CRITICAL SINGULAR PROBLEMS ON UNBOUNDED DOMAINS

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We present some results of existence for the following problem: $-\Delta u = a(x)g(u) + u|u|^{2^*-2}$, $x \in \mathbb{R}^N$ ($N \geq 3$), $u \in D^{1,2}(\mathbb{R}^N)$, where the function a is a sign-changing function with a singularity at the origin and g has growth up to the Sobolev critical exponent $2^* = 2N/(N-2)$.

1. Introduction

Recently many works have been devoted to the study of existence of positive solutions u of the equation

$$-\Delta u = a(|x|)g(u), \quad x \in \mathbb{R}^N \ (N \geq 3), \ u \in H^1(\mathbb{R}^N), \quad (1.1)$$

with a continuous function a and a subcritical growth function g . This type of equation includes the Makutuma equation, when $a(|x|) = 1/(1 + |x|^2)$ and $g(s) = |s|^{p-1}s$, with $1 < p < 2^* - 1 = (N+2)/(N-2)$, which appears in astrophysics and scalar curvature equations on $\mathbb{R}^N$ (see, e.g., [15, 17, 18])

In [16] Munyamarere and Willem obtained a result of multiplicity of nodal solutions for these equations, considering the function a nonnegative and radially symmetric. The authors worked with a subspace of radial functions of $H^1(\mathbb{R}^N)$ which has the compactness properties desired to handle a problem like this modelled on an unbounded domain. In the same direction, Alama and Tarantello [2] studied the following problem when a is not radially symmetric and changes sign (see also [1, 7, 21, 22]):

$$-\Delta u - \lambda u = a(x)g(u), \quad x \in \Omega \subseteq \mathbb{R}^N \ (N \geq 3), \ u \in H^1(\Omega), \quad (1.2)$$

where Ω is a bounded domain and g behaves at infinity like a power function, $g(s) \simeq |s|^{p-1}s$, with $1 < p < (N+2)/(N-2)$ (subcritical case).

The above results on a bounded domain were extended, in part, by Costa and Tehrani in [12] for the whole space $\mathbb{R}^N$. They considered a weighted eigenvalue problem, namely,

$$-\Delta u = \lambda h(x)u, \quad x \in \mathbb{R}^N, \ u \in H^1(\mathbb{R}^N) \ (N \geq 3), \quad (1.3)$$

with $0 \leq h \in L^{N/2}(\mathbb{R}^N) \cap L^\alpha(\mathbb{R}^N)$, $\alpha > N/2$, which has the same properties as the eigenvalue problem for $-\Delta$ in a bounded domain (see, e.g., [11]). With the aid of this information, they studied the problem

$$-\Delta u - \lambda h(x)u = a(x)g(u), \quad x \in \mathbb{R}^N (N \geq 3), \quad u \in H^1(\mathbb{R}^N). \quad (1.4)$$

Recently, still in the subcritical case, Tehrani in [23], considering problem (1.4) with $h = 0$ and Ω an unbounded exterior domain $\Omega = \mathbb{R}^N \setminus \overline{O}$ with $\overline{O} \neq \emptyset$, obtained similar results to those papers above.

Our main purpose in this work is to study the problem

$$-\Delta u = a(x)g(u) + u|u|^{2^*-2}, \quad x \in \mathbb{R}^N (N \geq 3), \quad u \in D^{1,2}(\mathbb{R}^N), \quad (1.5)$$

where the Hilbert space $D^{1,2}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ endowed with the norm $\|u\| = (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^{1/2}$.

The above kind of problem is important since it is related to conformal deformations of Riemannian structures on noncompact manifolds (see, e.g., [14]). Also, it is a physical model that appears when one describes the dynamics of galaxies (see, e.g., [4]).

It is relevant to remark that our concern to study this type of problem with a function a changing sign comes from the following fact: if $u \in D^{1,2}(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)$ is a positive solution of (1.5), using a generalized Pohazev identity (see [8, Proposition 1]), we have

$$\int_{\mathbb{R}^N} (a(x)g(u)u + u^{2^*}) dx = 2^* \int_{\mathbb{R}^N} \left(a(x)G(u) + \frac{1}{2^*} u^{2^*} \right) dx, \quad (1.6)$$

where $G(t) = \int_0^t g(s)ds$. Thus, for instance, if $g(s) = |s|^{p-2}s$, $2 < p < 2^*$, then a must change sign.

We would like to mention that when $a \in L^{2^*/(2^*-2)}(\mathbb{R}^N)$, Benci and Cerami in [6] studied the case $a \leq 0$ on $\mathbb{R}^N$, while in [19] Pan treated the case $a > 0$, and a case when a changes sign was handled by Ben-Naoum et al. in [5].

