ON LINEAR SINGULAR FUNCTIONAL-DIFFERENTIAL EQUATIONS IN ONE FUNCTIONAL SPACE

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We use a special space of integrable functions for studying the Cauchy problem for linear functional-differential equations with nonintegrable singularities. We use the ideas developed by Azbelev and his students (1995). We show that by choosing the function ψ generating the space, one can guarantee resolvability and certain behavior of the solution near the point of singularity.

1. Linear Volterra operators in Δ_ψ spaces

We consider the following n -dimensional functional-differential equation:

$$\mathcal{L}x \stackrel{\text{def}}{=} \dot{x} + (K + S)\dot{x} + Ax(0) = f, \quad (1.1)$$

where

$$(Ky)(t) = \int_0^t K(t,s)y(s)ds, \quad (1.2)$$

$$(Sy)(t) = \begin{cases} B(t)y[g(t)] & \text{if } g(t) \in [0,1], \\ 0 & \text{if } g(t) \notin [0,1]. \end{cases} \quad (1.3)$$

The case where K and S are continuous on $L_p[0,1]$ operators is well studied (see, e.g., [1] and the references therein). Here we suppose that the functions $K(t,s)$ and $B(t)$ may be nonintegrable at $t = 0$. More precisely, we will formulate conditions on operators K and S in Sections 2 and 3. Under such conditions, those operators are not bounded on $L[0,1]$ and one has to choose other functional spaces for studying (1.1). We propose a space of integrable functions on $[0,1]$ and show that it may be useful in such a case.

We call Δ_ψ^p space the space of all measurable functions $y: [0,1] \rightarrow \mathbb{R}^n$, for which

$$\|y\|_{\Delta_\psi^p} = \sup_{0 < h \leq 1} \frac{1}{\psi(h)} \left(\int_0^h |y(s)|^p ds \right)^{1/p} < \infty. \quad (1.4)$$

We assume everywhere below that ψ is a nondecreasing, absolutely continuous function, $\psi(0) = 0$.

THEOREM 1.1. *The space Δ_ψ^p is a Banach space.*

Let $X[a, b]$, $Y[a, b]$ be spaces of functions defined on $[a, b]$.

We will call $V : X[0, 1] \rightarrow Y[0, 1]$ the *Volterra operator* [3] if for every $\xi \in [0, 1]$ and for any $x_1, x_2 \in X[0, 1]$ such that $x_1(t) = x_2(t)$ on $[0, \xi]$, $(Vx_1)(t) = (Vx_2)(t)$ for $t \in [0, 1]$.

It is possible to say that each Volterra operator $V : X[0, 1] \rightarrow Y[0, 1]$ generates a set of operators $V_\xi : X[0, \xi] \rightarrow Y[0, \xi]$, where $\xi \in (0, 1]$. By y_ξ , we denote the restriction of function y defined on $[0, 1]$ onto segment $[0, \xi]$.

THEOREM 1.2. *Let $V : L \rightarrow L$ be a linear bounded operator. Then V is a linear bounded operator in Δ_ψ^p and $\|V\|_{\Delta_\psi^p} \leq \|V\|_{L^p}$.*

Proof. Let $y \in \Delta_\psi^p$. Then

$$\begin{aligned} \|Vy\|_{\Delta_\psi^p} &= \sup_{0 < h \leq 1} \frac{1}{\psi(h)} \|(V_\xi y_\xi)\|_{L[0, \xi]^p} \\ &\leq \sup_{0 < h \leq 1} \frac{1}{\psi(h)} \|V_\xi\|_{L[0, \xi]} \|y_\xi\|_{L[0, \xi]} \leq \|V\|_{L^p} \|y\|_{L^p}. \end{aligned} \quad (1.5)$$

