INTERNAL STABILIZATION OF MAXWELL'S EQUATIONS IN HETEROGENEOUS MEDIA

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We consider the internal stabilization of Maxwell's equations with Ohm's law with space variable coefficients in a bounded region with a smooth boundary. Our result is mainly based on an observability estimate, obtained in some particular cases by the multiplier method, a duality argument and a weakening of norm argument, and arguments used in internal stabilization of scalar wave equations.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^3$ with a boundary Γ of class C^2 . For the sake of simplicity we further assume that Ω is simply connected and that Γ is connected.

In this paper we study the stabilization of Maxwell's equations with Ohm's law:

$$D' - \operatorname{curl}(\mu B) + \sigma D = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1.1)$$

$$B' + \operatorname{curl}(\lambda D) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1.2)$$

$$\operatorname{div} B = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1.3)$$

$$D(0) = D_0, \quad B(0) = B_0 \quad \text{in } \Omega, \quad (1.4)$$

$$D \times \nu = 0, \quad B \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty), \quad (1.5)$$

where D, B are three-dimensional vector-valued functions of $t, x = (x_1, x_2, x_3)$; $\mu = \mu(x)$, $\lambda = \lambda(x)$, $\sigma = \sigma(x)$ are scalar functions in $C^1(\overline{\Omega})$ such that $\sigma(x) \geq 0$ and λ and μ are uniformly bounded from below by a positive constant, that is,

$$\lambda(x) \geq \lambda_0 > 0, \quad \mu(x) \geq \mu_0 > 0, \quad \forall x \in \overline{\Omega}. \quad (1.6)$$

D_0, B_0 are the initial data in a suitable space and ν denotes the outward unit normal vector to Γ . We further assume that σ satisfies

$$\sigma(x) \geq \sigma_0 > 0, \quad \forall x \in \omega, \quad (1.7)$$

for some non empty open subset ω of Ω .

In that paper we will give sufficient conditions on λ , μ and ω which guarantee the exponential decay of the energy

$$\mathcal{E}(t) := \frac{1}{2} \int_{\Omega} \left(\lambda(x) |D(x, t)|^2 + \mu(x) |B(x, t)|^2 \right) dx \quad (1.8)$$

of our system.

The exact boundary controllability and stabilization of Maxwell's equations have been studied by many authors [4, 6, 7, 8, 10, 13, 15, 17, 18, 19, 21] and are usually based on an observability estimate obtained by different methods like the multiplier method, microlocal analysis, the frequency domain method. A similar strategy leads to the internal controllability of Maxwell's equations, see for instance [17, 18, 22, 23].

But to our knowledge the internal stabilization of Maxwell's equations with Ohm's law is only considered for constant coefficients λ and μ [17]. Therefore our goal is to consider the internal stabilization of Maxwell's equations with Ohm's law for space variable coefficients λ and μ . We then give sufficient conditions guaranteeing the exponential decay of the energy. Our method actually combines arguments used in the stabilization of scalar wave equation with locally distributed (internal) damping [24] with the use of an internal observability estimate for the standard Maxwell equations obtained for constant coefficients by Phung [17] using microlocal analysis and extended here to some subsets ω of Ω and space variable coefficients. This observability estimate is obtained using a vectorial multiplier method (see [11] in the scalar case and [22] for constant coefficients), a duality argument from [1, 12] and a weakening of norm argument (as in [11] in the scalar case).

The schedule of the paper is the following one: Well-posedness of the problem is analysed in Section 2 under appropriate conditions on Ω , λ , μ and σ using semigroup theory. Section 3 is devoted to the proof of the observability estimate when ω is a (small) neighbourhood of the boundary. Finally we conclude in Section 4 by the exponential stability of our system.

2. Well-posedness of the problem

Introduce the Hilbert spaces

$$\begin{aligned} \hat{J}(\Omega) &:= \{B \in L^2(\Omega)^3 : \operatorname{div} B = 0 \text{ in } \Omega; B \cdot \nu = 0 \text{ on } \Gamma\}, \\ H &:= L^2(\Omega)^3 \times \hat{J}(\Omega), \end{aligned} \quad (2.1)$$

equipped with the inner product

$$\left(\begin{pmatrix} D \\ B \end{pmatrix}, \begin{pmatrix} D_1 \\ B_1 \end{pmatrix} \right)_H = \int_{\Omega} \{ \lambda D D_1 + \mu B B_1 \} dx. \quad (2.2)$$

Now define the operator A as follows:

$$D(A) = H_0(\operatorname{curl}, \Omega) \times (\hat{J}(\Omega) \cap H^1(\Omega)^3), \quad (2.3)$$

where, as usual,

$$H_0(\operatorname{curl}, \Omega) = \{D \in L^2(\Omega)^3 : \operatorname{curl} D \in L^2(\Omega)^3, D \times \nu = 0 \text{ on } \Gamma\}. \quad (2.4)$$

For any $\begin{pmatrix} D \\ B \end{pmatrix}$ in $D(A)$ we take

$$A \begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} \operatorname{curl}(\mu B) - \sigma D \\ -\operatorname{curl}(\lambda D) \end{pmatrix}. \quad (2.5)$$

We then see that formally problem (1.1) to (1.5) is equivalent to

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= A\Phi, \\ \Phi(0) &= \Phi_0, \end{aligned} \quad (2.6)$$

when $\Phi = \begin{pmatrix} D \\ B \end{pmatrix}$ and $\Phi_0 = \begin{pmatrix} D_0 \\ B_0 \end{pmatrix}$.

We will prove that this problem (2.6) has a unique solution using Lumer-Phillips' theorem [16] by showing the following lemma.

LEMMA 2.1. *A is a maximal dissipative operator.*

Proof. We start with the dissipativeness of A , in other words we need to show that

$$\Re(A\Phi, \Phi)_H \leq 0, \quad \forall \Phi \in D(A). \quad (2.7)$$

With the above notation we have

$$(A\Phi, \Phi)_H = \int_{\Omega} \{\lambda(\operatorname{curl}(\mu B) - \sigma D) \cdot D - \mu \operatorname{curl}(\lambda D) B\} dx. \quad (2.8)$$

By Green's formula and the boundary condition $D \times \nu = 0$ on Γ , we get

$$(A\Phi, \Phi)_H = - \int_{\Omega} \lambda \sigma |D|^2 dx \leq 0. \quad (2.9)$$

Let us now pass to the maximality. For that purpose it suffices to show that for all $\begin{pmatrix} f \\ g \end{pmatrix}$ in H , there exists a unique $\begin{pmatrix} D \\ B \end{pmatrix}$ in $D(A)$ such that

$$(I - A) \begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \quad (2.10)$$

Equivalently, we have

$$B = g - \operatorname{curl}(\lambda D), \quad (2.11)$$

$$D + \operatorname{curl}(\mu \operatorname{curl}(\lambda D)) + \sigma D = f + \operatorname{curl}(\mu g). \quad (2.12)$$

This last problem has a unique solution D in $H_0(\text{curl}, \Omega)$ because its variational formulation is

$$\begin{aligned} & \int_{\Omega} \{ \mu \text{curl}(\lambda D) \cdot \text{curl}(\lambda w) + \lambda(1 + \sigma) D \cdot w \} dx \\ &= \int_{\Omega} \{ \lambda f \cdot w + \mu g \cdot \text{curl}(\lambda w) \} dx, \quad \forall w \in H_0(\text{curl}, \Omega). \end{aligned} \quad (2.13)$$

This problem has a unique solution by the Lax-Milgram lemma because the bilinear form defined as the left-hand side is coercive on $H_0(\text{curl}, \Omega)$ because $\lambda(1 + \sigma) \geq \lambda_0$.

