following subcases:

(i) if \(\delta(x)\delta(y) \leq |x - y|^2\), then we have
\[
A(x, y) \sim |x - y|^n \leq |x - z|^n + |y - z|^n \leq A(x, z) + A(y, z);
\] (2.48)

(ii) the inequality \(|x - y|^2 \leq \delta(x)\delta(y)\) implies, from Lemma 2.7, that \(\delta(x) \sim \delta(y)\). So we deduce the following:

(a) if \(|x - z|^2 \leq \delta(x)\delta(z)\) or \(|y - z|^2 \leq \delta(y)\delta(z)\), then it follows from Lemma 2.7 that \(\delta(x) \sim \delta(y) \sim \delta(z)\). Hence,
\[
A(x, y) \sim |x - y|^{n - 2m}(\delta(x)\delta(y))^m \leq (\delta(x)\delta(y))^m(|x - z|^{n - 2m} + |y - z|^{n - 2m}) \leq |x - z|^{n - 2m}(\delta(x)\delta(z))^m + |y - z|^{n - 2m}(\delta(y)\delta(z))^m \leq A(x, z) + A(y, z);
\] (2.49)

(b) if \(|x - z|^2 \geq \delta(x)\delta(z)\) and \(|y - z|^2 \geq \delta(y)\delta(z)\), then using Lemma 2.7, we have
\[
(\delta(x) \vee \delta(z)) \leq |x - z|, \quad (\delta(y) \vee \delta(z)) \leq |y - z|.
\] (2.50)
So, we have
\[
A(x, y) \sim |x - y|^{n-2m} (\delta(x)\delta(y))^m \\
\leq (|x - z|^{n-2m} + |y - z|^{n-2m}) (\delta(x)\delta(y))^m \\
\leq |x - z|^{n-2m} (\delta(x))^2m + |y - z|^{n-2m} (\delta(y))^{2m} \\
\leq |x - z|^n + |y - z|^n \\
\leq A(x, z) + A(y, z).
\] (2.51)

**Case 2.** For \(2m = n\), using Proposition 2.2, we have
\[
A(x, y) \sim \frac{(\delta(x)\delta(y))^m}{\log(1 + (\delta(x)\delta(y))^m/|x - y|^{2m})}. \tag{2.52}
\]
Then, since for each \(t \geq 0\),
\[
\frac{t}{1 + t} \leq \log(1 + t) \leq t, \tag{2.53}
\]
we deduce that
\[
|x - y|^{2m} \leq A(x, y) \leq |x - y|^{2m} + (\delta(x)\delta(y))^m. \tag{2.54}
\]
So we distinguish the following subcases:

(i) if \(\delta(x)\delta(y) \leq |x - y|^2\), then by (1.19), we have
\[
A(x, y) \leq |x - y|^{2m} \leq |x - z|^{2m} + |y - z|^{2m} \leq A(x, z) + A(y, z); \tag{2.55}
\]

(ii) if \(|x - y|^2 \leq \delta(x)\delta(y)\), it follows by Lemma 2.7 that \(\delta(x) \sim \delta(y)\).
So, we distinguish the following two subcases:
(a) if \(|x - z|^2 \leq \delta(x)\delta(z)\) or \(|y - z|^2 \leq \delta(y)\delta(z)\), so from Lemma 2.7, we deduce that \(\delta(x) \sim \delta(y) \sim \delta(z)\).
Now, since
\[
|x - y|^{2m} \leq |x - z|^{2m} + |y - z|^{2m} \leq (|x - z|^{2m} \lor |y - z|^{2m}), \tag{2.56}
\]
then we obtain that
\[
\left( \log \left( 1 + \frac{(\delta(x)\delta(z))^m}{|x - z|^{2m}} \right) \land \log \left( 1 + \frac{(\delta(y)\delta(z))^m}{|y - z|^{2m}} \right) \right) \\
\leq \log \left( 1 + \frac{(\delta(x)\delta(y))^m}{|x - y|^{2m}} \right), \tag{2.57}
\]
which, together with (2.52), implies that
\[
A(x, y) \leq A(x, z) + A(y, z); \tag{2.58}
\]
(b) if $|x - z|^2 \geq \delta(x)\delta(z)$ and $|y - z|^2 \geq \delta(y)\delta(z)$, then by Lemma 2.7, it follows that

$$\left(\delta(x) \vee \delta(z)\right) \leq |x - z|,$$

$$\left(\delta(y) \vee \delta(z)\right) \leq |y - z|. \quad (2.59)$$

Hence, by (2.54), we have

$$A(x, y) \leq \left(\delta(x)\delta(y)\right)^m \leq (\delta(x))^{2m} + (\delta(y))^{2m} \leq |x - z|^{2m} + |y - z|^{2m} \leq A(x, z) + A(y, z). \quad (2.60)$$

