A DENSITY THEOREM FOR LOCALLY CONVEX LATTICES

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Let E be a real, locally convex, locally solid vector lattice of (AM)-type. First, we prove an approximation theorem of Bishop's type for a vector subspace of such a lattice. Second, using this theorem, we obtain a generalization of Nachbin's density theorem for weighted spaces.

1. Introduction

In this paper, we introduce the concept of antisymmetric ideal with respect to a pair (A,F), when A is a subset of the real part of the center of E, and F is a vector subspace of E. This notion is a generalization, for locally convex lattices, of the notion of antisymmetric set from the theory of function algebras.

Further, we study some properties of the family of antisymmetric ideals. For example, we show that every element of this family contains a unique minimal element belonging to this family.

The main result of this paper is Theorem 4.2 which states that for every $x \in E$ we have $x \in \overline{F}$ if and only if $\pi_I(x) \in \overline{\pi_I(F)}$ for any minimal (A,F)-antisymmetric ideal I, where π_I denotes the canonical mapping $E \to E/I$.

This theorem is a Bishop's type approximation theorem and generalizes a similar result for $C(X)$.

Finally, we show that if the pair (A,F) fulfils some supplementary conditions, then F is dense in E, and also show how Nachbin's density theorem for weighted spaces follows from this theorem.

2. Preliminaries

In the sequel, E denotes a real, locally convex, locally solid vector lattice of (AM)-type. For every closed ideal I of E, we will denote by π_I the canonical mapping $E \to E/I$ and by π_I' it's adjoint. The center $Z(E)$ of E is the algebra of all order-bounded endomorphisms on E, that is, those $U \in L(E,E)$ for which there exists $\lambda_U > 0$ such that $|U(x)| \leq \lambda_U |x|$, for all $x \in E$. The real part of the center is $\operatorname{Re} Z(E) = Z(E)_+ - Z(E)_+$.

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Definition 2.1. For every closed ideal I of E and every $U \in \operatorname{Re} Z(E)$, $\pi_I(U) : E/I \to E/I$ is defined by

$$\pi_I(U)(\pi_I(x)) = \pi_I(U(x)), \quad x \in E. \tag{2.1}$$

It is easily seen that the operator $\pi_I(U)$ is well defined. For every $A \subset Z(E)$, we denote

$$\pi_I(A) = \{\pi_I(U);\ U \in A\}. \tag{2.2}$$

Remark 2.2. If $A \subset \operatorname{Re} Z(E)$, then $\pi_I(A) \subset \operatorname{Re} Z(E/I)$.

Indeed, if $U \in A$, then, for every $x \in E$, we have

$$\begin{aligned} |\pi_I(U)(\pi_I(x))| &= |\pi_I(U(x))| = \pi_I(|U(x)|) \\ &\le \pi_I(\lambda_U |x|) = \lambda_U \pi_I(|x|) = \lambda_U |\pi_I(x)|, \end{aligned} \tag{2.3}$$

hence $\pi_I(U) \in Z(E/I)$.

Definition 2.3. Let I and J be two closed ideals of E such that $I \subset J$. Then the following two mappings can be defined: $\pi_{IJ} : E/I \to E/J$ given by

$$\pi_{IJ}[\pi_I(x)] = \pi_J(x), \quad x \in E, \tag{2.4}$$

and $M_{IJ} : \operatorname{Re} Z(E/I) \to \operatorname{Re} Z(E/J)$ given by

$$M_{IJ}(U)(\pi_J(x)) = \pi_{IJ}(U(\pi_I(x))), \quad U \in \operatorname{Re} Z(E/I). \tag{2.5}$$

As a consequence of the inequality,

$$\begin{aligned} |M_{IJ}(U)(\pi_J(x))| &= |\pi_{IJ}(U(\pi_I(x)))| \\ &= \pi_{IJ}(|U(\pi_I)(x)|) \le \pi_{IJ}(\lambda_U |\pi_I(x)|) \\ &= \lambda_U \pi_{IJ}(|\pi_I(x)|) = \lambda_U \pi_J(|x|) = \lambda_U |\pi_J(x)|, \end{aligned} \tag{2.6}$$

for every $x \in E$, the range of M_{IJ} is included in $\operatorname{Re} Z(E/J)$.