It then remains to show that B given by (2.11) belongs to $\hat{J}(\Omega) \cap H^1(\Omega)^3$. Indeed by (2.11), we see that

$$\text{curl}(\mu B) = (1 + \sigma)D - f, \quad (2.14)$$

which shows that $\text{curl} B \in L^2(\Omega)^3$. On the other hand $\text{div} B = \text{div} g = 0$ since g belongs to $\hat{J}(\Omega)$. Finally $B \cdot \nu = 0$ on Γ because the boundary condition $\lambda D \times \nu = 0$ on Γ implies that $\text{curl}(\lambda D) \cdot \nu = 0$ on Γ and because $g \in \hat{J}(\Omega)$. Altogether we have that $B \in H_T(\text{curl}, \text{div}, \Omega)$, where

$$\begin{aligned} H_T(\text{curl}, \text{div}, \Omega) &:= \{ B \in L^2(\Omega)^3 : \text{curl} B \in L^2(\Omega)^3, \\ &\quad \text{div} B \in L^2(\Omega); B \cdot \nu = 0 \text{ on } \Gamma \}. \end{aligned} \quad (2.15)$$

Since the boundary Γ is supposed to be smooth we have the continuous embedding $H_T(\text{curl}, \text{div}, \Omega) \hookrightarrow (H^1(\Omega))^3$ (see, e.g., [5, Section I.3.4]), which leads to the requested regularity on B . $\square$

Since it is well-known that $D(A)$ is dense in H (see [9, Section 7] or [10]), by Lumer-Phillips' theorem (see, e.g., [16, Theorem I.4.3]), we conclude that A generates a C_0 -semigroup of contraction $T(t)$. Therefore we have the following existence result.

THEOREM 2.2. *For all $\Phi_0 \in H$, the problem (2.6) has a weak solution $\Phi \in C([0, \infty), H)$ given by $\Phi = T(t)\Phi_0$. If moreover $\Phi_0 \in D(A)$, the problem (2.6) has a strong solution $\Phi \in C([0, \infty), D(A)) \cap C^1([0, \infty), H)$.*

For our further use we also need the next result.

THEOREM 2.3. *Fix $T > 0$. Then for all $f \in L^2(0, T; L^2(\Omega)^3)$, the problem*

$$D' - \text{curl}(\mu B) = f \quad \text{in } Q_T := \Omega \times (0, T), \quad (2.16)$$

$$B' + \text{curl}(\lambda D) = 0 \quad \text{in } Q_T, \quad (2.17)$$

$$\text{div} B = 0 \quad \text{in } Q_T, \quad (2.18)$$

$$D(0) = 0, \quad B(0) = 0 \quad \text{in } \Omega, \quad (2.19)$$

$$D \times \nu = 0, \quad B \cdot \nu = 0 \quad \text{on } \Sigma_T := \Gamma \times (0, T), \quad (2.20)$$

has a unique mild solution $(\begin{smallmatrix} D \\ B \end{smallmatrix}) \in C([0, T], H)$ which satisfies the estimate

$$\int_{Q_T} \{ |D(x, t)|^2 + |B(x, t)|^2 \} dx dt \leq CT^2 \int_{Q_T} |f(x, t)|^2 dx dt, \quad (2.21)$$

for some positive constant C depending on λ and μ .

Proof. Denoting by A_0 the above operator A corresponding to $\sigma = 0$, the above problem (2.16) to (2.20) is equivalent to

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= A_0 \Phi + F, \\ \Phi(0) &= 0, \end{aligned} \quad (2.22)$$

when $\Phi = (\begin{smallmatrix} D \\ B \end{smallmatrix})$ and $F = (\begin{smallmatrix} f \\ 0 \end{smallmatrix})$.

As A_0 generates a C_0 -semigroup of contraction $T_0(t)$, problem (2.22) has a unique mild solution $\Phi \in C([0, \infty), H)$ given by (see [16, Section 4.4.2])

$$\Phi(t) = \int_0^t T_0(t-s)F(s)ds. \quad (2.23)$$

This identity implies that

$$\|\Phi(t)\|_H \leq \int_0^t \|F(s)\|_H ds \leq \int_0^t \left(\int_{\Omega} \lambda(x) |f(x, s)|^2 dx \right)^{1/2} ds. \quad (2.24)$$

We conclude by integrating the square of this estimate in $t \in (0, T)$, using Cauchy-Schwarz's inequality and taking into account the assumption (1.6). $\square$

We end this section by showing that the energy of our system is decreasing.

LEMMA 2.4. *Let (D_0, B_0) be an initial pair in H and let (D, B) be the solution of the system (1.1), (1.2), (1.3), (1.4), and (1.5). Then the derivative of the energy (defined by (1.8)) is*

$$\mathcal{E}'(t) = - \int_{\Omega} \lambda \sigma |D|^2 dx \leq 0, \quad \forall t > 0. \quad (2.25)$$

Proof. Deriving (1.8) we obtain

$$\mathcal{E}' = \int_{\Omega} \{ \lambda D \cdot D' + \mu B \cdot B' \} dx, \quad (2.26)$$

then, by (1.1) and (1.2),

$$\mathcal{E}' = \int_{\Omega} \{ \lambda D \cdot (\operatorname{curl} \mu B - \sigma D) - \mu B \cdot \operatorname{curl} \lambda D \} dx. \quad (2.27)$$

We conclude by integrating by parts in the first term of this right-hand side and using the boundary condition (1.5). $\square$

From this lemma we directly conclude that the energy is non-increasing.

