We prove an existence result for solution to a class of nonlinear degenerate elliptic equation associated with a class of partial differential operators of the form $Lu(x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x))$, with $D_j = \partial/\partial x_j$, where $a_{ij} : \Omega \to \mathbb{R}$ are functions satisfying suitable hypotheses.

1. Introduction

In this paper, we prove the existence of solution in $D(A) \subseteq H_0(\Omega)$ for the following nonlinear Dirichlet problem:

$$
-Lu(x) + g(u(x))\omega(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x) \quad \text{on } \Omega,
$$

$$
u(x) = 0 \quad \text{on } \partial \Omega,
$$

where $L$ is an elliptic operator in divergence form

$$
Lu(x) = \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)), \quad \text{with } D_j = \frac{\partial}{\partial x_j}
$$

and the coefficients $a_{ij}$ are measurable, real-valued functions whose coefficient matrix $(a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$
|\xi|^2 \omega(x) \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq |\xi|^2 \nu(x)
$$

for all $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega \subset \mathbb{R}^n$ a bounded open set with piecewise smooth boundary (i.e., $\partial \Omega \in C^{0,1}$), and $\omega$ and $\nu$ two weight functions (i.e., locally integrable non-negative functions).
The basic idea is to reduce (1.1) to an operator equation

$$Au = T, \quad u \in D(A),$$

(1.4)

where $D(A) = \{u \in H_0(\Omega) : u(x)g(u(x)) \in L^1(\Omega, \omega)\}$, and apply the theorem below.

**Theorem 1.1.** Suppose that the following assumptions are satisfied.

(H1) Dual pairs. Let the dual pairs $\{X, X^+\}$ and $\{Y, Y^+\}$ be given, where $X, X^+, Y,$ and $Y^+$ are Banach spaces with corresponding bilinear forms $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ and the continuous embeddings $Y \subseteq X$ and $X^+ \subseteq Y^+$.

The dual pairs are compatible, that is,

$$\langle T, u \rangle_X = \langle T, u \rangle_Y, \quad \forall T \in X^+, \ u \in Y.$$  \hspace{1cm} (1.5)

Moreover, the Banach spaces $X$ and $Y$ are separable and $X$ is reflexive.

(H2) Operator $A$. Let the operator $A : D(A) \subseteq X \to Y^+$ be given, and let $K$ be a bounded closed convex set in $X$ containing the zero point as an interior point and $K \cap Y \subseteq D(A)$.

(H3) Local coerciveness. There exists a number $\alpha \geq 0$ such that $\langle Av, v \rangle_Y \geq \alpha$ for all $v \in Y \cap \partial K$, where $\partial K$ denotes the boundary of $K$ in the Banach space $X$.

(H4) Continuity. For each finite-dimensional subspace $Y_0$ of the Banach space $Y$, the mapping $u \mapsto \langle Au, v \rangle_Y$ is continuous on $K \cap Y_0$ for all $v \in Y_0$.

(H5) Generalized condition (M). Let $\{u_n\}$ be a sequence in $Y \cap K$ and let $T \in X^+$. Then, from

$$u_n - u \quad \text{in} \ X \text{ as } n \to \infty, \quad (1.6)$$

$$\langle Au_n, v \rangle_Y \to \langle T, v \rangle_X \quad \text{as } n \to \infty, \ \forall v \in Y,$$

$$\lim_{n \to \infty} \langle Au_n, u_n \rangle_Y \leq \langle T, u \rangle_X, \quad (1.7)$$

it follows that $Au = T$.

(H6) Quasiboundedness. Let $\{u_n\}$ be a sequence in $Y \cap K$. Then, from (1.6) and $\langle Au_n, u_n \rangle_Y \leq C\|u\|_X$ for all $n$, it follows that the sequence $\{Au_n\}$ is bounded in $Y^+$.

(H7) The operator $A$ is coercive, that is, $\langle Av, v \rangle_Y / \|v\|_X \to \infty$ as $\|v\|_X \to \infty$, $v \in Y$.

Then $X^+ \subseteq R(A)$, that is, the equation $Au = T$ has a solution $u$ for each $T \in X^+$.

