We study the best constant involving the $L^2$ norm of the $p$-derivative solution of Wente's problem in $\mathbb{R}^{2p}$. We prove that this best constant is achieved by the choice of some function $u$. We give also explicitly the expression of this constant in the special case $p = 2$.

1. Introduction and statement of the results

The Wente problem arises in the study of constant mean curvature immersions (see [6]). Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$. Given $u = (a, b)$ be function defined on $\Omega$. Consider the following problem:

$$
-\Delta \psi = \det \nabla u = a_{x_1} b_{x_2} - a_{x_2} b_{x_1} \text{ in } \Omega,
\psi = 0 \text{ on } \partial \Omega,
$$

(1.1)

where $x = (x_1, x_2)$ and $a_{x_i}$ denote the partial derivative with respect to the variable $x_i$, for $i = 1, 2$. If $\Omega = \mathbb{R}^2$, we consider the limit condition $\lim_{|x| \to +\infty} \psi(x) = 0$, where $|x| = r = (x_1^2 + x_2^2)^{1/2}$. When $u = (a, b) \in H^1(\Omega, \mathbb{R}^2)$, it is proven in [7] and [3] that $\psi$, the solution of (1.1) is in $L^\infty(\Omega)$. In particular, this provides control of $\nabla \psi$ in $L^2(\Omega)$ and continuity of $\psi$ by simple arguments. We also have

$$
\|\psi\|_\infty + \|\nabla \psi\|_2 \leq C_0(\Omega) \|\nabla a\|_2 \|\nabla b\|_2.
$$

(1.2)

Denote

$$
C_\infty(\Omega) = \sup_{\nabla a, \nabla b \neq 0} \frac{\|\psi\|_\infty}{\|\nabla a\|_2 \|\nabla b\|_2},
\quad C_1(\Omega) = \sup_{\nabla a, \nabla b \neq 0} \frac{\|\nabla \psi\|_2}{\|\nabla a\|_2 \|\nabla b\|_2}.
$$

(1.3)

It is proved in [1, 5, 7] that $C_\infty(\Omega) = 1/2\pi$ and in [4] that $C_1(\Omega) = \sqrt{3/16\pi}$.
Here, we are interested to study a generalization of problem (1.1) in higher dimensions. More precisely, let $p \in \mathbb{N}^*$ and $u \in W^{1,2p}(\mathbb{R}^{2p}, \mathbb{R}^{2p})$. Consider the following problem:

$$(-\Delta)^p \varphi = \det \nabla u \quad \text{in} \quad \mathbb{R}^{2p},$$

$$\lim_{|x| \to +\infty} \varphi(x) = 0. \quad (1.4)$$

It was proved in [2] that the solution $\varphi$ of (1.4) is in $L^\infty(\mathbb{R}^{2p})$ and $\tilde{\Delta}^{k/2} \varphi$ is in $L^{2p/k}(\mathbb{R}^{2p})$ for $1 \leq k \leq p$, with the following estimates:

$$\|\varphi\|_\infty + \|\tilde{\Delta}^{k/2} \varphi\|_{2p/k} \leq C \|\nabla u\|_{2p}^2, \quad (1.5)$$

where

$$\|\tilde{\Delta}^{k/2} \varphi\|_{2p/k} = \begin{cases} \|\Delta^{k/2} \varphi\|_{2p/k} & \text{if } k \text{ is even}, \\ \|\nabla (\Delta^{(k-1)/2}) \varphi\|_{2p/k} & \text{if } k \text{ is odd}. \end{cases} \quad (1.6)$$

Moreover, the best constant involving the $L^\infty$ norm was determined. Here, we will focus our attention to the quantity $\|\tilde{\Delta}^{p/2} \varphi\|_2$. We will introduce some notations, denote by $B^{2p}$ the unit ball in $\mathbb{R}^{2p}$, $S^{2p}$ the unit sphere in $\mathbb{R}^{2p+1}$ and $\sigma_{2p+1} = \text{vol}(S^{2p})$. Denote $\Psi$ the function defined on $(0, +\infty)$ by

$$\Psi(s) = \frac{1}{s^p} \left( \int_{\mathbb{R}^{2p}} (s |\nabla \varphi|^2 + |\nabla u|^2)^{p} \right)^{2p+1} = \frac{1}{s^p} \left( \sum_{k=0}^{p} C_p^k \|\nabla \varphi\|^k |\nabla u|^{p-k} \right)^{2p+1}. \quad (1.7)$$

