ON A BOUNDARY VALUE PROBLEM FOR SCALAR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Theorems on the Fredholm alternative and well-posedness of the linear boundary value problem $u'(t) = \ell(u)(t) + q(t)$, $h(u) = c$, where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ are linear bounded operators, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$, are established even in the case when ℓ is not a *strongly bounded* operator. The question on the dimension of the solution space of the homogeneous equation $u'(t) = \ell(u)(t)$ is discussed as well.

1. Introduction

The following notation is used throughout: $\mathbb{N}$ is the set of all natural numbers; $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0,+\infty[$; $\operatorname{Ent}(x)$ is an entire part of $x \in \mathbb{R}$; $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$; $C([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a,b]\}$; $\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$; $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$; $L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for } t \in [a,b]\}$; $\operatorname{mes} A$ is the Lebesgue measure of the set A; $\mathcal{M}_{ab}$ is the set of measurable functions $\tau : [a,b] \to [a,b]$; $\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$; $\widetilde{\mathcal{L}}_{ab}$ is the set of linear strongly bounded operators, that is, for each of the operators $\ell \in \mathcal{L}_{ab}$, there exists $\eta \in L([a,b];\mathbb{R}_+)$ such that

$$|\ell(v)(t)| \leq \eta(t)\|v\|_C \quad \text{for } t \in [a,b],\ v \in C([a,b];\mathbb{R}); \tag{1.1}$$

$\mathcal{P}_{ab}$ is the set of linear nonnegative operators, that is, operators $\ell \in \mathcal{L}_{ab}$ mapping the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$. If $\ell \in \mathcal{L}_{ab}$, then $\|\ell\| = \sup\{\|\ell(v)\|_L : \|v\|_C \leq 1\}$.

Let $t_0 \in [a,b]$. We will say that $\ell \in \mathcal{L}_{ab}$ is a t_0-Volterra operator if for arbitrary $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, and $u \in C([a,b];\mathbb{R})$ such that

$$u(t) = 0 \quad \text{for } t \in [a_1,b_1], \tag{1.2}$$

we have

$$\ell(u)(t) = 0 \quad \text{for } t \in [a_1,b_1]. \tag{1.3}$$

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On the segment $[a,b]$, consider the boundary value problem

$$u'(t) = \ell(u)(t) + q(t), \tag{1.4}$$

$$h(u) = c, \tag{1.5}$$

where $\ell \in \mathcal{L}_{ab}$, $h : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear bounded functional, $q \in L([a,b];\mathbb{R})$, and $c \in \mathbb{R}$.

By a solution of (1.4) we understand a function $u \in \widetilde{C}([a,b];\mathbb{R})$ satisfying the equality (1.4) almost everywhere on $[a,b]$. By a solution of the problem (1.4), (1.5), we understand a solution u of (1.4) which also satisfies the condition (1.5). Together with (1.4), (1.5), we will consider the corresponding homogeneous problem

$$u'(t) = \ell(u)(t), \tag{1.6}$$

$$h(u) = 0. \tag{1.7}$$

From the general theory of boundary value problems for functional differential equations, it is known that if $\ell \in \widetilde{\mathcal{L}}_{ab}$, then the problem (1.4), (1.5) has a Fredholm property (see, e.g., [1, 2, 7, 8, 10]). More precisely, the following assertion is valid.

Theorem 1.1. *Let $\ell \in \widetilde{\mathcal{L}}_{ab}$. Then the problem (1.4), (1.5) is uniquely solvable if and only if the corresponding homogeneous problem (1.6), (1.7) has only the trivial solution.*

Theorem 1.1 allows us to introduce the following definition.

Definition 1.2. Let $\ell \in \widetilde{\mathcal{L}}_{ab}$ and let the problem (1.6), (1.7) have only the trivial solution. An operator $\Omega : L([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ which assigns to every $q \in L([a,b];\mathbb{R})$ a solution u of the problem (1.4), (1.7) is called Green operator of the problem (1.6), (1.7).

It follows from Theorem 1.1 that if $\ell \in \widetilde{\mathcal{L}}_{ab}$ and the problem (1.6), (1.7) has only the trivial solution, then the Green operator is well defined. Evidently, Green operator is linear. Moreover, the following theorem is valid (see, e.g., [1, 2, 7, 8]).

Theorem 1.3. *Let $\ell \in \widetilde{\mathcal{L}}_{ab}$ and let the problem (1.6), (1.7) have only the trivial solution. Then the Green operator of the problem (1.6), (1.7) is a linear bounded operator.*

In [7, 8] the question on the well-posedness of linear boundary value problem for systems of functional differential equations is studied. Theorem 1.3 can also be derived as a consequence of more general results on well-posedness obtained therein.

Note that both Theorems 1.1 and 1.3 claim that $\ell \in \widetilde{\mathcal{L}}_{ab}$. This condition covers a quite wide class of linear operators; for example, the equation with a deviating argument

$$u'(t) = p(t)u\big(\tau(t)\big) + q(t), \tag{1.8}$$

where $p,q \in L([a,b];\mathbb{R})$, $\tau \in \mathcal{M}_{ab}$, is a special case of (1.4) with

$$\ell(v)(t) \overset{\text{def}}{=} p(t)v\big(\tau(t)\big) \quad \text{for } t \in [a,b]. \tag{1.9}$$

More generally, it is known (see [6, page 317]) that $\ell \in \widetilde{\mathcal{L}}_{ab}$ if and only if the operator ℓ admits the representation by means of a Stieltjes integral.

On the other hand, Schaefer proved that there exists an operator $\ell \in \mathcal{L}_{ab}$ such that $\ell \notin \widetilde{\mathcal{L}}_{ab}$ (see [9, Theorem 4]). Therefore, a question naturally arises to study boundary value problem (1.4), (1.5) without the additional requirement (1.1). In particular, the question whether Theorems 1.1 and 1.3 are valid for general operator $\ell \in \mathcal{L}_{ab}$ is interesting.

The first important step in this direction was made by Bravyi (see [3]), where Theorem 1.1 was proved for $\ell \in \mathcal{L}_{ab}$ (i.e., without the additional assumption $\ell \in \widetilde{\mathcal{L}}_{ab}$). Bravyi's proof essentially uses Nikol'ski's theorem (see, e.g., [5, Theorem XIII.5.2, page 504]) and it is concentrated on the question of Fredholm property. The question whether Theorem 1.3 is valid for the case when $\ell \in \mathcal{L}_{ab}$ remains open.

In the present paper, among others, we answer this question affirmatively. More precisely, in Section 2 we prove that the operator $T : C([a,b];R) \to C([a,b];\mathbb{R})$ defined by $T(v)(t) \overset{\text{def}}{=} \int_a^t \ell(v)(s)ds$ for $t \in [a,b]$ is compact provided that $\ell \in \mathcal{L}_{ab}$ (see Proposition 2.9). Based on this result and Riesz-Schauder theory, we give an alternative proof (different from that in [3]) of Theorem 1.1 for $\ell \in \mathcal{L}_{ab}$ (see Theorem 2.1).

On the other hand, the compactness of the operator T allows us to study a question on the well-posedness of boundary value problem (1.4), (1.5). Section 3 is devoted to this question. As a special case of theorem on well-posedness, we obtain the validity of Theorem 1.3 for $\ell \in \mathcal{L}_{ab}$ (see Corollary 3.3).

In Section 4, the question on dimension of solution space U of homogeneous equation (1.6) is discussed. Proposition 4.6 shows that if $\dim U \geq 2$, then there exists $q \in L([a,b];\mathbb{R})$ such that the nonhomogeneous equation (1.4) has no solution. This "pathological" behaviour of functional differential equations affirms the importance of the question whether the solution space of the homogeneous equation (1.6) is one dimensional. In Theorems 4.8 and 4.10, the nonimprovable effective sufficient conditions are established guaranteeing that $\dim U = 1$.

