We first introduce a modified proximal point algorithm for maximal monotone operators in a Banach space. Next, we obtain a strong convergence theorem for resolvents of maximal monotone operators in a Banach space which generalizes the previous result by Kamimura and Takahashi in a Hilbert space. Using this result, we deal with the convex minimization problem and the variational inequality problem in a Banach space.

1. Introduction

Let $E$ be a real Banach space and let $T \subset E \times E^*$ be a maximal monotone operator. Then we study the problem of finding a point $v \in E$ satisfying

\[ 0 \in Tv. \] (1.1)

Such a problem is connected with the convex minimization problem. In fact, if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then Rockafellar’s theorem [14, 15] ensures that the subdifferential mapping $\partial f \subset E \times E^*$ of $f$ is a maximal monotone operator. In this case, the equation $0 \in \partial f(v)$ is equivalent to $f(v) = \min_{x \in E} f(x)$.

In 1976, Rockafellar [17] proved the following weak convergence theorem.

**Theorem 1.1 (Rockafellar [17]).** Let $H$ be a Hilbert space and let $T \subset H \times H$ be a maximal monotone operator. Let $I$ be the identity mapping and let $J_r = (I + rT)^{-1}$ for all $r > 0$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in H$ and

\[ x_{n+1} = J_r x_n \quad (n = 1, 2, \ldots), \] (1.2)

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. If $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}0$.

This is called the proximal point algorithm, which was first introduced by Martinet [11]. If $T = \partial f$, where $f : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function,
then (1.2) is reduced to

\[ x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r_n} \| x_n - y \|^2 \right\} \quad (n = 1, 2, \ldots). \]  

Later, many researchers studied the convergence of the proximal point algorithm in a Hilbert space; see Brézis and Lions [4], Lions [10], Passty [12], Güler [7], Solodov and Svaiter [19] and the references mentioned there. In particular, Kamimura and Takahashi [8] proved the following strong convergence theorem.

**Theorem 1.2 (Kamimura and Takahashi [8]).** Let \( H \) be a Hilbert space and let \( T \subset H \times H \) be a maximal monotone operator. Let \( J_r = (I + rT)^{-1} \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in H \) and

\[ x_{n+1} = \alpha_n x + (1 - \alpha_n) J_r x_n \quad (n = 1, 2, \ldots), \]

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). If \( T^{-1}0 \neq \emptyset \), then the sequence \( \{x_n\} \) converges strongly to \( P_{T^{-1}0}(x) \), where \( P_{T^{-1}0} \) is the metric projection from \( H \) onto \( T^{-1}0 \).

Recently, using the hybrid method in mathematical programming, Kamimura and Takahashi [9] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which extended the result of Solodov and Svaiter [19] in a Hilbert space. On the other hand, Censor and Reich [6] introduced a convex combination which is based on Bregman distance and studied some iterative schemes for finding a common asymptotic fixed point of a family of operators in finite-dimensional spaces.

In this paper, motivated by Censor and Reich [6], we introduce the following iterative sequence for a maximal monotone operator \( T \subset E \times E^* \) in a smooth and uniformly convex Banach space: \( x_1 = x \in E \) and

\[ x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n) JJ_r x_n) \quad (n = 1, 2, \ldots), \]

where \( \{\alpha_n\} \subset [0, 1] \), \( \{r_n\} \subset (0, \infty) \), \( J \) is the duality mapping from \( E \) into \( E^* \), and \( J_r = (J + rT)^{-1}J \) for all \( r > 0 \). Then we extend Kamimura-Takahashi’s theorem to the Banach space (Theorem 3.3). It should be noted that we do not assume the weak sequential continuity of the duality mapping [1, 5, 13]. Finally, we apply Theorem 3.3 to the convex minimization problem and the variational inequality problem.

