WHICH SOLUTIONS OF THE THIRD PROBLEM FOR THE POISSON EQUATION ARE BOUNDED?

DAGMAR MEDKOVÁ

Received 10 September 2002

This paper deals with the problem $\Delta u = g$ on G and $\partial u / \partial n + u f = L$ on ∂G . Here, $G \subset \mathbb{R}^m$, $m > 2$, is a bounded domain with Lyapunov boundary, f is a bounded nonnegative function on the boundary of G , L is a bounded linear functional on $W^{1,2}(G)$ representable by a real measure μ on the boundary of G , and $g \in L_2(G) \cap L_p(G)$, $p > m/2$. It is shown that a weak solution of this problem is bounded in G if and only if the Newtonian potential corresponding to the boundary condition μ is bounded in G .

Suppose that $G \subset \mathbb{R}^m$, $m > 2$, is a bounded domain with Lyapunov boundary (i.e., of class $C^{1+\alpha}$). Denote by $n(y)$ the outer unit normal of G at y . If $f, g, h \in C(\partial G)$ and $u \in C^2(\text{cl } G)$ is a classical solution of

$$\begin{aligned} \Delta u &= g \quad \text{on } G, \\ \frac{\partial u}{\partial n} + u f &= h \quad \text{on } \partial G, \end{aligned} \tag{1}$$

then Green's formula yields

$$\int_G \nabla u \cdot \nabla v d\mathcal{H}_m + \int_{\partial G} u f v d\mathcal{H}_{m-1} = \int_{\partial G} h v d\mathcal{H}_{m-1} - \int_G g v d\mathcal{H}_m \tag{2}$$

for each $v \in \mathcal{D}$, the space of all compactly supported infinitely differentiable functions in $\mathbb{R}^m$. Here, ∂G denotes the boundary of G and $\text{cl } G$ is the closure of G ; $\mathcal{H}_k$ is the k -dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. Denote by $\mathcal{D}(G)$ the set of all functions from $\mathcal{D}$ with the support in G .

For an open set $V \subset \mathbb{R}^m$, denote by $W^{1,2}(V)$ the collection of all functions $f \in L_2(V)$, the distributional gradient of which belongs to $[L_2(V)]^m$.

Definition 1. Let $f \in L_\infty(\mathcal{H})$, $g \in L_2(G)$ and let L be a bounded linear functional on $W^{1,2}(G)$ such that $L(\varphi) = 0$ for each $\varphi \in \mathcal{D}(G)$. We say that $u \in W^{1,2}(G)$ is a weak solution

in $W^{1,2}(G)$ of the third problem for the Poisson equation

$$\begin{aligned}\Delta u &= g \quad \text{on } G, \\ \frac{\partial u}{\partial n} + uf &= L \quad \text{on } \partial G,\end{aligned}\tag{3}$$

if

$$\int_G \nabla u \cdot \nabla v d\mathcal{H}_m + \int_{\partial G} u f v d\mathcal{H} = L(v) - \int_G g v d\mathcal{H}_m\tag{4}$$

for each $v \in W^{1,2}(G)$.

Denote by $\mathcal{C}'(\partial G)$ the Banach space of all finite signed Borel measures with support in ∂G with the total variation as a norm. We say that the bounded linear functional L on $W^{1,2}(G)$ is representable by $\mu \in \mathcal{C}'(\partial G)$ if $L(\varphi) = \int \varphi d\mu$ for each $\varphi \in \mathcal{D}$. Since $\mathcal{D}$ is dense in $W^{1,2}(G)$, the operator L is uniquely determined by its representation $\mu \in \mathcal{C}'(\partial G)$.

For $x, y \in \mathbb{R}^m$, denote

$$h_x(y) = \begin{cases} (m-2)^{-1} A^{-1} |x-y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y, \end{cases}\tag{5}$$

where A is the area of the unit sphere in $\mathbb{R}^m$. For the finite real Borel measure ν , denote

$$\mathcal{U}\nu(x) = \int_{\mathbb{R}^m} h_x(y) d\nu(y)\tag{6}$$

the Newtonian potential corresponding to ν , for each x for which this integral has sense.

We denote by $\mathcal{C}'_b(\partial G)$ the set of all $\mu \in \mathcal{C}'(\partial G)$ for which $\mathcal{U}\mu$ is bounded on $\mathbb{R}^m \setminus \partial G$.