COROLLARY 2.5. *Let (D_0, B_0) be an initial pair in H and let (D, B) be the solution of the system (1.1), (1.2), (1.3), (1.4), and (1.5). Then, for all $0 \leq S < T < +\infty$, we have*

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \int_{\Omega} \lambda \sigma |D|^2 dx \geq 0. \quad (2.28)$$

3. An observability estimate

Let us consider the solution (D_h, B_h) of the standard Maxwell system:

$$D'_h - \operatorname{curl}(\mu B_h) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.1)$$

$$B'_h + \operatorname{curl}(\lambda D_h) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.2)$$

$$\operatorname{div} D_h = \operatorname{div} B_h = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.3)$$

$$D_h(0) = D_0, \quad B_h(0) = B_0 \quad \text{in } \Omega, \quad (3.4)$$

$$D_h \times \nu = 0, \quad B_h \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty). \quad (3.5)$$

For our next purposes, we need that the following internal observability estimate holds: The subset ω of Ω is such that there exist a time $T > 0$ and a constant $C > 0$ such that

$$\frac{1}{2} \int_{\Omega} (\lambda(x) |D_0(x)|^2 + \mu(x) |B_0(x)|^2) dx \leq C \int_0^T \int_{\omega} |D_h(x, t)|^2 dx dt, \quad \forall (D_0, B_0) \in H_1, \quad (3.6)$$

where

$$H_1 = \{(D, B) \in H : \operatorname{div} D = 0 \text{ in } \Omega\}. \quad (3.7)$$

This estimate was proved by Phung [17, Theorem 3.4] using microlocal analysis, when μ and λ are constant and $\omega = \bar{\omega} \cap \Omega$ such that $\bar{\omega}$ controls geometrically Ω . We will extend such an estimate to variable coefficients and some open subsets ω using the multiplier method. For that purpose, we further require that there exist $x_0 \in \Omega$ and a positive constant c_0 such that

$$\begin{aligned} \lambda(x) - \nabla \lambda(x) \cdot (x - x_0) &\geq c_0 \lambda(x), \\ \mu(x) - \nabla \mu(x) \cdot (x - x_0) &\geq c_0 \mu(x), \end{aligned} \quad (3.8)$$

for all $x \in \Omega$.

We first reduce the estimate to the estimate of the electric field.

LEMMA 3.1. *Fix $T > 0$. Let (D_h, B_h) be the solution of (3.1), (3.2), (3.3), (3.4), and (3.5) with initial datum $(D_0, B_0) \in H_1$. Then there exists $C > 0$ such that*

$$\frac{1}{2} \int_{\Omega} (\lambda(x) |D_0(x)|^2 + \mu(x) |B_0(x)|^2) dx \leq C \int_0^T \int_{\Omega} |D_h(x, t)|^2 dx dt, \quad \forall (D_0, B_0) \in H_1. \quad (3.9)$$

Proof. We adapt step 1 of the proof of [17, Theorem 3.4] to our setting. Recall that the Hilbert space $H_T(\text{curl}, \text{div}, \Omega)$, defined in (2.15), equipped with its natural norm is compactly embedded into $(L^2(\Omega))^3$ [20]. Therefore there exists a unique $\psi \in H_T(\text{curl}, \text{div}, \Omega)$ solution of

$$\begin{aligned} \text{curl}(\lambda \text{curl} \psi) &= B_h \quad \text{in } \Omega, \\ \text{div} \psi &= 0 \quad \text{in } \Omega, \\ \psi \cdot \nu &= 0, \quad \text{curl} \psi \times \nu = 0 \quad \text{on } \Gamma, \end{aligned} \quad (3.10)$$

in the sense that $\psi \in H_T(\text{curl}, \text{div}, \Omega)$ is the unique solution of

$$\int_{\Omega} \{\lambda \text{curl} \psi \cdot \text{curl} w + \text{div} \psi \text{div} w\} dx = \int_{\Omega} B_h \cdot w dx, \quad \forall w \in H_T(\text{curl}, \text{div}, \Omega). \quad (3.11)$$

Indeed the above compactness property and the hypotheses on Ω and Γ guarantee that the left-hand side of (3.11) is coercive on $H_T(\text{curl}, \text{div}, \Omega)$. On the other hand since $\text{div} B_h = 0$ in Ω we easily see that the solution ψ of (3.11) satisfies (3.10) (see [2, Theorem 1.1]). Setting $A = \text{curl} \psi$, we deduce that

$$B_h = \text{curl}(\lambda A) \quad \text{in } \Omega, \quad (3.12)$$

$$\text{div} A = 0 \quad \text{in } \Omega, \quad (3.13)$$

$$A \times \nu = 0 \quad \text{on } \Gamma. \quad (3.14)$$

Moreover taking $w = \psi$ in (3.11) we see that

$$\lambda_0 \|A\|_{L^2(\Omega)^3}^2 \leq \|B_h\|_{L^2(\Omega)^3} \|\psi\|_{L^2(\Omega)^3} \leq C \|B_h\|_{L^2(\Omega)^3} \|A\|_{L^2(\Omega)^3}, \quad (3.15)$$

this last estimate following from the compact embedding of $H_T(\text{curl}, \text{div}, \Omega)$ into $(L^2(\Omega))^3$. In other words we have

$$\|A\|_{L^2(\Omega)^3} \leq C \|B_h\|_{L^2(\Omega)^3}. \quad (3.16)$$

Using (3.2), (3.3), (3.5) and (3.12) to (3.14), we see that

$$\text{curl}(\lambda(A' + D_h)) = 0 \quad \text{in } \Omega, \quad (3.17)$$

$$\text{div}(A' + D_h) = 0 \quad \text{in } \Omega, \quad (3.18)$$

$$(A' + D_h) \times \nu = 0 \quad \text{on } \Gamma. \quad (3.19)$$

The first identity and the fact that Ω is simply connected imply that

$$\lambda(A' + D_h) = \nabla \varphi, \quad (3.20)$$

with $\varphi \in H^1(\Omega)$. The properties (3.18), (3.19) and the fact that Γ is connected imply that φ is constant and therefore we conclude that

$$A' + D_h = 0 \quad \text{in } \Omega. \quad (3.21)$$

Take $\Phi(t) = t(T - t)$ and consider

$$\int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt. \quad (3.22)$$

Then by (3.12) and Green's formula we get, owing to (3.14),

$$\int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt = \int_{Q_T} \Phi(t)^2 \operatorname{curl}(\mu B_h) \cdot \lambda A dx dt. \quad (3.23)$$

Therefore by (3.1) we obtain

$$\int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt = \int_{Q_T} \lambda \Phi(t)^2 D'_h \cdot A dx dt. \quad (3.24)$$

Now by integration by parts in t , we get

$$\int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt = - \int_{Q_T} \lambda (2\Phi \Phi' A + \Phi^2 A') \cdot D_h dx dt. \quad (3.25)$$

The identity (3.21) then yields

$$\int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt = -2 \int_{Q_T} \lambda \Phi \Phi' A \cdot D_h dx dt + \int_{Q_T} \lambda \Phi^2 |D_h|^2 dx dt. \quad (3.26)$$

