LAPLACE TRANSFORM GENERATION THEOREMS AND LOCAL CAUCHY PROBLEMS

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We give new criterions to decide if some vector-valued function is a local Laplace transform and apply this to the theory of local Cauchy problems. This leads to an improvement of known results and new Hille-Yosida-type theorems for local convoluted semigroups.

1. Introduction

Let X be a Banach space, $x \in X$, $T > 0$, $h \in L_1([0, T], \mathbb{K})$, and $f(t) := \int_0^t h(s)ds$ the anti-derivative of h .

Assume $A : D(A) \rightarrow X$ is some closed (linear) operator in X .

Consider the abstract Cauchy problem

(ACP) $Aw(t) = w'(t)$ if $0 \leq t \leq T$, $w(0) = x$, and $w \in C^1([0, T], X)$

and the f -regularized problem

(ACP _{f}) $Au(t) + f(t)x = u'(t)$ if $0 \leq t \leq T$, $u(0) = 0$, and $u \in C^1([0, T], X)$.

If w solves (ACP), then the convolution $u := w * f$ solves (ACP _{f}), since

$$u' = w' * f + f(\cdot)x = w * h \quad \text{on } [0, T]. \quad (1.1)$$

Any function u that solves (ACP _{f}) is called h -regularized solution of (ACP).

Now assume that $0 \in \text{supp } h$, which means that h does not vanish on any interval $[0, \varepsilon)$, and which is equivalent to $0 \in \text{supp } f$.

By the theorem of Titchmarsh-Foiaş (see [3]) the convolution operator

$$\begin{aligned} S_f : C([0, T], X) &\longrightarrow C_0([0, T], X) = \{g \in C([0, T], X) \mid g(0) = 0\}, \\ S_f g &= g * f, \end{aligned} \quad (1.2)$$

can be extended to an isometric isomorphism

$$\tilde{S}_f : C^{[f]}([0, T], X) \longrightarrow C_0([0, T], X), \quad (1.3)$$

where the space $C^{[f]}([0, T], X)$ of generalized functions is the completion of $C([0, T], X)$ with norm $\|g\| := \|S_f g\|_\infty$. On this construction see also [9]. Thus, if w solves (ACP), then $S_f w$ solves (ACP_f).

This gives a reason to extend the notion of solutions of (ACP): a generalized function $w \in C^{[f]}([0, T], X)$ is called h -generalized solution of

(ACP_{*}) $Aw(t) = w'(t)$ if $0 \leq t \leq T$, $w(0) = x$

if $u = \tilde{S}_f w$ solves (ACP_f), that is, if $\tilde{S}_f w$ is an h -regularized solution of (ACP).

The notation of generalized solutions was introduced by Cioranescu and Lumer [5, 6].

If w is an h -generalized solution of (ACP_{*}), there is a sequence $(v_n)_n \subset C([0, T], \overline{D(A)})$ with $\lim_n v_n = u'$ in $C([0, T], X)$ and with $\lim_n v_n^{[1]} = u$ in $C([0, T], \overline{D(A)})$ (where $\overline{D(A)}$ denotes the Banach space $D(A)$ with the graph norm).

Thus, $\lim_n v_n = u'$, and $\lim_n (Av_n^{[1]} + f(\cdot)x - v_n) = 0$ in $C([0, T], X)$.

If $(v_n)_n \subset C([0, T], X)$ converges uniformly and satisfies $v_n^{[1]}(t) \in D(A)$ for all t and $\lim_n (Av_n^{[1]} + f(\cdot)x - v_n) = 0$ uniformly, then $(v_n)_n$ is called h -approximate solution of (ACP). Thus to every h -generalized solution there is an h -approximate solution.

On the other hand, if $(v_n)_n$ is an h -approximate solution of (ACP), then $w_n = \tilde{S}_h^{-1}(v_n - v_n(0))$ converges in $C^{[h]}([0, T], X)$ to some w , and $u = \tilde{S}_f w = \lim_n v_n^{[1]}$ solves (ACP_f). On approximate solutions see [1, 2]. Consequently, the notations of generalized solution of (ACP_{*}), of approximate solutions of (ACP), and (classical) solutions of (ACP_f) are equivalent. That is the reason why it is interesting to study problems of type (ACP_f).

We first clarify the notations.

If X is some Banach space, $g \in L_1^{\text{loc}}([0, \infty), X)$, and $\alpha, \tau > 0$, we let

$$g^{(-\alpha)}(t) := g^{[\alpha]}(t) := \frac{1}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} g(t-s) ds \quad (1.4)$$

be the α th integral of g . If there is some $h \in C([0, \tau], X)$ with $g = h^{[\alpha]}$, then let $g^{(\alpha)} = h$ be the α th derivative of g .

The finite Laplace transform of g on $[0, \tau]$ is given by

$$\hat{g}_\tau(\lambda) = g_\tau^\wedge(\lambda) := \int_0^\tau e^{-\lambda t} g(t) dt. \quad (1.5)$$

If $\alpha > 0$ and g is exponentially bounded for large arguments, then $g^{[\alpha]}$ is exponentially bounded for large arguments, and

$$\lambda^\alpha \int_0^\infty e^{-\lambda t} g^{[\alpha]}(t) dt = \int_0^\infty e^{-\lambda t} g(t) dt =: \hat{g}(\lambda) = g^\wedge(\lambda) \quad (1.6)$$

for all large $\text{Re } \lambda$, which is the Laplace transform of g .

In fact we have, if $\tau > 0$,

$$\lambda^\alpha \int_0^\tau e^{-\lambda t} g^{[\alpha]}(t) dt \sim_\tau \int_0^\tau e^{-\lambda t} g(t) dt, \quad (1.7)$$

where

$$\begin{aligned} f(\lambda) \sim_{\tau} h(\lambda) &\iff \limsup_{\lambda \rightarrow \infty} \frac{\ln |f(\lambda) - h(\lambda)|}{\lambda} \leq -\tau \\ &\iff \lim_{\lambda \rightarrow \infty} e^{-\delta\lambda} e^{\lambda\tau} |f(\lambda) - h(\lambda)| = 0 \quad \forall \delta > 0. \end{aligned} \quad (1.8)$$

This follows from

$$\begin{aligned} \lambda^{\alpha} \int_0^{\tau} e^{-\lambda t} g^{[\alpha]}(t) dt &= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} (p_{\alpha-1} * g)^{\wedge}_{\tau}(\lambda), \quad \text{where } p_{\beta}(s) = s^{\beta}, \\ &\sim_{\tau} \frac{\lambda^{\alpha}}{\Gamma(\alpha)} (p_{\alpha-1})^{\wedge}(\lambda) \cdot g_{\tau}^{\wedge}(\lambda) = g_{\tau}^{\wedge}(\lambda). \end{aligned} \quad (1.9)$$

In trying to get a nice theory how to solve abstract Cauchy problems

$$\begin{aligned} Au_x(t) + f(t)x &= u'_x(t) \quad \text{if } 0 \leq t \leq \tau, \\ u_x(0) &= 0, \\ u_x &\in C^1([0, \tau], X), \end{aligned} \quad (1.10)$$

with continuous inhomogeneity f , it seems to be natural first to consider inhomogeneities of type t^{β} , where $\beta \in \mathbb{N}$ or, more generally, $\beta \geq 0$. The reason for this is that if the problem is well posed on $[0, \tau]$ with this inhomogeneity, that is, A generates a β -integrated semigroup on $[0, \tau]$, then it is well posed with inhomogeneities $h^{[\beta+1]}$, where $h \in L_1([0, \tau], \mathbb{K})$, that is, A generates a local $h^{[\beta+1]}$ -convoluted semigroup on $[0, \tau]$. This is shown in [Section 2](#).

[Section 3](#) starts with a generalization of the complex representation theorem, see [\[4, 12\]](#). This will lead to an improvement of [\[7, Theorem II\]](#).

In [Section 4](#), we generalize a representation theorem of Prüss [\[12\]](#) and develop a new Hille-Yosida-type theorem for integrated semigroups.

2. Integrated semigroups

In this section, we show that it is worthwhile to study integrated semigroups, that is, abstract Cauchy problems (ACP_f) with inhomogeneities $f(t) = t^{\beta}$.

If $\tau > 0$, we say that a subspace $Y \subset L_1([0, \tau], \mathbb{K})$ satisfies property (A) if Y is closed and there is some strictly decreasing null sequence $(\varepsilon_n)_n$ with $\mathbf{1}_{[0, \varepsilon_n]} \in Y$ for all n .

