The main aim of this paper is to prove that every non-$\sigma$-lower porous Suslin set in a topologically complete metric space contains a closed non-$\sigma$-lower porous subset. In fact, we prove a general result of this type on “abstract porosities.” This general theorem is also applied to ball small sets in Hilbert spaces and to $\sigma$-cone-supported sets in separable Banach spaces.

1. Introduction

This paper is a continuation of the work done in [9]. We are interested in the following question within the context of $\sigma$-ideals of $\sigma$-porous type.

Let $X$ be a metric space and let $\mathcal{I}$ be a $\sigma$-ideal of subsets of $X$. Let $S \subset X$ be a Suslin set with $S \notin \mathcal{I}$. Does there exist a closed set $F \subset S$ which is not in $\mathcal{I}$?

The answer is positive provided that $X$ is locally compact and $\mathcal{I}$ is a $\sigma$-ideal of $\sigma$-$P$-porous sets, where $P$ is a porosity-like relation satisfying some additional conditions (see the definitions below, and for the precise statement, see [9]). In the case of the $\sigma$-ideal of ordinary (i.e., upper) $\sigma$-porous sets, which satisfies the assumptions of the above-mentioned theorem in any locally compact metric space, even more is true: $X$ can be any topologically complete metric space (see [8]). The proofs are not easy; they use either some amount of descriptive set theory (see [9]) or a quite complicated construction (see [8]).

In this paper, we deal with $\sigma$-ideals of $\sigma$-$P$-porous sets again, but these $\sigma$-ideals are supposed to be generated by closed $P$-porous sets, that is, every $\sigma$-$P$-porous set is covered by countably many closed $P$-porous sets. Note that this property does not hold for ordinary $\sigma$-porous sets but does hold for $\sigma$-lower porous sets. Although we will also work in nonseparable spaces, it turns out that the situation is much simpler than in [9]. Under a simple additional condition on the porosity-like relation $P$, we prove that every such $\sigma$-ideal has the property that every non-$\sigma$-$P$-porous Suslin subset of a topologically complete metric space $X$ contains a closed non-$\sigma$-$P$-porous subset. As the main tool, we use a nonseparable version of Solecki’s theorem proved in [2].
The general result will be applied to the \( \sigma \)-ideals of \( \sigma \)-lower porous sets, of \( \sigma \)-cone-supported sets, and of ball small sets.

2. The general result

We start with notations and definitions. Let \((X, \rho)\) be a metric space. Then the open ball with center \(x \in X\) and radius \(r > 0\) is denoted by \(B(x, r)\). We will use the following terminology from [7, 9]. We say that \(R\) is a point-set relation on \(X\) if it is a relation between points of \(X\) and subsets of \(X\). Thus a point-set relation \(R\) is a subset of \(X \times 2^X\). The symbol \(R(x, A)\), where \(x \in X\) and \(A \subseteq X\), means that \((x, A) \in R\), that is, \(R\) holds for the pair \((x, A)\).

Let \(R\) be a point-set relation on \(X\). If \(A \subseteq X\) and \(B \subseteq X\), then \(R(A, B) \overset{\text{def}}{=} \forall a \in A : R(a, B)\). The point-set relation \(\neg R\) on \(X\) is defined by \((\neg R)(x, A) \iff \neg(R(x, A))\).

We consider the following properties of a point-set relation \(R\) on \(X\).

(A1) If \(A \subseteq B \subseteq X\), \(x \in X\), and \(R(x, B)\), then \(R(x, A)\).

(A2) \(R(x, A)\) if and only if there is \(r > 0\) such that \(R(x, A \cap B(x, r))\).

(A3) \(R(x, A)\) if and only if \(R(x, \overline{A})\).

We say that a point-set relation \(P\) on \(X\) is a porosity-like relation if \(P\) satisfies the “axioms” (A1)–(A3).

Let \(P\) be a porosity-like relation on \(X\). We say that \(A \subseteq X\) is

(i) \(P\)-porous at \(x \in X\) if \(P(x, A)\),

(ii) \(P\)-porous if \(P(x, A)\) for every \(x \in A\),

(iii) \(\sigma\)-\(P\)-porous if \(A\) is a countable union of \(P\)-porous sets.

If \(P\) is a porosity-like relation on \(X\) and \(A \subseteq X\), then the set of all points of \(A\), at which \(P\) is not \(P\)-porous, is denoted by \(N(P, A)\).

The proof of our result is based on the following nonseparable version (see [2, Corollary 3.6 and Remark 3.7]) of Solecki’s theorem (see [3]). We need the following definitions to formulate it.