Using Young's inequality we arrive at

$$\begin{aligned} \int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt &\leq \epsilon \int_{Q_T} \lambda \Phi^2 |A|^2 dx dt \\ &\quad + \frac{1}{\epsilon} \int_{Q_T} \lambda (\Phi')^2 |D_h|^2 dx dt + \int_{Q_T} \lambda \Phi^2 |D_h|^2 dx dt, \end{aligned} \quad (3.27)$$

for any $\epsilon > 0$. Using finally the estimate (3.16) we have proved that

$$\begin{aligned} \int_{Q_T} \mu(x) \Phi(t)^2 |B_h(x, t)|^2 dx dt &\leq \frac{C\epsilon}{\mu_0} \int_{Q_T} \Phi^2 \mu |B_h|^2 dx dt \\ &\quad + \frac{1}{\epsilon} \int_{Q_T} \lambda (\Phi')^2 |D_h|^2 dx dt + \int_{Q_T} \lambda \Phi^2 |D_h|^2 dx dt, \end{aligned} \quad (3.28)$$

for any $\epsilon > 0$. Choosing ϵ small enough we arrive at

$$\int_{Q_T} \mu \Phi^2 |B_h|^2 dx dt \leq C \int_{Q_T} \lambda |D_h|^2 dx dt. \quad (3.29)$$

Using the conservation of energy (identity (2.28) with $\sigma = 0$) we may write

$$\int_{Q_T} (\mu |B_h|^2 + \lambda |D_h|^2) dx dt = 3 \int_{T/3}^{2T/3} \int_{\Omega} (\mu |B_h|^2 + \lambda |D_h|^2) dx dt. \quad (3.30)$$

As $\Phi(t) \geq 2T^2/9$ on $[T/3, 2T/3]$ we get

$$\int_{Q_T} (\mu |B_h|^2 + \lambda |D_h|^2) dx dt \leq \frac{243}{4T^4} \int_{T/3}^{2T/3} \mu \Phi^2 |B_h|^2 dx dt + 3 \int_{Q_T} \lambda |D_h|^2 dx dt. \quad (3.31)$$

The conclusion follows from (3.29). $\square$

Since it remains to estimate $\int_0^T \int_{\Omega} |D_h(x, t)|^2 dx dt$ we are looking at D_h as solution of the following second order system:

$$D_h'' + \operatorname{curl}(\mu \operatorname{curl}(\lambda D_h)) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.32)$$

$$\operatorname{div} D_h = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (3.33)$$

$$D_h(0) = D_0, \quad D_h'(0) = D_1 = \operatorname{curl}(\mu B_0) \quad \text{in } \Omega, \quad (3.34)$$

$$D_h \times \nu = 0, \quad \operatorname{curl}(\lambda D_h) \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty). \quad (3.35)$$

Consider the set

$$\begin{aligned} H_N(\operatorname{curl}, \operatorname{div}, \Omega) \\ := \{D \in L^2(\Omega)^3 : \operatorname{curl} D \in L^2(\Omega)^3, \operatorname{div} D \in L^2(\Omega); D \times \nu = 0 \text{ on } \Gamma\}, \end{aligned} \quad (3.36)$$

continuously embedded into $H^1(\Omega)^3$ (see, e.g., [5, Section I.3.4]) and compactly embedded into $L^2(\Omega)^3$ [20]. Let us set

$$\begin{aligned} \mathcal{H} &:= \{D \in L^2(\Omega)^3 : \operatorname{div} D = 0 \text{ in } \Omega\}, \\ \mathcal{V} &:= \{D \in H_N(\operatorname{curl}, \operatorname{div}, \Omega) : \operatorname{div} D = 0 \text{ in } \Omega\}, \\ a(D, D_1) &:= \int_{\Omega} \mu \operatorname{curl}(\lambda D) \cdot \operatorname{curl}(\lambda D_1) dx, \quad \forall D, D_1 \in \mathcal{V}. \end{aligned} \quad (3.37)$$

The bilinear form a is symmetric and strongly coercive on $\mathcal{V}$, moreover $\mathcal{V}$ is compactly embedded into $\mathcal{H}$ (see [10]). By spectral analysis, the above problem has a unique solution $D_h \in C([0, T], \mathcal{V}) \cap C^1([0, T], \mathcal{H})$ if (D_0, D_1) belongs to $\mathcal{V} \times \mathcal{H}$. Obviously D_h is the same as the one from problem (3.1), (3.2), (3.3), (3.4), and (3.5) if $(D_0, B_0) \in \mathcal{V} \times (\hat{J}(\Omega) \cap H^1(\Omega)^3)$, because then $(D_0, D_1 = \operatorname{curl}(\mu B_0))$ belongs to $\mathcal{V} \times \mathcal{H}$.

The energy of the solution of that system is given by

$$E_D(t) := \frac{1}{2} \int_{\Omega} (\lambda(x) |D_h'(x, t)|^2 + \mu(x) |\operatorname{curl}(\lambda(x) D_h(x, t))|^2) dx. \quad (3.38)$$

A simple application of Green's formula shows that

$$E_D'(t) = 0, \quad (3.39)$$

and therefore the energy E_D is constant.

Using a vectorial multiplier method we first prove the following lemma. An analogous lemma was proved in [22] in the case of constant coefficients.

LEMMA 3.2. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$, and let $q : \overline{\Omega} \rightarrow \mathbb{R}^3$ a C^1 vector field. Then for any time $T > 0$ the following identity holds:*

$$\begin{aligned}
& \left[\int_{\Omega} 2(D'_h, q, \operatorname{curl}(\lambda D_h)) dx \right]_0^T \\
&= \int_0^T \int_{\Gamma} \left[\lambda(q \cdot \nu) |D'_h|^2 - \mu(q \cdot \nu) |\operatorname{curl}(\lambda D_h)|^2 \right] d\Gamma dt \\
&+ \int_0^T \int_{\Omega} \left[\left(\lambda |D'_h|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2 \right) \operatorname{div} q - 2\lambda \sum_{i,j=1}^3 (D'_h)_i (D'_h)_j \partial_i q_j \right. \\
&\quad \left. - 2\mu \sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j \right] dx dt \\
&- \int_0^T \int_{\Omega} \left[|D'_h|^2 q \cdot \nabla \lambda + |\operatorname{curl}(\lambda D_h)|^2 q \cdot \nabla \mu \right] dx dt,
\end{aligned} \tag{3.40}$$

where the notation $(a, b, c) = a \cdot (b \times c)$ means the mixed product of the vectors a, b, c .

