THE CAUCHY PROBLEM AND DECAY RATES FOR STRONG SOLUTIONS OF A BOUSSINESQ SYSTEM

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The Boussinesq equations describe the motion of an incompressible viscous fluid subject to convective heat transfer. Decay rates of derivatives of solutions of the three-dimensional Cauchy problem for a Boussinesq system are studied in this work.

1. Introduction

In this work we show some theoretical results about decay rates of strong solutions of the three-dimensional Cauchy problem for Boussinesq equations, described by the following partial differential equation problem (see [6]):

$$\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \nu\triangle\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla\pi &= \theta\mathbf{f} \quad \text{in } (0,T)\times\mathbb{R}^3,\\
\operatorname{div}\mathbf{u} &= 0 \quad \text{in } (0,T)\times\mathbb{R}^3,\\
\frac{\partial \theta}{\partial t} - \chi\triangle\theta + \mathbf{u}\cdot\nabla\theta &= 0 \quad \text{in } (0,T)\times\mathbb{R}^3,\\
\mathbf{u}(0,x) &= \mathbf{a}(x) \quad \text{in } \mathbb{R}^3,\\
\theta(0,x) &= b(x) \quad \text{in } \mathbb{R}^3,
\end{aligned} \tag{1.1}$$

where the unknown are $\mathbf{u}$, θ, π which denote, respectively, the velocity field, the scalar temperature and the scalar pressure. Data are the positive constants ν, χ, respectively, the viscosity and the thermal conductivity coefficients and the function $\mathbf{f}$ the external force field, and $\mathbf{a}(x)$, $b(x)$, respectively, represent the initial velocity and initial temperature.

The main objective of this work is to obtain a decay rate of derivatives for the strong solutions to the Cauchy problem (1.1). For this, we will consider the usual Lebesgue spaces $L^p(\mathbb{R}^3)$ with the usual norms $|\cdot|_p$. We will denote $L^p_\sigma(\mathbb{R}^3)$ the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3) = \{\mathbf{v}\in C^\infty_0; \operatorname{div}\mathbf{v} = 0\}$ in $L^p(\mathbb{R}^3)$. We will denote too by $L^P(0,T;L^q(\mathbb{R}^3))$ the Banach space, classes of functions defined a.e. in $[0,T]$ on $L^q(\mathbb{R}^3)$, that are L^p-integrable in the sense of Bochner. For more details see [1, 3].

We observe that this model of fluids includes as a particular case the classical Navier-Stokes equations, which has been thoroughly studied (see, e.g., [7, 8]). Rojas Medar and

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Lorca obtained results of uniqueness and existence of the local solutions and regularity of solutions for Boussinesq equations [9, 10].

Results of decay rates of strong solution were obtained by Cheng He and Ling Hsião [4]. In this paper, we are interested to get similar results for Boussinesq equations.

2. Results of decay rates

The main objective of this work is to establish the decay rates of derivatives about time variable and spaces variables for the strong solutions to the Cauchy problem of the Boussinesq equations (1.1). For this, we will consider a sequence of Cauchy problems for the linearized Boussinesq equations

$$\begin{gathered}
\frac{\partial \mathbf{u}^k}{\partial t} - \nu\triangle \mathbf{u}^k + (\mathbf{u}^{k-1}\cdot\nabla)\mathbf{u}^{k-1} + \nabla\pi^k = \theta^{k-1}\mathbf{f} \quad \text{on } (0,T)\times\mathbb{R}^3,\\
\operatorname{div}\mathbf{u}^k = 0 \quad \text{on } (0,T)\times\mathbb{R}^3,\\
\frac{\partial \theta^k}{\partial t} - \chi\triangle\theta^k + \mathbf{u}^{k-1}\cdot\nabla\theta^{k-1} = 0 \quad \text{on } (0,T)\times\mathbb{R}^3,\\
\mathbf{u}^k(0,x) = \mathbf{a}^k(x) \quad \text{on } \mathbb{R}^3,\\
\theta^k(0,x) = b^k(x) \quad \text{on } \mathbb{R}^3,
\end{gathered} \tag{2.1}$$

for $k \geq 1$, where $\mathbf{a}^k \in C^\infty_{0,\sigma}(\mathbb{R}^3)$ and $b^k \in C^\infty_0(\mathbb{R}^3)$ such that

