QUASILINEAR DEGENERATE ELLIPTIC UNILATERAL PROBLEMS

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We will be concerned with the existence result of a degenerate elliptic unilateral problem of the form $Au + H(x, u, \nabla u) = f$, where A is a Leray-Lions operator from $W^{1,p}(\Omega, w)$ into its dual. On the nonlinear lower-order term $H(x, u, \nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, but without assuming the sign condition. The right-hand side f belongs to $L^1(\Omega)$.

1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, let p be a real number such that $1 < p < \infty$, and let $w = \{w_i(x), 0 \leq i \leq N\}$ be a vector of weight functions on Ω , that is, each $w_i(x)$ is a measurable a.e. strictly positive function on Ω , satisfying some integrability conditions. And let $Au = -\operatorname{div}(a(x, u, \nabla u))$ be a Leray-Lions operator defined from the weighted Sobolev space $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$.

The aim of this paper is to study the degenerate unilateral problems associated to a nonlinear operator of the form

$$\begin{aligned} -\operatorname{div}(a(x, u, \nabla u)) + H(x, u, \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where H is a nonlinear lower-order term having natural growth with respect to $|\nabla u|$. With respect to $|u|$ we do not assume any growth restrictions (i.e., $H(x, s, \xi) \leq \gamma(x) + g(s) \sum_{i=1}^N w_i |\xi_i|^p$). The right-hand side f belongs to $L^1(\Omega)$. More precisely, we prove the existence of solutions for the following nonlinear Dirichlet problems:

$$\begin{aligned} u &\geq \psi \quad \text{a.e. in } \Omega, \\ T_k(u) &\in W_0^{1,p}(\Omega, w) \quad \forall k > 0, \\ \int_{\Omega} (a(x, u, \nabla u)) T_k(u - v) + \int_{\Omega} H(x, u, \nabla u) T_k(u - v) & \\ &\leq \int_{\Omega} f T_k(u - v) \quad \forall v \in K_{\psi} \cap L^{\infty}(\Omega), \end{aligned} \tag{P}$$

where $K_\psi = \{u \in W_0^{1,p}(\Omega, w), u \geq \psi \text{ a.e. in } \Omega\}$ and T_k is the usual truncation operator. Note that the existence result is proved without assuming the sign condition $H(x, s, \xi)s \geq 0$. For that we prove the strong convergence of the truncations $T_k(u_n)$ in $W_0^{1,p}(\Omega, w)$, where u_n is a solution of the approximate problem. If we take $\psi = -\infty$, we obtain the existence result of problem (1.1) in the case of equation.

Recently in [9] Porretta studied problem (1.1) in the classical Sobolev space $W_0^{1,p}(\Omega)$ where the right-hand side is a measure. We point out that another work in this direction can be found in [6] where problem (1.1) is studied with $f \in L^m(\Omega)$, for which the authors have proved that there exists a bounded weak solution for $m > (N/2)$, and an unbounded entropy solution for $(N/2) > m > (2N/(N+2))$. A different approach (without using the sign condition) was introduced also in [5] when $b(x, s, \xi) = \lambda s - |\xi|^2$ with $\lambda > 0$. Our paper can be seen as a generalization of [9] in the weighted case and as a continuation of [1, 2, 3, 4] where in [1] the case of degenerate variational equation is treated and in [2, 3] the variational degenerate inequalities are studied, while in [4] the authors were concerned with problem (1.1) with the right-hand side f assumed to belong either to $W^{-1,p'}(\Omega, w^*)$ or to $L^1(\Omega)$. In [1, 2, 3, 4] we suppose that the lower-order term satisfies the sign condition $H(x, s, \xi)s \geq 0$ for all $s \in \mathbb{R}$, while in [4] (where $f \in L^1(\Omega)$), we have assumed also the exact natural growth, that is, $|H(x, s, \xi)| \geq \rho \sum_{i=1}^N w_i |\xi_i|^p$.

2. Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 1$). Let $1 < p < \infty$, and let $w = \{w_i(x); i = 1, \dots, N\}$, $0 \leq i \leq N$, be a vector of weight functions, that is, every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω . Further, we suppose in all our considerations that, for $0 \leq i \leq N$,

$$w_i \in L_{\text{loc}}^1(\Omega), \quad w_i^{-1/(p-1)} \in L_{\text{loc}}^1(\Omega). \quad (2.1)$$

We define the weighted space with weight γ in Ω as

$$L^p(\Omega, \gamma) = \{u(x), u\gamma^{1/p} \in L^p(\Omega)\}, \quad (2.2)$$

which is endowed with the norm

$$\|u\|_{p,\gamma} = \left(\int_{\Omega} |u(x)|^p \gamma(x) dx \right)^{1/p}. \quad (2.3)$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \forall i = 1, \dots, N. \quad (2.4)$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_{\Omega} |u(x)|^p w_0 dx + \sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p}. \quad (2.5)$$

To deal with the Dirichlet problem, we use the space

$$X = W_0^{1,p}(\Omega, w), \quad (2.6)$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.5). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|\cdot\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}$, $i = 1, \dots, N$, and p' is the conjugate of p , that is, $p' = p/(p-1)$. For more details we refer the reader to [7].

3. Main general results

3.1. Basic assumptions and statement of result. We state the following assumptions.

(H₁) The expression

$$\| |u| \|_X = \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p}, \quad (3.1)$$

is a norm defined on X and is equivalent to the norm (2.5). Note that $(X, \|u\|_X)$ is a uniformly convex (and reflexive) Banach space.