Remark that $\mathcal{C}'_b(\partial G)$ is the set of all $\mu \in \mathcal{C}'(\partial G)$ for which there is a polar set M such that $\mathcal{U}\mu(x)$ is meaningful and bounded on $\mathbb{R}^m \setminus M$, because $\mathbb{R}^m \setminus \partial G$ is finely dense in $\mathbb{R}^m$ (see [1, Chapter VII, Sections 2, 6], [7, Theorems 5.10 and 5.11]) and $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set. Remark that $\mathcal{H}_{m-1}(M) = 0$ for each polar set M (see [7, Theorem 3.13]). (For the definition of polar sets, see [4, Chapter 7, Section 1]; for the definition of the fine topology, see [4, Chapter 10].)

Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ to ∂G .

LEMMA 2. *Let $\mu \in \mathcal{C}'(\partial G)$. Then the following assertions are equivalent:*

- (1) $\mu \in \mathcal{C}'_b(\partial G)$,
- (2) $\mathcal{U}\mu$ is bounded in G ,
- (3) $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

Proof. (2) $\Rightarrow$ (3). Since ∂G is a subset of the fine closure of G by [1, Chapter VII, Sections 2, 6] and [7, Theorems 5.10 and 5.11], $\mathcal{U}\mu = \mathcal{U}\mu^+ - \mathcal{U}\mu^-$ is finite and fine-continuous outside of a polar set M , and $\mathcal{H}_{m-1}(M) = 0$ by [4, Theorem 7.33] and [7, Theorem 3.13], then we obtain that $\mathcal{U}\mu \in L_\infty(\mathcal{H})$.

(3) $\Rightarrow$ (1). Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ . For $z \in G$, denote by μ_z the harmonic measure corresponding to G and z . If $y \in \partial G$ and $z \in G$, then

$$\int_{\partial G} h_y(x) d\mu_z(x) = h_y(z) \quad (7)$$

by [7, pages 264, 299]. Using Fubini's theorem, we get

$$\int \mathcal{U}\mu^+ d\mu_z = \int_{\partial G} \int_{\partial G} h_y(x) d\mu_z(x) d\mu^+(y) = \int_{\partial G} h_y(z) d\mu^+(y) = \mathcal{U}\mu^+(z). \quad (8)$$

Similarly, $\int \mathcal{U}\mu^- d\mu_z = \mathcal{U}\mu^-(z)$. Since $\mathcal{U}\mu \in L_\infty(\mathcal{H})$, μ_z is a nonnegative measure with the total variation 1 (see [4, Lemma 8.12]) which is absolutely continuous with respect to $\mathcal{H}$ by [2, Theorem 1], then we obtain that $|\mathcal{U}\mu(z)| \leq \|\mathcal{U}\mu\|_{L_\infty(\mathcal{H})}$.

If $z \in \mathbb{R}^m \setminus \text{cl } G$, choose a bounded domain V with smooth boundary such that $\text{cl } G \cup \{z\} \subset V$. Repeating the previous reasonings for $V \setminus \text{cl } G$, we get $|\mathcal{U}\mu(z)| \leq \|\mathcal{U}\mu\|_{L_\infty(\mathcal{H})}$. $\square$

LEMMA 3. *Let $f \in L_\infty(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Then $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$. Moreover, there is a bounded linear functional L on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}'_b(\partial G)$ such that $\mathcal{U}(g\mathcal{H}_m)$ is a weak solution in $W^{1,2}(G)$ of the third problem for the Poisson equation*

$$\Delta u = -g \quad \text{on } G, \quad \frac{\partial u}{\partial n} + uf = L \quad \text{on } \partial G. \quad (9)$$

Proof. Suppose first that g is nonnegative. Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m)$ by [3, Theorem A.6], the energy $\int g\mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m < \infty$. According to [7, Theorem 1.20], we have

$$\int |\nabla \mathcal{U}(g\mathcal{H}_m)|^2 d\mathcal{H}_m = \int g\mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m < \infty, \quad (10)$$

and therefore $\mathcal{U}(g\mathcal{H}_m) \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.1.4]).

Since $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m) \cap W^{1,2}(G)$, $f \in L_\infty(\mathcal{H})$ and the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], then the operator

$$L(\varphi) = \int_G \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f \varphi d\mathcal{H}_{m-1} - \int_G g \varphi d\mathcal{H}_m \quad (11)$$

is a bounded linear functional on $W^{1,2}(G)$.