$$\begin{gathered}
\mathbf{a}^k \longrightarrow \mathbf{a} \quad \text{in } L^3_\sigma(\mathbb{R}^3) \text{ strongly},\\
b^k \longrightarrow b \quad \text{in } L^3(\mathbb{R}^3) \text{ strongly},
\end{gathered} \tag{2.2}$$

with $|\mathbf{a}^k|_3 \leq |\mathbf{a}|_3$ and $|b^k|_3 \leq |b|_3$. The first term, $(\mathbf{u}^0, \pi^0, \theta^0)$, of this sequence is solution of the trivial Cauchy problem:

$$\begin{gathered}
\frac{\partial \mathbf{u}^0}{\partial t} - \nu\triangle \mathbf{u}^0 + \nabla\pi^0 = \mathbf{0} \quad \text{on } (0,T)\times\mathbb{R}^3,\\
\operatorname{div}\mathbf{u}^0 = 0 \quad \text{on } (0,T)\times\mathbb{R}^3,\\
\frac{\partial \theta^0}{\partial t} - \chi\triangle\theta^0 = 0 \quad \text{on } (0,T)\times\mathbb{R}^3,\\
\mathbf{u}^0(0,x) = \mathbf{a}^0(x) \quad \text{on } \mathbb{R}^3,\\
\theta^0(0,x) = b^0(x) \quad \text{on } \mathbb{R}^3.
\end{gathered} \tag{2.3}$$

Let $\Gamma_\nu(t,x;s,y) = (4\nu\pi t)^{-3/2}\exp(-|x|^2/4\nu t)$ be a fundamental solution of the heat equation in $\mathbb{R}^3$ (with viscosity coefficient ν). Then, the solution of the linearized Boussinesq

system (2.1) can be written as follows:

$$\begin{aligned} u_i^k(t,x) = &\int_{\mathbb{R}^3} \Gamma_\nu(t,x;0,y)a_i^k(y)dy \\ &- \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \sum_{j=1}^3 u_j^{k-1}(s,y) \frac{\partial u_i^{k-1}}{\partial x_j}(s,y)dy\,ds \\ &- \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \frac{\partial \pi^k}{\partial x_i}(s,y)dy\,ds \\ &+ \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y)\theta^{k-1}(s,y)f_i(s,y)dy\,ds, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \theta^k(t,x) = &\int_{\mathbb{R}^3} \Gamma_\chi(t,x;0,y)b^k(y)dy \\ &- \int_0^t \int_{\mathbb{R}^3} \Gamma_\chi(t,x;s,y) \sum_{j=1}^3 u_j^{k-1}(s,y) \frac{\partial \theta^{k-1}}{\partial x_j}(s,y)dy\,ds. \end{aligned} \tag{2.5}$$

The convergence for the above method can be seen in [2, 7].

Definition 2.1. A couple $(\mathbf{u},\theta)$ is called strong solution for the system (1.1), if

$$\begin{aligned} &\mathbf{u} \in L^p(0,\infty;L^q_\sigma(\mathbb{R}^3)) \cap L^\infty(0,\infty;L^3_\sigma(\mathbb{R}^3)), \\ &\theta \in L^p(0,\infty;L^q(\mathbb{R}^3)) \cap L^\infty(0,\infty;L^3(\mathbb{R}^3)) \end{aligned} \tag{2.6}$$

for some $p > 2$ and $q > 3$, and satisfying

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^3} \left(\mathbf{u} \cdot \frac{d\varphi}{dt} + \mathbf{u} \cdot \triangle\varphi + (\mathbf{u} \cdot \nabla\varphi)\mathbf{u}\right) dx\,dt \\ &\quad = -\int_{\mathbb{R}^3} \mathbf{a} \cdot \varphi(0,x)dx + \int_0^\infty \int_{\mathbb{R}^3} \theta\mathbf{f} \cdot \varphi\,dx\,dt \end{aligned} \tag{2.7}$$

for all $\varphi \in C_0^\infty(0,\infty;C_{0,\sigma}^\infty(\mathbb{R}^3))$ and

$$\int_0^\infty \int_{\mathbb{R}^3} \left(\theta\frac{d\psi}{dt} + \theta\triangle\psi + (\mathbf{u} \cdot \nabla\psi)\theta\right) dx\,dt = -\int_{\mathbb{R}^3} b\psi(0,x)dx \tag{2.8}$$

for all $\psi \in C_0^\infty(0,\infty;C_0^\infty(\mathbb{R}^3))$.