3. Antisymmetric ideals

Let A be a subset of $\operatorname{Re} Z(E)$ containing 0 and let F be a vector subspace of E.

Definition 3.1. A closed ideal I of E is said to be antisymmetric with respect to the pair (A,F) if, for every $U \in \pi_I(A)$ with the property $U[\pi_I(F)] \subset \pi_I(F)$, it follows that there exists a real number α such that $U = \alpha \mathbf{1}_{E/I}$, where $\mathbf{1}_{E/I}$ is the identity operator on E/I.

Of course, E itself is an antisymmetric ideal with respect to the pair (A,F) for every $A \subset \operatorname{Re} Z(E)$ and every vector subspace F of E.

Further, we denote by $\mathcal{A}_{A,F}(E)$ the family of all (A,F)-antisymmetric ideals of E.

Now we consider the particular case $E = C(X,\mathbb{R})$, where X is a compact Hausdorff space. It is well known that there is a one-to-one correspondence between the class of the closed ideals of $C(X,\mathbb{R})$ and the class of the closed subsets of X. Namely, for every closed

subset S of X, the set $I_S = \{f \in C(X,\mathbb{R});\ f|S = 0\}$ is a closed ideal of $C(X,\mathbb{R})$ and every closed ideal of $C(X,\mathbb{R})$ has this form.

Definition 3.2. Let A be a subset of $C(X,\mathbb{R})$ with $0 \in A$ and let F be a closed subset of $C(X,\mathbb{R})$. A closed subset S of X is said to be antisymmetric with respect to the pair (A,F) if every $f \in A$ with the property $f \cdot g|S \in F|S$ for every $g \in F$ is constant on S.

Remark 3.3. A closed subset S of X is (A,F)-antisymmetric if and only if the corresponding ideal I_S is (A,F)-antisymmetric in the sense of Definition 3.1.

Indeed, it is sufficient to observe that $\pi_{I_S}(a) = a|S$ for every subset S of X.

Lemma 3.4. *Let (I_α) be a family of elements of $\mathcal{A}_{A,F}(E)$ such that $J = \sum_\alpha I_\alpha \neq E$. Then*

$$I = \cap_\alpha I_\alpha \in \mathcal{A}_{A,F}(E). \tag{3.1}$$

Proof. If $U \in \pi_I(A)$ has the property $U[\pi_I(F)] \subset \pi_I(F)$, then

$$M_{II_\alpha}(U)(\pi_{I_\alpha}(F)) = \pi_{II_\alpha}[U(\pi_I(F))] \subset \pi_{II_\alpha}[\pi_I(F)] = \pi_{I_\alpha}(F). \tag{3.2}$$

Let $V \in A$ be such that $U = \pi_I(V)$. For every $x \in E$, we have

$$\begin{aligned} M_{II_\alpha}(U)(\pi_{I_\alpha}(x)) &= \pi_{II_\alpha}[U(\pi_I(x))] = \pi_{II_\alpha}[\pi_I(V)(\pi_I(x))] \\ &= \pi_{II_\alpha}[\pi_I(V(x))] = \pi_{I_\alpha}[V(x)] = \pi_{I_\alpha}(V)(\pi_{I_\alpha}(x)). \end{aligned} \tag{3.3}$$

Thus, $M_{II_\alpha}(U) = \pi_{I_\alpha}(V) \in \pi_{I_\alpha}(A) \subset \operatorname{Re} Z(E/I_\alpha)$ and $M_{II_\alpha}(U)(\pi_{I_\alpha}(F)) \subset \pi_{I_\alpha}(F)$. Since $I_\alpha \in \mathcal{A}_{A,F}(E)$, it follows that an $a_\alpha \in \mathbb{R}$ exists such that $M_{II_\alpha}(U) = a_\alpha \cdot \mathbf{1}_{E/I_\alpha}$.