According to [7, Theorem 4.2], there is a nonnegative $\nu \in \mathcal{C}'(\partial G)$ such that $\mathcal{U}\nu = \mathcal{U}(g\mathcal{H}_m)$ on $\mathbb{R}^m \setminus \text{cl } G$. Choose a bounded domain V with smooth boundary such that $\text{cl } G \subset V$. Since $\mathcal{U}\nu$ is bounded in $V \setminus \text{cl } G \subset \mathbb{R}^m \setminus \text{cl } G$, Lemma 2 yields that $\nu \in \mathcal{C}'_b(\partial(V \setminus \text{cl } G))$. Therefore, $\nu \in \mathcal{C}'_b(\partial G)$. According to [13, Lemma 4], there is $\tilde{\nu} \in \mathcal{C}'_b(\partial G)$ such that

$$\int_{\mathbb{R}^m \setminus \text{cl } G} \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m = \int_{\mathbb{R}^m \setminus \text{cl } G} \nabla \varphi \cdot \nabla \mathcal{U}\nu d\mathcal{H}_m = \int_{\partial G} \varphi d\tilde{\nu} \quad (12)$$

for each $\varphi \in \mathcal{D}$. Let $\mu = \tilde{\nu} - f\mathcal{U}(g\mathcal{H}_m)\mathcal{H}$. Since $\mathcal{U}(f\mathcal{U}(g\mathcal{H}_m)\mathcal{H}) \in \mathcal{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18] and $\mathcal{U}(f\mathcal{U}(g\mathcal{H}_m)\mathcal{H})(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we have $f\mathcal{U}(g\mathcal{H}_m)\mathcal{H} \in \mathcal{C}'_b(\partial G)$. Therefore, $\mu \in \mathcal{C}'_b(\partial G)$.

If $\varphi \in \mathcal{D}$, then $\varphi = \mathcal{U}((-\Delta\varphi)\mathcal{H}_m)$ by [3, Theorem A.2]. According to [7, Theorem 1.20],

$$\begin{aligned} \int_{\mathbb{R}^m} \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m &= \int_{\mathbb{R}^m} \nabla \mathcal{U}((-\Delta\varphi)\mathcal{H}_m) \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m \\ &= \int_{\mathbb{R}^m} g \mathcal{U}((-\Delta\varphi)\mathcal{H}_m) d\mathcal{H}_m \\ &= \int_{\mathbb{R}^m} g \varphi d\mathcal{H}_m. \end{aligned} \quad (13)$$

Since $\mathcal{H}_m(\partial G) = 0$,

$$\begin{aligned} &\int_G \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f \varphi d\mathcal{H}_{m-1} \\ &= \int_G g \varphi d\mathcal{H}_m + \int_{\partial G} \mathcal{U}(g\mathcal{H}_m) f \varphi d\mathcal{H}_{m-1} \\ &\quad - \int_{\mathbb{R}^m \setminus \text{cl } G} \nabla \varphi \cdot \nabla \mathcal{U}(g\mathcal{H}_m) d\mathcal{H}_m \\ &= \int_G g \varphi d\mathcal{H}_m + \int_{\partial G} \varphi d\mu. \end{aligned} \quad (14)$$

□

LEMMA 4. Let $f \in L_\infty(\mathcal{H})$ and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Let L be a bounded linear functional on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}'(\partial G)$. If $u \in L_\infty(G) \cap W^{1,2}(G)$ is a weak solution in $W^{1,2}(G)$ of problem (3), then $\mu \in \mathcal{C}'_b(\partial G)$.

Proof. Let $w = u - \mathcal{U}(g\mathcal{H}_m)$. According to Lemma 3, there is a bounded linear functional $\tilde{L}$ on $W^{1,2}(G)$ representable by $\nu \in \mathcal{C}'_b(\partial G)$ such that w is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned} \Delta w &= 0 \quad \text{on } G, \\ \frac{\partial w}{\partial n} + w f &= L - \tilde{L} \quad \text{on } \partial G. \end{aligned} \quad (15)$$

Fix $x \in G$. Choose a sequence G_j of open sets with C^∞ boundary such that $\text{cl } G_j \subset G_{j+1} \subset G$, $x \in G_1$, and $\cup G_j = G$. Fix $r > 0$ such that $\Omega_{2r}(x) \subset G_1$. Choose an infinitely differentiable function ψ such that $\psi = 0$ on $\Omega_r(x)$ and $\psi = 1$ on $\mathbb{R}^m \setminus \Omega_{2r}(x)$. According to Green's identity,

$$\begin{aligned} w(x) &= \lim_{j \rightarrow \infty} \left[\int_{\partial G_j} h_x(y) \frac{\partial w(y)}{\partial n} d\mathcal{H}_{m-1}(y) - \int_{\partial G_j} w(y) n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \right] \\ &= \lim_{j \rightarrow \infty} \left[\int_{G_j} \nabla w(y) \cdot \nabla (h_x(y) \psi(y)) d\mathcal{H}_m(y) \right. \\ &\quad \left. - \int_{G_j} \nabla (w(y) \psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y) \right] \end{aligned}$$

$$\begin{aligned}
&= \int_G \nabla w(y) \cdot \nabla (h_x(y)\psi(y)) d\mathcal{H}_m(y) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y) \\
&= \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \int_G \nabla (w(y)\psi(y)) \cdot \nabla h_x(y) d\mathcal{H}_m(y).
\end{aligned} \tag{16}$$