**Proof.** See [7, Theorem 27.B and Corollary 27.19].

We will apply this theorem to a sufficiently large ball $K$ in the Banach spaces $X = H_0(\Omega), X^+ = (H_0(\Omega))^*,$ and $Y^+ = Y^*$.

We make the following basic assumption on the weights $\omega$ and $\nu$.

The weighted Sobolev inequality (WSI). Let $\Omega$ be an open bounded set in $\mathbb{R}^n$. There is an index $q = 2\sigma$, $\sigma > 1$, such that for every ball $B$ and every $f \in \text{Lip}_0(B)$ (i.e., $f \in \text{Lip}(B)$ whose support is contained in the interior of $B$),

$$\left( \frac{1}{\nu(B)} \int_B |f|^q \nu \, dx \right)^{1/q} \leq CR_B \left( \frac{1}{\omega(B)} \int_B |\nabla f|^2 \omega \, dx \right)^{1/2},$$  \hspace{1cm} (1.8)
with the constant C independent of f and B, \(R_B\) the radius of B, and the symbol \(\nabla\) indicating the gradient, \(v(B) = \int_B v(x)dx\), and \(\omega(B) = \int_B \omega(x)dx\).

Thus, we can write

\[
\left(\int_B |f|^q v \, dx\right)^{\frac{1}{q}} \leq C_{B,\omega,v}(|\nabla f|^2 \omega)^{\frac{1}{2}},
\]

where \(C_{B,\omega,v}\) is called the Sobolev constant and

\[
C_{B,\omega,v} = \frac{C[v(B)]^{\frac{1}{q}} R_B}{[\omega(B)]^{\frac{1}{2}}}. \tag{1.10}
\]

For instance, the WSI holds if \(\omega\) and \(v\) are as in \([6, \text{Chapter X, Theorem 4.8}]\), or if \(\omega\) and \(v\) are as in \([1, \text{Theorem 1.5}]\).

The following theorem will be proved in Section 3.

**Theorem 1.2.** Let \(L\) be the operator (1.2) and satisfy (1.3). Suppose that the following assumptions are satisfied:

1. \((v, \omega) \in A_2\);
2. the function \(g : \mathbb{R} \to \mathbb{R}\) is continuous with \(xg(x) \geq 0\) for all \(x \in \mathbb{R}\);
3. \(f_0/v \in L^q(\Omega,v)\) and \(f_j/\omega \in L^2(\Omega,\omega), j = 1, 2, \ldots, n\) (where \(q\) is as in WSI). Then problem (1.1) has solution \(u \in D(A) \subseteq H_0^1(\Omega)\);
4. if the function \(g : \mathbb{R} \to \mathbb{R}\) is monotone increasing, then the solution is unique.

**Example 1.3.** Consider the domain \(\Omega = \{(x,y) \in \mathbb{R}^2 : |x| < 1\} \text{ and } |y| < 1\}. By Theorem 1.2, the problem

\[
-Lu(x) + u(x,y)e^{\omega(x,y)}|x|^{\frac{1}{2}} = 1 - \frac{\partial}{\partial x}(x^2|y|) - \frac{\partial}{\partial y}(y^2|x|) \quad \text{on } \Omega,
\]

\[
u(x,y) = 0 \quad \text{on } \partial \Omega,
\]

where

\[
Lu(x) = \left[\frac{\partial}{\partial x}(|x|^{\frac{1}{2}} \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(|x|^{-\frac{1}{2}} \frac{\partial u}{\partial y})\right] \tag{1.12}
\]

has a unique solution \(u \in D(A) = \{u \in H_0^1(\Omega) : g(u(x,y))u(x,y) \in L^1(\Omega,\omega)\}\), where \(g(t) = te^t\), \(\omega(x,y) = |x|^{\frac{1}{2}}\), \(\nu(x,y) = |x|^{-\frac{1}{2}}\), \(f_0(x,y) = 1\), \(f_1(x,y) = x^2|y|\), and \(f_2(x,y) = y^2|x|\).