(H₂) There exist a weight function σ on Ω and a parameter q , satisfying

$$1 < q < p + p', \quad (3.2)$$

$$\sigma^{1-q'} \in L_{\text{loc}}^1(\Omega), \quad (3.3)$$

with $q' = q/(q-1)$ and such that the Hardy inequality

$$\left(\int_{\Omega} |u|^q \sigma(x) dx \right)^{1/q} \leq C \left(\sum_{i=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p} \quad (3.4)$$

holds for every $u \in X$ with a constant $C > 0$ independent of u . Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^q(\Omega, \sigma) \quad (3.5)$$

determined by inequality (3.4) is compact.

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On the other hand, we consider the nonlinear elliptic differential operator in divergence form, defined from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ by

$$Au = -\operatorname{div}(a(x, u, \nabla u)), \quad (3.6)$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying the following conditions.

For all $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ and for almost every $x \in \Omega$,

$$|a_i(x, s, \xi)| \leq \beta w_i^{1/p}(x) \left[k(x) + \sigma^{1/p'} |s|^{q/p'} + \sum_{j=1}^N w_j^{1/p'}(x) |\xi_j|^{p-1} \right] \quad \text{for } i = 1, \dots, N, \quad (3.7)$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \quad \forall \xi \neq \eta, \quad (3.8)$$

$$a(x, s, \xi) \xi \geq \alpha \sum_{i=1}^N w_i(x) |\xi_i|^p, \quad (3.9)$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are positive constants.

Furthermore, let $H(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the growth condition

$$|H(x, s, \xi)| \leq \gamma(s) + g(s) \sum_{i=1}^N w_i(x) |\xi_i|^p \quad (3.10)$$

is satisfied, where $g : \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x)$ belongs to $L^1(\Omega)$.

Finally, let the convex set

$$K_\psi = \{u \in W_0^{1,p}(\Omega, w), u \geq \psi \text{ a.e. in } \Omega\}. \quad (3.11)$$

We will prove the following existence theorem.

THEOREM 3.1. *Assume that the assumptions (H_1) , (H_2) , and (3.7)–(3.11) hold and let f belong to $L^1(\Omega)$. Then, there exists a measurable function u which is a solution of the following problem:*

$$\begin{aligned} u &> \psi \quad \text{a.e. in } \Omega, \\ T_k(u) &\in W_0^{1,p}(\Omega, w), \\ \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - v) dx &+ \int_{\Omega} H(x, u, \nabla u) T_k(u - v) dx \\ &\leq \int_{\Omega} f T_k(u - v) dx \quad \forall v \in K_\psi \cap L^\infty(\Omega) \quad \forall k > 0. \end{aligned} \quad (\text{P})$$

Remark 3.2. The statement of Theorem 3.1 generalizes in the weighted case the analogous one in [9] with $\mu \in L^1(\Omega)$ (where the case of equation is treated).

Remark 3.3. We remark that in the case of $\psi = -\infty$, Theorem 3.1 gives the existence of solution in the case of equation, that is, the problem

$$\begin{aligned} T_k(u) &\in W_0^{1,p}(\Omega, w), \quad k > 0, \\ \int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - v) dx + \int_{\Omega} H(x, u, \nabla u) T_k(u - v) dx \\ &\leq \int_{\Omega} f T_k(u - v) dx \quad \forall v \in W_0^{1,p}(\Omega, w) \cap L^{\infty}(\Omega) \end{aligned} \quad (3.12)$$

has at least one solution.

3.2. Approximate problem. Let Ω_n be a sequence of compact subsets of Ω such that Ω_n is increasing to Ω as $n \rightarrow \infty$.

We consider the sequence of approximate problem

$$\begin{aligned} u_n &\in K_{\psi}, \\ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (u_n - v) dx + \int_{\Omega} H_n(x, u_n, \nabla u_n) (u_n - v) dx \\ &\leq \int_{\Omega} f_n (u_n - v) dx \quad \forall v \in K_{\psi}, \end{aligned} \quad (\mathcal{P}_n)$$

where f_n are regular functions which strongly converge to f in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ and

$$H_n(x, s, \xi) = \frac{H(x, s, \xi)}{1 + (1/n) |H(x, s, \xi)|} \chi_{\Omega_n}, \quad (3.13)$$

where χ_{Ω_n} is the characteristic function of Ω_n .

Note that $|H_n(x, s, \xi)| \leq |H(x, s, \xi)|$ and $|H_n(x, s, \xi)| \leq n$.

PROPOSITION 3.4. Assume that the assumptions (H_1) , (H_2) , and (3.7)–(3.11) hold. Then, the problem $(\mathcal{P}_n)$ has at least one solution $u_n \in K_{\psi}$.

Proof. We define the operator $G_n : X \rightarrow X^*$ by

$$\langle G_n u, v \rangle = \int_{\Omega} H_n(x, u, \nabla u) v dx. \quad (3.14)$$

Thanks to Hölder's inequality, we have, for all $u \in X$ and all $v \in X$,

$$\begin{aligned} \left| \int_{\Omega} H_n(x, u, \nabla u) v dx \right| &\leq \left(\int_{\Omega} |H_n(x, u, \nabla u)|^{q'} \sigma^{-q'/q} dx \right)^{1/q'} \left(\int_{\Omega} |v|^q \sigma dx \right)^{1/q} \\ &\leq n \left(\int_{\Omega_n} \sigma^{1-q'} dx \right)^{1/q'} \|v\|_{q, \sigma} \\ &\leq c_n \|v\|. \end{aligned} \quad (3.15)$$

The last inequality is due to (3.3) and (3.5). Consequently, in view of [3, Lemma 4.1], we deduce that the operator $B_n = A + G_n$ is pseudomonotone.