According to [16, Theorem 2.3.2], there is a sequence of infinitely differentiable functions w_n such that $w_n \rightarrow w\psi$ in $W^{1,2}(G)$. According to [6, Section 2],

$$\begin{aligned}
w(x) &= \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \lim_{n \rightarrow \infty} \int_G \nabla w_n(y) \cdot \nabla h_x(y) d\mathcal{H}_m(y) \\
&= \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \lim_{n \rightarrow \infty} \int_{\partial G} w_n(y)n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y).
\end{aligned} \tag{17}$$

Since the trace operator is a bounded operator from $W^{1,2}(G)$ to $L_2(\mathcal{H})$ by [8, Theorem 3.38], we obtain

$$w(x) = \mathcal{U}(\mu - \nu - fw\mathcal{H})(x) - \int_{\partial G} w(y)n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y). \tag{18}$$

Since $w \in L_\infty(G)$ by Lemma 3, the trace of w is an element of $L_\infty(\mathcal{H})$. Since

$$\begin{aligned}
&\left| \int_{\partial G} w(y)n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \right| \\
&\leq \|w\|_{L_\infty(\mathcal{H})} \int_{\partial G} |n(y) \cdot \nabla h_x(y)| d\mathcal{H}_{m-1}(y) \\
&\leq \|w\|_{L_\infty(\mathcal{H})} \left[\sup_{z \in \partial G} \int_{\partial G} |n(y) \cdot \nabla h_z(y)| d\mathcal{H}_{m-1}(y) + \frac{1}{2} \right] < \infty
\end{aligned} \tag{19}$$

by [6, Lemma 2.15 and Theorem 2.16] and the fact that ∂G is of class $C^{1+\alpha}$, the function

$$x \mapsto \int_{\partial G} w(y)n(y) \cdot \nabla h_x(y) d\mathcal{H}_{m-1}(y) \tag{20}$$

is bounded in G . Since $\mathcal{U}\nu$ is bounded in G and $\mathcal{U}(fw\mathcal{H})$ is bounded in G by [6, Corollary 2.17 and Lemma 2.18], the function $\mathcal{U}\mu$ is bounded in G by (18). Thus, $\mu \in \mathcal{C}'_b(\partial G)$ by Lemma 2. $\square$

Notation 5. Let X be a complex Banach space and T a bounded linear operator on X . We denote by $\text{Ker } T$ the kernel of T , by $\sigma(T)$ the spectrum of T , by $r(T)$ the spectral radius of T , by X' the dual space of X , and by T' the adjoint operator of T . Denote by I the identity operator.

THEOREM 6. *Let X be a complex Banach space and K a compact linear operator on X . Let Y be a subspace of X' and T a closed linear operator from Y to X such that $y(Tx) = x(Ty)$ for each $x, y \in Y$. Suppose that $K'(Y) \subset Y$ and $KTy = TK'y$ for each $y \in Y$. Let $\alpha \in \mathbb{C} \setminus \{0\}$, $\text{Ker}(K' - \alpha I)^2 = \text{Ker}(K' - \alpha I) \subset Y$, and $\{\beta \in \sigma(K'); (\beta - \alpha) \cdot \alpha \leq 0\} \subset \{\alpha\}$. If $x, y \in X$, $(K' - \alpha I)x = y$, then $x \in Y$ if and only if $y \in Y$.*

Proof. If $x \in Y$, then $y \in Y$. Suppose that $y \in Y$. Since K is a compact operator, the operator K' is a compact operator by [14, Chapter IV, Theorem 4.1]. Suppose first that $\alpha \in \sigma(K')$. Since K' is compact, then α is a pol of the resolvent by [5, Satz 50.4]. Since

$$\text{Ker}(K' - \alpha I)^2 = \text{Ker}(K' - \alpha I), \quad (21)$$

the ascent of $(K' - \alpha I)$ is equal to 1. Since α is a pol of the resolvent and the ascent of $(K' - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space X' is the direct sum of $\text{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X')$ and the descent of $(K' - \alpha I)$ is equal to 1. Since the descent of $(K' - \alpha I)$ is equal to 1, we have

$$(K' - \alpha I)^2(X') = (K' - \alpha I)(X'). \quad (22)$$

Since the space X' is the direct sum of $\text{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X') = (K' - \alpha I)^2(X')$, the operator $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$. If $\alpha \notin \sigma(K')$, then the space X' is the direct sum of $\text{Ker}(K' - \alpha I)$ and $(K' - \alpha I)(X')$, and the operator $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$. Therefore, there are $x_1 \in \text{Ker}(K' - \alpha I) \subset Y$ and $x_2 \in (K' - \alpha I)(X')$ such that $x_1 + x_2 = x$. We have $(K' - \alpha I)x_2 = y$.