2. **Definitions and basic results**

Let \(\omega\) be a locally integrable nonnegative function in \(\mathbb{R}^n\) and assume that \(0 < \omega < \infty\) almost everywhere. We say that \(\omega\) belongs to the Muckenhoupt class \(A_p, 1 < p < \infty\), or that \(\omega\) is an \(A_p\)-weight if there is a constant \(C_1 = C(p,\omega)\) such that

\[
\left(\frac{1}{|B|} \int_B \omega(x)dx\right)^p \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x)dx\right)^{p-1} \leq C_1, \tag{2.1}
\]
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for all balls $B \subset \mathbb{R}^n$, where $| \cdot |$ denotes the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [4, 5] for more information about $A_p$-weights). The weight $\omega$ satisfies the doubling condition if $\omega(2B) \leq C\omega(B)$, for all balls $B \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)dx$ and $2B$ denotes the ball with the same center as $B$ which is twice as large. If $\omega \in A_p$, then $\omega$ is doubling (see [5, Corollary 15.7]).

We say that the pair of weights $(v, \omega)$ satisfies the condition $A_p$ ($1 < p < \infty$ and $(v, \omega) \in A_p$) if and only if there is a constant $C_2$ such that

$$\left( \frac{1}{|B|} \int_B v(x)dx \right) \left( \frac{1}{|B|} \int_B \omega^{1/(1-p)}(x)dx \right)^{p-1} \leq C_2,$$

for every ball $B \subset \mathbb{R}^n$.

Remark 2.1. If $(v, \omega) \in A_p$ and $\omega \leq v$, then $\omega \in A_p$ and $v \in A_p$.

Given a measurable subset $\Omega$ of $\mathbb{R}^n$, we will denote by $L^p(\Omega, \omega)$, $1 \leq p < \infty$, the Banach space of all measurable functions $f$ defined on $\Omega$ for which

$$\|f\|_{L^p(\Omega, \omega)} = \left( \int_{\Omega} |f(x)|^p \omega(x)dx \right)^{1/p} < \infty.$$  \hspace{1cm} (2.3)

We will denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ such that the weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega, \omega)} = \left( \int_{\Omega} |u(x)|^p \omega(x)dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x)dx \right)^{1/p}. \hspace{1cm} (2.4)$$

If $\omega \in A_p$, then $W^{k,p}(\Omega, \omega)$ is the closure of $C^\infty(\bar{\Omega})$ with respect to the norm (2.4) (see [2, Proposition 3.5]). The space $W^{k,p}_0(\Omega, \omega)$ is the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{W^{k,p}_0(\Omega, \omega)} = \left( \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x)dx \right)^{1/p}. \hspace{1cm} (2.5)$$

When $k = 1$ and $p = 2$, the spaces $W^{1,2}(\Omega, \omega)$ and $W^{1,2}_0(\Omega, \omega)$ are Hilbert spaces. We will denote by $H_0(\Omega)$ the closure of $C^\infty_0(\bar{\Omega})$ with respect to the norm

$$\|u\|_{H_0(\Omega)} = \left( \int_{\Omega} \langle \mathcal{A}(x) \nabla u(x), \nabla u(x) \rangle dx \right)^{1/2}, \hspace{1cm} (2.6)$$

where $\mathcal{A}(x) = [a_{ij}(x)]$ (the coefficient matrix) and the symbol $\nabla$ indicates the gradient.
Remark 2.2. Using the condition (1.3), we have
\[\|u\|_{W^{1,2}_0(\Omega, \omega)} \leq \|u\|_{H^0(\Omega)} \leq \|u\|_{W^{1,2}_0(\Omega, \nu)},\] (2.7)
\[W^{1,2}_0(\Omega, \nu) \subset H^0(\Omega) \subset W^{1,2}_0(\Omega, \omega).\] (2.8)

Lemma 2.3. If \(\omega \in A_2\), then \(W^{1,2}_0(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega)\) is compact and
\[\|u\|_{L^2(\Omega, \omega)} \leq C_3 \|u\|_{W^{1,2}_0(\Omega, \omega)}.\] (2.9)

Proof. The proof follows the lines of [3, Theorem 4.6]. \(\square\)

We introduce the following definition of (weak) solutions for problem (1.1).