**Case 3.** For $2m > n$, from Proposition 2.2, we have

$$A(x, y) \sim (|x - y|^2 \vee (\delta(x)\delta(y)))^{n/2}. \quad (2.61)$$

Then the result holds by arguments similar to that of Case 2(i). \hfill \Box

### 3. The Kato class $K_{m,n}$

In this section, we will study properties of functions belonging to the class $K_{m,n}$. We first compare the classes $K_{j,n}$ for $j \geq 1$.

**Proposition 3.1.** For each $m \geq 1$, the following estimate is satisfied on $B^2$:

$$\left(\frac{\delta(y)}{\delta(x)}\right)^m G_{m,n}(x, y) \leq (\delta(y))^{2(m-1)} \left(\frac{\delta(y)}{\delta(x)}\right) G_{1,n}(x, y). \quad (3.1)$$

In particular, $K_{1,n} \subset (\delta(\cdot))^{2(m-1)} K_{m,n}$.

**Proof.** Using (1.2), we have

$$G_{m,n}(x, y) \leq |x - y|^{2m-n} \left(\frac{|x, y|^2}{|x - y|^2} - 1\right)^{m-1} \int_{|x, y|/|x - y|} \frac{dv}{\nu^{m-1}}. \quad (3.2)$$

Now, we remark by (1.25) that

$$\frac{|x, y|^2}{|x - y|^2} - 1 \sim \frac{\delta(x)\delta(y)}{|x - y|^2}. \quad (3.3)$$

So we deduce that

$$G_{m,n}(x, y) \leq (\delta(x)\delta(y))^{m-1} G_{1,n}(x, y), \quad (3.4)$$

which implies (3.1). The proof is complete by (1.13). \hfill \Box
Remark 3.2. Let $j, m \in \mathbb{N}$ such that $1 \leq j < m$, then we have

$$K_n(B) \subset K_{j,n} \subset K_{m,n}. \quad (3.5)$$

Indeed, by a similar argument as above, we prove that, on $B^2$,

$$\left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) \leq (\delta(y))^{2(m-j)} \left( \frac{\delta(y)}{\delta(x)} \right)^j G_{j,n}(x,y), \quad (3.6)$$

which implies that $K_{j,n} \subset K_{m,n}$. The first inclusion in (3.5) holds by putting $m = 1$ in Corollary 2.4.

Lemma 3.3. Let $\phi$ be a function in $K_{m,n}$. Then the function

$$x \rightarrow (\delta(x))^{2m} \phi(x) \quad (3.7)$$

is in $L^1(B)$.

Proof. Let $\phi \in K_{m,n}$, then by (1.13), there exists $\alpha > 0$ such that for each $x \in B$,

$$\int_{B(x,\alpha) \cap B} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) \phi(y) \, dy \leq 1. \quad (3.8)$$

Let $x_1, \ldots, x_p$ be in $B$ such that $B \subset \bigcup_{1 \leq i \leq p} B(x_i, \alpha)$. Then by (2.25), there exists $C > 0$ such that for all $i \in \{1, \ldots, p\}$ and $y \in B(x_i, \alpha) \cap B$, we have

$$(\delta(y))^{2m} \leq C \left( \frac{\delta(y)}{\delta(x_i)} \right)^m G_{m,n}(x_i,y). \quad (3.9)$$

Hence, we have

$$\int_B (\delta(y))^{2m} |\phi(y)| \, dy \leq C \sum_{1 \leq i \leq p} \int_{B(x_i,\alpha) \cap B} \left( \frac{\delta(y)}{\delta(x_i)} \right)^m G_{m,n}(x_i,y) |\phi(y)| \, dy$$

$$\leq Cp < \infty. \quad (3.10)$$

This completes the proof.