The smallest possible Y with property (A) is

$$Y = \left\{ \sum_{n=1}^{\infty} \alpha_n \mathbf{1}_{(\varepsilon_{n+1}, \varepsilon_n)} \mid (\alpha_n)_n \in \ell_{\infty} \right\}, \quad (2.1)$$

where $\mathbf{1}_{(a,b)}$ is the characteristic function on (a, b) .

THEOREM 2.1. *Let X be a real or complex Banach space, $A : D(A) \rightarrow X$ a closed linear operator, $\tau > 0$, and $Y \subset L_1([0, \tau], \mathbb{K})$ some subspace with property (A).*

Let $\beta \geq 0$, and assume A generates a local $h^{[\beta+1]}$ -convoluted semigroup on $[0, \tau]$ for all $h \in Y$.

Then $A_{\overline{D(A)}}$ generates a β -times-integrated semigroup on $[0, \tau]$.

Proof. If $x \in X$ and $h \in Y$, let $u_{x,h}$ denote the solution of the abstract Cauchy problem

$$\begin{aligned} Au_{x,h}(t) + h^{[\beta+1]}(t)x &= u'_{x,h}(t) \quad \text{if } 0 \leq t \leq \tau, \\ u_{x,h}(0) &= 0, \\ u_{x,h} &\in C^1([0, \tau], X). \end{aligned} \quad (2.2)$$

Using the closedness of A and the uniqueness property of the abstract Cauchy problem, it is easy to see that the linear operator

$$\begin{aligned} R_h : X &\longrightarrow C([0, \tau], X), \\ R_h x &= u'_{x,h}, \end{aligned} \quad (2.3)$$

has closed graph for all $h \in Y$.

Moreover, for all $x \in X$, the linear operator

$$\begin{aligned} S_x : Y &\longrightarrow C([0, \tau], X), \\ S_x h &= u'_{x,h}, \end{aligned} \quad (2.4)$$

has closed graph.

From the uniform boundedness principle, it follows that

$$\sup_{\|x\| \leq 1} \|S_x\| < \infty; \quad (2.5)$$

consequently

$$\sup_{\|x\| \leq 1} \sup_{\|h\|_Y \leq 1} \|u'_{x,h}\|_\infty < \infty. \quad (2.6)$$

Now, let $h_n := (1/\varepsilon_n) \cdot \mathbf{1}_{[0, \varepsilon_n]}$.

Then

$$\Gamma(\beta+1) \cdot h_n^{[\beta+1]}(t) = \begin{cases} \frac{1}{\varepsilon_n} \frac{t^{\beta+1}}{\beta+1} & \text{if } 0 \leq t \leq \varepsilon_n, \\ \frac{1}{\beta+1} \frac{t^{\beta+1} - (t - \varepsilon_n)^{\beta+1}}{\varepsilon_n} & \text{if } \varepsilon_n \leq t \leq \tau, \end{cases} \quad (2.7)$$

thus $\lim_{n \rightarrow \infty} h_n^{[\beta+1]}(t) = t^\beta / \Gamma(\beta+1)$ if $0 < t \leq \tau$.

Moreover, $\sup_n \|h_n^{[\beta+1]}\|_\infty < \infty$. From the dominated convergence theorem, it follows that

$$\int_0^\tau \left| h_n^{[\beta+1]}(t) - \frac{t^\beta}{\Gamma(\beta+1)} \right| dt \xrightarrow{n} 0. \quad (2.8)$$

Next, we show that

- (a) $\lim_{\lambda, n \rightarrow \infty} e^{\lambda\tau} |(h_n^{[\beta+1]})^\wedge_\tau(\lambda)| = \infty$,
- (b) $\liminf_{\lambda \rightarrow \infty} (\ln |(t^\beta/\Gamma(\beta+1))^\wedge_\tau(\lambda)|/\lambda) \geq 0$.

We prove (a). We have

$$\sup_{\lambda, n} e^{\lambda\tau} \left| (h_n^{[\beta+1]})^\wedge(\lambda) - (h_n^{[\beta+1]})^\wedge_\tau(\lambda) \right| < \infty \quad (2.9)$$

since

$$\left| \int_\tau^\infty e^{-\lambda t} h_n^{[\beta+1]}(t) dt \right| \leq \sup_n \|h_n^{[\beta+1]}\|_\infty \cdot \int_\tau^\infty e^{-\lambda t} dt. \quad (2.10)$$

Moreover,

$$e^{\lambda\tau} \left| (h_n^{[\beta+1]})^\wedge_\tau(\lambda) \right| \geq e^{\lambda\tau} \left| (h_n^{[\beta+1]})^\wedge(\lambda) \right| - e^{\lambda\tau} \left| (h_n^{[\beta+1]})^\wedge(\lambda) - (h_n^{[\beta+1]})^\wedge_\tau(\lambda) \right|. \quad (2.11)$$

Now, (a) follows from

$$(h_n^{[\beta+1]})^\wedge(\lambda) = \frac{\widehat{h_n}(\lambda)}{\lambda^{\beta+1}} = \frac{1}{\lambda^{\beta+2}} \frac{1 - e^{-\lambda\varepsilon_n}}{\varepsilon_n} \geq \frac{e^{-\lambda\varepsilon_n}}{\lambda^{\beta+1}} \geq \frac{e^{-\lambda\tau/2}}{\lambda^{\beta+1}} \quad \text{for all large } \lambda, n. \quad (2.12)$$

We prove (b). This follows from [10, Lemma 3.3.5] or [8, Lemma 2.6].

Now we can apply [11, Corollary 13, Theorem 14], which yields the desired results. $\square$

COROLLARY 2.2. *Let X be a real or complex Banach space, $A : D(A) \rightarrow X$ a densely defined closed linear operator, $\tau > 0$, and $\beta \geq 0$.*

Then the following assertions are equivalent:

- (1) *A generates a local β -times-integrated semigroup on $[0, \tau]$;*
- (2) *A generates a local $h^{[\beta+1]}$ -convoluted semigroup on $[0, \tau]$ whenever $h \in L_1([0, \tau], \mathbb{K})$;*
- (3) *for one (for all) subspace(s) $Y \subset L_1([0, \tau], \mathbb{K})$ with property (A), the operator A generates a local $h^{[\beta+1]}$ -convoluted semigroup on $[0, \tau]$ whenever $h \in Y$.*

Proof. The proof of (2) follows from (1) by a simple convolution argument even if $D(A)$ is not dense. $\square$

3. The complex representation theorem and its application

In the following, we generalize the complex representation theorem to the local case. The global version can be found in [4] or [12, Proposition 0.2].

THEOREM 3.1 (complex representation theorem). *Let X be a complex Banach space, $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$.*

Furthermore, let $q : M \rightarrow X$ be a holomorphic function satisfying

$$\sup_{\lambda \in M} \|\lambda q(\lambda)\| < \infty. \quad (3.1)$$

Then for every $b > 0$ there is some function $g_b \in C([0, ab), X)$ such that $t \mapsto ((ab - t)/t^b)g_b(t)$ is bounded on $(0, ab)$ and with

$$q(\lambda) \sim_{\xi} \lambda^b \cdot (g_b)_{\xi}^{\wedge}(\lambda) \quad \forall \xi \in (0, ab). \quad (3.2)$$

Remark 3.2. If $0 < b' < b$, then $g_b = g_{b'}^{[b-b']}$ on $[0, ab')$.

Proof. Let $\omega > \omega_1$, $b > 0$, and $\Gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \omega, |\operatorname{Im} z| \leq F(\operatorname{Re} z - \omega)\}$. Then $\Gamma = \bar{\Gamma} \subset M^0 = M$.