Our contribution to the study of these problems relay on the fact that we are working with a sign-changing discontinuous function a and with nonlinearities defined on the whole space $\mathbb{R}^N$ involving critical Sobolev exponent growth. These conditions imply a series of restrictions on the usual methods of dealing with these problems since the compactness of the Sobolev embedding is lost. In our case, a Hardy-type inequality is demanded. We would like to point out that our approach, with the corresponding changes, also works replacing $\mathbb{R}^N$ by a bounded or unbounded domain Ω . Finally we note that our work is precisely a version of the classical result of Brézis and Nirenberg (see [10]) considered under the aforementioned conditions.

Before stating our main theorem, we have to precise the set of assumptions on the functions g and a :

(i) $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying.

$$g(s) = o(|s|) \quad \text{as } s \rightarrow 0, \quad (1.7)$$

$$\lim_{|s| \rightarrow +\infty} \frac{sg(s)}{|s|^p} = 1, \quad \text{for some } 2 < p < 2^*, \quad (1.8)$$

$$g(s) > 0, \quad \forall s > 0, \quad (1.9)$$

$$\lim_{\varepsilon \rightarrow 0} \int_0^{\varepsilon^{-1/2}} G\left(\frac{\varepsilon^{-1/2}}{1+s^2}\right)^{(N-2)/2} s^{N-1} ds = \infty. \quad (1.10)$$

(ii) $a : \mathbb{R}^N \rightarrow \mathbb{R}$ is a sign-changing function such that

$$a(x) = O(|x|^{-\alpha}), \quad \text{as } |x| \rightarrow 0, \text{ for some } 0 < \alpha \leq \frac{2N - p(N-2)}{2}, \quad (1.11)$$

$$a(x) = O(|x|^{-2}), \quad \text{as } |x| \rightarrow +\infty, \quad (1.12)$$

$$a \text{ is a continuous function in } 1 \leq |x| \leq M \text{ for some } M > 0, \quad (1.13)$$

$$a \in L(\mathbb{R}^N - B_\rho(0)) \quad \text{for some } \rho > 0, \quad (1.14)$$

($B_r(a)$ denotes a ball with radius r centered at a)

$$a(x) < 0, \quad \text{for } |x| \geq R_0, \quad (1.15)$$

$$a(x) > 0, \quad \text{for } |x| \leq R_0 - \delta, \quad (1.16)$$

where $R_0 > M$ and $\delta > 0$ is small.

We also require that $\Omega^0 = \{x \in \mathbb{R}^N; a(x) = 0\}$ have “thick” zero measure, that is,

$$\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset, \quad (1.17)$$

where $\Omega^+ = \{x \in \mathbb{R}^N; a(x) > 0\}$, and $\Omega^- = \{x \in \mathbb{R}^N; a(x) < 0\}$.

Our main theorem is the following.

THEOREM 1.1. *Suppose that (1.7)–(1.17) hold. Then problem (1.5) has a positive solution.*

Remark 1.2. In addition to the hypotheses of Theorem 1.1, assuming that g is odd, problem (1.5) has infinitely many solutions. This follows by applying the classical genus theory, more exactly, a critical point theorem for even functional due to Rabinowitz (see [20]).

2. Variational framework

We are going to employ the variational methods to find a nontrivial weak solution for problem (1.5). To start, we define the Euler-Lagrange functional associated to it.

Let $\Psi : D^{1,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ be defined by

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} a(x)G(u)dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx, \quad (2.1)$$

where $G(s) = \int_0^s g(t)dt$.

In order to guarantee that Ψ is well defined, we need the following Hardy-type inequality (see [13]).

PROPOSITION 2.1. *For $N \geq 2$, there exists a constant $C = C(N)$ such that*

$$\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad (2.2)$$

for all $u \in D^{1,2}(\mathbb{R}^N)$.

We check that the functional Ψ is well defined. Hereafter, we denote by C a generic positive constant. By (1.7) and (1.8), we have that

$$\int_{\mathbb{R}^N} a(x)G(u)dx \leq C \left(\int_{\mathbb{R}^N} |a(x)||u|^2 dx + \int_{\mathbb{R}^N} |a(x)||u|^p dx \right) \equiv I_1 + I_2. \quad (2.3)$$

We check that I_1 is finite. Since $(2N - p(N - 2))/2 < 2$, we have by (1.11) and (2.2) that

$$\int_{|x| \leq 1} |a(x)||u|^2 dx \leq C \int_{|x| \leq 1} \frac{|u|^2}{|x|^\alpha} dx \leq C \int_{|x| \leq 1} \frac{|u|^2}{|x|^2} dx \leq C. \quad (2.4)$$

By (1.12) and (2.2),

$$\int_{|x| \geq M} |a(x)||u|^2 dx \leq \int_{|x| \geq M} (|a(x)||x|^2) \left(\frac{|u|^2}{|x|^2} \right) dx \leq C \int_{|x| \geq M} \frac{|u|^2}{|x|^2} dx \leq C. \quad (2.5)$$

Hence, by (2.4), (2.5), and (1.13), we have $I_1 < \infty$.