In the sequel, we use the notation

$$\|\phi\|_B := \sup_{x \in B} \int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |\phi(y)| \, dy. \quad (3.11)$$

Proposition 3.4. Let $\phi$ be a function in $K_{m,n}$, then $\|\phi\|_B < \infty$. 
Proof. Let \( \varphi \in K_{m,n} \) and \( \alpha > 0 \). Then we have

\[
\int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy \\
\leq \int_{B \cap |x-y| \leq \alpha} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy \\
+ \int_{B \cap |x-y| \geq \alpha} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy.
\]

(3.12)

Now, since by (2.24), we have

\[
\int_{B \cap |x-y| \geq \alpha} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy \leq \frac{1}{\alpha^n} \int_B (\delta(y))^{2m} |\varphi(y)| \, dy,
\]

(3.13)

then the result follows from (1.13) and Lemma 3.3. \( \square \)

Proposition 3.5. There exists a constant \( C > 0 \) such that, for all \( \varphi \in K_{m,n} \) and \( h \) a nonnegative harmonic function in \( B \),

\[
\int_B G_{m,n}(x, y) (\delta(y))^{m-1} h(y) |\varphi(y)| \, dy \leq C \|\varphi\|_B (\delta(x))^{m-1} h(x)
\]

(3.14)

for all \( x \) in \( B \).

Proof. Let \( h \) be a nonnegative harmonic function in \( B \). So by Herglotz representation theorem (see [9, page 29]), there exists a nonnegative measure \( \mu \) on \( \partial B \) such that

\[
h(y) = \int_{\partial B} P(y, \xi) \mu(d\xi),
\]

(3.15)

where \( P(y, \xi) = (1 - |y|^2)/|y - \xi|^n \), for \( y \in B \) and \( \xi \in \partial B \). So we need only to verify (3.14) for \( h(y) = P(y, \xi) \) uniformly in \( \xi \in \partial B \).

By (2.1) we have for each \( x, y \in B \),

\[
G_{m,n}(x, y) = \alpha_{m,n} \left( \frac{\theta(x, y)}{[x, y]^n} \right)^m (1 + o(1 - |y|^2)).
\]

(3.16)

Hence, for \( x, y, z \) in \( B \),

\[
\frac{G_{m,n}(y, z)}{G_{m,n}(x, z)} = \frac{(1 - |y|^2)^m [x, z]^n}{(1 - |x|^2)^m [y, z]^n} (1 + o(1 - |y|^2)),
\]

(3.17)
which implies that

\[
\lim_{z \to \xi} G_{m,n}(y,z) = \frac{(1 - |y|^2)^m |x - \xi|^n}{(1 - |x|^2)^m |y - \xi|^n} \sim \frac{(\delta(y))^{m-1} P(y, \xi)}{P(x, \xi)}
\]  

Thus by Fatou’s lemma and (1.12), we deduce that

\[
\int_B G_{m,n}(x,y) \left( \frac{\delta(y)}{\delta(x)} \right)^m P(y, \xi) |\varphi(y)| \, dy \leq \liminf_{z \to \xi} \left[ \int_B G_{m,n}(x,y) \frac{G_{m,n}(y,z)}{G_{m,n}(x,z)} |\varphi(y)| \, dy \right] 
\]

\[
\leq \liminf_{z \to \xi} \left[ \int_B \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |\varphi(y)| \, dy \right] + \left[ \int_B \left( \frac{\delta(y)}{\delta(z)} \right)^m G_{m,n}(z,y) |\varphi(y)| \, dy \right] 
\]

\[
\leq \|\varphi\|_{B},
\]

which completes the proof. \(\Box\)

**Corollary 3.6.** Let \(\varphi\) be in \(K_{m,n}\). Then

\[
\sup_{x \in B} \int_B G_{m,n}(x,y) (\delta(y))^{m-1} |\varphi(y)| \, dy < \infty.
\]

Moreover, the function \(x \mapsto (\delta(x))^{2m-1} \varphi(x)\) is in \(L^1(B)\).

**Proof.** Put \(h \equiv 1\) in (3.14) and using Proposition 3.4, we get (3.20).

Moreover, by (2.25), it follows that

\[
\int_B (\delta(y))^{2m-1} |\varphi(y)| \, dy \leq \int_B G_{m,n}(0,y) (\delta(y))^{m-1} |\varphi(y)| \, dy.
\]

Hence the result follows from (3.20). \(\Box\)

**Remark 3.7.** We recall (see [1]) that for \(m = 1\) and \(n \geq 3\), a radial function \(\varphi\) is in the classical Kato class \(K_n(B)\) if and only if

\[
\int_0^1 r |\varphi(r)| \, dr < \infty.
\]

Similarly, we will give in the sequel a characterization of the radial functions belonging to \(K_{m,n}\), which asserts, in particular, that inclusions (3.5) are proper. More precisely, we will prove in the next proposition that a radial function \(\varphi\) is in \(K_{m,n}\) if and only if (3.20) is satisfied.
Proposition 3.8. Let \( \varphi \) be a radial function in \( B \), then the following assertions are equivalent:

1. \( \varphi \in K_{m,n} \);
2. \( \sup_{x \in B} \int_B G_{m,n}(x,y)(\delta(y))^{m-1}|\varphi(y)|dy < \infty \);
3. for \( 2m < n \),
   \[
   \int_0^1 r^{2m-1}(1-r)^{2m-1}|\varphi(r)|dr < \infty. \tag{3.23}
   \]

For \( 2m = n \),
   \[
   \int_0^1 r^{n-1}(1-r)^{n-2}\log\left(\frac{1}{r}\right)|\varphi(r)|dr < \infty. \tag{3.24}
   \]

For \( 2m > n \),
   \[
   \int_0^1 r^{n-1}(1-r)^{2m-1}|\varphi(r)|dr < \infty. \tag{3.25}
   \]

Proof. Since the function \( x \to \int_{S^{n-1}} G_{m,n}(x,r\omega)d\sigma(\omega) \) is radial in \( B \), then we denote that \( t = |x| \) and

\[
\psi_{m,n}(t,r) = \int_{S^{n-1}} G_{m,n}(x,r\omega)d\sigma(\omega),
\]

where \( \sigma \) is the normalized measure on the unit sphere \( S^{n-1} \) of \( \mathbb{R}^n \).

Now, using Corollary 2.3 and the fact that for each \( y \in B \), \([0,y] = 1\), we deduce that

\[
\psi_{m,n}(0,r) \sim \begin{cases} r^{2m-n}(1-r)^m, & \text{for } 2m < n, \\ (1-r)^m \log\left(1 + \frac{1}{r^2}\right) \sim (1-r)^{m-1}\log\left(\frac{1}{r}\right), & \text{for } 2m = n, \\ (1-r)^m, & \text{for } 2m > n. \end{cases} \tag{3.27}
\]

So, assertion (3) is equivalent to

\[
(3') \int_0^1 r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|dr < \infty.
\]

We now prove the equivalences.

(1) \(\Rightarrow\) (2) follows from Corollary 3.6.

(2) \(\Leftrightarrow\) (3'). By virtue of [4, Theorem 2.4], we have that \( t \to \psi_{m,n}(t,r) \) is a non-increasing map on \([0,1]\), so that

\[
\sup_{x \in B} \int_B G_{m,n}(x,y)(\delta(y))^{m-1}|\varphi(y)|dy = \sup_{t \in [0,1]} \int_0^1 r^{n-1}(1-r)^{m-1}\psi_{m,n}(t,r)|\varphi(r)|dr \tag{3.28}
\]

\[
= \int_0^1 r^{n-1}(1-r)^{m-1}\psi_{m,n}(0,r)|\varphi(r)|dr.
\]
Thus, by elementary calculus, we obtain that
\[
\sup_{x \in B} \int_{B \cap B(x,a)} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) |\varphi(y)| \, dy
\leq \sup_{0 \leq t \leq 1} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1} \frac{(1-r)^m}{(1-t)^m} \psi_{m,n}(t,r) |\varphi(r)| \, dr
\leq \sup_{0 \leq t \leq 1/2} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1} \frac{(1-r)^m}{(1-t)^m} \psi_{m,n}(t,r) |\varphi(r)| \, dr
+ \sup_{1/2 \leq t \leq 1} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1} \frac{(1-r)^m}{(1-t)^m} \psi_{m,n}(t,r) |\varphi(r)| \, dr
= I_1 + I_2.
\]
Using [4, Theorem 2.4], we have
\[
I_1 \leq \sup_{0 \leq t \leq 1/2} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1} \frac{(1-r)^m}{(1-t)^m} \psi_{m,n}(0,r) |\varphi(r)| \, dr
\leq \sup_{0 \leq t \leq 1/2} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1}(1-r)^{m-1} \psi_{m,n}(0,r) |\varphi(r)| \, dr.
\]
On the other hand, by (3.1), we have
\[
I_2 \leq \sup_{1/2 \leq t \leq 1} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1}(1-r)^{2m-1} \psi_{1,n}(t,r) |\varphi(r)| \, dr.
\]
Now, by elementary calculus, we obtain that
\[
\psi_{1,n}(t,r) = \begin{cases} 
\frac{1}{n-2}(t \lor r)^{2-n}(1-(t \lor r)^{n-2}), & \text{for } n \geq 3, \\
\log \left( \frac{1}{t \lor r} \right), & \text{for } n = 2.
\end{cases}
\]
So, using (1.20) and the fact that \( \log(1/s) \leq (1-s) \) for \( s \geq 1/2 \), we have for each \( n \geq 2 \) and \( t \geq 1/2 \),
\[
\psi_{1,n}(t,r) \leq (1-t \lor r).
\]
Hence, from (3.27), we have
\[
I_2 \leq \sup_{1/2 \leq t \leq 1} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1}(1-r)^{2m-1} \frac{(1-t \lor r)}{1-t} \, dr
\leq \sup_{1/2 \leq t \leq 1} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1}(1-r)^{m-1} \psi_{m,n}(0,r) |\varphi(r)| \, dr.
\]
Thus, \( I_1 + I_2 \leq \sup_{0 \leq t \leq 1} \int_{(t-a) \vee 0}^{(t+a) \wedge 1} r^{n-1}(1-r)^{m-1} \psi_{m,n}(0,r) |\varphi(r)| \, dr.\)
Let \( \phi(s) = \int_s^1 r^{n-1}(1 - r)^{m-1} \psi_{m,n}(0,r) |\varphi(r)| dr \) for \( s \in [0,1] \).