First, we note that

$$C_1^2 := \sup_{r>0} \frac{1 + (F'(r))^2}{[(\omega + r)^2 + F(r)^2]^{b+1}} \cdot e^{2abr} < \infty. \quad (3.3)$$

Let $C_2 := \sup_{\lambda \in \Gamma} \|\lambda q(\lambda)\|$. We parameterize $\partial\Gamma$ by the following two functions:

$$\begin{aligned} \gamma^+ : [0, \infty) &\longrightarrow \mathbb{C}, & \gamma^+(t) &= \omega + t + iF(t), \\ \gamma^- : [0, \infty) &\longrightarrow \mathbb{C}, & \gamma^-(t) &= \omega + t - iF(t). \end{aligned} \quad (3.4)$$

Let $\gamma_n^+ := \gamma^+|_{[0, n]}$ and $\gamma_n^- := \gamma^-|_{[0, n]}$. Then the sequence

$$\begin{aligned} g_n^+ : [0, \infty) &\longrightarrow X, \\ g_n^+(t) &= \frac{1}{2\pi i} \int_{\gamma_n^+} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda, \end{aligned} \quad (3.5)$$

of continuous functions converges uniformly on $[0, q'b]$ whenever $q' \in (0, a)$, since if $t \in [0, q'b]$ and $n > m$, we obtain

$$\begin{aligned} 2\pi \|g_n^+(t) - g_m^+(t)\| &\leq \left\| \int_m^n e^{t\gamma^+(r)} \frac{q(\gamma^+(r))}{\gamma^+(r)^b} \cdot \frac{d}{dr} \gamma^+(r) dr \right\| \\ &\leq C_2 \int_m^n e^{t(\omega+r)} \frac{\sqrt{1 + F'(r)^2}}{[(\omega + r)^2 + F(r)^2]^{(b+1)/2}} dr \\ &\leq C_1 C_2 e^{t\omega} \int_m^n e^{tr} e^{-abr} dr \\ &\leq C_1 C_2 e^{t\omega} \int_m^n e^{rb(q'-a)} dr. \end{aligned} \quad (3.6)$$

Thus the function

$$\begin{aligned} g^+ : [0, ab) &\longrightarrow X, \\ g^+(t) &= \frac{1}{2\pi i} \int_{\gamma^+} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda, \end{aligned} \quad (3.7)$$

is continuous. In the same way we see that the function

$$g^- : [0, ab) \longrightarrow X, \\ g^-(t) = \frac{1}{2\pi i} \int_{-\gamma^-} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda = -\frac{1}{2\pi i} \int_{\gamma^-} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda, \quad (3.8)$$

is continuous.

Let $g(t) := g^+(t) + g^-(t)$ if $t \in [0, ab)$. We show that $t \mapsto ((ab - t)/t^b)g(t)$ is bounded on $(0, ab)$. To this end, let $R > 0$ and consider the three paths

$$\begin{aligned} \alpha : [R, \infty) &\longrightarrow \mathbb{C}, & \alpha(r) &= \omega + r - iF(r), \\ \beta : [R, \infty) &\longrightarrow \mathbb{C}, & \beta(r) &= \omega + r + iF(r), \\ \gamma : [-F(\omega + R), F(\omega + R)] &\longrightarrow \mathbb{C}, & \gamma(r) &= \omega + R + ir. \end{aligned} \quad (3.9)$$

Then, if $0 \leq t < ab$,

$$g(t) = \frac{1}{2\pi i} \int_{\beta + \gamma - \alpha} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda. \quad (3.10)$$

We have

$$\begin{aligned} \left\| \int_{\beta} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda \right\| &\leq C_1 C_2 e^{\omega t} \int_R^{\infty} e^{(t-ab)r} dr = C_1 C_2 e^{\omega t} \frac{e^{(t-ab)R}}{ab - t}, \\ \left\| \int_{-\alpha} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda \right\| &\leq C_1 C_2 e^{\omega t} \frac{e^{(t-ab)R}}{ab - t}. \end{aligned} \quad (3.11)$$

Finally,

$$\begin{aligned} \left\| \int_{\gamma} e^{\lambda t} \frac{q(\lambda)}{\lambda^b} d\lambda \right\| &\leq \int_{-\infty}^{\infty} e^{t(\omega+R)} C_2 \frac{dr}{[(\omega+R)^2 + r^2]^{(b+1)/2}} \\ &= \frac{C_2 e^{t(\omega+R)}}{(\omega+R)^b} \int_{-\infty}^{\infty} \frac{ds}{(1+s^2)^{(b+1)/2}}. \end{aligned} \quad (3.12)$$

If we let $R := 1/t$, we obtain the desired result.

Next, we show that, if $\lambda \in \Gamma$,

$$\frac{q(\lambda)}{\lambda^b} = \frac{1}{2\pi i} \lim_{n \rightarrow \infty} \int_{\gamma_n^+ - \gamma_n^-} \frac{(1/\mu^b)q(\mu)}{\lambda - \mu} d\mu. \quad (3.13)$$

To this end, consider a path β_R in $\mathbb{C}$ consisting of a part of a circle with center $\omega \in \mathbb{C}$ and radius R which connects a point on γ^+ with a point on γ^- . Its parameterization is given by

$$\beta_R : [-\psi, \psi] \longrightarrow \mathbb{C}, \quad \beta_R(\varphi) = \omega + R e^{i\varphi}, \quad (3.14)$$

with some $\psi \in (0, \pi/2)$ depending on R .

By Cauchy's formula we have to show that

$$\lim_{R \rightarrow \infty} \int_{\beta_R} \frac{(1/\mu^b)q(\mu)}{\mu - \lambda} d\mu = 0. \quad (3.15)$$

But, if R is large enough,

$$\begin{aligned} \left\| \int_{\beta_R} \frac{(1/\mu^b)q(\mu)}{\mu - \lambda} d\mu \right\| &\leq \int_{-\pi/2}^{\pi/2} \frac{\|q(\omega + \operatorname{Re}^{i\varphi})\|}{|\omega + \operatorname{Re}^{i\varphi}|^b \cdot |\lambda - (\omega + \operatorname{Re}^{i\varphi})|} R d\varphi \\ &\leq C_2 \int_{-\pi/2}^{\pi/2} \frac{R}{|\omega + \operatorname{Re}^{i\varphi}|^{b+1}} d\varphi \xrightarrow{R \rightarrow \infty} 0. \end{aligned} \quad (3.16)$$

Consequently, if $\xi \in (0, ab)$ and $\lambda \in \Gamma$,

$$\begin{aligned} \int_0^\xi e^{-\lambda t} g(t) dt &= \frac{1}{2\pi i} \int_0^\xi e^{-\lambda t} \lim_{n \rightarrow \infty} \int_{\gamma_n^+ - \gamma_n^-} e^{\mu t} \frac{q(\mu)}{\mu^b} d\mu dt \\ &= \frac{1}{2\pi i} \lim_n \int_{\gamma_n^+ - \gamma_n^-} \int_0^\xi e^{(\mu - \lambda)t} dt \frac{q(\mu)}{\mu^b} d\mu \\ &= \frac{1}{2\pi i} \lim_n \int_{\gamma_n^+ - \gamma_n^-} \frac{e^{(\mu - \lambda)\xi}}{\mu - \lambda} \frac{q(\mu)}{\mu^b} d\mu \\ &\quad + \frac{1}{2\pi i} \lim_n \int_{\gamma_n^+ - \gamma_n^-} \frac{(1/\mu^b)q(\mu)}{\lambda - \mu} d\mu. \end{aligned} \quad (3.17)$$

It remains to show that, for all $\sigma > 0$,

$$e^{-\lambda\sigma} e^{\lambda\xi} \lim_n \left\| \int_{\gamma_n^+ - \gamma_n^-} \frac{e^{(\mu - \lambda)\xi}}{\mu - \lambda} \frac{q(\mu)}{\mu^b} d\mu \right\| \xrightarrow{\lambda \rightarrow \infty} 0. \quad (3.18)$$

But, if $\lambda > \omega$,

$$\begin{aligned} e^{-\lambda\sigma} \left\| \int_{\gamma_n^+} \frac{e^{\mu\xi}}{\mu - \lambda} \frac{q(\mu)}{\mu^b} d\mu \right\| &\leq e^{-\lambda\sigma} \int_0^n \frac{e^{\xi \operatorname{Re} \gamma^+(t)}}{|\gamma^+(r) - \lambda|} \cdot \frac{\|q(\gamma^+(r))\|}{|\gamma^+(r)|^b} \cdot \left| \frac{d}{dr} \gamma^+(r) \right| dr \\ &\leq C_1 C_2 e^{-\lambda\sigma} e^{\xi\omega} \int_0^n \frac{e^{\xi r}}{|\gamma^+(r) - \lambda|} e^{-abr} dr \\ &\leq C_1 C_2 \left(\sup_{r \geq 0} \frac{1}{|\gamma^+(r) - \lambda|} \right) e^{\xi\omega} e^{-\lambda\sigma} \int_0^\infty e^{(\xi - ab)r} dr \xrightarrow{\lambda \rightarrow \infty} 0, \end{aligned} \quad (3.19)$$

where if λ is large enough,

$$\sup_{r \geq 0} \frac{1}{|\gamma^+(r) - \lambda|} \leq 1. \quad (3.20)$$

In the same way we obtain

$$e^{-\lambda\sigma} \left\| \int_{\gamma_n} \frac{e^{\mu\xi}}{\mu - \lambda} \frac{q(\mu)}{\mu^b} d\mu \right\| \xrightarrow{\lambda \rightarrow \infty} 0. \quad (3.21)$$

This shows

$$q(\lambda) \sim_{\xi} \lambda^b \hat{g}_{\xi}(\lambda) \quad \forall \xi \in (0, ab). \quad (3.22)$$

Finally, if $0 < b' < b$, then $g_b = g_{b'}^{[b-b']}$ on $[0, ab')$.