Choosing $r = 2N/(2N - p(N - 2))$, by Hölder's inequality and, respectively, by (1.11) and (1.12), we have

$$\int_{|x| \leq 1} a(x)|u|^p dx \leq \left(\int_{|x| \leq 1} \frac{1}{|x|^{\alpha r}} dx \right)^{1/r} \left(\int_{|x| \leq 1} |u|^{2^*} dx \right)^{p/2^*} \leq C, \quad (2.6)$$

$$\int_{|x| \geq M} a(x)|u|^p dx \leq \left(\int_{|x| \geq M} \frac{dx}{|x|^{2r}} \right)^{1/r} \left(\int_{|x| \geq M} |u|^{2^*} dx \right)^{p/2^*} \leq C. \quad (2.7)$$

By (2.6), (2.7), and (1.13), we achieve that $I_2 < \infty$.

Therefore, Ψ is well defined and under the assumptions on the nonlinearities, a straightforward computation yields that $\Psi \in C^1(D^{1,2}(\mathbb{R}^N))$ and that for $v \in D^{1,2}(\mathbb{R}^N)$, we have

$$\langle \Psi'(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v \, dx - \int_{\mathbb{R}^N} a(x)g(u)v \, dx - \int_{\mathbb{R}^N} vu|u|^{2^*-2} \, dx. \quad (2.8)$$

Hence, the critical points of Ψ are precisely the weak solutions for (1.5) and *vice versa*.

We also point out that with convenient hypotheses on the nonlinearities it is possible to obtain some regularization of the solutions.

3. Obtaining critical points Ψ

We are going to find a solution as a critical point of the functional Ψ . Before proceeding, we assure that the solution that we will find is indeed positive. Taking

$$\tilde{g}(u) = \begin{cases} g(u), & \text{if } u \geq 0, \\ g(-u), & \text{if } u < 0, \end{cases} \quad (3.1)$$

and using from now on the function $\tilde{g}(u)$, the critical point of Ψ is such that $u \geq 0$.

Now applying the maximum principle to the equation

$$-\Delta u - a^-(x)\tilde{g}(u) = a^+(x)\tilde{g}(u) + u_+^{2^*-1}, \quad x \in \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N), \quad (3.2)$$

we infer that u must be positive ($a^+ = \max\{a, 0\}$ and $a^- = a - a^+$).

For simplicity, in what follows, the function $\tilde{g}$ will be denoted by g .

Returning to the functional Ψ , let $E = D^{1,2}(\mathbb{R}^N)$ and we firstly check that under our hypotheses, Ψ has the mountain pass geometry, that is,

$$\exists \beta > 0, \rho > 0 \text{ s.t. } \Psi(u) \geq \rho \quad \text{if } \|u\| = \beta, \quad (3.3)$$

$$\Psi(0) = 0, \exists e \in E, \quad \|e\| > \beta \text{ s.t. } \Psi(e) \leq 0. \quad (3.4)$$

PROPOSITION 3.1. *If (1.7), (1.8), (1.11), (1.12), and (1.13) hold, then (3.3) and (3.4) also hold.*

Proof of (3.3). By (1.7) and (1.8), for any $\varepsilon > 0$, there exists a constant $C = C(\varepsilon, p) > 0$ such that

$$|G(s)| \leq \varepsilon |s|^2 + C |s|^p, \quad s \in \mathbb{R}. \quad (3.5)$$

Hence, by estimates (2.4), (2.5), (2.6), and (2.7), together with the last inequality, we have

$$\Psi(u) \geq \frac{\|u\|^2}{2} - C \left\{ \varepsilon \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx + \left(\int_{\mathbb{R}^N} |u|^{2^*} \, dx \right)^{p/2^*} \right\}. \quad (3.6)$$

By (2.2) and the Sobolev embedding, for $\|u\|$ sufficiently small, we achieve that

$$\Psi(u) \geq \frac{\|u\|^2}{2} - C(\varepsilon\|u\|^2 + \|u\|^p) \geq \tilde{C}\|u\|^2, \quad (3.7)$$

for some constant $\tilde{C} > 0$ and $\varepsilon > 0$ small enough. Therefore, (3.3) holds. $\square$