Then using (3'), we deduce that \( \phi \) is a continuous function on \([0,1]\), which implies that

\[
\int_{(t-\alpha) \lor 0}^{(t+\alpha) \land 1} r^{n-1}(1 - r)^{m-1} \psi_{m,n}(0,r) |\varphi(r)| dr = \phi((t+\alpha) \land 1) - \phi((t-\alpha) \lor 0)
\]

(3.35)

converges to zero as \( \alpha \to 0 \) uniformly for \( t \in [0,1] \). So, \( \lim_{\alpha \to 0}(I_1 + I_2) = 0 \), that is, \( \varphi \in K_{m,n} \). \( \square \)

Example 3.9. Let \( q \) be the function defined in \( B \) by

\[
q(x) = \frac{1}{(\delta(x))^\lambda}.
\]

(3.36)

By Proposition 3.8, \( q \in K_{m,n} \) if and only if \( \lambda < 2m \) and \( V_{m,n}q \) is bounded if and only if \( \lambda < m + 1 \). In fact, we give in the next proposition more precise estimates on the \( m \)-potential \( V_{m,n}q \).

**Proposition 3.10.** On \( B \), the following estimates hold:

(i) \( (\delta(x))^m \leq V_{m,n}q(x) \leq (\delta(x))^{2m-\lambda} \) if \( m < \lambda < m + 1 \);
(ii) \( (\delta(x))^m \leq V_{m,n}q(x) \leq (\delta(x))^m \log(2/\delta(x)) \) if \( \lambda = m \);
(iii) \( V_{m,n}q(x) \sim (\delta(x))^m \) if \( \lambda < m \).

**Proof.** Let \( \lambda < m + 1 \). Then from (2.25), we have

\[
(\delta(x))^m \int_B \frac{dy}{(\delta(y))^{\lambda-m}} \leq V_{m,n}q(x),
\]

(3.37)

which implies the lower estimates.

For the upper estimates, we have, from (3.1),

\[
V_{m,n}q(x) \leq \int_B (\delta(x))^{m-1}(\delta(y))^{m-1} G_{1,n}(x,y) q(y) dy \\
\leq (\delta(x))^{m-1} \int_0^1 \frac{r^{n-1}}{(1-r)^{\lambda+1-m}} \psi_{1,n}(|x|,r) dr.
\]

(3.38)

On the other hand, using (1.20) and the inequality \( t \log(1/t) \leq (1 - t) \), for \( t \in [0,1] \), we deduce from (3.32) that \( r^{n-1} \psi_{1,n}(|x|,r) \leq (1 - |x| \lor r) \) for each \( n \geq 2 \).
This implies that
\[
V_{m,n,q}(x) \leq (\delta(x))^{m-1} \int_{0}^{1} \frac{1 - (|x| \vee r)}{(1 - r)\lambda + 1 - m} dr
\leq (\delta(x))^{m} \int_{|x|}^{1} \frac{dr}{(1 - r)\lambda + 1 - m} + (\delta(x))^{m-1} \int_{|x|}^{1} \frac{dr}{(1 - r)^{\lambda - m}}
\]
\[= I_1 + I_2.
\]

(3.39)

So, by elementary calculus, we obtain that
\[
I_1 \leq (\delta(x))^m \begin{cases} 
(\delta(x))^{m-\lambda}, & \text{if } m < \lambda < m + 1, \\
\log \frac{2}{\delta(x)}, & \text{if } \lambda = m, \\
1, & \text{if } \lambda < m,
\end{cases}
\]
\[
I_2 \leq (\delta(x))^{2m-\lambda}.
\]

(3.40)

This completes the proof. □

Remark 3.11. By Proposition 3.10, we find again the result of Gilbarg and Trudinger in [6, Theorem 4.9] for the case \(m = 1\) and \(1 < \lambda < 2\).