Let \(\mathcal{A}\) be a system of subsets of a metric space \(X\). We say that \(\mathcal{A}\) is weakly locally determined if \(A \subseteq X\) belongs to \(\mathcal{A}\) whenever for each \(x \in X\) there exists \(a\), not necessarily open, neighbourhood \(U\) of \(x\) such that \(U \cap A \subseteq \mathcal{A}\).

Let \(\mathcal{F}\) be a family of closed subsets of a metric space \(X\). We say that \(\mathcal{F}\) is hereditary if for all sets \(F_1, F_2\) with \(F_1 \subseteq F_2, F_2 \in \mathcal{F}\), we have \(F_1 \in \mathcal{F}\).

PROPOSITION 2.1 (see [2]). Let \(X\) be a topologically complete metric space. Let \(\mathcal{F}\) be a hereditary weakly locally determined system of closed sets. Then each Suslin subset of \(X\) is either covered by countably many elements of \(\mathcal{F}\) or else contains a \(G_\delta\) set \(H\) such that \(H \cap G\) cannot be covered by countably many elements of \(\mathcal{F}\), whenever \(G\) is open and \(G \cap H \neq \emptyset\).

Definition 2.2. Let \(X\) be a metric space and let \(P\) be a porosity-like relation on \(X\). It is said that \(P\) has property \((*)\) if the following condition is satisfied.

\((*)\) If \(H \subseteq X\), \(x \in H'\), and \(H\) is not \(P\)-porous at \(x\), then there exists \(J \subseteq H\) such that \(J' = \{x\}\) and \(J\) is not \(P\)-porous at \(x\).

The symbol \(H'\) stands for the set of all points of accumulation of \(H\).

Now we can formulate our abstract theorem.
Theorem 2.3. Let $X$ be a topologically complete metric space and let $\mathbf{P}$ be a porosity-like relation on $X$ such that $\mathbf{P}$ satisfies $(\ast)$, and each $\sigma$-$\mathbf{P}$-porous set is covered by countably many closed $\mathbf{P}$-porous sets. If $S \subset X$ is a Suslin non-$\sigma$-$\mathbf{P}$-porous set, then there exists a closed non-$\sigma$-$\mathbf{P}$-porous set $F \subset S$.

The next lemma immediately follows by a Baire category argument.

Lemma 2.4. Let $X$ and $\mathbf{P}$ be as in Theorem 2.3. Let $F \subset X$ be a closed nonempty set such that $\mathbf{N}(\mathbf{P}, F)$ is dense in $F$. Then $F$ is not $\sigma$-$\mathbf{P}$-porous.

Proof of Theorem 2.3. We denote the $\sigma$-ideal of all $\sigma$-$\mathbf{P}$-porous sets by $\mathcal{F}$.

The system of all closed $\mathbf{P}$-porous sets is clearly hereditary and weakly locally determined by (A1) and (A2). According to Proposition 2.1, we may and do assume that $S$ is a $G_\delta$ set and $S \cap G \notin \mathcal{F}$ for every open $G \subset X$ intersecting $S$. If there is $x \in S \setminus S'$, then $\{x\} \notin \mathcal{F}$. In this case, $F := \{x\}$ can serve as the set for which we are looking. From now on, we assume that $S \subset S'$. Let $S = \bigcap_{n=1}^{\infty} G_n$, where $\{G_n\}_{n=1}^{\infty}$ is a decreasing sequence of open sets. We will construct a sequence $\{F_n\}_{n=0}^{\infty}$ of closed sets and a decreasing sequence $\{H_n\}_{n=1}^{\infty}$ of open sets such that $F_0 = \emptyset$ and for every $n \in \mathbb{N}$, we have

(a) $\emptyset \neq F_n \subset \mathbf{N}(\mathbf{P}, S)$,
(b) $F'_{n+1} = F_{n+1}$,
(c) $F_n \subset H_n \subset \overline{H_n} \subset G_n$,
(d) $(-\mathbf{P})(F_{n+1}, F_n)$.

We proceed by induction over $n$. Since $S \notin \mathcal{F}$, we can choose $x \in \mathbf{N}(\mathbf{P}, S)$. We put $F_1 = \{x\}$. We easily find an open set $H_1$ such that $x \in H_1$ and $\overline{H_1} \subset G_1$. The sets $F_1$ and $H_1$ satisfy (a)–(d) for $n = 1$.