Proof of (3.4). Hypothesis (1.8) implies that

$$0 < \theta G(s) \leq sg(s), \quad |s| \geq s_0, \text{ for some } s_0 > 0, \quad 2 < \theta < 2^*, \quad (3.8)$$

which, on its turn, implies that there exists $A > 0$ such that

$$|G(s)| \geq A|s|^\theta \quad \text{for } |s| \geq s_0. \quad (3.9)$$

Therefore, if $0 \leq \xi \in C_0^\infty(\Omega^+)$, by (3.9), we have that

$$\Psi(t\xi) = \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla \xi|^2 dx - \int_{\mathbb{R}^N} a(x)G(t\xi)dx - \frac{t^{2^*}}{2} \int_{\mathbb{R}^N} |\xi|^{2^*} dx \longrightarrow -\infty, \quad (3.10)$$

as $t \rightarrow \infty$. $\square$

Since (3.3) and (3.4) hold, by the mountain pass theorem without the Palais-Smale condition ((PS) condition, for short) (see [3]), if

$$\Gamma = \{\gamma \in C([0, 1], E); \gamma(0) = 0, \gamma(1) = e\}, \quad (3.11)$$

$$c := \inf_{P \in \Gamma} \max_{w \in P} \Psi(w) \geq \rho, \quad (3.12)$$

then there exists a sequence $(u_n) \subset E$ such that

$$\Psi(u_n) \longrightarrow c \quad \text{in } \mathbb{R}, \text{ as } n \longrightarrow \infty, \quad (3.13)$$

$$\Psi'(u_n) \longrightarrow 0 \quad \text{in } E', \text{ as } n \longrightarrow \infty, \quad (3.14)$$

where Ψ' is the Frechet derivative of Ψ and E' is the dual space of E .

We define

$$S = \inf_{u \in E \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{2/2^*}}. \quad (3.15)$$

In the following result, we are going to prove that there exists $w \in E$ such that the constant c in (3.12) may be chosen is such a way that $c < (1/N)S^{N/2}$.

PROPOSITION 3.2. *Suppose that (1.7)–(1.15) hold. Then there exists $u_0 \in E \setminus \{0\}$ such that*

$$\sup_{t \geq 0} \Psi(tu_0) < \frac{1}{N} S^{N/2} \quad (3.16)$$

Proof. Some ideas that follow in this proof were borrowed from [10]. We present them for completeness of the work.

We have that $a(x) > 0$ in $B_{R_0-\delta}(0)$. We choose a cutoff function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\text{supp } \varphi \subset B_{2R}(x_0) \subset (B_{R_0-\delta}(0) - \{0\})$, $\varphi \equiv 1$ on $B_R(x_0)$ and $0 \leq \varphi \leq 1$ on $B_{2R}(x_0)$, for some convenient open ball $B_{2R}(x_0)$. For $\varepsilon > 0$, if

$$U_\varepsilon(x) = \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}}, \quad (3.17)$$

it is well known that

$$\int_{\mathbb{R}^N} |\nabla U_\varepsilon|^2 dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{2^*} dx = S^{N/2}, \quad (3.18)$$

$$\int_{B_R(x_0)} |\nabla U_\varepsilon|^2 dx \leq \int_{B_R(x_0)} |U_\varepsilon|^{2^*} dx. \quad (3.19)$$

If we define $\eta_\varepsilon = \varphi U_\varepsilon$, it is easy to prove that

$$\int_{\mathbb{R}^N - B_R(x_0)} |\nabla \eta_\varepsilon|^2 dx = O(\varepsilon^{(N-2)/2}), \quad \text{as } \varepsilon \rightarrow 0. \quad (3.20)$$

To rewrite Ψ in a convenient way, let

$$v_\varepsilon = \frac{\eta_\varepsilon}{(\int_{B_{2R}(x_0)} |\eta_\varepsilon|^{2^*} dx)^{1/2^*}}, \quad \chi_\varepsilon = \int_{\mathbb{R}^N} |\nabla v_\varepsilon|^2 dx. \quad (3.21)$$

With this notation, it is forward to check that Ψ is bounded from above and that $\lim_{t \rightarrow \infty} \Psi(tv_\varepsilon) = -\infty$, for all $\varepsilon > 0$. So there exists $t_\varepsilon \geq 0$ such that

$$\sup_{t \geq 0} \Psi(tv_\varepsilon) = \Psi(t_\varepsilon v_\varepsilon). \quad (3.22)$$