□

THEOREM 1.3. *Let $V : \Delta_{\psi_1}^p \rightarrow \Delta_{\psi_1}^p$ be linear bounded operator and let*

$$\sup_{t \in [0, 1]} \frac{\psi_2(t)}{\psi_1(t)} < \infty. \quad (1.6)$$

Then V is linear and bounded in $\Delta_{\psi_2}^p$ and

$$\|V\|_{\Delta_{\psi_2}^p} \leq \|V\|_{\Delta_{\psi_1}^p} \sup_{\xi \in [0, 1]} \sup_{\tau \in [0, \xi]} \frac{\psi_1(\xi)\psi_2(\tau)}{\psi_2(\xi)\psi_1(\tau)}. \quad (1.7)$$

Proof. Let $y \in \Delta_{\psi_2}^p$. Then

$$\begin{aligned} \|Vy\|_{\Delta_{\psi_2}^p} &\leq \sup_{\xi \in [0, 1]} \frac{\|Vy_\xi\|_{L[0, \xi]} \psi_1(\xi)}{\psi_2(\xi)\psi_1(\xi)} \leq \sup_{\xi \in [0, 1]} \frac{\|Vy_\xi\|_{\Delta_{\psi_1}^p[0, \xi]} \psi_1(\xi)}{\psi_2(\xi)} \\ &\leq \|V\|_{\Delta_{\psi_1}^p} \sup_{\xi \in [0, 1]} \frac{\|y_\xi\|_{\Delta_{\psi_1}^p} \psi_1(\xi)}{\psi_2(\xi)} \\ &\leq \|V\|_{\Delta_{\psi_1}^p} \sup_{\xi \in [0, 1]} \sup_{\tau \in [0, \xi]} \frac{\|y_\tau\|_{L[0, \tau]} \psi_1(\xi)\psi_2(\tau)}{\psi_1(\tau)\psi_2(\xi)\psi_2(\tau)} \\ &\leq \|y\|_{\Delta_{\psi_2}^p} \|V\|_{\Delta_{\psi_1}^p} \sup_{\xi \in [0, 1]} \sup_{\tau \in [0, \xi]} \frac{\psi_1(\xi)\psi_2(\tau)}{\psi_2(\xi)\psi_1(\tau)}. \end{aligned} \quad (1.8)$$

□

COROLLARY 1.4. *If $V_1 : \Delta_{\psi_1}^p \rightarrow \Delta_{\psi_1}^p$ and $V_2 : \Delta_{\psi_2}^p \rightarrow \Delta_{\psi_2}^p$ are linear continuous Volterra operators, then $V = V_1 + V_2$ is continuous on space Δ_ψ^p generated by $\psi(t) = \min(\psi_1(t), \psi_2(t))$ and $\|V\|_{\Delta_\psi^p} \leq \|V_1\|_{\Delta_{\psi_1}^p} + \|V_2\|_{\Delta_{\psi_2}^p}$.*

2. Operator K

In this section, we consider the integral operator (1.2). We will show that under certain conditions on matrix $K(t, s)$, a function ψ may be indicated such that K is bounded on Δ_ψ and its norm is limited by a given number.

We say that matrix $K(t, s)$ satisfies the $\mathcal{N}$ condition if for some p and p_1 such that $1 \leq p \leq p_1 < \infty$ and for any $\varepsilon \in (0, 1]$,

$$\|K_\varepsilon(t, \cdot)\|_{L[0, t]} \in L_{p'}[\varepsilon, 1]. \quad (2.1)$$

Here $K_\varepsilon(t, s)$ is a restriction of $K(t, s)$ onto $[\varepsilon, 1] \times [0, t]$, $1/p + 1/p' = 1$.

The $\mathcal{N}$ condition admits a nonintegrable singularity at point $t = 0$.

LEMMA 2.1. *Let nonnegative function $\omega : [0, 1] \rightarrow \mathbb{R}$ be nonincreasing and having a nonintegrable singularity at $t = 0$.*

Then $\psi(t) = \exp[\int_1^t \omega(s)ds]$ is absolutely continuous on $[0, 1]$, does not decrease, and is a solution of the equation $\int_1^t \omega(s)x(s)ds = x(t)$.