Definition 2.4. A function \(u \in D(A) \subseteq H^0(\Omega)\) is (weak) solution to the problem (1.1) if
\[\int_\Omega a_{ij}(x)D_i u(x)D_j \varphi(x)dx + \int_\Omega g(u(x))\varphi(x)\omega(x)dx = \int_\Omega f_0(x)\varphi(x)dx + \sum_{j=1}^n \int_\Omega f_j(x)D_j \varphi(x)dx,\] (2.10)
for all \(\varphi \in Y = H^0(\Omega) \cap W^{k,p}(\Omega, \nu)\), where \(p > 4\), \(k > n/2\), and \(\|\varphi\|_Y = \|\varphi\|_{W^{k,p}(\Omega, \nu)}\), with \(D(A) = \{u \in H^0(\Omega) : g(u(x))u(x) \in L^1(\Omega, \omega)\}\).

Remark 2.5. Using that \(p > 4\), we have that \(\nu \in A_2 \subset A_{p/2}\) and
\[\|\cdot\|_{L^2(\Omega)} \leq \left[\nu^{1/(1-p/2)}(\Omega)\right]^{(p-2)/2p} \|\cdot\|_{L^p(\Omega, \nu)}.\] (2.11)

Thus, \(W^{k,p}(\Omega, \nu) \subset W^{k,2}(\Omega) \subset C(\hat{\Omega})\) (by the Sobolev embedding theorem).

Therefore \(\|\cdot\|_{C(\hat{\Omega})} \leq C\|\cdot\|_Y\) and the embedding \(Y \subset C(\hat{\Omega})\) is continuous.

3. Proof of Theorem 1.2

(I) Existence. For \(u \in D(A)\) and \(\varphi \in Y\), we define
\[B_1(u, \varphi) = \int_\Omega a_{ij}(x)D_i u(x)D_j \varphi(x)dx,\]
\[B_2(u, \varphi) = \int_\Omega g(u(x))\varphi(x)\omega(x)dx,\]
\[T(\varphi) = \int_\Omega f_0(x)\varphi(x)dx + \sum_{j=1}^n \int_\Omega f_j(x)D_j \varphi(x)dx.\] (3.1)

Then \(u \in D(A) \subseteq H^0(\Omega)\) is solution to problem (1.1) if
\[B_1(u, \varphi) + B_2(u, \varphi) = T(\varphi), \quad \forall \varphi \in Y.\] (3.2)
Step 1 ($T \in (H_0(\Omega))^*$). In fact, using hypothesis (iii), Lemma 2.3, the Hölder inequality, the WSI, and (2.7), we obtain

\[ |T(\varphi)| \leq \int_{\Omega} |f_0| |\varphi| dx + \sum_{j=1}^{n} \int_{\Omega} |f_j| |D_j \varphi| dx \]

\[ = \int_{\Omega} \left( \frac{|f_0|}{v} \right)^{1/q'} |\varphi| v^{1/q} dx + \sum_{j=1}^{n} \int_{\Omega} \left( \frac{|f_j|}{\omega} \right)^{1/2} |D_j \varphi| \omega^{1/2} dx \]

\[ \leq \left\| \frac{f_0}{v} \right\|_{L^q(\Omega,v)} \| \varphi \|_{L^q(\Omega,v)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega} \right\|_{L^2(\Omega,\omega)} \|D_j \varphi\|_{L^2(\Omega,\omega)} \]

\[ \leq C_{B,\omega,v} \left( \left\| \frac{f_0}{v} \right\|_{L^q(\Omega,v)} \| \nabla \varphi \|_{L^2(\Omega,\omega)} + \sum_{j=1}^{n} \left\| \frac{f_j}{\omega} \right\|_{L^2(\Omega,\omega)} \|\nabla \varphi\|_{L^2(\Omega,\omega)} \right) \| \varphi \|_{H_1(\Omega)} \leq C \| \varphi \|_{Y}, \quad \forall \varphi \in H_0(\Omega). \]