### 2. Preliminaries

Let \( E \) be a (real) Banach space with norm \( \| \cdot \| \) and let \( E^* \) denote the Banach space of all continuous linear functionals on \( E \). For all \( x \in E \) and \( x^* \in E^* \), we denote \( x^*(x) \) by \( \langle x, x^* \rangle \). We denote by \( \mathbb{R} \) and \( \mathbb{N} \) the set of all real numbers and the set of all positive integers, respectively. The *duality mapping* \( J \) from \( E \) into \( E^* \) is defined by

\[ J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \| x \|^2 = \| x^* \|^2 \right\} \]  

(2.1)
for all $x \in E$. If $E$ is a Hilbert space, then $J = I$, where $I$ is the identity mapping. We sometimes identify a set-valued mapping $A : E \to 2^{E^*}$ with its graph $G(A) = \{(x,x^*) : x^* \in Ax\}$. An operator $T \subset E \times E^*$ with domain $D(T) = \{x \in E : Tx \neq \emptyset\}$ and range $R(T) = \bigcup \{Tx : x \in D(T)\}$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x,x^*), (y,y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone operator $T \subset E \times E^*$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. If $T \subset E \times E^*$ is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. A proper function $f : E \to (-\infty, \infty]$ (which means that $f$ is not identically $\infty$) is said to be convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x,y \in E$ and $\alpha \in (0,1)$. The function $f$ is also said to be lower semicontinuous if the set $\{x \in E : f(x) \leq r\}$ is closed in $E$ for all $r \in \mathbb{R}$. For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, the subdifferential $\partial f$ of $f$ is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \quad \forall y \in E\}$$

for all $x \in E$. It is easy to verify that $0 \in \partial f(v)$ if and only if $f(v) = \min_{x \in E} f(x)$. It is known that the subdifferential of the function $f$ defined by $f(x) = \|x\|^2/2$ for all $x \in E$ is the duality mapping $J$. The following theorem is also well known (see Takahashi [21] for details).

**Theorem 2.1.** Let $E$ be a Banach space, let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function, and let $g : E \to \mathbb{R}$ be a continuous convex function. Then

$$\partial(f + g)(x) = \partial f(x) + \partial g(x)$$

for all $x \in E$.

A Banach space $E$ is said to be strictly convex if

$$\|x\| = \|y\| = 1, \quad x \neq y \implies \left\|\frac{x+y}{2}\right\| < 1.$$  

(2.5)

Also, $E$ is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \implies \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$  

(2.6)

It is also said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x,y \in \{z \in E : \|z\| = 1\}$. We know the following (see Takahashi [20] for details):

1. if $E$ is smooth, then $J$ is single-valued;
2. if $E$ is strictly convex, then $J$ is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x,x^*), (y,y^*) \in J$ with $x \neq y$;
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(3) if $E$ is reflexive, then $J$ is surjective;
(4) if $E$ is uniformly convex, then it is reflexive;
(5) if $E^*$ is uniformly convex, then $J$ is uniformly norm-to-norm continuous on each bounded subset of $E$.

Let $E$ be a smooth Banach space. We use the following function studied in Alber [1], Kamimura and Takahashi [9], and Reich [13]:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

(2.8) for all $x, y \in E$. It is obvious from the definition of $\phi$ that $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$. We also know that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$$

(2.9) for all $x, y, z \in E$. The following lemma was also proved in [9].

**Lemma 2.2 (Kamimura-Takahashi [9]).** Let $E$ be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n \to \infty} \phi(x_n, y_n) = 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $T \subset E \times E^*$ be a monotone operator. Then $T$ is maximal if and only if $R(J + rT) = E^*$ for all $r > 0$ (see Barbu [2] and Takahashi [21]). If $T \subset E \times E^*$ is a maximal monotone operator, then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in D(T)$ satisfying

$$J(x) \in J(x_r) + rTx_r.$$  

(2.10) We define the resolvent of $T$ by $J_r x = x_r$. In other words, $J_r = (I + rT)^{-1}J$ for all $r > 0$. The resolvent $J_r$ is a single-valued mapping from $E$ into $D(T)$. If $E$ is a Hilbert space, then $J_r$ is nonexpansive, that is, $\|J_r x - J_r y\| \leq \|x - y\|$ for all $x, y \in E$ (see Takahashi [20]). It is easy to show that $T^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ denotes the set of all fixed points of $J_r$. We can also define, for each $r > 0$, the Yosida approximation of $T$ by $A_r = (J - Jr)/r$. We know that $(J_r x, A_r x) \in T$ for all $r > 0$ and $x \in E$. Let $C$ be a nonempty closed convex subset of the space $E$. By Alber [1] or Kamimura and Takahashi [9], for each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $P_C(x)$) such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

(2.11) The mapping $P_C$ is called the generalized projection from $E$ onto $C$. If $E$ is a Hilbert space, then $P_C$ is coincident with the metric projection from $E$ onto $C$. We also know the following lemma.