Then, there exists a unique $\alpha = \alpha(\nabla \varphi, \nabla u) \in (0, +\infty)$ such that

$$\Psi(\alpha) = \inf_{s \in (0, +\infty)} \Psi(s) \quad (1.8)$$

satisfying

$$\sum_{k=0}^{p} \left[(2p+1)k - p \right] C_p^k \|\nabla \varphi\|^k |\nabla u|^{p-k} \alpha^k = 0. \quad (1.9)$$

Finally, let

$$C_p = \sup_{\nabla u \neq 0} \frac{\|\tilde{\Delta}^{p/2} \varphi\|^2}{\Psi^{1/(2p)}(\alpha)}. \quad (1.10)$$

Our main result is the following theorem.
Theorem 1.1. There exists
\[ C_p = \frac{1}{(2p+1)(2p+1)^{1/2}} \sigma_p. \] (1.11)

Moreover, the best constant \( C_p \) is achieved by a family of one parameter of functions \( \bar{\phi} \) and \( \bar{\psi} \) given by
\[ \bar{\phi}(x) = \frac{2}{(2p)! (1 + cr^2)} \left( \frac{1}{1 + cr^2} \right), \quad \bar{\psi}(x) = \frac{2 \sqrt{\bar{x}x}}{1 + cr^2}, \] (1.12)
where \( c > 0 \) is some arbitrary positive constant.

We can give for example more explicit expression of the best constant in the case where \( p = 2 \). Let \( u \in W^{1,4}(\mathbb{R}^4, \mathbb{R}^4) \) and \( \xi \) is the solution of
\[ \Delta^2 \xi = \det \nabla u \quad \text{in} \quad \mathbb{R}^4, \]
\[ \lim_{|x| \to +\infty} \xi(x) = 0. \] (1.13)
We get that
\[ \Psi(\alpha) = \frac{5^5 \| \nabla u \|_{4}^{12} \left( 5 \| \nabla \xi \| \| \nabla u \|_2^2 + \left( 9 \| \nabla \xi \| \| \nabla u \|_4^4 + 16 \| \nabla \xi \|_4 \| \nabla u \|_4^4 \right)^{1/2} \right)^5}{8^4 \left( 3 \| \nabla \xi \| \| \nabla u \|_2^2 + \left( 9 \| \nabla \xi \| \| \nabla u \|_4^4 + 16 \| \nabla \xi \|_4 \| \nabla u \|_4^4 \right)^{1/2} \right)^3}. \] (1.14)

Corollary 1.2. Let \( \xi \) be a solution of (1.13), then
\[ \sup_{\nabla u \neq 0} \frac{\| \Delta \xi \|_2^3 \left( 3 \| \nabla \xi \| \| \nabla u \|_2^2 + \left( 9 \| \nabla \xi \| \| \nabla u \|_4^4 + 16 \| \nabla \xi \|_4 \| \nabla u \|_4^4 \right)^{1/2} \right)^{3/4}}{8 \left( 15 \pi^2 \right)^{1/4}} = \frac{1}{28} \left( \frac{15}{8\pi^2} \right)^{1/4}, \] (1.15)
and the supremum is achieved by \( \tilde{\xi} \) and \( \tilde{\psi} \) given by
\[ \tilde{\xi}(x) = \frac{1}{12(1 + cr^2)}, \quad \tilde{\psi}(x) = \frac{2 \sqrt{\bar{x}x}}{1 + cr^2}, \] (1.16)
where \( c \) is some arbitrary positive constant.
2. Proof of results