Lemma 2.2. *For the pressure π^k the following estimate holds:*

$$\left|\nabla\pi^k\right|_r \le C\left|(\mathbf{u}^{k-1} \cdot \nabla)\mathbf{u}^{k-1} + \theta^{k-1}\mathbf{f}\right|_r \tag{2.9}$$

for $1 < r < \infty$, $k > 0$.

Lemma 2.3. *Let* $\mathbf{a} \in L^3_\sigma(\mathbb{R}^3)$, $b \in L^3(\mathbb{R}^3)$, $|\mathbf{f}(\mathbf{t})|_q \le C_0 t^{-1+3/2q}(|\mathbf{a}|_3 + |b|_3)$, $|\nabla \mathbf{f}(t)|_q \le C_0 t^{-3/2+3/2q}(|\mathbf{a}|_3 + |b|_3)$, *where the constant* C_0 *is independent of* t *and* q. *If* $C^* C_0(|\mathbf{a}|_3 + |b|_3) \le 1$ *for some constant* C^*, *then*

$$\begin{aligned} t^{1/2-3/2q}\Big(|\mathbf{u}^k(t)|_q + |\theta^k(t)|_q\Big) &\le CC_0(|\mathbf{a}|_3 + |b|_3), \\ t^{1-3/2q}\Big(|\nabla \mathbf{u}^k(t)|_q + |\nabla \theta^k(t)|_q\Big) &\le CC_0(|\mathbf{a}|_3 + |b|_3) \end{aligned} \tag{2.10}$$

for $3 \le q \le \infty$, $t \ge 0$ *and* $k \ge 0$.

Proof. We put

$$\begin{aligned} I^k_q &= t^{1/2-3/2q}\Big(|\mathbf{u}^k(t)|_q + |\theta^k(t)|_q\Big) \\ J^k_q &= t^{1-3/2q}\Big(|\nabla \mathbf{u}^k(t)|_q + |\nabla \theta^k(t)|_q\Big). \end{aligned} \tag{2.11}$$

We will assume by inductive hypotheses that the estimates (2.10) are true for $k-1$. By the Young inequality for convolution, we can estimate the terms of (2.4) as follows:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \Gamma_\nu(t,x;0,y) a^k_i(y)\,dy \right|_q &\le (4\nu\pi t)^{-3/2}\left(\int_{\mathbb{R}^3} e^{-|x-y|^2/4\nu t}\,dy\right)^{1/p} |a^k_i|_3 \\ &\le Ct^{-1/2+3/2q}|a^k_i|_3, \end{aligned} \tag{2.12}$$

where $1/p + 1/3 = 1 + 1/q$. Again, by the Young inequality we obtain

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \sum_{j=1}^3 u^{k-1}_j(s,y) \frac{\partial u^{k-1}_i}{\partial x_j}(s,y)\,dy\,ds \right|_q \\ &\quad \le C\int_0^t (t-s)^{-3/2(1/2-1/q)} |\mathbf{u}^{k-1}|_4 |\nabla \mathbf{u}^{k-1}|_4\,ds \\ &\quad \le CC_0^2(|\mathbf{a}|_3 + |b|_3)^2 t^{-1/2+3/2q}, \end{aligned} \tag{2.13}$$

where $1/p + 1/2 = 1 + 1/q$. Now, using (2.9) we have

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \frac{\partial \pi^k}{\partial x_i}(s,y)\,dy\,ds \right|_q &\le CC_0^2(|\mathbf{a}|_3 + |b|_3)^2 t^{-1/2+3/2q}, \\ \left| \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \theta^{k-1}(s,y) f_i(s,y)\,dy\,ds \right|_q &\le CC_0^2 t^{-1+3/2q}(|\mathbf{a}|_3 + |b|_3)^2. \end{aligned} \tag{2.14}$$

Moreover

$$|\mathbf{u}^k(t)|_q \le Ct^{-1/2+3/2q}\Big(C_0(|\mathbf{a}|_3 + |b|_3) + C_0^2(|\mathbf{a}|_3 + |b|_3)^2\Big). \tag{2.15}$$