Denote

$$\psi(t) = \exp \left[\frac{1}{C} \int_1^t \text{vraisup}_{s \in [0, \tau]} \|K(\tau, s)\| d\tau \right]. \quad (2.2)$$

THEOREM 2.2. *Let matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition with $p = 1$ and let C be some positive constant. Then operator K is bounded in Δ_ψ with function ψ defined by the equality (2.2) and $\|K\|_{\Delta_\psi} \leq C$.*

Proof. Let $x \in \Delta_\psi$ and $y = Kx$. From the $\mathcal{N}$ condition it follows that for almost all $t \in [0, 1]$, $K(\cdot, s) \in L_\infty$. Let $\omega(t) = \text{vraisup}_{s \in [0, \tau]} \|K(\tau, s)\| d\tau$. Then

$$\begin{aligned} \left(\int_0^t \|y(s)\| ds \right) &\leq \left[\int_0^t \left(\int_0^\tau \|K(\tau, s)\| \|x(s)\| ds \right) d\tau \right] \\ &\leq \int_0^t \left(\text{vraisup}_{s \in [0, \tau]} \|K(\tau, s)\| \right) \left(\int_0^\tau \|x(s)\| ds \right) d\tau \\ &\leq \|x\|_{\Delta_\psi} \int_0^t \omega(\tau) \psi(\tau) d\tau. \end{aligned} \quad (2.3)$$

According to Lemma 2.1, $\psi(t) = \exp[(1/C) \int_1^t \omega(s)ds]$ is a solution of the equation $\int_1^t \omega(s)\psi(s)ds = C\psi(t)$, does not decrease, is absolutely continuous, and $\psi(0) = 0$. That implies

$$\left(\int_0^t \|y(s)\| ds \right) \leq C \|x\|_{\Delta_\psi} \psi(t). \quad (2.4)$$

□

Remark 2.3. If $K(\cdot, s)$ has bounded variation on s , it is possible to indicate a “wider” space Δ_ψ for which conditions of [Theorem 2.2](#) are satisfied by defining function ψ as

$$\psi(t) = \exp \left[\frac{1}{C} \int_1^t \left(\|K(\tau, \tau)\| d\tau + \int_0^\tau d_s \operatorname{var}_{s \in [0, \tau]} \|K(\tau, s)\| \right) d\tau \right]. \quad (2.5)$$

THEOREM 2.4. Let matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition with $1 < p < \infty$ and let C be some positive constant. Then operator K is bounded in space Δ_ψ^p generated by

$$\psi(t) = \exp \left[\frac{1}{pC} \int_1^t \left(\int_0^\tau \|K(\tau, s)\|^{p'} ds \right)^{p_1/p'} d\tau \right] \quad (2.6)$$

and $\|K\|_{\Delta_\psi^p} \leq C$.

[Theorem 2.4](#) can be proved in a way similar to proof of [Theorem 2.2](#).

LEMMA 2.5. Let $K : \Delta_\psi^p \rightarrow \Delta_\psi^p$ ($1 < p < \infty$) be a bounded operator and let its matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition. Then $K : \Delta_\psi^p \rightarrow L_p$ is a compact operator.

Proof. For every $t \in [0, 1]$, $(Ky)(t)$ is a linear bounded functional on L_p . Let $\{y_i\}$ be a sequence weakly converging to y_0 in L_p . If $\{y_i\} \subset \Delta_\psi^p$ and $\|y_i\|_{\Delta_\psi^p} \leq 1$, then $\|y_0\|_{\Delta_\psi^p} \leq 1$. Indeed, if for some $t_1 \in [0, 1]$, $((1/\psi(t_1)) \int_0^{t_1} \|y(s)\|^p ds)^{1/p} > 1$, then the sequence $ly_i = \int_0^1 l(s)y_i(s)ds$ does not converge to ly_0 , where

$$l(s) = \begin{cases} 1, & \text{if } s \leq t_1, \\ 0, & \text{if } s > t_1. \end{cases} \quad (2.7)$$