Proof. By (3.32)

$$\begin{aligned}
0 &= \int_0^T \int_{\Omega} 2(D'_h'' + \operatorname{curl}(\mu \operatorname{curl}(\lambda D_h)), q, \operatorname{curl}(\lambda D_h)) dx dt \\
&= \left[\int_{\Omega} 2(D'_h, q, \operatorname{curl}(\lambda D_h)) dx \right]_0^T + \int_0^T \int_{\Gamma} 2\mu(\nu, \operatorname{curl}(\lambda D_h), q \times \operatorname{curl}(\lambda D_h)) d\Gamma dt \\
&+ \int_0^T \int_{\Omega} 2 \left[\mu \operatorname{curl}(\lambda D_h) \cdot \operatorname{curl}(q \times \operatorname{curl}(\lambda D_h)) - (D'_h, q, \operatorname{curl}(\lambda D'_h)) \right] dx dt.
\end{aligned} \tag{3.41}$$

Integrating by parts we obtain

$$\begin{aligned}
& \int_0^T \int_{\Omega} -2(D'_h, q, \operatorname{curl}(\lambda D'_h)) dx dt = \int_0^T \int_{\Omega} 2\lambda D'_h \cdot \operatorname{curl}(q \times D'_h) dx dt \\
&= \int_0^T \int_{\Omega} 2\lambda \left[D'_h \cdot (q \operatorname{div} D'_h - D'_h \operatorname{div} q) + \sum_{i,j=1}^3 (D'_h)_i (D'_h)_j \partial_i q_j \right. \\
&\quad \left. - \sum_{i,j=1}^3 (D'_h)_j q_i \partial_i (D'_h)_j \right] dx dt \\
&= \int_0^T \int_{\Omega} \left[2\lambda \sum_{i,j=1}^3 (D'_h)_i (D'_h)_j \partial_i q_j - 2\lambda |D'_h|^2 \operatorname{div} q - \lambda q \cdot \nabla (|D'_h|^2) \right] dx dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^T \int_{\Omega} \left[2\lambda \sum_{i,j=1}^3 (D'_h)_i (D'_h)_j \partial_i q_j - 2\lambda |D'_h|^2 \operatorname{div} q + |D'_h|^2 \operatorname{div}(\lambda q) \right] dx dt \\
&\quad - \int_0^T \int_{\Gamma} \lambda(q \cdot \nu) |D'_h|^2 d\Gamma dt,
\end{aligned} \tag{3.42}$$

and then

$$\begin{aligned}
&\int_0^T \int_{\Omega} -2(D'_h, q, \operatorname{curl}(\lambda D'_h)) dx dt \\
&= - \int_0^T \int_{\Gamma} \lambda(q \cdot \nu) |D'_h|^2 d\Gamma dt \\
&\quad + \int_0^T \int_{\Omega} \left[2\lambda \sum_{i,j=1}^3 (D'_h)_i (D'_h)_j \partial_i q_j - \lambda |D'_h|^2 \operatorname{div} q + |D'_h|^2 q \cdot \nabla \lambda \right] dx dt.
\end{aligned} \tag{3.43}$$

Analogously, we can rewrite

$$\begin{aligned}
&\int_0^T \int_{\Omega} 2\mu \operatorname{curl}(\lambda D_h) \cdot \operatorname{curl}(q \times \operatorname{curl}(\lambda D_h)) dx dt \\
&= \int_0^T \int_{\Omega} 2\mu \left\{ \operatorname{curl}(\lambda D_h) \cdot [q \operatorname{div} \operatorname{curl}(\lambda D_h) - \operatorname{curl}(\lambda D_h) \operatorname{div} q] \right. \\
&\quad \left. + \sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j \right. \\
&\quad \left. - \sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_j q_i \partial_i (\operatorname{curl}(\lambda D_h))_j \right\} dx dt \\
&= \int_0^T \int_{\Omega} \left\{ 2\mu \left[\sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j - |\operatorname{curl}(\lambda D_h)|^2 \operatorname{div} q \right] \right. \\
&\quad \left. - \mu q \cdot \nabla (|\operatorname{curl}(\lambda D_h)|^2) \right\} dx dt \\
&= \int_0^T \int_{\Omega} \left\{ 2\mu \left[\sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j - |\operatorname{curl}(\lambda D_h)|^2 \operatorname{div} q \right] \right. \\
&\quad \left. + |\operatorname{curl}(\lambda D_h)|^2 \operatorname{div}(\mu q) \right\} dx dt - \int_0^T \int_{\Gamma} \mu(q \cdot \nu) |\operatorname{curl}(\lambda D_h)|^2 d\Gamma dt,
\end{aligned} \tag{3.44}$$

and then

$$\begin{aligned}
 & \int_0^T \int_{\Omega} 2\mu \operatorname{curl}(\lambda D_h) \cdot \operatorname{curl}(q \times \operatorname{curl}(\lambda D_h)) dx dt \\
 &= - \int_0^T \int_{\Gamma} \mu(q \cdot \nu) |\operatorname{curl}(\lambda D_h)|^2 d\Gamma dt \\
 &+ \int_0^T \int_{\Omega} \left\{ 2\mu \left[\sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j - \mu |\operatorname{curl}(\lambda D_h)|^2 \operatorname{div} q \right] \right. \\
 &\quad \left. + |\operatorname{curl}(\lambda D_h)|^2 q \cdot \nabla \mu \right\} dx dt.
 \end{aligned} \tag{3.45}$$

Putting (3.43) and (3.45) in the first identity, we obtain

$$\begin{aligned}
 0 &= \left[\int_{\Omega} 2(D'_h, q, \operatorname{curl}(\lambda D_h)) dx \right]_0^T + \int_0^T \int_{\Omega} \left[|D'_h|^2 q \cdot \nabla \lambda + |\operatorname{curl}(\lambda D_h)|^2 q \cdot \nabla \mu \right] dx dt \\
 &+ \int_0^T \int_{\Gamma} \left[2\mu(\nu, \operatorname{curl}(\lambda D_h), q \times \operatorname{curl}(\lambda D_h)) - \lambda(q \cdot \nu) |D'_h|^2 \right. \\
 &\quad \left. - \mu(q \cdot \nu) |\operatorname{curl}(\lambda D_h)|^2 \right] d\Gamma dt \\
 &+ \int_0^T \int_{\Omega} \left[2\lambda \sum_{i,j=1}^3 (D'_h)_i (D'_h)_j \partial_i q_j + 2\mu \sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j \right. \\
 &\quad \left. - (\lambda |D'_h|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2) \operatorname{div} q \right] dx dt.
 \end{aligned} \tag{3.46}$$

Therefore (3.40) follows observing that the boundary term can be rewritten using

$$\begin{aligned}
 & 2\mu(\nu, \operatorname{curl}(\lambda D_h), q \times \operatorname{curl}(\lambda D_h)) \\
 &= 2\mu(q \cdot \nu) |\operatorname{curl}(\lambda D_h)|^2 \\
 &\quad - 2\mu(\nu \cdot \operatorname{curl}(\lambda D_h)) (q \cdot \operatorname{curl}(\lambda D_h)) \\
 &= 2\mu(q \cdot \nu) |\operatorname{curl}(\lambda D_h)|^2,
 \end{aligned} \tag{3.47}$$

recalling that $\operatorname{curl}(\lambda D_h) \cdot \nu = 0$ on $\Gamma \times (0, \infty)$. □