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On the other hand, we show that B_n is coercive, in the following sense: there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_n v, v - v_0 \rangle}{|||v|||} \longrightarrow +\infty \quad \text{if } |||v||| \longrightarrow \infty, \quad v \in K_\psi. \quad (3.16)$$

Let $v_0 \in K_\psi$. From Hölder's inequality, the growth condition (3.7), and the compact imbedding (3.5), we have

$$\begin{aligned} \langle Av, v_0 \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, v, \nabla v) \frac{\partial v_0}{\partial x_i} dx \\ &\leq \sum_{i=1}^N \left(\int_{\Omega} |a_i(x, v, \nabla v)|^{p'} w_i^{-p'/p} dx \right)^{1/p'} \left(\int_{\Omega} \left| \frac{\partial v_0}{\partial x_i} \right|^p w_i dx \right)^{1/p} \\ &\leq c_1 |||v_0||| \left(\int_{\Omega} k(x)^{p'} + |v|^q \sigma + \sum_{j=1}^N \left| \frac{\partial v}{\partial x_j} \right|^p w_j dx \right)^{1/p'} \\ &\leq c_2 (c_3 + |||v|||^{q/p'} + |||v|||^{p-1}), \end{aligned} \quad (3.17)$$

where c_i are various constants.

Thanks to (3.9), we obtain

$$\frac{\langle Av, v \rangle}{|||v|||} - \frac{\langle Av, v_0 \rangle}{|||v|||} \geq \alpha |||v|||^{p-1} - |||v|||^{p-2} - |||v|||^{(q/p')-1} - \frac{c}{|||v|||}. \quad (3.18)$$

In view of (3.2), we have $p-1 > (q/p')-1$, then,

$$\frac{\langle Av, v - v_0 \rangle}{|||v|||} \longrightarrow \infty \quad \text{as } |||v||| \longrightarrow \infty. \quad (3.19)$$

On the other hand,

$$\begin{aligned} |\langle G_n v, v \rangle| &= \left| \int_{\Omega_n} H_n(x, v, \nabla v) v dx \right| \\ &\leq \left(\int_{\Omega_n} |H_n(x, v, \nabla v)|^{q'} \sigma^{-q'/q} dx \right)^{1/q'} \left(\int_{\Omega_n} |v|^q \sigma dx \right)^{1/q} \\ &\leq n \left(\int_{\Omega_n} \sigma^{-q'/q} dx \right)^{1/q'} |||v||| \\ &\leq c_n |||v|||, \end{aligned} \quad (3.20)$$

hence (since $\langle G_n v, v \rangle / |||v|||$ and $\langle G_n v, v_0 \rangle$ are bounded), we have

$$\frac{\langle B_n v, v - v_0 \rangle}{|||v|||} = \frac{\langle Av, v - v_0 \rangle}{|||v|||} + \frac{\langle G_n v, v \rangle}{|||v|||} - \frac{\langle G_n v, v_0 \rangle}{|||v|||} \longrightarrow \infty \quad \text{as } |||v||| \rightarrow \infty. \quad (3.21)$$

Finally, B_n is pseudomonotone and coercive, hence by using [8, Chapter 2, Theorem 8.2], the approximate problem $(\mathcal{P}_n)$ has at least one solution. $\square$

3.3. A priori estimate

PROPOSITION 3.5. Assume that the assumptions (H_1) , (H_2) , and (3.7)–(3.11) hold, and let u_n be a solution of the approximate problem $(\mathcal{P}_n)$. Then, there exists a constant c (which does not depend on the n and k) such that

$$\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p w_i dx \leq ck, \quad (3.22)$$

where $k > 0$.

Proof. Let $v = u_n - \eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)$, where η is a real positive and $G(s) = \int_0^s (g(t)/\alpha) dt$ (note that the function g is the one that appeared in (3.10)). Since $v \in W_0^{1,p}(\Omega, w)$ and for η small enough, we have $v \geq \psi$, thus v is an admissible test function in $(\mathcal{P}_n)$, then

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) (\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)) dx \\ & \leq \int_{\Omega} f_n(\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)) dx \end{aligned} \quad (3.23)$$

which implies that

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (\exp(G(u_n)) T_k(u_n^+ - \psi^+)) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \leq \int_{\Omega} f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx. \end{aligned} \quad (3.24)$$

Then,

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n^+ - \psi^+) \exp(G(u_n)) dx \\ & \leq - \int_{\Omega} H_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \quad + \int_{\Omega} f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \leq \int_{\Omega} \gamma(x) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \quad + \int_{\Omega} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \quad + \int_{\Omega} f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx. \end{aligned} \quad (3.25)$$

In view of (3.9), we obtain

$$\begin{aligned}
& \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n^+ - \psi^+) \exp(G(u_n)) dx \\
& \leq \int_{\Omega} \gamma(x) \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
& \quad + \int_{\Omega} f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\
& \leq c_1 k,
\end{aligned} \tag{3.26}$$

where c_1 is a positive constant and does not depend on n .