**Lemma 2.3 ([1], see also [9]).** Let $E$ be a smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $x \in E$ and $x_0 \in C$. Then the following are equivalent:

1. $\phi(x_0, x) = \min_{y \in C} \phi(y, x);
2. $\langle y - x_0, Jx - Jx_0 \rangle \leq 0$ for all $y \in C$. 


3. Strong convergence theorem

The resolvents of maximal monotone operators have the following property, which was proved in the case of the resolvents of normality operators in Kamimura and Takahashi [9].

**Lemma 3.1.** Let $E$ be a strictly convex, smooth, and reflexive Banach space, let $T \subset E \times E^*$ be a maximal monotone operator with $T^{-1} 0 \neq \emptyset$, and let $J_r = (J + rT)^{-1}J$ for each $r > 0$. Then

$$\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x) \quad (3.1)$$

for all $r > 0$, $u \in T^{-1} 0$, and $x \in E$.

**Proof.** Let $r > 0$, $u \in T^{-1} 0$, and $x \in E$ be given. By the monotonicity of $T$, we have

$$\phi(u, x) = \phi(u, J_r x) + \phi(J_r x, x) + 2\langle u - J_r x, JJ_r x - Jx \rangle$$

$$= \phi(u, J_r x) + \phi(J_r x, x) + 2r\langle u - J_r x, -A, x \rangle \quad (3.2)$$

$$\geq \phi(u, J_r x) + \phi(J_r x, x).$$

□

Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $J$ be the duality mapping from $E$ into $E^*$. Then $J^{-1}$ is also single-valued, one-to-one, and surjective, and it is the duality mapping from $E^*$ into $E$. We make use of the following mapping $V$ studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \quad (3.3)$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping $g$ defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from $E^*$ into $\mathbb{R}$. We can prove the following lemma.

**Lemma 3.2.** Let $E$ be a strictly convex, smooth, and reflexive Banach space, and let $V$ be as in (3.3). Then

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x, x^* + y^*) \quad (3.4)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

**Proof.** Let $x \in E$ be given. Define $g(x^*) = V(x, x^*)$ and $f(x^*) = \|x^*\|^2$ for all $x^* \in E^*$. Since $J^{-1}$ is the duality mapping from $E^*$ into $E$, we have

$$\partial g(x^*) = \partial ( - 2\langle x, \cdot \rangle + f) (x^*) = -2x + 2J^{-1}(x^*) \quad (3.5)$$

for all $x^* \in E^*$. Hence, we have

$$g(x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq g(x^* + y^*), \quad (3.6)$$
that is,
\[ V(x,x^*) + 2\langle J^{-1}(x^*) - x, y^* \rangle \leq V(x,x^* + y^*) \quad (3.7) \]
for all \( x^*, y^* \in E^* \).

Now we can prove the following strong convergence theorem, which is a generalization of Kamimura-Takahashi’s theorem (Theorem 1.2).

**Theorem 3.3.** Let \( E \) be a smooth and uniformly convex Banach space and let \( T \subset E \times E^* \) be a maximal monotone operator. Let \( J_r = (J + rT)^{-1}J \) for all \( r > 0 \) and let \( \{x_n\} \) be a sequence defined as follows: \( x_1 = x \in E \) and
\[
x_{n+1} = J^{-1}(\alpha_nJx + (1 - \alpha_n)JJ_n^*x_n) \quad (n = 1, 2, \ldots),
\]
where \( \alpha_n \in [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). If \( T^{-1}0 \neq \emptyset \), then the sequence \( \{x_n\} \) converges strongly to \( P_{T^{-1}0}(x) \), where \( P_{T^{-1}0} \) is the generalized projection from \( E \) onto \( T^{-1}0 \).