Hence, for almost all $t \in [0, 1]$, $\{(Ky_i)(t)\}$ converges and the set Ky is compact in measure. Thus, for the operator $K : \Delta_\psi^p \rightarrow L_p$ to be compact, it is necessary and sufficient that the norms of Ky are equicontinuous for $\|y\|_{\Delta_\psi^p} \leq M$. Let $\delta \in (0, 1)$. As $K : \Delta_\psi^p \rightarrow \Delta_\psi^p$ is a bounded operator,

$$\left(\frac{1}{\psi(\delta)} \int_0^\delta \|(Ky)(s)\|^p ds \right)^{1/p} \leq \Delta_0. \quad (2.8)$$

This implies that for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $\delta < \delta_1$, then $(\int_0^\delta \|(Ky)(s)\|^p ds)^{1/p} \leq \varepsilon/2$.

Then, from the $\mathcal{N}$ condition, there exists δ_2 such that if $\operatorname{mes} e \leq \delta_2$ for some $e \subset [\delta, 1]$, then $(\int_e \|(Ky)(s)\|^p ds)^{1/p} \leq \varepsilon/2$.

Finally, for $e_1 \subset [\delta, 1]$ such that $\operatorname{mes} e_1 \leq \min\{\delta_1, \delta_2\}$,

$$\left(\int_{e_1} \|(Ky)(s)\|^p ds \right)^{1/p} \leq \left(\int_0^\delta \|(Ky)(s)\|^p ds \right)^{1/p} + \left(\int_\delta^1 \|(Ky)(s)\|^p ds \right)^{1/p} \leq \varepsilon. \quad (2.9)$$

□

LEMMA 2.6. Let $\{y_i\} \rightarrow y_0$ in L_p ($1 < p < \infty$) and let the sequence $\{(1/u)y_i\}$ be bounded in Δ_ψ^p for some continuous increasing function u , $u(0) = 0$. Then $\{y_i\} \rightarrow y_0$ in Δ_ψ^p .

Proof. We have

$$\left(\int_0^t \|y_i(s)\|^p ds \right)^{1/p} \leq u(t) \left(\int_0^t \left\| \frac{y_i(s)}{u(s)} \right\|^p ds \right)^{1/p} \leq Mu(t)\psi(t). \quad (2.10)$$

Thus, $y_i \in \Delta_\psi^p$. Beginning with some N for any $t \in [0, 1]$ and for any given $\varepsilon > 0$,

$$\left(\int_0^t \|y_i(s) - y_0(s)\|^p ds \right)^{1/p} \leq \varepsilon. \quad (2.11)$$

Hence,

$$\begin{aligned} \left(\int_0^t \|y_0(s)\|^p ds \right)^{1/p} &\leq \left(\int_0^t \|y_i(s) - y_0(s)\|^p ds \right)^{1/p} + \left(\int_0^t \|y_i(s)\|^p ds \right)^{1/p} \\ &\leq \varepsilon + Mu(t)\psi(t) \leq Mu(t)\psi(t), \\ \left(\int_0^t \|y_i(s) - y_0(s)\|^p ds \right)^{1/p} &\leq 2Mu(t)\psi(t), \end{aligned} \quad (2.12)$$

beginning with some N_δ for any $\delta > 0$, $\|y_0 - y_i\|_{\Delta_\psi^p} < \delta$. Indeed, [Lemma 2.5](#) guarantees the existence of $\tau \in (0, 1]$ such that for all $t \in [0, \tau]$,

$$\left(\int_0^t \|y_i(s) - y_0(s)\|^p ds \right)^{1/p} \leq \delta\psi(t). \quad (2.13)$$

Let $t \in [\tau, 1]$. Then for $\varepsilon = \delta\psi(\tau)$, (2.11) yields (2.13) for all $t \in [0, 1]$. $\square$

Let $u : [0, 1] \rightarrow \mathbb{R}$ be a continuous increasing function, $u(0) = 0$. Denote

$$\psi(t) = \exp \left[\int_1^t \frac{1}{u(\tau)} \left(\int_0^\tau \|K(\tau, s)\|^{p'} ds \right)^{p/p'} d\tau \right]. \quad (2.14)$$

Lemmas 2.5 and 2.6 imply the following theorem.