Consequently, we have

$$\begin{aligned}
& \int_{\{|u_n^+ - \psi^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n^+ \exp(G(u_n)) dx \\
& \leq \int_{\{|u_n^+ - \psi^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla \psi^+ \exp(G(u_n)) dx + c_1 k.
\end{aligned} \tag{3.27}$$

Thanks to (3.9) and Young's inequality, we obtain

$$\int_{\{|u_n^+ - \psi^+| \leq k\}} \sum_{i=1}^N \left| \frac{\partial u_n^+}{\partial x_i} \right|^p w_i dx \leq c_2 k. \tag{3.28}$$

Since $\{x \in \Omega, |u_n^+| \leq k\} \subset \{x \in \Omega, |u_n^+ - \psi^+| \leq k + \|\psi^+\|_{\infty}\}$, hence

$$\begin{aligned}
& \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(u_n^+)}{\partial x_i} \right|^p w_i dx \\
& = \int_{\{|u_n^+| \leq k\}} \sum_{i=1}^N \left| \frac{\partial u_n^+}{\partial x_i} \right|^p w_i dx \leq \int_{\{|u_n^+ - \psi^+| \leq k + \|\psi^+\|_{\infty}\}} \sum_{i=1}^N \left| \frac{\partial u_n^+}{\partial x_i} \right|^p w_i dx,
\end{aligned} \tag{3.29}$$

from which in addition to (3.10) we deduce that

$$\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(u_n^+)}{\partial x_i} \right|^p w_i dx \leq c_3 k \quad \forall k > 0, \tag{3.30}$$

where c_3 is a positive constant.

On the other hand, taking $v = u_n + \exp(-G(u_n))T_k(u_n^-)$ as the test function in $(\mathcal{P}_n)$, we obtain

$$\begin{aligned}
& - \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (\eta \exp(-G(u_n)) T_k(u_n^-)) dx \\
& \quad - \int_{\Omega} H_n(x, u_n, \nabla u_n) (\eta \exp(-G(u_n)) T_k(u_n^-)) dx \\
& \leq - \int_{\Omega} f_n (\eta \exp(-G(u_n)) T_k(u_n^-)) dx.
\end{aligned} \tag{3.31}$$

Using (3.10), we can write

$$\begin{aligned}
& \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(-G(u_n)) T_k(u_n^-) dx \\
& - \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n^-) \exp(-G(u_n)) dx \\
& \leq \int_{\Omega} \gamma(x) \exp(-G(u_n)) T_k(u_n^-) dx \\
& + \int_{\Omega} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \exp(-G(u_n)) T_k(u_n^-) dx \\
& - \int_{\Omega} f_n \exp(-G(u_n)) T_k(u_n^-) dx.
\end{aligned} \tag{3.32}$$

In virtue of (3.9) and since the functions γ and f_n lie in $L^1(\Omega)$ with $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$, we get

$$\begin{aligned}
& - \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n^-) \exp(-G(u_n)) dx \\
& = \int_{\{u_n \leq 0\}} a(x, u_n, \nabla u_n) \nabla T_k(u_n) \exp(-G(u_n)) dx \\
& \leq c_3 k.
\end{aligned} \tag{3.33}$$

By using, again, (3.9), we deduce that

$$\int_{\{u_n \leq 0\}} \sum_{i=1}^N \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p w_i dx \leq c_3 k, \tag{3.34}$$

that is,

$$\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(u_n^-)}{\partial x_i} \right|^p w_i dx \leq c_4 k, \tag{3.35}$$

where c_4 is a positive constant.

Combining (3.30) and (3.35), we conclude (3.22). $\square$

3.4. Strong convergence of truncations

PROPOSITION 3.6. *Let u_n be solutions of the problems $(\mathcal{P}_n)$, then there exists a measurable function u such that*

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega, w). \tag{3.36}$$

Proof.

Step 1. We prove that u_n converges to some function u locally in measure (and therefore we can always assume that the convergence is a.e. in Ω after passing to a suitable subsequence). We will show that u_n is a Cauchy sequence in measure in any ball B_R .

On the one hand, thanks to (3.22), we have

$$\begin{aligned} T_k(u_n) &\hookrightarrow v_k \quad \text{weakly in } W_0^{1,p}(\Omega, w), \\ T_k(u_n) &\longrightarrow v_k \quad \text{strongly in } L^q(\Omega, \sigma) \text{ and a.e. in } \Omega, \end{aligned} \quad (3.37)$$

for some function $v_k \in W_0^{1,p}(\Omega, w)$.