Assume that we have constructed $F_1, \ldots, F_m$ and $H_1, \ldots, H_m$ such that (a)–(d) hold for $n = 1, \ldots, m$. We find an open set $H_{m+1}$ with $F_m \subset H_{m+1} \subset \overline{H_{m+1}} \subset G_{m+1} \cap H_m$. The set $F_m \setminus F'_m$ is discrete in $X \setminus F'_m$, that is, for every $y \in X \setminus F'_m$, there exists $r > 0$ such that $B(y, r) \cap (F_m \setminus F'_m)$ contains at most one point. It is well known and easy to prove that, for each $z \in F_m \setminus F'_m$, we can choose $r_z > 0$ such that $B = (B(z, r_z))_{z \in F_m \setminus F'_m}$ is discrete in $X \setminus F'_m$, that is, for every $y \in X \setminus F'_m$, there exists $s > 0$ such that, for at most one $z \in F_m \setminus F'_m$, $B(y, s)$ intersects $B(z, r_z)$.

Since $S \cap G \notin \mathcal{F}$ for every open $G$ intersecting $S$, we have that $\mathbf{N}(\mathbf{P}, S)$ is dense in $S$. According to this, (A3), and (a), we have $F_m \subset \mathbf{N}(\mathbf{P}, \mathbf{N}(\mathbf{P}, S))$. Thus using the condition $(\ast)$ and (A2), we find for every $z \in F_m \setminus F'_m$ a set $J_z$ such that $J_z \subset B(z, r_z) \cap H_{m+1} \cap \mathbf{N}(\mathbf{P}, S)$, $(-\mathbf{P})(z, J_z)$, and $J'_z = \{z\}$.

We put $F_m + 1 = F_m \cup \bigcup \{J_z; z \in F_m \setminus F'_m\}$. Clearly, $F_{m+1} \subset \mathbf{N}(\mathbf{P}, S)$ and $F_{m+1} \subset H_{m+1}$. It is easy to see that $F_{m+1} = F_{m+1}$; in particular, $F_{m+1}$ is closed.

Let $x \in F_m$. We distinguish two possibilities. If $x \in F'_m = F_{m-1}$, then $(-\mathbf{P})(x, F_m)$ by the induction hypothesis, and so $(-\mathbf{P})(x, F_{m+1})$ by (A1). If $x \in F_m \setminus F'_m$, then $(-\mathbf{P})(x, J_x)$ and we also have $(-\mathbf{P})(x, F_{m+1})$. We get $(-\mathbf{P})(F_m, F_{m+1})$. Thus the sets $F_{m+1}$ and $H_{m+1}$ satisfy (a)–(d) for $n = m + 1$ and the construction of our sequences is finished.

The desired set $F$ is defined by $F = \bigcup_{n=1}^{\infty} F_n$. Using (c) and the monotonicity of the $H_n$’s, we get $F \subset S$. We have $(-\mathbf{P})(\bigcup_{n=1}^{\infty} F_n, F)$ by (d). The set $\bigcup_{n=1}^{\infty} F_n$ is dense in $F$. Hence $F \notin \mathcal{F}$, by Lemma 2.4. 

\[ \square \]
3. Applications

We will apply Theorem 2.3 to the \( \sigma \)-ideal of \( \sigma \)-lower porous sets (in a topologically complete metric space) and to two of its subsystems: to the \( \sigma \)-ideal of \( \sigma \)-cone-supported sets (in a separable Banach space) and to the \( \sigma \)-ideal of ball small sets (in an arbitrary Hilbert space).

Note that \( \sigma \)-lower porous sets (called frequently simply “\( \sigma \)-porous sets” and sometimes “\( \sigma \)-very porous sets”) were applied in a number of articles on exceptional sets in (sometimes also nonseparable) Banach spaces (cf. [6]). In [6], information on \( \sigma \)-cone-supported and ball small sets can also be found.

To verify condition \((\ast)\) in concrete cases, we will apply the following easy lemma.

**Lemma 3.1.** Let \( g : [0, \infty) \to [0, \infty) \) be a continuous increasing function with \( g(0) = 0 \). Let \((X, \rho)\) be a metric space, \( H \subset X \), and \( a \in H \). Then there exists \( J \subset H \setminus \{a\} \), such that for each \( 0 < r < r_0 \) such that for each \( 0 < r < r_0 \), there exists \( x^* \in J \) such that \( g(\rho(x, x^*)) < \min(\rho(x, a), \rho(x^*, a)) \).