We have

$$\begin{aligned} q(\lambda) &\sim_{\xi} \lambda^{b'} \cdot (g_{b'})_{\xi}^{\wedge}(\lambda) \quad \forall \xi \in (0, ab'), \\ q(\lambda) &\sim_{\xi} \lambda^b \cdot (g_b)_{\xi}^{\wedge}(\lambda) \quad \forall \xi \in (0, ab), \end{aligned} \quad (3.23)$$

thus if $\xi \in (0, ab')$,

$$(g_b)_{\xi}^{\wedge}(\lambda) \sim_{\xi} \frac{1}{\lambda^{b-b'}} (g_{b'})_{\xi}^{\wedge}(\lambda) \sim_{\xi} (g_{b'}^{[b-b']})_{\xi}^{\wedge}(\lambda). \quad (3.24)$$

The Phragmén Doetsch inversion formula (see, e.g., [4]) shows that $g_b = g_{b'}^{[b-b']}$ on $[0, ab')$. $\square$

THEOREM 3.3. *Let $\alpha \in \mathbb{R}$, $\omega, c, a > 0$, $F(t) = c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.*

Suppose $p : [0, \infty) \rightarrow \mathbb{C}$ is locally integrable with $|p(t)| \leq \text{const} \cdot e^{\omega t}$ for all large t and with

$$\liminf_{\lambda \rightarrow \infty} \frac{\ln |\hat{p}(\lambda)|}{\lambda} \geq 0, \quad (3.25)$$

$$\sup_{\lambda \in M} |\lambda|^{\alpha-1} \cdot |\hat{p}(\lambda)| < \infty. \quad (3.26)$$

Furthermore, let X be a complex Banach space, let $A : D(A) \rightarrow X$ be linear with $M \subset \rho(A)$, and let

$$\sup_{\lambda \in M} |\lambda|^{\alpha} \cdot \|\hat{p}(\lambda)R(\lambda, A)\| < \infty. \quad (3.27)$$

Then, for all $m \in (-\infty, \alpha - 1)$, the operator A generates a local $p^{(m-1)}$ -convoluted semi-group on $[0, a(\alpha - m - 1))$.

Remark 3.4. (i) Condition (3.25) can be omitted if one then assumes in addition that

$$\limsup_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda, A)\|}{\lambda} \leq 0. \quad (3.28)$$

(ii) From condition (3.25), it follows that

$$\limsup_{\lambda \rightarrow \infty} \frac{\ln |\hat{p}(\lambda)|}{\lambda} = 0, \quad (3.29)$$

which is equivalent to $0 \in \text{supp } p$, that is, p does not vanish on any interval $[0, \varepsilon)$, $\varepsilon > 0$.

This can be shown by using the Phragmén Doetsch inversion formula.

Proof. We first show that p is $C_0^{(m-1)}$ on $[0, a(\alpha - m - 1))$ if $m \geq 1$, that is, there is some $g \in C([0, a(\alpha - m - 1)), \mathbb{C})$ with $g(0) = 0$ and $p = g^{[m-1]}$.

To this end, define $\tilde{q} : M \rightarrow \mathbb{C}$ by $\tilde{q}(\lambda) = \lambda^{\alpha-2} \hat{p}(\lambda)$.

Then we have $\sup_{\lambda \in M} |\lambda \tilde{q}(\lambda)| < \infty$. Letting $b := \alpha - m - 1$, the complex representation theorem shows that there is some $g \in C([0, a(\alpha - m - 1)), \mathbb{C})$ with $g(0) = 0$ such that

$$\tilde{q}(\lambda) \sim_{\xi} \lambda^{\alpha-m-1} \hat{g}_{\xi}(\lambda) \quad \forall \xi \in (0, a(\alpha - m - 1)), \quad (3.30)$$

thus

$$\hat{p}(\lambda) \sim_{\xi} \frac{1}{\lambda^{m-1}} \hat{g}_{\xi}(\lambda) \sim_{\xi} \left(g^{[m-1]} \right)_{\xi}^{\wedge}(\lambda). \quad (3.31)$$

The Phragmén Doetsch inversion formula shows that $p^{[1]} = g^{[m]}$ on $(0, a(\alpha - m - 1))$.

Next, we define the holomorphic function

$$q : M \longrightarrow \mathcal{L}(X, \overline{D}(A)), \quad q(\lambda) = \lambda^{\alpha-2} \hat{p}(\lambda) R(\lambda, A), \quad (3.32)$$

where $\overline{D}(A)$ is the Banach space $D(A)$ with the norm $\|x\|_{\overline{D}(A)} = \|x\| + \|Ax\|$.

If $\lambda \in M$, we obtain

$$\begin{aligned} \|\lambda q(\lambda)\| &\leq |\lambda|^{\alpha-1} |\hat{p}(\lambda)| (\|R(\lambda, A)\| + \|\lambda R(\lambda, A) - \text{id}\|) \\ &\leq |\lambda|^{\alpha-1} |\hat{p}(\lambda)| (\|R(\lambda, A)\| + |\lambda| \cdot \|R(\lambda, A)\| + 1) \\ &\leq |\lambda|^{\alpha-1} |\hat{p}(\lambda)| + |\lambda|^{\alpha-1} |\hat{p}(\lambda)| \cdot (1 + |\lambda|) \cdot \|R(\lambda, A)\|. \end{aligned} \quad (3.33)$$

From conditions (3.26) and (3.27), it follows that

$$\sup_{\lambda \in M} \|\lambda q(\lambda)\| < \infty. \quad (3.34)$$

Letting $b := \alpha - m - 1$, the complex representation theorem yields some continuous function $H : [0, a(\alpha - m - 1)) \rightarrow \mathcal{L}(X, \overline{D}(A))$ with $H(0) = 0$ and with

$$q(\lambda) \sim_{\xi} \lambda^{\alpha-m-1} \hat{H}_{\xi}(\lambda) \quad \forall \xi \in (0, a(\alpha - m - 1)). \quad (3.35)$$

Consequently, if $\xi \in (0, a(\alpha - m - 1))$ and $x \in X$,

$$\lambda \left(p^{(m-1)} \right)_{\xi}^{\wedge}(\lambda) R(\lambda, A) x \sim_{\xi} \lambda^m \hat{p}_{\xi}(\lambda) R(\lambda, A) x \quad (3.36)$$

since $\|R(\lambda, A)\| \sim_0 0$ by conditions (3.25) and (3.27), and since $(p^{(m-1)})_{\xi}^{\wedge}(\lambda) \sim_{\xi} \lambda^{m-1} \hat{p}_{\xi}(\lambda)$.

Moreover,

$$\lambda^{\alpha-2} \hat{p}_\xi(\lambda) R(\lambda, A)x \sim_\xi q(\lambda)x \sim_\xi \lambda^{\alpha-m-1} (H(\cdot)x)_\xi^\wedge(\lambda). \quad (3.37)$$

Thus,

$$\begin{aligned} \lambda \left(p^{(m-1)} \right)_\xi^\wedge(\lambda) R(\lambda, A)x &\sim_\xi \lambda^{m-\alpha+2} \lambda^{\alpha-2} \hat{p}_\xi(\lambda) R(\lambda, A)x \\ &\sim_\xi \lambda (H(\cdot)x)_\xi^\wedge(\lambda) \\ &\sim_\xi \int_0^\tau e^{-\lambda t} d_t(H(t)x). \end{aligned} \quad (3.38)$$

Therefore, by the theory of local convoluted semigroups, the unique solution of the abstract Cauchy problem

$$\begin{aligned} Au_x(t) + p^{(m-1)}(t)x &= u'_x(t) \quad \forall t \in [0, a(\alpha - m - 1)), \\ u_x(0) &= 0, \\ u_x &\in C^1([0, a(\alpha - m - 1)), X), \end{aligned} \quad (3.39)$$

is given by

$$u_x(t) = H(t)x. \quad (3.40)$$

□

We now state and prove a lemma which we will need frequently in the sequel.

LEMMA 3.5. *Let $\tau > 0$, $\beta > 0$, let $h : [0, \infty) \rightarrow [0, \infty)$ be some function, and $\Gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0, |\operatorname{Im} z| \leq h(\operatorname{Re} z)\}$.*

Then

$$\lim_{\substack{\operatorname{Re} \lambda \rightarrow \infty \\ \lambda \in \Gamma}} \lambda^\beta e^{-\lambda \tau} = 0 \iff \lim_{t \rightarrow \infty} e^{-(\tau/\beta)t} h(t) = 0. \quad (3.41)$$

In one of these cases, even if $\beta \geq 0$,

$$\lim_{\substack{\operatorname{Re} \lambda \rightarrow \infty \\ \lambda \in \Gamma}} \lambda^{\beta+1} \int_\tau^\infty e^{-\lambda t} \left(d \cdot t^\beta + g^{[\beta+1]}(t) \right) dt = 0 \quad (3.42)$$

whenever $d \in \mathbb{C}$ and $g \in L_1^{\text{loc}}([0, \infty), \mathbb{C})$ is exponentially bounded for large arguments.