**Proof.** Put \( y_n = J_r^*x_n \) for all \( n \in \mathbb{N} \). We denote the mapping \( P_{T^{-1}0} \) by \( P \). We first prove that \( \{x_n\} \) is bounded. From Lemma 3.1, we have
\[
\phi(Px,x_{n+1}) = \phi(Px,J^{-1}(\alpha_nJx + (1 - \alpha_n)JJ_n^*x_n))
\]
\[
= V(Px,\alpha_nJx + (1 - \alpha_n)JJ_n^*x_n)
\]
\[
\leq \alpha_n V(Px,Jx) + (1 - \alpha_n) V(Px,J_n^*x_n)
\]
\[
= \alpha_n \phi(Px,x) + (1 - \alpha_n) \phi(Px,J_n^*x_n)
\]
\[
\leq \alpha_n \phi(Px,x) + (1 - \alpha_n) \phi(Px,x_n)
\]
for all \( n \in \mathbb{N} \). Hence, by induction, we have \( \phi(Px,x_n) \leq \phi(Px,x) \) for all \( n \in \mathbb{N} \). Since \( (\|u\| - \|v\|)^2 \leq \phi(u,v) \) for all \( u, v \in E \), the sequence \( \{x_n\} \) is bounded. Since \( \phi(Px,y_n) = \phi(Px,J_n^*x_n) \leq \phi(Px,x_n) \) for all \( n \in \mathbb{N} \), \( \{y_n\} \) is also bounded. We next prove
\[
\limsup_{n \to \infty} \langle x_n - Px, Jx - JPx \rangle \leq 0.
\]
Put \( z_n = x_{n+1} \) for all \( n \in \mathbb{N} \). Since \( \{z_n\} \) is bounded, we have a subsequence \( \{z_{n_i}\} \) of \( \{z_n\} \) such that
\[
\lim_{i \to \infty} \langle z_{n_i} - Px, Jx - JPx \rangle = \limsup_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle
\]
and \( \{z_{n_i}\} \) converges weakly to some \( v \in E \). From the definition of \( \{x_n\} \), we have
\[
Jz_n - Jy_n = \alpha_n(Jx - Jy_n)
\]
for all \( n \in \mathbb{N} \). Since \( \{y_n\} \) is bounded and \( \alpha_n \to 0 \) as \( n \to \infty \), we have
\[
\lim_{n \to \infty} ||Jz_n - Jy_n|| = \lim_{n \to \infty} \alpha_n||Jx - Jy_n|| = 0.
\]
Since $E$ is uniformly convex, $E^*$ is uniformly smooth, and hence the duality mapping $J^{-1}$ from $E^*$ into $E$ is uniformly norm-to-norm continuous on each bounded subset of $E^*$. Therefore, we obtain from (3.13) that

$$\lim_{n \to \infty} \|z_n - y_n\| = \lim_{n \to \infty} \|J^{-1}(Jz_n) - J^{-1}(Jy_n)\| = 0. \quad (3.14)$$

This implies that $y_{n_i} \rightharpoonup v$ as $i \to \infty$, where $\rightharpoonup$ implies the weak convergence. On the other hand, from $r_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n \to \infty} \|Ax_n\| = \lim_{n \to \infty} \frac{1}{r_n} \|Jx_n - Jy_n\| = 0. \quad (3.15)$$

If $(z, z^*) \in T$, then it holds from the monotonicity of $T$ that

$$\langle z - y_{n_i}, z^* - Ax_{n_i} \rangle \geq 0 \quad (3.16)$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z - v, z^* \rangle \geq 0$. Then, the maximality of $T$ implies $v \in T^{-1}0$. Applying Lemma 2.3, we obtain

$$\limsup_{n \to \infty} \langle z_n - Px, Jx - JPx \rangle = \lim_{i \to \infty} \langle z_{n_i} - Px, Jx - JPx \rangle = \langle v - Px, Jx - JPx \rangle \leq 0. \quad (3.17)$$