2. Fredholm property

Theorem 2.1. *Let $\ell \in \mathcal{L}_{ab}$. Then the problem (1.4), (1.5) is uniquely solvable if and only if the corresponding homogeneous problem (1.6), (1.7) has only the trivial solution.*

Analogously as in Section 1, we can introduce the notion of the Green operator of the problem (1.6), (1.7).

Definition 2.2. Let $\ell \in \mathcal{L}_{ab}$ and let the problem (1.6), (1.7) have only the trivial solution. An operator $\Omega : L([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ which assigns to every $q \in L([a,b];\mathbb{R})$ a solution u of the problem (1.4), (1.7) is called Green operator of the problem (1.6), (1.7).

Evidently, it follows from Theorem 2.1 that the Green operator is well defined.

Remark 2.3. From the proof of Theorem 2.1 and Riesz-Schauder theory, it follows that if the problem (1.6), (1.7) has a nontrivial solution, then for every $c \in \mathbb{R}$ there exists $q \in L([a,b];\mathbb{R})$, respectively, for every $q \in L([a,b];\mathbb{R})$ there exists $c \in \mathbb{R}$, such that the problem (1.4), (1.5) has no solution.

To prove Theorem 2.1 we will need several auxiliary propositions. First we recall some definitions.

Definition 2.4. Let X be a linear topological space, X^* its dual space. A sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is called weakly convergent if there exists $x \in X$ such that $\varphi(x) = \lim_{n\to+\infty} \varphi(x_n)$ for every $\varphi \in X^*$. The point x is called a weak limit of this sequence.

A set $M \subseteq X$ is called weakly relatively compact if every sequence of points from M contains a subsequence which is weakly convergent in X.

A sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ is called weakly fundamental if for every $\varphi \in X^*$, a sequence $\{\varphi(x_n)\}_{n=1}^{+\infty}$ is fundamental.

A space X is called weakly complete if every weakly fundamental sequence from X possesses a weak limit in X.

Let X and Y be Banach spaces and let $T : X \to Y$ be a linear bounded operator. The operator T is said to be weakly completely continuous if it maps a unit ball of X into weakly relatively compact subset of Y.

Definition 2.5. A set $M \subseteq L([a,b];\mathbb{R})$ has a property of absolutely continuous integral if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for an arbitrary measurable set $E \subseteq [a,b]$ satisfying the condition $\operatorname{mes} E \le \delta$, the following inequality is true:

$$\left| \int_E p(s)ds \right| \le \varepsilon \quad \text{for every } p \in M. \tag{2.1}$$

Proofs of the following three assertions can be found in [4].

Lemma 2.6 [4, Theorem IV.8.6]. *The space $L([a,b];\mathbb{R})$ is weakly complete.*

Lemma 2.7 [4, Theorem VI.7.6]. *A linear bounded operator mapping the space $C([a,b];\mathbb{R})$ into a weakly complete Banach space is weakly completely continuous.*

Lemma 2.8 [4, Theorem IV.8.11]. *If a set $M \subseteq L([a,b];\mathbb{R})$ is weakly relatively compact, then it has a property of absolutely continuous integral.*

The following proposition plays a crucial role in the proof of Theorem 2.1.

Proposition 2.9. *Let $\ell \in \mathcal{L}_{ab}$. Then the operator $T : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ defined by*

$$T(v)(t) \stackrel{\text{def}}{=} \int_a^t \ell(v)(s)ds \quad \text{for } t \in [a,b] \tag{2.2}$$

is compact.

Proof. Let $M \subseteq C([a,b];\mathbb{R})$ be a bounded set. According to Arzelá-Ascoli lemma, it is sufficient to show that the set $T(M) = \{T(v) : v \in M\}$ is bounded and equicontinuous. Obviously,

$$\begin{aligned} \|T(v)\|_C &= \max\left\{ \left| \int_a^t \ell(v)(s)ds \right| : t \in [a,b] \right\} \\ &\le \|\ell(v)\|_L \le \|\ell\| \cdot \|v\|_C \quad \text{for } v \in M, \end{aligned} \tag{2.3}$$

and thus, since $\ell \in \mathcal{L}_{ab}$ and M is bounded, the set $T(M)$ is bounded.

Further, Lemmas 2.6 and 2.7 imply that the operator ℓ is weakly completely continuous, that is, a set $\ell(M) = \{\ell(v) : v \in M\}$ is weakly relatively compact. Therefore, according

to Lemma 2.8, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \int_s^t \ell(v)(\xi)d\xi \right| \le \varepsilon \quad \text{for } s,t \in [a,b],\ |t-s| \le \delta,\ v \in M. \tag{2.4}$$

On the other hand,

$$|T(v)(t) - T(v)(s)| = \left| \int_s^t \ell(v)(\xi)d\xi \right| \quad \text{for } s,t \in [a,b],\ v \in C([a,b];\mathbb{R}), \tag{2.5}$$

which, together with (2.4), results in

$$|T(v)(t) - T(v)(s)| \le \varepsilon \quad \text{for } s,t \in [a,b],\ |t-s| \le \delta,\ v \in M. \tag{2.6}$$

Consequently, the set $T(M)$ is equicontinuous. □

Proof of Theorem 2.1. Let $X = C([a,b];\mathbb{R}) \times \mathbb{R}$ be a Banach space containing elements $x = (u,\alpha)$, where $u \in C([a,b];\mathbb{R})$ and $\alpha \in \mathbb{R}$, with a norm

$$\|x\|_X = \|u\|_C + |\alpha|. \tag{2.7}$$

Let

$$\hat{q} = \left(\int_a^t q(s)ds, c \right) \tag{2.8}$$

and define a linear operator $T : X \to X$ by setting

$$T(x) \stackrel{\text{def}}{=} \left(\alpha + u(a) + \int_a^t \ell(u)(s)ds, \alpha - h(u) \right). \tag{2.9}$$

Obviously, the problem (1.4), (1.5) is equivalent to the operator equation

$$x = T(x) + \hat{q} \tag{2.10}$$

in the space X in the following sense: if $x = (u,\alpha) \in X$ is a solution of (2.10), then $\alpha = 0$, $u \in \widetilde{C}([a,b];\mathbb{R})$, and u is a solution of (1.4), (1.5), and vice versa, if $u \in \widetilde{C}([a,b];\mathbb{R})$ is a solution of (1.4), (1.5), then $x = (u,0)$ is a solution of (2.10).

According to Proposition 2.9, we have that the operator T is compact. From Riesz-Schauder theory, it follows that (2.10) is uniquely solvable if and only if the corresponding homogeneous equation

$$x = T(x) \tag{2.11}$$

has only the trivial solution (see, e.g., [11, Theorem 2, page 221]). On the other hand, (2.11) is equivalent to the problem (1.6), (1.7) in the above-mentioned sense. □

Following [7, 8] we introduce the following notation.

Notation 2.10. Let $t_0 \in [a,b]$. Define operators $\ell^k : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ and numbers λ_k as follows:

$$\ell^0(v)(t) \stackrel{\text{def}}{=} v(t), \quad \ell^k(v)(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \ell(\ell^{k-1}(v))(s)ds \quad \text{for } t \in [a,b],\ k \in \mathbb{N}, \tag{2.12}$$

$$\lambda_k = h(\ell^0(1) + \ell^1(1) + \cdots + \ell^{k-1}(1)) \quad \text{for } k \in \mathbb{N}. \tag{2.13}$$

If $\lambda_k \neq 0$ for some $k \in \mathbb{N}$, then let

$$\begin{gathered} \ell^{k,0}(v)(t) \stackrel{\text{def}}{=} v(t) \quad \text{for } t \in [a,b], \\ \ell^{k,m}(v)(t) \stackrel{\text{def}}{=} \ell^m(v)(t) - \frac{h(\ell^k(v))}{\lambda_k} \sum_{i=0}^{m-1} \ell^i(1)(t) \quad \text{for } t \in [a,b],\ m \in \mathbb{N}. \end{gathered} \tag{2.14}$$

Theorem 2.11. *Let $\ell \in \mathcal{L}_{ab}$ and let there exist $k, m \in \mathbb{N}$, $m_0 \in \mathbb{N} \cup \{0\}$, and $\alpha \in [0,1[$ such that $\lambda_k \neq 0$ and for every solution u of the problem (1.6), (1.7), the inequality*

$$||\ell^{k,m}(u)||_C \leq \alpha ||\ell^{k,m_0}(u)||_C \tag{2.15}$$

is fulfilled. Then the problem (1.4), (1.5) has a unique solution.