Moreover, if $\beta > 0$,

$$\limsup_{\substack{\operatorname{Re} \lambda \rightarrow \infty \\ \lambda \in \Gamma}} |\lambda^\beta e^{-\lambda \tau}| < \infty \iff \limsup_{t \rightarrow \infty} e^{-(\tau/\beta)t} h(t) < \infty. \quad (3.43)$$

Proof. We prove (3.41).

“If” part. If $\lambda \in \Gamma$, we obtain

$$\begin{aligned} |\lambda^\beta e^{-\lambda\tau}|^{2/\beta} &= (|\operatorname{Im}\lambda|^2 + |\operatorname{Re}\lambda|^2) e^{-(2\tau/\beta)\operatorname{Re}\lambda} \\ &\leq (h(\operatorname{Re}\lambda) e^{-(\tau/\beta)\operatorname{Re}\lambda})^2 + |\operatorname{Re}\lambda|^2 e^{-(2\tau/\beta)\operatorname{Re}\lambda}. \end{aligned} \quad (3.44)$$

“Only if” part. Suppose there are some $\varepsilon > 0$ and numbers $x_n \in \mathbb{R}$ with $x_n \rightarrow \infty$ and with $|e^{-(\tau/\beta)x_n} h(x_n)| \geq \varepsilon$ for all n . Then $z_n := x_n + i\varepsilon e^{(\tau/\beta)x_n}$ is in Γ , and

$$0 = \lim_n \left| z_n^\beta e^{-z_n\tau} \right|^{2/\beta} = \lim_n (x_n^2 + \varepsilon^2 e^{(2\tau/\beta)x_n}) e^{-(2\tau/\beta)x_n} = \varepsilon^2, \quad (3.45)$$

which is a contradiction. This shows (3.41).

Now let $f(t) := dt^\beta + g^{[\beta+1]}(t)$. Then $f'(t) = d\beta t^{\beta-1} + g^{[\beta]}(t)$ is L_1^{loc} and exponentially bounded for large arguments.

Partial integration shows that

$$\lambda^{\beta+1} \int_\tau^\infty e^{-\lambda t} f(t) dt = \lambda^\beta e^{-\lambda\tau} f(\tau) + \lambda^\beta e^{-\lambda\tau} \int_0^\infty e^{-\lambda w} f'(w + \tau) dw. \quad (3.46)$$

□

COROLLARY 3.6. Let $\omega, a, c, \nu > 0$, $\mu \geq 0$, $\beta \in (\mu/a, \mu/a + \nu]$, $F(t) = c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.

Furthermore, let X be a complex Banach space, let $A : D(A) \rightarrow X$ be linear with $M \subset \rho(A)$, with

$$\limsup_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda, A)\|}{\lambda} \leq 0, \quad (3.47)$$

and with

$$\sup_{\lambda \in M} \left\| \lambda^{\beta-\nu+1} e^{-\lambda\mu} R(\lambda, A) \right\| < \infty. \quad (3.48)$$

Then A generates a ν -times-integrated semigroup on $[0, a\beta - \mu)$.

Proof. Apply Theorem 3.3 to $m := 0$, $\alpha := \beta + 1$, and $p(t) := \mathbf{1}_{(\mu, \infty)}(t) \cdot (t - \mu)^{\nu-1}$, where $\mathbf{1}_{(\mu, \infty)}$ is the characteristic function on (μ, ∞) . Then we have $\hat{p}(\lambda) = e^{-\lambda\mu}(\Gamma(\nu)/\lambda^\nu)$ if $\operatorname{Re}\lambda > 0$.

Condition (3.25) is omitted, see Remark 3.4.

Condition (3.26) is equivalent to $\sup_{\lambda \in M} |\lambda^{\beta-\nu} e^{-\lambda\mu}| < \infty$. This follows from Lemma 3.5. Here we need that $\beta \leq \mu/a + \nu$.

Condition (3.27) is equivalent to $\sup_{\lambda \in M} \|\lambda^{\beta+1-\nu} e^{-\lambda\mu} R(\lambda, A)\| < \infty$.

By Theorem 3.3 the operator A generates a local $p^{[1]}$ -convoluted semigroup on $[0, a\beta)$.

If $x \in X$, let u_x denote the (unique) solution of the abstract Cauchy problem

$$\begin{aligned} Au_x(t) + p^{[1]}(t)x &= u'_x(t) \quad \text{if } 0 \leq t < a\beta, \\ u_x(0) &= 0, \\ u_x &\in C^1([0, a\beta), X). \end{aligned} \quad (3.49)$$

The uniqueness property follows from the Ljubic uniqueness theorem.

Then $u_x(t) = 0$ if $0 \leq t \leq \mu$, since $p^{[1]}(t) = 0$ if $0 \leq t \leq \mu$. Thus if $x \in X$, then

$$\begin{aligned} v_x : [0, a\beta - \mu] &\longrightarrow X, \\ v_x(t) &= u_x(t + \mu), \end{aligned} \quad (3.50)$$

is the (unique) solution of the abstract Cauchy problem

$$\begin{aligned} Av_x(t) + \frac{t^\nu}{\nu \cdot x} &= v'_x(t) \quad \text{if } 0 \leq t < a\beta - \mu, \\ v_x(0) &= 0, \\ v_x &\in C^1([0, a\beta - \mu), X). \end{aligned} \quad (3.51)$$

□

COROLLARY 3.7. *Let X be a complex Banach space, let $A : D(A) \rightarrow X$ be linear, $\omega, c, a > 0$, $F(t) = c(e^{at} - 1)$, and $M = \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.*

Assume that $M \subset \rho(A)$ and

$$\sup_{\lambda \in M} \frac{\|R(\lambda, A)\|}{|\lambda|^\gamma} < \infty \quad \text{for some } \gamma \in [-1, \infty). \quad (3.52)$$

Then, for all $\varepsilon > 0$, the operator A generates a $(\gamma + 1 + \varepsilon)$ -integrated semigroup on $[0, a\varepsilon)$.

Proof. Apply [Corollary 3.6](#) to $\mu := 0$, $\nu := \gamma + \varepsilon + 1$, and $\beta := \varepsilon$. □

Remark 3.8. [Corollary 3.7](#) improves [\[7, Theorem II\]](#) if $D(A)$ is dense.

To see this we abbreviate “Cioranescu/Lumer” as “(CL)” and refer to the notation in [\[7\]](#).

In that theorem, put $\Phi^{(\text{CL})}(r) := \ln r$, $\alpha^{(\text{CL})} := 1/a$, $\gamma^{(\text{CL})} := \gamma$, $K^{(\text{CL})} := h^{[\gamma + \varepsilon + 1]}$, and $l^{(\text{CL})} := \gamma + \varepsilon + 1$. Then $\sigma^{(\text{CL})} = 1$ and $\chi^{(\text{CL})} = 0$.

If ω is large enough, then $M \subset \Gamma_{\alpha^{(\text{CL})}, \beta^{(\text{CL})}}^{(\text{CL})}$.

To apply [\[7, Theorem II\]](#) we have to make the assumptions that $\gamma > -1$, that $h^{[\gamma + \varepsilon + 1]}$ is exponentially bounded, and that $\widehat{K^{(\text{CL})}} \neq 0$ for all λ with large real part. Then, by [\[7, Theorem II\]](#), A generates a local $h^{[\gamma + \varepsilon + 2]}$ -convoluted semigroup on $[0, \tau^{(\text{CL})}) = [0, (l^{(\text{CL})} - \gamma^{(\text{CL})} - \sigma^{(\text{CL})})/\alpha^{(\text{CL})}) = [0, a\varepsilon)$.

Example 3.9. Let $X = C([0, 1], \mathbb{C})$, let $g \in X$ with $g(t) > 0$ for all $t \in [0, 1]$, and let $D(A) = \{f \in C^1([0, 1], \mathbb{C}) \mid f(0) = 0\}$.

Then the operator $Ah = -gh'$ generates a $(1 + \varepsilon)$ -integrated semigroup on $[0, \infty)$ for all $\varepsilon > 0$.

If $\lambda \in \mathbb{C}$, we have

$$R(\lambda, A)f(t) = e^{-\lambda G(t)} \int_0^t \frac{e^{\lambda G(s)} f(s)}{g(s)} ds, \quad (3.53)$$

where $G(t) = \int_0^t (ds/g(s))ds$.