Step 2. By condition (1.3) and the hypothesis that the matrix $\mathcal{A}$ is symmetric, we obtain

\[ |B_1(u,\varphi)| \leq \int_{\Omega} |\mathcal{A} \nabla u, \nabla \varphi| dx \]

\[ \leq \int_{\Omega} \langle \mathcal{A} \nabla u, \nabla \varphi \rangle^{1/2} \langle \mathcal{A} \nabla \varphi, \nabla \varphi \rangle^{1/2} dx \]

\[ \leq \|u\|_{H_0(\Omega)} \|\mathcal{A} \nabla \varphi\|_{H_0(\Omega)} \]

\[ \leq \|u\|_{H_0(\Omega)} \|\varphi\|_{W^{1,2}(\Omega,v)} \]

\[ \leq \|u\|_{H_0(\Omega)} \|\varphi\|_{Y}, \quad \forall u \in H_0(\Omega), \varphi \in Y. \]

Hence there exists exactly one linear continuous operator

\[ A_1 : H_0(\Omega) \rightarrow Y^*, \]

with

\[ \langle A_1 u, \varphi \rangle_Y = B_1(u,\varphi), \quad \forall u \in H_0(\Omega), \varphi \in Y. \]

Step 3. Note that $|g(x)| \leq xg(x) + C_4$, for all $x \in \mathbb{R}$. Therefore, if $u \in D(A)$, we have that $g(u(x)) \in L^1(\Omega,\omega)$. By using hypothesis (ii), Lemma 2.3, and Remark 2.5, we obtain for $u \in D(A)$ fixed

\[ |B_2(u,\varphi)| \leq \int_{\Omega} |g(u(x))| |\varphi(x)| \omega(x) dx \]

\[ \leq \|\varphi\|_{C(\Omega)} \int_{\Omega} |g(u(x))| \omega(x) dx \]

\[ \leq C \|\varphi\|_Y. \]
Thus, there exists a unique operator

$$A_2 : D(A) \subseteq H_0(\Omega) \longrightarrow Y^*, \quad (3.8)$$

with

$$\langle A_2 u, \varphi \rangle_Y = B_2(u, \varphi), \quad \forall u \in D(A), \varphi \in Y. \quad (3.9)$$

**Step 4.** We define the operator

$$A : D(A) \subseteq H_0(\Omega) \longrightarrow Y^*, \quad A = A_1 + A_2. \quad (3.10)$$

We have

$$\langle Au, \varphi \rangle_Y = \langle A_1 u, \varphi \rangle_Y + \langle A_2 u, \varphi \rangle_Y = B_1(u, \varphi) + B_2(u, \varphi). \quad (3.11)$$

Thus, $u \in D(A)$ is a solution to problem (1.1) if

$$\langle Au, \varphi \rangle_Y = T(\varphi), \quad \forall \varphi \in Y. \quad (3.12)$$

Then, the problem (1.1) corresponds to the operator equation (1.4).

**Step 5. Global coerciveness of operator $A$.** Using the condition (1.3) and hypothesis (ii), we obtain

$$\langle A\varphi, \varphi \rangle_Y = B_1(\varphi, \varphi) + B_2(\varphi, \varphi)$$

$$= \int_{\Omega} a_{ij}(x)D_i\varphi(x)D_j\varphi(x)dx + \int_{\Omega} g(\varphi(x))\varphi(x)\omega(x)dx$$

$$\geq \int_{\Omega} \langle \mathcal{A} \nabla \varphi, \nabla \varphi \rangle dx$$

$$= \|\varphi\|_{H_0(\Omega)}^2. \quad (3.13)$$

Thus

$$\lim_{\|\varphi\|_{H_0(\Omega)} \to \infty} \frac{\langle A\varphi, \varphi \rangle_Y}{\|\varphi\|_{L^2(\Omega)}} = +\infty. \quad (3.14)$$

**Step 6. Generalized condition (M).** Let $T \in (H_0(\Omega))^*$ and let $\{u_n\}$ be a sequence in $Y$ with

$$u_n \rightharpoonup u \text{ in } H_0(\Omega), \quad (3.15)$$

$$\langle Au_n, \varphi \rangle_Y \longrightarrow T(\varphi) \quad \text{as } n \to \infty, \forall \varphi \in Y, \quad (3.16)$$

$$\lim_{n \to \infty} \langle Au_n, u_n \rangle \leq T(u). \quad (3.17)$$

We want to show that this implies that $Au = T$.