Denote by Z the closure of Y . Since $K'(Y) \subset Y$, we obtain $K'(Z) \subset Z$. Denote by K'_Z the restriction of K' to Z . Then K'_Z is a compact operator in Z . Since $\text{Ker}(K' - \alpha I)^2 \subset Y$, we have

$$\text{Ker}(K'_Z - \alpha I)^2 = \text{Ker}(K' - \alpha I)^2 = \text{Ker}(K' - \alpha I) = \text{Ker}(K'_Z - \alpha I). \quad (23)$$

If $\alpha \notin \sigma(K'_Z)$, then the space Z is the direct sum of $\text{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z)$, and the operator $(K'_Z - \alpha I)$ is invertible on Z . Suppose that $\alpha \in \sigma(K'_Z)$. Since K'_Z is compact, then α is a pol of the resolvent by [5, Satz 50.4]. Since

$$\text{Ker}(K'_Z - \alpha I)^2 = \text{Ker}(K'_Z - \alpha I), \quad (24)$$

the ascent of $(K'_Z - \alpha I)$ is equal to 1. Since α is a pol of the resolvent and the ascent of $(K'_Z - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space Z is the direct sum of $\text{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z)$ and the descent of $(K'_Z - \alpha I)$ is equal to 1. Since the descent of $(K'_Z - \alpha I)$ is equal to 1, we have

$$(K'_Z - \alpha I)^2(Z) = (K' - \alpha I)(Z). \quad (25)$$

Since the space Z is the direct sum of $\text{Ker}(K'_Z - \alpha I)$ and $(K'_Z - \alpha I)(Z) = (K'_Z - \alpha I)^2(Z)$, the operator $(K'_Z - \alpha I)$ is invertible on $(K'_Z - \alpha I)(Z)$. Since $y \in Y \subset Z$, there are $y_1 \in \text{Ker}(K'_Z - \alpha I)$ and $y_2 \in (K'_Z - \alpha I)(Z)$ such that $y = y_1 + y_2$. Since X' is the direct sum of $\text{Ker}(K' - \alpha I) = \text{Ker}(K'_Z - \alpha I)$ and $(K' - \alpha I)(X') \supset (K'_Z - \alpha I)(Z)$ and $y \in (K' - \alpha I)(X')$, we obtain that $y_1 = 0$ and $y_2 = y$. Thus, $y \in (K'_Z - \alpha I)(Z)$. Since $(K'_Z - \alpha I)$ is invertible on $(K'_Z - \alpha I)(Z)$, there is $z \in (K'_Z - \alpha I)(Z)$ such that $(K'_Z - \alpha I)(z) = y$. Since $(K' - \alpha I)$ is invertible on $(K' - \alpha I)(X')$, we deduce that $x_2 = z \in (K'_Z - \alpha I)(Z) \subset Z$.

Now, let $w \in \text{Ker}(K' - \alpha I)$. Fix a sequence $\{z_k\} \subset Y$ such that $z_k \rightarrow z = x_2$. Then

$$\begin{aligned} w(Ty) &= y(Tw) = [(K' - \alpha I)x_2](Tw) = \lim_{k \rightarrow \infty} [(K' - \alpha I)z_k](Tw) \\ &= \lim_{k \rightarrow \infty} z_k((K - \alpha I)Tw) = \lim_{k \rightarrow \infty} z_k(T(K' - \alpha I)w) = \lim_{k \rightarrow \infty} z_k(0) = 0. \end{aligned} \quad (26)$$

Since $w(Ty) = 0$ for each $w \in \text{Ker}(K' - \alpha I)$, [15, Chapter 10, Theorem 3] yields $Ty \in (K - \alpha I)(X)$.