The assertion follows from [Corollary 3.7](#) since $\sup_{\operatorname{Re} \lambda > 0} \|R(\lambda, A)\| < \infty$.

COROLLARY 3.10. *Let X be a complex Banach space, let $A : D(A) \rightarrow X$ be linear, $\omega, c, a > 0$, $F(t) = c(e^{at} - 1)$, and $M = \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega)\}$.*

Assume that $M \subset \rho(A)$, that

$$\limsup_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda, A)\|}{\lambda} \leq 0, \quad (3.54)$$

and that

$$\sup_{\lambda \in M} \|e^{-\lambda \mu} R(\lambda, A)\| < \infty \quad \text{for some } \mu \geq 0. \quad (3.55)$$

Then for all $\varepsilon > \mu/a$, the operator A generates a $(\varepsilon + 1)$ -integrated semigroup on $[0, a\varepsilon - \mu]$.

Proof. Apply [Corollary 3.6](#) to $\nu := 1 + \varepsilon$ and $\beta := \varepsilon$. □

THEOREM 3.11. *Let $\beta \geq 0$, $\tau > 0$, $g \in L_1([0, \tau], \mathbb{K})$, $c \in \mathbb{K} \setminus \{0\}$, and $f(t) := ct^\beta + g^{[\beta+1]}(t)$.*

Furthermore, let $h : [0, \infty) \rightarrow [0, \infty)$ be some function with $h(t) = o(e^{(\tau/\beta)t})$ as $t \rightarrow \infty$ (this condition is always fulfilled if $\beta = 0$). In this case Γ is supposed to be some right half-plane in $\mathbb{C}$, and let

$$\Gamma := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \leq h(\operatorname{Re} z)\}. \quad (3.56)$$

Finally, let X be some real or complex Banach space and let $A : D(A) \rightarrow X$ be some linear and closed operator which generates a local f -convoluted semigroup on $[0, \tau]$.

Then there is some $\omega > 0$ such that $R(\lambda, A)$ exists if $\lambda \in \Gamma$ with $\operatorname{Re} \lambda > \omega$, and

$$\sup_{\substack{\operatorname{Re} \lambda > \omega \\ \lambda \in \Gamma}} \operatorname{Re} \lambda \cdot \frac{\|R(\lambda, A)\|}{|\lambda|^\beta} < \infty. \quad (3.57)$$

Proof. From [Lemma 3.5](#), it follows that

$$\lim_{\substack{\operatorname{Re} \lambda \rightarrow \infty \\ \lambda \in \Gamma}} \lambda^{\beta+1} \hat{f}_\tau(\lambda) = c \cdot \Gamma(\beta + 1). \quad (3.58)$$

Thus, again by [Lemma 3.5](#),

$$\left| \lambda e^{\lambda \tau} \hat{f}_\tau(\lambda) \right| = \left| \lambda^{-\beta} e^{\lambda \tau} \right| \cdot \left| \lambda^{\beta+1} \hat{f}_\tau(\lambda) \right| \longrightarrow \infty \quad \text{if } \operatorname{Re} \lambda \longrightarrow \infty, \lambda \in \Gamma, \quad (3.59)$$

which also holds if $\beta = 0$.

From the theory of local convoluted semigroups (see, e.g., [8, 11]), it follows that there is some $\omega > 0$ such that $R(\lambda, A)$ exists if $\operatorname{Re} \lambda > \omega$, $\lambda \in \Gamma$, and that

$$\|R(\lambda, A)\| \leq \frac{\text{const}}{\operatorname{Re} \lambda \cdot |\lambda \hat{f}_\tau(\lambda)|}. \quad (3.60)$$

To be more precise, let u_x denote the solution of the abstract Cauchy problem

$$\begin{aligned} Au_x(t) + f(t)x &= u'_x(t) \quad \text{if } 0 \leq t \leq \tau, \\ u_x(0) &= 0, \\ u_x &\in C^1([0, \tau], X). \end{aligned} \quad (3.61)$$

Let $Hx := Au_x(\tau)$ and $J_\lambda := \lambda \hat{f}_\tau(\lambda) \cdot \operatorname{id} - e^{-\lambda\tau}H$. From

$$\frac{J_\lambda}{\lambda \hat{f}_\tau(\lambda)} = \operatorname{id} - \frac{H}{\lambda e^{\lambda\tau} \hat{f}_\tau(\lambda)}, \quad (3.62)$$

it follows that J_λ is invertible if $\lambda \in \Gamma$ and $\operatorname{Re} \lambda$ is large, and

$$\left(\frac{J_\lambda}{\lambda \hat{f}_\tau(\lambda)} \right)^{-1} = \sum_{k=0}^{\infty} \left(\frac{H}{\lambda e^{\lambda\tau} \hat{f}_\tau(\lambda)} \right)^k. \quad (3.63)$$

Thus, $\|J_\lambda^{-1}\| \leq 2/|\lambda \hat{f}_\tau(\lambda)|$ if $\lambda \in \Gamma$ and $\operatorname{Re} \lambda$ is large.

From

$$(\lambda - A) \int_0^\tau e^{-\lambda t} du_x(t) = J_\lambda x \quad \forall x \in X, \lambda \in \mathbb{K}, \quad (3.64)$$

it follows that $R(\lambda, A)$ exists, and

$$\|R(\lambda, A)\| \leq \frac{\text{const}}{\operatorname{Re} \lambda \cdot |\lambda \hat{f}_\tau(\lambda)|}. \quad (3.65)$$

□

Example 3.12. Let $X = C([0, 1], \mathbb{C})$ and $D(A) = \{f \in C^2([0, 1], \mathbb{C}) \mid f(0) = f'(0) = 0\}$. Then $Ah = -h''$ does not generate a local β -integrated semigroup on $[0, \tau]$, independent of $\beta \geq 0$ and $\tau > 0$.

We have, if $\lambda \in \mathbb{C}$,

$$R(-\lambda^2, A)f(t) = \begin{cases} \frac{1}{\lambda} \int_0^t \sinh(\lambda s) f(t-s) ds & \text{if } \lambda \neq 0, \\ \int_0^t s f(t-s) ds & \text{if } \lambda = 0. \end{cases} \quad (3.66)$$

Let $\Gamma := \{z \in \mathbb{C} \mid |\operatorname{Im} z| \leq 4/3 \cdot |\operatorname{Re} z|\}$ and consider $\mu(\alpha) = -\alpha + 2i\alpha$ if $\alpha > 0$.

Then $-\mu^2 = 3\alpha^2 + 4i\alpha^2 \in \Gamma$, and

$$\frac{\|R(-\mu^2, A)\|}{|\mu|^{2\beta}} \geq \frac{\|R(-\mu^2, A)(\mathbf{1})\|}{|\mu|^{2\beta}} \xrightarrow{\alpha \rightarrow \infty} \infty, \quad (3.67)$$

where $\mathbf{1}(t) = 1$.

THEOREM 3.13. *Let X be a complex Banach space and $A : D(A) \rightarrow X$ a linear operator.*

Then the following assertions are equivalent.

(1) *There are $\beta \geq 0$ and $\xi > 0$ such that A generates a β -times integrated semigroup on $[0, \xi)$.*

(2) *There are $c, \omega, a > 0$ such that if $F(t) := c(e^{at} - 1)$ and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| \leq F(\operatorname{Re} z - \omega)\}$, the following is valid:*

$$\begin{aligned} M &\subset \rho(A), \\ \sup_{\lambda \in M} \left\| \frac{e^{-\lambda\mu}}{\lambda^{\tilde{\beta}}} R(\lambda, A) \right\| &< \infty \quad \text{for some } \mu, \tilde{\beta} \in \mathbb{R}, \\ \limsup_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda, A)\|}{\lambda} &\leq 0. \end{aligned} \quad (3.68)$$

(3) *There are $\omega, \delta > 0$ such that if $h(t) := e^{\delta t}$ and $\Gamma := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega, |\operatorname{Im} z| \leq h(\operatorname{Re} z)\}$, the following is valid:*

$$\begin{aligned} \Gamma &\subset \rho(A), \\ \sup_{\lambda \in \Gamma} \left\| \frac{R(\lambda, A)}{\lambda^{\bar{\beta}}} \right\| &< \infty \quad \text{for some } \bar{\beta} \in \mathbb{R}. \end{aligned} \quad (3.69)$$

Proof. (1) $\Rightarrow$ (3). Choose $\delta \in (0, \xi/\beta)$, where $\xi/\beta := \infty$ if $\beta = 0$. Apply [Theorem 3.11](#) to $g := 0$, $c := 1$, $\bar{\beta} := \beta$ and choose $\tau \in (\beta\delta, \xi)$. Then $h(t) = e^{\delta t}$ fulfills $h(t) = o(e^{(\tau/\beta)t})$, thus, by [Theorem 3.11](#), there is some $\omega > 0$ such that $R(\lambda, A)$ exists if $\operatorname{Re} \lambda > \omega$, $\lambda \in \Gamma$, and

$$\sup_{\substack{\operatorname{Re} \lambda > \omega \\ \lambda \in \Gamma}} \frac{\|R(\lambda, A)\|}{|\lambda|^{\bar{\beta}}} < \infty. \quad (3.70)$$

Here we needed the uniqueness property on $[0, \tau]$, see [\[10, Theorem 3.5.1\]](#).