Analogously, we can obtain the estimate for $|\theta^k(t)|_q$. Now, differentiating (2.4) and using the fact

$$\left| \frac{\partial}{\partial x_l} \Gamma_\nu(t,x;s,y) \right| \le C(t-s)^{-2} e^{-\lambda|x-y|^2/4\nu(t-s)} \tag{2.16}$$

for $i = 1,2,3$ and some constant $\lambda > 0$, follows

$$\left| \frac{\partial}{\partial x_l} \int_{\mathbb{R}^3} \Gamma_\nu(t,x;0,y) a_i^k(y) dy \right|_q \le C t^{-1+3/2q} |\mathbf{a}|_3. \tag{2.17}$$

Analogously, we obtain

$$\begin{aligned} &\left| \frac{\partial}{\partial x_l} \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \sum_{j=1}^3 u_j^{k-1}(s,y) \frac{\partial u_i^{k-1}}{\partial x_j}(s,y) dy\, ds \right|_q \\ &\qquad \le C C_0^2 \left(|\mathbf{a}|_3 + |b|_3\right)^2 t^{-1+3/2q}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} &\left| \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \frac{\partial \pi^k}{\partial x_i}(s,y) dy\, ds \right|_q \\ &\qquad \le C C_0^2 \left(|\mathbf{a}|_3 + |b|_3\right)^2 t^{-1+(3/2q)} \int_0^1 (1-w)^{-1/2-3/2r} w^{-3/2+3/2(1/r+1/q)} dw, \end{aligned} \tag{2.19}$$

where $r > 3$ and $1/r + 1/q > 1/3$, to obtain the convergence of the last integral in (2.19). Finally, we obtain

$$\left| \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \theta^{k-1} f_i\, dy\, ds \right|_q \le C \int_0^t (t-s)^{-1/2+3/2(1/q-1/l)} \left| \theta^{k-1} f_i \right|_l ds. \tag{2.20}$$

Without difficulty we can obtain for the equation of the temperature

$$J_q^k \le C_0 \left(|\mathbf{a}|_3 + |b|_3\right) + C C_0^2 \left(|\mathbf{a}|_3 + |b|_3\right)^2 \le 2 C_0 \left(|\mathbf{a}|_3 + |b|_3\right). \tag{2.21}$$

For $q = \infty$ we can obtain analogously as before

$$\begin{aligned} I_\infty^k &\le C\left(|\mathbf{a}|_3 + |b|_3\right) + C t^{1/2} \int_0^t (t-s)^{-3/4} \left|\mathbf{u}^{k-1}\right|_4 \left(\left|\nabla \mathbf{u}^{k-1}\right|_4 + \left|\nabla \theta^{k-1}\right|_4 \right) ds \\ &\quad + C t^{1/2} \int_0^t (t-s)^{-3/4} \left|\theta^{k-1}\right|_4 |\mathbf{f}|_4 ds \le C\left(|\mathbf{a}|_3 + |b|_3\right) + C C_0^2 \left(|\mathbf{a}|_3 + |b|_3\right)^2. \end{aligned} \tag{2.22}$$

Similarly, we obtain the estimative for J_∞^k. □

Lemma 2.4. *Let* $\mathbf{a} \in L_\sigma^3(\mathbb{R}^3)$, $b \in L^3(\mathbb{R}^3)$ *and* $|\mathbf{f}|_q \le C_0 t^{-1+3/2q}(|\mathbf{a}|_3 + |b|_3)$, $|\nabla \mathbf{f}|_q \le C_0 t^{-3/2+3/2q}(|\mathbf{a}|_3 + |b|_3)$. *If* $C^* C_0(|\mathbf{a}|_3 + |b|_3) \le 1$ *for some constant* C^*, *then for* $3 \le q \le \infty$, *the following estimate is true uniformly in* k:

$$t^{3/2-3/2q} \sum_{l,j=1}^3 \left(\left| \frac{\partial^2 \mathbf{u}^k}{\partial x_l \partial x_j} \right|_q + \left| \frac{\partial^2 \theta^k}{\partial x_l \partial x_j} \right|_q \right) \le 2 C_0 \left(|\mathbf{a}|_3 + |b|_3\right). \tag{2.23}$$