First, we introduce some notations which we will use later. Let \( \Omega \) be a bounded subset of \( \mathbb{R}^n \) and let \( W : \Omega \rightarrow \mathbb{R}^{n+1} \) be a regular function. Denote \( W = (w^1, w^2, \ldots, w^n, w^{n+1}) \) and \( W_i = (w^1, \ldots, w^{i-1}, w^{i+1}, \ldots, w^n, w^{n+1}) \), for \( i = 1, \ldots, n+1 \). Let \( V \) be the algebraic volume of the image of \( W \) in \( \mathbb{R}^{n+1} \) and denote by \( A \) the volume of the boundary of \( V \). Then, we have

\[
V = \frac{1}{n+1} \int_{\Omega} W_1 \times W_2 \times \cdots \times W_n, \quad (2.1)
\]

\[
A = \int_{\Omega} \left| W_1 \times W_2 \times \cdots \times W_n \right|, \quad (2.2)
\]

where \( W_1 \times W_2 \times \cdots \times W_n \) is some vector of \( \mathbb{R}^{n+1} \) given by

\[
W_1 \times W_2 \times \cdots \times W_n = \begin{vmatrix} e_1 & w_1^1 & \cdots & w_{x_1}^1 \\ e_2 & w_1^2 & \cdots & w_{x_2}^2 \\ \vdots & \vdots & \cdots & \vdots \\ e_{n+1} & w_{x_1}^{n+1} & \cdots & w_{x_1}^{n+1} \end{vmatrix} = \sum_{i=1}^{n+1} (-1)^{i-1} \det(\nabla W_i) e_i.
\quad (2.3)
\]

Here \( (e_i)_{1 \leq i \leq n+1} \) is the canonical base of \( \mathbb{R}^{n+1} \). We need the following Lemma.

**Lemma 2.1.** Let \( W : \Omega \rightarrow \mathbb{R}^{n+1} \) defined as above. Suppose that there exist \( 1 \leq i_0 \leq n \) such that \( w^{i_0} = 0 \) on \( \partial \Omega \), then

\[
\int_{\Omega} w^i \det(\nabla W_i) = (-1)^i \int_{\Omega} w^j \det(\nabla W_j), \quad (2.4)
\]

for \( 1 \leq i < j \leq n \).

2.1. Proof of Theorem 1.1. We will suppose that \( u \in C^\infty(\mathbb{R}^{2p}, \mathbb{R}^{2p}) \cap W^{1,2p}(\mathbb{R}^{2p}, \mathbb{R}^{2p}) \). The general case can be obtained by approximating \( u \) by regular functions. Then we define \( W \) in \( \mathbb{R}^{2p+1} \) as follows:

\[
W(x) = (u(x), t\varphi(x)), \quad (2.5)
\]

where \( t \) is a real parameter which will be chosen later. Using (2.4) the algebraic volume closed by the image of \( W \) in \( \mathbb{R}^{2p+1} \) is

\[
V = \int_{\mathbb{R}^{2p}} w^{2p+1} \det(\nabla W_{2p+1}) \, dx = t \int_{\mathbb{R}^{2p}} \varphi \det(\nabla u) \, dx = t \int_{\mathbb{R}^{2p}} \varphi(-\Delta)^p \varphi \, dx. \quad (2.6)
\]

Then we have

\[
V = t\|\tilde{\varphi}^{p/2}\varphi\|_2^2. \quad (2.7)
\]
Next, we will estimate $A$. We have by (2.2)

$$A \leq \int_{\mathbb{R}^{2p}} \prod_{i=1}^{2p} \left( |u_{x_i}|^2 + t^2 \varphi_{x_i}^2 \right)^{1/2}. \quad (2.8)$$

As $(\prod_{i=1}^{n} \alpha_i)^{1/n} \leq 1/n \sum_{i=1}^{n} \alpha_i$, we have

$$A \leq \frac{1}{(2p)^p} \int_{\mathbb{R}^{2p}} \left( \prod_{i=1}^{2p} \left( |u_{x_i}|^2 + t^2 \varphi_{x_i}^2 \right) \right)^p = \frac{1}{(2p)^p} \int_{\mathbb{R}^{2p}} \left( |\nabla u|^2 + t^2 |\nabla \varphi|^2 \right)^p. \quad (2.9)$$