(3) $\Rightarrow$ (2). Choose $a := \delta$, $c := 1$, and numbers $\mu \geq 0$, $\tilde{\beta} \in \mathbb{R}$ such that $\delta \cdot (\bar{\beta} - \tilde{\beta}) \leq \mu$. For example, take $\mu := 0$ and $\tilde{\beta} := \bar{\beta}$.

Then $M \subset \Gamma$, and if $\lambda \in M$,

$$\left\| \frac{e^{-\lambda\mu}}{\lambda^{\tilde{\beta}}} R(\lambda, A) \right\| = \left| \lambda^{\bar{\beta} - \tilde{\beta}} e^{-\lambda\mu} \right| \cdot \left\| \frac{R(\lambda, A)}{\lambda^{\bar{\beta}}} \right\|. \quad (3.71)$$

If $\bar{\beta} - \tilde{\beta} > 0$, then

$$\limsup_{\substack{\operatorname{Re} \lambda \rightarrow \infty \\ \lambda \in M}} \left| \lambda^{\bar{\beta} - \tilde{\beta}} e^{-\lambda \mu} \right| < \infty, \quad (3.72)$$

by Lemma 3.5, since

$$\limsup_{t \rightarrow \infty} e^{-(\mu/(\bar{\beta} - \tilde{\beta}))t} h(t) < \infty. \quad (3.73)$$

(2) $\Rightarrow$ (1). First, let $\tilde{\beta} \geq -\mu/a - 1$. Choose $\nu > \mu/a + 1 + \tilde{\beta}$ and $\hat{\beta} := \nu - \tilde{\beta} - 1$. Then $\hat{\beta} \in (\mu/a, \mu/a + \nu]$ and

$$\left\| \lambda^{\hat{\beta} - \nu + 1} e^{-\lambda \mu} R(\lambda, A) \right\| \leq \left\| \frac{e^{-\lambda \mu}}{\lambda^{\tilde{\beta}}} R(\lambda, A) \right\| \quad \forall \lambda \in M, \operatorname{Re} \lambda > \omega. \quad (3.74)$$

Corollary 3.6 shows that A generates a local ν -times integrated semigroup on $[0, a(\nu - 1 - \tilde{\beta}) - \mu)$.

Second, assume $\tilde{\beta} \leq -\mu/a - 1$. Let $\nu > 0$ and $\hat{\beta} = \mu/a + \nu$. Then $\hat{\beta} \in (\mu/a, \mu/a + \nu]$ and, if $\omega > 1$,

$$\left\| \lambda^{\hat{\beta} - \nu + 1} e^{-\lambda \mu} R(\lambda, A) \right\| \leq \left\| \frac{e^{-\lambda \mu}}{\lambda^{\tilde{\beta}}} R(\lambda, A) \right\| \quad \forall \lambda \in M, \operatorname{Re} \lambda > \omega, \quad (3.75)$$

since $\tilde{\beta} + \hat{\beta} - \nu + 1 \leq 0$.

Corollary 3.6 shows that A generates a local ν -times integrated semigroup on $[0, a\nu)$. $\square$

4. The theorem of Prüss and its application

Next we generalize a result from Prüss [12, Theorem 0.4] to the local case. In its original form, the function q has to be defined on some right half-plane and then is the Laplace transform of some continuous and exponentially bounded function.

This will lead to a new Hille-Yosida-type theorem for integrated semigroups.

THEOREM 4.1. *Let X be a complex Banach space, $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$.*

Furthermore, let $q : M \rightarrow X$ be a holomorphic function satisfying

$$\sup_{\lambda \in M} \|\lambda q(\lambda)\| < \infty, \quad \sup_{\lambda \in M} \|\lambda^2 q'(\lambda)\| < \infty. \quad (4.1)$$

Then there is some $H \in C((0, a), X)$ such that $t \mapsto (t - a)H(t)$ is bounded on $(0, a)$ and with

$$q(\lambda) \sim_{\xi} \hat{H}_{\xi}(\lambda) \quad \forall \xi \in (0, a). \quad (4.2)$$

Proof. From the complex representation theorem it follows that there are functions $g, h \in C([0, a), X)$ such that $g(t)/t$ and $h(t)/t$ are in $L^\infty_{\text{loc}}([0, a), X)$ and with

$$\begin{aligned} q(\lambda) &= \lambda \int_0^\xi e^{-\lambda t} g(t) dt + \varepsilon_\xi(\lambda), \\ \lambda q'(\lambda) &= \lambda \int_0^\xi e^{-\lambda t} h(t) dt + \psi_\xi(\lambda), \end{aligned} \quad (4.3)$$

for all $\xi \in (0, a)$, where ε_ξ and ψ_ξ are functions satisfying

$$\limsup_{\lambda \rightarrow \infty} \frac{\ln \|\varepsilon_\xi(\lambda)\|}{\lambda} \leq -\xi, \quad \limsup_{\lambda \rightarrow \infty} \frac{\ln \|\psi_\xi(\lambda)\|}{\lambda} \leq -\xi. \quad (4.4)$$

Thus, if $\eta > \omega > \omega_1$, then

$$\begin{aligned} \int_\omega^\eta q'(\lambda) d\lambda &= \int_\omega^\eta \int_0^\xi e^{-\lambda t} h(t) dt d\lambda + \int_\omega^\eta \frac{\psi_\xi(\lambda)}{\lambda} d\lambda \\ &= \int_0^\xi \frac{e^{-\omega t} - e^{-\eta t}}{t} h(t) dt + \int_\omega^\eta \frac{\psi_\xi(\lambda)}{\lambda} d\lambda \\ &\xrightarrow{\eta \rightarrow \infty} \int_0^\xi \frac{e^{-\omega t} h(t)}{t} dt + \int_\omega^\infty \frac{\psi_\xi(\lambda)}{\lambda} d\lambda. \end{aligned} \quad (4.5)$$

Thus, $y := \lim_{\eta \rightarrow \infty} q(\eta)$ exists, and

$$y - \omega \int_0^\xi e^{-\omega t} g(t) dt - \varepsilon_\xi(\omega) = \int_0^\xi \frac{e^{-\omega t} h(t)}{t} dt + \int_\omega^\infty \frac{\psi_\xi(\lambda)}{\lambda} d\lambda \quad (4.6)$$

for all $\omega > \omega_1$ and all $\xi \in (0, a)$.

It is not hard to see that for all $\delta > 0$ we have

$$e^{-\omega \delta} e^{\omega \xi} \left\| \int_\omega^\infty \frac{\psi_\xi(\lambda)}{\lambda} d\lambda \right\| \xrightarrow{\omega \rightarrow \infty} 0, \quad (4.7)$$

that is,

$$\limsup_{\omega \rightarrow \infty} \frac{\ln \left\| \int_\omega^\infty (\psi_\xi(\lambda)/\lambda) d\lambda \right\|}{\omega} \leq -\xi. \quad (4.8)$$

Thus

$$\int_0^\xi e^{-\omega t} (y - g(t)) dt \sim_\xi \int_0^\xi \frac{e^{-\omega t}}{\omega} \cdot \frac{h(t)}{t} dt \sim_\xi \int_0^\xi e^{-\omega t} H^{[1]}(t) dt, \quad (4.9)$$

where $H(t) := h(t)/t$.