Denote by $\tilde{K}'$ the restriction of K' to $(K' - \alpha I)(X)$. If we denote by P the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator K' , then $P(X') = (K' - \alpha I)(X')$ by [5, Satz 50.2] and $\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Therefore,

$$\sigma(\tilde{K}') = \sigma(K') \setminus \{\alpha\} \subset \{\beta; (\beta - \alpha) \cdot \alpha > 0\} \cup_{t>0} \{\beta; |\beta - \alpha - t\alpha| < |t\alpha|\}. \quad (27)$$

Since $\{\beta; |\beta - \alpha - t_1\alpha| < |t_1\alpha|\} \subset \{\beta; |\beta - \alpha - t_2\alpha| < |t_2\alpha|\}$ for $0 < t_1 < t_2$ and $\sigma(\tilde{K}')$ is a compact set (see [14, Chapter VI, Theorem 1.3, and Lemma 1.5]), there is $t > 0$ such that $\sigma(\tilde{K}') \subset \{\beta; |\beta - \alpha - t\alpha| < |t\alpha|\}$. Therefore, $r(\tilde{K}' - \alpha I - t\alpha I) < |t\alpha|$. Since we have $r(t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)) < 1$, the series

$$V = \sum_{k=0}^{\infty} (-1)^k [t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)]^k \quad (28)$$

converges. Easy calculation yields that V is the inverse operator of the operator $I + t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I) = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)$. Since $t^{-1}\alpha^{-1}y = t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I)x_2$, we have $x_2 = t^{-1}\alpha^{-1}Vy$. Denote $z_k = t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(\tilde{K}' - \alpha I - t\alpha I)]^k y$. Then

$$x_2 = \sum_{k=0}^{\infty} z_k. \quad (29)$$

Since $K'(Y) \subset Y$, $z_k \in Y$ for each k . Since $KT = TK'$ on Y , we have $Tz_k = t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(K - \alpha I - t\alpha I)]^k Ty$.

Since $(K - \alpha I)$, $(K - \alpha I)^2$, $(K' - \alpha I)$, and $(K' - \alpha I)^2$ are Fredholm operators with index 0 (see [14, Chapter V, Theorem 3.1]), [14, Chapter VII, Theorem 3.2] yields

$$\dim \text{Ker}(K - \alpha I)^2 = \dim \text{Ker}(K' - \alpha I)^2 = \dim \text{Ker}(K' - \alpha I) = \dim \text{Ker}(K - \alpha I), \quad (30)$$

and thus $\text{Ker}(K - \alpha I)^2 = \text{Ker}(K - \alpha I)$. If $\alpha \notin \sigma(K)$, then the space X is the direct sum of $\text{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$, and the operator $(K - \alpha I)$ is invertible on X . Suppose that $\alpha \in \sigma(K)$. Since K is compact, then α is a pol of the resolvent by [5, Satz 50.4]. Since

$$\text{Ker}(K - \alpha I)^2 = \text{Ker}(K - \alpha I), \quad (31)$$

the ascent of $(K - \alpha I)$ is equal to 1. Since α is a pol of the resolvent and the ascent of $(K - \alpha I)$ is equal to 1, [5, Satz 50.2] yields that the space X is the direct sum of $\text{Ker}(K - \alpha I)$ and $(K - \alpha I)(X)$ and the descent of $(K - \alpha I)$ is equal to 1. Since the descent of $(K - \alpha I)$ is equal to 1, we have $(K - \alpha I)^2(X) = (K - \alpha I)(X)$. Since the space X is the direct sum

of $\text{Ker}(K - \alpha I)$ and $(K - \alpha I)(X) = (K - \alpha I)^2(X)$, the operator $(K - \alpha I)$ is invertible on $(K - \alpha I)(X)$. Denote by $\hat{K}$ the restriction of K to $(K - \alpha I)(X)$. If we denote by Q the spectral projection corresponding to the spectral set $\{\alpha\}$ and the operator K , then $Q(X) = (K - \alpha I)(X)$ by [5, Satz 50.2] and $\sigma(\hat{K}) = \sigma(K) \setminus \{\alpha\}$ by [14, Chapter VI, Theorem 4.1]. Since $\sigma(K) = \sigma(K')$ by [14, Chapter VI, Theorem 4.6], we obtain $\sigma(\hat{K}) \subset \{\beta; |\beta - \alpha - t\alpha| < |t\alpha|\}$. Therefore, $r(\hat{K} - \alpha I - t\alpha I) < |t\alpha|$. Since $Ty \in (K - \alpha I)(X)$ and $r(t^{-1}\alpha^{-1}(\hat{K} - \alpha I - t\alpha I)) < 1$, the series

$$\sum_{k=0}^{\infty} Tz_k = \sum_{k=0}^{\infty} t^{-1}\alpha^{-1}[-t^{-1}\alpha^{-1}(\hat{K} - \alpha I - t\alpha I)]^k Ty \quad (32)$$

converges. Since T is closed, $x_2 = \sum z_k$, and $\sum Tz_k$ converges, then the vector x_2 lies in Y , the domain of T . $\square$

THEOREM 7. *Let $f \in L_{\infty}(\mathcal{H})$, $f \geq 0$, and $g \in L_2(G) \cap L_p(\mathbb{R}^m)$, where $p > m/2$, $g = 0$ on $\mathbb{R}^m \setminus G$. Let L be a bounded linear functional on $W^{1,2}(G)$ representable by $\mu \in \mathcal{C}'(\partial G)$. If u is a weak solution in $W^{1,2}(G)$ of problem (3), then $u \in L_{\infty}(G)$ if and only if $\mu \in \mathcal{C}'_b(\partial G)$.*

Proof. If $u \in L_{\infty}(G)$, then $\mu \in \mathcal{C}'_b(\partial G)$ by Lemma 4.