Proof. The identity (2.4) can be written as follows:

$$\begin{aligned} u_i^k(t,x) = {} & \int_{\mathbb{R}^3} \Gamma_\nu(t,x;0,y) a_i^k(y)\,dy - \int_0^{t/2} \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \\ & \times \left(\sum_{j=1}^{3} u_j^{k-1}(s,y) \frac{\partial u_i^{k-1}}{\partial x_j}(s,y) + \frac{\partial \pi^k}{\partial x_i}(s,y) - \theta^{k-1}(s,y) f_i(s,y) \right) dy\,ds \\ & - \int_{t/2}^{t} \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \left(\sum_{j=1}^{3} u_j^{k-1}(s,y) \frac{\partial u_i^{k-1}}{\partial x_j}(s,y) \right. \\ & \qquad \left. + \frac{\partial \pi^k}{\partial x_i}(s,y) - \theta^{k-1}(s,y) f_i(s,y) \right) dy\,ds \end{aligned} \tag{2.24}$$

analogously for temperature.

We will make the case $l = j$ (for the case $l \neq j$ the argument is analogous). By the Young inequality we obtain

$$\begin{aligned} & \left| \frac{\partial^2}{\partial x_l^2} \int_{\mathbb{R}^3} \Gamma_\nu(t,x;0,y) a_i^k(y)\,dy \right|_q \\ & \quad \leq C \frac{1}{t} \left| \int_{\mathbb{R}^3} \left(\Gamma_\nu(t,x;0,y) a_i^k(y) + \frac{(x_l - y_l)^2}{t^2} \Gamma_\nu(t,x;0,y) a_i^k(y) \right) dy \right|_q \\ & \quad \leq C t^{-3/2+3/2q} |\mathbf{a}|_3. \end{aligned} \tag{2.25}$$

By analogous computations, we have

$$\begin{aligned} & \left| \frac{\partial^2}{\partial x_l^2} \int_0^{t/2} \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \sum_{j=1}^{3} u_j^{k-1}(s,y) \frac{\partial u_i^{k-1}}{\partial x_j}(s,y)\,dy \right|_q \\ & \quad \leq C C_0^2 \left(|\mathbf{a}|_3 + |b|_3 \right)^2 t^{-3/2+3/2q} \int_0^{1/2} (1-w)^{-7/4+3/2q} w^{-3/4}\,dw. \end{aligned} \tag{2.26}$$

The estimation for the terms that involve $\int_0^{t/2}$ are obtained analogously. By other side

$$\begin{aligned} & \left| \frac{\partial^2}{\partial x_l^2} \int_{t/2}^{t} \int_{\mathbb{R}^3} \Gamma_\nu(t,x;s,y) \sum_{j=1}^{3} u_j^{k-1}(s,y) \frac{\partial u_i^{k-1}}{\partial x_j}(s,y)\,dy\,ds \right|_q \\ & \quad \leq C C_0^2 \left(|\mathbf{a}|_3 + |b|_3 \right)^2 t^{-3/2+3/2q} \int_{1/2}^{1} (1-w)^{-1/2} w^{-2+3/2q}\,dw \\ & \quad + C C_0 \left(|\mathbf{a}|_3 + |b|_3 \right) \int_{1/2}^{1} (1-w)^{-1/2} w^{-2+3/2q}\,dw \\ & \quad \times \sup_{s \in [t/2,t]} \left| s^{-3/2+3/2q} \frac{\partial^2 \mathbf{u}^{k-1}}{\partial x_l^2}(s) \right|_q. \end{aligned} \tag{2.27}$$

Analogously

$$\begin{aligned}&\left|\frac{\partial^2}{\partial x_l^2}\int_{t/2}^{t}\int_{\mathbb{R}^3}\Gamma_\nu(t,x;s,y)\frac{\partial\pi}{\partial x_i}dy\,ds\right|_q\\&\quad\le C\int_{t/2}^{t}(t-s)^{-2}\left(\left|\frac{\partial}{\partial x_l}((\mathbf{u}^{k-1}\cdot\nabla)\mathbf{u}^{k-1})\right|_q+\left|\frac{\partial\theta^{k-1}\mathbf{f}}{\partial x_l}\right|_q\right)ds\end{aligned}\tag{2.28}$$

and, finally,

$$\begin{aligned}&\left|\frac{\partial^2}{\partial x_l^2}\int_{t/2}^{t}\int_{\mathbb{R}^3}\Gamma_\nu(t,x;s,y)\theta^{k-1}f_i\,dy\,ds\right|_q\\&\quad\le CC_0^2(|\mathbf{a}|_3+|b|_3)^2t^{-3/2+3/2q}\int_{1/2}^{1}(1-w)^{-1/2}w^{-2+3/2q}dw.\end{aligned}\tag{2.29}$$