On the other hand, we have

$$M_{IJ}(U) = M_{I_\alpha J}[M_{II_\alpha}(U)] = a_\alpha \cdot \mathbf{1}_{E/I_\alpha}. \tag{3.4}$$

Since $J \neq E$, it follows that $a_\alpha = a$ (constant) for any α. Therefore,

$$M_{II_\alpha}(U) = a \cdot \mathbf{1}_{E/I_\alpha} = a \cdot M_{II_\alpha}(\mathbf{1}_{E/I}), \tag{3.5}$$

hence,

$$M_{II_\alpha}(U - a \cdot \mathbf{1}_{E/I}) = 0, \tag{3.6}$$

for any α, and this involves $U = a \cdot \mathbf{1}_{E/I}$. □

Corollary 3.5. *Every $I \in \mathcal{A}_{A,F}(E)$ contains a unique minimal ideal $\widetilde{I} \in \mathcal{A}_{A,F}(E)$.*

Proof. Let $I \in \mathcal{A}_{A,F}(E)$ be such that $I \neq E$ and let $\widetilde{I} = \cap\{J \in \mathcal{A}_{A,F}(E);\ J \subset I\}$. According to Lemma 3.4, $\widetilde{I} \in \mathcal{A}_{A,F}(E)$. It is now obvious that $\widetilde{I} \subset I$ and $\widetilde{I}$ is minimal. □

Further, we denote by $\widetilde{\mathcal{A}}_{A,F}(E)$ the family of all minimal closed ideals of E, antisymmetric with respect to the pair (A,F).

4. Bishop's type approximation theorem

Lemma 4.1. *Let A be a subset of* $\operatorname{Re}Z(E)$ *with* $0 \in A$, *let F be a vector subspace of E, and let V be a convex and solid neighborhood of the origin of E, which is also a sublattice. If* $f \in \operatorname{Ext}\{V^0 \cap F^0\}$ *and* $I = \{x \in E;\ |f|(|x|) = 0\}$, *then* $I \in \mathcal{A}_{A,F}(E)$.

Proof. Let $U \in \pi_I(A)$ be such that $U[\pi_I(F)] \subset \pi_I(F)$. We can suppose that $0 \le U \le \mathbf{1}_{E/I}$. Since $f \in I^0$, there exists $g \in (E/I)'$ such that $f = \pi_1' g$. Obviously, $g \in \{[\pi_I(V)]^0 \cap [\pi_I(F)]^0\}$. We denote $g_1 = U'g$, $g_2 = (\mathbf{1}_{E/I} - U)'g$, and $a_i = \inf\{\lambda > 0 :\ g_i \in \lambda[\pi_I(V)]^0\} = \sup\{|g_i(y)| :\ y \in \pi_I(V)\}$, for $i = 1,2$.

Since $g = g_1 + g_2 \in (a_1 + a_2)[\pi_I(V)]^0$, it follows that $f \in (a_1 + a_2)V^0$, hence $a_1 + a_2 \ge 1$.

On the other hand, for any $y_1, y_2 \in \pi_I(V)$, we have

$$\begin{aligned} |g_1(y_1)| + |g_2(y_2)| &= |g(U(y_1))| + |g(\mathbf{1}_{E/I} - U)(y_2)| \\ &\le |g|(U(|y_1| \vee |y_2|)) + (\mathbf{1}_{E/I} - U)(|y_1| \vee |y_2|) \\ &= |g|(|y_1| \vee |y_2|). \end{aligned} \tag{4.1}$$

Since $\pi_I(V)$ is a sublattice and $g \in [\pi_I(V)]^0$, it follows that $|y_1| \vee |y_2| \in \pi_I(V)$, hence $|g|(|y_1| \vee |y_2|) \le 1$.

Therefore, $|g_1(y_1)| + |g_2(y_2)| \le 1$ for any $y_1, y_2 \in \pi_I(V)$ and this yields $a_1 + a_2 \le 1$, hence $a_1 + a_2 = 1$.

Now, we observe that if $|g|(|y|) = 0$, then $y = 0$. Indeed, let $x \in E$ be such that $y = \pi_I(x)$.

We have $0 = |g|(|\pi_I(x)|) = |\pi_I' g|(|x|) = |f|(|x|)$.

If follows that $x \in I$, hence $y = \pi_I(x) = 0$.