4. Positive singular solutions of the equation \(\Delta^m u = (-1)^m f(\cdot, u)\)

In this section, we are interested in the existence of positive singular solutions for problem (1.7). We present in the next theorem the main result of this section.

Theorem 4.1. Assume (H_1), (H_2), and (H_3). Then problem (1.7) has infinitely many solutions. More precisely, there exists \(b_0 > 0\) such that for each \(b \in (0, b_0]\), there exists a solution \(u\) of (1.7) continuous on \(B \setminus \{0\}\) and satisfying for all \(x \in B\),

\[
\frac{b}{2} G_{m,n}(x,0) \leq u(x) \leq \frac{3b}{2} G_{m,n}(x,0)
\]

(4.1)

and, for \(2m \leq n\),

\[
\lim_{|x| \to 0} \frac{u(x)}{G_{m,n}(x,0)} = b.
\]

(4.2)

For the proof, we need the following lemmas.

Lemma 4.2. Let \(\varphi \in K_{m,n}\) and \(x_0 \in \overline{B}\). Then

\[
\lim_{\alpha \to 0} \left( \sup_{x,z \in B} \frac{1}{G_{m,n}(x,z)} \int_{B \cap B(x_0,\alpha)} G_{m,n}(x,y)G_{m,n}(y,z) \left| \varphi(y) \right| dy \right) = 0.
\]

(4.3)
Singular solutions for polyharmonic equation

Proof. Let $\varepsilon > 0$. Then by (1.13), there exists $r > 0$ such that

$$\sup_{\xi \in B} \int_{B \cap B(\xi, r)} \left( \frac{\delta(y)}{\delta(\xi)} \right)^m G_{m,n}(\xi, y) |\varphi(y)| \, dy \leq \varepsilon. \quad (4.4)$$

Let $\alpha > 0$. Then it follows, from Theorem 2.8, that for each $x, z \in B$,

$$\frac{1}{G_{m,n}(x, z)} \int_{B \cap B(x_0, \alpha)} G_{m,n}(x, y) G_{m,n}(y, z) |\varphi(y)| \, dy$$

$$\leq C_{m,n} \int_{B \cap B(x_0, \alpha)} \left[ \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) + \left( \frac{\delta(y)}{\delta(z)} \right)^m G_{m,n}(y, z) \right] |\varphi(y)| \, dy$$

$$\leq 2C_{m,n} \sup_{\xi \in B} \int_{B \cap B(x_0, \alpha)} \left( \frac{\delta(y)}{\delta(\xi)} \right)^m G_{m,n}(\xi, y) |\varphi(y)| \, dy. \quad (4.5)$$

On the other hand, by (2.24), we have

$$\int_{B \cap B(x_0, \alpha)} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy$$

$$\leq \int_{B \cap |x-y| \leq r} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy$$

$$+ \int_{B \cap B(x_0, \alpha) \cap |x-y| \geq r} \left( \frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x, y) |\varphi(y)| \, dy \quad (4.6)$$

$$\leq \sup_{\xi \in B} \int_{B \cap B(\xi, r)} \left( \frac{\delta(y)}{\delta(\xi)} \right)^m G_{m,n}(\xi, y) |\varphi(y)| \, dy$$

$$+ \int_{B \cap B(x_0, \alpha)} \left( \delta(y) \right)^{2m} |\varphi(y)| \, dy.$$  

Now, using Lemma 3.3 and (4.4), the result holds by letting $\alpha \to 0$. \qed

Put $F := \{ \omega \in C^+(\overline{B}) : \|\omega\|_{\infty} \leq 1 \}$, where $\| \cdot \|_{\infty}$ is the uniform norm. So we have the following result.

Lemma 4.3. Assume $(H_1)$, $(H_2)$, and $(H_3)$. Define the operator $T$ on $F$ by

$$T\omega(x) = \frac{1}{G_{m,n}(x, 0)} \int_B G_{m,n}(x, y) f(y, \omega(y)G_{m,n}(y, 0)) \, dy, \quad x \in B. \quad (4.7)$$

Then the family of functions $T(F)$ is relatively compact in $C(\overline{B})$.