The proof of Lemma 2.4 is a consequence from the above estimative. □

Lemma 2.5. *Let* $\mathbf{a}\in L^3_\sigma(\mathbb{R}^3)$, $b\in L^3(\mathbb{R}^3)$ *and* $|\mathbf{f}|_q\le C_0t^{-1+3/2q}(|\mathbf{a}|_3+|b|_3)$, $|\nabla\mathbf{f}|_q\le C_0t^{-3/2+3/2q}(|\mathbf{a}|_3+|b|_3)$. *If* $M(|\mathbf{a}|_3+|b|_3)\le 1$ *for some constant* M, *then for* $3\le q\le\infty$, *the following estimative are verified uniformly in* k:

$$t^{3/2-3/2q}\,|\nabla\pi^k|_q\le C(|\mathbf{a}|_3+|b|_3)^2,\tag{2.30}$$

$$t^{3/2-3/2q}\left(\left|\frac{\partial\mathbf{u}^k}{\partial t}\right|_q+\left|\frac{\partial\theta^k}{\partial t}\right|_q\right)\le C(|\mathbf{a}|_3+|b|_3)+C(|\mathbf{a}|_3+|b|_3)^2.\tag{2.31}$$

Proof. The proof is a consequence of Lemmas 2.2, 2.3, and 2.4, and the following facts

$$\begin{aligned}\frac{\partial\mathbf{u}^k}{\partial t}&=\nu\triangle\mathbf{u}^k-(\mathbf{u}^{k-1}\cdot\nabla)\mathbf{u}^{k-1}-\nabla\pi^k+\theta^{k-1}\mathbf{f},\\\frac{\partial\theta^k}{\partial t}&=\chi\triangle\theta^k-\mathbf{u}^{k-1}\cdot\nabla\theta^{k-1}.\end{aligned}\tag{2.32}$$

□

The main result in this paper is the following.

Theorem 2.6. *Let* $\mathbf{a}\in L^3_\sigma(\mathbb{R}^3)$, $b\in L^3(\mathbb{R}^3)$ *and* $|\mathbf{f}|_q\le C_0t^{-1+3/2q}(|\mathbf{a}|_3+|b|_3)$, $|\nabla\mathbf{f}|_q\le C_0t^{-3/2+3/2q}(|\mathbf{a}|_3+|b|_3)$. *Then, there exists a positive constant* ε *such that, if* $(|\mathbf{a}|_3+|b|_3)\le\varepsilon$, *there exists a unique solution* $(\mathbf{u},\theta)$ *for (1.1), which satisfy:*

$$\begin{aligned}t^{1/2-3/2q}\mathbf{u}&\in BC([0,\infty);L^q(\mathbb{R}^3)),\\t^{1/2-3/2q}\theta&\in BC([0,\infty);L^q(\mathbb{R}^3)),\\t^{1-3/2q}|\nabla\mathbf{u}|&\in BC([0,\infty);L^q(\mathbb{R}^3)),\\t^{1/2-3/2q}|\nabla\theta|&\in BC([0,\infty);L^q(\mathbb{R}^3)),\end{aligned}\tag{2.33}$$

for $3 \le q \le \infty$ *and moreover*

$$\begin{gathered} t^{3/2-3/2q}\sum_{i,j=1}^{3}\left|\frac{\partial^2\mathbf{u}}{\partial x_j\partial x_i}\right| \in BC([0,\infty);L^q(\mathbb{R}^3)), \\ t^{3/2-3/2q}\sum_{i,j=1}^{3}\left|\frac{\partial^2\theta}{\partial x_j\partial x_i}\right| \in BC([0,\infty);L^q(\mathbb{R}^3)), \\ t^{3/2-3/2q}|\nabla\pi| \in BC([0,\infty);L^q(\mathbb{R}^3)), \\ t^{3/2-3/2q}\frac{\partial\mathbf{u}}{\partial t} \in BC([0,\infty);L^q(\mathbb{R}^3)), \end{gathered} \tag{2.34}$$

for $3 \le q \le \infty$.