Consequently, $y - g(t) = H^{[1]}(t)$ for all $0 \leq t < a$. This shows that $y = 0$ and

$$g(t) = - \int_0^t \frac{h(t)}{t} dt \quad \forall 0 \leq t < a. \quad (4.10)$$

Thus

$$q(\lambda) \sim_{\xi} \int_0^{\xi} e^{-\lambda t} H(t) dt. \quad (4.11)$$

□

THEOREM 4.2. *Let $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$, and let $p : (0, \infty) \rightarrow \mathbb{C}$ be some continuous function satisfying $|p(t)| \leq \operatorname{const} \cdot e^{\omega_1 t}$ for all $t > 0$ and*

$$\liminf_{\lambda \rightarrow \infty} \frac{\ln |\hat{p}(\lambda)|}{\lambda} \geq 0, \quad (4.12)$$

$$\sup_{\lambda \in M} |\lambda \hat{p}(\lambda)| < \infty, \quad (4.13)$$

$$\sup_{\lambda \in M} \left| \lambda^2 \frac{d}{d\lambda} \hat{p}(\lambda) \right| < \infty. \quad (4.14)$$

Let X be a complex Banach space, $A : D(A) \rightarrow X$ linear with $M \subset \rho(A)$,

$$\sup_{\lambda \in M} |\lambda^3 \hat{p}(\lambda)| \cdot \|R(\lambda, A)^2\| < \infty, \quad (4.15)$$

$$\sup_{\lambda \in M} |\lambda^2 \hat{p}(\lambda)| \cdot \|R(\lambda, A)\| < \infty, \quad (4.16)$$

$$\sup_{\lambda \in M} \left| \lambda^3 \frac{d}{d\lambda} \hat{p}(\lambda) \right| \cdot \|R(\lambda, A)\| < \infty. \quad (4.17)$$

Then the following assertions hold:

- (1) A generates a local $p^{[1]}$ -convoluted semigroup on $[0, a)$;
- (2) the abstract Cauchy problem

$$\begin{aligned} Av(t) + p(t)x &= v'(t) \quad \text{if } 0 < t < a, \\ v(0) &= 0, \\ v &\in C^1((0, a), X) \cap C([0, a), X), \end{aligned} \quad (4.18)$$

has a unique solution v_x for all $x \in X$;

- (3) if p can be continuously extended in 0, then $A_{\overline{D(A)}}$ generates a local p -convoluted semigroup on $[0, a)$.

Remark 4.3. Condition (4.12) can be omitted if one then assumes in addition that

$$\limsup_{\lambda \rightarrow \infty} \frac{\ln \|R(\lambda, A)\|}{\lambda} \leq 0. \quad (4.19)$$

Proof. Consider the holomorphic function

$$q : M \longrightarrow \mathcal{L}(X, \overline{D(A)}), \quad q(\lambda) = \hat{p}(\lambda)R(\lambda, A). \quad (4.20)$$

Then $\lambda^2 q'(\lambda) = -\lambda^2 \hat{p}(\lambda)R(\lambda, A)^2 + \lambda^2 (d\hat{p}(\lambda)/d\lambda)R(\lambda, A)$, thus

$$\begin{aligned}
 \|\lambda^2 q'(\lambda)\| &\leq \|\lambda^2 \hat{p}(\lambda)R(\lambda, A)^2\| + \left\| \lambda^2 \frac{d\hat{p}(\lambda)}{d\lambda} R(\lambda, A) \right\| \\
 &\quad + \left\| (\lambda R(\lambda, A) - \text{id}) \left(\lambda^2 \hat{p}(\lambda)R(\lambda, A) - \lambda^2 \frac{d\hat{p}(\lambda)}{d\lambda} \cdot \text{id} \right) \right\| \\
 &\leq \|\lambda^2 \hat{p}(\lambda)R(\lambda, A)^2\| + \left\| \lambda^2 \frac{d\hat{p}(\lambda)}{d\lambda} R(\lambda, A) \right\| + \|\lambda^3 \hat{p}(\lambda)R(\lambda, A)^2\| \\
 &\quad + \left\| \lambda^3 \frac{d\hat{p}(\lambda)}{d\lambda} R(\lambda, A) \right\| + \|\lambda^2 \hat{p}(\lambda)R(\lambda, A)\| + \left\| \lambda^2 \frac{d\hat{p}(\lambda)}{d\lambda} \right\|.
 \end{aligned} \tag{4.21}$$

It follows from the assumptions that

$$\sup_{\lambda \in M} \|\lambda^2 q'(\lambda)\| < \infty. \tag{4.22}$$

Moreover,

$$\|\lambda q(\lambda)\| \leq \|\lambda \hat{p}(\lambda)R(\lambda, A)\| + \|\lambda \hat{p}(\lambda)(\lambda R(\lambda, A) - \text{id})\|, \tag{4.23}$$

thus

$$\sup_{\lambda \in M} \|\lambda q(\lambda)\| < \infty. \tag{4.24}$$

From [Theorem 4.1](#) it follows that there is some $H \in C((0, a), \mathcal{L}(X, \overline{D}(A)))$ such that $t \mapsto (t - a)H(t)$ is bounded on $(0, a)$ and with

$$\hat{p}(\lambda)R(\lambda, A) \sim_{\xi} \int_0^{\xi} e^{-\lambda t} H(t) dt \quad \forall \xi \in (0, a). \tag{4.25}$$

From conditions (4.12) and (4.16), it follows that $\limsup_{\lambda \rightarrow \infty} (\ln \|R(\lambda, A)\|/\lambda) \leq 0$, thus

$$\begin{aligned}
 \lambda(p^{[1]})_{\xi}^{\wedge}(\lambda)R(\lambda, A)x &\sim_{\xi} \hat{p}_{\xi}(\lambda)R(\lambda, A)x \\
 &\sim_{\xi} \int_0^{\xi} e^{-\lambda t} H(t)x dt \\
 &\sim_{\xi} \int_0^{\xi} e^{-\lambda t} d_t(H^{[1]}(t)x).
 \end{aligned} \tag{4.26}$$

This shows (1). To be precise, the solution of the abstract Cauchy problem

$$\begin{aligned}
 Au_x(t) + p^{[1]}(t)x &= u'_x(t), \quad 0 \leq t < a, \\
 u_x(0) &= 0, \\
 u_x &\in C^1([0, a), X),
 \end{aligned} \tag{4.27}$$

is $u_x(t) = \int_0^t H(s)x ds$.

Further on, we define $v_x : [0, a) \rightarrow X$ by

$$v_x(t) = \begin{cases} 0 & \text{if } t = 0, \\ H(t)x & \text{if } 0 < t < a. \end{cases} \quad (4.28)$$

Since $Av_x^{[1]} = (AH(\cdot)x)^{[1]}$ is in $C^1((0, a), X)$, we have $v_x \in C^1((0, a), X)$, and since A is closed, we obtain

$$Av_x(t) + p(t)x = v'_x(t) \quad \text{if } 0 < t < a. \quad (4.29)$$

This shows (2).

Finally if $x \in D(A)$, then

$$\begin{aligned} A\overline{D(A)}v_x(t) + p(t)x &= v'_x(t) \quad \text{if } 0 \leq t < a, \\ v_x(0) &= 0, \\ v_x &\in C^1([0, a), \overline{D(A)}). \end{aligned} \quad (4.30)$$

The result follows if we can show that

$$\sup_{0 \leq t \leq \xi} \sup_{\substack{y \in D(A) \\ \|y\| \leq 1}} \|v'_y(t)\|_X < \infty \quad \forall \xi \in (0, a). \quad (4.31)$$

If $0 < t \leq \xi$ and $y \in D(A)$ with $\|y\| \leq 1$, we obtain

$$\begin{aligned} \|v'_y(t)\| &\leq \|Av_y(t) + p(t)y\| = \|AH(t)y + p(t)y\| \\ &\leq \|AH(t)\|_{\mathcal{L}(X)} + \|p\|_{L_\infty[0, \xi]}. \end{aligned} \quad (4.32)$$

□

COROLLARY 4.4. *Let $c, a, \omega_1 > 0$, $F(t) := c(e^{at} - 1)$, and $M := \{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_1, |\operatorname{Im} z| < F(\operatorname{Re} z - \omega_1)\}$.*

Suppose X is a complex Banach space, $A : D(A) \rightarrow X$ is linear with $M \subset \rho(A)$, and that there is some $\gamma \geq 0$ with

$$\begin{aligned} \sup_{\lambda \in M} \frac{\|R(\lambda, A)^2\|}{|\lambda|^{\gamma-2}} &< \infty, \\ \sup_{\lambda \in M} \frac{\|R(\lambda, A)\|}{|\lambda|^{\gamma-1}} &< \infty. \end{aligned} \quad (4.33)$$

Then the following assertions hold:

- (1) *A generates a $(\gamma + 1)$ -integrated semigroup on $[0, a)$;*

(2) *the abstract Cauchy problem*

$$\begin{aligned}
 Av(t) + t^{\gamma} \cdot x &= v'(t) \quad \text{if } 0 < t < a, \\
 v(0) &= 0, \\
 v &\in C^1((0, a), X) \cap C([0, a), X),
 \end{aligned}
 \tag{4.34}$$

has a unique solution v_x for all $x \in X$;

(3) $A_{\overline{D(A)}}$ generates a γ -integrated semigroup on $[0, a)$.

Proof. Let $p(t) = t^{\gamma}$ and apply [Theorem 4.2](#).

□

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