On the other hand, letting $k > 0$ large enough, we have

$$\begin{aligned} k \operatorname{meas}(\{|u_n| > k\} \cap B_R) &= \int_{\{|u_n| > k\} \cap B_R} |T_k(u_n)| dx \leq \int_{B_R} |T_k(u_n)| dx \\ &\leq \left(\int_{\Omega} |T_k(u_n)|^q \sigma dx \right)^{1/q} \left(\int_{B_R} \sigma^{1-q'} dx \right)^{1/q'} \\ &\leq c_R \left(\int_{\Omega} \sum_{i=1}^N \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p} \quad (\text{due to (3.5)}) \\ &\leq ck^{1/p} \quad (\text{due to (3.22)}) \end{aligned} \quad (3.38)$$

which implies that

$$\operatorname{meas}(\{|u_n| > k\} \cap B_R) \leq \frac{c}{k^{1-(1/p)}} \quad \forall k > 1. \quad (3.39)$$

For every $\delta > 0$, we have

$$\begin{aligned} &\operatorname{meas}(\{|u_n - u_m| > \delta\} \cap B_R) \\ &\leq \operatorname{meas}(\{|u_n| > k\} \cap B_R) + \operatorname{meas}(\{|u_m| > k\} \cap B_R) \\ &\quad + \operatorname{meas}(\{|T_k(u_n) - T_k(u_m)| > \delta\}). \end{aligned} \quad (3.40)$$

Let $\varepsilon > 0$, combining (3.37), (3.39), and (3.40), we can deduce that there exists some $k(\varepsilon) > 0$ such that $\operatorname{meas}(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta, R)$. This proves that (u_n) is a Cauchy sequence in measure in B_R , thus it converges almost everywhere to some measurable function u . Then by using (3.37), we have

$$\begin{aligned} T_k(u_n) &\hookrightarrow T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega, w), \\ T_k(u_n) &\longrightarrow T_k(u) \quad \text{strongly in } L^q(\Omega, \sigma) \text{ and a.e. in } \Omega. \end{aligned} \quad (3.41)$$

Step 2. We claim that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{j \leq |u_n| \leq j+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \quad (3.42)$$

Consider the function $v = u_n - \eta \exp(G(u_n)) T_1(u_n - T_j(u_n))^+$.

For j large enough and η small enough, we can deduce that $v \geq \psi$, and since $v \in W_0^{1,p}(\Omega, w)$, hence v is a test function in $(\mathcal{P}_n)$. Then, we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \left(\exp(G(u_n)) T_1(u_n - T_j(u_n))^+ \right) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx \\ & \leq \int_{\Omega} f_n \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx. \end{aligned} \quad (3.43)$$

From the growth condition (3.10), we have

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_1(u_n - T_j(u_n))^+ \exp(G(u_n)) dx \\ & \leq \int_{\Omega} \gamma(x) \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx \\ & \quad + \int_{\Omega} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx \\ & \quad + \int_{\Omega} f_n \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx, \end{aligned} \quad (3.44)$$

which gives by using (3.9) the following inequality:

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_1(u_n - T_j(u_n))^+ \exp(G(u_n)) dx \\ & \leq \int_{\Omega} \gamma(x) \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx \\ & \quad + \int_{\Omega} f_n \exp(G(u_n)) T_1(u_n - T_j(u_n))^+ dx. \end{aligned} \quad (3.45)$$

By Lebesgue's theorem the right-hand side goes to zero as n and j tend to infinity. Therefore, passing to the limit firstly in n and secondly in j , we obtain from (3.45)

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{j \leq u_n \leq j+1\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \quad (3.46)$$

On the other hand, taking $v = u_n + \exp(-G(u_n)) T_1(u_n - T_j(u_n))^-$ in $(\mathcal{P}_n)$ (which is an admissible test function), we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla \left(-\exp(-G(u_n)) T_1(u_n - T_j(u_n))^- \right) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) \left(-\exp(-G(u_n)) T_1(u_n - T_j(u_n))^- \right) dx \\ & \leq \int_{\Omega} f_n \left(-\exp(-G(u_n)) T_1(u_n - T_j(u_n))^- \right) dx, \end{aligned} \quad (3.47)$$

which implies that

$$\begin{aligned}
& \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(-G(u_n)) T_1(u_n - T_j(u_n))^- dx \\
& - \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_1(u_n - T_j(u_n))^- \exp(-G(u_n)) dx \\
& \leq \int_{\Omega} H_n(x, u_n, \nabla u_n) \exp(-G(u_n)) T_1(u_n - T_j(u_n))^- dx \\
& - \int_{\Omega} f_n \exp(-G(u_n)) T_1(u_n - T_j(u_n))^- dx.
\end{aligned} \tag{3.48}$$

In virtue of (3.9) and (3.10), it is possible to conclude that

$$\begin{aligned}
& - \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_1(u_n - T_j(u_n))^- \exp(-G(u_n)) dx \\
& \leq \int_{\Omega} \gamma(x) \exp(-G(u_n)) T_1(u_n - T_j(u_n))^- dx \\
& - \int_{\Omega} f_n \exp(-G(u_n)) T_1(u_n - T_j(u_n))^- dx.
\end{aligned} \tag{3.49}$$

The second term in the right-hand side of the previous inequality can be neglected since it is nonnegative, and by Lebesgue's theorem the first term goes to zero as n and j tend to infinity. Then (3.49) becomes

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{-j-1 \leq u_n \leq -j\}} a(x, u_n, \nabla u_n) \nabla u_n dx = 0. \tag{3.50}$$

Finally, (3.42) follows from (3.46) and (3.50).

Step 3. We will show that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) (1 - h_j(u_n)) dx = 0, \tag{3.51}$$

where h_j is a real variable function defined as

$$\begin{aligned}
h_j(s) &= 1 & \text{if } |s| \leq j, \\
h_j(s) &= 0 & \text{if } |s| \geq j+1, \\
h_j(s) &= j+1-s & \text{if } j \leq s \leq j+1, \\
h_j(s) &= s+j+1 & \text{if } -j-1 \leq s \leq -j,
\end{aligned} \tag{3.52}$$

with j a nonnegative real parameter.