For any $\varepsilon > 0$ let us denote by $\mathcal{N}_{\varepsilon}(\Gamma)$ the neighborhood of Γ of radius ε , that is,

$$\mathcal{N}_{\varepsilon}(\Gamma) = \left\{ x \in \Omega : \inf_{y \in \Gamma} |x - y| < \varepsilon \right\}. \tag{3.48}$$

Using the previous identity we prove the following lemma:

LEMMA 3.3. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$. If $\tilde{\omega} = \mathcal{N}_{\epsilon/2}(\Gamma)$, for some $\epsilon > 0$ and λ, μ satisfy (1.6), (3.8), then there exist $T_0 > 0$ and $C > 0$ such that for $T > T_0$ we have*

$$(T - T_0)E_D(0) \leq C \int_0^T \int_{\tilde{\omega}} (|D'_h(x, t)|^2 + |D_h(x, t)|^2) dx dt. \quad (3.49)$$

Proof. From (3.40), using the standard multiplier $q(x) = m(x) = x - x_0$, we obtain for any $T > 0$

$$\begin{aligned} & \int_0^T \int_{\Gamma} (m \cdot \nu) [\lambda |D'_h|^2 - \mu |\operatorname{curl}(\lambda D_h)|^2] d\Gamma dt \\ &= \left[\int_{\Omega} 2(D'_h, m, \operatorname{curl}(\lambda D_h)) dx \right]_0^T - \int_0^T \int_{\Omega} [\lambda |D'_h|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2] dx dt \\ &+ \int_0^T \int_{\Omega} [|D'_h|^2 m \cdot \nabla \lambda + |\operatorname{curl}(\lambda D_h)|^2 m \cdot \nabla \mu] dx dt. \end{aligned} \quad (3.50)$$

Using the assumption (3.8), the above identity implies

$$\begin{aligned} & c_0 T \int_{\Omega} [\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2] dx - 2 \left[\int_{\Omega} (D'_h, m, \operatorname{curl}(\lambda D_h)) dx \right]_0^T \\ & \leq \int_0^T \int_{\Gamma} (m \cdot \nu) [\mu |\operatorname{curl}(\lambda D_h)|^2 - \lambda |D'_h|^2] d\Gamma dt. \end{aligned} \quad (3.51)$$

Note that by (1.6)

$$\left| \left[2 \int_{\Omega} (D'_h, m, \operatorname{curl}(\lambda D_h)) dx \right]_0^T \right| \leq \frac{2 \max_{\overline{\Omega}} |m|}{\sqrt{\lambda_0 \mu_0}} \int_{\Omega} [\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2] dx. \quad (3.52)$$

So, setting

$$\tilde{T} = \frac{2 \max_{\overline{\Omega}} |m|}{c_0 \sqrt{\lambda_0 \mu_0}}, \quad (3.53)$$

we obtain

$$\begin{aligned} & c_0 (T - \tilde{T}) \int_{\Omega} (\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2) dx \\ & \leq \int_0^T \int_{\Gamma} (m \cdot \nu) [\mu |\operatorname{curl}(\lambda D_h)|^2 - \lambda |D'_h|^2] d\Gamma dt. \end{aligned} \quad (3.54)$$

Now, set $\omega_0 = \mathcal{N}_{\epsilon/4}(\Gamma)$ and apply (3.40) using as multiplier $q(x) = \varphi(x)m(x)$ with $\varphi \in C^1(\overline{\Omega})$, $0 \leq \varphi(x) \leq 1$,

$$\varphi(x) \equiv 1, \quad x \in \mathcal{N}_{\epsilon/8}(\Gamma), \quad \varphi(x) \equiv 0, \quad x \in \Omega \setminus \omega_0. \quad (3.55)$$

We obtain

$$\begin{aligned}
 & \int_0^T \int_{\Gamma} (m \cdot \nu) [\mu |\operatorname{curl}(\lambda D_h)|^2 - \lambda |D_h'|^2] d\Gamma dt \\
 & \leq C \int_0^T \int_{\omega_0} (|D_h'|^2 + |\operatorname{curl}(\lambda D_h)|^2) dx dt \\
 & \quad + c_0 \tilde{T} \int_{\Omega} (\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2) dx,
 \end{aligned} \tag{3.56}$$

for a suitable constant $C > 0$. Then, from (3.54) and (3.56),

$$c_0(T - 2\tilde{T}) \int_{\Omega} (\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2) dx \leq C \int_0^T \int_{\omega_0} (|D_h'|^2 + |\operatorname{curl}(\lambda D_h)|^2) dx dt. \tag{3.57}$$

Now, let $g : \bar{\Omega} \rightarrow \mathbb{R}$ be a C^1 function with $0 \leq g(x) \leq 1$, and

$$g(x) \equiv 1, \quad x \in \omega_0, \quad g(x) \equiv 0, \quad x \in \Omega \setminus \bar{\omega}. \tag{3.58}$$

By (3.32), for any positive time T , by integration by parts, we have

$$\begin{aligned}
 0 &= \int_0^T \int_{\Omega} [D_h'' + \operatorname{curl}(\mu \operatorname{curl}(\lambda D_h))] \cdot (g \lambda D_h) dx dt = \left[\int_{\Omega} \lambda g D_h' \cdot D_h dx \right]_0^T \\
 & \quad - \int_0^T \int_{\Omega} \lambda g |D_h'|^2 dx dt + \int_0^T \int_{\Omega} \mu \operatorname{curl}(\lambda D_h) \cdot [-\lambda D_h \times \nabla g + g \operatorname{curl}(\lambda D_h)] dx dt.
 \end{aligned} \tag{3.59}$$

Then,

$$\begin{aligned}
 \int_0^T \int_{\Omega} \mu g |\operatorname{curl}(\lambda D_h)|^2 dx dt &= \int_0^T \int_{\Omega} \lambda g |D_h'|^2 dx dt - \left[\int_{\Omega} \lambda g D_h' \cdot D_h dx \right]_0^T \\
 & \quad + 2 \int_0^T \int_{\Omega} \mu \sqrt{g} \operatorname{curl}(\lambda D_h) \cdot (\lambda D_h \times \nabla \sqrt{g}) dx dt.
 \end{aligned} \tag{3.60}$$

By Young's inequality we can estimate

$$\begin{aligned}
 & \left| 2 \int_0^T \int_{\Omega} \mu \sqrt{g} \operatorname{curl}(\lambda D_h) \cdot (\lambda D_h \times \nabla \sqrt{g}) dx dt \right| \\
 & \leq \frac{1}{2} \int_0^T \int_{\bar{\omega}} \mu g |\operatorname{curl}(\lambda D_h)|^2 dx dt + C \int_0^T \int_{\bar{\omega}} |D_h|^2 dx dt.
 \end{aligned} \tag{3.61}$$