Recall the isoperimetric inequality on a domains $\Omega$ of $\mathbb{R}^{2p+1}$. Denote by $V = \text{Vol}(\Omega)$ and $A = \text{Vol}(\partial \Omega)$, respectively, the volume of $\Omega$ and $\partial \Omega$, then

$$(2p+1)^2 \sigma_{2p+1} V^{2p} \leq A^{2p+1}. \quad (2.10)$$

By (2.7) and (2.9), we have

$$(2p+1)^2 \sigma_{2p+1} V^{2p} \leq A^{2p+1}. \quad (2.11)$$

We conclude that

$$\left\| \tilde{\Delta}^{p/2} \varphi \right\|_2^2 \leq \frac{1}{(2p+1)(2p)^{(2p+1)/2} \sigma_{2p+1}^{1/2p}} \Psi (t^2)^{1/2p}. \quad (2.12)$$

Then we obtain

$$C_p \leq \frac{1}{(2p+1)(2p)^{(2p+1)/2} \sigma_{2p+1}^{1/2p}}. \quad (2.13)$$

Next, we will show that $C_p$ is achieved. We will consider a special case

$$u(x) = g(|x|) x, \quad (2.14)$$

where $g: \mathbb{R}_+ \to \mathbb{R}$ is a regular function which will be chosen later. Since

$$\det \nabla u = \frac{1}{2pr^{2p-1}} \frac{d}{dr} \left( r^{2p} g^{2p}(r) \right), \quad (2.15)$$

then, the solution $\varphi$ of (1.4) is a radial function. Let $\chi$ a general radial function on $\mathbb{R}^{2p}$ and $W(x) = (g(|x|) x, t \chi(|x|))$. After a computation, we can show easily that in this case

$$\left| W_{x_1} \times W_{x_2} \times \cdots \times W_{x_{2p}} \right|^2 \leq g^{4p-2}(r) [g^2(r) + 2rg(r)g'(r) + r^2g^2(r) + t^2 \chi^2(r)] \quad (2.16)$$
The $L^2$ norm in the higher-order Wente problem

and for $1 \leq i \leq 2p$,

$$
|W_{xi}|^2 = g^2(r) + [2rg(r)g'(r) + r^2g''(r) + t^2\chi'(r)] \frac{x^2}{r^2}.
$$

(2.17)

Next, we will suppose that $\chi$ and $g$ satisfy

$$
2rg(r)g'(r) + r^2g''(r) + t^2\chi'(r) = 0.
$$

(2.18)

If we chose $\chi$ as the solution $\phi$ of (1.4) when $u = g(|x|)x$, then by (2.16), (2.17) and under the hypothesis (2.18), the inequality (2.9) becomes an equality. Let now

$$
\tilde{u}(x) = \tilde{g}(|x|)x \quad \text{with} \quad \tilde{g}(r) = \frac{2\sqrt{c}}{1 + cr^2},
$$

(2.19)

where $c > 0$ is some positive constant. Then the solution $\tilde{\phi}$ of (1.4) is given by

$$
\tilde{\phi}(x) = \frac{1}{(2p)!} \frac{2}{1 + cr^2}.
$$

(2.20)

Indeed, the expression of $\Delta^k \phi$, for $1 \leq k \leq p$ is

$$
\Delta^k \tilde{\phi}(r) = \frac{2^{2k+1}(-1)^k k! c^k}{(2p)! (1 + cr^2)^{2k+1}}
$$

$$
\times \left( \prod_{l=0}^{k-1} (p + l) \prod_{l=0}^{k-1} (p - 2 - l) c^k r^{2k} + \sum_{j=1}^{k-1} C^j_k \prod_{l=0}^{k-1} (p + l) \prod_{q=k-j}^{k-1} (p - 2 - q) c^j r^{2j} \right).
$$

(2.21)

Remark that all the coefficients of $r^{2j}$ for $2 \leq j \leq k$ in the expression of $\Delta^k \tilde{\phi}$ have the term $(p - k)$. Also, since

$$
\det \nabla \tilde{u} = \frac{1}{2pr^{2p-1}} \frac{d}{dr} (r^{2p} \tilde{g}^{2p}(r)) = 2^{2p} c^p \frac{1 - cr^2}{(1 + cr^2)^{2p+1}},
$$

(2.22)

so, we have

$$
(-\Delta)^p \tilde{\phi} = \det \nabla \tilde{u} \quad \text{on} \quad \mathbb{R}^{2p}.
$$

(2.23)