**Proof.** Let \( M_1 := \{x \in X; 1 \leq \rho(x, a)\} \) and \( M_n := \{x \in X; 1/n \leq \rho(x, a) < 1/(n - 1)\} \) for \( n = 2, 3, \ldots \). For each natural \( n \), choose \( \epsilon_n > 0 \) such that \( g(\epsilon_n) < 1/n \) and in \( H \cap M_n \), find a maximal \( \epsilon_n \)-discrete subset \( D_n (\rho(u, v) \geq \epsilon_n \) for each \( u, v \in D_n, u \neq v \)). Put \( J := \bigcup_{n=1}^{\infty} D_n \). Clearly, \( J \subset H \setminus \{a\} \) and \( \epsilon_n \) be given. Find \( n \in \mathbb{N} \) with \( x \in M_n \). By maximality of \( D_n \), we can choose \( x^* \in D_n \subset J \) with \( \rho(x, x^*) < \epsilon_n \). Consequently,

\[
g(\rho(x, x^*)) < g(\epsilon_n) < \frac{1}{n} \leq \min(\rho(x, a), \rho(x^*, a)). \quad (3.1)
\]

**3.1. \( \sigma \)-lower porous sets**

**Definition 3.2.** Let \((X, \rho)\) be a metric space. It is said that \( A \subset X \) is lower porous at \( x \in X \) if there exist \( c > 0 \) and \( r_0 > 0 \) such that for every \( r \in (0, r_0) \), there exists \( y \in B(x, r) \) with \( B(y, cr) \subset B(x, r) \setminus A \). The corresponding porosity-like relation is denoted by \( P_1 \), and \( \sigma \)-\( P_1 \)-porous sets are called \( \sigma \)-lower porous.

It is a well known and an easy fact that the \( \sigma \)-ideal \( \mathcal{J}_1 \) of all \( \sigma \)-lower porous sets is generated by closed \( P_1 \)-porous sets (see, e.g., [6, Proposition 2.5]). The proof of the following lemma is also easy.

**Lemma 3.3.** Let \( X \) be a metric space. Then \( P_1 \) has property \((\ast)\).

**Proof.** Let \( x \in N(P, H) \cap H' \). Put \( g(h) := \sqrt{h} \) (then \( h = o(g(h)) \), \( h \to 0^+ \)) and find \( J \subset H \) by Lemma 3.1. Then \( J' = \{x\} \). We will prove \((\neg P_1)(x, J)\).

Suppose on the contrary that \( J \) is lower porous at \( x \). Then there exist \( c > 0 \) and \( r_0 > 0 \) such that for each \( 0 < r < r_0 \), there exists \( y \in X \) with \( B(y, cr) \subset B(x, r) \setminus J \). We can clearly choose \( r_1 > 0 \) such that \( g(h) > 2h/c \) for each \( 0 < h < r_1 \). Put \( \tilde{r} := \min(r_0, r_1) \), \( \tilde{c} := c/2 \), and consider an arbitrary \( 0 < r < \tilde{r} \). Choose \( y \in X \) such that \( B(y, cr) \subset B(x, r) \setminus J \). To obtain a contradiction with \( x \in N(P, H) \), it is sufficient to show that

\[
B(y, \tilde{c}r) \cap H = \emptyset. \quad (3.2)
\]
Suppose that it is not the case and choose \( z \in B(y, cr) \cap H \). By the choice of \( J \), we can find \( z^* \in J \) such that \( g(\rho(z, z^*)) < \rho(z, x) < r < r_1 \). Since \( \tilde{c} < c \), we have \( z \neq z^* \) and the definition of \( r_1 \) gives \( g(\rho(z, z^*)) > 2\rho(z, z^*)/c \). Consequently, \( \rho(z, z^*) < cr/2 \), which implies that \( z^* \in B(y, cr) \cap J \). This is a contradiction which proves (3.2).

Theorem 2.3 thus implies the following result.

**Corollary 3.4.** Let \( X \) be a topologically complete metric space and let \( S \subset X \) be a Suslin set which is not \( \sigma \)-lower porous. Then there exists a closed \( F \subset S \) which is not \( \sigma \)-lower porous.

**Remark 3.5.** We say that \( A \subset \mathbb{R} \) is lower symmetrically porous at \( x \in \mathbb{R} \) if there exist \( r_0 > 0 \) and \( c > 0 \) such that for each \( 0 < r < r_0 \), there exist \( h > 0 \) and \( t \geq 0 \) such that \( h/r > c \), \( t + h \leq r \), \( (x + t, x + t + h) \cap A = \emptyset \), and \( (x - t - h, x - t) \cap A = \emptyset \). The notions of a lower symmetrically porous set and a \( \sigma \)-lower symmetrically porous set are defined in the obvious way.