Let $v = u_n + \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n))$, v a test function in $(\mathcal{P}_n)$. Then we have

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (-\exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n))) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) (-\exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n))) dx \\ & \leq \int_{\Omega} f_n (-\exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n))) dx. \end{aligned} \quad (3.53)$$

By using (3.10), we have

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n)) dx \\ & \quad - \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n)^- \exp(-G(u_n))(1 - h_j(u_n)) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla h_j(u_n) \exp(-G(u_n))T_k(u_n)^- dx \\ & \quad - \int_{\Omega} \gamma(x) \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n)) dx \\ & \quad - \int_{\Omega} \exp(-G(u_n))g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i T_k(u_n)^-(1 - h_j(u_n)) dx \\ & \leq - \int_{\Omega} f_n \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n)) dx. \end{aligned} \quad (3.54)$$

Thanks to (3.9), we can deduce that

$$\begin{aligned} & - \int_{\{u_n \leq 0\}} a(x, u_n, \nabla u_n) \nabla T_k(u_n) \exp(-G(u_n))(1 - h_j(u_n)) dx \\ & \quad - \int_{\{-j-1 \leq u_n \leq -j\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(-G(u_n))T_k(u_n)^- dx \\ & \quad + \int_{\Omega} \gamma(x) \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n)) dx \\ & \geq \int_{\Omega} f_n \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n)) dx. \end{aligned} \quad (3.55)$$

In view of (3.35), the second integral tends to zero as n and j go to infinity. And by Lebesgue's theorem, it is possible to conclude that the third and fourth integrals converge to zero as n and j go to infinity. Then (3.41) implies that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{u_n \leq 0\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) (1 - h_j(u_n)) dx = 0. \quad (3.56)$$

On the other hand, take $v = u_n - \eta \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n))$ which is an admissible test function in $(\mathcal{P}_n)$, then, we have

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n))) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) (\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n))) dx \\ & \leq \int_{\Omega} f_n (\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n))) dx, \end{aligned} \quad (3.57)$$

which in addition to (3.10) implies that

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n)) dx \\ & \quad + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n^+ - \psi^+) \exp(G(u_n)) (1 - h_j(u_n)) dx \\ & \quad + \int_{\{j \leq u_n \leq j+1\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \leq \int_{\Omega} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n)) dx \\ & \quad + \int_{\Omega} f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n)) dx \\ & \quad + \int_{\Omega} \gamma(x) \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n)) dx, \end{aligned} \quad (3.58)$$

which takes, by using (3.9), the form

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n^+ - \psi^+) \exp(G(u_n)) (1 - h_j(u_n)) dx \\ & \leq - \int_{\{j \leq u_n \leq j+1\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) dx \\ & \quad + \int_{\Omega} \gamma(x) \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n)) dx \\ & \quad + \int_{\Omega} f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+) (1 - h_j(u_n)) dx \\ & = \varepsilon_1(j, n). \end{aligned} \quad (3.59)$$

In virtue of (3.42) and Lebesgue's theorem, we can conclude that $\varepsilon_1(j, n)$ converges to zero as n and j go to infinity.

From (3.59), we have

$$\begin{aligned} & \int_{\{|u_n^+ - \psi^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n^+ \exp(G(u_n)) (1 - h_j(u_n)) dx \\ & \leq \int_{\{|u_n^+ - \psi^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla \psi^+ \exp(G(u_n)) (1 - h_j(u_n)) dx + \varepsilon_1(j, n). \end{aligned} \quad (3.60)$$

Thanks to the growth condition (3.7) and Young's inequality, it is possible to conclude that

$$\int_{\{|u_n^+ - \psi^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n^+ \exp(G(u_n)) (1 - h_j(u_n)) dx \leq \varepsilon_2(j, n), \quad (3.61)$$

where $\varepsilon_2(j, n)$ tends to 0 as n and j go to infinity.

Since $\exp(G(u_n))$ is bounded, then

$$\int_{\{|u_n^+ - \psi^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n^+ \exp(G(u_n)) (1 - h_j(u_n)) dx \leq \varepsilon_3(j, n). \quad (3.62)$$

Since $\{x \in \Omega, |u_n^+| \leq h\} \subset \{x \in \Omega, |u_n^+ - \psi^+| \leq h + \|\psi^+\|_\infty\}$, hence

$$\begin{aligned} & \int_{\{|u_n^+| \leq k\}} a(x, u_n, \nabla u_n) \nabla u_n (1 - h_j(u_n)) dx \\ & \leq \int_{\{|u_n^+ - \psi^+| \leq k + \|\psi^+\|_\infty\}} a(x, u_n, \nabla u_n) \nabla u_n (1 - h_j(u_n)) dx \\ & \leq \varepsilon_3(j, n), \end{aligned} \quad (3.63)$$

which yields, for all $k > 0$,

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{u_n \geq 0\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) (1 - h_j(u_n)) dx = 0. \quad (3.64)$$

Using (3.56) and (3.64), we conclude (3.51).