Using that the operator $A_1$ is linear and continuous, we obtain

$$\langle A_1 u_n, \varphi \rangle_Y \longrightarrow \langle A_1 u, \varphi \rangle_Y, \quad \forall \varphi \in Y. \quad (3.18)$$
Because of (3.16), it is sufficient to prove that \( u \in D(A) \) and
\[
\langle A_2 u_n, \varphi \rangle_Y \longrightarrow \langle A_2 u, \varphi \rangle_Y, \quad \forall \varphi \in Y. \tag{3.19}
\]
Therefore, it is sufficient to show that
\[
\int_{\Omega} [g(u_n(x)) - g(u(x))] \varphi(x) \omega(x) dx \longrightarrow 0 \quad \text{as } n \to \infty. \tag{3.20}
\]
Using the same argument in Step 3, we obtain
\[
\left| \int_{\Omega} (g(u_n(x)) - g(u(x))) \varphi(x) \omega(x) dx \right|
\leq \int_{\Omega} |g(u_n(x)) - g(u(x))| |\varphi(x)| \omega(x) dx
\leq \|\varphi\|_{C(\overline{\Omega})} \int_{\Omega} |g(u_n(x)) - g(u(x))| \omega(x) dx
\leq C \|\varphi\|_Y \int_{\Omega} |g(u_n(x)) - g(u(x))| \omega(x) dx. \tag{3.21}
\]
Therefore, it is sufficient to show that
\[
g(u_n(x)) \to g(u(x)) \quad \text{in } L^1(\Omega, \omega). \tag{3.22}
\]
Note that it is sufficient to prove (3.22) for a subsequence of \( \{u_n\} \).

If \((v, \omega) \in A_2 \) and \( \omega \leq v \), then \( \omega \in A_2 \) (see Remark 2.1). By Lemma 2.3,
\[
W^{1,2}_0(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega) \tag{3.23}
\]
is compact and \( \|u\|_{L^2(\Omega, \omega)} \leq C_2 \|u\|_{W^{1,2}_0(\Omega, \omega)} \).
Using (2.7), we also have that
\[
H_0(\Omega) \hookrightarrow L^2(\Omega, \omega) \tag{3.24}
\]
is compact. This implies \( u_n \to u \) in \( L^2(\Omega, \omega) \). Using again that \( \omega \in A_2 \), we have \( u_n \to u \) in \( L^1(\Omega) \). Thus, there exists a subsequence, again denoted by \( \{u_n\} \), such that \( u_n(x) \to u(x) \) for almost all \( x \in \Omega \). The continuity of \( g \) implies that \( g(u_n(x)) \to g(u(x)) \) for almost all \( x \in \Omega \). Moreover, since \( u_n \to u \) in \( H_0(\Omega) \), it follows that
\[
\sup \|u_n\|_{H_0(\Omega)} \leq C, \quad \text{independent of } n. \tag{3.25}
\]
Hence, using (1.2), we obtain
\[
\langle A_1 u_n, u_n \rangle_Y \leq \Lambda \|u_n\|_{H_0(\Omega)}^2 \leq \Lambda C^2, \quad \text{with } C \text{ independent of } n. \tag{3.26}
\]
Therefore, using (3.16), we obtain
\[
\lim_{n \to \infty} \langle A_2 u_n, u_n \rangle_Y = \lim_{n \to \infty} \int_{\Omega} g(u_n(x)) u_n(x) \omega(x) dx \leq C, \tag{3.27}
\]
with \( C \) independent of \( n \).
The continuity of \( g \) implies that \( g(u_n(x))u_n(x)\omega(x) \to g(u(x))u(x)\omega(x) \) for almost all \( x \in \Omega \). Therefore, by Fatou lemma, we have
\[
\int_{\Omega} g(u(x))u(x)\omega(x)dx < \infty,
\]
(3.28)
that is, \( u \in D(A) \).

Now we want to show that \( g(u_n(x)) \to g(u(x)) \) in \( L^1(\Omega, \omega) \).