Remark 2.12. The proof of Theorem 2.11 is omitted since it is completely the same as the proof of [8, Theorem 1.3.1] (see also [7, Theorem 1.2]). The only difference is that instead of Theorem 1.1, Theorem 2.1 has to be used.

Theorem 2.11 implies the following corollary.

Corollary 2.13. *Let $\ell \in \mathcal{L}_{ab}$ be a t_0-Volterra operator. Then the problem*

$$u'(t) = \ell(u)(t) + q(t), \qquad u(t_0) = c, \tag{2.16}$$

with $q \in L([a,b];\mathbb{R})$ and $c \in \mathbb{R}$, is uniquely solvable.

To prove this corollary we need the following lemma.

Lemma 2.14. *Let $\ell \in \mathcal{L}_{ab}$ be a t_0-Volterra operator and let ℓ^k ($k \in \mathbb{N} \cup \{0\}$) be operators defined by (2.12). Then*

$$\lim_{k \to +\infty} ||\ell^k|| = 0. \tag{2.17}$$

Proof. Let $\varepsilon \in]0,1[$. According to Proposition 2.9, the operator ℓ^1, defined by (2.12) for $k = 1$, is compact. Therefore, by virtue of Arzelà-Ascoli lemma, there exists $\delta > 0$ such that

$$\left| \int_s^t \ell(v)(\xi)d\xi \right| = \left| \ell^1(v)(t) - \ell^1(v)(s) \right| \leq \varepsilon \|v\|_C \quad \text{for } |t-s| \leq \delta. \tag{2.18}$$

Let

$$\begin{gathered} n = \operatorname{Ent}\left(\frac{b-t_0}{\delta}\right), \qquad m = \operatorname{Ent}\left(\frac{t_0-a}{\delta}\right), \\ t_i = t_0 + i\delta \quad \text{for } i = -m, -m+1, \dots, -1, 1, 2, \dots, n, \\ t_{-m-1} = a, \qquad t_{n+1} = b, \end{gathered} \tag{2.19}$$

and introduce the notation

$$\|\ell^k(v)\|_i = \begin{cases} \|\ell^k(v)\|_{C([t_0,t_i];\mathbb{R})} & \text{for } i = \overline{1,n+1}, \\ \|\ell^k(v)\|_{C([t_i,t_0];\mathbb{R})} & \text{for } i = \overline{-m-1,-1}. \end{cases} \tag{2.20}$$

We will show that

$$\|\ell^k(v)\|_i \le \alpha_i(k)\varepsilon^k \|v\|_{C([a,b];\mathbb{R})} \quad \text{for } i = \overline{1,n+1},\ k \in \mathbb{N}, \tag{2.21}$$

where

$$\begin{gathered} \alpha_i(k) = \gamma_i k^{i-1} \quad \text{for } i = \overline{1,n+1}, \\ \gamma_1 = 1, \qquad \gamma_{i+1} = i\gamma_i + i + 1 \quad \text{for } i = \overline{1,n}. \end{gathered} \tag{2.22}$$

First note that

$$\|\ell^1(v)\|_i \le i\varepsilon \|v\|_{C([a,b];\mathbb{R})} \quad \text{for } i = \overline{1,n+1}. \tag{2.23}$$

Indeed, according to (2.18), it is clear that

$$\begin{aligned} \|\ell^1(v)\|_i &= \max\left\{ \left| \int_{t_0}^{t} \ell(v)(\xi)d\xi \right| : t \in [t_0, t_i] \right\} \\ &\le \sum_{j=0}^{i-1} \max\left\{ \left| \int_{t_j}^{t} \ell(v)(\xi)d\xi \right| : t \in [t_j, t_{j+1}] \right\} \\ &\le i\varepsilon \|v\|_{C([a,b];\mathbb{R})} \quad \text{for } i = \overline{1,n+1}. \end{aligned} \tag{2.24}$$

Further, on account of (2.18) and the fact that ℓ is a t_0-Volterra operator, we have

$$|\ell^{k+1}(v)(t)| = \left| \int_{t_0}^{t} \ell(\ell^k(v))(\xi)d\xi \right| \le \varepsilon \|\ell^k(v)\|_1 \quad \text{for } t \in [t_0, t_1],\ k \in \mathbb{N}. \tag{2.25}$$

Hence, by virtue of (2.23), we get

$$\|\ell^k(v)\|_1 \le \varepsilon^k \|v\|_{C([a,b];\mathbb{R})} \quad \text{for } k \in \mathbb{N}, \tag{2.26}$$

that is, (2.21) holds for $i = 1$.

Now let the inequality (2.21) hold for some $i \in \{1,2,\dots,n\}$. With respect to (2.18) and the fact that ℓ is a t_0-Volterra operator, we have

$$\begin{aligned}
\|\ell^{k+1}(v)\|_{i+1} &= \max\left\{\left|\int_{t_0}^{t} \ell(\ell^k(v))(\xi)d\xi\right| : t \in [t_0, t_{i+1}]\right\} \\
&\le \sum_{j=0}^{i-1} \max\left\{\left|\int_{t_j}^{t} \ell(\ell^k(v))(\xi)d\xi\right| : t \in [t_j, t_{j+1}]\right\} \\
&\quad + \max\left\{\left|\int_{t_i}^{t} \ell(\ell^k(v))(\xi)d\xi\right| : t \in [t_i, t_{i+1}]\right\} \\
&\le i\varepsilon\|\ell^k(v)\|_i + \varepsilon\|\ell^k(v)\|_{i+1} \\
&\le i\alpha_i(k)\varepsilon^{k+1}\|v\|_{C([a,b];\mathbb{R})} + \varepsilon\|\ell^k(v)\|_{i+1} \quad \text{for } k \in \mathbb{N}.
\end{aligned} \tag{2.27}$$

Hence we get

$$\begin{aligned}
\|\ell^{k+1}(v)\|_{i+1} &\le i\alpha_i(k)\varepsilon^{k+1}\|v\|_{C([a,b];\mathbb{R})} \\
&\quad + \varepsilon\left[i\alpha_i(k-1)\varepsilon^{k}\|v\|_{C([a,b];\mathbb{R})} + \varepsilon\|\ell^{k-1}(v)\|_{i+1}\right] \quad \text{for } k \in \mathbb{N}.
\end{aligned} \tag{2.28}$$

To continue this procedure, on account of (2.23), we obtain

$$\|\ell^{k+1}(v)\|_{i+1} \le \left[i+1+i(\alpha_i(1)+\cdots+\alpha_i(k))\right]\varepsilon^{k+1}\|v\|_{C([a,b];\mathbb{R})} \quad \text{for } k \in \mathbb{N}. \tag{2.29}$$

With respect to (2.22), we get

$$\begin{aligned}
i+1+i\sum_{j=1}^{k}\alpha_i(j) &= i+1+i\gamma_i\left(1^{i-1}+2^{i-1}+\cdots+k^{i-1}\right) \le i+1+i\gamma_i k k^{i-1} \\
&= i+1+i\gamma_i k^i \le (i+1+i\gamma_i)k^i = \gamma_{i+1}k^i \le \alpha_{i+1}(k+1).
\end{aligned} \tag{2.30}$$

Therefore, from (2.29), it follows that

$$\|\ell^{k+1}(v)\|_{i+1} \le \alpha_{i+1}(k+1)\varepsilon^{k+1}\|v\|_{C([a,b];\mathbb{R})} \quad \text{for } k \in \mathbb{N}. \tag{2.31}$$

Thus, by induction, we have proved that (2.21) holds.