Step 4. We prove that

$$\begin{aligned} & \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \\ & \quad \times (\nabla T_k(u_n) - \nabla T_k(u)) h_j(u_n) dx = 0. \end{aligned} \quad (3.65)$$

On the one hand, let $v = u_n - \eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)$ with h_j defined as in (3.52) and η small enough such that $v \in K_\psi$, then, we take v as the test function in $(\mathcal{P}_n)$ and we obtain

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (\eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) (\eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)) dx \\ & \leq \int_{\Omega} f_n(\eta \exp(G(u_n))(T_k(u_n) - T_k(u))^+ h_j(u_n)) dx. \end{aligned} \quad (3.66)$$

Similarly, using (3.9) and (3.10), we obtain

$$\begin{aligned}
& \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (T_k(u_n) - T_k(u))^+ \exp(G(u_n)) h_j(u_n) dx \\
& \quad - \int_{\{j \leq u_n \leq j+1\}} a(x, u_n, \nabla u_n) \nabla u_n \exp(G(u_n)) (T_k(u_n) - T_k(u))^+ dx \\
& \leq \int_{\Omega} \gamma(x) \exp(G(u_n)) (T_k(u_n) - T_k(u))^+ h_j(u_n) dx \\
& \quad + \int_{\Omega} f_n \exp(G(u_n)) (T_k(u_n) - T_k(u))^+ h_j(u_n) dx,
\end{aligned} \tag{3.67}$$

that is,

$$\int_{\{T_k(u_n) - T_k(u) \geq 0\}} a(x, u_n, \nabla u_n) \nabla (T_k(u_n) - T_k(u)) \exp(G(u_n)) h_j(u_n) dx \leq \varepsilon_4(j, n). \tag{3.68}$$

Applying again (3.42) and Lebesgue's theorem, we deduce that $\varepsilon_4(j, n)$ goes to zero as n and j tend to infinity. Moreover, (3.68) becomes

$$\begin{aligned}
& \int_{\{T_k(u_n) - T_k(u) \geq 0\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla (T_k(u_n) - T_k(u)) \exp(G(u_n)) h_j(u_n) dx \\
& \quad + \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(u) \exp(G(u_n)) h_j(u_n) dx \\
& \leq \varepsilon_4(j, n).
\end{aligned} \tag{3.69}$$

Since $h_j(u_n) = 0$ if $|u_n| > j + 1$, hence, we obtain

$$\begin{aligned}
& \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a(x, u_n, \nabla u_n) \nabla T_k(u) \exp(G(u_n)) h_j(u_n) dx \\
& = \int_{\{T_k(u_n) - T_k(u) \geq 0, |u_n| > k\}} a(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n)) \nabla T_k(u) \\
& \quad \times \exp(G(u_n)) h_j(u_n) dx \\
& \leq \varepsilon_5(j, n)
\end{aligned} \tag{3.70}$$

which gives

$$\begin{aligned}
& \int_{\{T_k(u_n) - T_k(u) \geq 0\}} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla (T_k(u_n) - T_k(u)) \\
& \quad \times \exp(G(u_n)) h_j(u_n) dx \\
& \leq \varepsilon_6(j, n),
\end{aligned} \tag{3.71}$$

where $\varepsilon_6(j, n) = c(\int_{\{|u_n| > k\}} |a(x, T_{j+1}(u_n), \nabla T_{j+1}(u_n))| |\nabla T_k(u)| \exp(G(u_n)) h_j(u_n) dx + \varepsilon_5(j, n))$ which goes to zero as n and j tend to infinity.

On the other hand, take $v = u_n + \exp(-G(u_n))(T_k(u_n) - T_k(u))^- h_j(u_n)$ as the test function in $(\mathcal{P}_n)$ and, reasoning as in (3.71), it is possible to conclude that

$$\lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\{T_k(u_n) - T_k(u) \leq 0\}} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \times (\nabla T_k(u_n) - \nabla T_k(u)) h_j(u_n) dx = 0. \quad (3.72)$$

Combining (3.71) and (3.72), we deduce (3.65).

Step 5. We show that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega, w). \quad (3.73)$$

Firstly, we have

$$\begin{aligned} & \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) dx \\ &= \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \\ & \quad \times (\nabla T_k(u_n) - \nabla T_k(u)) h_j(u_n) dx \\ &+ \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \\ & \quad \times (\nabla T_k(u_n) - \nabla T_k(u)) (1 - h_j(u_n)) dx. \end{aligned} \quad (3.74)$$

Thanks to (3.65) the first integral of the right-hand side converges to zero as n and j tend to infinity.

For the second term, we have

$$\begin{aligned} & \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) (1 - h_j(u_n)) dx \\ &= \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) (1 - h_j(u_n)) dx \\ & \quad - \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u) (1 - h_j(u_n)) dx \\ & \quad - \int_{\Omega} a(x, T_k(u_n), \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) (1 - h_j(u_n)) dx. \end{aligned} \quad (3.75)$$

By (3.51) the first integral of the right-hand side goes to zero as n and j tend to infinity, and since $(a(x, T_k(u_n), \nabla T_k(u_n)))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega, w_i^*)$ uniformly on n while $\nabla T_k(u)(1 - h_j(u_n))$ converges to zero, hence, the second integral converges to zero. For the third integral, it converges to zero because $\nabla T_k(u_n) - \nabla T_k(u)$ weakly in $\prod_{i=1}^N L^p(\Omega, w_i)$.