This remark involves that if $g_1 = U'g = 0$, then $U = 0$ and, analogously, $g_2 = (\mathbf{1}_{E/I} - u)'g = 0$ implies $U = \mathbf{1}_{E/I}$.

Therefore, we can suppose that $g_i \ne 0$ for $i = 1,2$, and hence $a_i > 0$, $i = 1,2$. Further, we have

$$g = a_1 \frac{g_1}{a_1} + a_2 \frac{g_2}{a_2}, \qquad \frac{g_i}{a_i} \in [\pi_I(V)]^0 \cap [\pi_I(F)]^0, \quad i = 1,2. \tag{4.2}$$

Since $g \in \operatorname{Ext}\{[\pi_1(V)]^0 \cap [\pi_I(F)]^0\}$, either $g = g_1/a_1$ or $g = g_2/a_2$. In the first case, $(U - a_1\mathbf{1}_{E/I})'(g) = 0$.

The last equality yields $U = a_1\mathbf{1}_{E/I}$. □

The main result concerning antisymmetric ideals is the following Bishop's type approximation theorem.

Theorem 4.2. *Let E be a real, locally convex, locally solid vector lattice of (AM)-type,* $A \subset \operatorname{Re}Z(E)$ *with* $0 \in A$, *and let F be a vector subspace of E. Then, for any* $x \in E$,

$$x \in \overline{F} \Longleftrightarrow \pi_I(x) \in \overline{\pi_I(F)} \tag{4.3}$$

for every $I \in \widetilde{\mathcal{A}}_{A,F}(E)$.

Proof. The necessity is clear. We suppose that $\pi_I(x) \in \overline{\pi_I(F)}$ for any $I \in \widetilde{\mathcal{A}}_{A,F}(E)$ and that $x \notin \overline{F}$. Then, there exists $f \in E'$ such that $f(x) \neq 0$ and $f(y) = 0$ for any $y \in F$.

Let V be a solid, convex neighborhood of the origin which is also a sublattice of E. By the Krein-Milman theorem, we may assume that $f \in \operatorname{Ext}\{V^0 \cap F^0\}$. If we denote $J = \{x \in E;\ |f|(|x|) = 0\}$, then, according to Lemma 4.1, we have $J \in \widetilde{\mathcal{A}}_{A,F}(E)$. On the other hand, by Corollary 3.5, it follows that there exists $J_0 \in \widetilde{\mathcal{A}}_{A,F}(E)$ such that $J_0 \subset J$. Since $\pi_{J_0}(x) \in \overline{\pi_{J_0}(F)}$ and $f \in J_0^0 \cap F^0$, we have $f(x) = 0$, and this contradicts the choice of f. □

Theorem 4.3. *Let E be a real, locally convex, locally solid vector lattice of (AM)-type, let A be a subset of* $\operatorname{Re} Z(E)$ *with $0 \in A$, and let F be a vector subspace of E with the properties*

(i) *$AF \subset F$,*
(ii) *F is not included in any maximal ideal of E,*
(iii) *every closed (A,F)-antisymmetric ideal I of E with the property $\pi_I(A) \subset \mathbb{R} \cdot \mathbf{1}_{E/I}$ is a maximal ideal.*

Then $\overline{F} = E$.

Proof. Let $x \in E$ and $I \in \widetilde{\mathcal{A}}_{A,F}(E)$. Hypothesis (i) involves that $\pi_I(A)[\pi_I(F)] \subset \pi_I(F)$, and since I is (A,F)-antisymmetric, we have $\pi_I(U) = \alpha_U \cdot \mathbf{1}_{E/I}$ for any $U \in A$. Now, from (iii), it results that I is a maximal ideal and thus that the dimension of $\pi_I(E)$ is one.

Since $F \subset E$, we have either $\pi_I(F) = \{0\}$ or $\pi_I(F) = \pi_I(E)$.

From (ii), it results that $\pi_I(F) \neq \{0\}$. Therefore, we have $\pi_I(F) = \pi_I(E)$ and thus $\pi_I(x) \in \pi_I(F)$ for any $I \in \widetilde{\mathcal{A}}_{A,F}(E)$. According to Theorem 4.2, it follows that $x \in \overline{F}$. □

5. The case of weighted spaces

Typical examples of locally convex lattices are the weighted spaces.