Moreover, using the inequality

$$\int_{\Omega} |D_h|^2 dx \leq C \int_{\Omega} |\operatorname{curl}(\lambda D_h)|^2 dx, \quad (3.62)$$

consequence of the compact embedding of $H_N(\operatorname{curl}, \operatorname{div}, \Omega)$ into $L^2(\Omega)^3$, we have

$$\left| \left[\int_{\Omega} \lambda g D'_h \cdot D_h dx \right]_0^T \right| \leq C \int_{\Omega} (\lambda |D'_h|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2) dx. \quad (3.63)$$

Therefore, using (3.61) and (3.63) in (3.60), we obtain

$$\begin{aligned} \int_0^T \int_{\omega_0} |\operatorname{curl}(\lambda D_h)|^2 dx dt &\leq \int_0^T \int_{\bar{\omega}} g |\operatorname{curl}(\lambda D_h)|^2 dx dt \\ &\leq C \int_{\Omega} (\lambda |D'_h|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2) dx + C' \int_0^T \int_{\bar{\omega}} (|D_h|^2 + |D'_h|^2) dx dt, \end{aligned} \quad (3.64)$$

for suitable positive constants C, C' . Finally, by (3.57) and (3.64) we have

$$(T - 2\bar{T})E_D(0) \leq C \int_0^T \int_{\bar{\omega}} (|D'_h|^2 + |D_h|^2) dx dt + CE_D(0), \quad (3.65)$$

for some constant $C > 0$. So, we can deduce the existence of a time T_0 such that for $T > T_0$

$$(T - T_0)E_D(0) \leq \int_0^T \int_{\bar{\omega}} (|D'_h|^2 + |D_h|^2) dx dt. \quad (3.66)$$

□

In a second step using a duality argument as in [1] (see also [12, Lemma 10]) we prove the following estimate.

LEMMA 3.4. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$. If $\omega = \mathcal{N}_{\epsilon}(\Gamma)$ and $\bar{\omega} = \mathcal{N}_{\epsilon/2}(\Gamma)$, for some $\epsilon > 0$, then there exists $C > 0$ such that for any $\eta > 0$ we have*

$$\int_0^T \int_{\bar{\omega}} |D_h(x, t)|^2 dx dt \leq \frac{C}{\eta} \int_0^T \int_{\omega} |D'_h(x, t)|^2 dx dt + \eta \int_0^T E_D(t) dt + CE_D(0). \quad (3.67)$$

Proof. Fix $\beta \in \mathcal{D}(\mathbb{R}^3)$ such that $\beta \equiv 1$ on $\tilde{\omega}$ with a support included into ω .

Consider $z \in H_N(\text{curl}, \text{div}, \Omega)$ the unique solution of

$$\int_{\Omega} \mu \text{curl}(\lambda z) \cdot \text{curl}(\lambda w) dx + \int_{\Omega} \text{div} z \text{div} w dx = \int_{\Omega} \beta \lambda D_h(x, t) \cdot w(x) dx, \quad (3.68)$$

for all $w \in H_N(\text{curl}, \text{div}, \Omega)$. This solution z satisfies (due to the compact embedding of $H_N(\text{curl}, \text{div}, \Omega)$ in $L^2(\Omega)^3$ and to the properties of Ω and Γ)

$$\|z\|_{L^2(\Omega)^3} \leq C \|\beta D_h\|_{L^2(\Omega)^3}, \quad (3.69)$$

for some $C > 0$.

Multiplying (3.32) by λz and integrating in Q_T we get

$$0 = \int_{Q_T} \lambda (D_h'' + \text{curl}(\mu \text{curl}(\lambda D_h))) \cdot z dx dt. \quad (3.70)$$

Applying Green's formula (in space and time) and taking into account the boundary condition $z \times \nu = 0$ on Γ we obtain

$$0 = - \int_{Q_T} \lambda D_h' z' dx dt + \left[\int_{\Omega} \lambda D_h' z dx \right]_0^T + \int_{Q_T} \mu \text{curl}(\lambda D_h) \cdot \text{curl}(\lambda z) dx dt. \quad (3.71)$$

Now taking into account (3.33) and using (3.68) with $w = D_h$ we arrive at

$$0 = - \int_{Q_T} \lambda D_h' z' dx dt + \left[\int_{\Omega} \lambda D_h' z dx \right]_0^T + \int_{Q_T} \beta \lambda |D_h|^2 dx dt. \quad (3.72)$$

By Cauchy-Schwarz's inequality and the fact that $\beta \equiv 1$ on $\tilde{\omega}$, we get

$$\begin{aligned} \int_0^T \int_{\tilde{\omega}} \lambda |D_h|^2 dx dt &\leq \int_{Q_T} \beta \lambda |D_h|^2 dx dt = \int_{Q_T} \lambda D_h' z' dx dt - \left[\int_{\Omega} \lambda D_h' z dx \right]_0^T \\ &\leq \left(\int_{Q_T} \lambda |D_h'|^2 dx dt \right)^{1/2} \left(\int_{Q_T} \lambda |z'|^2 dx dt \right)^{1/2} \\ &\quad + \left(\int_{\Omega} \lambda |D_h'(x, t)|^2 dx \right)^{1/2} \left(\int_{\Omega} \lambda |z(x, t)|^2 dx \right)^{1/2} \Big|_{t=0, T}. \end{aligned} \quad (3.73)$$

Using the estimates (3.69), (3.62) and the definition of the energy we get

$$\begin{aligned} \int_0^T \int_{\bar{\omega}} \lambda |D_h|^2 dx dt &\leq C \left(\int_{Q_T} \lambda |D'_h|^2 dx dt \right)^{1/2} \left(\int_{Q_T} \beta |D'_h|^2 dx dt \right)^{1/2} + CE_D(0) \\ &\leq C \left(\int_0^T E_D(t) dt \right)^{1/2} \left(\int_0^T \int_{\omega} |D'_h|^2 dx dt \right)^{1/2} + CE_D(0). \end{aligned} \quad (3.74)$$

We conclude by Young's inequality. $\square$

COROLLARY 3.5. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$. If $\omega = \mathcal{N}_\epsilon(\Gamma)$, for some $\epsilon > 0$ and λ, μ satisfy (1.6), (3.8), then there exist $T_1 > 0$ and $C > 0$ such that for $T > T_1$ we have*

$$(T - T_1)E_D(0) \leq C \int_0^T \int_{\omega} |D'_h(x, t)|^2 dx dt. \quad (3.75)$$

Proof. By (3.49) and (3.67) we may write

$$\begin{aligned} (T - T_0)E_D(0) &\leq C \int_0^T \int_{\bar{\omega}} |D'_h(x, t)|^2 dx dt \\ &\quad + \frac{C}{\eta} \int_0^T \int_{\omega} |D'_h(x, t)|^2 dx dt + C\eta \int_0^T E_D(t) dt + CE_D(0), \end{aligned} \quad (3.76)$$

for any $\eta > 0$. By the conservation of energy, this yields

$$(T - T_0)E_D(0) \leq C \int_0^T \int_{\bar{\omega}} |D'_h(x, t)|^2 dx dt + \frac{C}{\eta} \int_0^T \int_{\omega} |D'_h(x, t)|^2 dx dt + C(\eta T + 1)E_D(0). \quad (3.77)$$

The conclusion follows by choosing η small enough. $\square$

We now finish by adapting a weakening of norm argument from [11, Section VII.2.4].