Proceeding quite similarly as above, we can easily obtain that each analytic set \( S \subset \mathbb{R} \) which is not \( \sigma \)-lower symmetrically porous contains a closed set which is not \( \sigma \)-lower symmetrically porous.

### 3.2. Cone-supported sets

**Definition 3.6.** If \( X \) is a Banach space, \( v \in X \), \( \|v\| = 1 \), and \( 0 < c < 1 \), then define the cone \( A(v, c) := \bigcup_{\lambda > 0} \lambda \cdot B(v, c) \). Define the (clearly porosity-like) point-set relation \( P_s \) as follows: \( P_s(x, M) \) if there exist \( r > 0 \) and a cone \( A(v, c) \) such that \( M \cap (x + A(v, c)) \cap B(x, r) = \emptyset \). Sets which are \( P_s \)-porous (\( \sigma \)-\( P_s \)-porous) are called cone supported (\( \sigma \)-cone supported).

If \( X \) is separable, it is easy to prove (see [4, Lemma 1], cf. [6]) that \( M \subset X \) is \( \sigma \)-cone supported (i.e., \( \sigma \)-\( P_s \)-porous) if and only if \( M \) can be covered by countably many Lipschitz hypersurfaces. Since each Lipschitz hypersurface is clearly a closed \( P_s \)-porous set, every \( \sigma \)-\( P_s \)-porous set is covered by countably many closed \( P_s \)-porous sets.

**Lemma 3.7.** Let \( X \) be a Banach space. Then \( P_s \) has property (\( \ast \)).

**Proof.** Let \( x \in N(P_s, H) \cap H' \). Put \( g(h) := \sqrt{h} \) and find \( J \subset H \) by Lemma 3.1. Then \( J' = \{x\} \). We will prove \( (P_s)(x, J) \). We can and will suppose that \( x = 0 \).

Suppose on the contrary that \( P_s(0, J) \). Then there exist \( v \in X \), with \( \|v\| = 1 \), \( 1 > c > 0 \), and \( r > 0 \) such that \( J \cap A(v, c) \cap B(0, r) = \emptyset \). We can suppose that \( r < c/4 \). To obtain a contradiction with \( 0 \in N(P_s, H) \), it is sufficient to show that

\[
H \cap A\left(v, \frac{c}{2}\right) \cap B\left(0, \frac{r}{2}\right) = \emptyset.
\]

(3.3)

Suppose that this is not the case and choose \( z \in H \cap A(v, c/2) \cap B(0, r/2) \). By the choice of \( J \), we can find \( z^* \in J \) such that \( \|z - z^*\| \leq \|z\|^2 < \min(r/2, c/4 \cdot \|v\|) \). Thus clearly \( z^* \in B(0, r) \). Choose \( \lambda > 0 \) with \( \|\lambda z - v\| < c/2 \). Then

\[
\|\lambda z^* - v\| \leq \frac{c}{2} + \lambda \|z - z^*\| \leq \frac{c}{2} + \|\lambda z\| \cdot \frac{c}{4} < \frac{c}{2} + \left(1 + \frac{c}{2}\right) \cdot \frac{c}{4} < c,
\]

(3.4)

and thus \( z^* \in A(v, c) \cap B(0, r) \). This is a contradiction which proves (3.3).
Theorem 2.3 thus implies the following result.

**Corollary 3.8.** Let $X$ be a separable Banach space and let $S \subset X$ be an analytic set which cannot be covered by countably many Lipschitz hypersurfaces. Then there exists a closed set $F \subset S$ which cannot be covered by countably many Lipschitz hypersurfaces.

### 3.3. Ball small sets

**Definition 3.9.** Let $X$ be a Banach space and let $r > 0$. It is said that $A \subset X$ is $r$-ball porous at a point $x \in A$ if for each $\varepsilon \in (0,r)$, there exists $y \in X$ such that $\|x - y\| = r$ and $B(y,r - \varepsilon) \cap A = \emptyset$. A set $A \subset X$ is called $r$-ball porous if it is $r$-ball porous at each $x \in A$. It is said that $A \subset X$ is ball small if it can be written in the form $A = \bigcup_{n=1}^{\infty} A_n$, where each $A_n$ is $r_n$-ball porous for some $r_n > 0$.