Suppose now that $\mu \in \mathcal{C}'_b(\partial G)$. Let $w = u - \mathcal{U}(g\mathcal{H}_m)$. According to Lemma 3, there is a bounded linear functional $\tilde{L}$ on $W^{1,2}(G)$ representable by $\tilde{\mu} \in \mathcal{C}'_b(\partial G)$ such that w is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned} \Delta w &= 0 \quad \text{on } G, \\ \frac{\partial w}{\partial n} + wf &= \tilde{L} \quad \text{on } \partial G. \end{aligned} \quad (33)$$

Define for $\varphi \in L_{\infty}(\mathcal{H})$ and $x \in \partial G$,

$$T\varphi(x) = \frac{1}{2}\varphi(x) + \int_{\partial G} \varphi(y) \frac{\partial}{\partial n(y)} h_x(y) d\mathcal{H}(y) + \mathcal{U}(f\varphi\mathcal{H}). \quad (34)$$

Since $\mathcal{U}(f\mathcal{H}) \in \mathcal{C}(\mathbb{R}^m)$ by [6, Corollary 2.17 and Lemma 2.18], the operator T is a bounded linear operator on $L_{\infty}(\mathcal{H})$ by [11, Proposition 8] and [6, Lemma 2.15]. The operator $T - (1/2)I$ is compact by [12, Theorem 20] and [6, Theorem 4.1 and Corollary 1.11]. According to [10, Theorem 1], there is $\nu \in \mathcal{C}'(\partial G) \subset (L_{\infty}(\mathcal{H}))'$ such that $T'\nu = \tilde{\mu}$ and

$$\int_G \nabla \mathcal{U}\nu \cdot \nabla v d\mathcal{H}_m + \int_{\partial G} \mathcal{U}\nu f v d\mathcal{H} = \int v d\tilde{\mu}, \quad (35)$$

for each $v \in \mathcal{D}$.

Remark that $\mathcal{C}'(\partial G)$ is a closed subspace of $(L_{\infty}(\mathcal{H}))'$. According to [11, Proposition 8], we have $T'(\mathcal{C}'(\partial G)) \subset \mathcal{C}'(\partial G)$. Denote by τ the restriction of T' to $\mathcal{C}'(\partial G)$. According to [10, Lemma 11] and [14, Chapter VI, Theorem 1.2], we have $\sigma(\tau) \subset \{\beta; \beta \geq 0\}$. Since $\sigma(\tau') = \sigma(\tau)$ (see [15, Chapter VIII, Section 6, Theorem 2]), each $\beta \in \sigma(T)$ is an eigenvalue (see [14, Chapter VI, Theorem 1.2]), and T is the restriction of τ' to $L_{\infty}(\mathcal{H})$, we obtain that $\sigma(T') = \sigma(T) \subset \{\beta; \beta \geq 0\}$ by [15, Chapter VIII, Section 6, Theorem 2].

According to [9, Theorem 1.11], we have $\text{Ker } T' \subset \mathcal{C}'_b(\partial G)$. According to [9, Lemma 1.10] and [10, Lemmas 12 and 13], $\text{Ker } T' = \text{Ker}(T')^2$. Denote, for $\rho \in \mathcal{C}'_b(\partial G)$, by $V\rho$ the restriction of $\mathcal{U}\rho$ to ∂G . Then V is a closed operator from $\mathcal{C}'_b(\partial G)$ to $L_\infty(\mathcal{H})$ by [13, Lemma 5]. If $\rho \in \mathcal{C}'_b(\partial G)$, then $VT'\rho = TV\rho$ by [13, Lemma 4]. If $\rho_1, \rho_2 \in \mathcal{C}'_b(\partial G)$, then ρ_1 and ρ_2 have finite energy by [13, Proposition 23], [7, Theorem 1.20], and

$$\int \mathcal{U}\rho_1 d\rho_2 = \int_{\mathbb{R}^m} \nabla \mathcal{U}\rho_1 \cdot \nabla \mathcal{U}\rho_2 d\mathcal{H}_m = \int \mathcal{U}\rho_2 d\rho_1. \quad (36)$$