In an analogous way, it can be shown that

$$\|\ell^k(v)\|_i \le \alpha_i(k)\varepsilon^k\|v\|_{C([a,b];\mathbb{R})} \quad \text{for } i = \overline{-m-1,-1},\ k \in \mathbb{N}, \tag{2.32}$$

where

$$\begin{gathered}
\alpha_i(k) = \gamma_i k^{|i|-1} \quad \text{for } i = \overline{-m-1,-1}, \\
\gamma_{-1} = 1, \qquad \gamma_{i-1} = |i|\gamma_i + |i| + 1 \quad \text{for } i = \overline{-m,-1}.
\end{gathered} \tag{2.33}$$

Now from (2.21), (2.22), (2.32), and (2.33), it follows that there exists $\gamma \in \mathbb{N}$ (independent of k) such that

$$\begin{aligned} \|\ell^k(v)\|_{C([a,b];\mathbb{R})} &\le \|\ell^k(v)\|_{-m-1} + \|\ell^k(v)\|_{n+1} \\ &\le \gamma k^{n+m}\varepsilon^k \|v\|_{C([a,b];\mathbb{R})} \quad \text{for } k \in \mathbb{N}. \end{aligned} \tag{2.34}$$

Hence, since $\varepsilon < 1$, it follows that (2.17) holds. □

Proof of Corollary 2.13. Let $h(v) \overset{\text{def}}{=} v(t_0)$. Obviously, for every $k, m \in \mathbb{N}$, we have $\lambda_k = 1$,

$$h(\ell^k(v)) = 0, \quad \ell^{k,m}(v)(t) = \ell^m(v)(t) \quad \text{for } t \in [a,b],\ v \in C([a,b];\mathbb{R}). \tag{2.35}$$

According to Lemma 2.14, we can choose $m \in \mathbb{N}$ such that

$$\|\ell^m\| < 1. \tag{2.36}$$

Thus the inequality (2.15) holds with $m_0 = 0$ and $\alpha = \|\ell^m\|$. □

For t_0-Volterra operators, Theorem 2.11 can be inverted. More precisely, the following assertion is valid.

Theorem 2.15. *Let $\ell \in \mathcal{L}_{ab}$ be a t_0-Volterra operator. Then the problem (1.4), (1.5) has a unique solution if and only if there exist $k, m \in \mathbb{N}$ such that $\lambda_k \neq 0$ and*

$$\|\ell^{k,m}\| < 1. \tag{2.37}$$

Proof. Let inequality (2.37) hold for some $k, m \in \mathbb{N}$. Obviously, for every $u \in C([a,b];\mathbb{R})$ (consequently, also for every solution of (1.6), (1.7)), we have

$$\|\ell^{k,m}(u)\|_C \le \|\ell^{k,m}\| \|u\|_C. \tag{2.38}$$

Therefore, the assumptions of Theorem 2.11 are fulfilled with $m_0 = 0$ and $\alpha = \|\ell^{k,m}\|$. Consequently, the problem (1.4), (1.5) has a unique solution.

Assume now that the problem (1.6), (1.5) is uniquely solvable. According to Theorem 2.1, the problem (1.6), (1.7) has only the trivial solution.

Let u_0 be a solution of the problem

$$u'(t) = \ell(u)(t), \qquad u(t_0) = 1, \tag{2.39}$$

the existence of which is guaranteed by Corollary 2.13. Obviously,

$$h(u_0) \neq 0, \tag{2.40}$$

since otherwise the function u_0 would be a nontrivial solution of the problem (1.6), (1.7). Let

$$u_n(t) = \sum_{i=0}^{n-1} \ell^i(1)(t) \quad \text{for } t \in [a,b],\ n \in \mathbb{N}. \tag{2.41}$$

From (2.39) it follows that

$$u_0(t) = 1 + \ell^1(u_0)(t) \quad \text{for } t \in [a,b]. \tag{2.42}$$

Hence we have

$$u_0(t) = 1 + \ell^1(1 + \ell^1(u_0))(t) = \ell^0(1)(t) + \ell^1(1)(t) + \ell^2(u_0)(t) \quad \text{for } t \in [a,b]. \tag{2.43}$$

To continue this process, we obtain

$$u_0(t) = \sum_{i=0}^{n-1} \ell^i(1)(t) + \ell^n(u_0)(t) \quad \text{for } t \in [a,b],\ n \in \mathbb{N}. \tag{2.44}$$

Hence, on account of (2.41) and Lemma 2.14, we get

$$\lim_{n\to+\infty} \|u_0 - u_n\|_C = 0. \tag{2.45}$$

Since $\lambda_n = h(u_n)$ for $n \in \mathbb{N}$ and h is a continuous functional, we have, with respect to (2.40) and (2.45), that

$$\lim_{n\to+\infty} \lambda_n = h(u_0) \neq 0. \tag{2.46}$$

Therefore, there exist $k_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$|\lambda_i| \geq \delta \quad \text{for } i \geq k_0. \tag{2.47}$$

Hence, by virtue of (2.45), it follows that there exists $\rho \in]0,+\infty[$ such that

$$\frac{1}{|\lambda_i|}\|u_j\|_C\|h\| \leq \rho \quad \text{for } i \geq k_0,\ j \in \mathbb{N}. \tag{2.48}$$

According to Lemma 2.14, there exist $k > k_0$ and $m \in \mathbb{N}$ such that

$$\|\ell^k\| \leq \frac{1}{2\rho}, \qquad \|\ell^m\| < \frac{1}{2}. \tag{2.49}$$

Furthermore, in view of (2.14), we have

$$\|\ell^{k,m}\| \leq \|\ell^m\| + \frac{\|u_m\|_C}{|\lambda_k|}\|h\|\|\ell^k\|, \tag{2.50}$$

which, together with (2.48) and (2.49), implies that (2.37) holds. □

Remark 2.16. For the case when $\ell \in \widetilde{\mathcal{L}}_{ab}$, Theorem 2.15 is proved in [8] (see also [7]).

3. Well-posedness

Together with the problem (1.4), (1.5), for every $k \in \mathbb{N}$, consider the perturbed boundary value problem

$$u'(t) = \ell_k(u)(t) + q_k(t), \qquad h_k(u) = c_k, \tag{3.1}$$

where $\ell_k \in \mathcal{L}_{ab}$, $h_k : C([a,b];\mathbb{R}) \to \mathbb{R}$ is a linear bounded functional, $q_k \in L([a,b];\mathbb{R})$, and $c_k \in \mathbb{R}$.

The question on well-posedness of general linear boundary value problem for functional differential equation under the assumptions $\ell \in \widetilde{\mathcal{L}}_{ab}$ and $\ell_k \in \widetilde{\mathcal{L}}_{ab}$ is studied in [7, 8] (see also references in [8, page 70]). In this section we will show that the theorems on well-posedness established in [7, 8] are valid also for the case when $\ell \in \mathcal{L}_{ab}$ and $\ell_k \in \mathcal{L}_{ab}$.

Notation 3.1. Let $\ell \in \mathcal{L}_{ab}$. Denote by M_ℓ the set of functions $y \in \widetilde{C}([a,b];\mathbb{R})$ admitting the representation

$$y(t) = z(a) + \int_a^t \ell(z)(s)ds \quad \text{for } t \in [a,b], \tag{3.2}$$

where $z \in C([a,b];\mathbb{R})$ and $\|z\|_C = 1$.