Then differentiating $\Psi(tv_\varepsilon)$, we achieve that

$$t_\varepsilon \chi_\varepsilon - t_\varepsilon^{2^*} - \int_{B_{2R}(x_0)} a(x) g(t_\varepsilon v_\varepsilon) dx = 0 \quad (3.23)$$

and hence that

$$t_\varepsilon \leq \chi_\varepsilon^{1/(2^*-1)}. \quad (3.24)$$

Also note that by (3.18), (3.19), (3.20), and (3.24), it follows that

$$\chi_\varepsilon \leq S + O(\varepsilon^{(N-2)/2}). \quad (3.25)$$

On the other hand, the function $t \rightarrow t^2 t_0^{2^*-2}/2 - t^{2^*}/2^*$ is increasing on the interval $[0, t_0]$, where $t_0 = \chi_\varepsilon^{1/(2^*-2)}$. Then assertion (3.22) together with the above inequalities implies that

$$\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - \int_{B_{2R}(x_0)} a(x) G(t_\varepsilon v_\varepsilon) dx. \quad (3.26)$$

By (1.7) and (1.8), for all $\tau > 0$ sufficiently small, there exists $C > 0$ satisfying $|a(x)g(u)| \leq C|u| + \tau|u|^{2^*-1}$, for all $x \in B_{2R}(x_0)$.

Thus

$$\left| \int_{B_{2R}(x_0)} \frac{a(x)g(t_\varepsilon v_\varepsilon)}{t_\varepsilon} dx \right| \leq \tau t_\varepsilon^{2^*-1} |v_\varepsilon|_{2^*}^{2^*} + C |v_\varepsilon|_2^2, \quad (3.27)$$

for τ sufficiently small.

Recalling that

$$|v_\varepsilon|_2^2 = \begin{cases} O(\varepsilon) & \text{if } N \geq 5, \\ O(\varepsilon \log \varepsilon) & \text{if } N = 4, \\ O(\varepsilon^{1/2}) & \text{if } N = 3, \end{cases} \quad (3.28)$$

from (3.27), we obtain

$$\int_{B_{2R}(x_0)} \frac{a(x)g(t_\varepsilon v_\varepsilon)}{t_\varepsilon} dx \longrightarrow 0, \quad \text{as } \varepsilon \longrightarrow 0. \quad (3.29)$$

Using this fact in (3.23), we conclude that

$$t_\varepsilon \longrightarrow S^{1/(2^*-2)}, \quad \text{as } \varepsilon \longrightarrow 0. \quad (3.30)$$

Now, using (3.18)–(3.20) and (3.30), we have

$$\int_{\mathbb{R}^N} a(x)G(t_\varepsilon v_\varepsilon) dx \geq C \int_{B_{2R}(x_0)} G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}}\right) dx, \quad (3.31)$$

for some positive constants c and C . Substituting (3.31) in (3.26), we get

$$\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N} S^{N/2} + O(\varepsilon^{(N-2)/2}) - C \int_{B_{2R}(x_0)} G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}}\right) dx. \quad (3.32)$$

But

$$J_\varepsilon \equiv \frac{1}{\varepsilon^{(N-2)/2}} \int_{B_{2R}(x_0)} G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + |x - x_0|^2]^{(N-2)/2}}\right) dx \longrightarrow \infty, \quad \text{as } \varepsilon \longrightarrow 0. \quad (3.33)$$

In fact, since

$$J_\varepsilon = \frac{\omega_N}{\varepsilon^{(N-2)/2}} \int_0^R G\left(\frac{c\varepsilon^{(N-2)/4}}{[\varepsilon + r^2]^{(N-2)/2}}\right) r^{N-1} dr, \quad (3.34)$$

(ω_N is the area of S^{N-1}) making the change of variables $r = \varepsilon^{1/2}s$ and rescaling ε , we get

$$J_\varepsilon = \varepsilon \omega_N \int_0^{R\varepsilon^{-1/2}} G\left(\left[\frac{\varepsilon^{-1/2}}{1+s^2}\right]^{(N-2)/2}\right) s^{N-1} ds \quad (3.35)$$

for some constant $R > 0$.

Then, if $R \geq 1$, using (3.35), assertion (3.33) follows directly from hypothesis (1.10). If $R < 1$, consider

$$Z_\varepsilon = \varepsilon \int_{R\varepsilon^{-1/2}}^{\varepsilon^{-1/2}} G\left(\left[\frac{\varepsilon^{-1/2}}{1+s^2}\right]^{(N-2)/2}\right) s^{N-1} ds. \quad (3.36)$$

Hence, there is $c > 0$ such that

$$|Z_\varepsilon| \leq c\varepsilon G(c\varepsilon^{(N-2)/4}) \varepsilon^{-N/2} \quad (3.37)$$

which implies, due to the growth of g , that $|Z_\varepsilon|$ is bounded as $\varepsilon \rightarrow 0$. Consequently, in the case $R < 1$, since

$$\int_0^{R\varepsilon^{-1/2}} = \int_0^{\varepsilon^{-1/2}} - \int_{R\varepsilon^{-1/2}}^{\varepsilon^{-1/2}}, \quad (3.38)$$

and the last integral is bounded, as $\varepsilon \rightarrow 0$, it follows that (3.33) is a consequence of (3.37) and, again, of hypothesis (1.10).