Let \( a > 0 \) be fixed. For each \( x \in \Omega \), we have either
\[
|u_n(x)| \leq a \quad \text{or} \quad |g(u_n(x))| \leq a^{-1}g(u_n(x))u_n(x)
\]
(3.29)
(if \( x \neq 0 \), we can write \( g(x) = x^{-1}|g(x)x| \)). We get \( |g(x)| \leq c(a) \) if \( |x| \leq a \) (because \( g \) is continuous).

Let \( X \) be a measurable subset of \( \Omega \). Then
\[
\int_{X} g(u_n(x))|\omega(x)dx = \int_{X \cap \{x:|u_n(x)| \leq a\}} g(u_n(x))|\omega(x)dx \\
+ \int_{X \cap \{x:|u_n(x)| > a\}} g(u_n(x))|\omega(x)dx \\
\leq c(a)\omega(X) + a^{-1} \int_{X} g(u_n(x))u_n(x)\omega(x)dx \\
\leq c(a)\omega(X) + a^{-1}C \quad \text{(by (3.27))}.
\]
(3.30)
Hence, for all \( \varepsilon > 0 \), we have
\[
\int_{X} g(u_n(x))|\omega(x)dx \leq \frac{\varepsilon}{2}
\]
(3.31)
if \( a \) is sufficiently large and \( \omega(X) \) is sufficiently small. Therefore, for all \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) \) such that
\[
\int_{X} |g(u_n(x)) - g(u(x))|\omega(x)dx \\
\leq \int_{X} |g(u_n(x))|\omega(x)dx + \int_{X} |g(u(x))|\omega(x)dx \leq \varepsilon,
\]
(3.32)
with \( \omega(X) < \delta \). Thus, the Vitali convergence theorem tells us that (3.22) holds.

**Step 7. Quasiboundedness of the operator \( A \).** Let \( \{u_n\} \) be a sequence in \( Y \) with \( u_n \to u \) in \( H_0(\Omega) \) and suppose that
\[
\langle Au_n, u_n \rangle_Y \leq C\|u_n\|_{H_0(\Omega)}, \quad \forall n.
\]
(3.33)
We want to show that the sequence \( \{Au_n\} \) is bounded in \( Y^* \). In fact, the boundedness of \( \{u_n\} \) in \( H_0(\Omega) \) implies that
\[
\lim_{n \to \infty} \langle Au_n, u_n \rangle_Y \leq C.
\]
(3.34)
Suppose by contradiction that the sequence \( \{Au_n\} \) is unbounded in \( Y^* \). Then there exists a subsequence, again denoted by \( \{u_n\} \), such that
\[
\|Au_n\|_{Y^*} \to \infty \quad \text{as} \quad n \to \infty.
\] (3.35)
By the same arguments as in Step 6, we obtain that
\[
\langle Au_n, \varphi \rangle_Y \to \langle Au, \varphi \rangle_Y \quad \text{as} \quad n \to \infty, \quad \forall \varphi \in Y.
\] (3.36)
The uniform boundedness principle tells us that the sequence \( \{Au_n\} \) is bounded (which is a contradiction with (3.35)).

Therefore, by Theorem 1.1, the equation \( Au = T \), with \( T \in (H_0(\Omega))^* \), has a solution \( u \in D(A) \subseteq H_0(\Omega) \), and it is the solution for problem (1.1).

(II) Uniqueness. If the function \( g : \mathbb{R} \to \mathbb{R} \) is monotone increasing, we have that \( (g(a) − g(b))(a − b) \geq 0 \), for all \( a, b \in \mathbb{R} \). Then
\[
\langle Au − Av, u − v \rangle_Y = \int_{\Omega} \langle \mathcal{A}(u − v), \nabla(u − v) \rangle dx
\]
\[
+ \int_{\Omega} (g(u(x)) − g(v(x))) (u(x) − v(x)) \omega(x) dx \geq \int_{\Omega} \langle \mathcal{A}(u − v), \nabla(u − v) \rangle dx = \|u − v\|_{H^1(\Omega)}^2.
\] (3.37)
for all \( u, v \in D(A) \).
Therefore, if \( u, v \in D(A) \) and \( Au = Av = T \), we obtain that \( u = v \).

References
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