THEOREM 2.7. *Let matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition with $1 < p < \infty$. And let ψ be defined by (2.14). Then $K : \Delta_\psi^p \rightarrow \Delta_\psi^p$ is a compact operator and its spectral radius is equal to zero.*

3. Operator S

Denote

$$\begin{aligned} (S_g y)(t) &= \begin{cases} y[g(t)] & \text{if } g(t) \in [0, 1], \\ 0 & \text{if } g(t) \notin [0, 1], \end{cases} \\ (Sy)(t) &= B(t)(S_g)(t). \end{aligned} \quad (3.1)$$

In [2], it is shown that S_g is bounded in L_p if $r = (\sup(\text{mes } g^{-1}(E)/\text{mes } E))^{1/p} < \infty$ and $\|S_g\|_{L_p} = r$, where $\sup$ is taken on all measurable sets from $[0, 1]$.

Let Ω_m be a set of points from $[0, 1]$ for which $g(t) \geq mt$, $\beta(t)$ is a nonincreasing majorant of function $\|B(t)\|$, and

$$\varphi(t) = \lim_{\text{mes } e \rightarrow 0} \frac{\text{mes } g^{-1}(e)}{\text{mes } e}, \quad (3.2)$$

where e is a closed interval containing t .

We say that operator S_g satisfies the $\mathcal{M}$ condition if $\text{vraisup}_{t \in [\varepsilon, 1]} \varphi(t) < \infty$ for any

$$\varepsilon \in (0, 1] \text{ vraisup}_{t \in [\varepsilon, 1]} \|B(t)\| < \infty, \quad (3.3)$$

and there exists $m \in [0, 1)$ such that

$$\mu_m = \text{vraisup}_{t \in g(\Omega_m)} (\beta(t)^p \varphi(t)) < \infty. \quad (3.4)$$

LEMMA 3.1. *There exists nonincreasing function $u : (0, 1] \rightarrow \mathbb{R}$ such that $\beta(t)^p \varphi(t) \leq u(t)$ and the function*

$$\psi(t) = \begin{cases} t^{u(t)} & \text{if } t \in (0, 1], \\ 0 & \text{if } t = 0, \end{cases} \quad (3.5)$$

is absolutely continuous on $[0, 1]$.

Proof. Let $\{t_i\}$ be a decreasing sequence, $t_1 = 1$, $t_i \rightarrow 0$. Denote

$$n_i = \text{vraisup}_{t \in (t_{i+1}, t_i)} (\beta(t)^p \varphi(t)), \quad u(t) = \frac{n_{i+1} - n_i}{t_{i+1} - t_i} (t - t_i) + n_i, \quad (3.6)$$

where $t \in (t_{i+1}, t_i)$. Then $\beta(t)^p \varphi(t) \leq u(t)$, u increases and is absolutely continuous on $[0, 1]$. $\square$

Let

$$\nu_m = m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right]. \quad (3.7)$$

THEOREM 3.2. *Let operator S_g satisfy the $\mathcal{M}$ condition and let function u satisfy conditions of Lemma 3.1. Then S_g is bonded in Δ_ψ^p with $\psi(t) = t^{u(t)}$ and*

$$\|S_g\|_{\Delta_\psi^p} \leq (\nu_m + \mu_m)^{1/p}. \quad (3.8)$$

Proof. Let $y \in \Delta_\psi^p$, $\|y\|_{\Delta_\psi^p} = 1$, and $\delta \in (0, 1)$. Denote measures λ and μ on $[\delta, 1]$ by $\lambda(e) = \int_e \beta(s)^p ds$ and $\mu(e) = \int_{g^{-1}(e)} \beta(s)^p ds$. Then by the Radon-Nikodym [2] theorem, we have

$$\begin{aligned} \left\| \int_\delta^t |(S_g y)(t)|^p ds \right\| &\leq \int_{g^{-1}([0, t]) \cap [\delta, 1]} \|y[g(s)]\|^p d\lambda(s) \\ &= \int_{g^{-1}([0, t]) \cap [\delta, 1]} \|y(s)\|^p \frac{d\mu}{d\lambda}(s) d\lambda(s). \end{aligned} \quad (3.9)$$