Let X be a locally compact Hausdorff space and let V be a Nachbin family on X, that is, a set of nonnegative upper semicontinuous functions on X directed in the sense that, given $v_1, v_2 \in V$ and $\lambda > 0$, a $v \in A$ exists such that $v_i \leq \lambda v$, $i = 1,2$. We denote by $CV_0(X)$ the corresponding weighted spaces, that is,

$$CV_0(X) = \{f \in C(X,\mathbb{R});\ fv \text{ vanishes at infinity for any } v \in V\}. \tag{5.1}$$

The weighted topology on $CV_0(X)$ is denoted by ω_V and it is determined by the seminorms $\{p_v\}_{v \in V}$, where

$$p_v(f) = \sup\{|f(x)|v(x):\ x \in X\}, \quad \text{for any } f \in CV_0(X). \tag{5.2}$$

The topology ω_V is locally convex and has a basis of open neighborhoods of the origin of the form

$$D_v = \{f \in CV_0(x):\ p_v(f) < 1\}. \tag{5.3}$$

Clearly, $CV_0(X)$ is a locally convex, locally solid vector lattice of (AM)-type with respect to the topology ω_V and to the ordering $f \leq g$ if and only if $f(x) \leq g(x)$, $x \in X$.

A result of Goullet de Rugy [1, Lemma 3.8] states that for every closed ideal I of $CV_0(X)$ there exists a closed subset Y of X such that

$$I = \{f \in CV_0(X) : f|Y = 0\}. \tag{5.4}$$

Therefore, there exists a one-to-one map from the family of closed ideals of $CV_0(X)$ onto the family of closed subsets of X.

If X is a compact Hausdorff space and $V = \{1\}$, then $CV_0(X) = C(X,\mathbf{R})$ and the weighted topology ω_V coincides with the uniform topology of $C(X,\mathbb{R})$.

Further, we denote by $C_b(X,\mathbb{R})$ the algebra of all real bounded continuous functions on X.

As in the case of $C(X)$, we have the following definition.

Definition 5.1. Let A be a subset of $C_b(X)$ with $0 \in A$ and let F be a vector subspace of $CV_0(X)$. A closed subset S of X is called antisymmetric with respect to the pair (A,F) if and only if the corresponding ideal

$$I_S = \{f \in CV_0(X) : f|S = 0\} \tag{5.5}$$

is an (A,F)-antisymmetric ideal, and this means that every $a \in A$ with the property $\alpha \cdot h|S \in F|S$, for any $h \in F$, is constant on S.

It is easily seen that every $x \in X$ belongs to a maximal (A,F)-antisymmetric set S_x. At the same time, if $x \neq y$, we have either $S_x = S_y$ or $S_x \cap S_y = \varnothing$.

Theorem 4.2 then involves the following theorem.

Theorem 5.2. *Let A and F be as in Definition 5.1. Then, a function $f \in CV_0(X)$ belongs to F if and only if $f|S_x \in \overline{F|S_x}$ for any $x \in X$.*

The following theorem is a generalization of Nachbin's density theorem for weighted spaces in the real case.

Theorem 5.3. *Let A be a subset of $C_b(X,\mathbb{R})$ with $0 \in A$ and let F be a vector subspace of $CV_0(X)$ with the properties*

(i) $AF \subset F$,
(ii) *A separates the points of X,*
(iii) *for every $x \in X$, there is an $f \in F$ such that $f(x) \neq 0$.*

Then $\overline{F} = CV_0(X)$.

Proof. Since the centre of the lattice $E = CV_0(X)$ is the algebra $C_b(X)$ of all continuous bounded functions on X (see, e.g., [2]), it follows that $A \subset \operatorname{Re} Z(E)$. On the other hand, from (iii), it follows that F is not included in any maximal ideal. Since $AF \subset F$ and A separates the points of X, it results that every (A,F)-antisymmetric subset S of X is a singleton, and thus the corresponding ideal I_S is a maximal ideal. Thus the hypotheses of Theorem 4.3 are satisfied and so Theorem 5.3 is proved. □

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