Theorem 3.2. *Let the problem (1.4), (1.5) have a unique solution u,*

$$\sup\left\{ \left| \int_a^t [\ell_k(y)(s) - \ell(y)(s)]ds \right| : t \in [a,b],\ y \in M_{\ell_k} \right\} \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty, \tag{3.3}$$

and let, for every $y \in \widetilde{C}([a,b];\mathbb{R})$,

$$\lim_{k\to+\infty} (1 + \|\ell_k\|) \int_a^t [\ell_k(y)(s) - \ell(y)(s)]ds = 0 \quad \text{uniformly on } [a,b]. \tag{3.4}$$

Let, moreover,

$$\lim_{k\to+\infty} (1 + \|\ell_k\|) \int_a^t [q_k(s) - q(s)]ds = 0 \quad \text{uniformly on } [a,b], \tag{3.5}$$

$$\lim_{k\to+\infty} h_k(y) = h(y) \quad \text{for } y \in C([a,b];\mathbb{R}), \tag{3.6}$$

$$\lim_{k\to+\infty} c_k = c. \tag{3.7}$$

Then there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$ the problem (3.1) has a unique solution u_k and

$$\lim_{k\to+\infty} \|u_k - u\|_C = 0. \tag{3.8}$$

From Theorem 3.2, the following corollary immediately follows.

Corollary 3.3. *Let $\ell \in \mathcal{L}_{ab}$ and the problem (1.6), (1.7) have only the trivial solution. Then the Green operator of the problem (1.6), (1.7) is continuous.*

To prove Theorem 3.2, we need two lemmas, the first of them immediately follows from Arzelà-Ascoli lemma and Proposition 2.9.

Lemma 3.4. *Let $\ell \in \mathcal{L}_{ab}$ and*

$$\widehat{\ell}(y)(t) \stackrel{\text{def}}{=} \int_a^t \ell(y)(s)ds \quad \text{for } t \in [a,b]. \tag{3.9}$$

Let, moreover, $\{x_n\}_{n=1}^{+\infty} \subset C([a,b];\mathbb{R})$ be a bounded sequence. Then the sequence $\{\widehat{\ell}(x_n)\}_{n=1}^{+\infty}$ contains a uniformly convergent subsequence.

Lemma 3.5. *Let the problem (1.6), (1.7) have only the trivial solution and let the sequences of operators $\ell_k \in \mathcal{L}_{ab}$ and linear bounded functionals $h_k : C([a,b];\mathbb{R}) \to \mathbb{R}$ satisfy conditions (3.3) and (3.6). Then there exist $k_0 \in \mathbb{N}$ and $r > 0$ such that an arbitrary $z \in \widetilde{C}([a,b];\mathbb{R})$ admits the estimate*

$$\|z\|_C \le r\rho_k(z) \quad \text{for } k > k_0, \tag{3.10}$$

where

$$\rho_k(z) = |h_k(z)| + \max\left\{(1+\|\ell_k\|)\left|\int_a^t [z'(s) - \ell_k(z)(s)]ds\right| : t \in [a,b]\right\}. \tag{3.11}$$

Proof. Note first that according to Banach-Steinhaus theorem and the condition (3.6), the sequence $\{\|h_k\|\}_{k=1}^{+\infty}$ is bounded, that is, there exists $r_0 > 0$ such that

$$|h_k(y)| \le r_0\|y\|_C \quad \text{for } y \in C([a,b];\mathbb{R}). \tag{3.12}$$

Let, for $y \in C([a,b];\mathbb{R})$,

$$\widehat{\ell}(y)(t) = \int_a^t \ell(y)(s)ds, \qquad \widehat{\ell}_k(y)(t) = \int_a^t \ell_k(y)(s)ds \quad \text{for } k \in \mathbb{N}. \tag{3.13}$$

Obviously, $\widehat{\ell} : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ and $\widehat{\ell}_k : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ for $k \in \mathbb{N}$ are linear bounded operators and

$$\|\widehat{\ell}_k\| \le \|\ell_k\| \quad \text{for } k \in \mathbb{N}. \tag{3.14}$$

With respect to our notation, the condition (3.3) can be rewritten as follows:

$$\sup\{\|\widehat{\ell}_k(y) - \widehat{\ell}(y)\|_C : y \in M_{\ell_k}\} \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \tag{3.15}$$

Assume on the contrary that the lemma is not valid. Then there exist an increasing sequence of natural numbers $\{k_m\}_{m=1}^{+\infty}$ and a sequence of functions $z_m \in \widetilde{C}([a,b];\mathbb{R})$, $m \in \mathbb{N}$, such that

$$\|z_m\|_C > m\rho_{k_m}(z_m) \quad \text{for } m \in \mathbb{N}. \tag{3.16}$$

Let

$$y_m(t) = \frac{z_m(t)}{||z_m||_C}, \quad v_m(t) = \int_a^t [y_m'(s) - \ell_{k_m}(y_m)(s)]ds \quad \text{for } t \in [a,b], \tag{3.17}$$

$$y_{0m}(t) = y_m(t) - v_m(t) \quad \text{for } t \in [a,b], \tag{3.18}$$

$$w_m(t) = \widehat{\ell}_{k_m}(y_{0m})(t) - \widehat{\ell}(y_{0m})(t) + \widehat{\ell}_{k_m}(v_m)(t) \quad \text{for } t \in [a,b]. \tag{3.19}$$

Obviously,

$$||y_m||_C = 1 \quad \text{for } m \in \mathbb{N}, \tag{3.20}$$

$$y_{0m}(t) = y_m(a) + \widehat{\ell}_{k_m}(y_m)(t) \quad \text{for } t \in [a,b],\ m \in \mathbb{N}, \tag{3.21}$$

$$y_{0m}(t) = y_m(a) + \widehat{\ell}(y_{0m})(t) + w_m(t) \quad \text{for } t \in [a,b],\ m \in \mathbb{N}. \tag{3.22}$$

On the other hand, from (3.14) and (3.17), by virtue of (3.16), we get

$$||v_m||_C \le \frac{\rho_{k_m}(z_m)}{||z_m||_C(1+||\ell_{k_m}||)} < \frac{1}{m(1+||\ell_{k_m}||)} \quad \text{for } m \in \mathbb{N}, \tag{3.23}$$

$$||\widehat{\ell}_{k_m}(v_m)||_C \le ||\ell_{k_m}|| \cdot ||v_m||_C < \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \tag{3.24}$$

From (3.20) and (3.21), it follows that $y_{0m} \in M_{\ell_{k_m}}$, and therefore, in view of (3.15), we have

$$\lim_{m\to+\infty} ||\widehat{\ell}_{k_m}(y_{0m}) - \widehat{\ell}(y_{0m})||_C = 0. \tag{3.25}$$

On account of (3.24) and (3.25), equality (3.19) implies that

$$\lim_{m\to+\infty} ||w_m||_C = 0, \tag{3.26}$$

and with respect to (3.18), (3.20), and (3.23),

$$||y_{0m}||_C \le ||y_m||_C + ||v_m||_C \le 2 \quad \text{for } m \in \mathbb{N}. \tag{3.27}$$

According to Lemma 3.4, without loss of generality, we can assume that

$$\lim_{m\to+\infty} y_{0m}(t) = y_0(t) \quad \text{uniformly on } [a,b]. \tag{3.28}$$

With respect to (3.18), (3.20), (3.22), (3.23), and (3.26),

$$\lim_{m\to+\infty} ||y_m - y_0||_C = 0, \tag{3.29}$$

$$||y_0||_C = 1, \qquad y_0(t) = y_0(a) + \widehat{\ell}(y_0)(t) \quad \text{for } t \in [a,b]. \tag{3.30}$$

Consequently, y_0 is a nontrivial solution of (1.6).