Then as $g(t) \leq t$,

$$\frac{d\mu}{d\lambda}(s) = \lim_{\text{mes } e \rightarrow 0} \frac{\int_{g^{-1}(e)} \beta(s)^p ds}{\int_e \beta(s)^p ds} \leq \lim_{\text{mes } e \rightarrow 0} \frac{\text{vraisup}_{g^{-1}(e)} \beta(s)^p ds}{\text{vraisup}_e \beta(s)^p} \varphi(s) = \varphi(s) \quad (3.10)$$

or

$$\begin{aligned} \left\| \int_{\delta}^t |(S_g y)(t)|^p ds \right\| &\leq \int_{g^{-1}([0,t] \setminus \Omega_m) \cap [\delta,1]} \beta(s)^p \|y(s)\|^p \varphi(s) ds \\ &\quad + \int_{g^{-1}(\Omega_m) \cap [\delta,1]} \beta(s)^p \|y(s)\|^p \varphi(s) ds \\ &\leq \int_0^{mt} \beta(s)^p \|y(s)\|^p \varphi(s) ds + \int_0^t \|y(s)\|^p \mu_m ds \\ &\leq \int_0^{mt} \|y(s)\|^p u(s) ds + \mu_m \psi(t)^p. \end{aligned} \quad (3.11)$$

We denote function $u_k : (0, 1] \rightarrow \mathbb{R}$ by $u_k(t) = u(t_i)$, where $t_i = (2^k - i)/2^k$, $i = 0, 1, 2, \dots, 2^k - 1$. From $u_k \rightarrow u$, it follows that

$$\int_0^{mt} \|y(s)\|^p u(s) ds = \lim_{k \rightarrow 0} \int_0^{mt} \|y(s)\|^p u_k(s) ds. \quad (3.12)$$

We write function u_k in the form

$$u_k(t) = \begin{cases} u(t_0), & \text{if } t \in (t_1, t_0], \\ u(t_0) + [u(t_1) - u(t_0)], & \text{if } t \in (t_2, t_1], \\ \vdots & \vdots \\ u(t_{k-2}) + [u(t_{k-1}) - u(t_{k-2})], & \text{if } t \in (t_k, t_{k-1}]. \end{cases} \quad (3.13)$$

The condition $t < t_i$ implies that $\int_0^{mt} \|y(s)\|^p ds \leq \psi^p(mt) = (mt)^{u(mt)} \leq m^{pu(t_i)} \psi^p(t)$ and

$$\begin{aligned} \int_0^{mt} \|y(s)\|^p u(s) ds &\leq \sum_{i=1}^{2^k} m^{pu(t_i)} [u(t_i) - u(t_{i-1})] \psi^p(t) + u(1) m^{pu(1)} \psi^p(t) \\ &\leq \psi^p(t) \left[\int_{u(1)}^{\infty} m^s ds + m^{u(1)} u(1) \right] \\ &\leq \psi^p(t) m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right], \end{aligned} \quad (3.14)$$

simultaneously for all k . Finally,

$$\begin{aligned} \left\| \int_0^t |(S_g y)(s)|^p ds \right\| &= \lim_{\delta \rightarrow 0} \left\| \int_{\delta}^t |(S_g y)(s)|^p ds \right\| \\ &\leq \psi^p(t) m^{u(1)} \left[u(1) - \frac{1}{\ln m} \right] + \psi^p(t) \mu_m \\ &\leq \psi^p(t) (\nu_m + \mu_m) \end{aligned} \quad (3.15)$$

which proves the theorem. $\square$

Remark 3.3. From (3.7) and (3.8), it follows that if $\lim_{m \rightarrow 1} < 1$, then there exists function ψ such that the norm of operator $S_g : \Delta_\psi^p \rightarrow \Delta_\psi^p$ is less than 1.