Proof. By $(H_2)$, we have for all $\omega \in F$,

$$|T\omega(x)| \leq \frac{1}{G_{m,n}(x, 0)} \int_B G_{m,n}(x, y)G_{m,n}(y, 0)g(y) \, dy. \quad (4.8)$$
Since \( g(x) = q(x, G_{m,n}(x, 0)) \in K_{m,n} \), then, by Theorem 2.8, we deduce that

\[
\| T\omega \|_\infty \leq 2C_{m,n} \sup_{\xi \in B} \left( \frac{\delta(y)}{\delta(\xi)} \right)^m G_{m,n}(\xi, y) g(y) dy
\]

\[
\leq \| g \|_B.
\] (4.9)

Hence, the family \( T(F) \) is uniformly bounded. Now, we will prove the equicontinuity of \( T(F) \) in \( \overline{B} \). Let \( x_0 \in \overline{B} \) and \( \alpha > 0 \). Let \( x, x' \in B(x_0, \alpha) \cap B \) and \( \omega \in F \), then

\[
\| T\omega(x) - T\omega(x') \| \leq \int_{\Omega} \frac{G_{m,n}(x, y)}{G_{m,n}(x, 0)} \frac{G_{m,n}(x', y)}{G_{m,n}(x', 0)} \left| G_{m,n}(\xi, y) G_{m,n}(\xi, 0) \right| G_{m,n}(y, 0) g(y) dy.
\]

\[
(4.10)
\]

If \( |x_0 - y| \geq 2\alpha \), then \( |x - y| \geq \alpha \) and \( |x' - y| \geq \alpha \). So (1.12) and (2.24) imply that, for all \( x \in B(x_0, \alpha) \cap B \) and \( y \in \Omega := B^c(0, 2\alpha) \cap B^c(x_0, 2\alpha) \cap B \),

\[
\frac{G_{m,n}(x, y)}{G_{m,n}(x, 0)} \frac{G_{m,n}(x', y)}{G_{m,n}(x', 0)} \leq (\delta(y))^{2m}.
\]

\[
(4.11)
\]

Moreover, using (3.18), we deduce, when \( y \in \Omega \), that the function \( x \to \frac{G_{m,n}(x, y)}{G_{m,n}(x, 0)} \frac{G_{m,n}(x', y)}{G_{m,n}(x', 0)} \) is continuous in \( B(x_0, \alpha) \cap B \). Then, by Lemma 3.3 and the dominated convergence theorem, we obtain that

\[
\int_{\Omega} \left| \frac{G_{m,n}(x, y)}{G_{m,n}(x, 0)} \frac{G_{m,n}(x', y)}{G_{m,n}(x', 0)} \right| G_{m,n}(y, 0) g(y) dy \to 0
\]

(4.12)

as \( |x - x'| \to 0 \).

By Lemma 4.2, we deduce that

\[
| T\omega(x) - T\omega(x') | \to 0, \quad \text{as} \ |x - x'| \to 0,
\]

(4.13)

uniformly for all \( \omega \in F \). The result follows by Ascoli’s theorem. \( \square \)
Remark 4.4. Let $\alpha > 0$. Then for $2m \leq n$ and $y \in B^c(0,2\alpha) \cap B$, we have

$$\lim_{|x| \to 0} \frac{G_{m,n}(x,y)}{G_{m,n}(x,0)} = 0. \quad (4.14)$$

So, using the same argument as in the proof of Lemma 4.3, we deduce that for $2m \leq n$,

$$|T \omega(x)| \to 0, \quad \text{as } |x| \to 0, \quad (4.15)$$

uniformly for all $\omega \in F$.

Proof of Theorem 4.1. We aim to show that there exists $b_0 > 0$ such that for each $b \in (0,b_0]$, there exists a continuous function $u$ in $B \setminus \{0\}$ satisfying the following integral equation:

$$u(x) = bG_{m,n}(x,0) + \int_B G_{m,n}(x,y)f(y,u(y))dy, \quad x \in B \setminus \{0\}. \quad (4.16)$$

Let $\beta \in (0,1)$. Then by Lemma 4.3, the function

$$T_\beta(x) = \frac{1}{G_{m,n}(x,0)} \int_B G_{m,n}(x,y)G_{m,n}(y,0)q(y,\beta G_{m,n}(y,0))dy \quad (4.17)$$

is continuous in $\overline{B}$. Moreover, using $(1.12)$, $(H_2)$, and $(H_3)$, we have

$$\sup_{\zeta \in B} \int_B \left( \frac{\delta(y)}{\delta(\zeta)} \right)^m G_{m,n}(\zeta,y)g(y)dy \leq \|g\|_B. \quad (4.18)$$

So, we deduce by the dominated convergence theorem and $(H_2)$ that

$$\lim_{\beta \to 0} T_\beta(x) = 0 \quad \forall x \in \overline{B}. \quad (4.19)$$