On the other hand, from (3.12) and (3.16), we get

$$\begin{aligned} |h_{k_m}(y_0)| &\le |h_{k_m}(y_0 - y_m)| + |h_{k_m}(y_m)| \\ &\le r_0 \|y_0 - y_m\|_C + \frac{1}{\|z_m\|_C} |h_{k_m}(z_m)| \\ &\le r_0 \|y_0 - y_m\|_C + \frac{1}{m} \quad \text{for } m \in \mathbb{N}. \end{aligned} \tag{3.31}$$

Hence, on account of (3.6) and (3.29), we obtain

$$h(y_0) = 0. \tag{3.32}$$

Thus y_0 is a nontrivial solution of the problem (1.6), (1.7), which contradicts the assumption of Lemma 3.5. □

Proof of Theorem 3.2. Let r and k_0 be numbers, the existence of which is guaranteed by Lemma 3.5. Then, obviously, for every $k > k_0$, the problem

$$u'(t) = \ell_k(u)(t), \qquad h_k(u) = 0, \tag{3.33}$$

has only the trivial solution. According to Theorem 2.1, for every $k > k_0$, the problem (3.1) is uniquely solvable.

We will show that if u and u_k are solutions of the problems (1.4), (1.5), and (3.1), respectively, then (3.8) holds. Let

$$v_k(t) = u_k(t) - u(t) \quad \text{for } t \in [a,b]. \tag{3.34}$$

Then, for every $k > k_0$,

$$v_k'(t) = \ell_k(v_k)(t) + \tilde{q}_k(t) \quad \text{for } t \in [a,b], \qquad h_k(v_k) = \tilde{c}_k, \tag{3.35}$$

where

$$\begin{gathered} \tilde{q}_k(t) = \ell_k(u)(t) - \ell(u)(t) + q_k(t) - q(t) \quad \text{for } t \in [a,b], \\ \tilde{c}_k = c_k - h_k(u). \end{gathered} \tag{3.36}$$

Now, by virtue of (3.4), (3.5), (3.6), and (3.7), we have

$$\delta_k = (1 + \|\ell_k\|) \max\left\{ \left| \int_a^t \tilde{q}_k(s)ds \right| : t \in [a,b] \right\} \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty, \tag{3.37}$$

$$\lim_{k \to +\infty} \tilde{c}_k = 0. \tag{3.38}$$

According to Lemma 3.5, (3.35), and (3.37),

$$\|v_k\|_C \le r(|\tilde{c}_k| + \delta_k) \quad \text{for } k > k_0. \tag{3.39}$$

Hence, in view of (3.37) and (3.38), we obtain

$$\lim_{k\to+\infty} \|v_k\|_C = 0, \tag{3.40}$$

and, consequently, (3.8) holds. □

4. On dimension of the solution set of homogeneous equation

Notation 4.1. Let U be the solution set of the homogeneous equation (1.6). Obviously, U is a linear vector space.

According to Theorem 2.1, we have $U \neq \{0\}$, that is, $\dim U \geq 1$. Moreover, the following assertion is valid.

Theorem 4.2. *The space U is finite dimensional.*

Proof. Let $T : C([a,b];\mathbb{R}) \to C([a,b];\mathbb{R})$ be an operator defined by

$$T(v)(t) \stackrel{\text{def}}{=} v(a) + \int_a^t \ell(v)(s)ds \quad \text{for } t \in [a,b]. \tag{4.1}$$

Evidently, the operator T is linear. According to Proposition 2.9, the operator T is compact as well. Obviously, (1.6) is equivalent to the operator equation (2.11) in the following sense: if $u \in \widetilde{C}([a,b];\mathbb{R})$ is a solution of (1.6), then $x = u$ is a solution of (2.11), and vice versa, if $x \in C([a,b];\mathbb{R})$ is a solution of (2.11), then $x \in \widetilde{C}([a,b];\mathbb{R})$ and $u = x$ is a solution of (1.6). In other words, the set U is also a solution set of the operator equation (2.11).

On the other hand, since T is a linear compact operator, from Riesz-Schauder theory, it follows that the solution space of (2.11) is finite-dimensional. Therefore, $\dim U < +\infty$. □

Remark 4.3. Example 5.1 below shows that $\dim U$ can be any natural number, even in the case when $\ell \in \widetilde{\mathscr{L}}_{ab}$.

Proposition 4.4. *The equality* $\dim U = 1$ *holds if and only if there exists* $\xi \in [a,b]$ *such that the problem*

$$u'(t) = \ell(u)(t), \qquad u(\xi) = 0 \tag{4.2}$$

has only the trivial solution.

Proof. Let $\dim U = 1$ and let problem (4.2) have a nontrivial solution u_ξ for every $\xi \in [a,b]$. Choose $t_0 \in]a,b]$ such that $u_a(t_0) \neq 0$. Then, obviously, functions u_a and u_{t_0} are linearly independent solutions of (1.6), which contradicts the assumption $\dim U = 1$.

Now assume that there exists $\xi \in [a,b]$ such that the problem (4.2) has only the trivial solution and $\dim U \geq 2$. Let $u_1, u_2 \in U$ be linearly independent. Obviously,

$$u_1(\xi) \neq 0, \qquad u_2(\xi) \neq 0. \tag{4.3}$$

Let

$$u(t) = u_2(\xi)u_1(t) - u_1(\xi)u_2(t) \quad \text{for } t \in [a,b]. \tag{4.4}$$

Then u is a solution of the problem (4.2), and so

$$u_2(\xi)u_1(t) - u_1(\xi)u_2(t) = 0 \quad \text{for } t \in [a,b]. \tag{4.5}$$

However, the last equality, together with (4.3), contradicts the linear independence of u_1 and u_2. □

Remark 4.5. From Proposition 4.4 and Theorem 2.1, it follows that if $\dim U = 1$, then for every $q \in L([a,b]\mathbb{R})$, the nonhomogeneous equation (1.4) has at least one solution (in fact, it possesses infinitely many solutions). If $\dim U \geq 2$, the situation is substantially different. More precisely, the following assertion holds.

Proposition 4.6. *Let* $\dim U \geq 2$. *Then there exists* $q \in L([a,b];\mathbb{R})$ *such that the nonhomogeneous equation (1.4) has no solution.*

Proof. According to Proposition 4.4, and the condition $\dim U \geq 2$, for every $\xi \in [a,b]$ the problem (4.2) has a nontrivial solution u_ξ. Therefore, by virtue of Remark 2.3, for every $\xi \in [a,b]$, there exists $q_\xi \in L([a,b];\mathbb{R})$ such that the problem

$$u'(t) = \ell(u)(t) + q_\xi(t), \qquad u(\xi) = 0, \tag{4.6}$$

has no solution. Choose $t_0 \in]a,b]$ such that

$$u_a(t_0) \neq 0. \tag{4.7}$$

We will show that an equation

$$u'(t) = \ell(u)(t) + q_{t_0}(t) \tag{4.8}$$

has no solution. Assume on the contrary that u is a solution of (4.8). Obviously, a function v defined by

$$v(t) = u(t) - \frac{u(t_0)}{u_a(t_0)} u_a(t) \quad \text{for } t \in [a,b] \tag{4.9}$$

satisfies

$$\begin{gathered} v'(t) = \ell(v)(t) + q_{t_0}(t) \quad \text{for } t \in [a,b], \\ v(t_0) = 0, \end{gathered} \tag{4.10}$$

which is a contradiction with a choice of q_{t_0}. □

Remark 4.7. From Proposition 4.6 and Theorem 2.1, it follows that if $\dim U \geq 2$, then for every linear bounded functional $h : C([a,b];\mathbb{R}) \to \mathbb{R}$, the problem (1.6), (1.7) has a nontrivial solution.

THEOREM 4.8. *Let the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and*

$$\int_a^b \ell_0(1)(s)ds < 1, \tag{4.11}$$

$$\int_a^b \ell_1(1)(s)ds < 2 + 2\sqrt{1 - \int_a^b \ell_0(1)(s)ds}. \tag{4.12}$$

Then $\dim U = 1$.