Finally, we conclude that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\Omega} (a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))) \\ & \quad \times (\nabla T_k(u_n) - \nabla T_k(u)) dx = 0. \end{aligned} \quad (3.76)$$

Then [1, Lemma 3.1] implies that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega, w). \quad (3.77)$$

This completes the proof of Proposition 3.6. $\square$

3.5. Proof of Theorem 3.1

Let $\varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and taking $v = T_k(u_n - \varphi)$ as test function in $(\mathcal{P}_n)$, we have

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\ & \leq \int_{\Omega} f_n T_k(u_n - \varphi) dx, \end{aligned} \quad (3.78)$$

which implies that

$$\begin{aligned} & \int_{\Omega} a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \nabla T_k(u_n - \varphi) dx \\ & \quad + \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\ & \leq \int_{\Omega} f_n T_k(u_n - \varphi) dx. \end{aligned} \quad (3.79)$$

In view of Proposition 3.6 and the growth condition (3.7), we deduce that

$$\begin{aligned} & a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \\ & \longrightarrow a(x, T_{k+\|\varphi\|_\infty}(u), \nabla T_{k+\|\varphi\|_\infty}(u)) \quad \text{strongly in } \prod_{i=1}^N L^{p'}(\Omega, w_i). \end{aligned} \quad (3.80)$$

Moreover, using, again, Proposition 3.6, we have

$$T_k(u_n - \varphi) \longrightarrow T_k(u - \varphi) \quad \text{strongly in } W_0^{1,p}(\Omega, w). \quad (3.81)$$

Then,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\Omega} a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) T_k(u_n - \varphi) dx \\ & = \int_{\Omega} a(x, T_{k+\|\varphi\|_\infty}(u), \nabla T_{k+\|\varphi\|_\infty}(u)) T_k(u - \varphi) dx. \end{aligned} \quad (3.82)$$

On the other hand, we claim that

$$H_n(x, u_n, \nabla u_n) \longrightarrow H(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \quad (3.83)$$

Let $v = u_n + \exp(-G(u_n)) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds$. Since $v \in W_0^{1,p}(\Omega, w)$ and $v \geq \psi$, hence v is an admissible test function in $(\mathcal{P}_n)$. Then,

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla (-\exp(-G(u_n))) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx \\ & + \int_{\Omega} H_n(x, u_n, \nabla u_n) (-\exp(-G(u_n))) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx \\ & \leq \int_{\Omega} f_n (-\exp(-G(u_n))) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx \end{aligned} \quad (3.84)$$

which implies that

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \frac{g(u_n)}{\alpha} \exp(-G(u_n)) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx \\ & + \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \exp(-G(u_n)) g(u_n) \chi_{\{u_n < -h\}} dx \\ & \leq \int_{\Omega} \gamma(x) \exp(-G(u_n)) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx \\ & + \int_{\Omega} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \exp(-G(u_n)) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx \\ & - \int_{\Omega} f_n \exp(-G(u_n)) \int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds dx. \end{aligned} \quad (3.85)$$

By (3.9) and since $\int_{u_n}^0 g(s) \chi_{\{s < -h\}} ds \leq \int_{-\infty}^{-h} g(s) ds$, we have

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla u_n) \nabla u_n \exp(-G(u_n)) g(u_n) \chi_{\{u_n < -h\}} dx \\ & \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_{-\infty}^{-h} g(s) ds (\|\gamma\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}) \\ & \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right) \int_{-\infty}^{-h} g(s) ds (\|\gamma\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}). \end{aligned} \quad (3.86)$$

Using again (3.9), we obtain

$$\int_{\{u_n < -h\}} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i dx \leq c \int_{-\infty}^{-h} g(s) ds \quad (3.87)$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h \rightarrow +\infty} \sup_{n \in \mathbb{N}} \int_{\{u_n < -h\}} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i dx = 0. \quad (3.88)$$

On the other hand, letting $M = \exp(-G(u_n)) \int_0^{+\infty} g(s) ds$ and $h \geq M + \|\psi^+\|_{L^\infty(\Omega)}$, we consider $v = u_n - \exp(G(u_n)) \int_0^{u_n} g(s) \chi_{\{s>h\}} ds$. Since $v \in W_0^{1,p}(\Omega, w)$ and $v \geq \psi$, v is an admissible test function in $(\mathcal{P}_n)$. Then, similarly as in (3.88), we deduce that

$$\lim_{h \rightarrow +\infty} \sup_{n \in \mathbb{N}} \int_{\{u_n > h\}} g(u_n) \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i dx = 0. \quad (3.89)$$

Combining (3.88), (3.89), Proposition 3.6, and Vitali's theorem, we conclude (3.60).

On the other hand, letting $\varphi \in K_\psi \cap L^\infty(\Omega)$ and taking $v = u_n - T_k(u_n - \varphi)$ as test function in $(\mathcal{P}_n)$, we get

$$\begin{aligned} u_n &\in K_\psi \quad \forall k > 0, \\ \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi) dx &+ \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\ &\leq \int_{\Omega} f_n T_k(u_n - \varphi) dx \quad \forall \varphi \in K_\psi \cap L^\infty(\Omega), \end{aligned} \quad (3.90)$$

which implies that

$$\begin{aligned} u_n &\in K_\psi \quad \forall k > 0, \\ \int_{\Omega} a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \nabla T_k(u_n - \varphi) dx &+ \int_{\Omega} H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi) dx \\ &\leq \int_{\Omega} f_n T_k(u_n - \varphi) dx \quad \forall \varphi \in K_\psi \cap L^\infty(\Omega). \end{aligned} \quad (3.91)$$

Finally, from (3.82) and (3.83), we can pass to the limit in (3.91). This completes the proof of Theorem 3.1.

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