Since the function $\beta \to T_\beta(x)$ is nondecreasing in $(0,1)$, it follows by Dini’s lemma that

$$\lim_{\beta \to 0} \left( \sup_{x \in B} \frac{1}{G_{m,n}(x,0)} \int_B G_{m,n}(x,y)G_{m,n}(y,0)q(y,\beta G_{m,n}(y,0))dy \right) = 0. \quad (4.20)$$
Thus, there exists $\beta \in (0, 1)$ such that for each $x \in \mathcal{B}$,

$$
\frac{1}{G_{m,n}(x,0)} \int_{\mathcal{B}} G_{m,n}(x,y) G_{m,n}(y,0) q(y, \beta G_{m,n}(y,0)) \, dy \leq \frac{1}{3}.
$$

(4.21)

Let $b_0 = (2/3)\beta$ and $b \in (0, b_0]$. We will use a fixed-point argument. Let

$$
S = \left\{ \omega \in C(\mathcal{B}) : \frac{b}{2} \leq \omega(x) \leq \frac{3b}{2} \right\}.
$$

(4.22)

Then, $S$ is a nonempty, closed, bounded, and convex set in $C(\mathcal{B})$. We define the operator $\Gamma$ on $S$ by

$$
\Gamma \omega(x) = b + \frac{1}{G_{m,n}(x,0)} \int_{\mathcal{B}} G_{m,n}(x,y) f(y, \omega(y) G_{m,n}(y,0)) \, dy, \quad x \in \mathcal{B}.
$$

(4.23)

By Lemma 4.3, $\Gamma S \subset C(\mathcal{B})$. Moreover, let $\omega \in S$, then for any $x \in \mathcal{B}$, we have

$$
\left| \Gamma \omega(x) - b \right| \leq \frac{3b}{2} \frac{1}{G_{m,n}(x,0)} \int_{\mathcal{B}} G_{m,n}(x,y) G_{m,n}(y,0) q(y, \beta G_{m,n}(y,0)) \, dy
$$

(4.24)

\begin{align*}
\leq \frac{b}{2}.
\end{align*}

It follows that $b/2 \leq \Gamma \omega(x) \leq 3b/2$ and so $\Gamma S \subset S$.

Next, we will prove the continuity of $\Gamma$ in the uniform norm. Let $(\omega_k)_k$ be a sequence in $S$ which converges uniformly to $\omega \in S$. Then since $f$ is continuous with respect to the second variable, we deduce by the dominated convergence theorem that

$$
\Gamma \omega_k(x) \rightarrow \Gamma \omega(x) \quad \text{as} \ k \rightarrow \infty, \ \forall x \in \mathcal{B}.
$$

(4.25)

Now, since $\Gamma S$ is a relatively compact family in $C(\mathcal{B})$, then

$$
\| \Gamma \omega_k - \Gamma \omega \|_{\infty} \rightarrow 0 \quad \text{as} \ k \rightarrow \infty.
$$

(4.26)

So the Schauder fixed-point theorem implies the existence of $\omega \in S$ such that $\Gamma \omega = \omega$.

For all $x \in \mathcal{B}$, put $u(x) = \omega(x) G_{m,n}(x,0)$. Then, $u$ is a continuous function in $\mathcal{B} \setminus \{0\}$ satisfying (4.16).

Furthermore, if $2m \leq n$, then by Remark 4.4, we obtain that $\lim_{|x| \to 0} \omega(x) = b$, that is, $\lim_{|x| \to 0} u(x)/G_{m,n}(x,0) = b$. This ends the proof. $\square$

Example 4.5. Let $p > 0$, $\lambda < 2m$, and $\mu < n \wedge 2m$. Let $V$ be a measurable function in $\mathcal{B}$ such that for each $x \in \mathcal{B}$,

$$
|V(x)| \leq \frac{1}{(\delta(x))^{\lambda} |x|^\mu (G_{m,n}(x,0))^p}.
$$

(4.27)
Then there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, the nonlinear problem

$$
\Delta^m u = (-1)^m V(x) u^{p+1}(x) \quad \text{in } B \setminus \{0\} \quad \text{(in the sense of distributions)},
$$

$$
u = \frac{\partial}{\partial v} u = \cdots = \frac{\partial^{m-1}}{\partial v^{m-1}} u = 0 \quad \text{on } \partial B,
$$

has a positive solution $u$, continuous on $B \setminus \{0\}$ and satisfying for all $x \in B$,

$$
\frac{b}{2} G_{m,n}(x,0) \leq u(x) \leq \frac{3b}{2} G_{m,n}(x,0)
$$

and for $2m \leq n$, we have

$$
\lim_{|x| \to 0} \frac{u(x)}{G_{m,n}(x,0)} = b.
$$

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