To prove Theorem 4.8, we need the following lemma.

LEMMA 4.9. *Let the assumptions of Theorem 4.8 be fulfilled and let u be a nontrivial solution of (1.6) satisfying*

$$u(a) = u(b). \tag{4.13}$$

Then

$$u(t) \neq 0 \quad \text{for } t \in [a,b]. \tag{4.14}$$

Proof. Suppose on the contrary that (4.14) is not valid. Let

$$M = \max\{u(t) : t \in [a,b]\}, \qquad m = -\min\{u(t) : t \in [a,b]\}, \tag{4.15}$$

and choose $t_M, t_m \in [a,b]$ such that

$$u(t_M) = M, \qquad u(t_m) = -m. \tag{4.16}$$

Obviously, $t_M \neq t_m$, $M \geq 0$, and $m \geq 0$. Without loss of generality, we can assume that $t_M < t_m$. The integration of (1.6) from t_M to t_m, with respect to (4.15), (4.16), and the assumption $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, yields

$$M + m = \int_{t_M}^{t_m} \ell_1(u)(s)ds - \int_{t_M}^{t_m} \ell_0(u)(s)ds \leq MA + mB, \tag{4.17}$$

where

$$A = \int_{t_M}^{t_m} \ell_1(1)(s)ds, \qquad B = \int_{t_M}^{t_m} \ell_0(1)(s)ds. \tag{4.18}$$

On the other hand, the integration of (1.6) from a to t_M and from t_m to b, on account of (4.15), (4.16), and the assumption $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, results in

$$\begin{aligned} M - u(a) &= \int_a^{t_M} \ell_0(u)(s)ds - \int_a^{t_M} \ell_1(u)(s)ds \\ &\le M \int_a^{t_M} \ell_0(1)(s)ds + m \int_a^{t_M} \ell_1(1)(s)ds, \\ u(b) + m &= \int_{t_m}^{b} \ell_0(u)(s)ds - \int_{t_m}^{b} \ell_1(u)(s)ds \\ &\le M \int_{t_m}^{b} \ell_0(1)(s)ds + m \int_{t_m}^{b} \ell_1(1)(s)ds. \end{aligned} \tag{4.19}$$

Summing the last two inequalities and taking into account (4.13), we obtain

$$M + m \le MC + mD, \tag{4.20}$$

where

$$\begin{aligned} C &= \int_a^{t_M} \ell_0(1)(s)ds + \int_{t_m}^{b} \ell_0(1)(s)ds, \\ D &= \int_a^{t_M} \ell_1(1)(s)ds + \int_{t_m}^{b} \ell_1(1)(s)ds. \end{aligned} \tag{4.21}$$

From (4.17) and (4.20), we get

$$m(1 - B) \le M(A - 1), \qquad M(1 - C) \le m(D - 1). \tag{4.22}$$

According to (4.11), (4.18), and (4.21),

$$C < 1, \qquad B < 1. \tag{4.23}$$

Therefore, from (4.22), we obtain $m > 0$ and $M > 0$, since otherwise it would be $m = 0$ and $M = 0$. Further, from (4.22), it follows that

$$A > 1, \qquad D > 1, \tag{4.24}$$

$$(1 - B)(1 - C) \le (A - 1)(D - 1). \tag{4.25}$$

Note that

$$\begin{aligned} (1 - B)(1 - C) &\ge 1 - B - C = 1 - \int_a^b \ell_0(1)(s)ds, \\ (A - 1)(D - 1) &\le \frac{1}{4}(A + D - 2)^2 = \frac{1}{4}\left(\int_a^b \ell_1(1)(s)ds - 2\right)^2. \end{aligned} \tag{4.26}$$

Thus inequality (4.25) yields

$$1-\int_a^b \ell_0(1)(s)ds \le \frac{1}{4}\left(\int_a^b \ell_1(1)(s)ds-2\right)^2, \tag{4.27}$$

which, together with (4.18), (4.21), and (4.24), contradicts (4.12). □

Proof of Theorem 4.8. Assume that $\dim U \ge 2$. Then, according to Remark 4.7, the boundary value problems

$$\begin{aligned} u'(t) &= \ell(u)(t), \qquad u(a)=u(b),\\ v'(t) &= \ell(v)(t), \qquad v(b)=0, \end{aligned} \tag{4.28}$$

have nontrivial solutions u and v, respectively. With respect to Lemma 4.9, without loss of generality, we can assume that

$$\begin{aligned} u(t)&>0 \quad \text{for } t\in[a,b],\\ v(a)&>0. \end{aligned} \tag{4.29}$$

Let

$$w(t)=\lambda u(t)-v(t) \quad \text{for } t\in[a,b], \tag{4.30}$$

where

$$\lambda=\max\left\{\frac{v(t)}{u(t)} : t\in[a,b]\right\}. \tag{4.31}$$

Obviously, $w \not\equiv 0$,

$$w(t)\ge 0 \quad \text{for } t\in[a,b], \tag{4.32}$$

$$w(a)<w(b), \tag{4.33}$$

and there exist $t_0\in[a,b[$ and $t_1\in]a,b]$ such that $t_0\ne t_1$ and

$$w(t_0)=0, \qquad w(t_1)=\|w\|_C. \tag{4.34}$$

Since w is a solution of (1.6), $\ell_1\in\mathcal{P}_{ab}$, and inequality (4.32) holds, we have

$$w'(t)\le \ell_0(w)(t) \quad \text{for } t\in[a,b]. \tag{4.35}$$

First suppose that $t_0<t_1$. The integration of (4.35) from t_0 to t_1, on account of (4.34), yields

$$\|w\|_C \le \int_{t_0}^{t_1} \ell_0(w)(s)ds \le \|w\|_C \int_a^b \ell_0(1)(s)ds. \tag{4.36}$$

Hence, since $w\not\equiv 0$, it follows that $\int_a^b \ell_0(1)(s)ds\ge 1$, which contradicts (4.11).

Suppose now that $t_1 < t_0$. The integration of (4.35) from a to t_1 and from t_0 to b, in view of (4.34), results in

$$\begin{aligned} \|w\|_C - w(a) &\le \int_a^{t_1} \ell_0(w)(s)ds \le \|w\|_C \int_a^{t_1} \ell_0(1)(s)ds, \\ w(b) &\le \int_{t_0}^{b} \ell_0(w)(s)ds \le \|w\|_C \int_{t_0}^{b} \ell_0(1)(s)ds. \end{aligned} \tag{4.37}$$

Summing the last two inequalities and taking into account (4.33), we obtain

$$\|w\|_C \le \|w\|_C \left(\int_a^{t_1} \ell_0(1)(s)ds + \int_{t_0}^{b} \ell_0(1)(s)ds \right) \le \|w\|_C \int_a^b \ell_0(1)(s)ds. \tag{4.38}$$

Hence, since $w \not\equiv 0$, we get $\int_a^b \ell_0(1)(s)ds \ge 1$, which contradicts (4.11). □

The following assertion can be proved analogously.

Theorem 4.10. *Let the operator ℓ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, and*

$$\begin{gathered} \int_a^b \ell_1(1)(s)ds < 1, \\ \int_a^b \ell_0(1)(s)ds < 2 + 2\sqrt{1 - \int_a^b \ell_1(1)(s)ds}. \end{gathered} \tag{4.39}$$

Then $\dim U = 1$.

Remark 4.11. Theorems 4.8 and 4.10 are nonimprovable in a certain sense. More precisely, neither one of the strict inequalities (4.11), (4.12), and (4.39) can be replaced by the nonstrict one (see Example 5.2).

Remark 4.12. It is known (see [6, Theorem VII.1.27, page 234]) that $\ell \in \widetilde{\mathcal{L}}_{ab}$ if and only if ℓ admits the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$. Therefore Theorems 4.8 and 4.10 actually claim that $\ell \in \widetilde{\mathcal{L}}_{ab}$.