In some particular cases, it is possible to give less strict conditions on function ψ generating the space Δ_ψ^p . Direct calculations prove the following theorem.

THEOREM 3.4. Let $B(t) \leq C_1/t^\alpha$ and $g(t) = C_2 t^\beta$ with $\beta > 1$. Then $\|S_g\|_{\Delta_\psi^p} \leq C_1/C_2$, where $\psi(t) = t^\gamma$, $\gamma \geq (\alpha p + \beta - 1)/p(\beta - 1)$. If $\gamma > (\alpha p + \beta - 1)/p(\beta - 1)$, then the spectral radius of S_g is equal to zero.

4. The Cauchy problem

We consider the Cauchy problem for (1.1):

$$(\mathcal{L}x)(t) = f(t), \quad x(0) = \alpha. \quad (4.1)$$

The theorems of this section are immediate corollaries of Theorems 2.2, 2.4, 2.7, 3.2, and 3.4.

THEOREM 4.1. Let matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition and let operator S_g satisfy the $\mathcal{M}$ condition. Let also $\text{vraisup}_{t \in [0, 1]} u(t) = \infty$, $(\mu_m)^{1/p} \leq q < 1$, and let the function ψ_1 be given by (2.14). Then if $C < 1 - q$, the Cauchy problem (4.1) has a unique solution in Δ_ψ^p with $\psi(t) = \min\{\psi_1(t), t^{u(t)}\}$ for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.

Let ω be a solution of the equation

$$m^\omega \left(\omega - \frac{1}{\ln m} \right) \leq C_1^p - q, \quad \gamma = \sup_{t \in [0, 1]} \{u(t), \omega\}, \quad (4.2)$$

where $0 \leq q \leq C_1^p < 1$, and u satisfies conditions of Lemma 3.1.

THEOREM 4.2. Let matrix $K(t, s)$ and operator S_g satisfy the $\mathcal{N}$ and $\mathcal{M}$ conditions, respectively. Let $\text{vraisup}_{t \in [0, 1]} u(t) < \infty$ and $(\mu_m)^{1/p} \leq q < 1$. Then if $q < C_1$, $(C_1 + C_2) < 1$, then the Cauchy problem (4.1) has a unique solution in Δ_ψ^p with $\psi(t) = \min\{\psi_1(t), t^\gamma\}$ for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.

THEOREM 4.3. Let matrix $K(t, s)$ satisfy the $\mathcal{N}$ condition, $B(t) \leq C_1/t^\alpha$, $g(t) = C_2 t^\beta$ ($\beta > 1$), and $\gamma > (\alpha p + \beta - 1)/p(\beta - 1)$. Let also $C < 1$ and $\psi(t) = \min\{\psi_1(t), t^\gamma\}$. Then the Cauchy problem (4.1) has a unique solution for f and α such that $(f - \alpha A) \in \Delta_\psi^p$.

Example 4.4. The Cauchy problem

$$\begin{aligned} \dot{x}(t) + p(t) \frac{x[h(t)]}{t^k} + q(t) \dot{x}(t^2) &= f(t), \quad t \in [0, 1], \\ x(\xi) &= 0, \quad \text{if } h(\xi) \leq 0, \end{aligned} \quad (4.3)$$

where $h(t) \leq t$, $k > 1$, and p and q are bounded functions, has a solution if $\int_0^t |f(s)| ds \leq M \exp(-t^{1-k})$. If $(t - h(t)) \geq \tau > 0$, then it has a solution if $\int_0^t |f(s)| ds \leq M t^\gamma$ for $\gamma > 1$.

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