Since $T'\nu = \tilde{\mu} \in \mathcal{C}'_b(\partial G)$, Theorem 6 yields that $\nu \in \mathcal{C}'_b(\partial G)$. Since ν has finite energy $\int \mathcal{U}\nu d\nu$ and $\int \mathcal{U}\nu d\nu = \int |\nabla \mathcal{U}\nu|^2 d\mathcal{H}_m$ by [7, Theorem 1.20], we obtain that $\mathcal{U}\nu \in W^{1,2}(G)$ (see [7, Lemma 1.6] and [16, Theorem 2.14]). Since $\mathcal{D}$ is dense in $W^{1,2}(G)$ by [16, Theorem 2.3.2], relation (35) yields that the function $\mathcal{U}\nu$ is a weak solution in $W^{1,2}(G)$ of (33). Since $\nu = \mathcal{U}\nu - w$ is a weak solution in $W^{1,2}(G)$ of the problem

$$\begin{aligned} \Delta \nu &= 0 \quad \text{on } G, \\ \frac{\partial \nu}{\partial n} + \nu f &= 0 \quad \text{on } \partial G, \end{aligned} \quad (37)$$

and $f \geq 0$, we obtain

$$0 = \int_G \nabla \nu \cdot \nabla \nu d\mathcal{H}_m + \int_{\partial G} \nu f \nu d\mathcal{H} \geq \int_G |\nabla \nu|^2 d\mathcal{H}_m \geq 0. \quad (38)$$

Therefore, $\nabla \nu = 0$ on G and there is a constant c such that $\nu(x) = c$ for $\mathcal{H}_m$ -a.a. $x \in G$ by [16, Corollary 2.1.9]. Since $\nu \in \mathcal{C}'_b(\partial G)$, the function $\mathcal{U}\nu$ is bounded in G . Since $u(x) = \mathcal{U}(g\mathcal{H}_m)(x) + \mathcal{U}\nu(x) - c$ for $\mathcal{H}_m$ -a.a. $x \in G$ and $\mathcal{U}(g\mathcal{H}_m) \in \mathcal{C}(\mathbb{R}^m)$ by Lemma 3, we obtain $u \in L_\infty(G)$. $\square$

Acknowledgment

The author was supported by GAČR Grant no. 201/00/1515.

References

- [1] M. Brelot, *Éléments de la Théorie Classique du Potentiel*, Les Cours de Sorbonne. 3e cycle, Centre de Documentation Universitaire, Paris, 1959.
- [2] B. E. J. Dahlberg, *Estimates of harmonic measure*, Arch. Rational Mech. Anal. **65** (1977), no. 3, 275–288.
- [3] L. E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Cambridge Tracts in Mathematics, vol. 128, Cambridge University Press, Cambridge, 2000.
- [4] L. L. Helms, *Introduction to Potential Theory*, Pure and Applied Mathematics, vol. 22, Wiley-Interscience, New York, 1969.
- [5] H. Heuser, *Funktionalanalysis*, Mathematische Leitfäden, B. G. Teubner, Stuttgart, 1975.
- [6] J. Král, *Integral Operators in Potential Theory*, Lecture Notes in Mathematics, vol. 823, Springer-Verlag, Berlin, 1980.
- [7] N. S. Landkof, *Fundamentals of Modern Potential Theory*, Izdat. Nauka, Moscow, 1966.
- [8] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

- [9] D. Medková, *The third boundary value problem in potential theory for domains with a piecewise smooth boundary*, Czechoslovak Math. J. **47(122)** (1997), no. 4, 651–679.
- [10] ———, *Solution of the Robin problem for the Laplace equation*, Appl. Math. **43** (1998), no. 2, 133–155.
- [11] I. Netuka, *Generalized Robin problem in potential theory*, Czechoslovak Math. J. **22(97)** (1972), 312–324.
- [12] ———, *An operator connected with the third boundary value problem in potential theory*, Czechoslovak Math. J. **22(97)** (1972), 462–489.
- [13] ———, *The third boundary value problem in potential theory*, Czechoslovak Math. J. **22(97)** (1972), 554–580.
- [14] M. Schechter, *Principles of Functional Analysis*, Academic Press, New York, 1973.
- [15] K. Yosida, *Functional Analysis*, Die Grundlehren der mathematischen Wissenschaften, vol. 123, Springer-Verlag, Berlin, 1965.
- [16] W. P. Ziemer, *Weakly Differentiable Functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989.

Dagmar Medková: Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic

Current address: Department of Technical Mathematics, Faculty of Mechanical Engineering, Czech Technical University, Karlovo nám. 13, 121 35 Praha 2, Czech Republic

E-mail address: medkova@math.cas.cz