Finally, applying (3.33) in (3.32), we see that

$$\Psi(t_\varepsilon u_\varepsilon) \leq \frac{1}{N} S^{N/2} \quad (3.39)$$

for small $\varepsilon > 0$, as desired. $\square$

Next we prove the following.

PROPOSITION 3.3. *If $(u_n) \subset D^{1,2}(\mathbb{R}^N)$ is a sequence such that (3.13) and (3.14) hold, then there exists a subsequence $u_n \rightharpoonup u_0$ weakly in $D^{1,2}(\mathbb{R}^N)$, as $n \rightarrow \infty$, for some $u_0 \in D^{1,2}(\mathbb{R}^N)$.*

Proof. The proof finishes if we prove that (u_n) is bounded. Suppose, on the contrary, that (u_n) is not bounded in $D^{1,2}(\mathbb{R}^N)$. We may assume that

$$\|u_n\| \equiv t_n \longrightarrow +\infty, \quad \text{as } n \longrightarrow \infty. \quad (3.40)$$

Define $v_n = u_n/t_n$. By (3.13) and (3.14), we achieve that

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx - \int_{\mathbb{R}^N} \frac{aG(u_n)}{t_n^2} dx - \frac{1}{2^*} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx = o_n(1) \quad (3.41)$$

and for all $v \in D^{1,2}(\mathbb{R}^N)$, we get that

$$\int_{\mathbb{R}^N} \nabla v_n \nabla v dx - \int_{\mathbb{R}^N} a \frac{g(u_n)}{t_n} v dx - \int_{\mathbb{R}^N} \frac{|u_n|^{2^*-2} u_n v}{t_n} dx = \frac{o_n(1) \|v\|}{t_n}. \quad (3.42)$$

Since $\|v_n\| = 1$, by (3.41), we have

$$\int_{\mathbb{R}^N} \frac{aG(u_n)}{t_n^2} dx + \frac{1}{2^*} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx = \frac{1}{2} + o_n(1), \quad (3.43)$$

and taking $v = v_n$ in (3.42), we infer that

$$\int_{\mathbb{R}^N} \frac{ag(u_n)u_n}{t_n^2} dx + \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx = 1 + o_n(1). \quad (3.44)$$

Observe that, combining (3.43) and (3.44) together with (1.9), we may assume that

$$\text{supp } a \cap \text{supp } g(u_n) \neq \emptyset, \quad \text{supp } a \cap \text{supp } G(u_n) \neq \emptyset. \quad (3.45)$$

From (3.43) and (3.44), we get that

$$\left(\frac{2}{2^*} - 1\right) \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx = \int_{\mathbb{R}^N} \frac{a(g(u_n)u_n - 2G(u_n))}{t_n^2} dx + o_n(1). \quad (3.46)$$

Observe that, by (1.7) and (1.8), we have

$$|g(s)| \leq C_1|s| + C_1|s|^q, \quad s \in \mathbb{R}, \quad 1 < q < 2^* - 1, \quad (3.47)$$

and hence, for a given ε , there exists a $K > 0$ such that

$$|g(t)t - 2G(t)| < \varepsilon|t|^{2^*} \quad \text{for } |t| \geq K. \quad (3.48)$$

The last integral in (3.46) may be split as

$$\begin{aligned} & \int_{|u_n| \leq K} \frac{a(g(u_n)u_n - 2G(u_n))}{t_n^2} dx + \int_{|u_n| \geq K} \frac{a^+(g(u_n)u_n - 2G(u_n))}{t_n^2} dx \\ & - \int_{|u_n| \geq K} \frac{a^-(g(u_n)u_n - 2G(u_n))}{t_n^2} dx. \end{aligned} \quad (3.49)$$

We bound these integrals. Since (1.14) holds, the first integral is $o_n(1)$; the second, taking $K > s_0$ in (3.9), is nonnegative, and the last one, by (3.48), is bounded as follows:

$$\int_{|u_n| \geq K} \frac{a^-(g(u_n)u_n - 2G(u_n))}{t_n^2} dx \leq \varepsilon \|a^-\|_\infty \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2}. \quad (3.50)$$

Using these facts in (3.46), we have

$$\left(\left(\frac{2}{2^*} - 1 \right) + \varepsilon \|a^-\|_\infty \right) \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx \geq o_n(1). \quad (3.51)$$

Thereafter, picking a small ε , we conclude that

$$\int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx \longrightarrow 0. \quad (3.52)$$

We use this limit to contradict the fact that $\|u_n\| \rightarrow \infty$.