5. Examples

Example 5.1. Let $n \in \mathbb{N}$, $t_i \in [a,b]$ $(i = \overline{0,n})$ be such that $t_0 = a$, $t_n = b$, $t_{i-1} < t_i$ $(i = \overline{1,n})$, and let $\xi_i \in]t_{i-1}, t_i[$ and $u_i \in \widetilde{C}([a,b];\mathbb{R})$ $(i = \overline{1,n})$ be such that

$$u_i(\xi_i) = 1, \quad u_i(t) = 0 \quad \text{for } t \in [a,b] \setminus]t_{i-1}, t_i[, \ i = \overline{1,n}. \tag{5.1}$$

Let

$$p(t) = u_i'(t), \quad \tau(t) = \xi_i \quad \text{for } t \in]t_{i-1}, t_i[, \ i = \overline{1,n}. \tag{5.2}$$

Obviously, $p \in L([a,b];\mathbb{R})$, $\tau \in \mathcal{M}_{ab}$, and the functions $u_1, \dots, u_n$ are linearly independent solutions of an equation with a deviating argument

$$u'(t) = p(t)u(\tau(t)). \tag{5.3}$$

Let u be an arbitrary solution of (5.3). Clearly,

$$u'(t) = u_i'(t)u(\xi_i) \quad \text{for } t \in [t_{i-1}, t_i],\ i = \overline{1,n}. \tag{5.4}$$

The integration of these equalities from t_{i-1} to t, with respect to $u_i(t_{i-1}) = 0$, yields

$$u(t) = u(t_{i-1}) + u(\xi_i)u_i(t) \quad \text{for } t \in [t_{i-1}, t_i],\ i = \overline{1,n}. \tag{5.5}$$

Hence, for $t = \xi_i$, since $u_i(\xi_i) = 1$, we obtain $u(t_{i-1}) = 0$ $(i = \overline{1,n})$, and, consequently,

$$u(t) = u(\xi_i)u_i(t) \quad \text{for } t \in [t_{i-1}, t_i],\ i = \overline{1,n}. \tag{5.6}$$

Now it is obvious that

$$u(t) = \sum_{i=1}^{n} \alpha_i u_i(t) \quad \text{for } t \in [a,b], \tag{5.7}$$

where $\alpha_i = u(\xi_i)$. Thus the solution space of (5.3) has dimension n.

Example 5.2. Let W be a set of pairs $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that either

$$x < 1, \qquad y < 2 + 2\sqrt{1-x}, \tag{5.8}$$

or

$$y < 1, \qquad x < 2 + 2\sqrt{1-y}. \tag{5.9}$$

By virtue of Theorems 4.8 and 4.10, if $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ are such that $(\|\ell_0(1)\|_L, \|\ell_1(1)\|_L) \in W$, then $\dim U = 1$. We will show that for every $x_0, y_0 \in \mathbb{R}_+$ such that $(x_0, y_0) \notin W$ there exists $\ell \in \widetilde{\mathcal{L}}_{ab}$ satisfying $\ell = \ell_0 - \ell_1$, $\ell_0, \ell_1 \in \mathcal{P}_{ab}$,

$$x_0 = \int_a^b \ell_0(1)(s)ds, \qquad y_0 = \int_a^b \ell_1(1)(s)ds, \tag{5.10}$$

and $\dim U \geq 2$.

First, let $x_0 \in [0,1[$, $y_0 \geq 2 + 2\sqrt{1-x_0}$, $c_i \in]a,b[$ $(i = 1,2,3,4)$, $a < c_1 < c_2 < c_3 < c_4 < b$, and choose $p, g \in L([a,b];\mathbb{R}_+)$ such that

$$\begin{gathered}
\int_a^{c_1} g(s)ds = 1, \qquad \int_{c_1}^{c_2} g(s)ds = y_0 - (2 + 2\sqrt{1-x_0}), \qquad \int_{c_2}^{c_3} g(s)ds = 0, \\
\int_{c_3}^{c_4} g(s)ds = \sqrt{1-x_0}, \qquad \int_{c_4}^{b} g(s)ds = 1 + \sqrt{1-x_0}, \\
\int_a^{c_2} p(s)ds = 0, \qquad \int_{c_2}^{c_3} p(s)ds = x_0, \qquad \int_{c_3}^{b} p(s)ds = 0.
\end{gathered} \tag{5.11}$$

Let $\tau \equiv c_4$,

$$\mu(t) = \begin{cases} a & \text{for } t \in [a, c_1[, \\ c_1 & \text{for } t \in [c_1, c_2[, \\ b & \text{for } t \in [c_2, c_4[, \\ c_4 & \text{for } t \in [c_4, b], \end{cases} \tag{5.12}$$

and define

$$\begin{gathered} \ell_0(v)(t) \overset{\text{def}}{=} p(t)v(\tau(t)), \quad \ell_1(v)(t) \overset{\text{def}}{=} g(t)v(\mu(t)) \quad \text{for } t \in [a,b], \\ \ell(v)(t) \overset{\text{def}}{=} \ell_0(v)(t) - \ell_1(v)(t) \quad \text{for } t \in [a,b]. \end{gathered} \tag{5.13}$$

Then (5.10) is fulfilled and (1.6) has two linearly independent solutions:

$$\begin{gathered} u_1(t) = \begin{cases} -\sqrt{1-x_0} + \sqrt{1-x_0}\displaystyle\int_a^t g(s)ds & \text{for } t \in [a, c_1[, \\ 0 & \text{for } t \in [c_1, c_2[, \\ \displaystyle\int_{c_2}^t p(s)ds & \text{for } t \in [c_2, c_3[, \\ x_0 + \sqrt{1-x_0}\displaystyle\int_{c_3}^t g(s)ds & \text{for } t \in [c_3, c_4[, \\ 1 - \displaystyle\int_{c_4}^t g(s)ds & \text{for } t \in [c_4, b], \end{cases} \\ u_2(t) = \begin{cases} -\sqrt{1-x_0} + \sqrt{1-x_0}\displaystyle\int_a^t g(s)ds & \text{for } t \in [a, c_1[, \\ 0 & \text{for } t \in [c_1, b]. \end{cases} \end{gathered} \tag{5.14}$$

Now let $x_0 \geq 1$, $y_0 \geq 1$, and $c \in]a,b[$. Choose $p, g \in L([a,b];\mathbb{R}_+)$ such that

$$\begin{gathered} \int_a^c p(s)ds = x_0 - 1, \qquad \int_c^b p(s)ds = 1, \\ \int_a^c g(s)ds = 1, \qquad \int_c^b g(s)ds = y_0 - 1, \end{gathered} \tag{5.15}$$

and let

$$\tau(t) = \begin{cases} c & \text{for } t \in [a, c[, \\ b & \text{for } t \in [c, b], \end{cases} \qquad \mu(t) = \begin{cases} a & \text{for } t \in [a, c[, \\ c & \text{for } t \in [c, b]. \end{cases} \tag{5.16}$$

Define operators ℓ_0, ℓ_1, and ℓ by (5.13). Then (5.10) is fulfilled and (1.6) has two linearly independent solutions:

$$\begin{gathered} u_1(t) = \begin{cases} 0 & \text{for } t \in [a, c[, \\ \displaystyle\int_c^t p(s)ds & \text{for } t \in [c, b], \end{cases} \\ u_2(t) = \begin{cases} \displaystyle\int_t^c g(s)ds & \text{for } t \in [a, c[, \\ 0 & \text{for } t \in [c, b]. \end{cases} \end{gathered} \tag{5.17}$$

At last, mention that in the case when $y_0 \in [0,1[$ and $x_0 \geq 2+2\sqrt{1-y_0}$, the operator ℓ satisfying $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, such that (5.10) holds and (1.6) has two linearly independent solutions, can be constructed in an analogous way as above.

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