LEMMA 3.6. *Fix $T > T_1$. Let (D_h, B_h) be the solution of (3.1), (3.2), (3.3), (3.4), and (3.5) with initial datum $(D_0, B_0) \in H_1$. If $\omega = \mathcal{N}_\epsilon(\Gamma)$, for some $\epsilon > 0$, then there exists $C > 0$ (depending on T) such that*

$$\int_0^T \int_{\Omega} |D_h(x, t)|^2 dx dt \leq C \int_0^T \int_{\omega} |D_h(x, t)|^2 dx dt. \quad (3.78)$$

Proof. We only need to prove (3.78) for $(D_0, B_0) \in \mathcal{V} \times (\hat{J}(\Omega) \cap H^1(\Omega)^3)$ since this space is dense in H_1 ([9, 10]).

Consider $\chi \in H_N(\text{curl}, \text{div}, \Omega)$ the unique solution of (with $D_1 = \text{curl}(\mu B_0)$)

$$\begin{aligned} \text{curl}(\mu \text{curl}(\lambda \chi)) &= D_1 \quad \text{in } \Omega, \\ \text{div} \chi &= 0 \quad \text{in } \Omega, \\ \chi \times \nu &= 0, \quad \text{curl}(\lambda \chi) \cdot \nu = 0 \quad \text{on } \Gamma, \end{aligned} \quad (3.79)$$

in the sense that $\chi \in H_N(\text{curl}, \text{div}, \Omega)$ is the unique solution of

$$\int_{\Omega} \{\mu \text{curl}(\lambda \chi) \cdot \text{curl}(\lambda w) + \text{div} \chi \text{div} w\} dx = \int_{\Omega} \lambda D_1 \cdot w dx, \quad \forall w \in H_N(\text{curl}, \text{div}, \Omega). \quad (3.80)$$

Set

$$w(t) = \int_0^t D_h(s) ds + \chi. \quad (3.81)$$

Then from (3.32), (3.33), (3.34), and (3.35) and (3.79), we see that w satisfies (3.32), (3.33), (3.35) and the initial conditions

$$w(0) = \chi \in \mathcal{V}, \quad w'(0) = D_0 \in \mathcal{H}. \quad (3.82)$$

Therefore by Corollary 3.5 we have

$$\frac{T - T_1}{2T} \int_0^T \int_{\Omega} \left(\lambda(x) |w'(x, t)|^2 + \mu(x) |\text{curl}(\lambda w(x, t))|^2 \right) dx dt \leq C \int_0^T \int_{\omega} |w'(x, t)|^2 dx dt. \quad (3.83)$$

This estimate directly leads to the conclusion as $w' = D_h$. $\square$

By Lemmas 3.1 and 3.6 we directly conclude the following theorem.

THEOREM 3.7. *If $\omega = \mathcal{N}_{\epsilon}(\Gamma)$, for some $\epsilon > 0$, and λ, μ satisfy (1.6), (3.8), then (3.6) holds for T large enough.*

4. The stability result

Based on the stability estimate of the previous section, we deduce our main result.

THEOREM 4.1. *Let ω be a subset of Ω such that (3.6) holds. Assume that σ satisfies (1.7). Then there exist $C \geq 1$ and $\gamma > 0$ such that*

$$\mathcal{E}(t) \leq C e^{-\gamma t} \mathcal{E}(0), \quad (4.1)$$

for every solution (D, B) of the system (1.1), (1.2), (1.3), (1.4), and (1.5) with initial datum in H_1 .

Proof. As in [24, Theorem 1.1], we split up (D, B) , solution of (1.1), (1.2), (1.3), (1.4), and (1.5) as follows:

$$(D, B) = (D_h, B_h) + (D_{nh}, B_{nh}), \quad (4.2)$$

where (D_h, B_h) is solution of (3.1), (3.2), (3.3), (3.4), and (3.5) and (D_{nh}, B_{nh}) is the remainder which then satisfies

$$\begin{aligned} D'_{nh} - \operatorname{curl}(\mu B_{nh}) &= -\sigma D \quad \text{in } \Omega \times (0, +\infty), \\ B'_{nh} + \operatorname{curl}(\lambda D_{nh}) &= 0 \quad \text{in } \Omega \times (0, +\infty), \\ \operatorname{div} B_{nh} &= 0 \quad \text{in } \Omega \times (0, +\infty), \\ D_{nh}(0) &= 0, \quad B_{nh}(0) = 0 \quad \text{in } \Omega, \\ D_{nh} \times \nu &= 0, \quad B_{nh} \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty). \end{aligned} \quad (4.3)$$

Equivalently (D_{nh}, B_{nh}) satisfies (2.16), (2.17), (2.18), (2.19), and (2.20) with $f = -\sigma D$. Therefore by Theorem 2.3, it holds

$$\int_{Q_T} \left\{ |D_{nh}(x, t)|^2 + |B_{nh}(x, t)|^2 \right\} dx dt \leq CT^2 \int_{Q_T} |\sigma D(x, t)|^2 dx dt, \quad (4.4)$$

and since σ is bounded we get

$$\int_{Q_T} \left\{ |D_{nh}(x, t)|^2 + |B_{nh}(x, t)|^2 \right\} dx dt \leq CT^2 \max_{x \in \bar{\Omega}} \sigma(x) \int_{Q_T} \sigma |D(x, t)|^2 dx dt. \quad (4.5)$$

On the other hand by (3.6) we have

$$\begin{aligned} \mathcal{E}(T) &\leq \mathcal{E}(0) = \frac{1}{2} \int_{\Omega} \left(\lambda(x) |D_0(x)|^2 + \mu(x) |B_0(x)|^2 \right) dx \\ &\leq C \int_0^T \int_{\omega} |D_h(x, t)|^2 dx dt \\ &\leq C \int_0^T \int_{\omega} \left\{ |D(x, t)|^2 + |D_{nh}(x, t)|^2 \right\} dx dt \\ &\leq \frac{C}{\sigma_0} \int_0^T \int_{\omega} \sigma |D(x, t)|^2 dx dt + C \int_0^T \int_{\omega} |D_{nh}(x, t)|^2 dx dt. \end{aligned} \quad (4.6)$$

By (4.5) we conclude that

$$\mathcal{E}(T) \leq C \int_{Q_T} \sigma |D(x, t)|^2 dx dt, \quad (4.7)$$

which leads to the conclusion due to (2.25), using a standard argument (see, e.g., [3, Theorem 3.3] or [14, Section 3]). $\square$

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