Using the obvious fact that $B(z, \|z - x\| - \varepsilon) \subset B(y,\rho - \varepsilon)$ whenever $\|y - x\| = \rho > 0$, $z$ lies on the segment $xy$, and $\|z - x\| > \varepsilon > 0$, it is easy to verify the following facts.

- (i) If $A$ is $r$-ball porous at $a$ and $0 < r^* < r$, then $A$ is $r^*$-ball porous at $a$.
- (ii) If $A$ is $r$-ball porous, then $\overline{A}$ is $r/2$-ball porous.

For $A \subset X$ and $x \in X$, we will write $P_b(x,A)$ if $A$ is $r$-ball porous at $x$ for some $r > 0$. Using (i), it is easy to see that $P_b$ is a porosity-like relation on $X$ and that the $\sigma$-ideal $\mathcal{J}_b$ of all ball small sets coincides with the system of all $\sigma$-$P_b$-porous sets.

By (ii), we easily obtain that $\mathcal{J}_b$ is generated by closed $P_b$-porous sets.

The proof of the following lemma is not difficult but slightly technical.

**Lemma 3.10.** Let $X$ be a Hilbert space. Then $P_b$ has property (\(*\)).

**Proof (Sketch).** First, observe that an elementary (two-dimensional) computation gives the following fact.

- (F) If $b, v, x, x^*$ are points of $X$, $\|v\| = 1$, $0 < \rho < 1/10$, $x \in B(b + \rho/2 \cdot v, \rho/2)$, and $\|x^* - x\| \leq 4|b - x|^2$, then $x^* \in B(b + \rho v, \rho)$.

Now let $H \subset X$ and $a \in N(P_b, H) \cap H'$. Put $g(h) := \sqrt{h}$ and find $J \subset H$ by Lemma 3.1. Then $J' = \{a\}$. We will prove \((-P_b)(a,J))$. Suppose to the contrary that $J$ is $r$-ball porous at $a$ for some $r > 0$. By (i), we can suppose that $r < 1/10$. Then for each $0 < \varepsilon < r/4$, there exists $v \in X$ with $\|v\| = 1$ such that $B(a + rv, r - \varepsilon) \cap J = \emptyset$. It is sufficient to prove that

$$B\left(\frac{a + r}{2} \cdot v, r - 2\varepsilon\right) \cap H = \emptyset. \quad (3.5)$$

Then $H$ is $r/2$-ball porous at $a$, a contradiction.

To prove (3.5), suppose on the contrary that there exists $x \in B(a + r/2 \cdot v, r/2 - 2\varepsilon) \cap H$. By the choice of $J$, there exists $x^* \in J$ such that $\|x - x^*\| < \|x - a\|^2$. Denote $b := a + 2\varepsilon v$ and distinguish two cases.

If $\|x - b\| < 2\varepsilon$, then $\|x - a\| < 4\varepsilon$ and therefore $\|x - x^*\| < 16\varepsilon^2 < \varepsilon$ (since $\varepsilon < r/4 < 1/40$). Consequently, $x^* \in B(a + r/2 \cdot v, r/2 - \varepsilon) \subset B(a + rv, r - \varepsilon)$, a contradiction.

If $\|x - b\| \geq 2\varepsilon$, then $\|x - a\| \leq 2\varepsilon + \|x - b\| \leq 2\|x - b\|$ and thus $\|x - x^*\| \leq 4\|x - b\|^2$. Put $\rho := r - 4\varepsilon$. Since $x \in B(b + \rho/2 \cdot v, \rho/2) = B(a + r/2 \cdot v, r/2 - 2\varepsilon)$, fact (F) implies
that

$$x^* \in B(b + \rho v, \rho) = B(a + (r - 2\varepsilon)v, r - 4\varepsilon) \subset B(a + rv, r - \varepsilon), \quad (3.6)$$

a contradiction. \[\square\]

**Corollary 3.11.** Let $X$ be a Hilbert space and let $S \subset X$ be a Suslin set which is not ball small. Then there exists a closed set $F \subset S$ which is not ball small.

Finally, note that Theorem 2.3 can be easily applied also to the system of $\sigma$-cone porous sets in an arbitrary Banach space (by a cone porous set, we mean a set which is $\alpha$-cone porous for some $\alpha > 0$; see [5] for the definition and [1] for some properties of $\alpha$-cone porous sets in Hilbert spaces). On the other hand, it seems that Theorem 2.3 can be applied neither to the (more interesting) related system of cone small sets (cf. [6]) nor to the system of $\sigma$-cone supported sets in nonseparable Banach spaces.

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