We also may consider that there exists $v \in D^{1,2}(\mathbb{R}^N)$ such that

$$v_n \longrightarrow v \quad \text{a.e. in } \mathbb{R}^N, \quad (3.53)$$

and for all bounded sets $U \subset \mathbb{R}^N$ and for $1 \leq t < 2^*$,

$$\begin{aligned} v_n &\longrightarrow v \quad \text{in } L^t(U), \\ v_n(x) &\longrightarrow v(x), \quad \text{for } x \in U \text{ a.e.}, \\ |v_n(x)| &\leq h(x) \quad \text{for } h \in L^t(U), \text{ and a.e. in } U, \end{aligned} \quad (3.54)$$

as $n \rightarrow \infty$.

In the sequel, we need the following claim which will be proved at the end of this proof.

Claim. $v \equiv 0$.

Proceeding, we take $\xi \in C_0^\infty(\mathbb{R}^N)$. Inserting $v = v_n \xi$ in (3.42) and using the claim, we get

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 \xi dx - \int_{\mathbb{R}^N} a \frac{g(u_n)u_n}{t_n^2} \xi dx - \int_{\mathbb{R}^N} |u_n|^{2^*-2} v_n^2 \xi dx = o_n(1). \quad (3.55)$$

We choose the cutoff function $\xi \in C_0^\infty(\mathbb{R}^N)$ such that $\xi \equiv 1$ on Ω^+ and $\xi \equiv 0$ on Ω^- .

Using (3.8) and (3.55), together with (3.45), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{aG(u_n)\xi}{t_n^2} dx &= \int_{|u_n| \leq s_0} \frac{aG(u_n)}{t_n^2} \xi dx + \int_{|u_n| > s_0} \frac{a(G(u_n))}{t_n^2} \xi dx \\ &\leq o_n(1) + \frac{1}{\theta} \int_{|u_n| > s_0} \frac{ag(u_n)u_n \xi}{t_n^2} dx \\ &= o_n(1) + \frac{1}{\theta} \left[\int_{\mathbb{R}^N} |\nabla v_n|^2 \xi dx - \int_{\mathbb{R}^N} |u_n|^{2^*-2} v_n^2 \xi dx \right] \\ &\leq o_n(1) + \frac{1}{\theta} - \frac{1}{\theta} \int_{\mathbb{R}^N} \frac{u_n^{2^*} \xi}{t_n^2} dx. \end{aligned} \quad (3.56)$$

The above inequality together with (3.8) and (3.42) yields

$$\left(\frac{1}{2} - \frac{1}{\theta}\right) \leq \frac{1}{2^*} \int_{\mathbb{R}^N} \frac{|u_n|^{2^*}}{t_n^2} dx + o_n(1), \quad (3.57)$$

which contradicts (3.52).

Proof of the claim. We are going to prove that $v(x) = 0$ a.e. for $x \in \Omega^+$, arguing by contradiction. Let $F = \{x \in \mathbb{R}^N; v(x) \neq 0\}$ and we suppose that there exists $B_r(x_0)$ such that

$$|F \cap B_{2r}(x_0)| > 0, \quad (3.58)$$

where $|\cdot|$ denotes the Lebesgue measure defined in $\mathbb{R}^N$.

Pick $\xi \in C_0^\infty(B_{2r}(x_0))$ such that $\xi(x) = 1$ if $x \in B_r(x_0)$. Replacing $v = v_n \xi$ in (3.42), we have

$$\begin{aligned} & \int_{\mathbb{R}^N} |\nabla v_n|^2 \xi dx + \int_{\mathbb{R}^N} v_n \nabla v_n \nabla \xi dx - \int_{\mathbb{R}^N} |u_n|^{2^*-2} v_n^2 \xi dx \\ &= t_n^{p-2} \left[\int_{\mathbb{R}^N} a |v_n|^p \frac{g(t_n v_n)}{|t_n v_n|^p} t_n v_n \xi dx \right] + o_n(1). \end{aligned} \quad (3.59)$$

But, since $|t_n v_n(x)| \rightarrow \infty$, for $x \in F$, as $n \rightarrow \infty$, using (1.8), the growth conditions of g , (3.54), and the Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} a |v_n|^p \frac{g(t_n v_n)}{|t_n v_n|^p} t_n v_n \xi dx \longrightarrow \int_{\mathbb{R}^N} a |v|^p \xi dx \geq \int_{\text{supp } \xi} a |v|^p dx > 0, \quad (3.60)$$

as $n \rightarrow \infty$. Observe that the left-hand side integrals in equality (3.59) are all bounded, but on the other hand, passing to the limit as $n \rightarrow \infty$ in (3.59), the right-hand side goes to ∞ , since (3.60) holds. This is a contradiction. Hence, $v \equiv 0$ on Ω^+ . A similar reasoning yields that $v \equiv 0$ on Ω^- . $\square$

This completes the proof of the proposition. $\square$

4. Proof of Theorem 1.1

By Proposition 3.3, we may assume that $u_n \rightharpoonup u_0$. Before proceeding further in order to prove that u_0 is the wanted positive solution, we firstly assume for a while three facts that we will prove later.