Finally, we prove that $x_n \to Px$ as $n \to \infty$. Let $\varepsilon > 0$ be given. From (3.10), we have $m \in \mathbb{N}$ such that

$$\langle x_n - Px, Jx - JPx \rangle \leq \varepsilon \quad (3.18)$$

for all $n \geq m$. If $n \geq m$, then it holds from (3.18) and Lemmas 3.1 and 3.2 that

$$\phi(Px, x_{n+1}) = V(Px, \alpha_n Jx + (1 - \alpha_n)Jy_n)$$

$$\leq V(Px, \alpha_n Jx + (1 - \alpha_n)Jy_n - \alpha_n(Jx - JPx))$$

$$- 2\langle J^{-1}(\alpha_n Jx + (1 - \alpha_n)Jy_n) - Px, -\alpha_n(Jx - JPx) \rangle$$

$$= V(Px, (1 - \alpha_n)Jy_n + \alpha_n JPx) + 2\langle x_{n+1} - Px, \alpha_n(Jx - JPx) \rangle$$

$$\leq (1 - \alpha_n) V(Px, Jy_n) + \alpha_n V(Px, JPx) + 2\alpha_n \langle x_{n+1} - Px, Jx - JPx \rangle$$

$$\leq (1 - \alpha_n) \phi(Px, y_n) + \alpha_n \phi(Px, Px) + 2\alpha_n \varepsilon$$

$$= (1 - \alpha_n) \phi(Px, Jx_n) + 2\alpha_n \varepsilon$$

$$\leq (1 - \alpha_n) \phi(Px, x_n) + 2\alpha_n \varepsilon$$

$$= 2\varepsilon (1 - (1 - \alpha_n)) + (1 - \alpha_n) \phi(Px, x_n).$$
Therefore, we have
\[
\phi(Px, x_{n+1}) \\
\leq 2\varepsilon \{1 - (1 - \alpha_n)\} + \alpha_n \{1 - (1 - \alpha_{n-1})\} + (1 - \alpha_n) \phi(Px, x_{n-1}) \\
= 2\varepsilon \{1 - (1 - \alpha_n)(1 - \alpha_{n-1})\} + (1 - \alpha_n)(1 - \alpha_{n-1}) \phi(Px, x_{n-1}) \\
\leq \cdots \leq 2\varepsilon \left\{1 - \prod_{i=m}^{n} (1 - \alpha_i)\right\} + \prod_{i=m}^{n} (1 - \alpha_i) \phi(Px, x_m)
\]
(3.20)
for all \(n \geq m\). Since \(\sum_{i=1}^{\infty} \alpha_i = \infty\), we have \(\prod_{i=m}^{\infty} (1 - \alpha_i) = 0\) (see Takahashi [21]). Hence, we have
\[
\limsup_{n \to \infty} \phi(Px, x_n) = \limsup_{l \to \infty} \phi(Px, x_{m+l+1}) \\
\leq \limsup_{l \to \infty} \left[2\varepsilon \left\{1 - \prod_{i=m}^{m+l} (1 - \alpha_i)\right\} + \prod_{i=m}^{m+l} (1 - \alpha_i) \phi(Px, x_m)\right] = 2\varepsilon.
\]
(3.21)
This implies \(\limsup_{n \to \infty} \phi(Px, x_n) \leq 0\) and hence we get
\[
\lim_{n \to \infty} \phi(Px, x_n) = 0.
\]
(3.22)
Applying Lemma 2.2, we obtain
\[
\lim_{n \to \infty} \|Px - x_n\| = 0.
\]
(3.23)
Therefore, \(\{x_n\}\) converges strongly to \(P_{\partial f}^{-1}(0)(x)\).

\section{4. Applications}

In this section, we first study the problem of finding a minimizer of a proper lower semicontinuous convex function in a Banach space.