If we choose $\tilde{t} = (2p)!$ and $\tilde{\chi}(r) = \tilde{\phi}(r) - 1/(2p)!$, we remark that $\tilde{t}, \tilde{\chi}$ and $\tilde{g}$ satisfy (2.18). Since $\tilde{W} = (\tilde{u}, \tilde{t}\tilde{\chi}) : \mathbb{R}^{2p} \to S^{2p}$ and that the isoperimetric inequality (2.10) becomes equality, then we have

$$
\frac{||\Delta^{p/2} \tilde{\phi}||^2_{L^2}}{\Psi((\tilde{t})^{1/(2p)})} = \frac{1}{(2p + 1)(2p)^{1/(2p+1)^2} d_{2p+1}^{1/(2p)}},
$$

(2.24)

We conclude that $\tilde{\alpha} = \alpha(\nabla \tilde{\phi}, \nabla \tilde{u})$ defined by (1.8) in this case is just $\tilde{\alpha} = ((2p)!)^2$. 
2.2. Proof of Corollary 1.2. Following step by step the proof of Theorem 1.1, we have

\[ A = \int_{\mathbb{R}^4} \left| W_{x_1} \times W_{x_2} \cdots W_{x_4} \right| \leq \frac{1}{16} \left( t^4 \| \nabla \xi \|_4^4 + 2t^2 \| \nabla \xi \|_2^2 \| \nabla u \|_2^2 + \| \nabla u \|_4^4 \right). \]  

(2.25)

Choosing

\[ t^2 = \alpha = \frac{2\| \nabla u \|_4^4}{3\| \nabla \xi \| \nabla u \|_2^2 + \left( 9\| \nabla \xi \| \nabla u \|_2^4 + 16\| \nabla \xi \|_4^4 \| \nabla u \|_4^4 \right)^{1/2}}, \]  

(2.26)

and using the fact that

\[ 4\| \nabla \xi \|_4^4 \alpha^2 + 3\| \nabla \xi \| \nabla u \|_2^2 \alpha - \| \nabla u \|_4^4 = 0, \]  

(2.27)

we have

\[ \Psi(\alpha) = \frac{5^5 \| \nabla u \|_4^4 \left( 5\| \nabla \xi \| \nabla u \|_2^2 + \left( 9\| \nabla \xi \| \nabla u \|_2^4 + 16\| \nabla \xi \|_4^4 \| \nabla u \|_4^4 \right)^{1/2} \right)^5}{8^4 \left( 3\| \nabla \xi \| \nabla u \|_2^2 + \left( 9\| \nabla \xi \| \nabla u \|_2^4 + 16\| \nabla \xi \|_4^4 \| \nabla u \|_4^4 \right)^{1/2} \right)^3}, \]  

(2.28)

and then

\[ \sup_{\nabla u \neq 0} \frac{\| \Delta \xi \|_2^2 \left( 3\| \nabla \xi \| \nabla u \|_2^2 + \left( 9\| \nabla \xi \| \nabla u \|_2^4 + 16\| \nabla \xi \|_4^4 \| \nabla u \|_4^4 \right)^{1/2} \right)^{3/4}}{\| \nabla u \|_4^4 \left( 5\| \nabla \xi \| \nabla u \|_2^2 + \left( 9\| \nabla \xi \| \nabla u \|_2^4 + 16\| \nabla \xi \|_4^4 \| \nabla u \|_4^4 \right)^{1/2} \right)^{5/4}} \leq \frac{1}{2^8} \left( \frac{15}{8\pi^2} \right)^{1/4}. \]  

(2.29)

By taking

\[ \tilde{\xi}(x) = \frac{1}{12(1 + cr^2)}, \quad \tilde{u}(x) = \frac{2\sqrt{c}x}{1 + cr^2}, \]  

(2.30)

we find

\[ \| \nabla \tilde{u} \|_4^4 = \frac{2^6 \times 3 \times \pi^2}{7}, \]  
\[ \| \Delta \tilde{\xi} \|_2^2 = \frac{\pi^2}{3^2 \times 5}, \quad \| \nabla \tilde{\xi} \|_4^4 = \frac{\pi^2}{2^6 \times 3^4 \times 5 \times 7}, \quad \| \| \nabla \tilde{\xi} \| \nabla \tilde{u} \|_2^2 = \frac{11\pi^2}{3^3 \times 5 \times 7}. \]  

(2.31)

Finally (1.15) follows.
References


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