(1)

$$\int_{\mathbb{R}^N} ag(u_n) v dx \longrightarrow \int_{\mathbb{R}^N} ag(u) v dx, \quad \forall v \in E, \text{ as } n \longrightarrow \infty. \quad (4.1)$$

(2) and (3) If $u_0 \equiv 0$, then

$$\int_{\mathbb{R}^N} ag(u_n) u_n dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad (4.2)$$

$$\int_{\mathbb{R}^N} aG(u_n) dx \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \quad (4.3)$$

By a Brézis-Lieb result (see [9]), we have that

$$\int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n v \, dx \longrightarrow \int_{\mathbb{R}^N} |u_0|^{2^*-2} u_0 v \, dx, \quad \text{as } n \longrightarrow \infty. \quad (4.4)$$

Hence, by (4.1), passing to the limit in (3.14), we achieve that

$$\langle \Psi'(u_0), v \rangle = 0, \quad \forall v \in E, \quad (4.5)$$

that is, u_0 is a weak solution for (1.5).

To see that $u_0 \neq 0$, suppose on the contrary, that $u_0 \equiv 0$. If

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \longrightarrow l, \quad \text{as } n \longrightarrow \infty, \quad (4.6)$$

then by (4.2) and (3.14) with $v = u_n$, we get

$$\int_{\mathbb{R}^N} |u_n|^{2^*} \, dx \longrightarrow l, \quad \text{as } n \longrightarrow \infty. \quad (4.7)$$

Using (4.3), (4.6), (4.7) and passing to the limit in (3.13) yields

$$l = Nc > 0 \quad (4.8)$$

with the choice that

$$c < \frac{1}{N} S^{N/2}, \quad (4.9)$$

since (3.16) holds.

Passing to the limit in definition (3.15) with u_n , and regarding (4.6) and (4.7), we get that $l \geq S^{N/2}$. But this inequality contradicts (4.9).

Proof of (4.1). Using (1.7) and (1.8), we see that for a given $\varepsilon > 0$,

$$|g(s)| \leq \varepsilon |s| + C |s|^{p-1}, \quad \forall s \in \mathbb{R}, \quad (4.10)$$

for some $C > 0$. Hence, combining (4.10), (1.11), and a similar reasoning for u_n such as that made in (3.54), there exists $h \in L^t(U)$, $U \subset \mathbb{R}^N$, $1 \leq t < 2^*$, such that

$$|ag(u)| \leq C \left(\frac{|h|v}{|x|^\alpha} + \frac{|h|^{p-1}v}{|x|^\alpha} \right) \in L^1(B_R(0)), \quad \text{for some } R, C > 0. \quad (4.11)$$

Thus, applying the Lebesgue dominated convergence theorem yields

$$\int_{|x| \leq R} ag(u_n) v \, dx \longrightarrow \int_{|x| \leq R} ag(u) v \, dx, \quad \text{as } n \longrightarrow \infty. \quad (4.12)$$

The proof finishes if we prove that

$$\lim_{R \rightarrow \infty} \int_{|x| > R} |ag(u_n)v| dx = 0, \quad \text{uniformly in } n. \quad (4.13)$$

By (1.12), (2.2), (4.10), and Hölder's inequality, we have

$$\begin{aligned} \int_{|x| > R} |ag(u_n)v| dx &\leq \varepsilon \int_{|x| > R} |a||u_n||v| dx + \int_{|x| > R} |a||u_n|^{p-1}|v| dx \\ &\leq \varepsilon C \|u_n\| \|v\| + C \left(\int_{|x| > R} |a|^r dx \right)^{1/r} \|u_n\|^{p-1} \|v\|^{2^*}, \end{aligned} \quad (4.14)$$

where $r = 2^*/(2^* - p)$. Since the sequence (u_n) is bounded in E norm, if $R > 0$ is chosen in the above inequality, such that

$$\left(\int_{|x| > R} |a|^r dx \right)^{1/r} < \varepsilon, \quad (4.15)$$

we assure that (4.13) holds. $\square$

Proof of (4.2) and (4.3). The proof is made using similar reasoning as those made in the previous proof. $\square$

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