Proof. Using Lemma 2.3 with $q = 3$, we obtain

$$\begin{gathered} |\mathbf{u}^k|_3 + |\theta^k|_3 \le C(|\mathbf{a}|_3 + |b|_3), \\ |\nabla\mathbf{u}^k|_3 + |\nabla\theta^k|_3 \le Ct^{-1/2}(|\mathbf{a}|_3 + |b|_3) \end{gathered} \tag{2.35}$$

then, for $1 < p < 2$ it is easy to show

$$\begin{gathered} \mathbf{u}^k \in L^\infty(0,\infty;L^3(\mathbb{R}^3)), \\ \theta^k \in L^\infty(0,\infty;L^3(\mathbb{R}^3)), \\ \nabla\mathbf{u}^k \in L^p_{\mathrm{loc}}(0,\infty;L^3(\mathbb{R}^3)), \\ \nabla\theta^k \in L^p_{\mathrm{loc}}(0,\infty;L^3(\mathbb{R}^3)). \end{gathered} \tag{2.36}$$

Let q and q^* such that $1/q + 1/q^* = 1$ and we consider the following estimative:

$$\begin{gathered} \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle\triangle\mathbf{u}^k,\mathbf{v}\rangle| \le \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\nabla\mathbf{u}^k|_q|\nabla\mathbf{v}|_{q^*} \le |\nabla\mathbf{u}^k|_q, \\ \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle(\mathbf{u}^{k-1}\cdot\nabla)\mathbf{u}^{k-1},\mathbf{v}\rangle| \le \left||\mathbf{u}^{k-1}|^2\right|_q, \\ \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle\nabla\pi^k,\mathbf{v}\rangle| \le |\pi^k|_q. \end{gathered} \tag{2.37}$$

Now, using Hölder and Sobolev inequalities, we have

$$\begin{aligned} \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle\theta^{k-1}\mathbf{f},\mathbf{v}\rangle| &\le C\sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\theta^{k-1}\mathbf{f}|_{3q/(q+3)}|\nabla\mathbf{v}|_{q^*} \\ &= C|\theta^{k-1}\mathbf{f}|_{3q/(q+3)}. \end{aligned} \tag{2.38}$$

Thus, by using Lemma 2.4 together with Sobolev and Hölder inequalities, we obtain

$$\left|\frac{\partial\mathbf{u}^k}{\partial t}\right|_{W^{-1,q}} \le |\nabla\mathbf{u}^k|_q + C|\mathbf{u}^{k-1}|_3|\nabla\mathbf{u}^{k-1}|_q + C|\theta^{k-1}|_3|\mathbf{f}|_q, \tag{2.39}$$

consequently,

$$\left|\frac{\partial \mathbf{u}^k}{\partial t}\right|_{W^{-1,q}} \leq Ct^{-1+3/2q}. \tag{2.40}$$

Analogously the following inequality can be proved for the temperature θ^k

$$\left|\frac{\partial \theta^k}{\partial t}\right|_{W^{-1,q}(\mathbb{R}^3)} \leq Ct^{-1+3/2q}. \tag{2.41}$$

Therefore, for $1 < r < 2q/(2q-3)$, we have

$$\begin{aligned} \frac{\partial \mathbf{u}^k}{\partial t} &\in L^r_{\mathrm{loc}}(0,\infty; W^{-1,q}(\mathbb{R}^3)), \\ \frac{\partial \theta^k}{\partial t} &\in L^r_{\mathrm{loc}}(0,\infty; W^{-1,q}(\mathbb{R}^3)). \end{aligned} \tag{2.42}$$

By using the compact embedding of $W^{1,3}(\mathbb{R}^3)$ on $L^3_{\mathrm{loc}}(\mathbb{R}^3)$ and the Compactness Theorem in [11, Cap. 3], we obtain that there exists $(\mathbf{u},\theta)$ such that

$$\begin{aligned} \mathbf{u}^k &\longrightarrow \mathbf{u} \quad \text{in } L^2_{\mathrm{loc}}(0,\infty; L^3_{\mathrm{loc}}(\mathbb{R}^3)) \text{ strongly}, \\ \theta^k &\longrightarrow \theta \quad \text{in } L^2_{\mathrm{loc}}(0,\infty; L^3_{\mathrm{loc}}(\mathbb{R}^3)) \text{ strongly}. \end{aligned} \tag{2.43}$$

Now, using the standard arguments, it is easily to show that $(\mathbf{u},\theta)$ is a unique solution of (1.1) (see [5]). □

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