\textbf{Theorem 4.1.} Let \(E\) be a smooth and uniformly convex Banach space and let \(f : E \to (-\infty, \infty]\) be a proper lower semicontinuous convex function such that \((\partial f)^{-1}(0) \neq \emptyset\). Let \(\{x_n\}\) be a sequence defined as follows: \(x_1 = x \in E\) and
\[
y_n = \arg\min_{y \in E} \left\{f(y) + \frac{1}{2r_n} \|y\|^2 - \frac{1}{r_n} \langle y, Jx_n\rangle\right\} \quad (n = 1, 2, \ldots),
\]
\[
x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n) Jy_n) \quad (n = 1, 2, \ldots),
\]
(4.1)
where \(\{\alpha_n\} \subset [0, 1]\) and \(\{r_n\} \subset (0, \infty)\) satisfy \(\lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,\) and \(\lim_{n \to \infty} r_n = \infty\). Then the sequence \(\{x_n\}\) converges strongly to \(P_{(\partial f)^{-1}(0)}(x)\).

\textbf{Proof.} By Rockafellar's theorem [14, 15], the subdifferential mapping \(\partial f \subset E \times E^*\) is maximal monotone (see also Borwein [3], Simons [18], or Takahashi [21]). Fix \(r > 0, z \in E\), and let \(J_r\) be the resolvent of \(\partial f\). Then we have
\[
J_z \in J(J_r z) + r \partial f (J_r z)
\]
(4.2)
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and hence,

\[ 0 \in \partial f(Jrz) + \frac{1}{r} J(Jrz) - \frac{1}{r} Jz = \partial \left( f + \frac{1}{2r} \| \cdot \|_2^2 - \frac{1}{r} Jz \right) (Jrz). \]  

(4.3)

Thus, we have

\[ Jrz = \arg\min_{y \in E} \left\{ f(y) + \frac{1}{2r} \| y \|_2^2 - \frac{1}{r} \langle y, Jz \rangle \right\}. \]  

(4.4)

Therefore, \( y_n = Jr_n x_n \) for all \( n \in \mathbb{N} \). Using Theorem 3.3, \{\( x_n \)\} converges strongly to \( P_{(\partial f)^{-1}(0)}(x) \). \( \square \)

We next study the problem of finding a solution of a variational inequality. Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) and let \( A : C \to E^* \) be a single-valued, monotone, and hemicontinuous operator such that \( \text{VI}(C, A) \neq \emptyset \). Let \{\( x_n \)\} be a sequence defined as follows: \( x_1 = x \in E \) and

\[ y_n = \text{VI}
\]

\[ \left( C, A + \frac{1}{r_n} (J - Jx_n) \right) \]  

\[ (n = 1, 2, \ldots), \]  

(4.7)

where \( \{\alpha_n\} \subset [0, 1] \) and \( \{r_n\} \subset (0, \infty) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty \), and \( \lim_{n \to \infty} r_n = \infty \). Then, the sequence \{\( x_n \)\} converges strongly to \( P_{\text{VI}(C, A)}(x) \).

Proof. By Rockafellar’s theorem [16], the mapping \( T \in E \times E^* \) defined by

\[ Tx = \begin{cases} 
A(x) + N_C(x), & \text{if } x \in C, \\
\emptyset, & \text{otherwise},
\end{cases} \]  

(4.8)

is maximal monotone and \( T^{-1}0 = \text{VI}(C, A) \). Fix \( r > 0, z \in E \), and let \( J_r \) be the resolvent of \( T \). Then we have

\[ Jz \in J(Jrz) + rT(Jrz) \]  

(4.9)
and hence,
\[ -A(Jr, z) + \frac{1}{r} (Jz - J(Jr, z)) \in N_C(Jr, z). \]  
(4.10)

Thus, we have
\[ \left\langle y - Jr, z, A(Jr, z) + \frac{1}{r} (J(Jr, z) - Jz) \right\rangle \geq 0 \]  
(4.11)

for all \( y \in C \), that is,
\[ Jr, z = VI(C, A + \frac{1}{r} (J - Jz)). \]  
(4.12)

Therefore, \( y_n = Jr, x_n \) for all \( n \in \mathbb{N} \). Using Theorem 3.3, \( \{x_n\} \) converges strongly to \( P_{VI(C, A)}(x) \). □

References


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